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Rating transitions forecasting: a filtering approach

Areski Cousin* J. Lelong† R. Norberg‡ T. Picard§

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Abstract

Analyzing the effect of business cycle on rating transitions has been a subject of great interest these last fifteen years, particularly due to the increasing pressure coming from regulators for stress testing. In this paper, we consider that the dynamics of rating migrations is governed by an unobserved latent factor. Under a point process filtering framework, we explain how the current state of the hidden factor can be efficiently inferred from observations of rating histories. We then adapt the classical Baum-Welsh algorithm to our setting and show how to estimate the latent factor parameters. Once calibrated, we may reveal and detect economic changes affecting the dynamics of rating migration, in real-time. To this end we adapt a filtering formula which can then be used for predicting future transition probabilities according to economic regimes without using any external covariates. We propose two filtering frameworks: a discrete and a continuous version. We demonstrate and compare the efficiency of both approaches on fictive data and on a corporate credit rating database. The methods could also be applied to retail credit loans.

1 Introduction

Credit risk research has been on the rise over the last 20 years. In particular, the challenges that arose from the previous financial crisis prompted researchers to develop credit risk valuation models that take into account the evolution of the business cycle. The evolution of the banking supervisor regulations and accounting rules follow this trend: the Point-in-time estimations recommended in official guidelines of IFRS9 as [1] to integrate explanatory economical components in the credit risk assessment process. The EBA guidelines [16], on LGD downturn, require to identify economic downturn periods to adjust the initial LGD estimations. Moreover, stress testing described in [17], may be based on past economical scenarios.

A credit rating system evaluates the confidence in the ability of the borrower to comply with the credit's terms. A default probability is associated to each rating, which under the capital rule, Basel regulations (see [2]), impacts the amount of capital required for a credit. Such ratings may be generated by internal rating systems (IRB) or issued when it is available, by external rating agencies (like Moody's, Standard and Poor's, Fitch Ratings...). After the assignment of the initial credit rating, reviews are performed either periodically or based on market events. In that way an entity's rating may evolve through time according to its health and the economic cycle. Therefore, predicting the evolution of rating migrations is of primary importance

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for every financial institution. The migrations of a group of credit entities can be described by transition matrices, defining the probabilities to move from one rating state to another in a given period of time. Given recent evolution in banking supervisory and accounting rules, the challenge is to explain changes in transition probabilities due to changes in the business cycle.

Factor-based migration models provide a nice framework for capturing migration sensitivities to macro-economic changes. Factor migration models allow transition probabilities to depend on dynamic factors. Two main families of models are usually considered in the credit risk literature : the “ordered Probit” (or structural approach) introduced by [47], popularized by [39] and studied for credit ratings , e.g., in [26], [4], [40], [21] and the “multi-state latent factor intensity model” (or intensity approach) studied, e.g., in [34], [22], [33]. This paper focuses on the second approach.

In the basic reduced intensity form model, a credit event corresponds to the first jump time of a Poisson process with a constant hazard rate. The reduced form approach has been widely studied in the credit risk literature, see, e.g., [31], [15]... Nevertheless we can point out a limit of this naive modeling: many studies show that rating migrations’ dynamics first exhibit the non-Markovian behavior (migration data exhibit correlation among rating change dates, known as “rating drift”, time-dependent default and transition probabilities, contagion effect ...) that cannot be captured by this model. Other papers such as [3], [30] and [23] highlighted that migration intensities vary over time. In their research, [36] and [30] came to the conclusion that the rating transition probabilities depend on whether the bond entered its current rating by an upgrade or a downgrade. [36] also noticed that the probability to leave a rating category tends to decrease with the time spent at that rating. Above all, [40], [4] gave strong evidence that credit risk exposure is considerably affected by the macroeconomic conditions and differ across different economic regimes. The first paper highlighted quantified rating migrations’ dependency on the industry, domicile of the obligor and on the stage of the business cycle.

In the factor intensity approach, the migration dynamics of each credit entity is described by a stochastic intensity matrix (or generator matrix) whose components depend on a pool of common factors.

In both models the factors may be considered observable or not. The second approach has emerged in response to criticisms made against the first. As [26] pointed out, the risk in selecting covariates lies in excluding other ones which could be more relevant. [11] provided an overview of usual modelling and estimation approaches and compared the estimation and predictive performance of each approach on real data. When the underlying factors are unobservable, they adapted a method given in [26] to represent the considered factor migration model as a linear Gaussian model, and apply a Kalman filter to predict the state of the underlying latent factor. This approximation works on the hypothesis that the transition probabilities’ estimator relies on asymptotic normality. This assumption may be too restrictive and typically does not hold when the data set is too small. In this respect, [11] obtained poor calibration and predictive results using this approximation compared to the estimation performed under the observable factor approach.

Our approach consists in directly filtering the hidden factor given rating transitions’ past history. For a bond portfolio, the dynamics of rating migrations can be mathematically represented as a multidimensional counting process, each component representing the cumulative number of transitions from one rating category to another. This framework has been already used in the credit risk literature for other applications: [24] used a counting process filtering approach to estimate the latent factor given default observation and use this estimation to price credit derivatives. Filtering theory with a counting process has been also applied to credit risk for example in [37] and [28].

Assuming that the counting process has a latent intensity determined by the hidden factor’s state, this project will look into estimating this intensity only using observations of the counting process. In our framework, we assume that the unobserved driving factor is given as a finite state

Markov chain and rating transitions process as Mixture of Markov Chain (MMC). Previous studies already advocated a MMC model to account for stochastic business cycle effect in credit rating migration risk ([25], [20]...). [20] demonstrated the efficiency of such approach which resulted in considerably decreasing the capital requirement from BASEL III. They used the trajectory of a two states economic factor (one of contraction and one of expansion) provided by The National Bureau of Economic Research (NBER) to estimate the factor's dynamic. However this modeling is totally dependant on the information provided by the institution and might be restrictive. A more reliable alternative is to consider the Markov chain not observable. Hidden Markov Chain modeling (HMM) is a popular approach in credit risk analysis ([10], [46], [18], [19], ...). Among others, [27], used an EM algorithm, the classical Baum-Welsh algorithm, for estimation of a two-state hidden factor driving occurrence of defaults. They applied the model to all entities and to different sectors to compare global and industry-specific risks. [35] supposed that a Markov chain governing the 'true' credit quality evolution is hidden in 'noisy' or incomplete observations represented by credit ratings. As we do in this paper, they used filtering and expected maximization approach to calibrate their model. In the same vein [13] and [41] identified two hidden states as one of expansion and the other of contraction. The second is the closest study from ours. As we do in this paper, they recently used an extension of the Baum-Welsh algorithm adapted to "regime switching hidden Markov model" (RSMC) to propose a forecasting of sovereign credit rating transitions. In a different scope, they also assumed that every rating processes are governed by an unique discrete hidden Markov chain but issued from different trajectories of this Markov chain. Those studies are theoretically different from our models in which a realization of the observable factor is common to every firm. We believe that our approach which rather keeps the dependencies within the observations sample, is more reliable and realistic. Indeed rating entities should be affected by the same realization of the economic factor. This different consideration changes the way to calibrate and to filter totally: our filtering framework uses the whole history of aggregated number of jumps. Furthermore for the sake of interpretability, our approach is more reliable if we want to consider the hidden factor as the universal and shared factor describing the economic cycle.

Behind every model mentioned, choosing a continuous or discrete approach is crucial and is a matter of debate. According to many studies like [36], using continuous-time analysis of the rating transition data is essential to meaningfully estimate probabilities of rare transitions. When those transitions are not directly and indirectly observed in our data set but are still possible, the discrete model might send null values which does not reflect reality. The continuous-time estimator is not so radical and is able to capture those rare events. Moreover they defended that censoring is handled easily within continuous-time framework.

On the other hand the discrete filter is more relevant with the daily format of the data. Rating transitions are usually registered on a daily basis, so that transitions of different entities may be observed at the exact same date, whereas construction of the continuous-time filtering formula strongly relies on the assumption of no simultaneous jumps in the migration counting processes.

This paper aims to participate to this debate by presenting different results: it assesses and compares two alternative stochastic migration models, a continuous factor-based intensity approach and its discrete-time version, on their ability to link the rating transition probabilities to an unobservable dynamic risk factor. The method, presented here, consists in filtering the underlying driving factor through the observations of a multivariate point process which counts migration events.

The contributions of the paper are both theoretical and practical. Two theoretical results are noteworthy: we first derive and generalize an extension of the classical univariate continuous point process filtering formula to multivariate case. Then we derive a discrete version of the filtering equation. Both formulas are adapted and applied to credit rating migrations context. An EM algorithm is extended to estimate parameters involved in both settings. Applying the recursive equations, we update the filtered factor to forecast rating transitions. Our approach may be considered as a new Point-in-time (PIT) rating transitions modeling which does not use any macro-economic factors. We assess and compare both approaches on a fictive data set and on a

Moody's ratings history [01/2000-05/2021] of a diversified portfolio of 5030 corporate entities. To the best of our knowledge, this study is the first attempt to apply and compare both discrete and continuous multivariate counting process filtering to predict future rating transitions.

The paper is organised as follows. We present theoretical results, the continuous and discrete point process filtering methods and their application to credit migration processes throughout Section (2) and Section (3). In order to calibrate both versions of filters, we propose in Section (4), two adaptations of the classical Baum-Welsh algorithm. The proposed adaptations aim to estimate parameters involved while respecting the different constraints of the models. Section (5), shows and compares the two filtering results on fictive data. This kind of experience presents many advantage but is not sufficient to deeply challenge models. Consequently in section (6), we test the two filters on real data sets. Finally Section (7), presents the conclusions and possible extensions of the paper.

2 Continuous-time version of the filter

2.1 Framework and statements

Let $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, be a filtered probability space satisfying the "usual conditions" of right-continuity and completeness needed to justify all operations to be made. All stochastic processes encountered are assumed to be adapted to the filtration \mathbf{F} and integrable on $[0, T]$. In particular, we have $\mathbf{A} = \mathcal{F}_T$. The time horizon T is supposed to be finite.

The problem is to estimate the states of an unobserved process Θ using only the information \mathbf{F}^N , resulting from the observation of a counting process N . We also define \mathbf{F}^Θ , the natural filtration of Θ . By definition of the conditional expectation,

$$\hat{\Theta}_t = \mathbb{E}[\Theta_t | \mathcal{F}_t^N].$$

is the L^2 approximation of Θ knowing N . With the same notation, all the processes O filtered by \mathcal{F}_t^N are written

$$\hat{O}_t = \mathbb{E}[O_t | \mathcal{F}_t^N].$$

Unfortunately, the explicit computation of $\hat{\Theta}$ (by the Bayes formula) is only possible in very simple cases, for example, when N is a mixture of Poisson processes. Moreover, numerical approaches are extremely heavy to implement as soon as non-trivial models are considered.

The filtering approach is different. It consists in writing $\hat{\Theta}$ as the solution of a stochastic differential equation (SDE). The main result on univariate point process filtering can be stated in the following way (see [7], [32], [37], [48]).

Proposition 1. *Let Θ be a process of the form*

$$d\Theta_t = a_t dt + dM_t,$$

where M is a martingale with no jumps in common with N . Then, the process $\hat{\Theta}$ satisfies the following equation

$$d\hat{\Theta}_t = \hat{a}_t dt + \eta_t (dN_t - \hat{\nu}_{t-} dt), \quad (1)$$

with

$$\eta_t = \frac{\widehat{(\Theta \nu)}_{t-}}{\hat{\nu}_{t-}} - \hat{\Theta}_{t-}, \quad (2)$$

and where ν is the \mathbf{F} -intensity of N .

Proof. see [7] ■

In order to study the dynamics of ratings in a credit portfolio or the dynamics of rating migrations, this filtering equation must be extended to a multivariate case.

2.2 Filtering multivariate point process

In this section, we show that the previous filtering equation admits a natural multivariate extension. Let $N = (N^1, \dots, N^\rho)$ be a multivariate counting process where $N^j = (N_t^j)_{t \in [0, T]}$, $j = 1, \dots, \rho$, is a set of simple counting processes, such that, $N_t^j = \sum_{0 < s \leq t} \Delta N_s^j < \infty$ and $\Delta N_s^j \in \{0, 1\}$, for any $j = 1, \dots, \rho$. It is assumed that these processes admit a predictable \mathbf{F} -intensity $\nu^j = (\nu_t^j)_{t \in [0, T]}$, and that they do not have any common jumps, i.e., $\Delta N_t^j \Delta N_t^k = \delta_{jk} \Delta N_t^j$ ie $[N^j, N^k]_t = 0$ (the continuous martingale part of a counting process being null). We introduce $\mathbf{F}^N = (\mathcal{F}_t^N)_{t \in [0, T]}$ the natural filtration of the multivariate counting process $N = (N^1, \dots, N^\rho)$.

Proposition 2. *Let Θ be a square integrable process of the form*

$$\Theta_t = \int_0^t a_s ds + M_t, \quad (3)$$

where a is a \mathbf{F} -adapted process and M is a \mathbf{F} -square integrable martingale with no jumps in common with N . Therefore, the process $\hat{\Theta}$ is solution of the SDE

$$d\hat{\Theta}_t = \hat{a}_t dt + \sum_j \eta_t^j (dN_t^j - \hat{\nu}_{t-}^j dt), \quad (4)$$

with

$$\eta_t^j = \frac{(\widehat{\Theta \nu^j})_{t-}}{\hat{\nu}_{t-}^j} - \hat{\Theta}_{t-}, \quad (5)$$

and initial condition

$$\hat{\Theta}_0 = \mathbb{E}[\Theta_0]. \quad (6)$$

Proof. Let g and h be two \mathbf{F} predictable processes such that $\mathbb{E} \left[\int_0^T (g_s^2 + h_s^2) \nu_s^j ds \right] \leq \infty$. We introduce the processes X and Y defined by $X_t = \int_0^t g_r (dN_r^j - \nu_r^j dr)$ and $Y_t = \int_0^t h_s (dN_s^k - \nu_s^k ds)$ for all $t \leq T$. X and Y are two \mathbf{F} -martingales. The Itô formula applied to XY yields

$$d(X_t Y_t) = X_{t-} dY_t + Y_{t-} dX_t + \Delta X_t \Delta Y_t.$$

Since N^j and N^k have no common jumps, $\Delta X_t \Delta Y_t = g_t h_t \Delta N_t^j \delta_{jk}$. Then, we obtain

$$\mathbb{E}[X_T Y_T - X_t Y_t | \mathcal{F}_t] = \delta_{jk} \mathbb{E} \left[\int_t^T g_s h_s dN_s^j | \mathcal{F}_t \right]. \quad (7)$$

Note that $X_T Y_T - X_t Y_t = (X_T - X_t)(Y_T - Y_t) - 2X_t Y_t + X_t Y_T + X_T Y_t$. Then,

$$\mathbb{E}[X_t Y_T - X_t Y_t | \mathcal{F}_t] = \mathbb{E}[(X_T - X_t)(Y_T - Y_t) | \mathcal{F}_t]. \quad (8)$$

Note that the process $Z = \left(\int_0^t g_s h_s (dN_s^j - \nu_s^j ds) \right)_{0 \leq t \leq T}$ is a \mathbf{F} -martingale, so $\mathbb{E}[Z_T - Z_t | \mathcal{F}_t] = 0$. Combining this remark with Equations (7) and (8), we finally obtain

$$\mathbb{E} \left[\int_t^T g_r (dN_r^j - \nu_r^j dr) \int_t^T h_s (dN_s^k - \nu_s^k ds) \middle| \mathcal{F}_t \right] = \delta_{jk} \mathbb{E} \left[\int_t^T g_s h_s \nu_s^j ds \middle| \mathcal{F}_t \right]. \quad (9)$$

