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RIGIDITY IN GENERALIZED ISOTHERMAL FLUIDS

RÉMI CARLES*

Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

KLEBER CARRAPATOSO AND MATTHIEU HILLAIRET

Institut Montpelliérain Alexander Grothendieck, CNRS, Univ. Montpellier, France

ABSTRACT. We investigate the long-time behavior of solutions to the isothermal Euler equation. By writing the system with a suitable time-dependent scaling we prove that the densities of global solutions display universal dispersion rate and asymptotic profile. This result extends to Korteweg or quantum Navier Stokes equations, as well as generalizations of these equations where the convex pressure law is asymptotically linear near vacuum.

1. Introduction.

1.1. Isentropic Euler equation: existence of singularities. In the isentropic case $\gamma > 1$, the Euler equation on \mathbb{R}^d , $d \geq 1$,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & \rho|_{t=0} = \rho_0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \kappa \nabla(\rho^\gamma) = 0, & \rho u|_{t=0} = J_0, \end{cases} \quad (1.1)$$

enjoys the formal conservations of mass,

$$M(t) = \int_{\mathbb{R}^d} \rho(t, x) dx \equiv M(0),$$

and entropy (or energy),

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} \rho(t, x) |u(t, x)|^2 dx + \frac{1}{\gamma - 1} \int_{\mathbb{R}^d} \rho(t, x)^\gamma dx \equiv E(0).$$

In general, smooth solutions are defined only locally in time (see [11, 7, 14]). Indeed, as first noticed in [11], considering the new unknown

$$(a, v) = \left(\rho^{\frac{\gamma-1}{2}}, v \right)$$

turns (1.1) into

$$\begin{cases} \partial_t a + v \cdot \nabla a + \frac{\gamma-1}{2} a \operatorname{div} v = 0, & a|_{t=0} = \rho_0^{\frac{\gamma-1}{2}}, \\ \partial_t v + v \cdot \nabla v + \kappa \frac{2\gamma}{\gamma-1} a \nabla a = 0, & v|_{t=0} = \frac{J_0}{\rho_0}. \end{cases} \quad (1.2)$$

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* Corresponding author: Rémi Carles.

This system is hyperbolic symmetric (with a constant symmetrizer), so there exists a unique local solution $(a, u) \in C(0, T; H^s)^{1+d}$, provided that $s > 1 + d/2$ and $\rho_0^{\frac{\gamma-1}{2}}, v_0 = \frac{J_0}{\rho_0} \in H^s(\mathbb{R}^d)$.

It is also proven in [11] (and generalized in [14]) that if $a|_{t=0}$ and $v|_{t=0}$ are smooth and compactly supported, then no matter how small they may be (and unless both are identically zero), the solution to (1.2) will develop a singularity in finite time. The proof relies on two key arguments:

- As long as the solution is smooth, its speed of propagation is zero (for instance, view the equations like ODEs).
- A virial computation shows that if the solution is global, then it is dispersive:

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 \rho(t, x) dx \geq E(t) \inf(2, 3(\gamma - 1)) = E_0 \inf(2, 3(\gamma - 1)) > 0,$$

where we have used (one more time) the assumption $\gamma > 1$ and the conservation of the energy (which is granted in the case of smooth solutions).

Suppose that the solution remains smooth for all time. Integrating the above estimate twice yields

$$\int |x|^2 \rho(t, x) dx \gtrsim t^2.$$

This is incompatible with the fixed compact support of ρ and the conservation of mass, since these properties imply, for some $K > 0$ independent of time,

$$\int |x|^2 \rho(t, x) dx \leq \int_{|x| < K} |x|^2 \rho(t, x) dx \lesssim K^{2d} \int \rho(t, x) dx = K^{2d} M(0).$$

Therefore, a singularity appears in finite time.

