Universal Dynamics for the Logarithmic Schrödinger Equations

Rémi Carles

UNIVERSAL DYNAMICS FOR THE LOGARITHMIC SCHRÖDINGER EQUATION

Rémi CARLES[1]

Abstract We consider the nonlinear Schrödinger equation with a logarithmic nonlinearity, whose sign is such that no non-trivial stationary solution exists. Explicit computations show that in the case of Gaussian initial data, the presence of the nonlinearity affects the large time behaviour of the solution, on at least three aspects. The dispersion is faster than usual by a logarithmic factor in time. The positive Sobolev norms of the solution grow logarithmically in time. Finally, after rescaling in space by the dispersion rate, the modulus of the solution converges to a universal Gaussian profile (whose variance is independent of the initial variance). In the case of general initial data, we show that these properties remain, up to weakening the third point (weak convergence instead of strong convergence). One of the key steps of the proof for the last point consists in using the Madelung transform. It reduces the equation to a variant of the isothermal compressible Euler equation, whose large time behaviour turns out to be governed by a parabolic equation involving a Fokker–Planck operator.

Keywords Nonlinear Schrödinger equation, logarithmic nonlinearity, large time behaviour, Euler equation, Fokker-Planck operator
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1 Introduction

1.1 Two linear equations leading to different asymptotic behaviours

We begin by recalling some basic results, which can be found for instance in [37].

Consider the heat equation on $\mathbb{R}^d$,

$$\partial_t u = \frac{1}{2}\Delta u, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{1.1}$$

This equation can be solved thanks to Fourier analysis. We normalize the Fourier transform as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

so that it is exactly an isometry on $L^2(\mathbb{R}^d)$ by Plancherel theorem. We solve (1.1) for initial data $u_{|t=0} = u_0 \in \mathcal{S}(\mathbb{R}^d)$ (a larger class of initial data can easily be considered),

$$\hat{u}(t,\xi) = \hat{u}_0(\xi) e^{-\frac{t}{2}|\xi|^2},$$

[1]CNRS, IMAG, 34095 Montpellier, France.
E-mail: Remi.Carles@math.cnrs.fr

where the Fourier transform is taken in the space variable only (and not in the time variable). We note the following decomposition and the associated estimates, for any $p \in [1,\infty]$:

$$\hat{u}(t,\xi) = \underbrace{e^{-\frac{t}{2}|\xi|^2}\hat{u}_0(0)}_{\text{order } t^{-d/(2p)} \text{ in } L^p} + \underbrace{e^{-\frac{t}{2}|\xi|^2}\left(\hat{u}_0(\xi)-\hat{u}_0(0)\right)}_{\mathcal{O}\left(|\xi|e^{-\frac{t}{2}|\xi|^2}\right)\text{: order } t^{-(d+1)/(2p)} \text{ in } L^p}.$$

Therefore, if $\hat{u}_0(0) \neq 0$, we have

$$\hat{u}(t,\xi) \underset{t\to\infty}{\sim} e^{-\frac{t}{2}|\xi|^2}\hat{u}_0(0) \text{ in } L^p(\mathbb{R}^d),\ \forall p \in [1,\infty).$$

Using the Fourier inverse formula, and recalling that $\hat{u}_0(0) = \int_{\mathbb{R}^d} u_0(x)dx =: m$ denotes the total mass (which is conserved by the heat flow), we infer that if $m \neq 0$,

$$u(t,x) \underset{t\to\infty}{\sim} \frac{m}{(2\pi t)^{d/2}} e^{-|x|^2/(2t)} \quad \text{in } L^2 \cap L^\infty(\mathbb{R}^d),$$

where we have used Hausdorff–Young inequality. We note that at leading order, the only role played by the initial data is the presence of the total mass m. The asymptotic profile is *universal*, and corresponds to the Gaussian

$$\gamma(x) := e^{-|x|^2/2}, \tag{1.2}$$

up to a scaling in time corresponding to the diffusive properties of the heat flow. The unusual factor $\frac{1}{2}$ in front of the Laplacian in (1.1) was there for the sake of consistency with the convention used for the Schrödinger equation below. It turns out that this Gaussian, whose precise normalization stems from this factor $\frac{1}{2}$, will be central in the rest of the analysis.

Consider now the linear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d. \tag{1.3}$$

It can be solved by Fourier analysis in the same fashion as for the heat equation, and we find the classical formula

$$u(t,x) = \frac{1}{(2i\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} u_0(y)dy.$$

By factorizing the Schrödinger group as

$$u(t,x) = M_t D_t \mathcal{F} M_t u_0(x),$$

where the three operators,

$$M_t = e^{i|x|^2/(2t)}, \quad D_t\varphi(x) = \frac{1}{(it)^{d/2}}\varphi\left(\frac{x}{t}\right), \quad \text{and } \mathcal{F},$$

are unitary on $L^2(\mathbb{R}^d)$, we get the leading order asymptotics in $L^2(\mathbb{R}^d)$,

$$\|u(t) - A(t)u_0\|_{L^2(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0, \quad \text{where} \quad A(t)u_0(x) := \frac{1}{(it)^{d/2}}\hat{u}_0\left(\frac{x}{t}\right)e^{i|x|^2/(2t)}. \tag{1.4}$$

This large time asymptotic solution shows two features:

- The main oscillation, $e^{i|x|^2/(2t)}$, is universal, in the sense that it does not depend on the initial data u_0.
- The asymptotic profile, $\hat{u}_0$, which appears rescaled by the dispersive rate D_t, does depend on the initial data, and is not universal, as opposed to the heat flow case.

Concerning the second point, one may argue that for fixed x, we have

$$\hat{u}_0\left(\frac{x}{t}\right) \underset{t\to\pm\infty}{\longrightarrow} \hat{u}_0(0) = \int_{\mathbb{R}^d} u_0(x)dx,$$

somehow like for the heat equation. However, this asymptotics ruins the preservation of the L^2 norm (the asymptotic profile obtained after this limit is not even in $L^2(\mathbb{R}^d)$), which is an important conserved quantity for the Schrödinger equation,

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^d)} = 0.$$

This is why it is sensible to consider the quantity $A(t)u_0$ for the leading order asymptotic behaviour of $u(t,x)$.

1.2 Nonlinear Schrödinger equation with a power nonlinearity

We now recall results which can be found in e.g. [11]. The most standard nonlinear pertubation of (1.3) (at least for the mathematicians) is

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{2\sigma}u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \tag{1.5}$$

with $\lambda\in\mathbb{R}$ and $\sigma>0$. The case $\lambda\in\mathbb{R}$ yields a Hamiltonian equation, for which in addition the total L^2 norm (usually referred to as "mass" in mathematics) is conserved. This means that we have, at least formally,

$$\frac{d}{dt}\underbrace{\|u(t)\|^2_{L^2(\mathbb{R}^d)}}_{\text{mass}} = \frac{d}{dt}\left(\underbrace{\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{\lambda}{\sigma+1}\|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^d)}}_{\text{energy}}\right) = 0.$$

Consider for instance initial data $u_{|t=0} = u_0 \in H^1(\mathbb{R}^d)$, and $\sigma < 2/(d-2)_+$, so that the (initial) energy is well-defined thanks to Sobolev embedding. The sign of λ may be crucial as far as the formation of singularity is concerned:

- If $\lambda \geqslant 0$, then the corresponding Cauchy problem has a unique, global solution $u \in C(\mathbb{R};H^1(\mathbb{R}^d))$.
- If $\lambda<0$ and $\sigma<2/d$ (L^2-subcritical nonlinearity), then the same conclusion holds, $u\in C(\mathbb{R};H^1(\mathbb{R}^d))$.
- On the other hand, if $\lambda<0$ and $\sigma\geqslant 2/d$, then finite blow-up may occur, in the sense that there may exist $T^*>0$ (or $T_*<0$ with a similar conclusion) such that the Cauchy problem has a unique solution $u\in C([0,T^*);H^1(\mathbb{R}^d))$ and

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \underset{t\to T^*}{\longrightarrow} \infty.$$

The case $\lambda > 0$ is refered to as "defocusing" or "repulsive", while the case $\lambda < 0$ is called "focusing" or "attractive". In the defocusing case, we have an extra piece of information regarding the large time behaviour of the solution, provided that the power σ of the nonlinearity is sufficiently large (this distinguishes between short range and long range scattering). Typically, if $\sigma \geqslant 2/d$ and

$$u_0 \in \Sigma = H^1 \cap \mathcal{F}(H^1) = \{f \in H^1(\mathbb{R}^d), \quad x \mapsto |x| f(x) \in L^2(\mathbb{R}^d)\},$$

then there exist $u_\pm \in \Sigma$ such that

$$\|e^{-i\frac{t}{2}\Delta} u(t) - u_\pm\|_\Sigma \underset{t\to\pm\infty}{\longrightarrow} 0.$$

Since $e^{-i\frac{t}{2}\Delta}$ is unitary on H^1 (it is not unitary on $\mathcal{F}(H^1)$, see e.g. [11]), we infer from (1.4) that in $L^2(\mathbb{R}^d)$, we have

$$u(t,x) \underset{t\to\pm\infty}{\sim} \frac{1}{(it)^{d/2}} \hat{u}_\pm\left(\frac{x}{t}\right) e^{i|x|^2/(2t)}.$$

In fact this result remains true under weaker assumptions on σ, and even in some cases when $\lambda < 0$, but entering into such details is not the goal of these notes; see e.g. [11, 22].

1.3 The logarithmic Schrödinger equation

From now on, we consider

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0\,, \tag{1.6}$$

with $x \in \mathbb{R}^d$, $d \geqslant 1$, $\lambda \in \mathbb{R} \setminus \{0\}$. It was introduced as a model of nonlinear wave mechanics and in nonlinear optics ([4], see also [6, 28, 29, 30, 35]). The mathematical study of this equation goes back to [12, 10] (see also [11]). The sign $\lambda < 0$ seems to be the more interesting from a physical point of view, and this case has been studied formally and rigorously (see [15, 29] for instance).

On a formal level at least, (1.6) satisfies conservation laws which are similar to the standard counterpart (1.5): for

$$\begin{aligned} M(u(t)) &:= \|u(t)\|^2_{L^2(\mathbb{R}^d)}\,,\\ E(u(t)) &:= \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln|u(t,x)|^2 dx\,, \end{aligned}$$

we have,

$$\frac{d}{dt} M(u(t)) = \frac{d}{dt} E(u(t)) = 0\,.$$

The last identity reveals the Hamiltonian structure of (1.6). Note that unlike in the case of (1.5), the energy for (1.6) never has a definite sign, due to the logarithm. Therefore, it is not obvious to derive a priori estimates for the solution, whichever the sign of λ. In addition, since the map

$$u \mapsto u \ln|u|^2$$

is not Lipschitz continuous, constructing even a local solution is not straightforward; one should typically forget about the classical fixed point argument which turns out to be so powerful in the case of (1.5).

