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Limit point buckling of a finite beam on a nonlinear foundation

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Abstract In this paper, we consider an imperfect finite beam lying on a nonlinear foundation, whose dimensionless stiffness is reduced from 1 to k as the beam deflection increases. Periodic equilibrium solutions are found analytically and are in good agreement with a numerical resolution, suggesting that localized buckling does not appear for a finite beam. The equilibrium paths may exhibit a limit point whose existence is related to the imperfection size and the stiffness parameter k through an explicit condition. The limit point decreases with the imperfection size while it increases with the stiffness parameter. We show that the decay/growth rate is sensitive to the restoring force model. The analytical results on the limit load may be of particular interest for engineers in structural mechanics.

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Keywords buckling, imperfection, finite beam, nonlinear foundation, limit point

An elastic beam on a foundation is a model that can be found in a broad range of applications: railway tracks, buried pipelines, sandwich panels, coated solids in material, network beams, floating structures, etc. The usual way to model the interaction between the beam and the foundation is to replace the latter with a set of independent springs whose restoring force is a linear^{1–8} or a nonlinear^{9–19} function of the local deflection of the beam. In both cases, the nonlinear effects, from the beam's deformation and/or from the restoring force, play a crucial role in the buckling and the post-buckling behaviors. In particular, for a softening nonlinear foundation, the equilibrium curves of the beam may exhibit a maximum load (i.e., limit point) at which the structure loses its stability. Small imperfections, arising from various sources, usually have an appreciable effect on this maximum load. The papers on deterministic imperfection sensitivity include those of Refs. 4, 5, 11, 15, 20–24 and extensive references for the stochastic imperfection sensitivity are compiled in Ref. 25. As a general rule, the maximum load at which the beam becomes unstable diminishes with increasing imperfection size. Considering a finite beam on a bi-linear/exponential foundation, Ref. 24 has shown the existence of a critical imperfection size A_{0c} . For $A_0 < A_{0c}$, the maximum load diminishes with the imperfection size, from the critical buckling load predicted by the classical linear analysis¹⁰ for $A_0 = 0$ to the Euler load for $A_0 = A_{0c}$. In this case, the decay rate is sensitive to the restoring force model. For $A_0 > A_{0c}$, the maximum load is the Euler load (i.e., buckling load of a beam with no foundation).

In the present paper we aim to extend these results to two restoring force models with more general softening behaviors. We derive an analytical expression for A_{0c} and study the evolution

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of the maximum load with the imperfection size and the stiffness reduction.

We consider the effects of a compressive load P on a beam of length L , with bending stiffness EI , lying on a foundation that provides a restoring force per unit length \bar{P} (Fig. 1). The beam and the foundation are assumed to be well bonded at their interface and remain bonded during deformation. Thus, interfacial slip or debonding is not considered. The mobilization of the foundation (also named the yield point) is noted Δ , its linear stiffness K_0 and its nonlinear stiffness K . In its initial configuration, the beam has an imperfect shape $W_0 = A_0 \sin(\pi X/L)$, where A_0 is the imperfection size and X is the longitudinal coordinate.

We introduce the characteristic length $L_c = (EI/K_0)^{1/4}$ and the non-dimensional quantities $l = L/L_c$, $x = X/L_c$, $w = W/\Delta$, $w_0 = W_0/\Delta$, $a_0 = A_0/\Delta$, $\lambda = PL_c^2/(EI)$, $k = K/K_0$, $p_k = \bar{P}/(K_0\Delta)$ as the dimensionless beam length, longitudinal coordinate, lateral deflection (measured from the initial configuration), imperfection shape, imperfection size, compressive load, stiffness ratio, restoring force respectively.

Two models for the restoring force p_k are considered in this letter (Fig. 2). The first one is

$$p_k(w) = -w - (1-k)(\text{sgn}(w) - w)H(|w| - 1), \quad (1)$$

where sgn denotes the sign function and H is the Heaviside function, defined as $H(|w| - 1) = 0$ for $|w| < 1$ and 1 for $|w| > 1$. This bi-linear restoring force refers to a foundation whose stiffness is instantaneously reduced from 1 to $k \leq 1$ for $w > 1$. The particular case $k = 1$ corresponds to a linear foundation. The particular case of $k = 0$ has been considered in Ref. 24. Here, we extend the study to $k \leq 1$, which leads to more general results.