1.2. Isentropic Euler equation: some global solutions and their asymptotic behavior. A first global existence of smooth solutions was obtained by D. Serre [12], under an extra geometric assumption involving a special structure for the initial velocity. For $1 < \gamma \leq 1 + 2/d$, change the unknown functions

$$\rho(t, x) = \frac{1}{(1+t)^d} R \left(\frac{t}{1+t}, \frac{x}{1+t} \right), \quad u(t, x) = \frac{1}{1+t} U \left(\frac{t}{1+t}, \frac{x}{1+t} \right) + \frac{x}{1+t},$$

and assume that $R_0^{\frac{\gamma-1}{2}}, U_0 \in H^s$, for some $s > 1 + d/2$. This means $\rho_0^{\frac{\gamma-1}{2}} \in H^s$ (like before), and $u_0(x) - x \in H^s$ (hence $u_0 \notin L^2$).

Theorem 1.1 (D. Serre, [12]). *There exists $\eta > 0$ such that if*

$$\|(\rho_0^{(\gamma-1)/2}, U_0)\|_{H^s(\mathbb{R}^d)} \leq \eta,$$

then there is a unique global solution, in the sense that $(R, U) \in C([0, \infty); H^s(\mathbb{R}^d))^{1+d}$. In addition, there exists $R_\infty, U_\infty \in H^s(\mathbb{R}^d)$ such that

$$\left\| \left(\rho(t, x) - \frac{1}{t^d} R_\infty \left(\frac{x}{t} \right), u(t, x) - \frac{x}{1+t} - \frac{1}{1+t} U_\infty \left(\frac{x}{t} \right) \right) \right\|_{L^\infty(\rho_0^{\frac{\gamma-1}{2}})} \xrightarrow{t \rightarrow \infty} 0.$$

Back to the initial unknown functions, we infer (this is a rather straightforward consequence of the proof in [12], see [6]):

Corollary 1.2. *Let $1 < \gamma \leq 1 + 2/d$ and $s > d/2 + 1$. If $R_\infty, U_\infty \in H^s(\mathbb{R}^d)$ are such that $\|(R_\infty^{(\gamma-1)/2}, U_\infty)\|_{H^s(\mathbb{R}^d)} \leq \eta$, then there exists Cauchy data $\rho_0, u_0 \in H^s(\mathbb{R}^d)$ such that the solution is global in time in the same sense as above, and*

$$\left\| \left(\rho(t, x) - \frac{1}{t^d} R_\infty \left(\frac{x}{t} \right), u(t, x) - \frac{x}{1+t} - \frac{1}{1+t} U_\infty \left(\frac{x}{t} \right) \right) \right\|_{L^\infty(\mathbb{R}^d)} \xrightarrow{t \rightarrow \infty} 0.$$

Some comments are in order:

- The assumptions on the velocity is reminiscent of the “good case” in Burgers’ equation: particles spread out. Generalizations of this result can be found in [9, 8].
- In this regime, the density is dispersive, and dispersive rate is universal,

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{t^d}.$$

- However, the asymptotic profile R_∞ may be any smooth, small function.

1.3. Isothermal case. In the case $\gamma = 1$,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \kappa \nabla \rho = 0, \end{cases} \quad (1.3)$$

with $\kappa > 0$, the mass is still conserved,

$$M(t) = \int_{\mathbb{R}^d} \rho(t, x) dx \equiv M(0),$$

as well as the entropy, which now reads

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} \rho(t, x) |u(t, x)|^2 dx + \kappa \int_{\mathbb{R}^d} \rho(t, x) \ln \rho(t, x) dx \equiv E(0).$$

Now, the energy has no definite sign: no a priori estimate like in the argument of [11] is available. We show that rigidity results are available in this case though, involving a large time behavior in sharp contrast with the case $1 < \gamma \leq 1 + 2/d$ of [12].

2. A large family of explicit solutions. To simplify the presentation, we assume $d = 1$ in this section, and refer to [6] for the general case. Consider

$$\rho_0(x) = b_0 e^{-\alpha_0 x^2}, \quad u_0(x) = \beta_0 x.$$

As noticed by M. Yuen [15], the above structure is preserved by the flow:

$$\rho(t, x) = b(t) e^{-\alpha(t) x^2}, \quad u(t, x) = \beta(t) x,$$

and solving the PDE (1.3) becomes equivalent to solving the ODEs

$$\dot{\alpha} + 2\alpha\beta = 0, \quad \dot{\beta} + \beta^2 - 2\kappa\alpha = 0, \quad \dot{b} = -\beta b.$$

Following T. Li and D. Wang [10], seek

$$\alpha(t) = \frac{\alpha_0}{\tau(t)^2}, \quad \beta(t) = \frac{\dot{\tau}(t)}{\tau(t)}.$$

We come up with the ODE

$$\ddot{\tau} = \frac{2\kappa\alpha_0}{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = \beta_0.$$

At this stage, the surprising fact is the universal behavior of solutions to the above equation, regardless of initial data.

