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STACKS

NICOLE MESTRANO AND CARLOS SIMPSON

ABSTRACT. We explore several different facets of the notion of stack, and how they relate to the applications of stacks in geometry.

1. INTRODUCTION

The theory of stacks is part of a more general trend in modern geometry, to combine geometrical structures with structures from algebraic topology. The notion of stack allows us to make such a combination in novel ways. Homotopy types from algebraic topology are allowed to show up on a punctual scale with respect to the geometry.

In this chapter, we describe some of the motivating examples and considerations leading towards the theory of higher Artin stacks. The guiding idea undergoes a transformation, changing from using algebraic topology to introduce fine structural information into algebraic geometry, to the idea of using algebraic geometry to attach new structure to homotopy types and thereby permit their utilisation in algebraic contexts. At the end we shall see, with an example, how shape theory implied by the idea of nonabelian cohomology, permits to gain insight about the homotopical structure of non-simply connected spaces.

The discussion here is not uniform in level. It is not our intention to provide a mathematical introduction to the main definitions inherent in the theory. On the other hand, we would like to illustrate the genesis of stack theory by referring to the most basic examples. Therefore, some parts of the discussion will treat examples a student would learn very early in any study of the theory, whereas other parts of the discussion will float rather lightly over some of the most advanced concepts.

We would also like to say that we shall be dealing with the notion of stack as it occurs in algebraic geometry. Many closely related if not identical notions are current in nearby fields such as differential geometry. These are the subjects

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of other chapters in the book and our present emphasis on the vantage point of algebraic geometry is meant to be complementary to them.

We don't attack the question of philosophy head-on. Instead, we hope that our discussion will illustrate some of the basic philosophical ideas that arose as consequences of fundamental constructions and examples.

2. SPACES AND COORDINATE CHARTS

A basic element of the modern notion of “space” is the idea of a *coordinate chart*. This is something that allows us to give a precise measurement of the relationship between nearby points. One should think of a coordinate chart as expressing a view of the space, from a certain given viewpoint. For example, if we fix a point $p \in X$, then a neighborhood $p \in U \subset X$ corresponds to a collection of “points near to p ”, and a coordinate system $U \rightarrow k^n$ provides a way of viewing nearby points as being determined by their numerical coordinates $(x_1, \dots, x_n) \in k^n$ where p corresponds to the origin $(0, \dots, 0)$.

Seen from a different point, with a different neighborhood $q \in V \subset X$, we would have a different system of coordinates (y_1, \dots, y_n) . The global structure of the space is obtained from the collection of coordinate change expressions which say, for example, how the new coordinates are related to the old ones:

$$(y_1(x_1, \dots, x_n), y_2(x_1, \dots, x_n), \dots, y_n(x_1, \dots, x_n)).$$

Differentiating this expression gives the well-known Jacobian matrix, in turn providing glueing data to define the tangent bundle.

We have just described the most down-to-earth way that the structure of a space is built up out of its coordinate charts. One of the original objectives of the notion of stack was to give a fully general setting to this idea. Among other things, the notion of “point” itself can become secondary: rather than viewing coordinate charts as corresponding to neighborhoods of specific points, we just remember the maps $U \rightarrow X$. On the other side, the coordinate functions $(x_1, \dots, x_n) : U \rightarrow k^n$ are abstracted to any kind of *concrete algebraic structure* which allows for a computational description of U .

A space is now viewed as something which is *covered* by a collection of maps $p_\alpha : U_\alpha \rightarrow X$, where the U_α themselves have a concrete algebraic structure. For the notion of scheme, the covering objects are *affine schemes* of the form $U_\alpha = \text{Spec}(A_\alpha)$ with A_α a commutative ring, usually of finite presentation:

$$A_\alpha = \frac{k[z_1, \dots, z_n]}{(f_1, \dots, f_m)}.$$

It can be convenient to collect all of the charts together into a single object $U := \coprod_\alpha U_\alpha$. The coordinate change transformations correspond to the cartesian diagram

$$\begin{array}{ccc} R := U \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & X. \end{array}$$

The object R together with its map $R \rightarrow U \times U$ is the equivalence relation glueing U to itself to obtain X . In other words, the above cartesian diagram is also cocartesian in an appropriate sense.

Defining the correct notion of pushout in order to make that work requires some kind of descent. Typically, this is where the notion of *Grothendieck site* comes in: we fix a convenient category with a Grothendieck topology, and our objects are considered as sheaves on the site. Then the glueing property says that X should be the coequalizer of our two maps $R \rightrightarrows U$, in the category of sheaves on the site. The passage to sheaves insures that the glueing notion is invariant with respect to the choice of covering: a different covering should give rise to the same object—that wouldn’t be the case if we took just quotient presheaves.

This level of generality leads, for example, to Artin’s notion of *algebraic space* which is something obtained by glueing together affine schemes by etale equivalence relations.

Once we have written things in this way, the motivation for introducing a notion of *stack* is easy to see: what happens if we try to construct a coequalizer

$$R \rightrightarrows U \rightarrow X$$

but where the map $R \rightarrow U \times U$ is not injective? Geometrically, given two points r and s in R which map to the same pair (u, u') it means that we would like to glue u to u' in two different ways. Now, one might like to keep track of which way was used to glue. Then, the composition of these two glueings will look like an *automorphism* of our point u .

The interpretation of non-injectivity of our relation in terms of groups, is also seen when we consider the transitivity condition for an equivalence relation R . In the case $R \hookrightarrow U \times U$, transitivity may be viewed as a composition operation

$$R \times_U R \rightarrow R$$

relative to $U \times U$, using the first map of the first factor R and the second map of the second factor; the remaining two maps are absorbed in the fiber product expression.

In order to generalize to the case of a not-necessarily-injective map $R \rightarrow U \times U$, it is most natural to keep the composition map as a part of the data, then to impose associativity (which was automatic in the injective case). Existence of inverses corresponds to the symmetry property of a relation, and existence of identities corresponds to reflexivity. Altogether, U , R and these maps now have a structure of *groupoid*. That is to say, they form an internal category with the invertibility property of a groupoid, in whatever category of spaces we are looking at.

Sometimes, a stack X is given to us most naturally as a functor from the base category¹ to the category of groupoids, or more generally as a fibered category.² This is what happens for most moduli problems. In this case, we then choose in some way charts covering the functor. The union of charts gives us U , and we may put $R := U \times_X U$. If R is representable, and if the map $U \rightarrow \mathcal{F}$ has the right smoothness properties, then we get a groupoid $R \rightrightarrows U$, and the stack X is the “quotient” of this groupoid in the same way as discussed above.

What we have just described is the concrete way of looking at a stack: it is viewed as a quotient of a groupoid. The more abstract approach will be described in Section 5.3 below.

¹For us in algebraic geometry the objects of the base category are usually some kind of schemes and the category is provided with a Grothendieck topology making it into a site.

²See Section 5 for further discussion of the definitional aspects.

3. STACKS IN ALGEBRAIC GEOMETRY

Consider a very simple example of a stack in algebraic geometry. Suppose we are given the action of a finite group, say $G = \mathbb{Z}/2$, on a smooth curve X . Suppose $p \in X$ is an isolated fixed point. The quotient curve $Y := X/G$ will also be smooth, but it has a point $y \in Y$ the image of p , which is special in some way because the map $X \rightarrow Y$ is ramified over y . The inverse image of y consists only of the point p , whereas the inverse image of a nearby point y' would contain two points x_1 and x_2 exchanged by the group action. We would then like to include the group-theoretical information of the stabilizer group $\mathbb{Z}/2$ of the inverse image point p , located at the point $y \in Y$. This information is included in the *quotient stack* $\mathcal{Y} := X//G$. We view the stabilizer group as corresponding to the homotopy type of a space $K(\mathbb{Z}/2, 1)$ located over $y \in Y$.

The groupoid presentation of this quotient stack is easy to describe, it is an example of the general notion of *action groupoid*. Let $R := X \times G$ with two maps $(x, g) \mapsto x$ and $(x, g) \mapsto gx$ to X . The groupoid structure map $R \times_X R \rightarrow R$ is given by $((x, g), (x', g'))_{gx=x'} \mapsto (x, g'g)$, and the resulting groupoid represents our quotient stack \mathcal{Y} .

This kind of example is known as an orbifold in differential or analytic geometry, and as a *Deligne-Mumford stack* in algebraic geometry.

3.1. The moduli stack of curves of genus g . Deligne and Mumford introduced the theory in their paper [13], motivated by what is undoubtedly the first main example: their moduli stack \mathcal{M}_g of smooth projective curves of genus $g \geq 2$, and its completion $\overline{\mathcal{M}}_g$ parametrizing stable nodal curves. In order to put this example in its natural setting, Deligne and Mumford looked at stacks in the etale topology, with coordinate charts such that the transformations between different charts are etale maps.

Over the complex numbers, a smooth projective curve is the same thing as a compact Riemann surface. These quite naturally come in “families”: a family of curves is a flat map $X \rightarrow S$ whose fibers are curves. The *moduli problem* consists in finding a scheme M_g such that families of curves $X \rightarrow S$ correspond to morphisms $S \rightarrow M_g$. The optimal way of solving such a moduli problem would be to have a *universal family* $X_g \rightarrow M_g$ such that for any X/S corresponding to a parametrization map $S \rightarrow M_g$, we have $X = S \times_{M_g} X_g$.

While there does exist a moduli space M_g in the category of schemes, it doesn’t admit a universal family. This is mainly due to the existence of curves with automorphisms, and that is one of the main motivations to introduce the *moduli stack* \mathcal{M}_g . Working in the world of stacks, there is a universal family $\mathcal{X}_g \rightarrow \mathcal{M}_g$ solving the moduli problem in the best possible way.

In order to get back to a variety or scheme, we use the following definition.

Definition 3.1. Suppose X is a stack. We say that a morphism

$$X \rightarrow X^{\text{coarse}}$$

from X to a scheme X^{coarse} is the coarse moduli space if, for any other map to a scheme $X \rightarrow V$ there is a unique factorization $X \rightarrow X^{\text{coarse}} \rightarrow V$.

Deligne and Mumford construct both the moduli stack of curves and its coarse moduli space. While the coarse moduli space $M_g = (\mathcal{M}_g)^{\text{coarse}}$ is a scheme, it doesn’t admit a universal family.

Another indication of the advantage of using \mathcal{M}_g is the question of smoothness. The moduli stack, and even its natural compactification $\overline{\mathcal{M}}_g$ are *smooth* Deligne-Mumford stacks of dimension $3g - 3$. However, at points corresponding to curves with automorphisms, the moduli scheme M_g is quite singular in general: it has finite quotient singularities, the local charts being indeed the quotients of the smooth charts in \mathcal{M}_g by the finite automorphism groups.

The complex analytic stack associated to \mathcal{M}_g is a smooth orbifold (except for $g = 2$ where it also involves a gerbe, see below). The local charts are open balls in \mathbb{C}^{3g-3} , as we shall understand further in the upcoming example. In the analytic or differential geometric setting, this moduli stack has a global quotient structure

$$(\mathcal{M}_g)^{\text{an}} = \mathcal{T}_g / \Gamma$$

where \mathcal{T}_g is the famous *Teichmüller space* [32] and Γ is the mapping class group of oriented automorphisms of the topological surface up to isotopy.

3.2. Natural charts at fully degenerate curves. The principle of natural local charts that are smooth, is perfectly illustrated by looking at the degenerate boundary points in the compactification $\overline{\mathcal{M}}_g$. These boundary points are the stable curves which are unions of smooth curves meeting at nodes. Typical examples would be the “stick figures” in \mathbb{P}^3 composed of rational lines, if we assume that no more than two lines intersect at any given point.

Deligne and Mumford said that a curve is *stable* if any rational component has at least three nodes. A curve that isn’t rational can be deformed, and given a rational component with four or more nodes, it can also be deformed or “broken” into a union of rational curves with less nodes. Thus, the most degenerate points in the boundary, corresponding to 0-dimensional strata, are the ones where each component curve is rational, i.e. \mathbb{P}^1 , with exactly three nodes.

Recall that the universal deformation space of a nodal curve singularity is one-dimensional. The universal family may be written down very easily as just $xy = t$, with t being the deformation parameter. At $t = 0$ we have a nodal curve $xy = 0$ whereas for $t \neq 0$ the curve is smooth. We could cut out this deformation along a sphere centered at the origin, and then glue it into a global picture.

