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# Equivalent one-dimensional first-order linear hyperbolic systems and range of the minimal null control time with respect to the internal coupling matrix

Long Hu\*      Guillaume Olive†

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## Abstract

In this paper, we are interested in the minimal null control time of one-dimensional first-order linear hyperbolic systems by one-sided boundary controls. Our main result is an explicit characterization of the smallest and largest values that this minimal null control time can take with respect to the internal coupling matrix. In particular, we obtain a complete description of the situations where the minimal null control time is invariant with respect to all the possible choices of internal coupling matrices. The proof relies on the notion of equivalent systems, in particular the backstepping method, a canonical  $LU$ -decomposition for boundary coupling matrices and a compactness-uniqueness method adapted to the null controllability property.

**Keywords.** Hyperbolic systems, Minimal null control time, Equivalent systems, Backstepping method,  $LU$ -decomposition, Compactness-uniqueness method

**2010 Mathematics Subject Classification.** 35L40, 93B05

## 1 Introduction and main result

### 1.1 Problem description

In this article we are interested in the null controllability properties of the following class of one-dimensional first-order linear hyperbolic systems, which appears for instance in linearized Saint-Venant equations and many other physical models of balance laws (see e.g. [BC16, Chapter 1] and many references therein):

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) + \Lambda(x) \frac{\partial y}{\partial x}(t, x) = M(x)y(t, x), \\ y_-(t, 1) = u(t), \quad y_+(t, 0) = Qy_-(t, 0), \\ y(0, x) = y^0(x). \end{cases} \quad (1)$$

In (1),  $t > 0$ ,  $x \in (0, 1)$ ,  $y(t, \cdot)$  is the state at time  $t$ ,  $y^0$  is the initial data and  $u(t)$  is the control at time  $t$ . We denote by  $n \geq 2$  the total number of equations of the system. The matrix

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$\Lambda \in C^{0,1}([0, 1])^{n \times n}$  is assumed to be diagonal:

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n), \quad (2)$$

with  $m \geq 1$  negative speeds and  $p \geq 1$  positive speeds ( $m + p = n$ ) such that:

$$\lambda_1(x) < \dots < \lambda_m(x) < 0 < \lambda_{m+1}(x) < \dots < \lambda_{m+p}(x), \quad \forall x \in [0, 1]. \quad (3)$$

Finally, the matrix  $M \in L^\infty(0, 1)^{n \times n}$  couples the equations of the system inside the domain and the constant matrix  $Q \in \mathbb{R}^{p \times m}$  couples the equations of the system on the boundary  $x = 0$ .

All along this paper, for a vector (or vector-valued function)  $v \in \mathbb{R}^n$  and a matrix (or matrix-valued function)  $A \in \mathbb{R}^{n \times n}$ , we use the notation

$$v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix}, \quad A = \begin{pmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{pmatrix},$$

where  $v_- \in \mathbb{R}^m$ ,  $v_+ \in \mathbb{R}^p$  and  $A_{--} \in \mathbb{R}^{m \times m}$ ,  $A_{-+} \in \mathbb{R}^{m \times p}$ ,  $A_{+-} \in \mathbb{R}^{p \times m}$ ,  $A_{++} \in \mathbb{R}^{p \times p}$ .

We recall that the system (1) is well posed in  $(0, T)$  for every  $T > 0$ : for every  $y^0 \in L^2(0, 1)^n$  and  $u \in L^2(0, T)^m$ , there exists a unique solution

$$y \in C^0([0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(0, T)^n)$$

to the system (1). By solution we mean “solution along the characteristics”, this will be detailed in Section 2 below.

The regularity  $C^0([0, T]; L^2(0, 1)^n)$  of the solution allows us to consider control problems in the space  $L^2(0, 1)^n$ :

**Definition 1.1.** Let  $T > 0$ . We say that the system (1) is:

- **exactly controllable in time  $T$**  if, for every  $y^0, y^1 \in L^2(0, 1)^n$ , there exists  $u \in L^2(0, T)^m$  such that the corresponding solution  $y$  to the system (1) in  $(0, T)$  satisfies  $y(T, \cdot) = y^1$ .
- **null controllable in time  $T$**  if the previous property holds at least for  $y^1 = 0$ .

Clearly, exact controllability implies null controllability, but the converse is not true in general.