Lemma 2.1 ([5]). *Let $a_0, \tilde{\kappa} > 0$, $\beta_0 \in \mathbb{R}$. Consider the ordinary differential equation*

$$\ddot{\tau} = \frac{2\tilde{\kappa}}{\tau}, \quad \tau(0) = a_0, \quad \dot{\tau}(0) = \beta_0.$$

It has a unique solution $\tau \in C^2(0, \infty)$, and it satisfies, as $t \rightarrow \infty$,

$$\tau(t) = 2t\sqrt{\tilde{\kappa} \ln t} (1 + \mathcal{O}(\ell(t))), \quad \dot{\tau}(t) = 2\sqrt{\tilde{\kappa} \ln t} (1 + \mathcal{O}(\ell(t))),$$

where $\ell(t) := \frac{\ln \ln t}{\ln t}$.

Back to (1.3), this yields

$$\rho(t, x) \underset{t \rightarrow \infty}{\sim} \frac{b_0}{2t\sqrt{\alpha_0 \kappa \ln t}} e^{-x^2/(2t\sqrt{\kappa \ln t})^2}, \quad u(t, x) \underset{t \rightarrow \infty}{\sim} \frac{x}{t}.$$

Here, we emphasize the property

$$b_0/\sqrt{\alpha_0} \propto \|\rho_0\|_{L^1}.$$

Note that since the velocity is linear in x , this provides explicit solutions for the (Newtonian) isothermal Navier-Stokes equation.

Remark 2.2. It is possible to consider an initial Gaussian density which is not centered at the origin, or, equivalently,

$$\rho_0(x) = b_0 e^{-\alpha_0 x^2}, \quad u_0(x) = \beta_0 x + c_0.$$

Then

$$\rho(t, x) = b(t) e^{-\alpha(t)(x - \bar{x}(t))^2}, \quad u(t, x) = \beta(t)x + c(t),$$

with b , α and β like before, and

$$\bar{x}(t) = c_0 t, \quad c(t) = c_0 \left(1 - \frac{\dot{\tau}(t)}{\tau(t)} t \right).$$

Remark 2.3 (Universal dynamics for the density). In this Gaussian case, the function ρ exhibits two interesting features. We get a new dispersive rate, different from the one proved in [12] in the case $1 < \gamma \leq 1 + 2/d$ (logarithmic correction). More suprisingly, the density enjoys a universal asymptotic profile: no matter what the initial variance is, the asymptotic one is always the same.

Remark 2.4 (Generalization to other equations). The same approach can be extended to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \kappa \nabla \rho = \frac{\varepsilon^2}{2} \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \nu \operatorname{div}(\rho D(u)), \end{cases}$$

with $\varepsilon, \nu \geq 0$ and $D(u) = \frac{1}{2}(\nabla u + {}^t \nabla u)$. The term in ε corresponds to Korteweg equation (this term moldes capillarity). The term in ν corresponds to quantum Navier–Stokes equation, to take dissipative effects into account (see [4] for the derivation). Essentially, we proceed like before, and get the ODE

$$\ddot{\tau}^{\varepsilon, \nu} = \frac{2\kappa\alpha_0}{\tau^{\varepsilon, \nu}} + \varepsilon^2 \frac{\alpha_0^2}{(\tau^{\varepsilon, \nu})^3} - \nu \alpha_0 \frac{\dot{\tau}^{\varepsilon, \nu}}{(\tau^{\varepsilon, \nu})^2}, \quad \tau^{\varepsilon, \nu}(0) = 1, \quad \dot{\tau}^{\varepsilon, \nu}(0) = \beta_0.$$

It turns out that ε and ν do not alter the large time behavior, and we observe the same universal large time dynamics as for the isothermal Euler equation; see [6] for details.