Suppose $Y = \bigcup_{i=1}^k Y_i$ is a union of smooth rational curves meeting at nodes z_1, \dots, z_r . Let Y^- be the curve obtained by cutting out small disks around each of the nodes, and glue the universal deformation spaces considered in the previous paragraph at each of the nodes. We get a family of curves parametrized by a smooth parameter space with r parameters t_1, \dots, t_r , say for example parametrized by Δ^r where Δ is the unit disk.

The genus of Y is given by

$$g(Y) = 1 + \# \text{ nodes} - \# \text{ components} = 1 + r - k.$$

In the fully degenerate case where each component has three nodes, a node is contained in two components so $2r = 3k$. Thus $g(Y) = 1 + r - \frac{2}{3}r = 1 + \frac{1}{3}r$, in other words

$$r = 3g(Y) - 3.$$

This tells us that our deformation space obtained by combining together the universal deformations of each of the singularities, has the same dimension as the moduli space M_g of genus g curves, and indeed one may calculate that the tangent map

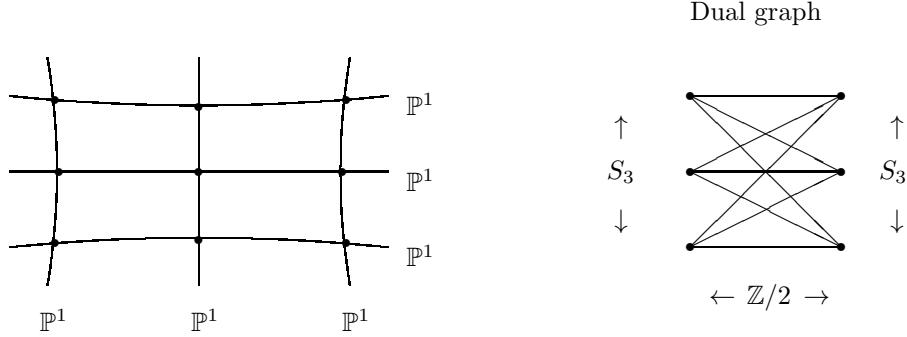


FIGURE 1. A curve Y of genus $g = 4$ with $9 = 3g - 3$ nodes and automorphism group $G = \text{Aut}(Y) = (S_3 \times S_3) \rtimes \mathbb{Z}/2$

from \mathbb{C}^r to the deformation space $\text{Def}(Y)$ is an isomorphism. Therefore, we would like to think of our family as providing a chart for the moduli space.

It is a particularly nice chart. The boundary divisor of singular curves appears transparently as a divisor with normal crossings, being given by the equation $t_1 t_2 \cdots t_{r-1} t_r = 0$.

In general, our nodal curve Y will have automorphisms. One can form the *dual graph* \mathcal{D} with k vertices one for each component Y_i , and r edges joining two components when they intersect in a node. In our fully degenerate case, $\text{Aut}(Y) = \text{Aut}(\mathcal{D})$, and it is easy to write down graphs with automorphisms (Figure 1).

We now have a completely concrete situation showing the benefits of the notion of stack. In the moduli stack $\overline{\mathcal{M}}_g$, the family defined above is a coordinate chart, showing how $\overline{\mathcal{M}}_g$ has a structure of “smooth space” at the point Y . The automorphisms of Y induce automorphisms of the coordinate chart since they permute the nodes, but these maps are etale.

The only kind of variety we can get, the coarse moduli space \overline{M}_g will have a quotient singularity at the point $[Y]$ (isomorphism class of Y), given by taking the quotient of Δ^r by the automorphism group of Y (see Figure 2).

It is clear that we would rather like to look at the smooth neighborhoods of Y with highly natural coordinate systems discussed above. On the smooth charts, the coordinate functions, defining the boundary divisor components, correspond to the nodes. The possibility of looking at such charts is afforded by the notion of Deligne-Mumford stack.

This example was for us primordial. Explained by the first author to the second, it showed in a concrete way what the notion of stack was all about.

3.3. Orbifolds and DM-stacks. Such considerations date back quite a long ways. The classical Kuranishi theory provided versal deformation spaces, and the problem of automorphisms led quite early on to the definition of *V-manifold* by Satake. A *V*-manifold is essentially the same thing as a smooth Deligne-Mumford stack, but the generic stabilizer group is assumed trivial. In topology, the corresponding notion was called an *orbifold*.

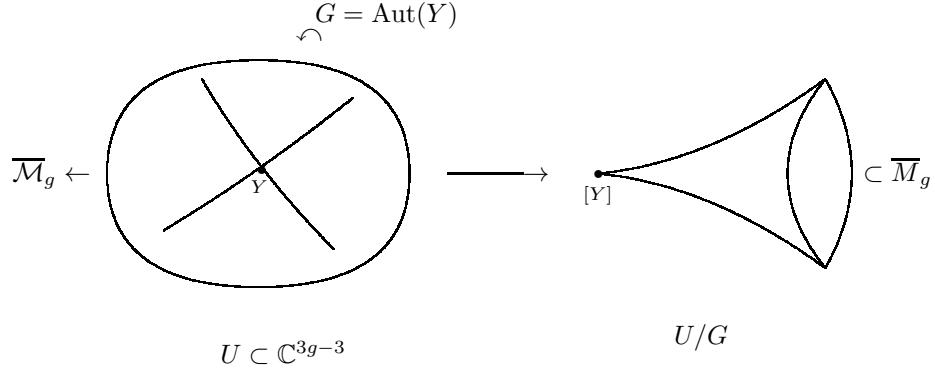


FIGURE 2. The chart over $\overline{\mathcal{M}}_g$ and its image in $\overline{\mathcal{M}}_g$

The definition of Deligne and Mumford provides us with a few useful generalizations going beyond the notion of orbifold. For example, the stack doesn't need to be smooth or even reduced. This reflects the fact that we might very well, for some reason, have wanted to consider a compatible collection of singular subschemes of the coordinate charts.

Singular DM-stacks can fit as points in moduli spaces, giving new kinds of compactifications. Abramovich and Vistoli [1] introduced the notion of *twisted curve*, a nodal DM-curve with orbifold points at the normal crossings where the cyclic stabilizer groups of the orbifold points on the two branches are identified. Several authors [2] [12] [30] have constructed moduli stacks of twisted curves giving interesting compactifications of \mathcal{M}_g .

The other direction of generalization is quite interesting and leads to a more intricate topological structure. The local automorphism group in the stack, coming from the stabilizer group of a group action or the automorphism group of the objects whose moduli we are looking at, is not necessarily required to be trivial at the generic point. In other words, we can have “stacky” behavior over the whole space rather than just concentrated at some points lower down in a stratification. This is reflected in the notion of *gerbe*.

4. GERBES

The generalization to gerbes is quite natural and would in due course have necessarily been included in any development of the original notion of V -manifold. Indeed, consider some subvariety of a V -manifold over which the automorphism groups are nontrivial. Then this subspace has an induced stack structure, but the generic stabilizer is nontrivial.

When there is a nontrivial generic stabilizer group, but that group stays “the same” over all points of the stack, then we have a *gerbe*. A gerbe over a point is just a stack of the form BG where G is a group. If G is a finite group (or a group scheme etale over the base field in positive characteristic) then BG is a Deligne-Mumford stack. More general cases include for example G an algebraic group scheme, in which case BG is an Artin stack.

Let us look at the easiest case when G is a finite group. Suppose X is a variety. Then a gerbe over X with group G is a stack \mathcal{Y} together with a map $p : \mathcal{Y} \rightarrow X$, such that X has an etale covering $\{a_i : U_i \rightarrow X\}$ and $U_i \times_X \mathcal{Y} \cong U_i \times BG$. There will not exist in general a global trivialization, nor even a trivialization over a Zariski open covering. And there need not even exist local sections. The G -gerbes over X are classified by the 2-stack $B \text{Aut}(BG)$. This is a connected 2-stack whose π_1 is $\text{Out}(G) = \text{Aut}(G)/G$ the group of outer automorphisms, and whose π_2 is the center $Z(G)$. Thus, a G -gerbe over X corresponds first to a cohomology class $\alpha \in H^1(X, \text{Out}(G))$ which is to say an $\text{Out}(G)$ -torsor over X , and secondly if this is trivial a class in $H^2(X, Z(G))$.

4.1. Stratification by automorphism group in $\overline{\mathcal{M}}_g$. Such structures may be found within the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus g . We feel that this phenomenon is an interesting question for study. It furthermore illustrates how gerbes occur “in nature”.

A first example is, of course, the whole moduli stack $\overline{\mathcal{M}}_2$ of curves of genus 2: any genus 2 curve is hyperelliptic so it has an involution, and the generic stabilizer group on $\overline{\mathcal{M}}_2$ is $\mathbb{Z}/2\mathbb{Z}$.

More generally, there is a decomposition into locally closed subsets

$$\overline{\mathcal{M}}_g = \coprod_G S_G$$

where S_G is the locus of curves whose automorphism group is isomorphic to G . Now S_G is itself a stack, and it is a G -gerbe over its coarse moduli space S_G , where we have the corresponding decomposition $\overline{\mathcal{M}}_G = \coprod_G S_G$ of the coarse moduli space.

It is an interesting question to try to understand the structure of these gerbes. A very combinatorial version is obtained by restricting to the dimension 1 pieces of the boundary stratification, corresponding to nodal unions of rational curves such that one component has four nodes and the rest have only three nodes.

These questions have been studied by many authors. Some references include, non-exhaustively: [14] [15] [16] [27] [44] [45].

4.2. Classification of gerbes. The theory of stacks is a bridge from geometry and algebra to topology. A basic building block is the gerbe BG over a point. It is the stack classifying G -torsors, but we may also think of it as the unique (up to weak homotopy equivalence) connected pointed topological space with $\pi_1(BG, o) = G$. The correspondence between groupoids and 1-truncated homotopy types is the link between algebra and topology in this simplest case.

Theorem 4.1 (Giraud [17]). *Suppose G is a sheaf of groups over a site. There is a 2-stack $B \text{Aut}(BG)$ over the site, and a universal G -gerbe over it, such that for any object (or indeed stack) X , the 2-groupoid of G -gerbes over X is naturally equivalent to the 2-groupoid of maps $X \rightarrow B \text{Aut}(BG)$. If G is a constant group then and this 2-stack has a fibration sequence*

$$K(Z(G), 2) \rightarrow B \text{Aut}(BG) \rightarrow B \text{Out}(G).$$

The obstruction classes in cohomology of X discussed previously are direct consequences.

Giraud gave the proof [17] in purely categorical terms. One can phrase the statement and give its proof in terms of cocycles and 2-cocycles. Breen discusses the generalization to 2-gerbes in this light [10].

In order to give a rather more simple view of the idea, we discuss the topological version of the statement, classifying fibrations with fiber BG , in subsection 4.4 below. We consider the case of a constant discrete group G . The case of a sheaf of groups is more general, and indeed one should note that Giraud's theory treats the most general situation of gerbes under a *lien*, which basically means a sheaf of “groups up to inner automorphism” [17] [43]. This results in the complicated algebraic structures discussed in [17], that will not really be covered by our topological discussion.

4.3. Structure theorem for DM-curves. The classification of gerbes works equally well if the base X itself is a stack. This is useful since we have the following structure theorem for 1-dimensional Deligne-Mumford stacks:

Theorem 4.2. *Suppose \mathcal{X} is a smooth 1-dimensional Deligne-Mumford stack. Then there is a smooth 1-dimensional orbifold X^{orb} , that is to say a DM-stack with trivial generic stabilizer or equivalently a V -manifold, and a map $p : \mathcal{X} \rightarrow X^{\text{orb}}$ such that p is a G -gerbe for G the generic stabilizer group of \mathcal{X} . Furthermore, X^{orb} is a root stack*

$$X^{\text{orb}} = X\left[\frac{1}{n_1}D_1, \dots, \frac{1}{n_k}D_k\right]$$

over the coarse moduli space X a smooth curve, for points $D_1, \dots, D_k \in X$ with integer multiplicities $n_i \geq 1$.

The local structures are determined by the integer multiplicities, so the new topological information is contained in the gerbe p .