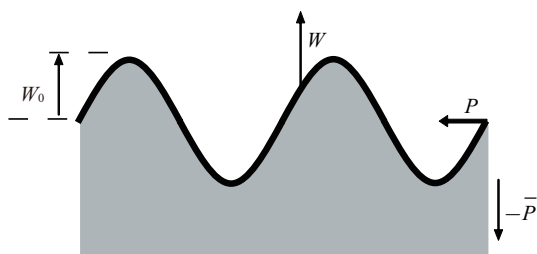


Fig. 1. Sketch of a beam on a nonlinear foundation. The beam has an imperfect shape W_0 and its lateral displacement is W . The compressive force is P and the restoring force per unit length is $-\bar{P}$.

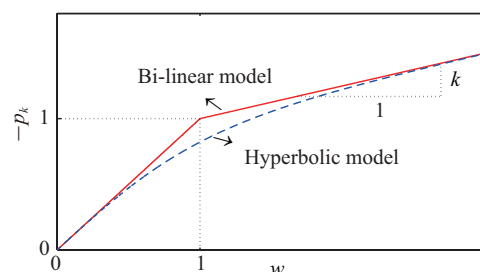


Fig. 2. Dimensionless restoring force p_k . The stiffness ratio is $k \leq 1$.

To reflect the experimental tests on railway tracks performed in Ref. 26, also reported in Refs. 27 and 28, which showed that the lateral friction force acting on a track is a smooth function of the lateral displacement, we introduce a hyperbolic profile defined as

$$p_k(w) = -kw - (1-k)\tanh(w), \quad (2)$$

which is a regularization of the bi-linear model as they share the same asymptotic behaviors.

We assume that λ and p_k are conservative forces, then strains are small compared to unity and

the kinematics of the beam is given by the classical Euler–Bernoulli assumption. The imperfection is also assumed to be small so that terms with higher powers of w_0 or its derivatives are neglected in the expression of the potential energy. Under these assumptions, the potential energy V with low-order geometrically nonlinear terms is¹⁰

$$V = \int_0^l \left[w'^2/2 - \lambda (w'^2/2 + w'_0 w') - \int_0^w p_k(t) dt \right] dx, \quad (3)$$

where a prime denotes differentiation with respect to x . The first term in the integral is the elastic bending energy, the second is the work done by the load λ , and the last term is the energy stored in the elastic foundation.

The equilibrium states are given by the critical values of V . Assuming a simply supported beam, the boundary conditions are $w(0) = w(l) = 0$. Variations of Eq. (3) for an arbitrary kinematically admissible virtual displacement δw yields the weak formulation of the equilibrium problem

$$\int_0^l [w'''' + \lambda (w'' + w''_0) - p_k(w)] \delta w dx = 0, \quad (4)$$

which is equivalent to the stationary Swift–Hohenberg equation

$$w'''' + \lambda (w'' + w''_0) - p_k(w) = 0, \quad (5)$$

along with static boundary conditions $w''(0) = w''(l) = 0$.

This boundary value problem is nonlinear because of the restoring force and its solutions are highly sensitive to the length l , as shown in Ref. 4. Therefore, it is unrealistic to describe the behavior of the system over a large range of variation for l . As done in Ref. 24, this study is restricted to a finite length beam where $l < \sqrt{2}\pi$. For such values of l , a classical linear analysis¹⁰ shows that the first buckling mode is the most unstable one and appears for $\lambda_c = \lambda_e + \lambda_e^{-1}$, where $\lambda_e = (\pi/l)^2$ is the Euler load.