These notions also depend on the time  $T$  and, since controllability in time  $T_1$  implies controllability in time  $T_2$  for every  $T_2 \geq T_1$ , it is natural to try to find the smallest possible control time, the so-called “minimal control time”. This problem was recently completely solved in [HO21a] for the notion of exact controllability and we will investigate here what happens for the null controllability.

**Definition 1.2.** For any  $\Lambda, M$  and  $Q$  as above, we denote by  $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$  the minimal null control time of the system (1), that is

$$T_{\text{inf}}(\Lambda, M, Q) = \inf \{ T > 0 \mid \text{the system (1) is null controllable in time } T \}.$$

The time  $T_{\text{inf}}(\Lambda, M, Q)$  is named “minimal” null control time according to the current literature, despite it is not always a minimal element of the set. We keep this naming here, but we use the notation with the “inf” to avoid eventual confusions. The time  $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$  is thus the unique time that satisfies the following two properties:

- If  $T > T_{\text{inf}}(\Lambda, M, Q)$ , then the system (1) is null controllable in time  $T$ .
- If  $T < T_{\text{inf}}(\Lambda, M, Q)$ , then the system (1) is not null controllable in time  $T$ .

Finally, let us introduce the elementary times  $T_1(\Lambda), \dots, T_n(\Lambda)$  defined by

$$T_i(\Lambda) = \begin{cases} \int_0^1 \frac{1}{-\lambda_i(\xi)} d\xi & \text{if } i \in \{1, \dots, m\}, \\ \int_0^1 \frac{1}{\lambda_i(\xi)} d\xi & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (4)$$

For the rest of this article it is important to keep in mind that the assumption (3) implies the following order relation among  $T_i(\Lambda)$ :

$$\begin{cases} T_1(\Lambda) \leq \dots \leq T_m(\Lambda), \\ T_{m+p}(\Lambda) \leq \dots \leq T_{m+1}(\Lambda). \end{cases} \quad (5)$$

## 1.2 The LCU decomposition

An important feature of the present article is that no assumption will be required on the boundary coupling matrices  $Q$ . To be able to handle such a general case and state our main result we introduce a notion of canonical form.

**Definition 1.3.** We say that a matrix  $Q^0 \in \mathbb{R}^{p \times m}$  is in canonical form if either  $Q^0 = 0$  or there exist an integer  $\rho \geq 1$ , row indices  $r_1, \dots, r_\rho \in \{1, \dots, p\}$  with  $r_1 < \dots < r_\rho$  and distinct column indices  $c_1, \dots, c_\rho \in \{1, \dots, m\}$  such that

$$\begin{cases} q_{ij}^0 = 1 & \text{if } (i, j) \in \{(r_1, c_1), \dots, (r_\rho, c_\rho)\}, \\ q_{ij}^0 = 0 & \text{otherwise.} \end{cases}$$

For  $Q^0 = 0$  we set  $\rho = 0$  for convenience.

Note that we necessarily have  $\rho = \text{rank } Q^0$ .

**Example 1.4.** *The matrices*

$$Q_1^0 = \begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ \boxed{1} & 0 & 0 \end{pmatrix}, \quad Q_2^0 = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{pmatrix},$$

are both in canonical form, with

$$\begin{aligned} \text{for } Q_1^0: & \quad (r_1, c_1) = (1, 2), \quad (r_2, c_2) = (2, 3), \quad (r_3, c_3) = (4, 1), \\ \text{for } Q_2^0: & \quad (r_1, c_1) = (1, 1), \quad (r_2, c_2) = (2, 2), \quad (r_3, c_3) = (4, 3). \end{aligned}$$

Using the Gaussian elimination we can transform any matrix into a canonical form, this is what we will call in this article the “LCU decomposition” (for Lower–Canonical–Upper decomposition). More precisely, we have

**Proposition 1.5.** *For every  $Q \in \mathbb{R}^{p \times m}$ , there exists a unique  $Q^0 \in \mathbb{R}^{p \times m}$  such that the following two properties hold:*

- (i) *There exists an upper triangular matrix  $U \in \mathbb{R}^{m \times m}$  with only ones on its diagonal and there exists an invertible lower triangular matrix  $L \in \mathbb{R}^{p \times p}$  such that*

$$LQU = Q^0.$$

(ii)  $Q^0$  is in canonical form.

We call  $Q^0$  the canonical form of  $Q$ .

We mention that, because of possible zero rows or columns of  $Q^0$ , the matrices  $L$  and  $U$  are in general not unique.