4.4. Classification of fibrations. In order to gain some insight into what is going on in Theorem 4.1, let us consider a fixed discrete group G and look at how to classify fibrations $Y \rightarrow X$ such that the fiber is isomorphic to the space BG . Our discussion takes place in the world of topology.

Theorem 4.3. *Suppose F is a space. There is a space $B \text{Aut}(F)$ classifying fibrations with fiber F . It has a universal fibration, and given a fibration Y/X with fiber F , the space of homotopy classes of pairs (f, ζ) where $f : X \rightarrow B \text{Aut}(F)$ and ζ is a homotopy equivalence between Y/X and the pullback of the universal fibration, is contractible. In the case when G is a discrete group and $F = BG$ then $B \text{Aut}(BG)$ is a connected, 2-truncated space with*

$$\pi_1(B \text{Aut}(BG)) = \text{Out}(G) \text{ and } \pi_2(B \text{Aut}(BG)) = Z(G).$$

This theorem comes from Segal's theory of classifying spaces [37]. The “group” of self-homotopy equivalences $\text{Aut}(F)$ may be viewed as a grouplike Segal space in the following way. Define a simplicial set T with $T_0 = *$, and $T_1 \subset \text{Map}(F, F)$ equal to the union of connected components corresponding to maps which are weak homotopy equivalences. Let $T_n := T_1 \times \cdots \times T_1$ with the various face and degeneracy maps defined by the monoid structure of T_1 . This tautologically satisfies the Segal conditions. Since the maps in T_1 are invertible up to homotopy, this simplicial space also satisfies Segal's grouplike condition. We may therefore look at the realization

$$B \text{Aut}(F) := |T|,$$

and Segal shows that it is a space such that the tautological map

$$\text{Aut}(F) = T_1 \rightarrow \Omega_*|T|$$

is a homotopy equivalence. One constructs the universal family and shows that $B \text{Aut}(F)$ is the classifying space.³

In the case $F = BG$,

$$\begin{aligned}\pi_1(B \text{Aut}(BG)) &= \pi_0(\text{Aut}(BG)) = \text{Out}(G), \\ \pi_2(B \text{Aut}(BG)) &= \pi_1(\text{Aut}(BG)) = Z(G).\end{aligned}$$

The fibrations Y/X with fibers BG are essentially the same thing as G -gerbes. The 2-stack in Theorem 4.1, for the case of a constant group G , is just the constant 2-stack with value equal to the Poincaré 2-stack of the space $B \text{Aut}(F)$ of the previous theorem.

In this topological framework we can easily understand how to get obstruction classes for gerbes. Suppose given a fibration Y/X with fiber BG . The classifying map $X \rightarrow B \text{Aut}(BG)$ composes with the map from $B \text{Aut}(BG)$ to $B \text{Out}(G)$, to give a class in $H^1(X, \text{Out}(G))$ or equivalently an $\text{Out}(G)$ -torsor over X . This torsor measures the way in which the fundamental groups of the fibers of Y/X change as we move around loops in the base. Since Y/X doesn't necessarily have a section, the fundamental groups of the fibers are not well-defined, but one can choose sections locally and a change of section corresponds to an inner automorphism of G . The monodromy of the system of fundamental groups is therefore an element of the group of outer automorphisms of G .

Suppose this torsor is trivial. It means that the map $X \rightarrow B \text{Aut}(BG)$ may be viewed as going into the fiber of the projection to $B \text{Out}(G)$ (more precisely, a choice of trivialization of the torsor corresponds to a choice of homotopy to a map into the fiber). That fiber is the Eilenberg-MacLane space $B^2 S(G) = K(Z(G), 2)$ of degree 2. Thus, we get a cohomology class in $H^2(X, Z(G))$. This class is the obstruction to existence of a section. If this class is trivial too, then there exists a section and our fibration is trivial, $Y = X \times BG$.

More generally, given an $\text{Out}(G)$ torsor α we get a local system $Z(G)_\alpha$; and a gerbe with that $\text{Out}(G)$ torsor gives a classifying element of $H^2(X, Z(G)_\alpha)$.

This very brief tour of obstruction theory and classification of fibrations is designed to correspond to Giraud's classification of gerbes within the theory of stacks [17]. Some kind of work needs to be done in order to obtain the analogous classification theory for stacks over a site, for example to treat the case when G is sheaf of groups or even a lie.

A modern treatment of the proof of Theorem 4.1 might appeal to the theory of simplicial presheaves in order to transpose somewhat more directly the topological classification theory presented here to the relative situation. We are not sure if a full treatment in this spirit has yet been proposed.

5. THE ABSTRACT NOTION OF HIGHER STACK

The discussion of the previous section concerned the classification of fibrations over a base topological space. In algebraic geometry, gerbes and more generally

³Suppose we have a fibration with fiber F over the realization of a simplicial set $X = |X_\cdot|$. For $x \in X_0$ choose a weak equivalence $Y_x \cong BG$. For any $u_1 \in X_1$ we have a fibration over Δ^1 , and choice of a trivialization compared with the previous choices at the endpoints gives a self homotopy equivalence of BG , hence a point in T_1 . For $u_2 \in X_2$, its faces are $u_2(01), u_2(12), u_2(02) \in T_1$. The 2-cell u_2 may be seen as a homotopy in T_1 between $u_2(12) \circ u_2(01)$ and $u_2(02)$. Continuing in this way, we can build a homotopy coherent map from X to the simplicial space T_\cdot , yielding the classifying map $X \rightarrow |T_\cdot|$ for the fibration.

stacks exist over base algebraic varieties. For example we have considered the natural gerbes over strata $S_G \subset \overline{M}_g$ in the moduli space of stable curves. In such a situation, the notion of fibration of topological spaces is replaced by the notion of stack itself, and stacks are viewed as families of spaces over algebraic varieties.

Such a dialogue between algebraic geometry and algebraic topology has proven fundamental to recent developments including the gradual move towards the idea of ∞ -stacks:

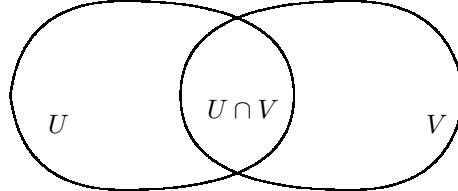
sets	groupoids	n -groupoids	∞ -groupoids
discrete spaces	$K(\pi, 1)$'s	n -truncated homotopy types	all homotopy types
presheaves	prestacks	n -prestacks	∞ -prestacks
sheaves	stacks	n -stacks	∞ -stacks

5.1. Glueing and descent data. The goal of this section 5 is to discuss the more abstract definitional aspects of the notion of stack, those having been postponed in the earlier more geometric sections.

We start by explaining with some pictures the basic idea for going from the presheaf/prestack line to the sheaf/stack line in the above table. Recall that the property characterizing *sheaves* among all presheaves, is that they are required to satisfy a glueing condition. In this subsection, we'll assume that we are talking about objects (sheaves, stacks, ...) over a base topological space X .

A *presheaf* \mathcal{F} over X is just a functor from the opposite of the category of opens of X , to sets. Thus, it consists of a collection of sets $\mathcal{F}(U)$ for any open set $U \subset X$, together with restriction functors $a \mapsto a|_{U'}$ whenever $U' \subset U$, satisfying the natural transitivity condition.

Suppose given two intersecting open subsets:



For a presheaf \mathcal{F} , the set of locally defined sections, or *descent data* with respect to the covering of $U \cup V$ by these two open sets, is the set of pairs (a, b) where $a \in \mathcal{F}(U)$ and $b \in \mathcal{F}(V)$ such that $a|_{U \cap V} = b|_{U \cap V}$ in $\mathcal{F}(U \cap V)$.

A presheaf \mathcal{F} is a *sheaf*, if $\mathcal{F}(U \cup V)$ is always isomorphic to the set of these locally defined sections. This is viewed as a *glueing condition*: given sections a and b over U and V respectively, such that they “agree” on $U \cap V$, then they should “glue” i.e. come from a (unique) section over $U \cup V$. In other words, the diagram

$$(5.1) \quad \begin{array}{ccc} \mathcal{F}(U \cup V) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V) \end{array}$$

is cartesian, the upper left corner being the limit of the lower right angle diagram.

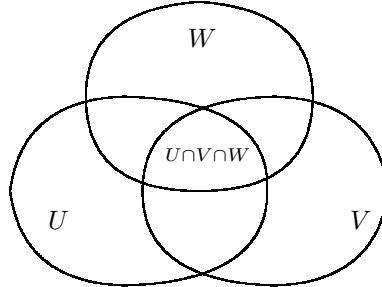
Going rightward in the above table, we would like to generalize from presheaves that are families of sets $\mathcal{F}(U)$ indexed by opens $U \subset X$, to 1-, n - or ∞ -prestacks that are families of spaces. For now let's just think of one of these as being a functor from the opposite category of opens of X , to spaces. The case of 1-stacks is, as indicated in the second column of the table, the case where these spaces are $K(\pi, 1)$'s.

In case of two open sets intersecting as pictured above, we would like to define a space of locally defined sections in a way that is adapted to homotopy theory. So, instead of requiring an equality of points $a|_{U \cap V} = b|_{U \cap V}$ in the space $\mathcal{F}(U \cap V)$, it is better to ask for a path. Since the choice of path is not unique, it should be considered as part of the data. Thus, we arrive at the space of locally defined sections as being the space of triples (a, b, p) where $a \in \mathcal{F}(U)$ and $b \in \mathcal{F}(V)$ are points, and $p : [0, 1] \rightarrow \mathcal{F}(U \cap V)$ is a path joining $a|_{U \cap V} = p(0)$ to $b|_{U \cap V} = p(1)$.

The condition for being a *stack* (relative to two-subset coverings, at least) is that the map from $\mathcal{F}(U \cup V)$ to the above space of locally defined sections, should be a weak homotopy equivalence. This says that we can glue together sections a over U and b over V , if a and b agree *up to homotopy* over $U \cap V$, and the glueing is uniquely defined up to homotopy once the path p has been specified. In other words, the diagram (5.1) should be *homotopy cartesian*, the upper left being the homotopy limit of the lower right.

In the case of finite coverings of a space they can always be considered as gotten from a sequence of two-subset coverings as in the picture, and the above definition of the descent condition is sufficient.

Let us look, nonetheless, at what the natural notion of locally defined section should be for a more complicated covering, say just one with three open subsets:



Our descent data will start with:

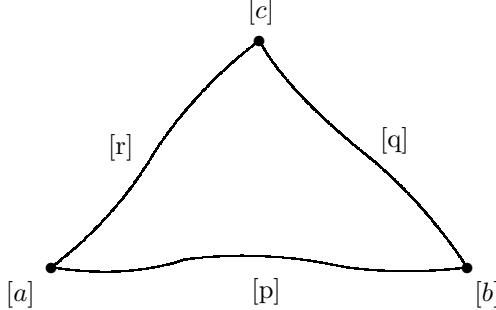
$$a \in \mathcal{F}(U), \quad b \in \mathcal{F}(V), \quad c \in \mathcal{F}(W),$$

then it should also have:

- a path p between $a|_{U \cap V}$ and $b|_{U \cap V}$ in $\mathcal{F}(U \cap V)$,
- a path q between $b|_{V \cap W}$ and $c|_{V \cap W}$ in $\mathcal{F}(V \cap W)$, and
- a path r between $a|_{U \cap W}$ and $c|_{U \cap W}$ in $\mathcal{F}(U \cap W)$.

So far, if we project all of those things into the space $\mathcal{F}(U \cap V \cap W)$ by the appropriate restriction maps, denoting their images by $[a] := a|_{U \cap V \cap W}$ etc., we get

the following picture:



This suggests what the last piece of the descent data should be: a triangle mapping into $\mathcal{F}(U \cap V \cap W)$ whose boundary is $[p] \cup [q] \cup [r]$.

For 1-prestacks where the spaces are $K(\pi, 1)$'s, existence of the triangle is a condition often called the *cocycle condition*. For the case of higher prestacks, this homotopy is itself part of the data.

More abstractly, the space of descent data as above may be expressed as a *homotopy limit (holim)*, namely the homotopy limit of the diagram formed by the spaces $\mathcal{F}(U)$, $\mathcal{F}(V)$, $\mathcal{F}(W)$, $\mathcal{F}(U \cap V)$, $\mathcal{F}(V \cap W)$, $\mathcal{F}(U \cap W)$ and $\mathcal{F}(U \cap V \cap W)$ with the restriction maps between them. With more open sets, the reader may imagine the collection of higher-dimensional homotopy data called *higher coherencies* entering into the homotopy limit space of descent data.