3. Isothermal Euler equation and universal large time dynamics. Motivated by the remarkable observations in the one-dimensional Gaussian case, we introduce another change of unknown functions, in the case of the general space dimension $d \geq 1$. Consider the universal dispersion τ ,

$$\ddot{\tau} = \frac{2\kappa}{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0, \quad (3.1)$$

and change the unknown functions

$$\rho(t, x) = \frac{1}{\tau(t)^d} R \left(t, \frac{x}{\tau(t)} \right) \frac{\|\rho_0\|_{L^1}}{\|\Gamma\|_{L^1}}, \quad u(t, x) = \frac{1}{\tau(t)} U \left(t, \frac{x}{\tau(t)} \right) + \frac{\dot{\tau}(t)}{\tau(t)} x,$$

where

$$\Gamma(y) = e^{-|y|^2}$$

is the Gaussian that appeared in the previous section. The system in $(\rho, \rho u)$ is equivalent to

$$\begin{cases} \partial_t R + \frac{1}{\tau^2} \operatorname{div}(RU) = 0, \\ \partial_t(RU) + \frac{1}{\tau^2} \operatorname{div}(RU \otimes U) + 2\kappa y R + \kappa \nabla R = 0. \end{cases} \quad (3.2)$$

Naturally, the conservation of mass remains. The good news is that we gain some positivity in the entropy. Indeed, define the pseudo-energy

$$\mathcal{E}(t) := \frac{1}{2\tau^2} \int R|U|^2 + \kappa \int (R|y|^2 + R \ln R).$$

Formally, it satisfies

$$\dot{\mathcal{E}}(t) = -\mathcal{D}(t) =: -\frac{\dot{\tau}}{\tau^3} \int R|U|^2.$$

We also have

$$\mathcal{E}(t) := \frac{1}{2\tau^2} \int R|U|^2 + \kappa \int R \ln \frac{R}{\Gamma},$$

and since $\int R = \int \Gamma$, the Csiszár-Kullback inequality (see e.g. [1])

$$\|f - g\|_{L^1(\mathbb{R}^d)}^2 \leq 2\|f\|_{L^1(\mathbb{R}^d)} \int f(x) \ln \left(\frac{f(x)}{g(x)} \right) dx$$

shows that \mathcal{E} is the sum of two non-negative terms. As a matter of fact, Csiszár-Kullback inequality is not used: it is just a hint that more estimates are available in terms of (R, U) than in terms of (ρ, u) .

Lemma 3.1. *Suppose that $\int_{\mathbb{R}^d} R(t, y) dy$ is bounded and $\mathcal{E}(t) \leq \Lambda$ for all $t \geq 0$. There exists $C_0 > 0$ such that*

$$\frac{1}{\tau^2} \int_{\mathbb{R}^d} R|U|^2 + \int_{\mathbb{R}^d} R(1 + |y|^2 + |\ln R|) \leq C_0, \quad \forall t \geq 0.$$

Proof. We decompose \mathcal{E} in order to introduce a sum of positive terms,

$$\mathcal{E}_+(t) := \frac{1}{2\tau^2} \int R|U|^2 + \kappa \left(\int R|y|^2 + \int_{R \geq 1} R \ln R \right).$$

Since \mathcal{E} is non-increasing, we have

$$\mathcal{E}_+(t) \leq \Lambda + \kappa \int_{R < 1} R \ln \frac{1}{R} \lesssim 1 + \int_{\mathbb{R}^d} R^{1-\eta}.$$

By interpolation,

$$\int_{\mathbb{R}^d} R^{1-\eta} \leq C_\eta \|R\|_{L^1(\mathbb{R}^d)}^{1-\eta-d\eta/2} \| |y|^2 R \|_{L^1(\mathbb{R}^d)}^{d\eta/2}, \quad 0 < \eta < \frac{2}{d+2}.$$

Therefore, since the mass is conserved,

$$\mathcal{E}_+(t) \leq \Lambda + C\mathcal{E}_+(t)^{d\eta/4}$$

and since $d\eta/4 < 1$, $\mathcal{E}_+(t)$ is uniformly bounded for $t \geq 0$. As \mathcal{E}_+ is the sum of three non-negative terms, each one is uniformly bounded, and the only remaining term in \mathcal{E} is also bounded. \square

Recall that

$$\dot{\mathcal{E}}(t) = -\mathcal{D}(t) =: -\frac{\dot{\tau}}{\tau^3} \int R|U|^2 \leq 0.$$

Therefore, \mathcal{E} is naturally bounded from above. The previous lemma shows that \mathcal{E} is bounded from below, hence

$$\int_0^\infty \mathcal{D}(t) dt < \infty.$$

We can now state our main result.