To solve Eq. (4) we apply a Galerkin method with a trigonometric test function w of amplitude y , $w = y \sin(\pi x/l)$, assuming that the deflection has the same shape as the first buckling mode and the initial imperfection. For more details about the principle of the method, the reader is referred to Ref. 24, where the procedure has already been used. In that paper, this method has been shown to be reliable in the prediction of the equilibrium paths of the system, for $k = 0$. We shall see in the present paper that it is actually reliable for any $k \leq 1$, thereby extending the results of Ref. 24.

The insertion of $\delta w = \delta y \sin(\pi x/l)$ in Eq. (4) yields

$$\int_0^l \sin(\pi x/l) [w'''' + \lambda (w'' + w''_0) - p_k(w)] dx = 0. \quad (6)$$

Splitting the restoring force in a linear and a nonlinear term $N(w)$ leads to $p_k(w) = -w - (1-k)N(w)$. With this decomposition and $w = y \sin(\pi x/l)$, Eq. (6) can be rewritten as

$$\lambda_k = [\lambda_c y + (1-k)Q(y)/\lambda_e] / (a_0 + y), \quad (7)$$

where the subscript k denotes the dependance of λ on the parameter k . The function Q takes into account the nonlinear behavior of the restoring force and is given by

$$Q(y) = (2/l) \int_0^l \sin(\pi x/l) N(y \sin(\pi x/l)) dx. \quad (8)$$

For the restoring force models (1) and (2), Q is negative and decreases monotonically to the asymptote $-y + 4/\pi$ for $y \rightarrow +\infty$. Thus λ_k is maximum for $k = 1$ (linear foundation) and has an horizontal asymptote $\lambda_k^\infty = \lambda_c + k\lambda_c^{-1}$ for $y \rightarrow +\infty$.

Equilibrium paths predicted by Eq. (7) are traced out in the plane $(y = \max(w), \lambda)$ by gradually incrementing y and evaluating λ_k , k and a_0 being fixed. Predictions are compared with a numerical solution of Eq. (5), using MATLAB's boundary value solver `bvp4c` (this is a finite difference code that implements a collocation formula, details of which can be found in Ref. 29).

The equilibrium paths predicted by the Galerkin method and the numerical solution are shown in Fig. 3. A perfect agreement in the predictions is found for both restoring force models (the relative error between the two methods being less than 0.1%). Since the Galerkin method was initiated with a test function having the same shape as the imperfection, we conclude that the deflection is just an amplification of the initial curvature. In other words, in the range $l < \sqrt{2}\pi$, no localized buckling is observed for a beam on a bi-linear or hyperbolic foundation. This behavior has also been reported in Ref. 4 for a linear foundation, showing a tendency toward localization when increasing the beam length.