The *descent condition*⁴ for a prestack to be a stack, is that for any covering the map from the space assigned to the union of open sets, to the homotopy limit of the corresponding diagram, should be a weak homotopy equivalence.

5.2. Topology relative to a site. If the pictures in the previous subsection were drawn with a topological space as base, it is most often necessary to consider stacks over a more general kind of abstract structure designed precisely to allow the kinds of arguments that proceed from the intuition of glueing.

So, the first ingredient in the whole theory is the notion of *site*, a category \mathcal{C} provided with a *Grothendieck topology* τ . It means that for any object $X \in \mathcal{C}$ we know when a family $U_i \rightarrow X$ is said to “cover” X . This information is supposed to obey appropriate axioms. When \mathcal{C} admits disjoint unions it suffices to consider one-object coverings $U \rightarrow X$ obtained from the previous by setting $U := \coprod U_i$, and we shall generally assume this notational simplification. The topology allows us to say that something is happening “locally on X ”, meaning that it happens on some covering, in a coherent way.

A topological space relative to a site (\mathcal{C}, τ) should be viewed as a “presheaf of spaces” over the category \mathcal{C} . Such presheaves could come in many flavors. For instance, in many treatments of stack theory (the reader is referred to [41] in particular) the fundamental objects are fibered categories, whose fibers are groupoids. A fibered category is a kind of weak presheaf of groupoids: the assignment of a

⁴The reader is encouraged to consult [40] for some more technical and philosophical discussion about descent.

functor between fibers to every map in the base, is not necessarily strictly compatible with composition but only compatible up to a natural equivalence satisfying further axioms. Nonetheless, a standard strictification process allows one to replace such weak presheaves by strict presheaves, so another valid viewpoint is to start from the beginning with strict presheaves of spaces over the category underlying our Grothendieck site.

The Grothendieck topology is taken into account via the condition of *descent*. One generalizes over a site the discussion we had in the previous subsection, where the base was the category of open subsets of a topological space.

For a presheaf of discrete or 0-truncated spaces, corresponding to a presheaf of sets \mathcal{F} , the descent condition is the condition of being a *sheaf*: for a covering $U \rightarrow X$ the diagram

$$\mathcal{F}(X) \rightarrow \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U)$$

should be an equalizer diagram of sets. In other words, an element of $\mathcal{F}(X)$ is the same thing as $a \in \mathcal{F}(U)$ such that $p_1^*(a) = p_2^*(a)$ in $\mathcal{F}(U \times_X U)$. This latter condition is what we meant when we said “in a coherent way” three paragraphs ago.

For a 1-truncated presheaf of spaces, corresponding to a presheaf of groupoids \mathcal{F} , the descent condition is that of a *stack*. It means that the diagram

$$\mathcal{F}(X) \rightarrow \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U) \xrightarrow{3} \mathcal{F}(U \times_X U \times_X U)$$

should express the groupoid $\mathcal{F}(X)$ as being equivalent to the “homotopy limit”, or *groupoid of descent data* of the sequence on the right. An object of the homotopy limit consists of an object $a \in \mathcal{F}(U)$ and an isomorphism $\eta : p_1^*(a) \cong p_2^*(a)$ in $\mathcal{F}(U \times_X U)$, such that the natural diagram we can write in $\mathcal{F}(U \times_X U \times_X U)$ commutes (corresponding to the triangle in 5.1). Notice that now the “coherence” becomes a datum, that of the isomorphism η , which is itself required to satisfy a higher coherence condition.

The descent condition says that any such “descent datum” should be equivalent to an object coming from $\mathcal{F}(X)$. The morphisms in the groupoid of descent data are defined using the sheaf condition of the previous paragraph, and we also require that this set of morphisms be isomorphic to the set of morphisms in $\mathcal{F}(X)$.

The discussion of subsection 4.4 brings out an important phenomenon: if 1-stacks came up when we look for classifying spaces of regular objects having automorphisms, similarly when we try to classify 1-stacks we end up talking about a 2-stack such as $B\text{Aut}(BG)$. In general, the classifier for n -stacks is an $(n+1)$ -stack. In this sense we can’t avoid going upwards in the ladder of degrees, and it becomes natural to consider the notion of n -stack for any n . Of course in many practical situations it suffices to stop after going up a level or two: for instance, when speaking of 1-stacks, it is occasionally useful to talk about 2-stacks but we usually don’t need to look at 3-stacks in the same context.

Another force pushing upwards is the following question: what happens with a groupoid

$$\mathcal{R} \rightrightarrows U$$

where U is a scheme, but the morphism object \mathcal{R} is itself a 1-stack? In this case the quotient is a 2-stack, as we shall see in the explicit example of 6.6 below. Continuing leads to n -stacks for any n , in a way closely related to the Artin condition to be discussed in the next section.

The descent conditions generalize directly to the case of n -truncated⁵ presheaves, saying that the space of sections over an object is equal to the homotopy limit of the space of sections over the nerve of a covering. An n -stack in n -groupoids is an n -truncated presheaf of spaces satisfying this descent condition. When we go to $n = \infty$ with arbitrary, non-truncated spaces, it turns out that there are two natural flavors of descent: hyperdescent and finite descent. This distinction was pointed out by Lurie, among others, and he closely investigated the difference [26]. The case of hyperdescent corresponds to the locality condition in the Jardine closed model structure on simplicial presheaves [21].

The idea of doing topology relative to a site connects naturally with the notion of *nonabelian cohomology*. Giraud's classification of gerbes was one of the first instances. From the topological version of this discussion described above, we see that nonabelian cohomology really means looking at maps into some space. In algebraic geometry, this is happening relative to a site: spaces are replaced by presheaves of spaces required to satisfy a "descent condition". We'll get back to the idea of nonabelian cohomology in this setting later.

5.3. Definitions of stacks. We present, for comparison, some of the possible definitions following the discussion of the previous section. Suppose \mathcal{C} is a category provided with a Grothendieck topology τ . We restrict here to the case of stacks of groupoids. There are now many different definitions of n -category, which have been shown to be equivalent by the axiomatic approach of Barwick and Schommer-Pries [5], but for groupoids this can be understood easily in that an n -groupoid may be viewed as the same thing, up to homotopy, as an n -truncated space.

Definition 5.1. A 1-prestack over \mathcal{C} is a functor $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ with the property of being a fibered category, such that the fibers \mathcal{F}_x are 1-groupoids.

An n -prestack over \mathcal{C} is a functor of n -categories $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ with the property of being a fibered category, such that the fibers \mathcal{F}_x are n -groupoids.

Definition 5.2. A 1-prestack over \mathcal{C} is a functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{1Gpd}$, that is a presheaf of groupoids.

An n -prestack over \mathcal{C} is a functor $F : \mathcal{C}^{\text{op}} \rightarrow n\text{Gpd}$, that is a presheaf of n -groupoids.

Definition 5.3. A 1-prestack over \mathcal{C} is a presheaf of spaces $F : \mathcal{C}^{\text{op}} \rightarrow \text{Top}$, such that $F(x)$ is a 1-truncated space.

An n -prestack over \mathcal{C} is a presheaf of spaces $F : \mathcal{C}^{\text{op}} \rightarrow \text{Top}$, such that $F(x)$ is an n -truncated space.

Here, Top may profitably be replaced by the category of simplicial sets, so 1- or n -prestacks may be viewed as simplicial presheaves over \mathcal{C} .

Given a prestack $\mathcal{F} \rightarrow \mathcal{C}$ in the sense of 5.1, let $F(x)$ be the 1-groupoid of sections of $\mathcal{C}/x \rightarrow \mathcal{F}$. This has strict pullbacks corresponding to restriction of sections, so it is a presheaf of groupoids in the sense of 5.2. In the other direction, given a presheaf of groupoids F we can let $\mathcal{F} := \int_{\mathcal{C}} F$ be the *Grothendieck integral*, also known as the

⁵Recall that a space Y is said to be n -truncated if $\pi_i(Y)$ is trivial for $i > n$. Allowing $n = \infty$, an ∞ -groupoid is a space with no truncation condition. The theory of ∞ -categories is not yet well developed, but for (∞, n) -categories, those ∞ -categories in which morphisms are invertible in degrees $> n$, the various theories and their equivalence are fairly well understood. We leave it to the reader to investigate the numerous references on this aspect.

category of elements of F . The objects of \mathcal{F} are the pairs (x, u) where $x \in \text{Ob}(\mathcal{C})$ and $u \in F(x)$, and morphisms are defined in a natural way. These constructions are inverse up to equivalence. The same discussion relates 5.1 and 5.2 for n -prestacks.

To go between 5.2 and 5.3, recall that to a groupoid G we can associate its classifying space $|G|$ which is a 1-truncated space, described for example as the realization of the simplicial set nerve of G . Given a 1-truncated space, its Poincaré groupoid is the corresponding groupoid. Again, these constructions extend to the equivalence between n -groupoids and n -truncated spaces, if one is not actually taking n -truncated spaces as the definition of n -groupoids. Being functorial, these constructions extend to the case of presheaves.

We may now use interchangeably any of the above definitions for prestacks. Given a prestack \mathcal{F}/\mathcal{C} , we obtain its higher groupoid of sections $\Gamma(\mathcal{C}, \mathcal{F})$, which should be calculated in a suitable way to give an answer invariant under homotopy equivalences relative to \mathcal{C} . The best way to do that is to use Quillen model category structures, although for sections of a 1-prestack there is an algebraic definition predating model category theory. Taking the point of view that ∞ -groupoids correspond to homotopy types, the higher groupoid of sections is the homotopy limit:

$$\Gamma(\mathcal{C}, \mathcal{F}) = \text{holim}_{\mathcal{C}} \mathcal{F}.$$

A prestack is called a *stack* if it satisfies the *descent condition*. There are many different viewpoints on this notion, and it is not our purpose to give an extensive discussion here. The idea is to encode the property we first saw in subsection 5.1. One way to say it is as follows:

Definition 5.4. Suppose $x \in \mathcal{C}$, and suppose given a covering of x for the topology τ . The covering determines a sieve which is a subcategory $\mathcal{B} \subset \mathcal{C}/x$. The descent condition says that the map

$$\mathcal{F}(x) \sim \Gamma(\mathcal{C}/x, \mathcal{F}|_{\mathcal{C}/x}) \rightarrow \Gamma(\mathcal{B}, \mathcal{F}|_{\mathcal{B}})$$

should be an equivalence (of 1-groupoids or n -groupoids or spaces), for any x and any covering sieve \mathcal{B} . A prestack satisfying this condition is called a stack.

The homotopy theory of n -stacks is obtained by using the homotopy theory of n -prestacks i.e. diagrams of spaces, restricted to the subcategory of n -stacks. Jardine first constructed this homotopy theory by constructing a closed model category of simplicial presheaves, where the weak equivalences are “local” weak equivalences. An n -stack⁶ is any n -truncated simplicial presheaf which is levelwise equivalent to a fibrant object in Jardine’s model structure.

5.4. Homotopy group sheaves. A local weak equivalence is a map that induces weak equivalences on the “stalks” (defined, over sites with enough points, using a homotopy colimit in the same way as for sheaves). It can also be measured using the *homotopy group sheaves*. If \mathcal{F} is a prestack, then the presheaf $x \mapsto \pi_0(\mathcal{F}(x))$ has associated sheaf denoted $\pi_0(\mathcal{F})$. Similarly if $u \in \Gamma(\mathcal{C}, \mathcal{F})$ is a section then

⁶If $n = \infty$ then there is a distinction [26] between descent for coverings, and descent for hypercoverings. We presented here descent for a covering. The descent condition for a hypercovering can be expressed in the same way, but the sieve is replaced by the category of elements of the hypercovering which has a functor $\mathcal{B} \rightarrow \mathcal{C}/x$.

Without a truncation condition, in other words for $n = \infty$, the fibrant objects in Jardine’s model structure are the prestacks which satisfy hyperdescent; Lurie constructs a model category where the fibrant objects are the ones satisfying just descent for coverings [26].