Another striking difference between (1.5) and (1.6) concerns the effect of a scaling factor. It is readily checked that if u solves (1.5), then for $k > 0$,

$$u_k(t,x) = ku\,(t,x)\,,$$

solves (1.5) with λ replaced by $\lambda k^{2\sigma}$. In the case $\lambda < 0$, the standard virial argument (see e.g. [11] implies for instance that for any $u_0 \in \mathcal{S}(\mathbb{R}^d)$, the solution to (1.5) with initial datum ku_0 will be global in time if $k > 0$ is sufficiently small, and blow up in finite time if $k > 0$ is sufficiently large. This is an easy way to see that scaling factors strongly affect the nonlinear dynamics. In the case of (1.6) however, scaling does not affect the dynamics, except for a somehow irrelevant time dependent oscillatory factor. Indeed, if u solves (1.6), then for $k > 0$, ku solves

$$i\partial_t(ku) + \frac{1}{2}\Delta(ku) = \lambda \ln\left(|ku|^2\right) ku - 2\lambda(\ln k)ku.$$

The scaling factor thus corresponds to a purely time-dependent gauge transform:

$$ku(t,x)e^{2it\lambda \ln k}$$

solves (1.6) (with initial datum ku_0). In particular, the L^2-norm of the initial datum does not influence the dynamics of the solution. This can be compared to the large time of the solution of the linear heat equation (1.1), and might suggest that nonlinear effects in (1.6) are rather weak. We will see that on the contrary, they alter the dynamics in a fairly unique fashion.

We now recall the main results in the case $\lambda < 0$. The Cauchy problem is studied in [12] (see also [11]). Define

$$W := \left\{ u \in H^1(\mathbb{R}^d)\,,\, x \mapsto |u(x)|^2 \ln |u(x)|^2 \in L^1(\mathbb{R}^d) \right\}.$$

Proposition 1.1 (Théorème 2.1 from [12], see also Theorem 9.3.4 from [11]) *Let the initial data u_0 belong to W. In the case when $\lambda < 0$, there exists a unique, global solution $u \in C(\mathbb{R}; W)$ to* (1.6). *In particular, for all $t \in \mathbb{R}$, $|u(t,\cdot)|^2 \ln |u(t,\cdot)|^2$ belongs to $L^1(\mathbb{R}^d)$, and the mass $M(u)$ and the energy $E(u)$ are independent of time.*

In the case $\lambda < 0$, it can be proved that there is no dispersion for large times, whichever the initial data. This is already a striking difference with the power case (1.5). Indeed the following result holds.

Lemma 1.1 (Lemma 3.3 from [10]) *Let $\lambda < 0$ and $k < \infty$ such that*

$$L_\kappa := \left\{ u \in W,\ \|u\|_{L^2(\mathbb{R}^d)} = 1,\ E(u) \leqslant \kappa \right\} \neq \emptyset\,.$$

Then

$$\inf_{\substack{u \in L_\kappa \\ 1 \leqslant p \leqslant \infty}} \|u\|_{L^p(\mathbb{R}^d)} > 0\,.$$

This lemma, along with the conservation of the energy for (1.6), indicates that in the case $\lambda < 0$, no solution to (1.6) is dispersive: typically, the L^∞ norm is bounded from below. Actually in the case of Gaussian initial data, some solutions are even known to be periodic in time, as proved in [15] (and already noticed in [4]).

Proposition 1.2 ([15]) *In the case $\lambda < 0$, the* Gausson $\exp(-2i\lambda\omega t + \omega + d/2 + \lambda|x|^2)$ *is a solution to* (1.6) *for any period $\omega \in \mathbb{R}$.*

We emphasize that several results address the existence of stationary solutions to (1.6) in the case $\lambda < 0$, and the orbital stability of the Gausson; see e.g. [4, 10, 15, 3]. Note that Theorem 1.2 provides a C^∞ solution to (1.6) also in the case $\lambda > 0$, but then, the solution is not even in $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions.

1.4 Main results

Throughout the rest of this paper, we assume $\lambda > 0$. The method followed in [12] does not seem to extend to the case $\lambda > 0$ to prove the existence of a solution. On the other hand, a beautiful estimate established there yields uniqueness. To state our first result, introduce, for $0 < \alpha \leqslant 2$, the weighted L^2 space

$$\mathcal{F}(H^\alpha) = \left\{ u \in L^2(\mathbb{R}^d)\,,\, x \mapsto \langle x\rangle^\alpha u(x) \in L^2(\mathbb{R}^d) \right\},$$

where $\langle x\rangle := \sqrt{1+|x|^2}$, with norm

$$\|u\|_{\mathcal{F}(H^\alpha)} := \|\langle x\rangle^\alpha u(x)\|_{L^2(\mathbb{R}^d)}\,.$$

Note that for any $\alpha > 0$, $\mathcal{F}(H^\alpha) \cap H^1 \subset W$. The Cauchy problem for (1.6) is investigated in [25], where in three space dimensions, the existence of a unique solution in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \cap C(\mathbb{R}; L^2(\mathbb{R}^3))$ is proved as soon as the initial data belongs to $\mathcal{F}(H^1) \cap H^1(\mathbb{R}^3)$. Actually it is possible to improve that result into the following theorem.

Theorem 1.1 *Let the initial data u_0 belong to $\mathcal{F}(H^\alpha)\cap H^1(\mathbb{R}^d)$ with $0 < \alpha \leqslant 1$. In the case when $\lambda > 0$, there exists a unique, global solution $u \in L^\infty_{\rm loc}(\mathbb{R}; \mathcal{F}(H^\alpha) \cap H^1)$ to* (1.6)*. Moreover the mass $M(u)$ and the energy $E(u)$ are independent of time. If in addition $u_0 \in H^2(\mathbb{R}^d)$, then $u \in L^\infty_{\rm loc}(\mathbb{R}; H^2)$.*

Note that due to the lack of regularity of the nonlinearity, propagating higher regularity (H^s, with $s > 2$) is a challenging question. We will see that the case of H^2 can be treated thanks to Kato's trick, which allows for little regularity on the nonlinearity.

Once a unique, global solution is available, the natural question concerns its large time behaviour. A striking feature of (1.6) is that important hints are given by explicit computations in the case of Gaussian initial data. As noticed already in [4], an important property of (1.6) is that the evolution of initial Gaussian data remains Gaussian for all time. Since (1.6) is invariant by translation in space, we may consider centered Gaussian initial data. The following result is a crucial guide for the general case. We define from now on the function

$$\ell(t) := \frac{\ln\ln t}{\ln t}\cdot \tag{1.7}$$

Theorem 1.2 *Let $\lambda > 0$, and consider the initial data*

$$u_0(x) = b_0 \exp\Big(-\frac{1}{2}\sum_{j=1}^d a_{0j} x_j^2 \Big), \tag{1.8}$$

with $b_0, a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. *Then the solution* u *to* (1.6) *is given by*

$$u(t,x) = b_0 \prod_{j=1}^{d} \frac{1}{\sqrt{r_j(t)}} \exp\Big(i\phi_j(t) - \alpha_{0j} \frac{x_j^2}{2r_j^2(t)} + i \frac{\dot r_j(t)}{r_j(t)} \frac{x_j^2}{2} \Big)$$

for some real-valued functions ϕ_j, r_j *depending on time only, such that, as* $t \to \infty$,

$$r_j(t) = 2t\sqrt{\lambda \alpha_{0j} \ln t}\,\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big), \quad \dot r_j(t) = 2\sqrt{\lambda\alpha_{0j}\ln t}\,\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big). \tag{1.9}$$

In particular, as $t \to \infty$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \sim \frac{1}{\Big(t\sqrt{\ln t}\Big)^{d/2}} \frac{\|u_0\|_{L^2}}{\big(2\lambda\sqrt{2\pi}\big)^{d/2}}\,.$$

On the other hand u *belongs to* $L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1(\mathbb{R}^d))$ *and as* $t \to \infty$

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d \|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t.$$

At least three aspects of this result differ from the more standard Schrödinger equations, as discussed in more detail below:

- The dispersion is of order $(t\sqrt{\ln t})^{-d/2}$, as opposed to $t^{-d/2}$ in the case of the free Schrödinger equation (1.3), or of defocusing nonlinear Schrödinger equations with sufficiently short range nonlinearity ((1.5) with $\lambda > 0$, and $\sigma \geqslant 2/d$ typically). The nonlinearity therefore has an effect on the dispersion rate.
- Even though the solution is dispersive, its H^1-norm is unbounded. This is due to the discrepancy between the dispersive rate and the rate of the main oscillations, since

$$\frac{\dot r_j(t)}{r_j(t)} \frac{x_j^2}{2} \underset{t\to\infty}{\sim} \frac{x_j^2}{2t}.$$

- Up to a rescaling, the modulus of u converges for large time to a *universal* Gaussian profile,

$$(2t\sqrt{\lambda \ln t})^{d/2} \left| u\left(t, x \times 2t\sqrt{\lambda \ln t}\right)\right| \underset{t\to\infty}{\longrightarrow} \frac{\|u_0\|_{L^2}}{\pi^{d/4}} e^{-|x|^2/2}\,,$$

  that is, regardless of the value of the variance of the Gaussian initial datum (a more precise statement is given in Corollary 1.2 below).

We see that like in the case of the Schrödinger equation (1.3), the asymptotic behaviour displays a universal oscillation, which is the same as for (1.3), of the form

$$e^{i|x|^2/(2t)}.$$

Leaving out the exotic dispersive rate, we see that like for the heat equation, the asymptotic profile of the solution is the universal Gaussian γ defined in (1.2).

The parameter functions r_j are given by ordinary differential equations, whose large time behaviour turns out to be independent of the initial data. This motivates us to introduce the universal dispersion rate τ through the following lemma.

Lemma 1.2 (Universal dispersion) *Consider the ordinary differential equation*

$$\ddot{\tau} = \frac{2\lambda}{\tau}\,, \quad \tau(0) = 1\,, \quad \dot{\tau}(0) = 0\,. \tag{1.10}$$

It has a unique solution $\tau \in C^2(0,\infty)$, and it satisfies, as $t \to \infty$,

$$\tau(t) = 2t\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big)\,, \quad \dot{\tau}(t) = 2\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big)\,.$$

In view of the previous discussion on scaling factors, we may suppose $\|u_0\|_{L^2(\mathbb{R}^d)} = \|\gamma\|_{L^2(\mathbb{R}^d)}$, an assumption that we make in the next statement in order to lighten the notations.

Theorem 1.3 *Let $u_0 \in \Sigma = H^1 \cap \mathcal{F}(H^1)$, with $\|u_0\|_{L^2(\mathbb{R}^d)} = \|\gamma\|_{L^2(\mathbb{R}^d)}$, and rescale the solution provided by Theorem* 1.1 *to $v = v(t,y)$ by setting*

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\Big(i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}\Big)\,. \tag{1.11}$$

There exists C such that for all $t \geqslant 0$,

$$\int_{\mathbb{R}^d} \big(1 + |y|^2 + \big|\ln|v(t,y)|^2\big|\big)\,|v(t,y)|^2 dy + \frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C\,. \tag{1.12}$$

We have moreover

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy\,. \tag{1.13}$$

Finally,

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \textit{weakly in } L^1(\mathbb{R}^d)\,. \tag{1.14}$$

The above result shows that all solutions are dispersive in the case $\lambda > 0$, which is the reason why this case may be called "defocusing", by analogy with the case of (1.5), even though the energy functional has no definite sign.

Remark 1.1 If the initial data is not normalized in $L^2(\mathbb{R}^d)$ then the result (1.14) becomes

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \frac{\|u_0\|^2_{L^2}}{\pi^{d/2}}\,\gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d)\,.$$

To the best of our knowledge, this is the first time that a universal profile is observed for the large time behaviour of solutions to a dispersive, Hamiltonian equation.