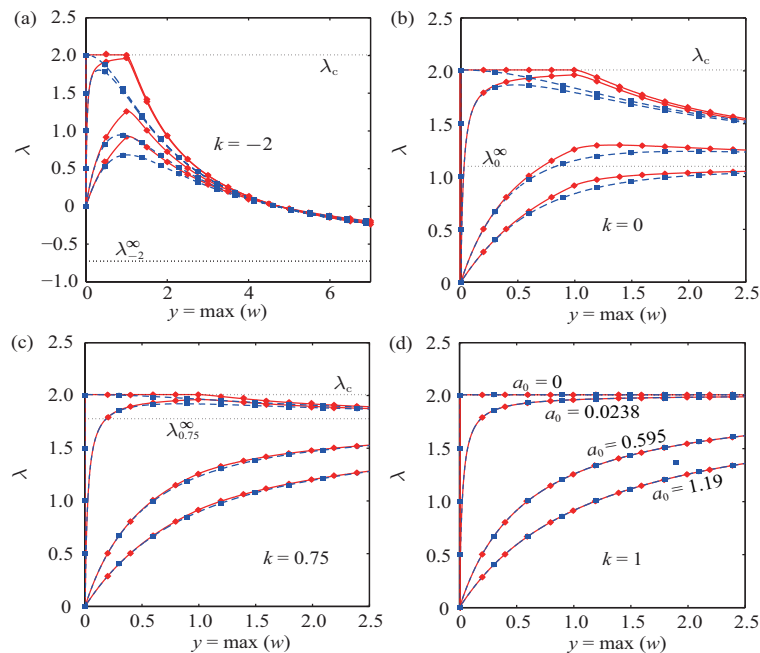


Fig. 3. Equilibrium paths of a finite length beam on a nonlinear foundation. Markers: numerical predictions. Lines: Galerkin solution. Rhombus: bi-linear restoring force model. Square: hyperbolic model. On each subfigure, the equilibrium paths are plotted (from top to bottom) for $a_0 = 0$, $a_0 = 0.0238$, $a_0 = 0.595$, and $a_0 = 1.19$, as shown in (d). The length of the beam is $l = 3$.

As expected, the equilibrium paths traced out for the hyperbolic restoring force are below those traced out for the bi-linear force, the hyperbolic profile modeling a softer foundation than the bi-linear one. However, the choice of the restoring force has little influence on the shape of the equilibrium paths.

For small a_0 , the equilibrium paths first increase to a maximum λ_m that is smaller (or equals to in the case of $a_0 = 0$) than the buckling load λ_c . Then, the paths asymptotically decrease to λ_k^∞ . In the case of $a_0 = 0, k = 1$, they remain equal to λ_c . For high a_0 , the equilibrium paths increase monotonically to the asymptote $\lambda_k^\infty \leq \lambda_c$. The asymptotic value $\lambda_k^\infty = \lambda_c$ is reached for $a_0 = 0$ and $k = 1$.

Note that for $k < -\lambda_e^2$, λ_k^∞ is negative, so that equilibrium states with $\lambda < 0$ are predicted (Fig. 3(a)). Physically, for $k < -\lambda_e^2$, the restoring force p_k may become negative so that springs are compressed, pushing up the beam. In this situation, the restoring force has a destabilizing effect on the beam. To counteract this effect, a tensile force $\lambda < 0$ has to be applied.

The evolution of λ_m versus a_0 is shown in Fig. 4. A gradual drop in the maximum load admissible by the structure from λ_c to λ_k^∞ is observed when increasing a_0 (resp. decreasing k). This gradual drop is highly sensitive to the restoring force model. A log scale applied on Fig. 4 shows that, for small imperfection sizes, the decay rate does not depend on k . $\lambda_m - \lambda_c$ scales as $-a_0$ for the bi-linear model and as $-a_0^{2/3}$ for the hyperbolic model.

For a_0 larger than a critical value a_{0c} , the equilibrium paths do not have a limit point anymore. Actually, a path with no limit point may be seen as a path with a limit point at $(\infty, \lambda_k^\infty)$. Thereby, a_{0c} may be obtained from Eq. (7) by enforcing $y \rightarrow \infty$ in $d\lambda_k/dy = 0$. Both restoring force models leads to

$$a_{0c} = (4/\pi)(1 - k)/(\lambda_e^2 + k), \quad (9)$$

whose dimensional equivalent form is $A_{0c} = (4/\pi)[(K_0 - K)\Delta(\pi^4 EI/L^4 + K)^{-1}]$.

The critical imperfection size predicted by Ref. 24 is therefore recovered in the particular case $K = 0$, showing that A_{0c} only depends on the limiting plateau $K_0\Delta$ of the restoring force.³⁰ Finally, since a_0 is larger than 0, Eq. (9) shows that if k is less than $-\lambda_e^2$ then the equilibrium paths always have a limit point with $\lambda_k^\infty < \lambda_m < \lambda_c$.

In this paper, we considered the buckling of an imperfect finite beam on a bi-linear/hyperbolic foundation. The imperfection has been introduced as an initial curvature of size a_0 and the foundation stiffness ratio as a parameter $k \leq 1$, extending the result of Ref. 24 derived for $k = 0$. Equilibrium paths of the beam have been predicted using a Galerkin method initiated with a single trigonometric function which has the same shape as the imperfection. Predictions compare well with a numerical solution and lead to the conclusion that only periodic buckling can arise for a finite beam on a bi-linear/hyperbolic foundation, as also observed in Ref. 4 for an linear foundation.