The innovation theorem says that the \mathbf{F}^N -intensities of the counting processes N^j exist and are

$$\hat{\nu}_{t-}^j = \mathbb{E} \left[\nu_{t-}^j | \mathcal{F}_{t-}^N \right] = \mathbb{E} \left[\nu_t^j | \mathcal{F}_{t-}^N \right]. \quad (10)$$

For any \mathbf{F}^N predictable process h satisfying $\mathbb{E} \left[\int_0^T |h_s \nu_s| ds \right] < \infty$, we have

$$\mathbb{E} \left[\int_0^\infty h_t dN_t \right] = \mathbb{E} \left[\int_0^\infty h_t \nu_t dt \right] = \mathbb{E} \left[\int_0^\infty h_t \mathbb{E} \left[\nu_t | \mathcal{F}_{t-}^N \right] dt \right] = \mathbb{E} \left[\int_0^\infty h_t \hat{\nu}_{t-} dt \right].$$

Now, rewrite (3) as

$$\Theta_t = \int_0^t \hat{a}_s ds + L_t + M_t, \quad (11)$$

with

$$L_t = \int_0^t (a_s - \hat{a}_s) ds. \quad (12)$$

Taking conditional expectation, w.r.t. \mathcal{F}_t^N , in (11) yields

$$\hat{\Theta}_t = \int_0^t \hat{a}_s ds + \hat{L}_t + \hat{M}_t. \quad (13)$$

While L need not be a \mathbf{F} -martingale, it is clear that \hat{L} is an \mathbf{F}^N -martingale. For $r < t$,

$$\mathbb{E}[\hat{L}_t - \hat{L}_r | \mathcal{F}_r^N] = \mathbb{E} \left[\int_r^t (a_s - \mathbb{E}[a_s | \mathcal{F}_s^N]) ds \middle| \mathcal{F}_r^N \right] = 0.$$

From the tower property, we deduce that \hat{M} is also a \mathbf{F}^N -martingale. Introduce

$$K_t = L_t + M_t = \Theta_t - \int_0^t \hat{a}_s ds. \quad (14)$$

Since Θ and N have no common jumps, we can deduce that K and N have any either. It has been shown that $\hat{K} = \hat{L} + \hat{M}$ is a \mathbf{F}^N -martingale. Therefore it has a predictable representation,

$$\hat{K}_t = \gamma + \sum_j \int_0^t \eta_s^j (dN_s^j - \hat{\nu}_{s-}^j ds), \quad (15)$$

where $\gamma = \hat{K}_0$ is \mathcal{F}_0^N -measurable and the η^j are \mathbf{F}^N -predictable processes (see [7]). Note that $\hat{K}_0 = \mathbb{E}[\Theta_0]$. Now, any integrable \mathbf{F}^N -measurable random variable has a representation $g + \sum_j \int_0^t h_s^j (dN_s^j - \hat{\nu}_{s-}^j ds)$, with g constant and the h^j are \mathbf{F}^N -predictable. Therefore, since \hat{K}_t is the L^2 projection of K_t onto the space of square integrable \mathcal{F}_t^N -measurable random variables, the coefficients in the representation (15) are uniquely determined by the normal equations

$$\mathbb{E} \left[\left(K_t - \gamma - \sum_j \int_0^t \eta_s^j (dN_s^j - \hat{\nu}_{s-}^j ds) \right) \left(g + \sum_j \int_0^t h_s^j (dN_s^j - \hat{\nu}_{s-}^j ds) \right) \right] = 0,$$

for all constants g and all \mathbf{F}^N -predictable processes h^j . Setting $g = 0$ and using (9) give

$$\mathbb{E} \left[K_t \sum_j \int_0^t h_s^j (dN_s^j - \hat{\nu}_{s-}^j ds) - \sum_j \int_0^t h_s^j \eta_s^j \hat{\nu}_{s-}^j ds \right] = 0. \quad (16)$$

For $j \in \{1, \dots, \rho\}$, we compute $\mathbb{E} \left[K_t \int_0^t h_s^j dN_s^j \right]$. Using that \hat{K} is a \mathbf{F}^N -martingale and that K and N^j have no common jumps, we have

$$\begin{aligned} \mathbb{E} \left[K_t \int_0^t h_s^j dN_s^j \right] &= \mathbb{E} \left[K_t \sum_{s \leq t} h_s^j \Delta N_s^j \right] = \sum_{s \leq t} \mathbb{E} \left[\mathbb{E} [K_t | \mathcal{F}_s^N] h_s^j \Delta N_s^j \right] \\ &= \sum_{s \leq t} \mathbb{E} \left[\hat{K}_s h_s^j \Delta N_s^j \right] = \mathbb{E} \left[\int_0^t K_{s-} h_s^j dN_s^j \right] \\ &= \mathbb{E} \left[\int_0^t K_{s-} h_s^j \nu_s^j ds \right] = \int_0^t \mathbb{E} \left[h_s^j \mathbb{E} [K_{s-} \nu_s^j | \mathcal{F}_{s-}^N] \right] ds \\ &= \mathbb{E} \left[\int_0^t h_s^j \widehat{\Theta \nu^j}_{s-} ds \right] - \mathbb{E} \left[\int_0^t h_s^j \hat{\nu}_{s-}^j \int_0^s \hat{a}_u du ds \right]. \end{aligned}$$

Using similar arguments, we compute the second term

$$\begin{aligned}\mathbb{E}\left[K_t \int_0^t h_s^j \hat{\nu}_{s-}^j ds\right] &= \int_0^t \mathbb{E}\left[h_s^j \mathbb{E}\left[K_t | \mathcal{F}_{s-}^N\right] \hat{\nu}_{s-}^j\right] ds = \int_0^t \mathbb{E}\left[h_s^j \hat{K}_{s-} \hat{\nu}_{s-}^j\right] ds \\ &= \mathbb{E}\left[\int_0^t h_s^j \hat{\Theta}_{s-} \hat{\nu}_{s-}^j ds\right] - \mathbb{E}\left[\int_0^t h_s^j \hat{\nu}_{s-}^j \int_0^s \hat{a}_u du ds\right].\end{aligned}$$

Inserting these expressions into (16), gives

$$\sum_j \mathbb{E}\left[\int_0^t h_s^j \left(\widehat{\Theta \nu}_{s-}^j - \hat{\Theta}_{s-} \hat{\nu}_{s-}^j - \eta_s^j \hat{\nu}_{s-}^j\right) ds\right] = 0.$$

Choosing h_s^j equal to the expression in the parentheses, gives $\sum_j \mathbb{E}\left[\int_0^t (h_s^j)^2 ds\right] = 0$ hence all h^j vanish and $\forall j = 1, \dots, \rho$:

$$\eta_s^j = \frac{\widehat{\Theta_{s-} \nu_{s-}^j}}{\hat{\nu}_{s-}^j} - \hat{\Theta}_{s-}. \quad (17)$$

From (15), (13), and the equality $\hat{K} = \hat{L} + \hat{M}$, it follows that

$$\hat{\Theta}_t = \mathbb{E}[\Theta_0] + \int_0^t \hat{a}_s ds + \sum_j \int_0^t \eta_s^j (dN_s^j - \hat{\nu}_{s-}^j ds) \quad (18)$$

with the η^j are given by (17). This finishes the proof of the proposition. \blacksquare

2.3 Finite latent factor model and a credit risk application

This section aims to apply previous filtering framework to credit rating migrations. In order to remain realistic and to fix the terminology, a bond market containing a finite number of individual bonds is considered. All bonds are affected by variable and random market conditions represented by the same latent process Θ . A bond q of the sample is observed between the dates s^q and u^q , $0 \leq s^q \leq u^q \leq T$. At all times, the bond can only be in a state belonging to a finite set of states, $\mathbb{J} = \{1, \dots, p\}$. This space represents different credit risk scores or ratings in descending order, p being the default state. For example, Standard and Poor's long-term investment ratings can be translated to AAA = 1, AA = 2, A = 3, BBB = 4, ..., D (Default) = 10.

Let us define $\tilde{\mathbb{J}} = \{(j, k) \in \mathbb{J}^2, j \neq k\}$, the space of possible migrations. Let $Z_t^q \in \mathbb{J}$ be the state of bond q at time t and $Z^q = (Z_t^q)_{t \in [s^q, u^q]}$ be the rating process describing its evolution. The migration counting process associated with Z^q , which counts the number of jumps of the entity q from rating j to k , is denoted by $N^{q,jk}$ and is such that, $\forall t \in [s^q, u^q]$,

$$\Delta N_t^{q,jk} = \mathbf{1}_{[Z_{t-}^q = j, Z_t^q = k]}. \quad (19)$$

In this study, we assume that $(Z_t^q, t \in [0, T])_q$ are described within a factor migration model. More specifically, knowing \mathcal{F}_T^Θ , the rating processes Z^q are assumed to be conditionally independent Markov chains with the same generator matrix. In reality the change of rating of a bond may also induce the change of state of other bonds but this contagion effect is not considered in this paper. Moreover, the censorship mechanism governing (s^q, u^q) is assumed to be non-informative and can therefore be considered deterministic and belonging to \mathcal{F}_0^N . Under this exchangeable setting, to infer information on the underlying hidden factor Θ , it is sufficient to observe the aggregated counting processes N^{jk} , $(j, k) \in \tilde{\mathbb{J}}$, defined by

$$N_t^{jk} = \sum_q N_t^{q,jk}. \quad (20)$$

and the exposure processes Y^j , $j \in \mathbb{J}$ defined by

$$Y_t^j = \sum_{q; s^q \leq t < u^q} \mathbf{1}_{[Z_t^q=j]}. \quad (21)$$

Note that the exposure process Y^j is left continuous. It increases by 1 when N^{kj} jumps for any $k \neq j$, or when a new bond enters the pool with rating j . It decreases by 1 when a bond jumps outside rating j , i.e., whenever N^{jk} jumps for $k \neq j$ or when a bond expires with rating j .

We will now consider the filtering problem of Θ given observation of the multivariate processes $N := (N^{jk}, (j, k) \in \tilde{\mathbb{J}})$, using Prop. 2. We denote by $(\nu_t^{jk})_{(j,k) \in \tilde{\mathbb{J}}}$ the \mathbf{F} intensity of N . The dynamic (4) suggests a recursive algorithm to update the process $\hat{\Theta}$. A consequence of (4) is that the dynamics of $\hat{\Theta}$ depends on $(\widehat{\Theta\nu})$ which in turn depends on the term $(\widehat{\Theta\nu^2})$, and so on... This filtering equation induces an infinite nesting problem.

To solve the issue of infinite imbrication, we assume that the intensities $(\nu^{j,k})_{(j,k) \in \tilde{\mathbb{J}}}$ are governed by a finite state Markov chain. Consequently the hidden factor driving process Θ is assumed to be a Markov chain with finite number of states in $\mathbb{T} = \{1, \dots, m\}$ and with constant transition intensities k^{hi} , $i \neq h$, such that $k^{hh} = -\sum_{i; i \neq h} k^{hi}$ and for small enough dt

$$\forall i \neq h \in \mathbb{T}^2 : \mathbb{P}(\Theta_{t+dt} = h | \Theta_t = i) \approx k^{ih} dt. \quad (22)$$

The initial distribution of Θ is defined

$$\forall i \in \mathbb{T} : \Pi_i = \mathbb{P}(\Theta_0 = i). \quad (23)$$

Let us introduce the processes I^h , $h \in \mathbb{T}$, defined by

$$I_t^h = \mathbf{1}_{[\Theta_t=h]}, \quad h \in \mathbb{T}. \quad (24)$$

The process Θ can be represented as a finite sum of indicator processes

$$\Theta_t = \sum_{h \in \mathbb{T}} h I_t^h. \quad (25)$$

With these assumptions, the processes $(Z^q)_q$ are governed as described before by their common intensity matrices $(l^h)_{h \in \mathbb{T}}$, such as for small enough dt

$$\mathbb{P}[Z_{t+dt}^q = k | \Theta_t = h, Z_t^q = j] \approx \ell^{h,jk} dt. \quad (26)$$

Then the counting processes N^{jk} are governed by the \mathbf{F} intensities

$$\nu_t^{jk} = Y_t^j \sum_h \ell^{h,jk} I_{t-}^h. \quad (27)$$

As Y^j is \mathbf{F}^N -adapted, the \mathbf{F}^N -intensity of N , the counting process which counts the cumulative number of transitions, may be written

$$\hat{\nu}_{t-}^{jk} = Y_t^j \sum_h \ell^{h,jk} \hat{I}_{t-}^h, \quad (28)$$

By applying the previous multivariate filtering formula (4) to the process $(I_t)_{t \in [0, T]}$, we obtain the following proposition.

Proposition 3. *With the previous assumptions, the unobserved process indicator filtered with rating jumps satisfies the following recursive equation*

$$d\hat{I}_t^h = \sum_{i=1}^m k^{ih} \hat{I}_{t-}^i dt + \hat{I}_{t-}^h \sum_{j \neq k} \left(\frac{l^{h,jk}}{\sum_r l^{r,jk} \hat{I}_{t-}^r} - 1 \right) \left(dN_t^{jk} - Y_t^j \sum_{i=1}^m l^{i,jk} \hat{I}_{t-}^i dt \right) \quad (29)$$

Proof. In order to apply proposition 2, one needs to find the representation (3) for I_t^h . Let Ψ^{hi} , $h \neq i$, $h, i \in \mathbb{T}$, be the counting processes defined by

$$\Psi_t^{hi} = \#\{s \in (0, t]; \Theta_{s-} = h, \Theta_s = i\}.$$

The starting point is the expression

$$I_t^h = I_0^h + \sum_{i:i \neq h} (\Psi_t^{ih} - \Psi_t^{hi}),$$

which comes from the obvious dynamics

$$dI_t^h = \sum_{i:i \neq h} (d\Psi_t^{ih} - d\Psi_t^{hi})$$

The counting processes Ψ^{hi} have intensities of the form $I_{t-}^h \kappa^{hi}$. Reshaping the last expression as

$$\begin{aligned} I_t^h &= I_0^h + \int_0^t \sum_{i:i \neq h} (I_{s-}^i \kappa^{ih} - I_{s-}^h \kappa^{hi}) ds \\ &+ \int_0^t \sum_{i:i \neq h} [(d\Psi_s^{ih} - I_{s-}^i \kappa^{ih} ds) - (d\Psi_s^{hi} - I_{s-}^h \kappa^{hi} ds)], \end{aligned}$$

shows that I^h is of the form

$$I_t^h = \int_0^t a_s^h ds + M_t^h,$$

with

$$a_t^h = \sum_{i:i \neq h} (I_{t-}^i \kappa^{ih} - I_{t-}^h \kappa^{hi}) = \sum_i \kappa^{ih} I_{t-}^i \quad (30)$$

and M^h is a martingale commencing at $M_0^h = I_0^h$. This martingale has no jumps in common with N .