With this proposition, we can extend the definition of the indices  $(r_1, c_1), \dots, (r_\rho, c_\rho)$  to any nonzero matrix:

**Definition 1.6.** For any nonzero matrix  $Q \in \mathbb{R}^{p \times m}$ , we denote by  $(r_1, c_1), \dots, (r_\rho, c_\rho)$  the positions of the nonzero entries of its canonical form ( $r_1 < \dots < r_\rho$ ).

As previously mentioned, the existence of the *LCU* decomposition is a direct consequence of the Gaussian elimination, where the matrix  $U$  corresponds to rightward column substitutions, whereas the matrix  $L$  corresponds to downward row substitutions and then normalization to 1 of the remaining nonzero entries. Let us present some examples which will make this point clearer.

**Example 1.7.** We illustrate how to find the decomposition of Proposition 1.5 in practice. Consider

$$Q_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 5 \\ 0 & 1 & 2 \\ 4 & -4 & 4 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 5 & -1 & 8 \\ 0 & 1 & 1 & 1 \\ -1 & 3 & 6 & 4 \end{pmatrix}.$$

Let us deal with  $Q_1$  first. We look at the first row, we take the first nonzero entry as pivot. We remove the entries to the right on the same row by doing the column substitution  $C_3 \leftarrow C_3 - 2C_2$ , which gives

$$Q_1 U_1 = Q_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ 4 & -4 & 12 \end{pmatrix}. \quad (6)$$

We now look at the next row and take as new pivot the first nonzero entry that is not in the column of the previous pivot, that is, not in  $C_2$ . Since there is no entry to the right of this new pivot, there is nothing to do and we move to the next row. Since this next row has no nonzero element which is not in  $C_2, C_3$ , we move again to the next and last row. We take as new pivot the first nonzero entry that is not in  $C_2$  or  $C_3$  and we remove the entries to the right on the same row by doing the column substitutions  $C_2 \leftarrow C_2 + C_1$  and  $C_3 \leftarrow C_3 - 3C_1$ , which gives

$$Q_1 U_1 U_2 = Q_1 U_1 \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Working then on the rows with downward substitutions only (starting with the first row) and finally normalizing to 1 the remaining nonzero entries, we see that  $Q_1$  becomes  $Q_1^0$  of Example 1.4. Similarly, it can be checked that the canonical form of  $Q_2$  is in fact  $Q_2^0$  of Example 1.4.

**Remark 1.8.** Observe that we only need to compute the matrix  $U$  in order to find the indices  $(r_1, c_1), \dots, (r_\rho, c_\rho)$ .

The uniqueness of the *LCU* decomposition is less straightforward and we refer for instance to the arguments in the proof of [DJM06, Theorem 1] or to [HO21a, Appendix A] for a proof.

**Remark 1.9.** In the Gaussian elimination process described above, we absolutely do not want to perform any permutation of the rows. This is because we have ordered the speeds of our system in a particular way (recall (3)). The fact that we use right multiplication by upper triangular matrices and left multiplication by lower triangular matrices is also dictated by this choice of order (for instance in [HO21a] the speeds were ordered differently and right multiplication by lower triangular matrices was considered instead).

### 1.3 Literature

Boundary controllability of one-dimensional first-order hyperbolic systems has been widely investigated since the late 1960s. Two pioneering works are [Rus67] and the celebrated survey paper [Rus78], in which the author established the null controllability of the system (1) in any time  $T \geq T_{[\text{Rus}]}(\Lambda)$ , where  $T_{[\text{Rus}]}(\Lambda)$  is the sum of the largest transport times from opposite directions, that is,

$$T_{[\text{Rus}]}(\Lambda) = T_{m+1}(\Lambda) + T_m(\Lambda).$$

An important feature of this result is that it is valid whatever are the internal and boundary coupling matrices  $M$  and  $Q$ . In other words, the time  $T_{[\text{Rus}]}(\Lambda)$  gives an upper bound for the minimal null control time  $T_{\text{inf}}(\Lambda, M, Q)$  with respect to these matrices. It is also easy to see that that upper bound is reached (simply take  $M = 0$  and  $Q$  the matrix whose entries are all equal to zero except for  $q_{1,m} = 1$ ). However, for most of the matrices  $M$  and  $Q$ , this upper bound is too large. Indeed, by just slightly restricting the class of such matrices (in particular, for  $Q$ ), it is possible to have a strictly better upper bound than  $T_{[\text{Rus}]}(\Lambda)$ .