We have shown the existence of a critical imperfection size $a_{0c} = 4(1 - k)[\pi(\lambda_e^2 + k)]^{-1}$, independent of the restoring force model. (1) For $a_0 < a_{0c}$, the maximum load diminishes with increasing imperfection size, from $\lambda_c = \lambda_e + \lambda_e^{-1}$ for $a_0 = 0$ to $\lambda_e + k\lambda_e^{-1}$ for $a_0 = a_{0c}$, $\lambda_e = (\pi/l)^2$ being the Euler load. The decay rate has been shown to be sensitive to the restoring force model.

In the limit of small a_0 , $(\lambda_m - \lambda_c)$ scales as to $-a_0$ for the bi-linear model and $(\lambda_m - \lambda_c)$ scales as to $-a_0^{2/3}$ for the hyperbolic model. (2) For $a_0 > a_{0c}$, the maximum load simply corresponds to $\lambda_e + k\lambda_e^{-1}$.

Finally, we have shown that for $k < -\lambda_e^2$ an imperfect finite beam on a bi-linear/hyperbolic foundation can support a compressive load larger than λ_k^∞ , and smaller than λ_c , whatever the imperfection size is. This feature is highly interesting for an engineer since a_0 is usually hard to evaluate. The main results from this study are summarized in Fig. 5.

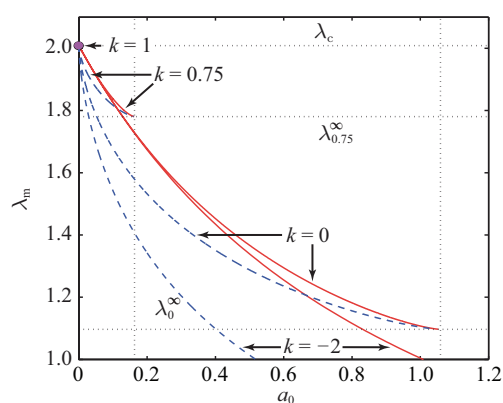


Fig. 4. Maximum load λ_m that the beam may support versus the imperfection size a_0 and the foundation stiffness ratio k . Solid line: bi-linear restoring force model. Dashed line: hyperbolic model. Vertical dotted lines correspond to the critical imperfection sizes a_{0c} for $k = 0.75$ and $k = 0$. The length of the beam is $l = 3$.

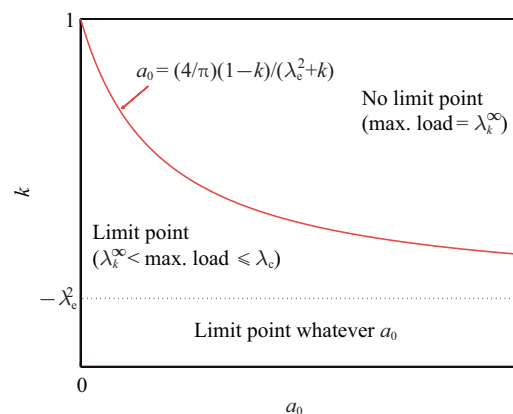


Fig. 5. Diagram of existence of a limit point for an imperfect finite beam on a bi-linear/hyperbolic foundation. k is the stiffness ratio of the foundation and a_0 the imperfection size. $\lambda_e = (\pi/l)^2$ is the Euler load, $\lambda_c = \lambda_e + \lambda_e^{-1}$ and $\lambda_k^\infty = \lambda_e + k\lambda_e^{-1}$.

In the present paper, a bi-linear restoring force model for the foundation has been used but plasticity effects that would emerge from loading/unloading cycles have not been considered. Future works will have to highlight the way those effects could modify the maximum load that the beam can support. A basic model would consist of considering a permanent deflection as an imperfection whose size would grow up after each cycle. In that case, from the present study, it is expected a decrease of the maximum load after each cycle, at least as long as the accumulated deflection remains smaller than a threshold equivalent to a_{0c} .

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