Now, let I^h take the role of Θ in Proposition 2. Then the role of a_t is taken by a_t^h in (30), the role of $(\Theta \nu^j)_{t-}$ is taken by

$$(-I^h \nu^{jk})_{t-} = I_{t-}^h Y_t^j \sum_i \ell^{i,jk} I_{t-}^i = Y_t^j \ell^{h,jk} I_{t-}^h,$$

and the F^N -intensities of N are given in (28). Inserting these expressions into (4), gives

$$d\hat{I}_t^h = \sum_{i=1}^m k^{ih} \hat{I}_{t-}^i dt + \hat{I}_{t-}^h \sum_{j \neq k} \left(\frac{l^{h,jk}}{\sum_r l^{r,jk} \hat{I}_{t-}^r} - 1 \right) \left(dN_t^{jk} - Y_t^j \sum_{i=1}^m l^{i,jk} \hat{I}_{t-}^i dt \right) \quad (31) \quad \blacksquare$$

Remark 4. *Despite being non-linear, the filtering equations (29) can be decomposed into a prediction term and an innovation term, as for the classical Kalman filter. Indeed, one can show that:*

$$\mathbb{E}[I_t^h | F_{t-}^N] = \hat{I}_{t-}^h + \sum_i k^{ih} \hat{I}_{t-}^i dt \quad (32a)$$

$$\mathbb{E}[I_t^h | F_t^N] = \underbrace{\mathbb{E}[I_t^h | F_{t-}^N]}_{\text{prediction}} + \hat{I}_{t-}^h \sum_{j \neq k} \left(\frac{l^{h,jk}}{\sum_r l^{r,jk} \hat{I}_{t-}^r} - 1 \right) \underbrace{\left(dN_t^{jk} - Y_t^j \sum_i l^{i,jk} \hat{I}_{t-}^i dt \right)}_{\text{new Information}} \quad (32b)$$

We can identify (32a) as the prediction term and (32b) as the correction term.

Usually, information on rating migrations are only available to public on a daily basis. For large credit portfolios, it is then frequent to observe multiple transitions (of several entities) occurring at the same day. In addition, clustering of rating migrations may also happen following the disclosure of a major economic events. Then, the presented continuous-time filtering approach is not fully compliant with migration data since it precludes simultaneous jumps. This has prompted us to develop an alternative discrete-time version of the filter.

3 Discrete version of the filter

We present in this section a discrete version of the filter. To apply filtering to credit rating migrations, discrete time environment tends to be more compliant with daily format of rating data and releases the strong hypothesis of jumps simultaneity: jumps can occur at the same time.

3.1 Univariate Form

Let $(t_k)_{k \in [0, \Gamma]}$ be a time grid of $[0, T]$ with a number Γ of time periods. We consider the discrete time filtered probability space $(\Omega, \mathbf{A}, \mathbf{F} = (\mathcal{F}_k)_{k \in [0, \Gamma]}, \mathbb{P})$, with $\mathbf{A} = \mathcal{F}_\Gamma$. Let N be a discrete-time counting process satisfying

$$\forall n \in [1, \Gamma], \Delta N_n = N_n - N_{n-1}, \quad (33)$$

$$\exists J \in \mathbb{N} \text{ s.t. } \forall n, \mathbb{P}(\Delta N_n \leq J) = 1 \text{ and } \forall j \leq J, \mathbb{P}(\Delta N_n = j) > 0. \quad (34)$$

We introduce the canonical notations $\hat{O}_k = \mathbb{E}[O_k | \mathcal{F}_k^N]$, $\bar{O}_{k-1} = \mathbb{E}[O_k | \mathcal{F}_{k-1}^N]$ and $\tilde{O}_{k-1} = \mathbb{E}[O_k | \mathcal{F}_{k-1}]$.

We also introduce the following processes

$$\epsilon_k^j = \mathbf{1}_{[\Delta N_k = j]}, \lambda_{k-1}^j = \mathbb{E}[\epsilon_k^j | \mathcal{F}_{k-1}], \hat{\lambda}_{k-1}^j = \mathbb{E}[\lambda_{k-1}^j | \mathcal{F}_{k-1}^N] = \mathbb{E} \left[\mathbb{E}[\epsilon_k^j | \mathcal{F}_{k-1}] | \mathcal{F}_{k-1}^N \right]. \quad (35)$$

Note that, for all k , we have $\sum_{j=0}^J \epsilon_k^j = \sum_{j=0}^J \lambda_k^j = \sum_{j=0}^J \hat{\lambda}_k^j = 1$.

Under this setting, we present a discrete adaptation of the univariate filtering equation (1).

Proposition 5. *Let Θ be a square integrable process of the form*

$$\Theta_n = A_n + M_n, \quad (36)$$

where A is a \mathbf{F} -predictable process and M is a \mathbf{F} -martingale.

Then, the filtered process $\hat{\Theta}$ satisfies the following equation

$$\hat{\Theta}_n = \sum_{s=0}^J \frac{\widehat{(\tilde{\Theta} \lambda^s)}_{n-1}}{\hat{\lambda}_{n-1}^s} \mathbf{1}_{[\Delta N_n = s]}. \quad (37)$$

Remark 6. *The decomposition (36) always exists (and is unique) for any square integrable and \mathbf{F} adapted process (see, e.g [14]). There exists A and M such*

$$\Theta_n = A_n + M_n, \quad (38)$$

where A , \mathbf{F} -predictable and M , \mathbf{F} -martingale. We may identify $A_n = \sum_{k=1}^n (\mathbb{E}[\Theta_k | \mathcal{F}_{k-1}] - \Theta_{k-1})$ and $M_n = \Theta_0 + \sum_{k=1}^n (\Theta_k - \mathbb{E}[\Theta_k | \mathcal{F}_{k-1}])$.

The proof of Proposition 5 heavily relies on the following lemma.

Lemma 7. Let K be a \mathbf{F} adapted process such that \hat{K} is a \mathbf{F}^N -martingale. Then K is solution of the following recursive equation

$$\hat{K}_n = K_0 + \sum_{k=1}^n \sum_{j=0}^J \frac{\widehat{(\tilde{K}\lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \quad (39)$$

Remark 8. If K is a \mathbf{F} -martingale then the previous equation is verified with $\tilde{K}_{k-1} = K_{k-1}$.

Proof (of Lemma 7). Let P be a \mathbf{F}^N -martingale. Then P_n is \mathcal{F}_n^N measurable. By using a measurability characteristic stated in [7]: $\exists g$ measurable function such as $P_n = g(n, N_0, \dots, N_n)$.

With $R_j = N_j - N_{j-1}$, $\exists h$ measurable function such as: $P_n = h(n, R_0, \dots, R_n)$.

We can write

$$P_n = \sum_{j=0}^J h(n, R_0, \dots, R_{n-1}, j) \epsilon_n^j.$$

Since P is a \mathbf{F}^N -martingale then

$$P_n - P_{n-1} = P_n - \mathbb{E}[P_n | \mathcal{F}_{n-1}^N] = \sum_{j=0}^J h(n, R_0, \dots, j) (\epsilon_n^j - \hat{\lambda}_{n-1}^j).$$

Then P has the following martingale representation

$$P_n = P_0 + \sum_{k=1}^n \sum_{j=0}^J H_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j), \quad (40)$$

with H_{k-1}^j \mathcal{F}_{k-1}^N measurable. Reciprocally, it is easy to check that a process written like that is a \mathbf{F}^N -martingale.

Since $\epsilon_0^j = 1 - \sum_{j=1}^J \epsilon_k^j$ and $\hat{\lambda}_0^j = 1 - \sum_{j=1}^J \hat{\lambda}_k^j$, we can rewrite P with the following martingale representation

$$P_n = P_0 + \sum_{k=1}^n \sum_{j=1}^J W_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j), \quad (41)$$

with $W_{k-1}^j = H_{k-1}^j - H_{k-1}^0$ in the previous decomposition, still \mathcal{F}_{k-1}^N measurable.

Let K be \mathbf{F} adapted such that \hat{K} is a \mathbf{F}^N -martingale. Using the tower property, we deduce that \hat{K} is a \mathbf{F}^N -martingale. According to (41), we can write $\hat{K}_n = m + \sum_{k=1}^n \sum_{j=1}^J W_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j)$, where W_{k-1} is a \mathcal{F}_{k-1}^N -measurable random variable.

For all \mathbf{F}^N -adapted process X , we have $\mathbb{E}[(K_n - \hat{K}_n)X_n] = 0$. Choosing X to be a \mathbf{F}^N -martingale with the decomposition $g + \sum_{k=1}^n \sum_{j=1}^J G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j)$, we obtain

$$\mathbb{E} \left[\left(K_n - m - \sum_{j,k \geq 1} W_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right) \left(g + \sum_{j,k \geq 1} G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right) \right] = 0. \quad (42)$$

Choosing $G_{k-1}^j = 0 \forall j$, leads to $m = \mathbb{E}[\hat{K}_n] = \mathbb{E}[K_n] = K_0$.

With no loss of generality, we can consider that $M_0 = 0$. By choosing $g = 0$ we obtain:

$$\mathbb{E} \left[K_n \sum_{j,k \geq 1} G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] - \mathbb{E} \left[\sum_{j_1, j_2 \geq 1} \sum_{k_1, k_2 \geq 1} W_{k_1-1}^{j_1} (\epsilon_{k_1}^{j_1} - \hat{\lambda}_{k_1-1}^{j_1}) G_{k_2-1}^{j_2} (\epsilon_{k_2}^{j_2} - \hat{\lambda}_{k_2-1}^{j_2}) \right] = 0.$$

For $k_1 < k_2$,

$$\mathbb{E} \left[\mathbb{E} \left[W_{k_1-1}^{j_1} (\epsilon_{k_1}^{j_1} - \hat{\lambda}_{k_1-1}^{j_1}) G_{k_2-1}^{j_2} (\epsilon_{k_2}^{j_2} - \hat{\lambda}_{k_2-1}^{j_2}) | \mathcal{F}_{k_2-1}^N \right] \right] = 0$$

Noticing that $\mathbb{E} [K_n | \mathcal{F}_k^N] = \mathbb{E} [\mathbb{E} [K_n | \mathcal{F}_n^N] | \mathcal{F}_k^N] = \hat{K}_k$, we compute the term

$$\begin{aligned} S_{n,k} &= \mathbb{E} \left[K_n G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] = \mathbb{E} \left[\mathbb{E} [K_n | \mathcal{F}_k^N] G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] \\ &= \mathbb{E} \left[\hat{K}_k G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right] = \mathbb{E} \left[K_k G_{k-1}^j (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \right]. \end{aligned}$$

Now let us compute

$$\begin{aligned} \mathbb{E} [K_k (\epsilon_k^j - \hat{\lambda}_{k-1}^j) | \mathcal{F}_{k-1}^N] &= \mathbb{E} [K_k \epsilon_k^j | \mathcal{F}_{k-1}^N] - \mathbb{E} [K_k \hat{\lambda}_{k-1}^j | \mathcal{F}_{k-1}^N] \\ &= \mathbb{E} \left[\mathbb{E} [K_k \epsilon_k^j | \mathcal{F}_{k-1}] | \mathcal{F}_{k-1}^N \right] - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j. \end{aligned}$$

We notice at this point that K_k and ϵ_k are independent knowing the σ -field \mathcal{F}_{k-1} and $\mathbb{E} [K_k | \mathcal{F}_{k-1}^N] = \hat{K}_{k-1}$. Then,

$$\begin{aligned} \mathbb{E} \left[\mathbb{E} [K_k \epsilon_k^j | \mathcal{F}_{k-1}] | \mathcal{F}_{k-1}^N \right] - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j &= \mathbb{E} \left[\mathbb{E} [K_k | \mathcal{F}_{k-1}] \mathbb{E} [\epsilon_k^j | \mathcal{F}_{k-1}] | \mathcal{F}_{k-1}^N \right] - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j \\ &= \widehat{(\tilde{K} \lambda^j)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j. \end{aligned}$$

So by conditioning by \mathcal{F}_{k-1}^N inside the expectation we obtain

$$\mathbb{E} \left[\sum_{k=1}^n \sum_{j=1}^J G_{k-1}^j \left(\widehat{(\tilde{K} \lambda^j)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j - \mathbb{E} \left[\sum_{i=1}^J W_{k-1}^i (\epsilon_k^i - \hat{\lambda}_{k-1}^i) (\epsilon_k^j - \hat{\lambda}_{k-1}^j) | \mathcal{F}_{k-1}^N \right] \right) \right] = 0$$

With this form, we choose

$$G_{k-1}^j = \widehat{(\tilde{K} \lambda^j)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j - \mathbb{E} \left[\sum_i W_{k-1}^i (\epsilon_k^i - \hat{\lambda}_{k-1}^i) (\epsilon_k^j - \hat{\lambda}_{k-1}^j) | \mathcal{F}_{k-1}^N \right].$$

We deduce that $E^{(k)} W_{k-1} = (\widehat{(\tilde{K} \lambda)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1})$, with the matrix $E^{(k)}$ defined $\forall (i, j) \in [1..J]^2$ by $E_{i,j}^{(k)} = \mathbb{E} \left[(\epsilon_k^i - \hat{\lambda}_{k-1}^i) (\epsilon_k^j - \hat{\lambda}_{k-1}^j) | \mathcal{F}_{k-1}^N \right]$. We have

$$\begin{aligned} \text{For } i \neq j : \mathbb{E} \left[(\epsilon_k^j - \hat{\lambda}_{k-1}^j) (\epsilon_k^i - \hat{\lambda}_{k-1}^i) | \mathcal{F}_{k-1}^N \right] &= -\hat{\lambda}_{k-1}^i \hat{\lambda}_{k-1}^j. \\ \text{For } i = j : \mathbb{E} \left[(\epsilon_k^j - \hat{\lambda}_{k-1}^j) (\epsilon_k^i - \hat{\lambda}_{k-1}^i) | \mathcal{F}_{k-1}^N \right] &= \hat{\lambda}_{k-1}^i \sum_{\substack{j=0 \\ j \neq i}}^J \hat{\lambda}_{k-1}^j. \end{aligned}$$

With simple manipulations on columns, we easily compute $\det(E^{(k)}) = \prod_{j=0}^J \hat{\lambda}_{k-1}^j$. If $\exists j \leq J$ such that $\hat{\lambda}_{k-1}^j = 0$ then $\mathbb{P}(\Delta N_k = j) = 0$ which contradicts the assumption (34). Therefore, we can invert the matrix to obtain

$$\left(E^{(k)-1} (\widehat{(\tilde{K} \lambda)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}) \right)_j = \frac{1}{\hat{\lambda}_{k-1}^j} (\widehat{(\tilde{K} \lambda^j)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}^j) + \frac{1}{\hat{\lambda}_{k-1}^0} \sum_{i \geq 1} (\widehat{(\tilde{K} \lambda^i)}_{k-1} - \hat{K}_{k-1} \hat{\lambda}_{k-1}^i).$$

Finally, by replacing $\sum_{j=1}^J \epsilon_k^j$, $\sum_{j=1}^J \hat{\lambda}_{k-1}^j$ by $1 - \epsilon_k^0$ and $1 - \hat{\lambda}_{k-1}^0$, we derive the general filtering formula

$$\begin{aligned}\hat{K}_n &= K_0 + \sum_{k=1}^n \sum_{j=1}^J \left(\frac{\widehat{(\tilde{K}\lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} - \frac{\widehat{(\tilde{K}\lambda^0)}_{k-1}}{\hat{\lambda}_{k-1}^0} \right) (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \\ &= K_0 + \sum_{k=1}^n \sum_{j=0}^J \frac{\widehat{(\tilde{K}\lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} (\epsilon_k^j - \hat{\lambda}_{k-1}^j)\end{aligned}$$

Finally, K is solution of the following recursion formula

$$\hat{K}_n = K_0 + \sum_{k=1}^n \sum_{j=0}^J \frac{\widehat{(\tilde{K}\lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \quad \blacksquare \quad (43)$$

Proof (of Proposition 5). We define

$$a_n = A_n - A_{n-1} = \mathbb{E}[\Theta_n | \mathcal{F}_{n-1}] - \Theta_{n-1}, \quad (44)$$

$$B_n = \sum_{k=1}^n \mathbb{E}[a_k | \mathcal{F}_{k-1}^N] = \sum_{k=1}^n \bar{a}_{k-1}, \quad (45)$$

$$L_n = \sum_{k=1}^n a_k - \mathbb{E}[a_k | \mathcal{F}_{k-1}^N] = \sum_{k=1}^n a_k - \bar{a}_{k-1}, \quad (46)$$