Remark 1.2 As a straightforward consequence, we infer the slightly weaker property that $|v(t,\cdot)|^2$ converges to γ^2 in Wasserstein distance:

$$W_2\left(\frac{|v(t,\cdot)|^2}{\|u_0\|^2_{L^2}}, \frac{\gamma^2}{\pi^{d/2}}\right) \underset{t\to\infty}{\longrightarrow} 0,$$

where we recall that the Wasserstein distance is defined, for ν_1 and ν_2 probability measures, by

$$W_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\mu(x,y)\right)^{1/p}\,; \quad (\pi_j)_\sharp \mu = \nu_j\right\},$$

where μ varies among all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$, and $\pi_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the canonical projection onto the j-th factor (see e.g. [38]).

In the context of nonlinear Hamiltonian partial differential equations, a general question is the evolution of Sobolev norms, as emphasized in [5]. In some cases, it can be shown that Sobolev norms are unbounded at least in some sense, but without growth rate; see e.g. [13, 24, 23] for the nonlinear Schrödinger equation. For other equations (cubic Szegö equation or half-wave equation), with *specific initial data*, a growth rate can be exhibited, possibly along a sequence of time; see [20, 21, 36]. We show that in the case of (1.6), the Sobolev norms of all solutions grow, and we give a sharp rate.

Corollary 1.1 *Let $u_0 \in \Sigma = H^1 \cap \mathcal{F}(H^1)$, and $0 < s \leqslant 1$. The solution to* (1.6) *satisfies, as $t \to \infty$,*

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d \|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t,$$

and

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2},$$

where $\dot{H}^s(\mathbb{R}^d)$ denotes the standard homogeneous Sobolev space.

The proof of the first case (which is a refined version of the general statement, in the case $s = 1$) is straightforward in view of Theorem 1.3. Indeed, using (1.11), we have

$$\begin{aligned}\nabla u(t,x) &= \frac{1}{\tau(t)^{d/2}} \nabla_x \left(v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}} \right) \\ &= \frac{1}{\tau(t)} \frac{1}{\tau(t)^{d/2}} \nabla_y v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}} + i\dot\tau \frac{1}{\tau(t)^{d/2}} \frac{x}{\tau} v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}}.\end{aligned}$$

The estimate (1.12) shows that the first term is bounded in $L^2(\mathbb{R}^d)$, uniformly in time. On the other hand, (1.13) shows that the L^2 norm of the second term satisfies

$$\begin{aligned}\left\| \dot\tau \frac{1}{\tau^{d/2}} \frac{x}{\tau} v\left(t, \frac{x}{\tau}\right) \right\|_{L^2(\mathbb{R}^d)} &= \dot\tau(t) \|y v(t,y)\|_{L^2(\mathbb{R}^d)} \\ &\underset{t\to\infty}{\sim} \dot\tau(t) \|y\gamma\|_{L^2(\mathbb{R}^d)} = \dot\tau(t) \sqrt{\frac{d}{2}} \|\gamma\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} \sqrt{2d\lambda \ln t} \|u_0\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

where we have used Lemma 1.2 and the assumption $\|u_0\|_{L^2} = \|\gamma\|_{L^2}$. The case $0 < s < 1$ is a consequence of Remark 1.2 and general properties related to Wasserstein distance (which can be found in e.g. [38]), as well as a technical lemma adapted to monokinetic oscillations, borrowed from [2]. Note that in view of the first point and the conservation of the L^2 norm, we have, by interpolation,

$$\|u(t)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2}.$$

In the Gaussian case, the Csiszár-Kullback inequality enables us to obtain the strong convergence of $|v|^2$ to γ^2 in L^1. This is made precise in the next statement.

Corollary 1.2 (Strong convergence in the Gaussian case) *Suppose that the initial data u_0 is a Gaussian as in* (1.8)*, with $\|u_0\|_{L^2(\mathbb{R}^d)} = \|\gamma\|_{L^2(\mathbb{R}^d)}$. Then, with v given by* (1.11)*, the relative entropy of $|v|^2$ goes to zero for large time:*

$$\int_{\mathbb{R}^d} |v(t,y)|^2 \ln \left| \frac{v(t,y)}{\gamma(y)} \right|^2 dy \underset{t\to\infty}{\longrightarrow} 0,$$

and the convergence of $|v|^2$ to γ^2 is strong in L^1:

$$\big\||v(t,\cdot)|^2-\gamma^2\big\|_{L^1(\mathbb{R}^d)}\underset{t\to\infty}{\longrightarrow}0\,.$$

2 Cauchy problem

We now explain how to prove Theorem 1.1.

2.1 Uniqueness

Consider u_1 and u_2 two solutions of (1.6) in $L^\infty(\mathbb{R};L^2(\mathbb{R}^d))$. Then the function $u:=u_1-u_2$ satisfies

$$i\partial_t u+\frac{1}{2}\Delta u=\lambda\big(\ln\big(|u_1|^2\big)\,u_1-\ln\big(|u_2|^2\big)\,u_2\big).$$

The standard energy estimate consists in multiplying by $\overline{u}$, integrating in space, and taking the imaginary part of the outcome. It yields

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^d)}=\lambda\,\mathrm{Im}\int_{\mathbb{R}^d}\big(\ln\big(|u_1|^2\big)\,u_1-\ln\big(|u_2|^2\big)\,u_2\big)(\bar{u}_1-\bar{u}_2)(t)dx.$$

The end of the argument follows from a nice algebraic property:

Lemma 2.1 (Lemme 1.1.1 from [12]; see also Lemma 9.3.5 from [11]) *We have*

$$\big|\mathrm{Im}\left(\left(z_2\ln|z_2|^2-z_1\ln|z_1|^2\right)(\bar{z}_2-\bar{z}_1)\right)\big|\leqslant 4|z_2-z_1|^2\,,\quad\forall z_1,z_2\in\mathbb{C}\,.$$

We infer

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^d)}\leqslant 4\lambda\|u(t)\|^2_{L^2(\mathbb{R}^d)},$$

so if $u(0)=0$, then $u(t)=0$ for all time thanks to Gronwall lemma, hence the uniqueness part of Theorem 1.1.

2.2 Existence

In [12], the existence of a (weak) solution in the case $\lambda<0$ was established by replacing the nonlinearity $z\ln|z|^2$ by a suitable polynomial for $|z|\leqslant\varepsilon$, and eventually letting ε go to zero. However, the argument does not seem to be useful in the case $\lambda>0$, so we adopt a different strategy, and approximate the equation as follows: consider for all $\varepsilon\in(0,1)$ the equation

$$i\partial_t u_\varepsilon+\frac{1}{2}\Delta u_\varepsilon=\lambda\ln\left(\varepsilon+|u_\varepsilon|^2\right)u_\varepsilon\,,\quad u_{\varepsilon|t=0}=u_0\,. \tag{2.1}$$

Now for $\varepsilon>0$ fixed, the new nonlinearity is Lipschitzean. Equation (2.1) is easily solved in $C(\mathbb{R};L^2(\mathbb{R}^d))$ since it is subcritical in L^2 (see [11]). It remains therefore to prove uniform bounds for $u_\varepsilon(t)$ in $\mathcal{F}(H^\alpha)\cap H^1(\mathbb{R}^d)$, which will provide compactness in space for the sequence u_ε. Since time compactness (in $H^{-2}(\mathbb{R}^d)$) is a direct consequence of the equation, the Ascoli theorem will then give the result. Actually once a bound in $L^\infty_{\mathrm{loc}}(\mathbb{R};H^1(\mathbb{R}^d))$ is derived, then the $L^\infty_{\mathrm{loc}}(\mathbb{R};\mathcal{F}(H^\alpha))$ bound can be obtained directly thanks to the following computation: define

$$I_{\varepsilon,\alpha}(t):=\int_{\mathbb{R}^d}\langle x\rangle^{2\alpha}|u_\varepsilon|^2(t,x)\,dx\,.$$

Then multiplying the equation by $\langle x\rangle^{2\alpha}\overline{u}_\varepsilon$ and integrating in space provides

$$\begin{aligned}\frac{d}{dt}I_{\varepsilon,\alpha}(t) &= 2\alpha\,\mathrm{Im}\int \frac{x\cdot\nabla u_\varepsilon}{\langle x\rangle^{2-2\alpha}}\,\overline{u}_\varepsilon(t)dx \leqslant 2\alpha\|\,\langle x\rangle^{2\alpha-1}\,u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\|\nabla u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\\ &\leqslant 2\alpha\|\,\langle x\rangle^{\alpha}\,u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\|\nabla u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\,,\end{aligned}$$

where the last estimate stems from the property $\alpha\leqslant 1$. Therefore,

$$\|u_\varepsilon(t)\|^2_{\mathcal{F}(H^\alpha)} \leqslant \|u_0\|^2_{\mathcal{F}(H^\alpha)} + 2\alpha\int_0^t \|u_\varepsilon(t')\|_{\mathcal{F}(H^\alpha)}\|\nabla u_\varepsilon(t')\|_{L^2(\mathbb{R}^d)}\,dt'\,.$$

For any $1\leqslant j\leqslant d$ one has

$$i\partial_t\partial_j u_\varepsilon + \frac{1}{2}\Delta\partial_j u_\varepsilon = \lambda\ln\left(\varepsilon+|u_\varepsilon|^2\right)\partial_j u_\varepsilon + 2\lambda\frac{1}{\varepsilon+|u_\varepsilon|^2}\,\mathrm{Re}(\bar{u}_\varepsilon\partial_j u_\varepsilon)u_\varepsilon, \tag{2.2}$$

which is again subcritical in L^2 since $\left|\dfrac{1}{\varepsilon+|u_\varepsilon|^2}2\,\mathrm{Re}(\bar{u}_\varepsilon\partial_j u_\varepsilon)u_\varepsilon\right| \leqslant 2|\partial_j u_\varepsilon|$. We therefore conclude that u_ε belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R};H^1(\mathbb{R}^d))$. The conservation of mass, angular momentum, and energy is established in the same way as in [12] (see also [11]). The first part of Theorem 1.1 follows.

2.3 Higher regularity

As in [12], the idea is to consider time derivatives. This fairly general idea in the context of nonlinear Schrödinger equations (see [11]) is all the more precious in the present framework that the logarithmic nonlinearity is very little regular. In particular, we emphasize that if $u_0\in H^k(\mathbb{R}^d)$, $k\geqslant 3$, we cannot guarantee in general that this higher regularity is propagated.

To complete the proof of Theorem 1.1, assume that $u_0\in\mathcal{F}(H^\alpha)\cap H^2$, for some $\alpha>0$. We already know that a unique, global, weak solution $u\in L^\infty_{\mathrm{loc}}(\mathbb{R};\mathcal{F}(H^\alpha)\cap H^1)$ is obtained by the procedure described in the previous subsection, that is, as the limit of u_ε solution to (2.1). The idea is that for all $T>0$, there exists $C=C(T)$ independent of $\varepsilon\in(0,1)$ such that

$$\sup_{-T\leqslant t\leqslant T}\|\partial_t u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\leqslant C\,.$$

Indeed, we know directly from (2.1) that

$$\partial_t u_{\varepsilon|t=0} = \frac{i}{2}\Delta u_0 - i\lambda\ln\left(\varepsilon+|u_0|^2\right)u_0\in L^2(\mathbb{R}^d)\,,$$

uniformly in ε, in view of the pointwise estimate

$$\left|\ln\left(\varepsilon+|u_0|^2\right)u_0\right| \leqslant C\left(|u_0|^{1+\eta}+|u_0|^{1-\eta}\right)\,,$$

where $\eta>0$ can be chosen arbitrarily small, and C is independent of $\varepsilon\in(0,1)$. Then we can replace the spatial derivative ∂_j in (2.2) with the time derivative ∂_t, and infer that $\partial_t u_\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};L^2(\mathbb{R}^d))$, uniformly in ε: by passing to the limit (up to a subsequence), $\partial_t u\in L^\infty_{\mathrm{loc}}(\mathbb{R};L^2(\mathbb{R}^d))$. Using the equation (1.6), we conclude that $\Delta u\in L^\infty_{\mathrm{loc}}(\mathbb{R};L^2(\mathbb{R}^d))$. This concludes the proof of Theorem 1.1.