This fact was already observed in [Rus78], where the author tried to find the minimal null control time in the particular case of conservation laws ( $M = 0$ ), rightly by looking more closely at the properties of the boundary coupling matrix. He could not solve this problem though and he left it as an open problem ([Rus78, Remark p. 656]). This was eventually solved few years later in [Wec82], where the author gave an explicit expression of the minimal null control time in terms of some indices related to  $Q$ .

Concerning systems of balance laws ( $M \neq 0$ ), finding the minimal null control time for arbitrary  $M$  and  $Q$  is still an open challenging problem. Recently, there has been a resurgence on the characterization of such a time. A first result in this direction has been obtained in [CN19] with an improvement of the upper bound  $T_{[\text{Rus}]}(\Lambda)$  for a certain class of boundary coupling matrices  $Q$ . More precisely, they considered the class  $\mathcal{B}$  defined by

$$\mathcal{B} = \{Q \in \mathbb{R}^{p \times m} \mid (8) \text{ is satisfied for every } i \in \{1, \dots, \min\{p, m-1\}\}\}, \quad (7)$$

where the condition (8) is:

$$\text{the } i \times i \text{ matrix formed from the first } i \text{ rows and the first } i \text{ columns of } Q \text{ is invertible,} \quad (8)$$

(it is understood that the set  $\mathcal{B}$  is empty when  $m = 1$ ). For this class of boundary coupling matrices, the authors then showed that the upper bound  $T_{[\text{Rus}]}(\Lambda)$  can be reduced to the time  $T_{[\text{CN}]}(\Lambda)$  defined by

$$T_{[\text{CN}]}(\Lambda) = \begin{cases} \max \left\{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_k(\Lambda), T_m(\Lambda) \right\} & \text{if } m \geq p, \\ \max_{k \in \{1, \dots, m\}} T_{m+k}(\Lambda) + T_k(\Lambda) & \text{if } m < p. \end{cases} \quad (9)$$

This was first done for some generic internal coupling matrices or under rather stringent conditions ([CN19, Theorem 1.1 and 1.5]) but the same authors were then able to remove these restrictions in [CN21a, Theorem 1 and 3].

On the other hand, when the boundary coupling matrix  $Q$  is full row rank, the problem of finding the minimal null control time, and not only an upper bound, has also been recently completely solved in [HO21a]. More precisely, it is proved in [HO21a, Theorem 1.12 and Remark 1.3] that

$$\text{rank } Q = p \implies T_{\text{inf}}(\Lambda, M, Q) = \max \left\{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), T_m(\Lambda) \right\},$$

where we recall that the indices  $c_k$  are defined in Definition 1.6. We see in this case that the minimal null control time has the remarkable property to be independent of the internal coupling matrix  $M$ . In particular, this is the same time as the one found for conservation laws in [Wec82], yet with a more explicit expression. For  $m > p$ , this generalizes the aforementioned results of [CN19, CN21a] in two ways: firstly, this is a result for arbitrary full row rank boundary coupling matrices (not only for  $Q \in \mathcal{B}$ ) and, secondly, this obviously establishes that no better time can be obtained (even for  $Q \in \mathcal{B}$  this is not proved in [CN19, CN21a]). We mention this because the results of the present paper will share these two features.

For the special case of  $2 \times 2$  systems, the minimal null control time has also been found in [CVKB13] when the boundary coupling matrix (which is then a scalar) is not zero and in [HO21b] when the boundary coupling is reduced to zero. Notably, in the second situation, the minimal null control time depends on the behavior of the internal coupling matrix  $M$  ([HO21b, Theorem 1.5]).

Finally, we would like to mention the related works [CHOS21, CN21b, AKM21] concerning time-dependent systems and [Li10, LR10, Hu15, CN20a, CN20b] for quasilinear systems.

As we have discussed, finding what exactly is the minimal null control time turns out to be a difficult task. Instead, in this article we propose to look for the smallest and largest values that the minimal null control time  $T_{\text{inf}}(\Lambda, M, Q)$  can take with respect to the internal coupling matrix  $M$ . Our main result is an explicit and easy-to-compute formula for both of these times. We will also completely characterize all the parameters  $\Lambda$  and  $Q$  for which  $T_{\text{inf}}(\Lambda, M, Q)$  is invariant with respect to all  $M \in L^\infty(0, 1)^{n \times n}$ . We will show that our results generalize all the known works that have been previously quoted. In the course of the proof we will obtain some new results even for conservation laws ( $M = 0$ ), notably with an explicit feedback law stabilizing the system in the minimal time.