We notice that

$$\Theta_n - B_n = M_n + L_n. \quad (47)$$

We know that \hat{M} is a \mathbf{F}^N -martingale and \hat{L} is clearly a \mathbf{F}^N -martingale too. As the sum of two martingales, we can apply the previous Lemma (7) to $\Theta - B$. $\hat{B}_{k-1} = B_k$ and B is \mathbf{F}^N predictable, hence, $\widehat{B\lambda^j}_{k-1} = B_k \hat{\lambda}_{k-1}^j$. Then

$$\begin{aligned}\hat{\Theta}_n - \hat{B}_n &= \hat{\Theta}_0 - \hat{B}_0 + \sum_{k=1}^n \sum_{j=0}^J \left(\frac{\widehat{((\tilde{\Theta} - \tilde{B})\lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} \right) (\epsilon_k^j - \hat{\lambda}_{k-1}^j) \\ &= \hat{\Theta}_0 - \hat{B}_0 + \sum_{k=1}^n \sum_{j=0}^J \frac{\widehat{(\tilde{\Theta}\lambda^j)}_{k-1}}{\hat{\lambda}_{k-1}^j} (\epsilon_k^j - \hat{\lambda}_{k-1}^j)\end{aligned}$$

We compute $\hat{\Theta}_n - \hat{\Theta}_{n-1} = \bar{a}_{n-1} + f(\lambda_{n-1}, \epsilon_n, \tilde{\Theta}_{n-1})$. From (44), $\bar{a}_{n-1} = \bar{\Theta}_{n-1} - \hat{\Theta}_{n-1}$ and using the expression of

$$f(\lambda_{n-1}, \epsilon_n, \tilde{\Theta}_{n-1}) = \sum_{j=0}^J \frac{\widehat{(\tilde{\Theta}\lambda^j)}_{n-1}}{\hat{\lambda}_{n-1}^j} (\epsilon_n^j - \hat{\lambda}_{n-1}^j) = \sum_{j=0}^J \frac{\widehat{(\tilde{\Theta}\lambda^j)}_{n-1}}{\hat{\lambda}_{n-1}^j} \mathbf{1}_{[\Delta N_n=j]} - \bar{\Theta}_{n-1},$$

we deduce the final form of the filtering formula

$$\hat{\Theta}_n = \sum_{j=0}^J \frac{\widehat{(\tilde{\Theta}\lambda^j)}_{n-1}}{\hat{\lambda}_{n-1}^j} \mathbf{1}_{[\Delta N_n=j]}. \quad \blacksquare \quad (48)$$

As in the continuous framework, this formula can be extended to a multivariate setting.

3.2 Multivariate form

Let $N = (N^1, \dots, N^\rho)$ is a multivariate counting process where $N^j = (N_t^j)_{t \in [0, \Gamma]}$, $j = 1, \dots, \rho$, be a set of simple counting processes, such that, $N_t^j = \sum_{s=0}^t \Delta N_s^j < \infty$ and $\Delta N_s^j \in [0, \dots, J]$, for any $j = 1, \dots, \rho$. Under this setting, the no simultaneous jump assumption is not required.

Proposition 9. *Let Θ be a process of the form*

$$\Theta_n = A_n + M_n, \quad (49)$$

where A , is a \mathbf{F} -predictable process and M , is a \mathbf{F} -martingale. Then

$$\hat{\Theta}_n = \sum_{s \in [0 \dots J]^\rho} \frac{\widehat{(\tilde{\Theta} \lambda^s)}_{n-1}}{\hat{\lambda}_{n-1}^s} \mathbb{1}_{[\Delta N_n = s]}. \quad (50)$$

Proof. We leave the proof to the reader as it goes along the same lines as the proof of Proposition 5. \blacksquare

We now apply this filtering approach to a rating migration process. As for the continuous case, this formula presents infinite nesting problem that can be solved when a finite state process Θ is consider.

3.3 Application to rating migrations

As for the continuous framework, we apply this general formula to credit rating transitions. We first present the formula in the context of a single pair of rating categories; thus a single transition from one given rating to another (the transition to default for instance). Then we extend the approach to multiple rating transitions.

3.3.1 Unique rating transition

In this framework Θ is assumed to be a Markov chain with finite number of states in $\mathbb{T} = \{1, \dots, m\}$. We present here the recursive equation verified by $\hat{I}_n^h = \mathbb{E}[\mathbb{1}_{[\Theta_n = h]} | \mathcal{F}_n^N]$, where N represents the cumulative number of the considered transition. The process Z^q takes the value 1 when the entity q realizes the transition and 0 otherwise. We define respectively $\forall n \in [1, \Gamma]$:

$$K^{ih} = \mathbb{P}(\Theta_n = h | \Theta_{n-1} = i), \quad L^h = \mathbb{P}(Z_n^q = 1 | Z_{n-1}^q = 0, \Theta_{n-1} = h),$$

the hidden factor's transition probabilities and the common conditional transition probabilities of entities.

The process Y represents the number of entities that may jump. This process is assumed to be \mathbf{F}^N -predictable. We assume that the random variable $(\Delta N_k | \Theta_k = h, Y_k = y_k)$ follows a binomial distribution with parameters (y_k, L^h) . Similar settings can be found in [8] and [27].

Proposition 10. *With these assumptions, the filtered process \hat{I}_n^h solves the following recursive equation*

$$\hat{I}_n^h = \sum_{\delta N_n = 0}^J \frac{\sum_i K^{ih} (L^i)^{\delta N_n} (1 - L^i)^{Y_n - \delta N_n} \hat{I}_{n-1}^i}{\sum_s (L^s)^{\delta N_n} (1 - L^s)^{Y_n - \delta N_n} \hat{I}_{n-1}^s} \mathbb{1}_{[\Delta N_n = \delta N_n]}. \quad (51)$$

Proof. We have

$$\mathbb{E}(\mathbb{1}_{[\Delta N_k = j]} | \mathcal{F}_{k-1}) = \mathbb{E}(\mathbb{1}_{[\Delta N_k = j]} | N_{k-1}, \Theta_{k-1}, Y_k), \quad (52)$$

and

$$\mathbb{P}(\Delta N_k = j | \Theta_{k-1} = h, Y_k = y_k) = \binom{y_k}{j} (L^h)^j (1 - L^h)^{y_k - j}. \quad (53)$$

Then we compute the following expressions

$$\hat{\lambda}_{k-1}^j = \mathbb{E} [\mathbf{1}_{[\Delta N_k=j]} | \mathcal{F}_{k-1}^N] = \sum_h \binom{Y_k}{j} (L^h)^j (1-L^h)^{Y_k-j} \hat{I}_{k-1}^h, \quad (54)$$

$$\widehat{(I^h \lambda^j)}_{k-1} = \binom{Y_k}{j} (L^h)^j (1-L^h)^{Y_k-j} \hat{I}_{k-1}^h \quad (55)$$

and

$$\begin{aligned} \widehat{(\tilde{I}^h \lambda^{\delta_n})}_{n-1} &= \mathbb{E} [\mathbb{E}[I_n^h | \mathcal{F}_{n-1}] \mathbb{E}[\mathbf{1}_{[\Delta N_n=\delta_n]} | \mathcal{F}_{n-1}] | \mathcal{F}_{n-1}^N] \\ &= \mathbb{E} \left[\left(\sum_i K^{ih} I_{n-1}^i \right) \left(\sum_s \binom{Y_n}{\delta N_n} (L^s)^{\delta N_n} (1-L^s)^{Y_n-\delta N_n} I_{n-1}^s \right) | \mathcal{F}_{n-1}^N \right] \\ &= \sum_i K^{ih} \binom{Y_n}{\delta N_n} (L^i)^{\delta N_n} (1-L^i)^{Y_n-\delta N_n} \hat{I}_{n-1}^i. \end{aligned}$$

Finally using (36) we obtain

$$\hat{I}_n^h = \sum_{\delta N_n=0}^J \frac{\sum_i K^{ih} (L^i)^{\delta N_n} (1-L^i)^{Y_n-\delta N_n} \hat{I}_{n-1}^i}{\sum_s (L^s)^{\delta N_n} (1-L^s)^{Y_n-\delta N_n} \hat{I}_{n-1}^s} \mathbf{1}_{[\Delta N_n=\delta N_n]}. \quad (56) \quad \blacksquare$$

In order to apply our framework to rating migrations, the previous setting is extended using the multivariate filtering formula (50).

3.3.2 Multiple Rating transitions

Let $Z_n^q \in \mathbb{J} = \{1, \dots, p\}$ be the state of bond q at time $n \in [0, \Gamma]$ and $Z^q = (Z_n^q)_{n \in [0, \Gamma]}$ be the migration process that describes its evolution. The counting process associated to Z^q , which counts the number of jumps of the entity q , from rating j to k , $j \neq k \in [1..p]^2$, is denoted by $N^{q,jk}$ and is such that, $\forall n \in [1, \Gamma]$,

$$\Delta N_n^{q,jk} = \mathbf{1}_{[Z_{n-1}^q=j, Z_n^q=k]}. \quad (57)$$

We also define, $\forall j \in [1..p]$, the natural counting process $N^{q,jj}$ such that,

$$\Delta N_n^{q,jj} = \mathbf{1}_{[Z_{n-1}^q=j, Z_n^q=j]} = \sum_{k \neq j} \mathbf{1}_{[Z_{n-1}^q=j, Z_n^q=k]}. \quad (58)$$

The associated multivariate counting process is $N = (N^{jk})_{(j,k) \in [1,p]^2}$, defined $\forall n \in [1, \Gamma]$ by $N_n^{jk} = \sum_q N_n^{q,jk}$. We define the transition probabilities of Θ as

$$\forall (i, j) \in \mathbb{T}^2, \forall n \in [1, \Gamma], K^{ij} = \mathbb{P}(\Theta_n = j | \Theta_{n-1} = i).$$

We also define the conditional transition probabilities of $(Z^q)_q$

$$\forall (i, j) \in [1..p]^2, \forall h \in \mathbb{T}, \forall n \in [1, \Gamma], L^{h,ij} = \mathbb{P}(Z_n^q = j | Z_{n-1}^q = i, \Theta_{n-1} = h).$$

We can notice that the processes Z^1, \dots, Z^d are not independent. Indeed, only one realisation of trajectory of Θ governs observed rating processes.

The process Y^j represents the number of entities that belong to rating j . This process is assumed to be \mathbf{F}^N -predictable. The multivariate random variable $(\Delta N_n^{jk} | \Theta_n = h, Y_n^j = y_n^j)_{k \in [1..p]}$ follows a multinomial distribution with parameters $(y_n^j, (L^{h,jk})_k)$.

Proposition 11. *With such assumptions, the filtered process \hat{I}_n^h is solution of the following recursive equation*

$$\hat{I}_n^h = \sum_{\delta N_n \in [0 \dots J]^{p^2}} \frac{\sum_i K^{ih} \prod_{j,k=1}^p (L^{i,jk})^{\delta N_n^{jk}} \hat{I}_{n-1}^i}{\sum_s \prod_{j,k=1}^p (L^{s,jk})^{\delta N_n^{jk}} \hat{I}_{n-1}^s} \mathbb{1}_{[\Delta N_n = \delta N_n]} \quad (59)$$

Proof. We leave the proof to the reader as it goes along the same lines as the proof of Proposition 10. ■

Once the hidden factor filtered state is obtained, we are able to predict the future migration probabilities.

3.3.3 Transition probability prediction

We define for $(i, j) \in [1..p]^2$, the process ν^{ij} , which forecasts for the next time step, according to the hidden factor, the transition probability from rating i to rating j.

$$\forall (i, j) \in [1, p]^2, \forall n \in [1, \Gamma] : \nu_{n-1}^{ij} = \mathbb{E} [\mathbb{1}_{[Z_n^q=j]} | Z_{n-1}^q = i, \mathcal{F}_{n-1}] = \sum_h L^{h,ij} I_{n-1}^h \quad (60)$$

With the filtered current hidden factor, we can forecast the future transition probabilities

$$\forall (i, j) \in [1, p]^2, \forall n \in [1, \Gamma] : \hat{\nu}_{n-1}^{ij} = \sum_h L^{h,ij} \hat{I}_{n-1}^h \quad (61)$$

4 Calibration

In this section, we propose two adaptations of the classical HMM Baum-Welsh algorithm in order to use the parameter estimations in the filtering formulas (29) and (59).

4.1 A Baum-welsh algorithm adapted for a discrete framework

The proposed method is a maximisation expectation (EM) algorithm for hidden Markov chains (HMM), adapted to the model. We can find studies on the classical model in [45], [5], [42] and [43].

However the classical algorithm is not totally adapted to calibrate the discrete filtering equation (59). We highlight two inconsistencies between the classical algorithm and our model:

- In the famous framework, rating status law is only governed by the state of the hidden factor. In the considered case, it also depends on the previous rating.
- Observations used (rating process trajectories) must be independent whereas in our framework, they are dependent through the common factor Θ .

The first step of the algorithm assigns initial values to the parameters we want to estimate. Then the algorithm replaces the missing data (states of Θ) with Bayesian estimators using the observations and the current parameters values.

The second one consists on improving a conditional likelihood in order to obtain better estimates of parameters. Then these new estimates are used to repeat the first step along the next iteration to converge to a local maximum.

In our previous discrete framework, we decided that the jumps of Θ at time t can not directly

affect rating process Z at time t (there should be a response time between). That's why, we change the conventions compared to classical HMM models, by supposing that the law of the rating process at current time t is governed by the hidden factor Θ at the previous time. Obviously, this consideration has no impact on the filtering. Along with the definition of a self financing portfolio, one can find the same kind of considerations (see [6]).

We must also integrate that the new rating does not only depend on the economic cycle (state of Θ) but on the previous rating as well. Therefore, we rather adapt the classical Baum-Welsh algorithm by using that

$$\forall t \in [1, \Gamma] : \mathbb{P}(Z_t | Z_0, \dots, Z_{t-1}, \Theta_0, \dots, \Theta_{t-1}) = \mathbb{P}(Z_t | Z_{t-1}, \Theta_{t-1}). \quad (62)$$

Furthermore, the observations are all dependent on the same realization of Θ . Then, we must adapt our algorithm differently from [41] who considered that each rating process is governed by its own and independent trajectory of Θ .

4.2 Initialization

The calibration algorithm presented is based on iterative improvement of a likelihood. This expectation maximization algorithm (EM) (see [12]), as most of iterative maximisation algorithms, might be trapped in a local maximum. Obtained parameters may not be relevant when the global maximum is not found. This success is deeply dependant of the initialization. Several empirical and analytical methods have been proposed to deal with this matter. [38] decided to use frequency estimators for transition probabilities. They succeed to considerably reduce the number of iterations to find their local maximum. By noticing that the transition matrices have strong diagonals, [41] initialized their model by adding small perturbations to identity or uniform matrices. In our study we choose a third option which seems to be more reliable: we test a high number of initial values (picked at random) in order to find the global maximum. In order to converge almost surely to the global maximum, we must pick initial values uniformly. However a uniform law can not be defined on a value space which has zero Lebesgue-measure: a model transfer to a smaller space dimension, a compact with no empty interior, is then achieved.

4.3 Bayesian estimators

This part only presents the main results of the algorithm. Proofs and derivations of estimators are provided in Appendix (A).