3 Gaussian initial data

3.1 From (1.6) to ordinary differential equations

As noticed in [4], the flow of (1.6) preserves any initial Gaussian structure. We emphasize that this phenomenon is quite rare, even for linear equations. Consider for instance the wave equation

$$\partial_t^2 u - \Delta u = 0.$$

The fundamental solution is explicit (see e.g. [37]), and if the initial data are Gaussian functions, we check that the solution ceases to be Gaussian. On the other hand, it is well-known that the evolution of Gaussian under the heat flow (1.1) or the Schrödinger flow (1.3), is still a Gaussian, which can be computed explicitly; see e.g. [37]. Still, the case of (1.6) seems to be a rather rare case where the evolution of a Gaussian under a nonlinear flow remains Gaussian.

We consider the data given by (1.8), and we seek the solution u to (1.6) under the form

$$u(t,x) = b(t)\exp\Big(-\frac{1}{2}\sum_{j=1}^d a_j(t)x_j^2\Big)\,, \tag{3.1}$$

with $\operatorname{Re} a_j(t) > 0$. With u of this form, (1.6) becomes equivalent to

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda\Big(\ln|b(t)|^2 - \sum_{j=1}^d \operatorname{Re} a_j(t)x_j^2\Big)u\,, \quad u_{|t=0} = u_0\,.$$

This is a linear Schrödinger equation with a time-dependent harmonic potential, and an initial Gaussian. It is well-known in the context of the propagation of coherent states (see [27, 14]) that the evolution of a Gaussian wave packet under a time-dependent harmonic oscillator is a Gaussian wave packet. Therefore, it is consistent to look for a solution to (1.6) of this form. Notice in particular that

$$\|u(t)\|_{L^p(\mathbb{R}^d)} = \left(\frac{2\pi}{p}\right)^{d/(2p)} \frac{|b(t)|}{\Big(\prod_{j=1}^d \operatorname{Re} a_j(t)\Big)^{1/(2p)}}\,, \quad 1\leqslant p\leqslant\infty\,, \tag{3.2}$$

and

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 = \frac{1}{2}\pi^{d/2}\frac{|b(t)|^2}{\Big(\prod_{j=1}^d \operatorname{Re} a_j(t)\Big)^{1/2}}\sum_{j=1}^d \frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))}\,. \tag{3.3}$$

To prove Theorem 1.2 we therefore need to find the asymptotic behaviour in time of $b(t)$ and $a_j(t)$.

Plugging (3.1) into (1.6), we obtain, after simplification by the Gaussian,

$$i\dot b - i\sum_{j=1}^d \dot a_j\frac{x_j^2}{2}b - \sum_{j=1}^d \frac{a_j b}{2} + \sum_{j=1}^d a_j^2\frac{x_j^2}{2}b = \lambda\Big(\ln\big(|b|^2\big) - \sum_{j=1}^d (\operatorname{Re} a_j)\,x_j^2\Big)b\,.$$

Equating the constant in x and the factors of x_j^2, we get

$$i\dot a_j - a_j^2 = 2\lambda\operatorname{Re} a_j\,, \quad a_{j|t=0} = a_{0j}\,, \tag{3.4}$$

$$i\dot b - \sum_{j=1}^d \frac{a_j b}{2} = \lambda b\ln\big(|b|^2\big)\,, \quad b_{|t=0} = b_0\,. \tag{3.5}$$

We can express the solution to (3.5) directly as a function of the a_j's: indeed

$$b(t) = b_0 \exp\left(-i\lambda t \ln\left(|b_0|^2\right) - \frac{i}{2}\sum_{j=1}^{d} A_j(t) - i\lambda \sum_{j=1}^{d} \operatorname{Im}\int_0^t A_j(s) s ds\right),$$

where we have set

$$A_j(t) := \int_0^t a_j(s) ds\,.$$

In particular, the dispersive properties of u are measured by

$$|b(t)|^2 = |b_0|^2 e^{\sum_{j=1}^d \operatorname{Im} A_j(t)} = |b_0|^2 e^{\sum_{j=1}^d \operatorname{Im}\int_0^t a_j(s) ds}\,.$$

Since the equations (3.4) are decoupled as j varies, we simply consider from now on

$$i\dot{a} - a^2 = 2\lambda \operatorname{Re} a\,, \quad a_{|t=0} = a_0 = \alpha_0 + i\beta_0\,, \tag{3.6}$$

which amounts to assuming $d = 1$ in (1.6). Note that β_0 is actually zero in our context but it is not more difficult to deal with that more general case. Following [32], we seek a of the form

$$a = -i\frac{\dot{\omega}}{\omega}\cdot$$

Then (3.6) becomes

$$\ddot{\omega} = 2\lambda\omega \operatorname{Im}\frac{\dot{\omega}}{\omega}\cdot$$

Introducing the polar decomposition $\omega = re^{i\theta}$, we get

$$\begin{cases} \ddot{r} - (\dot{\theta})^2 r = 2\lambda r\dot{\theta} \\ \ddot{\theta} r + 2\dot{\theta}\dot{r} = 0\,. \end{cases}$$

Notice that

$$\dot{\theta}_{|t=0} = \alpha_0\,, \quad \left(\frac{\dot{r}}{r}\right)_{|t=0} = -\beta_0\,.$$

We therefore have a degree of freedom to set $r(0)$, and we decide $r(0) = 1$ so

$$\dot{\theta}(0) = \operatorname{Re} a_0 = \alpha_0\,, \quad \dot{r}(0) = -\operatorname{Im} a_0 = -\beta_0\,.$$

The equation in θ yields

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0\,,$$

so $r^2\dot{\theta}$ is constant and we can express the problem in terms of r only: we write

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot{r}(t)}{r(t)}\,, \tag{3.7}$$

with

$$\ddot{r} = \frac{\alpha_0^2}{r^3} + 2\lambda\frac{\alpha_0}{r}\,, \quad r(0) = 1\,,\ \dot{r}(0) = -\beta_0\,. \tag{3.8}$$

Multiplying by $\dot{r}$ and integrating, we infer

$$(\dot{r})^2 = \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r^2}\right) + 4\lambda\alpha_0 \ln r\,. \tag{3.9}$$

Back to the solution u, in the case when $d=1$ then writing in view of (3.2) and (3.7),

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} = |b(t)| = |b_0| \exp\Big(\frac{1}{2}\int_0^t \operatorname{Im} a(s)ds\Big) = \frac{|b_0|}{\sqrt{r(t)}},$$

we find that the study of $r(t)$ is enough to find the dispersion rate of $u(t)$. Once the rate in one space dimension is known, the result in d space dimensions follows directly.

Moreover recalling (3.3), we have

$$\begin{aligned}
\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} &= \frac{1}{2}\pi^{d/2}\frac{|b(t)|^2}{\Big(\prod_{j=1}^d \operatorname{Re} a_j(t)\Big)^{1/2}}\sum_{j=1}^d \frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))}\\
&= \frac{\pi^{d/2}|b_0|^2}{2\Big(\prod_{j=1}^d r_j(t)\Big)\Big(\prod_{j=1}^d \operatorname{Re} a_j(t)\Big)^{1/2}}\sum_{j=1}^d \frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))}\\
&= \frac{\pi^{d/2}|b_0|^2}{2\sqrt{\prod_{j=1}^d \alpha_{0j}}}\sum_{j=1}^d \Big((\dot r_j)^2 + \frac{\alpha_0^2}{r_j^2}\Big)\frac{1}{\alpha_{0j}}\\
&= c + 2\lambda\frac{\pi^{d/2}|b_0|^2}{2\sqrt{\prod_{j=1}^d \alpha_{0j}}}\sum_{j=1}^d \ln r_j(t)\,.
\end{aligned}$$

As soon as $r_j(t)\to\infty$ when $|t|\to\infty$, the H^1 norm therefore becomes unbounded. This is proved to be the case below (with an explicit rate): actually it can be seen from the rate provided in Lemma 3.8 below that the energy remains bounded because the unbounded contributions of both parts of the energy cancel exactly.

3.2 Study of $r(t)$

The aim of this paragraph is to prove the following result. Recall that as defined in (1.7), $\ell(t) = \frac{\ln\ln t}{\ln t}$.

Lemma 3.1 *Let r solve (3.8). Then as $t\to\infty$, there holds*

$$r(t) = 2t\sqrt{\lambda\alpha_0\ln t}\,\Big(1+\mathcal{O}\big(\ell(t)\big)\Big)\,.$$

The proof of the lemma is achieved in three steps: first, we prove that $r(t)\to\infty$ as $t\to\infty$. In view of that result it is natural to approximate the solution to (3.8) by

$$\ddot r_{\rm eff} = 2\lambda\frac{\alpha_0}{r_{\rm eff}}\,,\quad r_{\rm eff}(T) = r(T)\,,\ \dot r_{\rm eff}(T) = \dot r(T)\,, \tag{3.10}$$

for $T\gg 1$. This approximation is justified in the second step, along with a first estimate on the large time behaviour of $r_{\rm eff}$. The conclusion of the proof is achieved in a third step, by proving Lemma 1.2.

First step: $r(t)\to\infty$. We readily see from (3.9) that r is bounded from below:

$$\exists\delta>0\,,\quad r(t)\geqslant\delta\,,\quad \forall t\in\mathbb{R}\,.$$

Indeed, if it were not so, there would exist a sequence t_n such that $r(t_n)\to 0$: for n large, the right hand side of (3.9) then becomes negative, hence a contradiction.

Now let us prove that $r(t) \to +\infty$ as $t \to +\infty$. Assume first that $\dot r(0) > 0$. Then (3.8) yields $\ddot r \geqslant 0$, hence $\dot r(t) \geqslant \dot r(0)$ for all $t \geqslant 0$, and

$$r(t) \geqslant \dot r(0)t + 1 \underset{t\to+\infty}{\longrightarrow} +\infty\,. \tag{3.11}$$

On the other hand, for $\dot r(0) \leqslant 0$, assume that r is bounded, $r(t) \leqslant M$. Then (3.8) yields

$$\ddot r(t) \geqslant \frac{\alpha_0^2}{M^3} + 2\lambda\frac{\alpha_0}{M}\,,$$

hence a contradiction for t large enough. We infer that for T sufficiently large, there holds $r(T) \geqslant 1$ and $\dot r(T) > 0$. The first case then implies $r(t) \to +\infty$.

Note that we have proved in particular that

$$\exists T \geqslant 1\,, \quad \dot r(T) > 0 \quad \text{and} \quad \forall t \geqslant T\,, \quad r(t) \geqslant \dot r(T)(t-T) + 1\,. \tag{3.12}$$

Second step: $r(t) \sim r_{\rm eff}(t)$ with a rough bound. Let us prove the following result.