$\pi_i(\mathcal{F}, u)$ is the sheafification of the presheaf $x \mapsto \pi_i(\mathcal{F}(x), u(x))$. These notions are often most useful when restricted to slice categories \mathcal{C}/x . A map is a local weak equivalence if and only if it is an *Illusie weak equivalence*: it induces isomorphisms of homotopy group sheaves. The sheafification process used to define the homotopy group sheaves is the key aspect of this condition. In the same way as for sheaves, the stack associated to a prestack is the canonical model of its local weak equivalence class, the universal element that the other ones map to.

6. ARTIN STACKS

Once we have the theoretical framework to talk about stacks—and those could now mean ordinary 1-stacks, n -stacks, or similar generalizations—we can return to the original geometrical motivation. We wanted to represent objects presented as “quotients” of some kind of chart, by a generalized equivalence relation. The quotient is viewed as an object in our category of stacks.⁷ The question can therefore be rephrased, what stacks \mathcal{F} should be considered to admit a good presentation of the form

$$R \rightrightarrows U \rightarrow \mathcal{F}$$

as discussed in Section 2?

For 1-stacks this condition was originally formulated by Artin [4]. He called such stacks *algebraic stacks*, but nowadays the terminology *Artin stacks* may be preferred, or—particularly in the higher context—*geometric stacks*.

As we have seen before, the dynamical relationship between the idea of a stack as a quotient of an equivalence relation, and the idea of a stack as a functor with values in groupoids, n -groupoids or ∞ -groupoids (spaces or simplicial sets) can go in both directions. We might start with a natural chart and hence with a groupoid, or we might start with a functor and proceed later to look for a chart. In either case, we do eventually reach the situation of having a chart, and should therefore ask what good conditions such a chart might satisfy in order to make it useful for studying the properties of the stack.

The basic idea is that the map $U \rightarrow X$ from the chart to the stack should be *smooth*. That is, as shall be seen from the example of BG below, the best possible condition we can hope for in general, allowing to study the infinitesimal structure of X . Artin’s observation is that smoothness of $U \rightarrow X$ is reflected in smoothness of the two maps $R \rightarrow U$.

By looking at Artin’s condition in the right way, it generalizes immediately to the case of n -stacks. This was pointed out to us by Charles Walter, during discussions we were having with André Hirschowitz. It led to the preprint [38]. The discussion there is straightforward and self-contained; we will just sketch the basic outline, after the motivating example of BG in the next subsection.

⁷To define the quotient technically, suppose we are given a groupoid $R \rightrightarrows U$ in schemes. It represents a functor $\mathcal{C}^{\text{op}} \rightarrow \text{1Gpd}$. From this 1-prestack we take the associated stack. In terms of simplicial presheaves, take the Jardine fibrant replacement, but that can also be done by composing three times the functor of taking the groupoid of descent data. The result is a 1-stack, the universal 1-stack to which our prestack maps. It may also be viewed as the homotopy colimit of the groupoid (considered as a simplicial scheme) in the 2-category of stacks. Taking the associated stack makes it so that the resulting object stays the same even if we replace U by a different chart, say a smooth cover $U' \rightarrow U$.

6.1. A chart for BG . Consider first one of the basic cases, the stack BG for G a group. If G is finite, BG is a Deligne-Mumford stack. It is natural to want to expand the collection of groups we can consider here, while staying within the algebro-geometric context. So let's look at BG for an algebraic group scheme G . To fix ideas for simplicity say we work over a field k of characteristic zero. The most canonical map from a scheme to BG is just the basepoint

$$* \rightarrow BG.$$

This is a map whose fiber is G , indeed suppose $f : X \rightarrow BG$ is any map. Then f corresponds to a G -torsor or principal G -bundle $F \rightarrow X$. The fiber product $X \times_{BG} *$ is the space representing trivializations of this principal bundle, so in fact

$$X \times_{BG} * = F$$

itself. We see that the map $X \times_{BG} * \rightarrow X$ has fiber isomorphic (non-canonically) to G , and this is what it means that $* \rightarrow BG$ is “a map with fiber G ”.

If G has positive dimension, this map is not etale, but since G is a group scheme and we are working in characteristic zero, it is smooth. Indeed, in the above formula $X \times_{BG} * = F$ the projection to X is the structural map of the torsor which is smooth, and since we know it for any $X \rightarrow BG$, it means that the map $* \rightarrow BG$ is smooth.

The idea of Artin stacks is to allow “coordinate charts” which are smooth maps, rather than just etale ones. An important further property of our map $* \rightarrow BG$ is that it is *representable*, meaning that for any map $X \rightarrow BG$ the fiber product as above is representable (it is best to require representability in the category of algebraic spaces). Now we can say that our stack BG is *locally a scheme*, that is to say it has a smooth representable map from a scheme. Call this chart $U := * \rightarrow BG$.

The scheme of relations is

$$R := U \times_{BG} U = * \times_{BG} * = G.$$

Both maps $R \rightarrow U = *$ are of course the same; and the map $R \times_U R \rightarrow R$ is just the composition in the group G . We get the groupoid

$$(G \rightrightarrows *) \cong BG.$$

6.2. The geometricity condition. With the example of BG as motivation, we can now describe rather easily the general definition of an Artin stack. The point of view we shall present here was explained to us by Walter, and he suggested that it would generalize immediately to the case of higher stacks. The classical terminology used by Artin was “algebraic stack”, but as Walter pointed out, it seems more intuitive to call the condition “geometricity”.

Notice that if F is a 1-stack, and if X and Y are schemes, in particular they are 0-stacks, then the fiber product $X \times_F Y$ is 0-truncated⁸.

⁸More generally, if F is n -truncated and X, Y are k -truncated for $k < n$ then $X \times_F Y$ is $n - 1$ -truncated. At the topological level this may be seen by considering the exact sequences for homotopy groups (cf 5.4), first of the fibration X/F :

$$\pi_{n+1}(F) \rightarrow \pi_n(\text{fiber}(X/F)) \rightarrow \pi_n(X)$$

showing $\pi_n(\text{fiber}(X/F))$ trivial; then of the fibration $X \times_F Y/Y$, noting that the fiber is the same as before:

$$\pi_n(\text{fiber}(X/F)) = \pi_n(\text{fiber}(X \times_F Y/Y)) \rightarrow \pi_n(X \times_F Y) \rightarrow \pi_n(Y)$$

showing that $\pi_n(X \times_F Y)$ is trivial.

We say that a map of 1-stacks $A \rightarrow B$ is *representable* if for any map from a scheme $X \rightarrow B$, the fiber product $X \times_B A$ is representable by an algebraic space. We say that a representable map is *smooth* if for any such X , the map $X \times_B A \rightarrow X$ is a smooth map of algebraic spaces. Now, the geometricity condition for a 1-stack F is that there exists a “smooth chart”, that is to say a surjective smooth representable map from a scheme $X \rightarrow F$.

If that exists, then we can give a presentation for F as a quotient of a nice equivalence relation: representability applies in particular to our chart itself, so the fiber product $R := X \times_F X$ is an algebraic space, and the two projection maps $R \rightarrow X$ are smooth. There is a natural composition $R \times_X R \rightarrow R$ and (X, R) gains a structure of groupoid in the category of algebraic spaces. Our stack F is the quotient $F = X // R$.

Following Walter’s suggestion, the above definition generalizes immediately to the case of n -stacks. We define by induction on n simultaneously the notions of geometricity for an n -stack, for a morphism of n -stacks, and smoothness for a geometric morphism of n -stacks. Assuming these are known for $(n - 1)$ -stacks, the definitions for n are exactly as before: a morphism $A \rightarrow B$ from an $(n - 1)$ -stack to an n -stack is *geometric* (resp. smooth geometric) if for any map from a scheme $X \rightarrow B$, the $(n - 1)$ -stack (see the previous footnote) $X \times_B A$ is geometric (resp. $X \times_B A$ is geometric and the map $X \times_B A \rightarrow X$ is a smooth map to a scheme); an n -stack F is *geometric* if it admits a geometric smooth surjective map $X \rightarrow F$ from a scheme of finite type, called a “chart”; and a map $F \rightarrow Y$ from a geometric n -stack to a scheme is smooth if for any chart $X \rightarrow F$, the composed map $X \rightarrow Y$ is smooth. The reader may check that these definitions work together inductively to give the required notions, as is discussed in more detail in [38].

Notice that we start with a 0-geometric stack being an algebraic space. One can naturally go from just the notion of scheme, to the notion of algebraic space, by adding an additional step of the same form in the induction.

It has now become usual terminology to say that an *Artin n -stack* is an n -stack satisfying the above geometricity condition.⁹

6.3. First examples. The first examples of Artin 1-stacks are BG for G an algebraic group scheme. One may consider these as quotients $BG = * // G$ for the trivial action of G on a point. More generally, if G acts on an algebraic variety X , then the stack quotient $X // G$ is an Artin 1-stack. The map $X \rightarrow X // G$ is a smooth

⁹**The Grothendieck-Pridham condition:** There is another way to look at the condition of being an Artin stack, much more directly related to the simplicial point of view. Pridham [36] has shown the very nice characterization that an n -stack F is Artin if and only if it admits a presentation as a simplicial scheme X . (take the simplicial presheaf associated to the simplicial scheme and take the associated stack, or equivalently replace it by a fibrant object in the Jardine model structure) such that the pieces X_i are schemes of finite type, and such that the attaching maps

$$X_i \rightarrow \lim_{i \rightarrow j < i} X_j = (\text{cs}k_{i-1}(X.))_i$$

are smooth. This condition was mentioned by Grothendieck in [18] as a possible definition of schematic homotopy type, so Pridham’s theorem says that Grothendieck’s intuition gives rise to the same very natural notion of Artin n -stack for which an inductive definition was discussed above.

Many of the basic properties are easy to see from the geometricity definition, but not immediately clear from Grothendieck’s definition. One could therefore view Pridham’s theorem as a way of showing that Grothendieck’s definition satisfies the properties one would want.

chart, and the relation is

$$R = X \times_{X//G} X \cong G \times X$$

with the two maps $G \times X \rightarrow X$ being the projection, and the group action respectively.

The quotient construction gives a huge array of examples, indeed most moduli spaces are obtained by a quotient construction using Mumford's GIT. The corresponding moduli stacks are then quotient stacks of the above form.

One can play around with small examples. An interesting and useful one is the quotient

$$\mathcal{A} := \mathbb{A}^1/\mathbb{G}_m.$$

This stack has two points, corresponding to the two orbits [0] and [1]. The stabilizer group of [0] is \mathbb{G}_m while the stabilizer group of [1] is trivial. This stack has many different uses, for example families $V \rightarrow \mathcal{A}$ may be viewed as *filtered objects* with $V_{[1]}$ being the underlying object and $V_{[0]}$ being the “associated graded” with its \mathbb{G}_m -action. This viewpoint is very useful for nonabelian filtrations occurring in nonabelian Hodge theory.

In a somewhat different direction, \mathcal{A} is a classifier for Cartier divisors: given a scheme X , there is a one-to-one correspondence between Cartier divisors on X and morphisms $X \rightarrow \mathcal{A}$, with the divisor being the pullback of [0].

6.4. The Artin-Lurie representability theorem. Artin's approximation theorem is essential for translating charts constructed in an analytic or formal way, such as the Kuranishi deformation spaces, into algebro-geometric charts. This typical procedure for transforming a moduli problem, given as a functor, into an Artin stack, was codified in the *Artin representability theorem* [4]. The statement combines the local infinitesimal considerations needed to get a formal chart, with the algebraicity assumptions needed for the approximation theorem, to get to maps from global schemes.

Lurie proves a vast generalization to higher stacks [25], and particularly to higher *derived stacks*. The introduction of derived structure is crucial for getting a good control of the local infinitesimal theory. It would go outside of our scope to delve into the details. See for example Pridham's paper [35] for discussion and simplification.

6.5. The higher stack of perfect complexes. A motivating example for going to higher stacks and looking at Artin n -stacks is the stack **Perf** of perfect complexes. Fix a function $b : \mathbb{Z} \rightarrow \mathbb{N}$ which is zero outside of finitely many values, and let **Perf** $^{\leq b}$ be the higher stack whose value on a scheme X is the ∞ -category of perfect complexes C^\bullet over X , such that at any point $x \in X$ we have $h^i(C_x) \leq b(i)$. It is an n -stack, where n is the length of the interval on which b is nonzero. A main theorem says that **Perf** $^{\leq b}$ is an Artin n -stack of finite type. This was stated without proof in [20] and first proven by Toën and Vaquié [42]. They proved, in fact, a vast generalization concerning the moduli stack of perfect complexes relative to any projective scheme, and they considered its natural structure as a derived stack.