Let $Z = (Z^q)_q$ be the multivariate rating process and we call for $(t_1, t_2) \in [0, \Gamma]^2$, $(Z)_{t \in [t_1, t_2]} = Z_{t_1|t_2}$. We define the forward probability

$$\forall j \in \mathbb{T}, \forall t \in [1, \Gamma] : \alpha_t(j) = \mathbb{P}(Z_{0|t} = z_{0|t}, \Theta_{t-1} = j) \quad (63)$$

and the backward probability

$$\forall j \in \mathbb{T}, \forall t \in [1, \Gamma - 1] : \beta_t(j) = \mathbb{P}(Z_{t+1|T} = z_{t+1|T} | Z_t = z_t, \Theta_{t-1} = j). \quad (64)$$

We use the following recursive formulas in order to compute the two previous probabilities

$$\alpha_t(j) = \sum_{k=1}^m \alpha_{t-1}(k) K^{kj} \prod_d L^{j, z_{t-1}^d z_t^d}, \quad (65)$$

$$\beta_t(j) = \sum_{l=1}^m \beta_{t+1}(l) K^{jl} \prod_d L^{l, z_t^d z_{t+1}^d}. \quad (66)$$

For $t \in [1, \Gamma]$, we introduce two random variables useful to describe Θ

$$u_t(j) = \mathbb{1}_{[\Theta_t=j]}, \quad (67)$$

$$v_t(k, j) = \mathbb{1}_{[\Theta_t=j, \Theta_{t-1}=k]}. \quad (68)$$

The forward and backward probabilities are helpful to compute the following Bayesian estimators

$$\check{u}_t(j) = \mathbb{P}(\Theta_t = j | Z_{0:T} = z_{0:T}) = \frac{\beta_{t+1}(j)\alpha_{t+1}(j)}{L_T}, \quad (69)$$

and

$$\check{v}_t(k, j) = \mathbb{P}(\Theta_t = j, \Theta_{t-1} = k | Z_{0:T} = z_{0:T}) = \frac{\beta_{t+1}(j)K^{kj}\alpha_t(k) \prod_d L^{j, z_t^d z_{t+1}^d}}{L_T}, \quad (70)$$

with L_t the likelihood of the whole sample,

$$L_T = \mathbb{P}(Z_{0:T} = z_{0:T}) = \sum_j \alpha_T(j). \quad (71)$$

4.4 Parameters estimation

The maximization phase consists in finding better parameters than those of the previous iteration. We call $M^{(s)} = (\Pi^{(s)}, L^{(s)}, K^{(s)})$, the parameters obtained at the iteration (s) .

To compute the parameters at iteration $(s+1)$, we define

$$w_i = \mathbb{P}(Z, \Theta = \theta_i | M^{(s)}), \quad (72)$$

$$r_i = \mathbb{P}(Z, \Theta = \theta_i | M^{(s+1)}). \quad (73)$$

The new parameters are deemed to improve the likelihood according to:

$$\mathbb{P}(Z | M^{(s+1)}) \geq \mathbb{P}(Z | M^{(s)}). \quad (74)$$

To achieve that, we are looking for maximizing $\log \frac{\mathbb{P}(Z | M^{(s+1)})}{\mathbb{P}(Z | M^{(s)})}$, which is equivalent to maximizing

$$Q(M^{(s)}, M^{(s+1)}) = \sum_{i=1}^m \mathbb{P}(\Theta = i, Z | M^{(s)}) \log(\mathbb{P}(\Theta = i, Z | M^{(s+1)})). \quad (75)$$

After optimization, we obtain the following estimators

$$\Pi_i = \check{u}_0(i), \quad (76)$$

$$L^{i,kr} = \frac{\sum_d \sum_t \check{u}_t(i) \mathbb{1}_{[z_t^d=r, z_{t-1}^d=k]}}{\sum_d \sum_t \check{u}_t(i) \mathbb{1}_{[z_{t-1}^d=k]}}, \quad (77)$$

$$K^{ji} = \frac{\sum_t \check{v}_t(j, i)}{\sum_t \check{u}_{t-1}(j)}. \quad (78)$$

Proof. for more details, see Appendix (A) ■

4.5 Continuous adaptation: compliance with discrete data

This part aims to calibrate parameters involved in the continuous filtering formula (29). We propose an adaptation of the calibration algorithm in order to be compliant with the continuous filtering formula (29). Previous adaptations done for the discrete framework are still required but are not sufficient: Baum-Welsh algorithm is an estimation in discrete time which is not compliant with the continuous version of the filter. Furthermore simultaneous jumps are forbidden when the processes are assumed to be continuous.

For the first deviation, we propose to realize the calibration in discrete time anyway. Then it is necessary to switch to continuous time dimension for filtering. The transition between probabilities to intensities turns out to be easy when the time interval chosen is small enough according to (22) and (26).

The second deviation is also essential: there is no common jumps between processes. This hypothesis mirrors the reality since any event occurs exactly at the same moment. However, ratings are not natural processes. Human decisions and algorithm appreciations are reported at the same moment in a day. Therefore we have to deal with simultaneous daily observations.

Our solution consists in considering a different time grid. We cut each day into small intervals and spread each jump randomly on these time steps. We insure to cut enough finely to have a maximum of one jump per interval. This manipulation has two drawbacks:

- We might loose the conditional independence between rating entities if we force the non-simultaneity
- We might modify and add information by distributing it randomly.

The second effect is studied in a testing benchmark at (5.1.1).

Once the data have been modified, we adapt our calibration algorithm to respect continuous structure. To this end, we use a prior law of jumps which respects the constraint of no simultaneity (that the number of observed jumps is only 0 or 1). The main idea consists on assuming that one entity is randomly chosen to be allowed to jump. Then, the entity may jump according the common migration matrices $(L^{h,ij})_h, (i, j) \in \{1, \dots, p\}$.

More details of this adaptation can be found in Appendix (B).

5 Filtering on simulated data

The purpose of this section is to test and validate the continuous and discrete versions of the filters with simulated data: we build migration ratings database by simulating a common latent Markov chain. We want to use favourable and adapted environment to observe the maximum ability of the filters to predict transitions. Since the structure of the two models are totally different, we do not use the same data sample for the two models. This experiment is also relevant to measure the effects of spreading jumps.

5.1 Discrete time filtering approach

To build the database we assume that the hidden factor is described by a finite-state space Markov chain with 7-states. We choose the Markov chain parameters and we associate to each state a rating transition matrix. We try to choose matrices compliant with filtering: we must have sufficient variability among conditional transition probabilities $(L^{h,ij})_h, (i, j) \in \{1, \dots, p\}$. We work with 3 ratings categories and initialize our sample with 1000 entities per rating. Then we simulate the hidden Markov chain on 300 time steps. According to the hidden factor's trajectory, we simulate transitions using conditional transition matrices.

At this stage, we can only use the cumulative number of transitions as inputs. We use the first 200 observations to calibrate using Section 4 and the remainder to test the model.

We perform the calibration described above. Most of the time, we obtain the parameters of the simulation (with permutations).

Figure 1 presents the real trajectory versus the filtered trajectory of the hidden factor on the testing sample (of 100 points).



Figure 1: Real and filtered trajectories

We can notice that the filter is quite efficient. It faithfully follows the real trajectory and respects the different phases and trends but it does not exactly mimic the true value since the filter formula is a weighted average of the state values. We can also observe a delay in the estimation. The explanation is theoretical: it is caused by the effect of delay in the filtering model: Θ_{t-1} has an impact on the ratings at time t . Therefore, when we try to estimate the hidden factor at time t , we can only use the freshest observations available at this time: the rating jumps at time t which have been governed by the hidden factor at time $t-1$, Θ_{t-1} . Consequently we infer the current hidden factor state with information generated by its previous value. We can easily understand that the calibration plays a crucial role to make the filtering efficient. In order to forecast in time, states need to be strongly linked at least to one other state. Let's imagine a rare and very unstable state. Since it is hardly reachable from other states, it will never influence the direction of the filter and will be never predicted in time. Once the filter realizes that the hidden factor jumps to this state, it is too late, the hidden factor has already returned to another state. Finally, the filter is unable to capture rare events to unstable states. This remains acceptable since our purpose is to detect transitions to stable regimes. Visiting a state for a brief period time does not represent useful information for long term forecasting.

The following Figures 2, 3, 4, 5, 6, 7, represent real ratios of observed transitions with the predicted transitions dynamics, obtained with formulas (59) and (61), between three ratings.

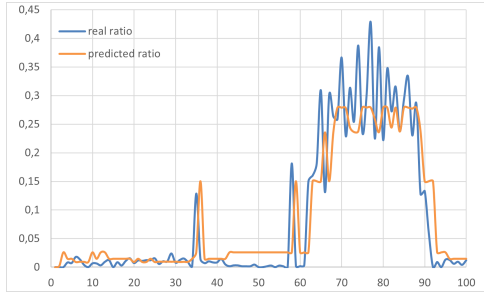


Figure 2: Real and predicted ratios from 0 to 1

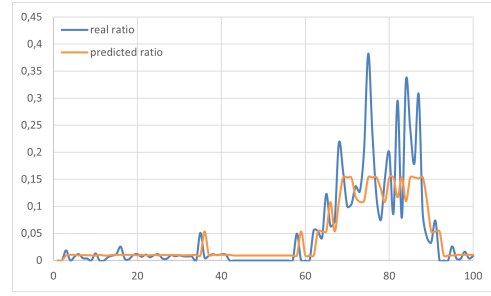


Figure 3: Real and predicted ratios from 0 to 2

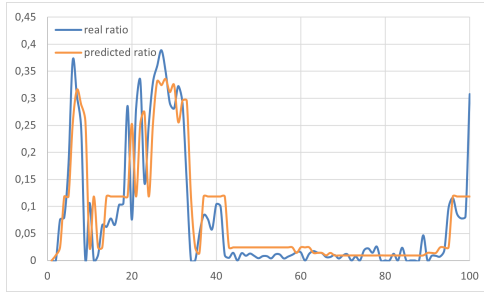


Figure 4: Real and predicted ratios from 1 to 0

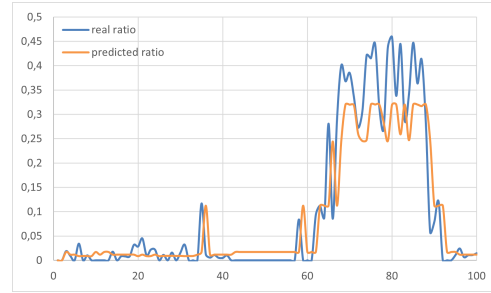


Figure 5: Real and predicted ratios from 1 to 2

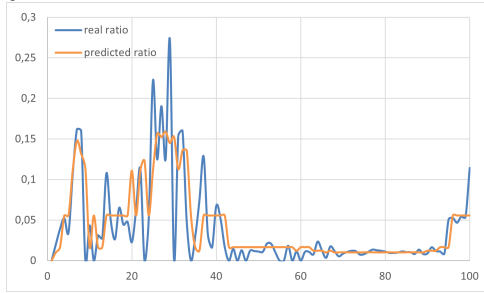


Figure 6: Real and predicted ratios from 2 to 0

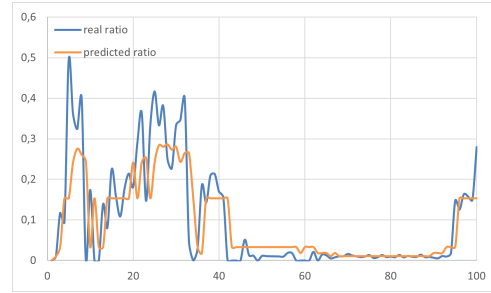


Figure 7: Real and predicted ratios from 2 to 1

The results are very encouraging. The filter is able to provide good predictions of future jumps without using covariates. The predictions vary as a function of the regime cycle. Even when the real ratios sharply increase or decrease, the prediction are immediately corrected. Although, one of our previous intuitions is confirmed, the predictions can not capture extreme variations since the filter is an expectation of trajectory.

5.1.1 Continuous framework

In order to test the continuous filtering, we simulate new data, which are consistent with the continuous structure, by respecting the format of non simultaneity of jumps. We keep 3 rating categories and 1000 entities per class but for sake of simplicity, we reduce the number of states of the latent factor to 4. We realize a benchmark by comparing two different models: the continuous filter is directly applied on simulated data first. Then we will challenge the re-distribution through a second experience. We aggregate jumps and randomly spread them before filtering. This experience is motivated by

- validating the continuous framework
- observing and measuring the effect of the random re-distribution of jumps.

Figures 8 and 9 show the dynamics of the proportion of transitions predicted against the real observed ratios from rating 1 to rating 2, respectively without and with redistribution.

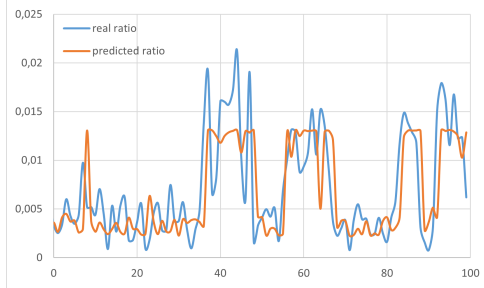


Figure 8: Real and predicted ratios from 1 to 2 without redistribution

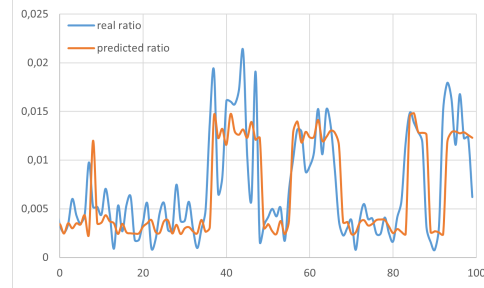


Figure 9: Real and predicted ratios from 1 to 2 with redistribution

The plots of forecasting in Figure 8, validate the use of the continuous filtering: the curve of predictions respects the general trends of the real trajectory.

Even if the data samples used are different, we can notice that the transitions of predictions are less brutal than in the discrete framework. The continuous filter uses information progressively (due to the absence of simultaneous jumps) and is more flexible to anticipate regime changes. Assimilating jumps one by one, seems to improve the quality of predictions. Nevertheless, the effect of delay is still observable.

By comparing with Figure 9, we deduce that spreading information (to avoid the simultaneity of jumps) does not alter the predictions. Thanks to this comparison exercise, we can apply continuous framework to real data without concern that the results are altered by this action.

6 Application on real data

This section is dedicated to present the results of our different models applied on a real rating database. We will perform two discrete versions of filtering (univariate and multivariate) and a continuous multivariate filtering approach.

6.1 Data Description

Ratings issue credit rating are forward-looking opinions about the creditworthiness of an obligor with respect to a specific financial obligation. Since June 15, 2015, any nationally recognized statistical rating organisation is obliged by the US Securities and Exchange Commission (SEC) to disclose any credit rating actions from June 15, 2012. We build a transitions rating database from Moody's credit rating disclosure. We use only aggregated data (number of transitions). The considered sample contains 7791 days from January 2000 to May 2021. We study the evolution of Long Term ratings, without sector consideration of 5030 corporate entities during this period. For specific experiments (validation, comparison...) we use the whole sample to calibrate and to apply our models. For others such as testing the predictive power of model, we will proceed to a cross validation.

Moody's rating system relates 21 ratings categories. Keeping this granularity means estimating more than 420 transitions. That's why, like many other studies ([34], [19]...), we reduce the

number of rating categories. We decide to aggregate the 22 ratings to 6 : A, Baa, Ba, B, C and W.

An obligation is rated W when it has no rating. We will also rate W the entity whose rating is not observed. This happens when the data is missing, censored or when it is not appeared yet. There are many ways to manage not rated status (W). We can consider it as bad information, good information, no information for the credit or not considering them at all. According to [9], only few (roughly 13 percent) of the migration to the not rated category are related to changes in credit quality. This argument motivated [40] to use the last method, that consists simply on removing from the sample all the entities that experience the not rated status. But this approach is dubious in regard of the loss of information. In this study, we will consider no rated status as censorship. This is achieved by progressively eliminating companies whose rating is withdrawn and adding them when a new rating is provided.