Lemma 3.2 *There is T large enough so that defining $r_{\rm eff}$ the solution of* (3.10) *then as $t \to \infty$, there holds*

$$|r_{\rm eff}(t)| = 2t\sqrt{\lambda\alpha_0 \ln t} + \epsilon(t\sqrt{\ln t})\,, \quad \textit{and} \quad |r(t) - r_{\rm eff}(t)| \leqslant C(T)t\,, \quad \forall t \geqslant T\,,$$

where $\epsilon(t)/t$ goes to zero as t goes to infinity.

Proof. Let us start by studying $r_{\rm eff}$. Multiplying (3.10) by $\dot r_{\rm eff}$ and integrating, we get

$$\begin{aligned}(\dot r_{\rm eff}(t))^2 &= (\dot r(T))^2 + 4\lambda\alpha_0 \ln r_{\rm eff}(t) - 4\lambda\alpha_0 \ln r(T)\\ &= 4\lambda\alpha_0 \ln r_{\rm eff}(t) + \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r(T)^2}\right)\,,\end{aligned}$$

where we have used (3.9) at time $t = T$. Denote by

$$C_0 := \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r(T)^2}\right).$$

By similar arguments as in the proof of (3.12), we have $\dot r_{\rm eff}(t) > 0$ for all $t \geqslant T$, and

$$r_{\rm eff}(t) \geqslant \dot r(T)(t-T) + 1,$$

hence

$$\dot r_{\rm eff}(t) = \sqrt{4\lambda\alpha_0 \ln r_{\rm eff}(t) + C_0}\,.$$

Separating the variables,

$$\frac{dr_{\rm eff}}{\sqrt{4\lambda\alpha_0 \ln r_{\rm eff} + C_0}} = dt\,,$$

so we naturally consider the anti-derivative

$$I := \int \frac{dr}{\sqrt{4\lambda\alpha_0 \ln r + C_0}}\cdot$$

The change of variable

$$y := \sqrt{4\lambda\alpha_0 \ln r + C_0}$$

yields

$$I = \frac{1}{2\lambda\alpha_0}\int e^{(y^2-C_0)/(4\lambda\alpha_0)}dy\,.$$

Since for x large (Dawson function, see e.g. [1]),

$$\int e^{x^2}dx \sim \frac{1}{2x}e^{x^2}\,,$$

we infer

$$I \sim \frac{r}{\sqrt{4\lambda\alpha_0\ln r + C_0}}\,.$$

In particular,

$$\frac{r_{\rm eff}(t)}{\sqrt{4\lambda\alpha_0\ln r_{\rm eff}(t) + C_0}} \underset{t\to+\infty}{\sim} t\,,$$

hence

$$\frac{r_{\rm eff}(t)}{\sqrt{\ln r_{\rm eff}(t)}} \underset{t\to+\infty}{\sim} 2t\sqrt{\lambda\alpha_0}\,.$$

We conclude that

$$r_{\rm eff}(t) \underset{t\to+\infty}{\sim} 2t\sqrt{\lambda\alpha_0\ln t}\,.$$

Now let us prove that r can be well approximated by $r_{\rm eff}$. We define $h := r - r_{\rm eff}$ and we want to prove that if T is chosen large enough, then $h(t) \lesssim t$ when $t\to\infty$. We have

$$\begin{aligned}\dot h(t) &= \sqrt{4\lambda\alpha_0\ln r(t) + \beta_0^2 + \alpha_0^2\Big(1-\frac{1}{r(t)^2}\Big)} - \sqrt{4\lambda\alpha_0\ln r_{\rm eff}(t) + \beta_0^2 + \alpha_0^2\Big(1-\frac{1}{r(T)^2}\Big)}\\ &\leqslant \sqrt{4\lambda\alpha_0\Big|\ln\frac{r(t)}{r_{\rm eff}(t)}\Big| + \alpha_0^2\Big(\frac{1}{r(T)^2}-\frac{1}{r(t)^2}\Big)}\,.\end{aligned}$$

Given $\varepsilon\in(0,1/2)$, let $T\geqslant 1$ be large enough so that for all $t\geqslant T$

$$r_{\rm eff}(t) \geqslant t\sqrt{\lambda\alpha_0\ln t} \tag{3.13}$$

and

$$\alpha_0^2\Big(\frac{1}{r(T)^2}-\frac{1}{r(t)^2}\Big) \leqslant \varepsilon^2\,. \tag{3.14}$$

We shall also need that

$$\left(2\frac{(\lambda\alpha_0)^{\frac14}}{\sqrt{\ln T}} + \varepsilon\right) \leqslant \frac12\,. \tag{3.15}$$

Then noticing that

$$\Big|\ln\frac{r(t)}{r_{\rm eff}(t)}\Big| = \Big|\ln\Big(1+\frac{h(t)}{r_{\rm eff}(t)}\Big)\Big| \leqslant \frac{|h(t)|}{r_{\rm eff}(t)} \leqslant \frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln t}} \leqslant \frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln T}}$$

as soon as $t\geqslant T$ thanks to (3.13), we infer that

$$\forall t\geqslant T\,,\quad \dot h(t) \leqslant \varepsilon + 2\sqrt{\lambda\alpha_0}\left(\frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln T}}\right)^{\frac12}\,,\quad \text{with}\quad h(T)=0\,.$$

Our goal is to prove that the function $t \mapsto h(t)/t$ is bounded for large t, so let $T^* > T$ be the maximal time such that

$$\forall t \in [T, T^*)\,, \quad |h(t)| \leqslant t\,.$$

Then for $t \in [T, T^*)$,

$$\dot h(t) \leqslant \varepsilon + 2(\lambda\alpha_0)^{\frac14} \frac{1}{\sqrt{\ln T}}$$

so thanks to (3.15)

$$h(t) \leqslant \Big(\varepsilon + 2(\lambda\alpha_0)^{\frac14} \frac{1}{\sqrt{\ln T}}\Big)(t-T) \leqslant \frac{t}{2}\,,$$

which contradicts the maximality of T^*. The result follows, and Lemma 3.2 is proved. □

Third step: $r(t) \sim r_{\rm eff}(t)$ with improved bound. Let us end the proof of Lemma 3.1. By (3.9) and as in the previous paragraph, we have for T sufficiently large so that $\dot r(t) \geqslant \dot r(T) > 0$ for $t \geqslant T$:

$$\dot r = \sqrt{C_0 + \alpha_0^2 \left(\frac{1}{r(T)^2} - \frac{1}{r^2}\right) + 4\lambda\alpha_0 \ln r}\,,$$

with the same constant C_0 as above: recall that

$$\dot r_{\rm eff} = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\rm eff}}\,.$$

To lighten notation let us recall that $h := r - r_{\rm eff}$ and let us define

$$R_{\rm eff} := C_0 + 4\lambda\alpha_0 \ln r_{\rm eff}\,.$$

Then using a Taylor expansion for $\dot r$, we have:

$$\begin{aligned}\dot r &= \sqrt{R_{\rm eff} + \alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\lambda\alpha_0 \ln\big(1 + \frac{h}{r_{\rm eff}}\big)} \\ &= \sqrt{R_{\rm eff}}\sqrt{1 + \frac{1}{R_{\rm eff}}\alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\frac{\lambda\alpha_0}{R_{\rm eff}} \ln\big(1 + \frac{h}{r_{\rm eff}}\big)}\,.\end{aligned}$$

On the one hand we know that $R_{\rm eff} \to \infty$ and by Lemma 3.2 we have $h \lesssim t$ and $r_{\rm eff} \underset{t\to\infty}{\sim} t\sqrt{\ln t}$ so we infer that

$$\dot r \underset{t\to\infty}{\sim} \sqrt{R_{\rm eff}}\Big(1 + \frac{1}{2R_{\rm eff}}\Big(\alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\lambda\alpha_0 \ln\big(1 + \frac{h}{r_{\rm eff}}\big)\Big)\Big)\,.$$

As a consequence

$$\dot r - \dot r_{\rm eff} \underset{t\to\infty}{\sim} \frac{1}{2\sqrt{R_{\rm eff}}}\Big(\alpha_0^2\big(\frac{1}{r(T)^2} - \frac{1}{r^2}\big) + 4\lambda\alpha_0 \ln\big(1 + \frac{h}{r_{\rm eff}}\big)\Big)$$

and since $h/r_{\rm eff} = \mathcal{O}(1/\sqrt{\ln t})$ we infer that

$$\dot r - \dot r_{\rm eff} \underset{t\to\infty}{\sim} \frac{C(T)}{\sqrt{\lambda \ln t}}\,.$$

By integration, and comparison of diverging integrals, we find

$$h(t) \underset{t\to\infty}{\sim} C_1 \frac{t}{\sqrt{\ln t}}\,,$$

hence

$$r(t) = 2t\sqrt{\lambda\alpha_0 \ln t}\left(1 + \mathcal{O}\big(\ell(t)\big)\right),$$

as soon as we know that this holds for $r_{\rm eff}$. Lemma 3.1 is therefore proved, up to the study of the universal dispersion τ.

3.3 Study of the universal dispersion $\tau(t)$: proof of Lemma 1.2

It remains to prove Lemma 1.2. By scaling, we may assume $\lambda = 1$, to lighten the notations. Introduce the approximate solution

$$\tau_{\rm eff}(t) := 2t\sqrt{\ln t}.$$

We have clearly

$$\sqrt{\ln t} = \sqrt{\ln \tau_{\rm eff}}\left(1 + \mathcal{O}\left(\frac{\ln\ln t}{\ln \tau_{\rm eff}}\right)\right).$$

In view of a comparison with (1.10), which reads

$$\dot\tau = 2\sqrt{\ln \tau}\,,$$

write

$$\dot\tau_{\rm eff} = 2\sqrt{\ln t} + \frac{1}{\sqrt{\ln t}} = 2\sqrt{\ln \tau_{\rm eff}}\left(1 + \mathcal{O}\left(\frac{\ln\ln t}{\ln \tau_{\rm eff}}\right)\right) = 2\sqrt{\ln \tau_{\rm eff}} + \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right).$$

Thus,

$$\begin{aligned}\dot\tau - \dot\tau_{\rm eff} &= 2\left(\sqrt{\ln \tau} - \sqrt{\ln \tau_{\rm eff}}\right) + \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right)\\ &= 2\sqrt{\ln \tau_{\rm eff} + \ln\frac{\tau}{\tau_{\rm eff}}} - 2\sqrt{\ln \tau_{\rm eff}} + \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right).\end{aligned}$$

Since we already know from Lemma 3.2 that $\tau/\tau_{\rm eff} \to 1$ as $t\to\infty$, we obtain

$$\dot\tau - \dot\tau_{\rm eff} = \mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right), \quad \text{and} \quad \tau - \tau_{\rm eff} = \mathcal{O}\left(t\frac{\ln\ln t}{\sqrt{\ln t}}\right),$$

by integration. This proves Lemma 1.2. □

Back to the previous section, we simply note that

$$\dot r_{\rm eff} - \sqrt{\alpha_0}\dot\tau = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\rm eff}} - \sqrt{4\lambda\alpha_0\ln\tau}\,,$$

with $C_0 \neq 0$ in general, so the same computation as above yields

$$\dot r_{\rm eff} - \sqrt{\alpha_0}\dot\tau = \mathcal{O}\left(\frac{1}{\sqrt{\ln r_{\rm eff}}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right),$$

hence

$$r_{\rm eff} - \sqrt{\alpha_0}\tau = \mathcal{O}\left(\frac{t}{\sqrt{\ln t}}\right),$$

by integration. This completes the proof of Lemma 3.1.