It is interesting to contemplate the structure of **Perf** $^{\leq b}$. For one thing, it has only finitely many points, much like in the example \mathcal{A} . We can enumerate the points of **Perf** $^{\leq b}$. Indeed, a perfect complex over an algebraically closed field of characteristic zero is determined up to equivalence by the dimensions of its cohomology groups.

Thus, the points of $\mathbf{Perf}^{\leq b}$ are in one-to-one correspondence with the functions $h : \mathbb{Z} \rightarrow \mathbb{N}$ such that $h^i \leq b(i)$.

On the other hand, the local coordinate charts of $\mathbf{Perf}^{\leq b}$ are the very classical *Buchsbaum-Eisenbud schemes*. If we fix a point, corresponding to a function (h^i) , then let V^i be the vector space of dimension h^i , and let $BE(h^\cdot)$ be the scheme which parametrizes collections of differentials $d^i : V^i \rightarrow V^{i+1}$ such that $d^{i+1}d^i = 0$. Clearly $BE(h^\cdot)$ parametrizes perfect complexes in the sense that there is a tautological perfect complex over it, and by semicontinuity the cohomology dimensions are $\leq h^i \leq b(i)$ so we get a map

$$BE(h^\cdot) \rightarrow \mathbf{Perf}^{\leq b}.$$

These very natural charts were what we had in mind in [20], but they don't appear explicitly in the much more general situation of [42]. Benzeghlé [6] proved that they do indeed provide smooth charts for the Artin n -stack, and furthermore that these charts may be naturally completed to a simplicial scheme satisfying Pridham's smoothness criterion for giving an Artin n -stack.

Thus we may view the stack $\mathbf{Perf}^{\leq b}$ as being the natural quotient of the collection of Buchsbaum-Eisenbud schemes, by the relation of weak equivalence. Notice that if one wanted to remain in the realm of 1-stacks, by taking the 1-truncation of the quotient, the geometricity property would no longer hold. We need to use the notion of n -stack in order to give this quotient its most natural nice property.

The theory of perverse coherent sheaves [3] [7] allows one to isolate a 1-truncated substack for which the previous comment doesn't apply, yielding an Artin 1-stack of moduli.

Kapranov and Pimenov have recently constructed derived schemes generalizing the Buchsbaum-Eisenbud schemes [22]. It is natural to expect that these should provide charts for the derived moduli stack of perfect complexes as it was constructed by Toën and Vaquié (applying [42] with X being a single point). We don't know if that has been done yet.

The Artin property of $\mathbf{Perf}^{\leq b}$ means that it can be considered as a geometric object of the same kind as a usual algebraic variety. One may propose the following

Question 6.1. *What does the cohomology $H^*(\mathbf{Perf}^{\leq b}, \mathbb{Q})$ look like? What is its mixed Hodge structure?*

Cohomology classes $\eta \in H^i(\mathbf{Perf}^{\leq b}, \mathbb{Q})$ provide characteristic classes for perfect complexes whose cohomology is bounded by b . One should note that if we take the direct limit over bigger and bigger bounding functions b , then the structure of the cohomology becomes rather more simple and the only characteristic classes are the usual Chern classes of the cohomology sheaves. Preliminary calculations suggest that if we maintain a fixed bound b there can be more.

6.6. An example: perfect complexes of type $\leq (1, 1)$. Let us close this section by examining explicitly the case of perfect complexes with Betti numbers $(1, 1)$ and $(0, 0)$. We will get something that looks a lot like the 1-stack \mathcal{A} seen at the start of the section, but with an Artin 2-stack structure instead. This example, accessible without too much technical baggage, provides a useful window on the ideas behind higher Artin stacks.

Consider the simplest function defined by $b(0) = b(1) = 1$ but $b(i) = 0$ elsewhere. Use the superscript $\leq (1, 1)$. The Euler characteristic is a locally constant

function on $\mathbf{Perf}^{\leq(1,1)}$ so we can fix the open and closed substack $\mathcal{A}'' \subset \mathbf{Perf}^{\leq(1,1)}$ of complexes with Euler characteristic zero.¹⁰ This stack has two closed points corresponding to cohomology dimensions $(0,0)$ and $(1,1)$ respectively (the dimensions are required to be zero outside of degrees $i = 0, 1$). These points may be viewed as the complexes

$$e_1 : k \xrightarrow{1} k$$

and

$$e_0 : k \xrightarrow{0} k$$

respectively. The complex e_1 is quasiisomorphic to $0 \rightarrow 0$ i.e. it is the zero complex, in particular its space of automorphisms is reduced to a single point. It is the open point of \mathcal{A}'' , very analogous to the point $1 \in \mathcal{A}$ considered at the beginning of the section.

The group of automorphisms of the complex e_0 , on the other hand, is $\mathbb{G}_m \times \mathbb{G}_m$. This is different from the automorphism group \mathbb{G}_m of the point $0 \in \mathcal{A}$. In order to understand this difference, we should note that \mathcal{A}'' is a 2-stack rather than a 1-stack, so we should consider not the group of all automorphisms of e_0 but rather the 1-stack of automorphisms modulo homotopy. The identity endomorphism $i : e_0 \rightarrow e_0$ has self-homotopies, and in fact the sheaf of homotopies is \mathbb{G}_a , corresponding to the functions $e_0^1 \rightarrow e_0^0$. The automorphism group $\mathbb{G}_m \times \mathbb{G}_m$ acts on the sheaf of homotopies via the product map. We therefore obtain a picture of the 1-stack of automorphisms of e_0 as having homotopy group sheaves (cf 5.4) represented by $\pi_0 = \mathbb{G}_m \times \mathbb{G}_m$ and $\pi_1 = \mathbb{G}_a$. In terms of dimensions everything works out because this automorphism group stack still has dimension 1, so e_0 has dimension -1 and it is still a codimension 1 closed substack of \mathcal{A}'' as was the case for $0 \in \mathcal{A}$.

The Buchsbaum-Eisenbud scheme providing a neighborhood of the point e_0 is just the affine line \mathbb{A}^1 parametrizing complexes of the form

$$e_t : k \xrightarrow{t} k$$

so our Artin 2-stack \mathcal{A}'' may be thought of as a different way of taking a quotient of the affine line with the open subset of nonzero points as single open orbit.

Consider the relation $R \rightarrow \mathbb{A}^1 \times \mathbb{A}^1$ defining \mathcal{A}'' . Itself a 1-stack, R is the quotient of a variety R_0 by an action of the group scheme \mathbb{G}_a as follows. The variety R_0 is the space of quadruples (s, t, g, h) where (s, t) is a pair of elements in the parametrizing Buchsbaum-Eisenbud scheme \mathbb{A}^1 , and (g, h) form a map of complexes

$$\begin{array}{ccc} k & \xrightarrow{s} & k \\ g \downarrow & & \downarrow h \\ k & \xrightarrow{t} & k \end{array}$$

meaning that $tg = hs$, such that furthermore (g, h) is a quasiisomorphism. The quasiisomorphism condition means that if either $s = 0$ or $t = 0$ then g and h are nonzero. Thus, R_0 is the open subset of the affine variety $tg = sh$ in \mathbb{A}^4 , defined by removing the two planes $(s = 0, g = 0)$ and $(t = 0, h = 0)$. The action of \mathbb{G}_a is by $a : (s, t, g, h) \mapsto (s, t, g + as, h + at)$. The quotient $R = R_0/\mathbb{G}_a$ is a 1-stack, and this 1-stack projects by (s, t) to $\mathbb{A}^1 \times \mathbb{A}^1$. It has a groupoid structure and the quotient 2-stack is \mathcal{A}'' .

¹⁰The two other components corresponding to Betti numbers $(1, 0)$ and $(0, 1)$ are just $B\mathbb{G}_m$'s.

We see again that the automorphism group of e_1 , the fiber of R over $s = t = 1$, is given by equation $g = h$ modulo $g \mapsto g + a$ and $h \mapsto h + a$; this quotient of \mathbb{A}^1 by \mathbb{G}_a is trivial. The automorphism group of e_0 , the fiber of R over $s = t = 0$, is the “quotient” of the space of (g, h) with $g \neq 0$ and $h \neq 0$, by the trivial action of \mathbb{G}_a . Thus the automorphism group of e_0 is isomorphic as a stack to $\mathbb{G}_m \times \mathbb{G}_m \times B\mathbb{G}_a$ although the group multiplication law is twisted.

This simplest example gives an idea of what the n -stack $\mathbf{Perf}^{\leq b}$ looks like in general, and of what it means to have a higher Artin structure. It illustrates the idea that we might have a groupoid where the relation object is itself a stack, leading to a 2-stack.

7. NONABELIAN COHOMOLOGY

The introduction of higher algebraic stacks paves the way towards a new possibility of using them to investigate the structure of spaces, a technique that could go by the generic name of “shape theory”.

Very roughly speaking the shape of a space means the nonabelian cohomology functor it defines. In this section, we start by explaining the motivation, then we discuss an extended example that shows how shape theory using nonabelian cohomology leads to a new perspective on non-simply connected spaces, and then we get to a general discussion of the categories of coefficients that might be used.

7.1. From topologizing algebraic geometry to algebro-geometrizing topology. Our discussion up until now has focused on the idea that new more complicated kinds of algebraic varieties, known as stacks, have local and even global topological structures attached to them, and indeed these topological structures are essential for the definition and theory of stacks. Here, topology is viewed as a tool needed to understand structures which arise naturally in algebraic geometry, and we can say that algebraic-geometry becomes “topologized”.

Developing this idea leads to the quite general point of view of presheaves of spaces (simplicial presheaves) over a Grothendieck site. But now, such a theory allows us to envision a new and different direction: algebro-geometrizing topology. By this, we mean envisioning new and more complicated kinds of “enriched spaces”, where the additional structure is of an algebro-geometric nature. A first and typical example is the stack BGL_n . It is an Artin 1-stack classifying rank n vector bundles. Giving a morphism from an algebraic variety X to BGL_n is the same thing as giving a rank n vector bundle over X . Natural transformations between morphisms correspond to isomorphisms between bundles.¹¹

One salient feature is that $\pi_1(BGL_n, o) = GL_n$ is a sheaf of groups, the sheaf represented by the group scheme GL_n . Thus, we can view BGL_n as a new kind of “space”, whose homotopy groups are group schemes rather than just discrete

¹¹Here is the construction of the stack BGL_n as a presheaf of spaces. For each affine scheme $\text{Spec}(A)$, consider the group $GL_n(A)$ and form its classifying simplicial set $B(GL_n(A))$. The set of k -simplices here is $GL_n(A)^{k-1}$ with face maps given by projections to factors and multiplication. These simplicial sets are organized into a simplicial presheaf over our site. It is a 1-prestack because each simplicial set is 1-truncated. Then BGL_n is the associated stack, which may for example be described as the fibrant replacement of the prestack in the Jardine model structure. A more explicit description of the “stackification” is possible, generalizing the explicit description of the sheafification of a presheaf.

groups. Generalizing this idea to higher stacks leads to the *schematization of homotopy*. The schematization operation was one of Grothendieck's overall aims in *Pursuing Stacks* [18]. Breen had an early paper on the schematic sphere [9], and one should note that his paper on cohomology calculations [8] provides in retrospect the foundation for schematization. More recently, Katzarkov Pantev and Toën [23] [24], Pridham [34] and others [28] defined the schematization in full generality and developed Hodge theory for it.

In the simply connected case, the schematization recovers the usual rational homotopy type. As indicated in the notation, if X is a simply connected homotopy type then the schematization $X \otimes k$ corresponds to tensoring the rational homotopy type, considered for example as a dg commutative algebra, with the field k over \mathbb{Q} . Even in this case, additional insight is gained by thinking of $X \otimes k$ as a higher stack, rather than just “the homotopy type whose dgca would be the tensor product one”.