6.2 Univariate Discrete filtering

In this part, we assume that each transition is governed by its own hidden factor. Under this assumption, each transition evolves according to the evolution of its own latent factor, independently from the others. This modeling is meaningful to integrate specific behaviours in regime switching dynamics.

On the data set described above, we focus on a single transition: from rating B to C. We choose this transition because it could be identified as a "transition to default" and witness of crisis. This will entail the use and calibration of the univariate form of the filter (51).

A first step consists on calibrating the models with the past of the involved transition. We choose a 5 states hidden factor. The time step at stake in every transition is 30 days. We highlight the efficiency of our approach without cross validation: same sample for the calibration is used for the filtering phase.

We obtain in Table 1, the calibrated transition matrix of the hidden factor. Table 2 presents the conditional transition probabilities from rating B to C.

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$\Theta = 0$	0.90598	0.074109	0.018316	0	0.001595
$\Theta = 1$	0.230415	0.715919	0.040626	0	0.013040
$\Theta = 2$	0.000304	0.381375	0.540412	0.077909	0
$\Theta = 3$	0	0	0.740452	0.259548	0
$\Theta = 4$	0.491597	0	0	0.508403	0

Table 1: Θ 's transition matrix

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$B \rightarrow C$	0.001814	0.0050001	0.0158818	0.0451715	0.085771

Table 2: Conditional transition probabilities B to C

The obtained calibration reveals a scale of risk between the states of Θ . Table 1 highlights two stable states, 0 and 1 and an unstable and rare state, state 4. By analysing Table 2, we notice that state 4 is clearly identified as the riskiest state with a downgrade probability fifty time greater than in 0, the most favourable state. State 3 is also a state of crisis but more stable. State 2 can be interpreted as an intermediate state between favourable and unfavourable situation. Consequently we can expect that the economy often remains in a calm and favourable situation and experiences sometimes brief transitions to stressed states when downgrade probability B to C increases a lot. Figure 11 shows the dynamics of 30 days forecast of transition B to C. Figure 10 presents the filtered indicator function trajectories of its own hidden factor, without cross validation.

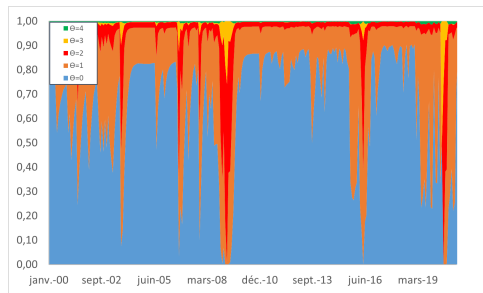


Figure 10: Filtered trajectories of the hidden factor indicator functions

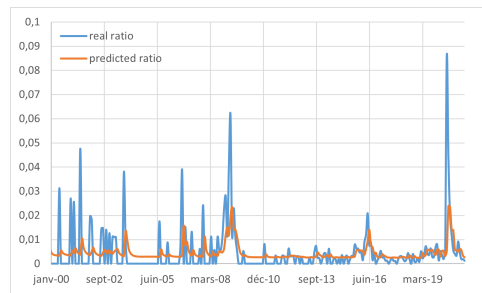


Figure 11: Real and predicted ratios for transition B to C

Figure 10 highlights regime switches between the states of the latent factor. Our intuitions are confirmed, the filter is often closed to favourable states 0 and 1. It sometimes experiences brief transitions to 2, an intermediate state, when the downgrade probability increases. Sometimes this state transits to state 3, a state of true crisis. Transitions to state 4 are sudden from state 0 or 1 but remain rare. Fortunately this state of extreme "crisis" is only visited for brief periods. By analysing 11, it can be noted that the forecasts reflect the general trend of the real ratio with the same effect of delay or "lag" effect observed than on fictive data. The filter is able to detect regimes and transitions phases but can not handle with brutal and short transitions. The behaviour expected during calibration analyses of long periods of favourable situations and brief transitions to stress states, is now confirmed.

Note that here, the hidden factor is specific to the involved transition. It may cover systemic risk but also the risk which might be specific to the ratings at stake. An economical regime process should be unique and affect all ratings. In that sens, a transition specific hidden factor can not be faithfully interpreted as global economic factor.

We will now consider the multivariate case where the hidden factor is shared by several transitions.

6.3 Multivariate discrete Filtering

Using multiple transitions to infer the hidden factor assumes that this last is shared by those transitions. This approach should bring more information to forecast but presents several difficulties:

- The algorithm of calibration finds centroids which might be far from each other due to the high dimension of the space. Consequently the true number of transitions may be very different from predictions.
- Rating transition events may not be sufficiently correlated. Idiosyncratic behaviour might be very important in risk valuation.

We achieve two experiments. With a time reference step of 30 days, we filter in sample, the hidden factor, to observe the behaviour of the multivariate model. Then, along a second experiment, using a time reference step of 50 days, we proceed to a cross validation to faithfully assess the predictive power of the model.

6.3.1 Single calibration on the whole period

To see the benefits of this approach and the differences with the univariate one, we keep 5 states for the hidden factor, a time step of 30 days and we do not proceed to cross validation. Certain transitions are weakly correlated and might bring noise. We must only consider the most correlated transitions to extract the global factor dynamics. Therefore we decide to only focus on adjacent downgrade transitions (the upper diagonal). Indeed empirical results from [11] show that the

upgrades are more subject to idiosyncratic shocks than downgrades. To remove the impact of the remaining transitions on the model, we assign them the same probability for each state of the hidden factor: we use the time-homogeneous intensity estimators to compute these probabilities (see, e.g., [31], [36], [15], [34], [11]...). Consequently we reduce the number of transitions to calibrate to four.

We obtain in Table 3, the calibrated transition matrix of the hidden factor. Table 4 presents the conditional downgrade probabilities for a step time of 30 days.

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$\Theta = 0$	0.9499	0.0418	0.0010	0	0.0073
$\Theta = 1$	0.1075	0.7661	0.1264	0	0
$\Theta = 2$	0.0004	0.2685	0.6340	0.0503	0.0469
$\Theta = 3$	0	0	0.5133	0.4867	0
$\Theta = 4$	0	0	1	0	0

Table 3: Θ 's transition matrix on march 2021

	$\Theta = 0$	$\Theta = 1$	$\Theta = 2$	$\Theta = 3$	$\Theta = 4$
$A \rightarrow Baa$	0.00297589	0.00224944	0.00838262	0.00801885	0.0194804
$Baa \rightarrow Ba$	0.00125687	0.00146192	0.00492593	0.00985172	0.031583
$Ba \rightarrow B$	0.00326413	0.00633207	0.0150595	0.0282736	0.0228716
$B \rightarrow C$	0.00189228	0.00492691	0.0128149	0.0641203	0.0114155

Table 4: Downgrade probabilities

By analysing the tables, we highlight various profiles of the economic states. States 0 and 1 are stable states which are favourable for the credit. States 3 and 4 are states of crisis, where specific downgrade probabilities are higher. Note that state 4 is totally unstable and transitory. The transition between favourable and stable stressed period (state 3) is exclusively achieved through state 2, an intermediate state.

In this experiment, we decide to only present the forecasts for transition B to C. Figure 12 and Figure 13 respectively present the latent factor trajectory of indicator functions and the dynamics of the predicted ratio from the rating B to C, within a multivariate framework, without cross validation.

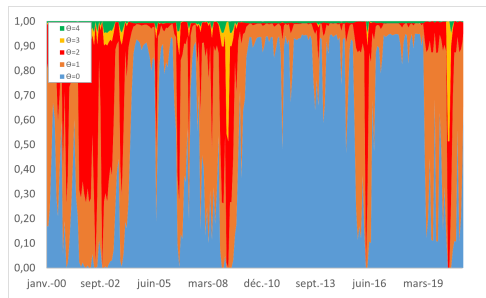


Figure 12: Filtered trajectories of hidden factor indicator functions

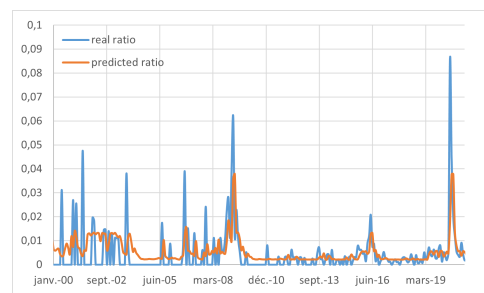


Figure 13: Real and predicted ratios for transition B to C

Figure 12 brings us new information on the economical dynamics. Periods of crisis in state 3 and 4 are pretty rare and brief. By analysing Figure 13, we can first notice that the multivariate framework is also a good predictor. The forecasts dynamics follow the general trend of observed ratios and is enough flexible to fit with different regimes. Comparing with the univariate curve, the multivariate model seems to be more sensitive to events and captures more information: the multivariate model notices the crisis of sep-2000 while the univariate one does not. The forecasts of a transition are not only based on its own past evolution but also stem from the past of others.

On the other hand, rating transition dynamics might also send noise while inferring the common latent factor.

6.3.2 Annual recalibration with past information

In this experiment, we change the time reference step to 50 days. In order to truly test the multivariate model we must predict on time intervals not used for the calibration. To this end, we use the first eight years of the sample to perform a first calibration and to initialize our parameters. Then from January 2008 to may 2021, we predict the dynamics of the 50 days transition rates. The model is re-calibrated every year, integrating the observations of the past year. Since we re-calibrate the model yearly, parameters and states structure vary over time. Presenting filtered trajectory and calibration results would make no sens. Figures 14, 15, 16, 17 compare the dynamics of predictions to the observed ratios with annual re-calibration.

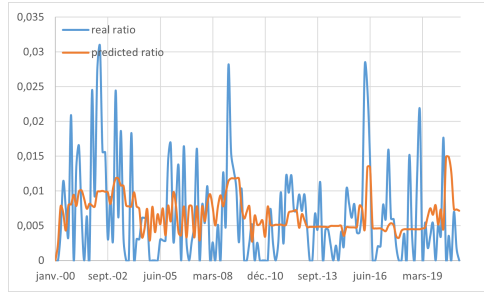


Figure 14: Real and predicted ratios dynamics from rating A to Baa

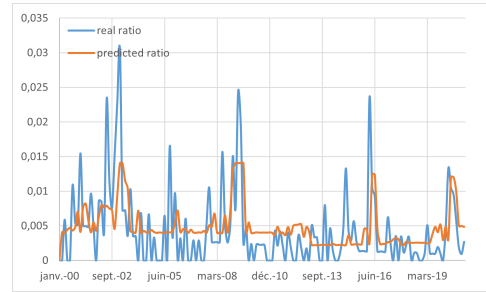


Figure 15: Real and predicted ratios dynamics from rating Baa to Ba

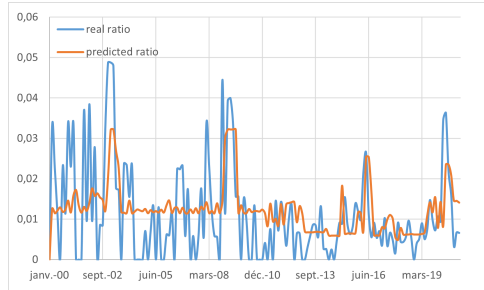


Figure 16: Real and predicted ratios dynamics from rating Ba to B

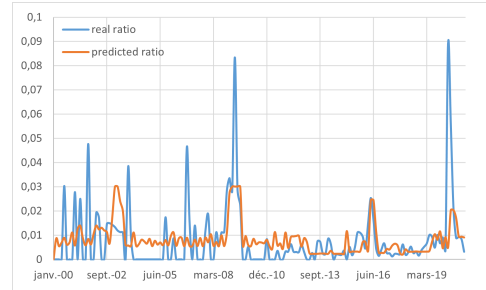


Figure 17: Real and predicted ratios dynamics from rating B to C

We notice that transitions are correlated during specific periods like crisis. Four are noteworthy: a first small one around 2002, a moderated one in 2016 and two significant in 2008 and 2020. These latter are clearly identified as the subprime crisis and the health crisis caused by the covid 19. The two others, moderated, would be respectively the consequences of the dot-com bubble in 2000 and the China stock market crash in June, 2015. During these periods the downgrades probabilities increase.

Our model is able to detect the evolution of the economical situation by using the evolution of migrations. Then it adapts the forecasts according to the inferred economic cycle. During crisis periods, the model is able to propose adapted and higher downgrade probabilities.

Nevertheless we can still notice that this model struggles to capture brief and brutal variations. Continuous framework has the advantage of assimilating jumps one by one and of being more flexible and suitable to anticipate sudden transitions. In that sens, we test the continuous framework on real data in order to have a final comparison.

6.4 Multivariate continuous filtering

In this section, we apply the continuous filtering framework (2) and its adaptations (4.5) to real data. The model is applied on the same sample used for the discrete filtering described in Subsection (6.3.2). We keep a time reference step equal to 50 days. We directly present in Figures 18, 19, 20 and 21, the dynamics of predictions against the true ratios of transitions.

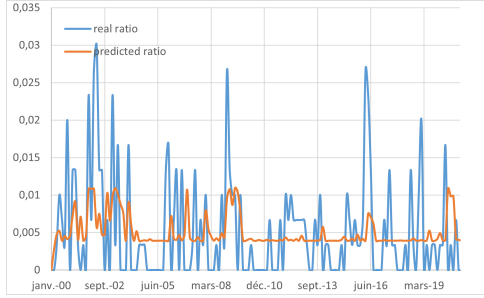


Figure 18: Real and predicted ratios dynamics from rating A to Baa

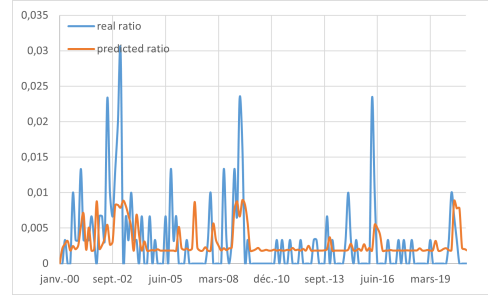


Figure 19: Real and predicted ratios dynamics from rating Baa to Ba

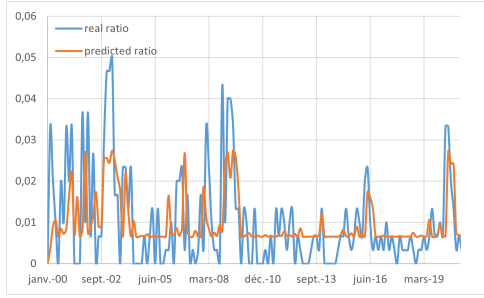


Figure 20: Real and predicted ratios dynamics from rating Ba to B

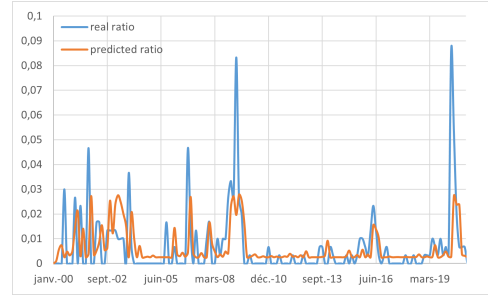


Figure 21: Real and predicted ratios dynamics from rating B to C

We can directly notice that the results are quite similar than with the discrete model. The dynamics of predictions follow the general trend of true ratios. The forecasts are also adapted when noteworthy crisis occurs. The continuous version presents three main advantages:

- the effect of delay (or lag effect) is less significant than in discrete framework. By spreading jumps in small time interval, we make last information used for filtering fresher than it actually is. This fictive operation, however, improves the predictions.
- the absence of simultaneous jumps reduces considerably the dimension. Indeed, our calibration algorithm, as an expected maximisation algorithm (EM), turns out to be a lot affected by the famous "curse of the dimensionality".
- the filter seems to be more flexible to predict sharp transitions. Since information is spread and displayed in fictive intervals, the filter assimilates slower and progressively information and therefore can adapt its predictions.