4 General a priori estimates and first convergence results

From now on, we suppose that the assumptions of Theorem 1.3 are satisfied: $u_0 \in \Sigma$ is such that $\|u_0\|_{L^2} = \|\gamma\|_{L^2}$, where we recall that $\gamma(y) = e^{-|y|^2/2}$.

4.1 First a priori estimates

Recall that by definition, v is related to u through the relation

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right), \tag{4.1}$$

where τ is the solution to

$$\ddot\tau = \frac{2\lambda}{\tau}, \quad \tau(0) = 1, \quad \dot\tau(0) = 0.$$

Then v solves

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2 - \lambda v \ln\tau, \quad v_{|t=0} = u_0.$$

Using a gauge transform (by replacing v with $ve^{i\theta(t)}$ for $\dot\theta = \lambda\ln\tau$), we may assume that the last term is absent, and we focus our attention on

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2, \quad v_{|t=0} = u_0. \tag{4.2}$$

Because we now have a non-autonomous equation, the Hamiltonian structure of (1.6) is lost. We compute

$$E(t) := \mathrm{Im}\int_{\mathbb{R}^d} \bar v(t,y)\partial_t v(t,y)dy = E_{\rm kin}(t) + \lambda E_{\rm ent}(t),$$

where

$$E_{\rm kin}(t) := \frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2$$

is the kinetic energy and

$$E_{\rm ent}(t) := \int_{\mathbb{R}^d} |v(t,y)|^2 \ln\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy$$

is a relative entropy. The transform (4.1) is unitary on $L^2(\mathbb{R}^d)$ so the conservation of mass for u trivially corresponds to the conservation of mass for v:

$$\|v(t)\|_{L^2} = \|u_0\|_{L^2} = \|\gamma\|_{L^2}. \tag{4.3}$$

Thanks to (4.3), the Csiszár-Kullback inequality yields

$$E_{\rm ent}(t) \gtrsim \left\||v(t)|^2 - \gamma^2\right\|_{L^1(\mathbb{R}^d)},$$

hence in particular $E_{\rm ent} \geqslant 0$, which is another way of justifying the term "defocusing" for the case $\lambda > 0$. We easily compute

$$\dot E = -2\frac{\dot\tau}{\tau}E_{\rm kin}. \tag{4.4}$$

We now prove the first part of Theorem 1.3, that is, (1.12) which is recast and complemented in the next lemma.

Lemma 4.1 *Under the assumptions of Theorem* 1.3*, there holds*

$$\sup_{t\geqslant 0}\left(\int_{\mathbb{R}^d}\left(1+|y|^2+\left|\ln|v(t,y)|^2\right|\right)|v(t,y)|^2dy+\frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)}\right)<\infty$$

and

$$\int_0^\infty \frac{\dot\tau(t')}{\tau^3(t')}\|\nabla_y v(t')\|^2_{L^2(\mathbb{R}^d)}dt'<\infty. \tag{4.5}$$

Proof. Write

$$E_{\rm ent}=\int_{\mathbb{R}^d}|v|^2\ln|v|^2+\int_{\mathbb{R}^d}|y|^2|v|^2\,,$$

and

$$\int_{\mathbb{R}^d}|v|^2\ln|v|^2=\int_{|v|>1}|v|^2\ln|v|^2+\int_{|v|<1}|v|^2\ln|v|^2.$$

We have

$$E_+:=E_{\rm kin}+\lambda\int_{|v|>1}|v|^2\ln|v|^2+\lambda\int_{\mathbb{R}^d}|y|^2|v|^2\leqslant E(0)+\lambda\int_{|v|<1}|v|^2\ln\frac{1}{|v|^2}\cdot$$

The last term is controlled by

$$\int_{|v|<1}|v|^2\ln\frac{1}{|v|^2}\lesssim\int_{\mathbb{R}^d}|v|^{2-\varepsilon}\,,$$

for all $\varepsilon>0$. We conclude thanks to the estimate

$$\int_{\mathbb{R}^d}|v|^{2-\varepsilon}\lesssim\|v\|_{L^2}^{2-(1+d/2)\varepsilon}\|yv\|_{L^2}^{d\varepsilon/2}\,,$$

for $\varepsilon>0$ sufficiently small ($0<\varepsilon<\frac{4}{d+2}$), which can be readily proved by an interpolation method (cutting the integral into $|y|<R$ and $|y|>R$, using Hölder inequality and optimizing over R; see e.g. [9]). This implies

$$E_+\lesssim 1+E_+^{d\varepsilon/4}\,,$$

and thus $E_+\in L^\infty(\mathbb{R})$. Finally, (4.5) follows from (4.4), since $E(t)\geqslant 0$ for all $t\geqslant 0$. □

At this stage, we can already infer the first part of Corollary 1.1. Indeed, recalling the conservation of the energy for u, we have, in view of (4.1),

$$\begin{aligned}\frac12\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}&=\underbrace{\frac12\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}+\lambda\int_{\mathbb{R}^d}|u(t,x)|^2\ln|u(t,x)|^2dx}_{E(u(t))}-\lambda\int_{\mathbb{R}^d}|u(t,x)|^2\ln|u(t,x)|^2dx\\
&=E(u_0)-\lambda\int_{\mathbb{R}^d}|u(t,x)|^2\ln|u(t,x)|^2dx\\
&=E(u_0)-\lambda\int_{\mathbb{R}^d}\frac{1}{\tau^d}\left|v\left(t,\frac{x}{\tau}\right)\right|^2\ln\left(\frac{1}{\tau^d}\left|v\left(t,\frac{x}{\tau}\right)\right|^2\right)dx\\
&=E(u_0)-\lambda\int_{\mathbb{R}^d}|v\,(t,y)|^2\ln\left(\frac{1}{\tau^d}\,|v\,(t,y)|^2\right)dy\\
&=E(u_0)-\lambda\int_{\mathbb{R}^d}|v\,(t,y)|^2\ln\left(|v\,(t,y)|^2\right)dy+\lambda d\ln\tau(t)\|v(t)\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

The first two terms are bounded (from Lemma 4.1), so Lemma 1.2 and the conservation of the mass yield

$$\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} \lambda d \ln t \|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

We emphasize that the proof of the second part of Corollary 1.1 (estimates in $\dot{H}^s$ for $0 < s < 1$) requires the convergence in Wasserstein distance, which in turn relies on the convergence of the momenta (along with the weak convergence of $|v|^2$), which is established in the next subsection.

4.2 Convergence of some quadratic quantities

Let us prove (1.13), as stated in the next lemma.

Lemma 4.2 *Under the assumptions of Theorem* 1.3*, there holds*

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy\,.$$

Proof. The first line is trivial, in view of the conservation of mass. Introduce

$$I_1(t) := \mathrm{Im} \int_{\mathbb{R}^d} \overline{v}(t,y) \nabla_y v(t,y) dy\,, \quad I_2(t) := \int_{\mathbb{R}^d} y|v(t,y)|^2 dy\,.$$

We compute:

$$\dot{I}_1 = -2\lambda I_2\,, \qquad \dot{I}_2 = \frac{1}{\tau^2(t)} I_1\,. \tag{4.6}$$

Set $\tilde{I}_2 := \tau I_2$: we have $\ddot{\tilde{I}}_2 = 0$, hence (unless the data are well prepared in the sense that $I_1(0) = 0$)

$$I_2(t) = \frac{1}{\tau(t)} \left(\dot{\tilde{I}}_2(0) t + \tilde{I}_2(0) \right) = \frac{1}{\tau(t)} \left(-I_1(0) t + I_2(0) \right) \underset{t\to\infty}{\sim} \frac{c}{\sqrt{\ln t}}\,,$$

and

$$I_1(t) \underset{t\to\infty}{\sim} \tilde{c} \frac{t}{\sqrt{\ln t}}\,.$$

In particular,

$$\int_{\mathbb{R}^d} y|v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} 0 = \int_{\mathbb{R}^d} y\gamma(y)^2 dy\,.$$

In order to obtain estimates for higher order quadratic observables, we follow a more direct strategy than in [8]. The price to pay is that the convergence rate that we obtain is weaker than in [8].

We go back to some conserved quantities for u and translate them into estimates on v.

- Mass: $\dfrac{d}{dt}\|u(t)\|^2_{L^2} = 0\,;$
- Energy:
$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u(t)\|^2_{L^2} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx \right) = 0\,.$$

We recall the mass conservation for v stated in (4.3). Substituting (4.1) into the conservation of the energy of u, we get

$$\frac{d}{dt}\Big(\frac{1}{2\tau^2}\|\nabla v(t)\|_{L^2}^2+\frac{(\dot\tau)^2}{2}\|yv\|_{L^2}^2-\frac{\dot\tau}{\tau}\operatorname{Im}\int_{\mathbb{R}^d}v(t,y)y\cdot\nabla\bar v(t,y)dy \\ +\lambda\int|v|^2\ln|v|^2-\lambda d\ln\tau\|v(t)\|_{L^2}^2\Big)=0\,.$$

In view of Cauchy–Schwarz inequality, Lemma 4.1 and Lemma 1.2,

$$\left|\frac{\dot\tau}{\tau}\operatorname{Im}\int_{\mathbb{R}^d}v(t,y)y\cdot\nabla\bar v(t,y)dy\right|\leqslant\dot\tau\|yv(t,y)\|_{L^2}\frac{1}{\tau}\|\nabla v(t)\|_{L^2}\lesssim\dot\tau(t)\lesssim\sqrt{\ln t}.$$

Therefore, in the above expression, all the terms are either bounded or $\mathcal{O}\left(\sqrt{\ln t}\right)$, but two:

$$\frac{(\dot\tau)^2}{2}\int|y|^2|v|^2\quad\text{and}\quad-\lambda d\ln\tau\int|v|^2\,.$$

We infer

$$\frac{(\dot\tau)^2}{2}\int|y|^2|v|^2-\lambda d\ln\tau\int|v|^2=\mathcal{O}\left(\sqrt{\ln t}\right)\,.$$

Integrating (1.10), we find

$$\frac{(\dot\tau)^2}{2}=2\lambda\ln\tau\,,$$

hence

$$\int|y|^2|v|^2-\frac{d}{2}\|v\|_{L^2}^2=\mathcal{O}\left(\frac{\sqrt{\ln t}}{\ln\tau}\right)=\mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right)\,.$$

Now we recall a property of γ:

$$\|v(t)\|_{L^2}^2=\|\gamma\|_{L^2}^2=\frac{2}{d}\|y\gamma\|_{L^2}^2.$$

The lemma is proved. □

At this stage, we therefore have proved Theorem 1.3, up to the final point regarding the asymptotic profile for $|v|^2$.

Remark 4.1 In [8], the above convergence is improved to

$$\|yv(t)\|_{L^2(\mathbb{R}^d)}^2-\|y\gamma\|_{L^2(\mathbb{R}^d)}^2=\mathcal{O}\left(\frac{1}{\ln t}\right)\,.$$

5 Proof of the main result

We now address the final point of Theorem 1.3. We emphasize that it concerns the modulus of v only, an aspect which explains the decomposition introduced in the next subsection.