In the non-simply connected case, the schematization $X \otimes k$ provides a stack whose fundamental group is the *pro-algebraic completion* of $\pi_1(X, x)$. That is to say, the inverse limit of linear k -algebraic group schemes G over the index category of representations $\rho : \pi_1 \rightarrow G$. The map

$$\pi_1(X, x) \rightarrow \pi_1(X \otimes k, x)$$

is the universal representation to an affine group scheme over k . The higher homotopy of $X \otimes k$ records the relative Malcev completions of the homotopy of X , at all of the representations ρ .

When X is not simply connected, the schematization $X \otimes k$ doesn't provide a fully satisfactory answer to the problem of finding homotopical invariants “over k ”. It can be a very big object when $\pi_1(X)$ admits families of representations. In that case $\pi_1(X \otimes k)$ contains a big direct product of all the target groups for the representations in the family, but the direct product doesn't reflect the continuous structure in the variation of these representations. If $k' : k$ is a transcendental field extension then $X \otimes k'$ is not just the extension of scalars of the stack $X \otimes k$ from k to k' , because we have a lot of new representations defined over k' so the algebraic fundamental group gets a lot of new factors.

7.2. Shape theory. One possible answer to this question is to adopt a point of view of *shape theory* using nonabelian cohomology as the basic structure.

We let \mathcal{C} be an appropriate site of schemes. For our purposes we consider a field k of characteristic zero and look at the site $\mathcal{C} := \text{Aff}_k^{\text{ft}}$ of affine k -schemes of finite type, with the etale topology. Let Stack denote the ∞ -category of stacks of ∞ -groupoids over \mathcal{C} , with Stack_n denoting the subcategory of n -stacks.

Suppose \mathcal{P} is an ∞ -category with a functor ζ to Stack . If X is a space, let \underline{X} denote the constant ∞ -stack whose values are the Poincaré ∞ -groupoid of X . In terms of simplicial presheaves, it means to take the (fibrant replacement of) the constant simplicial presheaf with values X . We define the \mathcal{P} -*shape* of X to be the functor

$$\mathcal{P}\text{Shape}_X : \mathcal{P} \rightarrow \text{Stack}, \quad T \mapsto \mathbf{Hom}(\underline{X}, \zeta(T)).$$

We'll discuss more about the choice of \mathcal{P} below (and once chosen, it may usually be dropped from the notation).

Getting back to the simply-connected case to begin, the k -rational homotopy type, or schematization, of a simply connected space may be viewed as an object representing the shape for an appropriate choice of category \mathcal{P} . For this, let \mathcal{P} be

the full subcategory of \mathbf{Stack}_n consisting of the n -stacks which are 1-connected and whose higher homotopy group sheaves are of the form \mathbb{G}_a^k . Recall that $\mathbb{G}_a \cong \mathbb{A}^1$ is the group scheme represented by the affine line, which as a sheaf is just the structure sheaf \mathcal{O} . So we are asking that the higher homotopy group sheaves be of the form \mathcal{O}^k .

Then, for any simply connected finite CW-complex X there is an n -stack $(X \otimes k)_{\leq n} \in \mathcal{P}$ together with a map

$$X_C \rightarrow (X \otimes k)_{\leq n}$$

representing the functor $\mathcal{P}\text{Shape}_X$. For $i \leq n$ we have

$$\pi_i((X \otimes k)_{\leq n}) = \pi_i(X) \otimes_{\mathbb{Z}} k$$

in the sense that the left side is the sheaf of groups over C represented by the k -vector space on the right considered as an affine k -scheme.

This gives a functorial point of view to rational homotopy theory for simply-connected spaces, although it doesn't lead to the introduction of any really new invariants.

7.3. An example of shape in the non-simply connected case. One of the main motivations for introducing nonabelian cohomology and for looking at the associated shape theory, is the well-known problem that rational homotopy theory doesn't behave very well for non-simply connected spaces. Shape theory gives us new structures to use in this context, leading to a rather good improvement in some aspects.

This is an area where new ideas from the theory of stacks can shed light on phenomena in algebraic topology. Our goal in this subsection is to illustrate with an example.

We should clearly enlarge the category \mathcal{P} of target stacks to include some non-simply connected ones. In the present subsection, we look at a first example. Our relatively simple example will have abelian fundamental group, so for simplicity we consider target stacks whose fundamental group scheme is just \mathbb{G}_m .

Let $T_1 := K(\mathbb{G}_m, 1)$. Let us consider 3-stacks T with maps $T \rightarrow T_1$ inducing $\tau_{\leq 1}(T) \cong T_1$. We impose the condition that $\pi_2(T)$ and $\pi_3(T)$ be abelian group schemes of the form \mathbb{G}_a^n . Throughout this subsection, fix \mathcal{P} to be the 4-category of stacks of this form.

Consider the space $X := S^1 \vee S^2$. We note that $\Gamma := \pi_1(X) = \mathbb{Z}$, and the universal cover \tilde{X} is a wedge sum of one copy of S^2 for each element of Γ . Thus, $\pi_2(X) = H_2(\tilde{X}) = \mathbb{Z}[\Gamma]$. On the other hand, the higher homology groups vanish so

$$\pi_3(X) \otimes \mathbb{Q} = \text{Sym}^2(\pi_2(X) \otimes \mathbb{Q}) = \text{Sym}^2(\mathbb{Q}[\Gamma]).$$

As a $\mathbb{Q}[\Gamma]$ -module, it has infinite type.

Our goal in this paragraph is to illustrate how nonabelian cohomology with stack coefficients provides an invariant which has better finiteness properties yet still allows us to distinguish different classes in $\pi_3(X) \otimes \mathbb{Q}$.

The functor

$$\text{Shape}_X : \mathcal{P} \rightarrow \text{Stack}_3$$

is defined by $\text{Shape}_X(T) := \mathbf{Hom}(\underline{X}, T)$ where \underline{X} is the constant prestack with value X .

The image of this functor lies in the subcategory $\text{Artin}_3 \subset \text{Stack}_3$ of Artin 3-stacks of finite type, see Proposition 7.2 in subsection 7.4 below.

The Artin finite-type property is what we mean by a better finiteness condition. The functor Shape_X relates categories of objects of finite type in this sense.

We would now like to see how the shape functor distinguishes elements of $\pi_3(X)$. The shape functor has a pointed version: fix a basepoint $x \in X$, and look at pointed stacks (T, t) . Let $\text{Shape}_X^*(T)$ be the space of pointed maps $(X, x) \rightarrow (T, t)$.

An element of $\pi_3(X, x)$ is a pointed map $\varphi : S^3 \rightarrow X$ inducing a natural transformation

$$\text{Shape}_\varphi^*(T) : \text{Shape}_X^*(T) \rightarrow \text{Shape}_{S^3}^*(T).$$

That may also be extended to the case when $\varphi \in \pi_3(X, x) \otimes \mathbb{Q}$.

Proposition 7.1. *Different elements $\varphi \neq \psi$ in $\pi_3(X, x) \otimes \mathbb{Q}$ give different natural transformations: $\text{Shape}_\varphi^* \neq \text{Shape}_\psi^*$ from the X -shape to the S^3 -shape.* ¹²

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Proof. Consider pointed stacks $T^{a,b}$ which are as follows:

$$\begin{array}{ccc} F^{a,b} & \longrightarrow & T^{a,b} \\ & & \downarrow \\ & & B\mathbb{G}_m \end{array}$$

where, in turn,

$$\begin{array}{ccc} K(V^{a+b}, 3) & \longrightarrow & F^{a,b} \\ & & \downarrow \\ & & K(V^a \oplus V^b, 2) \end{array}$$

with V^a being the sheaf represented by an affine line which as a \mathbb{G}_m -module of rank one has t acting by t^a . The structure of $F^{a,b}$ is the standard map

$$\mu : \text{Sym}^2(V^a \oplus V^b) \rightarrow V^{a+b}.$$

The basepoint t comes from the standard ones in the Eilenberg-Maclane stacks.

A pointed map $(X, x) \rightarrow (T, t)$ is given by a pair of pointed maps $S^1 \rightarrow T$ and $S^2 \rightarrow T$, so up to homotopy it is just given by a triple (t, v', v'') where $t \in \mathbb{G}_m$, $v' \in V^a$ and $v'' \in V^b$. Thus

$$\pi_0 \text{Shape}_X^*(T^{a,b}) = \mathbb{G}_m \times V^a \times V^b.$$

Similarly a pointed map $S^3 \rightarrow T$ is given by $w \in V^{a+b}$,

$$\pi_0 \text{Shape}_{S^3}^*(T^{a,b}) = V^{a+b}.$$

Recall that we denote $\Gamma := \pi_1(X) \cong \mathbb{Z}$. An element of $\pi_3(X)$ may be written, rationally, as

$$\varphi = \sum_{i,j} p_{ij} \gamma^i \gamma^j \in \text{Sym}_{\mathbb{Q}}^2(\mathbb{Q}[\Gamma])$$

with $p_{ij} = p_{ji}$, and where $\gamma^i \in \Gamma$ is the i -th multiple (power) of the generator. We get an element of $\pi_3(X) \otimes \mathbb{Q}$ by letting p_{ij} be any rational coefficients.

The action of such a map φ on the shape is

$$\begin{aligned} \pi_0 \text{Shape}_\varphi^*(T^{a,b}) &: \mathbb{G}_m \times V^a \times V^b \rightarrow V^{a+b} \\ (t, v', v'') &\mapsto \sum_{i,j} p_{ij} \mu((t^i \cdot (v', v''))(t^j \cdot (v', v''))) \\ &= \sum_{i,j} p_{ij} (t^{ai+bj} + t^{aj+bi}) v' v'' = 2 \sum_{i,j} p_{ij} t^{ai+bj}. \end{aligned}$$

The last equation is due to the symmetry $p_{ij} = p_{ji}$.

In this example the shape functor, with reasonable finiteness properties, is rich enough to distinguish the elements of the infinite-type group $\pi_3(X, x) \otimes \mathbb{Q}$.

It will be interesting to see how the result of the proposition generalizes to other fundamental groups and homotopy groups in higher degrees. The shape can certainly fail to see some parts of the fundamental group, such as the intersection of subgroups of finite index. Characterizing the homotopical information carried by the shape is undoubtedly a subtle problem.

7.4. Coefficients for nonabelian cohomology. Our new kinds of geometrized spaces allow us to envision a rich theory of nonabelian cohomology. As we can see from our discussion of the classification of gerbes, a nonabelian cohomology theory involves, in its simplest incarnation, fixing a target space T and looking at the functor

$$X \mapsto \text{Hom}(X, T).$$

The abelian case is when $T = K(A, n)$ and $H^n(X, A) = \pi_0 \text{Hom}(X, T)$. The first degree nonabelian cohomology with coefficients in a group G is given by taking $T = BG = K(G, 1)$, and $\text{Hom}(X, T)$ is the 1-truncated space corresponding to the nonabelian cohomology groupoid $\mathcal{H}^1(X, G)$ of G -torsors over X .

In usual cohomology theory we pretty quickly want to pass to cohomology with coefficients over, say, a field k . Thus in the examples of the previous paragraph, A would be a k -vector space (such as k itself) and G would be an algebraic k -group scheme. In these cases we can say explicitly how the cohomology retains a similar algebraic structure over k .

A general nonabelian cohomology situation will be when T is some kind of space with homotopy groups in various different degrees. What does it mean for T to be “algebraic over a field k ”? This is where the idea of algebro-geometrizing topology comes in: rather than looking for just a space T , we look for a simplicial presheaf, or n -stack, over a conveniently chosen site such as $\text{Aff} = \text{Aff}_k^{\text{ft}, \text{et}}$ the affine schemes of finite type over k with the etale topology. This is the site we shall refer to in the subsequent discussion.