Nevertheless we can see that this framework is not fully adapted to rating migrations. The discrete version is easier and faster to compute: manipulations described in (4.5) increases considerably the number of data, the complexity and remains laborious. Above all the discrete model is more consistent with the data. The continuous approach should be improved by using a true continuous EM algorithm. The effectiveness of this method would rather be highlighted by filtering

a continuous phenomenon, where observations can not be simultaneous and exact dates are known.

This intuition is confirmed by the following experiment. We compute the R^2 coefficient in the sample, described in [29] to access the forecasting power of predictive models. We keep a time reference step of 50 days. We respectively compare the coefficients of the constant generator intensity model, the univariate and the multivariate discrete models and the continuous model in Table 5.

	$A \rightarrow Baa$	$Baa \rightarrow Ba$	$Ba \rightarrow B$	$B \rightarrow C$
Constant	0.463012	0.250668	0.483855	0.250514
Univ.Discrete	0.431184	0.346886	0.608684	0.22263515
Mult.Discrete	0.494395	0.479324	0.644975	0.367094
Mult.Continuous	0.2022	0.331062	0.49396	0.279736

Table 5: R^2 in the sample

We can directly notice that the discrete filter is the most accurate whatever the transition. The R^2 of the continuous filter is closed to discrete filter for transitions Baa to Ba and B to C but is low for the transitions A to Ba, even lower than those issued from the constant generator model. This transition being the less stable (sudden and brief fluctuations), the filter has difficulties to forecast the variations in time. This phenomenon can also be explained by the poorer calibration done for the continuous filtering (for a sake of rapidity) and the inconsistency with the format of the data.

7 Conclusion

Finally we have presented a new way to predict rating transition probabilities. Our method has the advantage of basing its predictions on a systemic factor shared by all transitions, identified as global economical status, without using any economical covariates. Throughout this article, we have compared two approaches, a continuous and a discrete one, on fictive and real data sets. Obviously, both approaches can not capture all information: indeed [44] found that only 18-26% of global default risk variation is systematic while the reminder is idiosyncratic. The share of systematic default risk is higher (39-51%) if industry-specific variation is counted as systematic. Nevertheless our methods remain to be good predictors adapted to economical cycle. We believe that our approaches can be used to improve current PIT-estimations in the sens that we based our predictions on the business cycle without concern for whose and how many covariates we should use.

Unfortunately the effect of delay is inevitable and our models is less efficient when transitions are sudden. Nevertheless using the continuous framework relieves a bit this undesired effect. It also presents many other advantages. Applying a model on gradual data decreases considerably the dimension of the model and makes the filtering more sensible to capture regime changes. However, applying the continuous framework to discrete environment is laborious and presents a risk of altering information and the quality of predictions. This continuous adaptation is also much more complex to apply and suffers of a poorer calibration compared to other models to truly highlight its benefits. For that reason, we recommend to only retain the continuous model when the observed phenomenon is also continuous. In that sense we advise to use the discrete model for credit rating predictions.

Appendices

A Calibration of the discrete Version

We provide here more details about the two calibration frameworks (discrete and continuous cases). This part describes the derivation for the Baum-Welsh algorithm adaptation used to calibrate the multivariate discrete filtering equation (56). We compute $\forall j \in [1, m], \forall t \in [1, \Gamma]$, the forward probability $\alpha_t(j) = \mathbb{P}(Z_{0|t} = z_{0|t}, \Theta_{t-1} = j)$ and $\forall j \in [1, m], \forall t \in [1, \Gamma - 1]$, the backward probability $\beta_t(j) = \mathbb{P}(Z_{t+1|\Gamma} = z_{t+1|\Gamma} | Z_t = z_t, \Theta_{t-1} = j)$.

We derive $\forall t \in [2, \Gamma - 1]$ and $\forall j \in [1..m]$,

$$\begin{aligned}
\alpha_t(j) &= \mathbb{P}(Z_{0|t} = z_{0|t}, \Theta_{t-1} = j) = \sum_k \mathbb{P}(Z_{0|t} = z_{0|t}, \Theta_{t-1} = j, \Theta_{t-2} = k) \\
&= \sum_k \prod_d \mathbb{P}(Z_t^d = z_t^d | Z_{0|t-1} = z_{0|t-1}, \Theta_{t-1} = j, \Theta_{t-2} = k) \mathbb{P}(Z_{0|t-1} = z_{0|t-1}, \Theta_{t-1} = j, \Theta_{t-2} = k) \\
&= \sum_k \prod_d \mathbb{P}(Z_t^d = z_t^d | Z_{t-1} = z_{t-1}, \Theta_{t-1} = j) \mathbb{P}(\Theta_{t-1} = j | \Theta_{t-2} = k, Z_{0|t-1} = z_{0|t-1}) \alpha_{t-1}(k) \\
&= \sum_k \prod_d (\mathbb{P}(Z_t^d = z_t^d | Z_{t-1}^d = z_{t-1}^d, \Theta_{t-1} = j)) \mathbb{P}(\Theta_{t-1} = j | \Theta_{t-2} = k) \alpha_{t-1}(k) \\
&= \sum_k K^{kj} \alpha_{t-1}(k) \prod_d L^{j, z_{t-1}^d, z_t^d}.
\end{aligned}$$

For $t=1$, we do not know the state of the individuals before the simulation. To tackle this issue, we use the initial proportion of the ratings. We have

$$\begin{aligned}
\alpha_1(i) &= \mathbb{P}(Z_1 = z_1, \Theta_0 = i) \\
&= \prod_d \mathbb{P}(Z_1^d = z_1^d | \Theta_0 = i) \mathbb{P}(\Theta_0 = i) \\
&= \prod_d (\sum_j \mathbb{P}(Z_1^d = z_1^d | Z_0^d = j, \Theta_0 = i) \mathbb{P}(Z_0^d = j | \Theta_1 = i)) \Pi(i) \\
&= \prod_d (\sum_j \mathbb{P}(Z_1^d = z_1^d | Z_0^d = j, \Theta_0 = i) \mathbb{P}(Z_0^d = j)) \Pi(i) \\
&= \prod_d (\sum_j L^{i, j, z_1^d} \mathbb{P}(Z_0^d = j)) \Pi(i).
\end{aligned}$$

Similarly, we recursively derive the backward probability $\forall t \in [1, \Gamma - 2]$ and $\forall j \in [1..m]$

$$\begin{aligned}
\beta_t(j) &= \mathbb{P}(Z_{t+1|\Gamma} = z_{t+1|\Gamma} | \Theta_{t-1} = j, Z_t = z_t) \\
&= \sum_l \mathbb{P}(Z_{t+1|\Gamma} = z_{t+1|\Gamma}, \Theta_t = l | \Theta_{t-1} = j, Z_t = z_t) \\
&= \sum_l \mathbb{P}(Z_{t+2|\Gamma} = z_{t+2|\Gamma} | \Theta_t = l, Z_{t+1} = z_{t+1}, Z_t = z_t, \Theta_{t-1} = j) \mathbb{P}(Z_{t+1} = z_{t+1}, \Theta_t = l | \Theta_{t-1} = j, Z_t = z_t) \\
&= \sum_l \mathbb{P}(Z_{t+2|\Gamma} = z_{t+2|\Gamma} | \Theta_t = l, Z_{t+1} = z_{t+1}) \prod_d (\mathbb{P}(Z_{t+1}^d = z_{t+1}^d | \Theta_t = l, Z_t^d = z_t^d)) * \mathbb{P}(\Theta_t = l | \Theta_{t-1} = j) \\
&= \sum_l \beta_{t+1}(l) K^{jl} \prod_d L^{l, z_t^d, z_{t+1}^d}.
\end{aligned}$$

For $t = \Gamma - 1$, we take $\forall j \in [1..m]$, $\beta_{\Gamma-1}(j) = 1$.

Both estimators will be used to replace the missing data during the maximization phase. The missing data describing the hidden factor are defined $\forall j \in [1, m], \forall t \in [1, \Gamma]$, $u_t(j) = \mathbb{1}_{[\Theta_t=j]}$, and $v_t(k, j) = \mathbb{1}_{[\Theta_t=j, \Theta_{t-1}=k]}$. We define the associated Bayesian estimators $\check{u}_t(j) = \mathbb{P}(\Theta_t = j | Z_{0:\Gamma} = z_{0:\Gamma})$, and $\check{v}_t(k, j) = \mathbb{P}(\Theta_t = j, \Theta_{t-1} = k | Z_{0:\Gamma} = z_{0:\Gamma})$.

We derive expression of these Bayesian estimators with the forward and the backward probabilities

$$\begin{aligned} \forall j \in [1, m], \forall t \in [1, \Gamma - 2], \check{u}_t(j) &= \mathbb{P}(\Theta_t = j | Z_{0:\Gamma} = z_{0:\Gamma}) \\ &= \frac{\mathbb{P}(Z_{t+2|\Gamma} = z_{t+2|\Gamma} | \Theta_t = j, Z_{0|t+1} = z_{0|t+1}) \alpha_{t+1}(j)}{\mathbb{P}(Z_{0|\Gamma} = z_{0|\Gamma})} \\ &= \frac{\mathbb{P}(Z_{t+2|\Gamma} = z_{t+2|\Gamma} | \Theta_t = j, Z_{t+1} = z_{t+1}) \alpha_{t+1}(j)}{L_\Gamma} \\ &= \frac{\beta_{t+1}(j) \alpha_{t+1}(j)}{L_\Gamma}. \end{aligned}$$

With L_t being the likelihood on the whole sample, $L_\Gamma = \mathbb{P}(Z_{0:\Gamma} = z_{0:\Gamma}) = \sum_j \alpha_\Gamma(j)$.

$$\begin{aligned} \forall j \in [1, m], \forall t \in [1, \Gamma - 2] : \check{v}_t(k, j) &= \mathbb{P}(\Theta_t = j, \Theta_{t-1} = k | Z_{0:\Gamma} = z_{0:\Gamma}) \\ &= \frac{\mathbb{P}(\Theta_t = j, Z_{t+1|\Gamma} = z_{t+1|\Gamma} | Z_{0|t} = z_{0|t}, \Theta_{t-1} = k) \alpha_t(k)}{L_\Gamma} \\ &= \frac{\mathbb{P}(Z_{t+2|\Gamma} = z_{t+2|\Gamma} | Z_{0|t+1} = z_{0|t+1}, \Theta_{t-1} = k, \Theta_t = j) \mathbb{P}(\Theta_t = j, Z_{t+1} = z_{t+1} | Z_{0|t} = z_{0|t}, \Theta_{t-1} = k) \alpha_t(k)}{L_\Gamma} \\ &= \frac{\mathbb{P}(Z_{t+2|\Gamma} = z_{t+2|\Gamma} | Z_{t+1} = z_{t+1}, \Theta_t = j) \prod_d (\mathbb{P}(Z_{t+1}^d = z_{t+1}^d | \Theta_t = j, Z_t^d = z_t^d)) \mathbb{P}(\Theta_t = j | \Theta_{t-1} = k) \alpha_t(k)}{L_\Gamma} \\ &= \frac{\beta_{t+1}(j) K^{kj} \alpha_t(k) \prod_d L^{j, z_t^d, z_{t+1}^d}}{L_\Gamma}. \end{aligned}$$

Using the concavity of the log function, we establish a useful inequality for the next step of the derivations. For two strictly positive suites w_i and r_i

$$\begin{aligned} \log \left(\frac{\sum_i r_i}{\sum_k w_k} \right) &= \log \left(\sum_i \frac{w_i r_i}{\sum_k w_k w_i} \right) \\ &\geq \sum_i \frac{w_i}{\sum_k w_k} \log(r_i) - \frac{w_i}{\sum_k w_k} \log(w_i) \\ &= \frac{1}{\sum_k w_k} \left(\sum_i (w_i \log(r_i) - w_i \log(w_i)) \right). \end{aligned}$$

The maximization step consists in finding better parameters than those of the previous iteration. We call $M^{(s)} = (\Pi^{(s)}, L^{(s)}, K^{(s)})$, the parameters of the current iteration (s).

We are seeking new parameters $M^{(s+1)} = (\Pi^{(s+1)}, L^{(s+1)}, K^{(s+1)})$.

Using $u_i = \mathbb{P}(Z, \Theta = \Theta_i | M^{(s)})$ and $v_i = \mathbb{P}(Z, \Theta = \Theta_i | M^{(s+1)})$ in the previous established inequality and defining $Q(M^{(s)}, M^{(s+1)}) = \sum_{\Theta_i} w_i \log(r_i)$ and $Q(M^{(s)}, M^{(s)}) = \sum_{\Theta_i} w_i \log(w_i)$, we obtain

$$\begin{aligned} \log \left(\frac{\sum_{\Theta_i} r_i}{\sum_{\Theta_i} w_i} \right) &= \log \frac{\mathbb{P}(Z | M^{(s+1)})}{\mathbb{P}(Z | M^{(s)})} \\ &\geq \frac{1}{\mathbb{P}(Z | M^{(s)})} \left(\sum_{\Theta_i} w_i \log(r_i) - \sum_{\Theta_i} w_i \log(w_i) \right) \\ &= \frac{1}{\mathbb{P}(Z | M^{(s)})} (Q(M^{(s)}, M^{(s+1)}) - Q(M^{(s)}, M^{(s)})). \end{aligned}$$

This last inequality shows that we obtain $\mathbb{P}(Z|M^{(s+1)}) \geq \mathbb{P}(Z|M^{(s)})$ by maximizing

$$Q(M^{(s)}, M^{(s+1)}) = \sum_{\Theta} \mathbb{P}(\Theta = \Theta, Z|M^{(s)}) \log(\mathbb{P}(\Theta = \Theta, Z|M^{(s+1)})).$$

We cut $\log(\mathbb{P}(Z, \Theta|M^{(s+1)})) = \log(\mathbb{P}(Z|\Theta, M^{(s+1)})) + \log(\mathbb{P}(\Theta|M^{(s+1)}))$. Since the processes $(Z^d)_d$ are independent knowing the unobserved factor, we have

$$\begin{aligned} \log(\mathbb{P}(Z, \Theta|M^{(s+1)})) &= \log(\mathbb{P}(\Theta_0)) + \sum_t \log(\mathbb{P}(\Theta_t|\Theta_{t-1})) + \sum_d \log(\mathbb{P}(Z_{0\Gamma}^d|\Theta)) \\ &= \log(\mathbb{P}(\Theta_0)) + \sum_t \log(\mathbb{P}(\Theta_t|\Theta_{t-1})) + \sum_d \sum_t \log(\mathbb{P}(Z_t^d|Z_{t-1}^d, \Theta_{t-1})). \end{aligned}$$