Theorem 3.2. *Let (R, U) be a global weak solution, with constant mass.*

1. *If $\sup_{t \geq 0} \mathcal{E}(t) < \infty$, then*

$$\int_{\mathbb{R}^d} yR(t, y) dy \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{and} \quad \left| \int_{\mathbb{R}^d} (RU)(t, y) dy \right| \xrightarrow[t \rightarrow \infty]{} \infty,$$

unless $\int yR(0, y) dy = \int (RU)(0, y) dy = 0$ (a case where each of these quantities remains identically zero).

2. *If $\sup_{t \geq 0} \mathcal{E}(t) < \infty$ and the energy E satisfies $E(t) = o(\ln t)$ as $t \rightarrow \infty$, then*

$$\int_{\mathbb{R}^d} |y|^2 R(t, y) dy \xrightarrow[t \rightarrow \infty]{} \int_{\mathbb{R}^d} |y|^2 \Gamma(y) dy.$$

3. *If $\sup_{t \geq 0} \mathcal{E}(t) + \int_0^\infty \mathcal{D}(t) dt < \infty$, then $R(t, \cdot) \rightharpoonup \Gamma$ weakly in $L^1(\mathbb{R}^d)$ as $t \rightarrow \infty$.*

Remark 3.3 (Wasserstein distance). Theorem 3.2 implies the large time convergence of R to Γ in the Wasserstein distance W_2 , defined, for ν_1 and ν_2 probability measures, by

$$W_p(\nu_1, \nu_2) = \inf \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\mu(x, y) \right)^{1/p}; \quad (\pi_j)_\# \mu = \nu_j \right\},$$

where μ varies among all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$, and $\pi_j : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ denotes the canonical projection onto the j -th factor. This implies, for instance, the convergence of fractional momenta (see e.g. [13, Theorem 7.12])

$$\int |y|^{2s} R(t, y) dy \xrightarrow[t \rightarrow \infty]{} \int |y|^{2s} \Gamma(y) dy, \quad 0 \leq s \leq 1.$$

In [6], we prove the above result in the following generalized framework:

- The result remains valid in the presence of capillarity (Korteweg equation) and quantum dissipation (quantum Navier-Stokes).

- The result remains valid also with a generalized pressure law:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \frac{\varepsilon^2}{2} \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \nu \operatorname{div}(\rho Du), \end{cases}$$

with P convex and $P'(0) = \kappa > 0$.

Remark 3.4. In the case where $0 \leq \varepsilon \leq \nu$ and $\nu > 0$, with $P(\rho) \equiv \kappa \rho$, we have another rigidity result: up to an extraction, sequences of solutions on $(0, T)$ enjoying uniformly the conservation of mass, and “natural” inequalities (energy dissipation, BD-entropy dissipation, Mellet-Vasseur inequality in (R, U)), converge to a weak solution on $(0, T)$. The proof, presented in [6], follows the same argument as in [2].

4. Elements of proof. We present the main arguments to prove Theorem 3.2. Details can be found in [6]. Many features of the proof are similar to the arguments given in [5] in the context of a Schrödinger equation with logarithmic nonlinearity.