If T is an n -stack over Aff and X is a space, then $\text{Hom}(X, T)$ is again an n -stack over Aff , by the formula

$$\text{Hom}(X, T)(\text{Spec } A) := \text{Hom}(X, T(\text{Spec } A)).$$

Suppose $\psi = \sum_{i,j} q_{ij} \gamma^i \gamma^j$ is a different element, such that

$$\pi_0 \text{Shape}_{\varphi}^*(T^{a,b}) = \pi_0 \text{Shape}_{\psi}^*(T^{a,b})$$

for any a, b . It means that

$$\sum_{i,j} p_{ij} t^{ai+bj} = \sum_{i,j} q_{ij} t^{ai+bj}$$

for any $t \in \mathbb{G}_m$. Separating terms, we get that for any integer k ,

$$(7.1) \quad \sum_{ai+bj=k} (p_{ij} - q_{ij}) = 0.$$

We claim this implies $p_{ij} = q_{ij}$. Indeed, let $\Sigma \subset \mathbb{Z} \times \mathbb{Z}$ be the set of pairs where $(p_{ij} - q_{ij}) \neq 0$. If it is nonempty, then we can choose a corner (i_0, j_0) of its convex hull, and fix a rational slope of line which meets the convex hull only at that corner. That determines a, b, k such that the intersection of $ai + bj = k$ with Σ consists of just one point (i_0, j_0) . The above relation (7.1) says $p_{i_0 j_0} - q_{i_0 j_0} = 0$ but that contradicts the choice of $(i_0, j_0) \in \Sigma$. Hence Σ is empty and $p_{ij} = q_{ij}$, so $\varphi = \psi$ in the rational homotopy group. This completes the proof of the proposition. \square

Thus, if T is an n -stack over k , so is the nonabelian cohomology stack $\mathbf{Hom}(X, T)$. This satisfies our requirement for having a theory of coefficients relative to k .

For Hodge theory with nonabelian coefficients, it is particularly necessary to have a theory of coefficients relative to the ground field of complex numbers \mathbb{C} , indeed the natural structures of abelian Hodge theory exist only on cohomology with complex coefficients and the same is true in the nonabelian case.

Consideration of some more or less pathological examples shows that we need to do more in order to obtain a nice theory of coefficients. We would like our cohomology stacks $\mathbf{Hom}(X, T)$ to have a geometrical structure. For example, the first nonabelian cohomology $\mathcal{H}^1(X, G) = \mathbf{Hom}(X, BG)$ with coefficients in a linear algebraic group G , has a structure of Artin algebraic stack whose coarse moduli stack is the classical *character variety* of $\pi_1(X)$ with coefficients in G .

So it is natural to look for an appropriate kind of geometrical structure to impose on the coefficients, in such a way that the cohomology stack maintains the same kind of structure. The notion of Artin n -stack provides such a structure, as is shown by the following proposition.

Proposition 7.2. *The category of Artin n -stacks of finite type is closed under finite products and fiber products. If X is a finite CW-complex and T is an n -stack, then $\mathbf{Hom}(X, T)$ is in the category of n -stacks generated by products and fiber products starting with T . Therefore, if T is an Artin n -stack of finite type, so is the nonabelian cohomology stack $\mathbf{Hom}(X, T)$.*

7.5. Nonabelian de Rham cohomology. Hodge theory for a complex algebraic variety X is about the relationship between Betti cohomology, the cohomology of the usual topological space X^{top} , and other cohomologies defined using the algebraic structure of X , such as de Rham or Dolbeault cohomology.

Very briefly, we can define the nonabelian de Rham cohomology of X by introducing the sheaf X_{dR} on the site $\text{Aff}_{\mathbb{C}}^{\text{ft}, \text{et}}$ defined by

$$X_{\text{dR}}(Y) := X(Y^{\text{red}}).$$

When X is smooth, the de Rham stack X_{dR} is defined by a formal groupoid whose object object is X and whose morphism object is the completion $\widehat{X \times X}$ of the diagonal in $X \times X$. The morphism object injects into $X \times X$ so it is really just a relation whose quotient is a sheaf: one should think of it as “glueing together infinitesimally near points”.

If T is an n -stack on $\text{Aff}_{\mathbb{C}}^{\text{ft}, \text{et}}$, the nonabelian de Rham cohomology¹³ of X with coefficients in T is $\mathbf{Hom}(X_{\text{dR}}, T)$. One may check that this recovers the usual notions of nonabelian de Rham cohomology. When $T = BG$ for a complex linear algebraic group, a map $X_{\text{dR}} \rightarrow BG$ consists, firstly, of a map $X \rightarrow BG$ i.e. a principal G -bundle over X ; and secondly, of glueing data over the relation $\widehat{X \times X}$ with a coherence condition, amounting to providing the principal bundle with a flat connection. Therefore, $\mathbf{Hom}(X_{\text{dR}}, BG)$ is the Artin 1-stack of principal G -bundles with flat connection.

¹³The natural extension to Dolbeault cohomology, and the Hodge filtration, are obtained by considering the deformation of this formal groupoid to its normal cone. The attachments between infinitesimally near points are deformed to loops located at each point, so X_{Dol} is the formal completion of the zero-section in the tangent bundle of X . Nonabelian Dolbeault cohomology is the moduli space of Higgs bundles in degree one, and reflects the Dolbeault cohomology of Higgs bundles in higher degrees.

When $T = K(\mathbb{G}_a, n)$, $\mathbf{Hom}(X_{\text{dR}}, T)$ yields Grothendieck's algebraic de Rham cohomology $H_{\text{dR}}^n(X, \mathbb{C})$.

One of the main tools used to link Betti and de Rham cohomology is Serre's GAGA theorem, saying that cohomology of a projective algebraic variety is the same as cohomology of the corresponding analytic variety. If we want to promote our abelian theory to one that involves nonabelian coefficients, then not only should the coefficients be "defined over \mathbb{C} " corresponding to n -stacks over $\text{Aff}_{\mathbb{C}}^{\text{ft}, \text{et}}$, but also there should hold an appropriate GAGA principle.

Consider a basic example: suppose $T = K(\mathbb{G}_m, n)$. This is an Artin n -stack. But for $n \geq 2$, $\text{Hom}(X_{\text{dR}}, K(\mathbb{G}_m, n))$ on the algebraic site, is very different from the same space computed on the analytic site. The algebraic cohomology is torsion, whereas the analytic one is isomorphic to the Betti cohomology with \mathbb{C}/\mathbb{Z} coefficients.

This example shows that we need to impose some condition on the higher homotopy group schemes of our coefficient stack T . A similar example in degree $n = 1$ shows that we can't use an abelian variety A : again $\text{Hom}(X_{\text{dR}}, K(A, 1))$ on the algebraic site would be very different from the corresponding analytic cohomology group.

With these examples in mind, we make the following definition.

Definition 7.3. *A connected very presentable stack T is an n -stack on $\text{Aff}_{\mathbb{C}}^{\text{ft}, \text{et}}$ such that $\pi_0(T) = *$, $\pi_1(T) = G$ is a linear algebraic group, and for $2 \leq i \leq n$, $\pi_i(T)$ is a direct sum of copies of \mathbb{G}_a .*

For connected very presentable coefficient stacks T , which are in particular Artin n -stacks, GAGA works in the usual way, as may be seen by induction on the Postnikov tower.

If X is a smooth projective complex algebraic variety, we obtain an algebraic de Rham cohomology stack $\mathbf{Hom}(X_{\text{dR}}, T)$ which is an Artin n -stack whose analytification is the Betti cohomology:

$$\mathbf{Hom}(X_{\text{dR}}, T)^{\text{an}} \cong \mathbf{Hom}(X^{\text{top}}, T)^{\text{an}}.$$

Recall that X^{top} is a finite CW complex so the Betti cohomology is an Artin n -stack by Proposition 7.2.

If X is only quasiprojective, then $\mathbf{Hom}(X_{\text{dR}}, T)$ will in general be considerably bigger, due to the presence of maps $X_{\text{dR}} \rightarrow BG$ corresponding to G -bundles with irregular connections. For example there are nontrivial such maps when $X = \mathbb{A}^1$. The de Rham shape, restricted to the 1-truncated stacks, recovers the information of the differential Galois group. We feel that the higher de Rham shape should lead to an interesting extension of differential Galois theory.

One might want to consider coefficient stacks T that are not necessarily "connected". Then $\pi_0(T)$ is a sheaf of sets on $\text{Aff}_{\mathbb{C}}^{\text{ft}, \text{et}}$. It is somewhat more subtle to write down the conditions analogous to Definition 7.3. One may in some cases still obtain a good structure on the de Rham cohomology, such as the case suggested by Bertrand Toën of coefficients in the stack $\mathbf{Perf}^{\leq b}$ that we looked at above. The cohomology stack

$$\mathbf{Hom}(X_{\text{dR}}, \mathbf{Perf}^{\leq b})$$

has a rich structure not yet completely elucidated [39]. For quasiprojective X it should give a new differential Galois invariant.

8. PROSPECTS

The theory of stacks has shown its usefulness in a wide variety of geometrical situations. Following current thinking that integrates a homotopical direction into geometry, higher stacks will find their place and contribute.

The theory of higher stacks involves a combination of working relative to a Grothendieck site, and looking at functors with values in higher homotopical structures such as ∞ -categories. This allows us to perceive a rich and intricate interplay between homotopy, algebra and geometry.

These inputs come at different scales. In the application of stacks to moduli problems in algebraic geometry, the homotopical information is often concentrated at the infinitesimally tiny scale of points or subvarieties where the parametrized objects acquire extra automorphisms; but in the case of a gerbe we may think of a small scale homotopical phenomenon spread out over the variety and manifesting global cohomological behavior. In the other direction, the theory of stacks allows us to enrich homotopy theory by providing algebraic structure to homotopy types. Here, the component pieces of a homotopy type, for example the homotopy groups, gain structure of algebraic group schemes.

The mixture of geometrical and homotopical directions leads to new ideas: higher stacks become the natural coefficient systems for higher nonabelian cohomology, leading to new geometrical structures spreading across the homotopical directions. The enrichment to an algebraic structure allows us to consider nonabelian de Rham cohomology. The shape theory implied by the schematization of homotopy types can allow us to get a new and useful viewpoint on the structure of non-simply connected homotopy types.

We have seen a number of specific questions that seem interesting for future research. These include, of course, the abstract questions about further development of the foundations and theory. Let us think instead about some more explicit geometrical questions.

One is the structure of substacks of the moduli space of stable curves. Substacks can be generically nontrivial gerbes, and little is understood about the classification of what possibilities can occur here. On the boundary, the same question takes on a distinctly combinatorial feel.

What does the moduli 2-stack of DM-curves look like?

Similar questions for moduli stacks of vector bundles and other sheaves move already from the realm of Deligne-Mumford stacks to Artin 1-stacks where the stabilizer groups can be positive dimensional. Understanding the local geometry here is already significantly more complicated; yet there still remain important questions about classification of gerbes that can occur.

The fine structure of Artin 1-stacks gives an internal approach to various aspects of geometric invariant theory [19].

As we move on to higher stacks, a first main question is to understand what kinds of Artin n -stacks can be produced by natural constructions such as moduli of objects in dg-categories.

The moduli stacks of perfect complexes $\mathbf{Perf}^{\leq b}$ are very natural first examples of n -stacks with only finitely many closed points. In this case, the stabilizer groups of the closed points are all of the form GL_r (more precisely they are higher groups whose 0-truncations are GL_r). Are there other nice examples of n -stacks with finitely many closed points, say with other groups appearing as stabilizers?

Once we have some basic examples, it opens up the question of whether more general Artin n -stacks can be mapped into our example stacks such as $\mathbf{Perf}^{\leq b}$, and if such maps can form an embedding, say when combined with maps to ordinary projective space. For Deligne-Mumford stacks, this is the notion of *generating sheaf* [31] [29].

The classification question relates to nonabelian cohomology: we can ask what kinds of Artin n -stacks occur as nonabelian cohomology stacks, depending on the domain variety and the target coefficient stack. The notions of shifted symplectic and Poisson structures [33] [11] provide important constraints.

We can then ask how the position of nonabelian cohomology stacks in this classification relates to the geometry and homotopy theory of the domain variety. Indeed, this classification interacts with the construction problem for homotopy types of algebraic varieties. Information on the range of possibilities for the nonabelian cohomology stacks of algebraic varieties (satisfying various conditions such as projective and smooth) should give information about the special properties of their homotopy theory.

This aspect is closely related to nonabelian Hodge theory: higher stacks, as natural coefficient systems for nonabelian cohomology, need to be given further data in order to generate nonabelian Hodge structures on the cohomology. The foundational structure of what this data should look like, and then the geometry of the resulting cohomology, are basic areas open for further study.

A fundamental question raised by the present project is, how do stacks and the various roles they play in the study of geometry, interact with the new notions of space presented in the other chapters of this book?

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