So

$$\begin{aligned} Q(M^{(s)}, M^{(s+1)}) &= \sum_{\Theta} \log(\mathbb{P}(\Theta_0)) \mathbb{P}(\Theta, Z|M^{(s)}) + \sum_{\Theta} \sum_t \log(\mathbb{P}(\Theta_t|\Theta_{t-1})) \mathbb{P}(\Theta, Z|M^{(s)}) \\ &\quad + \sum_{\Theta} \sum_d \sum_t \log(\mathbb{P}(Z_t^d|Z_{t-1}^d, \Theta_{t-1})) \mathbb{P}(\Theta, Z|M^{(s)}) \\ &= \sum_{\Theta \setminus \Theta_0} \sum_i \log(\mathbb{P}(\Theta_0 = i)) \mathbb{P}(\Theta_0 = i, \Theta \setminus \Theta_0, Z|M^{(s)}) \\ &\quad + \sum_t \sum_{\Theta \setminus (\Theta_t, \Theta_{t-1})} \sum_{i,j} \log(\mathbb{P}(\Theta_t = i|\Theta_{t-1} = j)) \mathbb{P}(\Theta \setminus (\Theta_t, \Theta_{t-1}), \Theta_t = i, \Theta_{t-1} = j, Z|M^{(s)}) \\ &\quad + \sum_t \sum_{\Theta \setminus \Theta_{t-1}} \sum_i \sum_d \log(\mathbb{P}(Z_t^d|Z_{t-1}^d, \Theta_{t-1} = i)) \mathbb{P}(\Theta \setminus \Theta_{t-1}, \Theta_{t-1} = i, Z|M^{(s)}) \\ &= \sum_i \log(\mathbb{P}(\Theta_0 = i, Z|M^{(s)})) \Pi_i + \sum_t \sum_{i,j} \log(K^{ji}) \mathbb{P}(\Theta_t = i, \Theta_{t-1} = j, Z|M^{(s)}) \\ &\quad + \sum_d \sum_t \sum_i \sum_{k,r} \log(L^{i,kr}) \mathbb{P}(\Theta_{t-1} = i, Z|M^{(s)}) \mathbb{1}_{[Z_t^d=r, Z_{t-1}^d=k]}. \end{aligned}$$

We can then maximize by considering the three terms independently. We obtain

$$\Pi_i = \frac{\mathbb{P}(\Theta_0 = i, Z|M^{(s)})}{\sum_j \mathbb{P}(\Theta_0 = j, Z|M^{(s)})} = \mathbb{P}(\Theta_1 = i|Z, M^{(s)}) = \check{u}_0(i),$$

$$\begin{aligned} L^{i,kr} &= \frac{\sum_d \sum_t \mathbb{P}(\Theta_{t-1} = i, Z|M^{(s)}) \mathbb{1}_{[Z_t^d=r, Z_{t-1}^d=k]}}{\sum_d \sum_t \sum_r \mathbb{P}(\Theta_{t-1} = i, Z|M^{(s)}) \mathbb{1}_{[Z_t^d=r, Z_{t-1}^d=k]}} = \frac{\sum_d \sum_t \mathbb{P}(\Theta_{t-1} = i, Z|M^{(s)}) \mathbb{1}_{[Z_t^d=r, Z_{t-1}^d=k]}}{\sum_d \sum_t \mathbb{P}(\Theta_{t-1} = i, Z|M^{(s)}) \mathbb{1}_{[Z_{t-1}^d=k]}} \\ &= \frac{\sum_d \sum_t \mathbb{P}(\Theta_{t-1} = i|Z, M^{(s)}) \mathbb{1}_{[Z_t^d=r, Z_{t-1}^d=k]}}{\sum_d \sum_t \mathbb{P}(\Theta_{t-1} = i|Z, M^{(s)}) \mathbb{1}_{[Z_{t-1}^d=k]}} = \frac{\sum_d \sum_t \check{u}_{t-1}(i) \mathbb{1}_{[Z_t^d=r, Z_{t-1}^d=k]}}{\sum_d \sum_t \check{u}_{t-1}(i) \mathbb{1}_{[Z_{t-1}^d=k]}}. \end{aligned}$$

$$\begin{aligned} K^{ji} &= \frac{\sum_t \mathbb{P}(\Theta_{t-1} = j, \Theta_t = i, Z|M^{(s)})}{\sum_t \sum_i \mathbb{P}(\Theta_{t-1} = j, \Theta_t = i, Z|M^{(s)})} = \frac{\sum_t \mathbb{P}(\Theta_{t-1} = j, \Theta_t = i|Z, M^{(s)})}{\sum_t \mathbb{P}(\Theta_{t-1} = j|Z, M^{(s)})} \\ &= \frac{\sum_t \check{v}_t(j, i)}{\sum_t \check{u}_{t-1}(j)}. \end{aligned}$$

B Calibration of the continuous version

We present in this Appendix, the adaptations of the calibration for the continuous filtering framework. Let n_t be the total number of entities at time t and $\bar{n} = \max_t n_t$, the maximum reached over the whole calibration time period.

In order to make the model identifiable while considering the impact of the size of the sample (which may evolve), we define an independent process I , on $[0, \bar{n}]$, which chooses uniformly the entity which may jumps. If I picks an entity which is not in the current sample (because not already rated or censored...), we do not observe jumps. Otherwise the entity can jump according to the transition matrices $(L^h)_h$.

We have $\forall t \in [0, \Gamma]$, $(j, k) \in (1, p)^2$, $h \in \mathbb{T}$, $q \in [0, \bar{n}]$

$$\mathbb{P}(Z_t^q = k | Z_{t-1}^q = j, I_{t-1} = q, \Theta_{t-1} = h) = L^{h,jk}.$$

We compute

$$\begin{aligned} \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, \Theta_{t-1}) &= \sum_d \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1}) \mathbb{P}(I_{t-1} = d) \\ &= \sum_{d=1}^{n_{t-1}} \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1}) \mathbb{P}(I_{t-1} = d) \\ &\quad + \sum_{d=n_{t-1}}^{\bar{n}} \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1}) \mathbb{P}(I_{t-1} = d) \end{aligned}$$

Let focus on the first sum, describing the situation when the chosen entity has a rating at current time.

$$\begin{aligned} \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1}) &= \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1}) \mathbb{1}_{[z_t^d = z_{t-1}^d, \forall l \neq d: z_t^l = z_{t-1}^l]} \\ &\quad + \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1}) \mathbb{1}_{[z_t^d \neq z_{t-1}^d, \forall l \neq d: z_t^l = z_{t-1}^l]} \\ &= L^{\Theta_{t-1}, z_{t-1}^d, z_t^d} \mathbb{1}_{[z_t = z_{t-1}]} + L^{\Theta_{t-1}, z_{t-1}^d, z_t^d} \mathbb{1}_{[z_t^d \neq z_{t-1}^d, \forall l \neq d: z_t^l = z_{t-1}^l]} \end{aligned}$$

For the second sum, we have: $\mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, I_{t-1} = d, \Theta_{t-1}) = \mathbb{1}_{[z_t = z_{t-1}]}$.

So finally

$$\mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, \Theta_{t-1}) = \left(1 - \frac{n_{t-1}}{\bar{n}}\right) \mathbb{1}_{[z_t = z_{t-1}]} + \sum_{d=1}^{n_{t-1}} \frac{1}{\bar{n}} \mathbb{1}_{[|z_t - z_{t-1}|_0 \leq 1]} \mathbb{1}_{[\forall l \neq d: z_t^l = z_{t-1}^l]} L^{\Theta_{t-1}, z_{t-1}^d, z_t^d},$$

where $|x|_0 = \#\{x_i \neq 0\}$.

By calling $W_{t-1}^h = \mathbb{P}(Z_t = z_t | Z_{t-1} = z_{t-1}, \Theta_{t-1} = h)$, it is easy to check that the previous algorithm can be adapted to the new framework

$$\alpha_t(j) = \sum_{k=1}^m \alpha_{t-1}(k) K^{kj} W_{t-1}^j,$$

$$\beta_t(j) = \sum_{l=1}^m \beta_{t+1}(l) K^{jl} W_t^l,$$

$$\check{\alpha}_t(j) = \mathbb{P}(\Theta_t = j | Z_{0|\Gamma} = z_{0|\Gamma}) = \frac{\beta_{t+1}(j) \alpha_{t+1}(j)}{L_\Gamma},$$

$$\check{\nu}_t(k, j) = \mathbb{P}(\Theta_t = j, \Theta_{t-1} = k | Z_{0|\Gamma} = z_{0|\Gamma}) = \frac{\beta_{t+1}(j) K^{kj} \alpha_t(k) W_t^j}{L_\Gamma}.$$

The forms of the transitions matrices $(L^h)_h$, are a lot impacted by this adaptation. The maximisation does not run as simply as it does for the discrete setting. Explicit forms are heavy to derive. Then, these parameters are directly estimated with optimization algorithms.

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References

- [1] Amending regulation (ec) no 1126/2008 adopting certain international accounting standards in accordance with regulation (ec) no 1606/2002 of the european parliament and of the council as regards international financial reporting standard 9. *Official Journal of the European Union*, 2016.
- [2] Basel committee on banking supervision basel iii: Finalising post-crisis reforms. *Official Journal of the European Union*, 2017.
- [3] Edward I Altman and Duen Li Kao. The implications of corporate bond ratings drift. *Financial Analysts Journal*, 48(3):64–75, 1992.
- [4] Anil Bangia, Francis X Diebold, André Kronimus, Christian Schagen, and Til Schuermann. Ratings migration and the business cycle, with application to credit portfolio stress testing. *Journal of banking & finance*, 26(2-3):445–474, 2002.
- [5] C. Bishop. *Pattern Recognition and Machine Learning (Information Science and Statistics)*. Springer, 2006.
- [6] Tomas Björk. *Arbitrage theory in continuous time*. Oxford university press, 2009.
- [7] P. Brémaud. *Point Processes and Queues: Martingale Dynamics*. New-York: Springer-Verlag, 1981.
- [8] Anisa Caja, Quentin Guibert, and Frédéric Planchet. Influence of economic factors on the credit rating transitions and defaults of credit insurance business. 2015.
- [9] L. Carty. Moody’s rating migration and credit quality correlation. *Moody’s investor services, New-York*, 1997.
- [10] Wai-Ki Ching, Tak Kuen Siu, Li-min Li, Tang Li, and Wai-Keung Li. Modeling default data via an interactive hidden markov model. *Computational Economics*, 34(1):1–19, 2009.
- [11] A. Cousin and M. R. Kheliouen. A comparative study on the estimation of factor migration models. *Bulletin français d’actuariat*, 2015.
- [12] Arthur P Dempster, Nan M Laird, and Donald B Rubin. Maximum likelihood from incomplete data via the em algorithm. *Journal of the Royal Statistical Society: Series B (Methodological)*, 39(1):1–22, 1977.
- [13] Servaas Deroose, Werner Roeger, and Sven Langedijk. Reviewing adjustment dynamics in emu: from overheating to overcooling. *European Economy Economic Paper*, (198), 2008.
- [14] Joseph Leo Doob. *Stochastic processes*, volume 101. New York Wiley, 1953.

- [15] Darrell Duffie, Leandro Saita, and Ke Wang. Multi-period corporate default prediction with stochastic covariates. *Journal of Financial Economics*, 83(3):635–665, 2007.
- [16] European Banking Authorities EBA. Guidelines on pd estimation, lgd estimation and the treatment of defaulted exposures. 2017.
- [17] European Banking Authorities EBA. Guidelines on institutions’ stress testing. 2018.
- [18] Robert J Elliott, Lakhdar Aggoun, and John B Moore. *Hidden Markov models: estimation and control*, volume 29. Springer Science & Business Media, 2008.
- [19] Robert J Elliott, Tak Kuen Siu, and Eric S Fung. A double hmm approach to altman z-scores and credit ratings. *Expert Systems with Applications*, 41(4):1553–1560, 2014.
- [20] Fei Fei, Ana-Maria Fuertes, and Elena Kalotychou. Credit rating migration risk and business cycles. *Journal of Business Finance & Accounting*, 39(1-2):229–263, 2012.
- [21] Dingan Feng, Christian Gouriéroux, and Joann Jasiak. The ordered qualitative model for credit rating transitions. *Journal of Empirical Finance*, 15(1):111–130, 2008.
- [22] Stephen Figlewski, Halina Frydman, and Weijian Liang. Modeling the effect of macroeconomic factors on corporate default and credit rating transitions. *International Review of Economics & Finance*, 21(1):87–105, 2012.
- [23] Peter Fledelius, David Lando, and Jens Perch Nielsen. Non-parametric analysis of rating transition and default data. *Journal of Investment Management*, 2(2), 2004.
- [24] Rüdiger Frey and Thorsten Schmidt. Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. *Finance and Stochastics*, 16(1):105–133, 2012.
- [25] Halina Frydman and Til Schuermann. Credit rating dynamics and markov mixture models. *Journal of Banking & Finance*, 32(6):1062–1075, 2008.
- [26] P. Gagliardini and C. Gouriéroux. Stochastic migration models with application to corporate risk. *Journal of Financial Econometrics*, 3(2):188–226, 2005.
- [27] Giacomo Giampieri, Mark Davis, and Martin Crowder. Analysis of default data using hidden markov models. *Quantitative Finance*, 5(1):27–34, 2005.
- [28] Kay Giesecke and Gustavo Schwenkler. Filtered likelihood for point processes. *Journal of Econometrics*, 204(1):33–53, 2018.
- [29] Shihao Gu, Bryan Kelly, and Dacheng Xiu. Empirical asset pricing via machine learning. Technical report, National bureau of economic research, 2018.
- [30] David T Hamilton. Rating transitions and defaults conditional on watchlist, outlook and rating history. *Outlook and Rating History (February 2004)*, 2004.
- [31] Robert A Jarrow, David Lando, and Stuart M Turnbull. A markov model for the term structure of credit risk spreads. *The review of financial studies*, 10(2):481–523, 1997.
- [32] A. Karr. *Point Processes and their Statistical Inference*. Marcel Dekker Inc., New York, Basel, Hong Kong, 1991.
- [33] Dimitrios Kavvathas. Estimating credit rating transition probabilities for corporate bonds. 2001.
- [34] Siem Jan Koopman, André Lucas, and André Monteiro. The multi-state latent factor intensity model for credit rating transitions. *Journal of Econometrics*, 142(1):399–424, 2008.

- [35] Małgorzata W Korolkiewicz and Robert J Elliott. A hidden markov model of credit quality. *Journal of Economic Dynamics and Control*, 32(12):3807–3819, 2008.
- [36] David Lando and Torben M Skødeberg. Analyzing rating transitions and rating drift with continuous observations. *Journal of banking & finance*, 26(2-3):423–444, 2002.
- [37] V. Leijdekker and P. Spreij. Explicit computations for a filtering problem with point process observations with applications to credit risk, 2011.
- [38] Tingting Liu, Jan Lemeire, and Lixin Yang. Proper initialization of hidden markov models for industrial applications. In *2014 IEEE China summit & international conference on signal and information processing (ChinaSIP)*, pages 490–494. IEEE, 2014.
- [39] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of finance*, 29(2):449–470, 1974.
- [40] Pamela Nickell, William Perraudin, and Simone Varotto. Stability of rating transitions. *Journal of Banking & Finance*, 24(1-2):203–227, 2000.
- [41] Sung Youl Oh, Jae Wook Song, Woojin Chang, and Minhyuk Lee. Estimation and forecasting of sovereign credit rating migration based on regime switching markov chain. *IEEE Access*, 7:115317–115330, 2019.
- [42] Fritsche C Ozkan E, Lindsten F and Fredrik Gustafsson. Recursive maximum likelihood identification of jump markov nonlinear systems. *IEEE Transactions on Signal Processing*, 2013.
- [43] L. R. Rabinet. A tutorial on hidden markov models and selected applications in speech recognition. *Proceedings of the IEEE*, 1989.
- [44] Bernd Schwaab, Siem Jan Koopman, and André Lucas. Global credit risk: World, country and industry factors. *Journal of Applied Econometrics*, 32(2):296–317, 2017.
- [45] A. Tenyakov. Estimation of hidden markov models and their applications in finance. *Electronic Thesis and Dissertation Repository. 2348*, 2014.
- [46] Lyn C Thomas, David E Allen, and Nigel Morkel-Kingsbury. A hidden markov chain model for the term structure of bond credit risk spreads. *International Review of Financial Analysis*, 11(3):311–329, 2002.
- [47] James Tobin. Estimation of relationships for limited dependent variables. *Econometrica: journal of the Econometric Society*, pages 24–36, 1958.
- [48] J.H. van Schuppen. Filtering, prediction, and smoothing for counting process observations – a martingale approach. *SIAM J. Appl. Math.*, 1997.