4.1. Main Theorem: proof of the first point. Define

$$\mathcal{I}_1(t) = \int_{\mathbb{R}^d} (RU)(t, y) dy, \quad \mathcal{I}_2(t) = \int_{\mathbb{R}^d} y R(t, y) dy.$$

We compute

$$\dot{\mathcal{I}}_1 = -\frac{1}{\tau^2} \int_{\mathbb{R}^d} \operatorname{div}(RU \otimes U) - 2\kappa \mathcal{I}_2 - \kappa \int_{\mathbb{R}^d} \nabla R = -2\kappa \mathcal{I}_2,$$

thanks to Lemma 3.1. Similarly, we compute

$$\dot{\mathcal{I}}_2 = -\frac{1}{\tau^2} \int_{\mathbb{R}^d} y \operatorname{div}(RU) = \frac{1}{\tau^2} \int_{\mathbb{R}^d} RU \equiv \frac{1}{\tau^2} \mathcal{I}_1,$$

since, from Lemma 3.1 and Cauchy-Schwarz inequality,

$$R|y||U| \in L_{\text{loc}}^\infty(0, \infty; L^1(\mathbb{R}^d)).$$

Then $J_2 := \tau \mathcal{I}_2$ satisfies $\ddot{J}_2 = 0$, hence

$$\mathcal{I}_2(t) = \frac{-\mathcal{I}_1(0)t + \mathcal{I}_2(0)}{\tau(t)}, \quad \mathcal{I}_1(t) = \mathcal{I}_1(0) - 2\kappa \int_0^t \mathcal{I}_2(s) ds.$$

The first point is then an easy consequence of Lemma 2.1.

4.2. Main Theorem: proof of the second point. Recall that the energy (entropy) E is formally conserved,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} \rho(t, x) |u(t, x)|^2 dx + \kappa \int_{\mathbb{R}^d} \rho(t, x) \ln \rho(t, x) dx.$$

In view of the change of unknown functions $(\rho, u) \mapsto (R, U)$, rewrite E :

$$\begin{aligned} E(t) &= \frac{1}{2\tau^2} \int R|U|^2 dy + \frac{(\dot{\tau})^2}{2} \int R|y|^2 dy + \frac{\dot{\tau}}{\tau} \int Ry \cdot U dy + \kappa \int R \ln R dy \\ &\quad - \kappa \ln(\tau^d) \int R dy. \end{aligned}$$

We already know that the first and fourth terms are uniformly bounded (Lemma 3.1), and that the third term is $\mathcal{O}(\sqrt{\ln t})$ (Lemma 3.1, Cauchy-Schwarz inequality, and

Lemma 2.1), while each of the second and last term is potentially of order $\ln t$ (Lemma 2.1). Therefore, if $E(t) = o(\ln t)$,

$$2 \int R|y|^2 dy - d \int R dy = o(1), \quad \text{hence} \quad \int R|y|^2 dy \xrightarrow[t \rightarrow \infty]{d} \frac{d}{2} \int \Gamma dy = \int |y|^2 \Gamma(y) dy.$$

4.3. Main Theorem: proof of the last point. Discarding terms which seem negligible for large time in (3.2), we get

$$\begin{cases} \partial_t R + \frac{1}{\tau^2} \operatorname{div}(RU) = 0, \\ \partial_t(RU) + 2\kappa y R + \kappa \nabla R = 0, \end{cases}$$

hence

$$\partial_t (\tau^2 \partial_t R) = \kappa \mathcal{L}R, \quad \text{where} \quad \mathcal{L}f = \Delta f + 2 \operatorname{div}(yf)$$

is a Fokker–Planck operator. Since $\tau^2 \ll (\dot{\tau})^2$ as $t \rightarrow \infty$ (Lemma 2.1), we expect

$$\partial_t (\tau^2 \partial_t R) = \tau^2 \partial_t^2 R + 2\dot{\tau} \tau \partial_t R \approx 2\dot{\tau} \tau \partial_t R,$$

hence, for large time, $\partial_s R \approx \mathcal{L}R$, for the new time variable

$$s(t) = \kappa \int \frac{1}{\tau \dot{\tau}} = \frac{1}{2} \int \frac{\ddot{\tau}}{\dot{\tau}} = \frac{1}{2} \ln \dot{\tau}(t) \underset{t \rightarrow \infty}{\sim} \frac{1}{4} \ln \ln t.$$

The large time behavior is thus expected to be dictated by the Fokker–Planck equation

$$\partial_s R_\infty = \mathcal{L}R_\infty, \quad \mathcal{L}f = \Delta f + 2 \operatorname{div}(yf).$$