5.1 Hydrodynamical approach

We recall that the Madelung transform is a classical tool (see e.g. [34, 31, 19], or the survey [7]) to relate the (nonlinear) Schrödinger equation to fluid dynamics equations, via the change of unknown

$$v(t,y) = a(t,y)e^{i\phi(t,y)}\,, \quad \text{with } a,\phi \in \mathbb{R}\,. \tag{5.1}$$

Formally one obtains in our case the system of equations

$$\begin{cases} \partial_t \phi + \dfrac{1}{2\tau^2}|\nabla_y \phi|^2 + \lambda \ln \left|\dfrac{a}{\gamma}\right|^2 = \dfrac{1}{2\tau^2}\dfrac{\Delta_y a}{a} \\ \partial_t a + \dfrac{1}{\tau^2}\nabla_y \phi \cdot \nabla_y a + \dfrac{1}{2\tau^2} a \Delta_y \phi = 0\,, \end{cases}$$

which is easily related to the compressible Euler equations by using the change of unknown

$$\rho(t,y) := a^2 \quad \Lambda := a\nabla\phi\,, \quad J := a\Lambda\,. \tag{5.2}$$

In terms of these hydrodynamical variables, the above system becomes

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla \cdot J = 0 \\ \partial_t J + \dfrac{1}{\tau^2}\nabla\cdot(\Lambda\otimes\Lambda) + \lambda\nabla\rho + 2\lambda y\rho = \dfrac{1}{4\tau^2}\Delta\nabla\rho - \dfrac{1}{\tau^2}\nabla\cdot(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}) \\ \partial_j J^k - \partial_k J^j = 2\Lambda^k\partial_j\sqrt{\rho} - 2\Lambda^j\partial_k\sqrt{\rho}\,, \quad j,k\in\{1,\dots,d\}\,. \end{cases} \tag{5.3}$$

In the case where the initial data for (5.3) are well prepared, in the sense that they stem from the polar decomposition of an initial wave function as in (5.1)–(5.2), then the approach presented in [7, Section 5] can readily be adapted to show that (5.3) holds true in the distributional sense. We shall however retain simply one property related to this system: as soon as we have a solution v to (4.2), it can be decomposed as in (5.1)–(5.2) so as to produce a solution to (5.3). The most delicate issue to prove this is to give a suitable meaning to the phase ϕ when v vanishes; we refer to [7, Section 5] for details.

We shall prove that

$$\rho(t) \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d)\,.$$

This will stem from the fact that the weak limit of ρ evolves according to a Fokker–Planck operator. We note that a formal link between the hydrodynamical formulation of (1.6) and the Fokker–Planck equation can be found in [33, 26].

5.2 Heuristics

Let us explain the heuristics of the proof, which will be made rigorous in the next section. Formally only retaining the higher order terms (in terms of growth in time) in (5.3) we are led to studying the following simple model

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla\cdot J = 0 \\ \partial_t J + \lambda\nabla\rho + 2\lambda y\rho = 0\,. \end{cases} \tag{5.4}$$

Note that in the explicit case of the evolution of a Gaussian (recall the computations of Section 3), we can check that in the above simplification, we have indeed eliminated negligible terms. By elimination of J, (5.4) implies that

$$\partial_t \left(\tau^2 \partial_t \rho\right) = \lambda \nabla \cdot (\nabla + 2y)\, \rho = \lambda L \rho\,,$$

where

$$L := \Delta_y + \nabla_y \cdot (2y\, \cdot)$$

is a Fokker–Planck operator (associated to the harmonic potential). On the other hand,

$$\partial_t \left(\tau^2 \partial_t \rho\right) = \tau^2 \partial_t^2 \rho + 2\dot\tau \tau \partial_t \rho\,,$$

so since $1 \ll \tau^2 \ll (\dot\tau\tau)^2$, it is natural to expect the first order time derivative to dominate over the second order time derivative, and change scales in time accordingly. Define s such that

$$\frac{\dot\tau\tau}{\lambda}\partial_t = \partial_s\,,$$

or in other words define the following change of variables:

$$s = \int \frac{1}{\lambda \dot\tau \tau} = \int \frac{\ddot\tau}{2\dot\tau} = \frac{1}{2}\ln \dot\tau(t)\,. \tag{5.5}$$

Notice that

$$s \sim \frac{1}{4}\ln\ln t\,, \quad t \to \infty\,. \tag{5.6}$$

Then again discarding formally lower order terms we find

$$\partial_s \rho = L\rho\,,$$

for which it is well-known (see for instance [18]) that in large times the solution converges strongly to an element of the kernel of L, hence a Gaussian. Notice that the convergence is exponentially fast in s variables, so returning to t variables produces a logarithmic decay due to (5.6): we recover the logarithmic convergence rate observed in the Gaussian case (Section 3).

The difficulty to make this argument rigorous is the justification that the lower order terms may indeed be discarded, since we have very little control on higher norms on v to guarantee compactness in space of the solution: we have more precisely a sharp control of the momenta of v, but rather poor estimates in H^1. We do expect v to oscillate rapidly in time (in view of the Gaussian case), but $\sqrt{\rho}$ should be bounded in H^1, a property that does not seem easy to prove (because of the prefactor $1/\tau^2$ in the equation). This is the main obstacle to proving strong convergence to a Gaussian in the general case, and explains why in the end we only obtain a weak convergence result in L^1. This is made precise in the next section.

5.3 End of the proof

Let us follow the steps of the previous paragraph, this time neglecting no term. First, we consider a variant of the hydrodynamical formulation of (4.2), by recalling that the two nonlinear

terms in (5.3) correspond exactly to $\tau^{-2}\nabla|\nabla v|^2$, after the polar decomposition of v. Therefore, we simply use the fact that if $v = \sqrt{\rho}e^{i\phi}$, then we have

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla \cdot J = 0 \\ \partial_t J + \lambda \nabla \rho + 2\lambda y \rho = \dfrac{1}{4\tau^2}\Delta\nabla\rho - \dfrac{1}{\tau^2}\nabla|\nabla v|^2 \,. \end{cases} \tag{5.7}$$

By elimination of J,

$$\partial_t \left(\tau^2 \partial_t \rho\right) = -\partial_t \nabla \cdot J = \lambda L\rho - \frac{1}{4\tau^2}\Delta^2\rho - \frac{1}{\tau^2}\Delta|\nabla v|^2 \,,$$

with again $L := \Delta + \nabla \cdot (2y\,\cdot)$. With the change of variable (5.5) we introduce the notation $\tilde{\rho}(s(t), y) := \rho(t, y)$, and we find for $\tilde{\rho}$ the following equation:

$$\partial_s \tilde{\rho} - \frac{2\lambda}{(\dot{\tau})^2}\partial_s\tilde{\rho} + \frac{\lambda}{(\dot{\tau})^2}\partial_s^2\tilde{\rho} = L\tilde{\rho} - \frac{1}{4\lambda\tau^2}\Delta^2\tilde{\rho} - \frac{1}{\lambda\tau^2}\Delta|\nabla\tilde{v}|^2 \,, \tag{5.8}$$

where one should keep in mind that the functions τ and $\dot{\tau}$ also have undergone the change of time variable. In terms of s, Lemma 1.2 yields

$$\dot{\tau}(s) \underset{s\to\infty}{\sim} 2\sqrt{\lambda}e^{2s} \,, \quad \tau(s) \underset{s\to\infty}{\sim} 2\sqrt{\lambda}e^{2s+e^{4s}} \,.$$

In terms of s, the time integrability property of $E_{\rm kin}$ provided in (4.5) becomes

$$\int_0^\infty \left(\frac{\dot{\tau}(s)}{\tau(s)}\right)^2 \|\nabla\tilde{v}(s)\|_{L^2}^2 ds < \infty \,. \tag{5.9}$$

On the other hand, Lemma 4.1 yields

$$\sup_{s\geqslant 0}\int_{\mathbb{R}^d} \tilde{\rho}(s,y)\left(1 + |y|^2 + |\ln\tilde{\rho}(s,y)|\right) dy < \infty \,. \tag{5.10}$$

Mimicking the general approach of e.g. [16, 17], for $s \in [-1, 2]$ and $s_n \to \infty$, set

$$\tilde{\rho}_n(s, y) := \tilde{\rho}(s + s_n, y) \,.$$

From (5.10) along with the de la Vallée-Poussin and Dunford–Pettis Theorems, we get up to extracting a subsequence

$$\tilde{\rho}_n \rightharpoonup \tilde{\rho}_\infty \quad \text{in } L^p_s(-1, 2; L^1_y) \,,$$

for all $p \in [1, \infty)$. Up to another subsequence,

$$\tilde{\rho}_n(0) \rightharpoonup \tilde{\rho}_{0,\infty} \quad \text{in } L^1_y \,.$$

In view of (5.8):

$$\begin{cases} \partial_s\tilde{\rho}_\infty = L\tilde{\rho}_\infty \text{ in } \mathcal{S}'\left((-1,2)\times\mathbb{R}^d\right), \\ \tilde{\rho}_{\infty|s=0} = \tilde{\rho}_{0,\infty} \in L^1 \,. \end{cases} \tag{5.11}$$

We now go back to (5.7) and show that $\tilde{\rho}_\infty$ is independent of s. In the s variable, we have

$$\begin{cases} \partial_s\tilde{\rho} + \dfrac{\dot{\tau}}{\lambda\tau}\nabla\cdot\tilde{J} = 0 \\ \partial_s\tilde{J} + \tau\dot{\tau}\left(\nabla + 2y\right)\tilde{\rho} - \dfrac{\dot{\tau}}{4\lambda\tau}\nabla\Delta\tilde{\rho} = -\dfrac{\dot{\tau}}{\lambda\tau}\nabla|\nabla\tilde{v}|^2 \,. \end{cases} \tag{5.12}$$

Since $J = \mathrm{Im}\,\bar v \nabla_y v$, (5.9) implies

$$\frac{\dot\tau}{\tau}\tilde J \in L^2_s L^1_y\,.$$

With $\tilde J_n(s) := \tilde J(s+s_n)$, we have

$$\frac{\dot\tau}{\lambda\tau}\nabla\cdot\tilde J_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(-1,2;W^{-1,1})\,,$$

hence

$$\partial_s\tilde\rho_\infty = 0\,. \tag{5.13}$$

Putting (5.11) and (5.13) together, we have

$$L\tilde\rho_{\infty|s=1} = 0\,,$$

and since $\tilde\rho_{\infty|s=1}$ is a smooth function, we infer $\tilde\rho_\infty = \alpha\gamma^2$, for some $0\leqslant\alpha\leqslant 1$.