It was established in [3] that any solution to this equation, obeying the bounds given by Lemma 3.1, satisfies

$$\|R_\infty(t) - \Gamma\|_{L^1(\mathbb{R}^d)} \xrightarrow[t \rightarrow \infty]{} 0.$$

To make the argument rigorous, set $s(t) = \frac{1}{2} \ln \dot{\tau}(t)$. At this stage, we emphasize that this rescaled time turns out to be rather natural: in view of Lemma 2.1,

$$s(t) \underset{t \rightarrow \infty}{\sim} \frac{1}{4} \ln \ln t.$$

This property conciles the fact that R_∞ converges to Γ exponentially fast in s (due to a spectral gap), and the fact that the convergence of the above quadratic quantities involved a logarithmic convergence in t .

Now denote by $\alpha : s \mapsto \alpha(s) = t$ its inverse mapping. Set $\bar{R}(s, y) = R(t, y)$, $\bar{U}(s, y) = U(t, y)$:

$$\partial_s \bar{R} - \frac{2\kappa}{(\dot{\tau} \circ \alpha)^2} \partial_s \bar{R} + \frac{\kappa}{(\dot{\tau} \circ \alpha)^2} \partial_s^2 \bar{R} = \mathcal{L} \bar{R} + \frac{1}{(\tau \circ \alpha)^2} \nabla^2 : (\bar{R} \bar{U} \otimes \bar{U}).$$

As a consequence of Lemma 3.1, $\int_0^\infty \mathcal{D}(t) dt < \infty$, which now reads

$$\int_0^\infty \left(\frac{\dot{\tau} \circ \alpha}{\tau \circ \alpha} \right)^2 \left\| \sqrt{\bar{R} \bar{U}} \right\|_{L^2(\mathbb{R}^d)}^2 ds < \infty.$$

For $s \in [0, 1]$ and $s_n \rightarrow \infty$, let $\bar{R}_n(s, y) = \bar{R}(s + s_n, y)$, $\bar{U}_n(s, y) = \bar{U}(s + s_n, y)$:

$$\sup_{n \in \mathbb{N}} \sup_{s \in [0, 1]} \int_{\mathbb{R}^d} \bar{R}_n (1 + |y|^2 + |\ln \bar{R}_n|) dy \leq C,$$

$$\lim_{n \rightarrow \infty} \int_0^1 \left(\frac{\dot{\tau} \circ \alpha_n}{\tau \circ \alpha_n} \right)^2 \left\| \sqrt{\bar{R}_n \bar{U}_n} \right\|_{L^2(\mathbb{R}^d)}^2 ds = 0.$$

Dunford–Pettis criterion implies that there exists $R_\infty \in L^1((0, 1) \times \mathbb{R}^d)$, such that, up to extracting a subsequence,

$$\bar{R}_n \rightharpoonup R_\infty \text{ weakly in } L^1((0, 1) \times \mathbb{R}^d) \text{ as } n \rightarrow \infty,$$

and $\int_{\mathbb{R}^d} R_\infty = \int_{\mathbb{R}^d} \bar{R}_n = \int_{\mathbb{R}^d} \Gamma$ (tightness).

$$\lim_{n \rightarrow \infty} \int_0^1 \left(\frac{\dot{\tau} \circ \alpha_n}{\tau \circ \alpha_n} \right)^2 \left\| \sqrt{\bar{R}_n} \bar{U}_n \right\|_{L^2(\mathbb{R}^d)}^2 ds = 0$$

yields

$$\frac{1}{(\tau \circ \alpha_n)^2} \nabla^2 : (\bar{R}_n \bar{U}_n \otimes \bar{U}_n) \rightharpoonup 0, \quad \text{hence} \quad \partial_s R_\infty = \mathcal{L}R_\infty.$$

On the other hand, we can show that R_∞ is stationary, $\partial_s R_\infty = 0$, and we conclude thanks to the result of Arnold, Markowich, Toscani and Unterreiter [3]. The limit is unique, so no extraction is actually needed.

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E-mail address: Remi.Carles@math.cnrs.fr

E-mail address: Kleber.Carrapatoso@umontpellier.fr

E-mail address: Matthieu.Hillairet@umontpellier.fr