Using (5.10) again, we see that the family $(\tilde\rho(1+s_n,\cdot))_n$ is tight, and so $\alpha = 1$. The limit being unique, no extraction of a subsequence is needed, and we conclude

$$\tilde\rho(s) \underset{s\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d)\,.$$

6 About the corollaries

6.1 Proof of Corollary 1.1

We have seen already how to treat the case $s = 1$, so we now fix $0 < s < 1$. We may assume that $\|u_0\|_{L^2} = \|\gamma\|_{L^2}$ and use the conclusion of Theorem 1.3, since we are not tracking the multiplicative constants. The convergence in the Wasserstein distance W_2 (Remark 1.2) implies (see e.g. [38, Theorem 7.12])

$$\int |y|^{2s}|v(t,y)|^2dy \underset{t\to\infty}{\longrightarrow} \int |y|^{2s}\gamma^2(y)dy. \tag{6.1}$$

The idea is then to apply a fractional derivative to (1.11), that is

$$u(t,x) = \frac{1}{\tau(t)^{d/2}}v\left(t,\frac{x}{\tau(t)}\right)\exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right).$$

In order to shortcut this step, we recall a lemma employed in the context of semi-classical limit. We simplify the initial statement and leave out the dependence on the semi-classical parameter:

Lemma 6.1 (Lemma 5.1 from [2]) *There exists C such that if $\psi\in H^1(\mathbb{R}^d)$ and w is such that $\nabla w\in L^\infty(\mathbb{R}^d)$,*

$$\||w|^s\psi\|_{L^2} \leqslant \|\psi\|_{\dot H^s} + \|(\nabla - iw)\psi\|_{L^2}^s\|\psi\|_{L^2}^{1-s} + C\left(1+\|\nabla w\|_{L^\infty}\right)\|\psi\|_{L^2}.$$

In [2], w corresponds to the gradient of rapid oscillations carried by an exponential, so we naturally consider $\psi = u$ and introduce

$$w(t,x) = \frac{\dot\tau(t)}{\tau(t)}x.$$

In the present framework, Lemma 6.1 yields, in view of (1.11) and the conservation of the mass:

$$(\dot\tau)^s \||y|^s v(t)\|_{L^2} \leqslant \|u(t)\|_{\dot H^s} + \left\|\frac{1}{\tau}\nabla v(t)\right\|_{L^2}^s \|u_0\|_{L^2}^{1-s} + C\left(1+\frac{\dot\tau}{\tau}\right)\|u_0\|_{L^2}.$$

The result follows readily: the behaviour of the left hand side is given by Lemma 1.2 and (6.1), and all the terms of the right hand side are bounded, but the first one.

6.2 Proof of Corollary 1.2

In view of the tensorization in Theorem 1.2, we prove Corollary 1.2 in the case $d = 1$ to lighten the notations, and we assume

$$u_0(x) = b_0 \exp\left(-a_0(x-x_0)^2/2\right),$$

with $b_0, a_0 \in \mathbb{C}$, $\operatorname{Re} a_0 = \alpha_0 > 0$. We start with an initial center x_0 to show that in terms of v, the center is eventually zero (like in [18]). Recall that we have

$$u(t,x) = b_0 \frac{1}{\sqrt{r(t)}} e^{i\phi(t)} \exp\left(-\alpha_0 \frac{(x-x_0)^2}{2r^2(t)} + i\frac{\dot r(t)}{r(t)}\frac{(x-x_0)^2}{2}\right),$$

with r solution to (3.8), $r(0) = 1$, $\dot r(0) = -\operatorname{Im} a_0$. We thus have

$$\begin{aligned} v(t,y) = b_0 \sqrt{\frac{\tau(t)}{r(t)}} e^{i\phi(t)} &\exp\left(-\alpha_0\frac{\tau^2}{r^2}\frac{y^2}{2} + \alpha_0\frac{\tau}{r^2}yx_0 - \alpha_0\frac{x_0^2}{2r^2}\right) \\ &\times \exp\left(i\left(\frac{\dot r}{r} - \frac{\dot\tau}{\tau}\right)\tau^2\frac{y^2}{2} - i\frac{\dot r}{r}\tau y x_0 + i\frac{\dot r}{r}\frac{x_0^2}{2}\right). \end{aligned}$$

In particular,

$$|v(t,y)|^2 = |b_0|^2 \frac{\tau(t)}{r(t)} \exp\left(-\alpha_0\frac{\tau^2}{r^2}y^2 + 2\alpha_0\frac{\tau}{r^2}yx_0 - \alpha_0\frac{x_0^2}{r^2}\right).$$

On the other hand,

$$\|u_0\|_{L^2} = |b_0|\left(\frac{\pi}{\alpha_0}\right)^{1/4} = \pi^{1/4},$$

where the last equality corresponds to our assumption motivated by the effect of scaling factors. Therefore, the relative entropy is

$$\begin{aligned} E_{\rm ent}(t) &= \int_{\mathbb{R}} |v(t,y)|^2 \ln\left(\frac{|v(t,y)|^2}{\gamma^2(y)}\right) dy \\ &= \ln\left(\sqrt{\alpha_0}\frac{\tau(t)}{r(t)}\right)\|u_0\|_{L^2}^2 - \left(\alpha_0\frac{\tau(t)^2}{r(t)^2} - 1\right)\int_{\mathbb{R}} y^2|v(t,y)|^2 dy \\ &\quad + 2\alpha_0 x_0 \frac{\tau(t)}{r^2(t)}\int_{\mathbb{R}} y|v(t,y)|^2 dy - \alpha_0\frac{x_0^2}{r^2(t)}\|u_0\|_{L^2}^2 \underset{t\to\infty}{\longrightarrow} 0, \end{aligned}$$

where we have used the properties of the solutions to (3.8) and (1.10), established in Section 3. The end of the corollary simply stems from the standard Csiszár-Kullback inequality

$$\left\||v(t,\cdot)|^2 - \gamma^2\right\|_{L^1} \lesssim E_{\rm ent}.$$

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References

[1] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] T. Alazard and R. Carles, *Loss of regularity for super-critical nonlinear Schrödinger equations*, Math. Ann. **343** (2009), no. 2, 397–420.

[3] Alex H. Ardila, *Orbital stability of Gausson solutions to logarithmic Schrödinger equations*, Electron. J. Differential Equations 2016, Paper No. 335, 9 pp.

[4] Iwo Białynicki-Birula and Jerzy Mycielski, *Nonlinear wave mechanics*, Ann. Physics **100** (1976), no. 1-2, 62–93.

[5] J. Bourgain, *On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE*, Internat. Math. Res. Notices (1996), no. 6, 277–304.

[6] H. Buljan, A. Šiber, M. Soljačic, T. Schwartz, M. Segev, and D. N. Christodoulides, *Incoherent white light solitons in logarithmically saturable non-instantaneous nonlinear media*, Phys. Rev. E **68** (2003), 036697.

[7] R. Carles, R. Danchin, and J.-C. Saut, *Madelung, Gross-Pitaevskii and Korteweg*, Nonlinearity **25** (2012), no. 10, 2843–2873.

[8] R. Carles and I. Gallagher, *Universal dynamics for the defocusing logarithmic Schrödinger equation*, preprint. Archived at `https://hal.archives-ouvertes.fr/hal-01398526`, 2016.

[9] R. Carles and L. Miller, *Semiclassical nonlinear Schrödinger equations with potential and focusing initial data*, Osaka J. Math. **41** (2004), no. 3, 693–725.

[10] T. Cazenave, *Stable solutions of the logarithmic Schrödinger equation*, Nonlinear Anal. **7** (1983), no. 10, 1127–1140.

[11] ———, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[12] T. Cazenave and A. Haraux, *Équations d'évolution avec non linéarité logarithmique*, Ann. Fac. Sci. Toulouse Math. (5) **2** (1980), no. 1, 21–51.

[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Invent. Math. **181** (2010), no. 1, 39–113.

[14] M. Combescure and D. Robert, *Coherent states and applications in mathematical physics*, Theoretical and Mathematical Physics, Springer, Dordrecht, 2012.

[15] Pietro d'Avenia, Eugenio Montefusco, and Marco Squassina, *On the logarithmic Schrödinger equation*, Commun. Contemp. Math. **16** (2014), no. 2, 1350032, 15.

[16] L. Desvillettes, *Convergence to equilibrium in large time for Boltzmann and B.G.K. equations*, Arch. Rational Mech. Anal. **110** (1990), no. 1, 73–91.

[17] Eduard Feireisl and Hana Petzeltová, *Large-time behaviour of solutions to the Navier-Stokes equations of compressible flow*, Arch. Ration. Mech. Anal. **150** (1999), no. 1, 77–96.

[18] Thierry Gallay and C. Eugene Wayne, *Global stability of vortex solutions of the two-dimensional Navier-Stokes equation*, Comm. Math. Phys. **255** (2005), no. 1, 97–129.

[19] I. Gasser and P. A. Markowich, *Quantum hydrodynamics, Wigner transforms and the classical limit*, Asymptot. Anal. **14** (1997), no. 2, 97–116.

[20] Patrick Gérard and Sandrine Grellier, *Effective integrable dynamics for a certain nonlinear wave equation*, Anal. PDE **5** (2012), no. 5, 1139–1155.

[21] ———, *An explicit formula for the cubic Szegö equation*, Trans. Amer. Math. Soc. **367** (2015), no. 4, 2979–2995.

[22] J. Ginibre, *An introduction to nonlinear Schrödinger equations*, Nonlinear waves (Sapporo, 1995) (R. Agemi, Y. Giga, and T. Ozawa, eds.), GAKUTO International Series, Math. Sciences and Appl., Gakkōtosho, Tokyo, 1997, pp. 85–133.

[23] M. Guardia, E. Haus, and M. Procesi, *Growth of Sobolev norms for the analytic NLS on* $\mathbb{T}^2$, Adv. Math. **301** (2016), 615–692.

[24] M. Guardia and V. Kaloshin, *Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 1, 71–149.

[25] P. Guerrero, J. L. López, and J. Nieto, *Global* H^1 *solvability of the 3D logarithmic Schrödinger equation*, Nonlinear Anal. Real World Appl. **11** (2010), no. 1, 79–87.

[26] Pilar Guerrero, José Luis López, and Jesús Montejo-Gámez, *A wavefunction description of quantum Fokker-Planck dissipation: derivation and numerical approximation of transient dynamics*, J. Phys. A **47** (2014), no. 3, 035303, 21.

[27] G. A. Hagedorn, *Semiclassical quantum mechanics. I. The $\hbar \to 0$ limit for coherent states*, Comm. Math. Phys. **71** (1980), no. 1, 77–93.

[28] E. F. Hefter, *Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics*, Phys. Rev. A **32** (1985), 1201.

[29] E. S. Hernandez and B. Remaud, *General properties of Gausson-conserving descriptions of quantal damped motion*, Physica A **105** (1980), 130.

[30] W. Krolikowski, D. Edmundson, and O. Bang, *Unified model for partially coherent solitons in logaritmically nonlinear media*, Phys. Rev. E **61** (2000), 3122.

[31] L. Landau and E. Lifschitz, *Physique théorique ("Landau-Lifchitz"). Tome III: Mécanique quantique. Théorie non relativiste*, Éditions Mir, Moscow, 1967, Deuxième édition, Traduit du russe par Édouard Gloukhian.

[32] Tianhong Li and Dehua Wang, *Blowup phenomena of solutions to the Euler equations for compressible fluid flow*, J. Differential Equations **221** (2006), no. 1, 91–101.

[33] José L. López and J. Montejo-Gámez, *A hydrodynamic approach to multidimensional dissipation-based Schrödinger models from quantum Fokker-Planck dynamics*, Phys. D **238** (2009), no. 6, 622–644.

[34] E. Madelung, *Quanten theorie in Hydrodynamischer Form*, Zeit. F. Physik **40** (1927), 322.

[35] S. De Martino, M. Falanga, C. Godano, and G. Lauro, *Logarithmic Schrödinger-like equation as a model for magma transport*, Europhys. Lett. **63** (2003), 472.

[36] Oana Pocovnicu, *Explicit formula for the solution of the Szegö equation on the real line and applications*, Discrete Contin. Dyn. Syst. **31** (2011), no. 3, 607–649.

[37] J. Rauch, *Partial differential equations*, Graduate Texts in Math., vol. 128, Springer-Verlag, New York, 1991.

[38] Cédric Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003.