Our proof relies on the notion of equivalent systems, in particular the backstepping method with the results of [HDMVK16, HVDMK19], the introduction of a canonical  $LU$ -decomposition for boundary coupling matrix  $Q$  in the same spirit as in [HO21a], as well as a compactness-uniqueness method adapted to the null controllability inspired from the works [CN21a, DO18].

## 1.4 Main result and comments

As we have seen in the previous section, to explicitly characterize  $T_{\text{inf}}(\Lambda, M, Q)$  for arbitrary  $M$  and  $Q$  is still a challenging open problem. Instead, we propose to find the smallest and largest values that it can take with respect to the internal coupling matrix  $M$ .

**Definition 1.10.** We define

$$\begin{aligned} T_{\text{inf}}(\Lambda, Q) &= \inf \left\{ T_{\text{inf}}(\Lambda, M, Q) \mid M \in L^\infty(0, 1)^{n \times n} \right\}, \\ T_{\text{sup}}(\Lambda, Q) &= \sup \left\{ T_{\text{inf}}(\Lambda, M, Q) \mid M \in L^\infty(0, 1)^{n \times n} \right\}. \end{aligned}$$

The main result of the present paper is the following explicit characterization of these two quantities:

**Theorem 1.11.** Let  $\Lambda \in C^{0,1}([0, 1])^{n \times n}$  satisfy (2)-(3) and let  $Q \in \mathbb{R}^{p \times m}$  be fixed.

(i) We have

$$T_{\inf}(\Lambda, Q) = \max \left\{ \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+1}(\Lambda), \quad T_m(\Lambda) \right\}, \quad (10)$$

where we recall that the indices  $(r_k, c_k)$  are defined in Definition 1.6.

(ii) We have

$$T_{\sup}(\Lambda, Q) = \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\}, \quad (11)$$

where  $\rho_0$  is the largest integer  $i \in \{1, \dots, p\}$  such that

the  $i \times m$  matrix formed from the first  $i$  rows of  $Q$  has rank  $i$ ,

with  $\rho_0 = 0$  if the first row of  $Q$  is equal to zero.

In the statement of Theorem 1.11, we use the convention that the undefined quantities are simply not taken into account, which more precisely gives:

- If  $\rho = 0$ , then  $T_{\inf}(\Lambda, 0) = \max \{T_{m+1}(\Lambda), \quad T_m(\Lambda)\}$ .
- If  $\rho_0 = 0$ , then  $T_{\sup}(\Lambda, Q) = T_{m+1}(\Lambda) + T_m(\Lambda)$ .
- If  $\rho_0 = p$ , then  $T_{\sup}(\Lambda, Q) = \max \{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_m(\Lambda) \}$ .

An equivalent definition of  $\rho_0$  is (when the first row of  $Q$  is not equal to zero)

$$\rho_0 = \max \{i \in \{1, \dots, p\} \mid r_k = k, \quad \forall k \in \{1, \dots, i\}\}. \quad (12)$$

We emphasize that  $\rho_0$  is defined for any  $Q$ , it is not a condition like  $Q \in \mathcal{B}$ .

By investigating the possibilities of equality  $T_{\inf}(\Lambda, Q) = T_{\sup}(\Lambda, Q)$ , we obtain the following important consequence of Theorem 1.11:

**Corollary 1.12.** Let  $\Lambda \in C^{0,1}([0, 1])^{n \times n}$  satisfy (2)-(3) and let  $Q \in \mathbb{R}^{p \times m}$  be fixed. The map  $M \mapsto T_{\inf}(\Lambda, M, Q)$  is constant on  $L^\infty(0, 1)^{n \times n}$  if, and only if,  $\Lambda$  and  $Q$  satisfy

$$\rho_0 = p \quad \text{or} \quad \left( 0 < \rho_0 < p \quad \text{and} \quad \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda) \geq T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right). \quad (13)$$

**Remark 1.13.** In the proof of Theorem 1.11, we will show in fact that the infimum in  $T_{\inf}(\Lambda, Q)$  and the supremum in  $T_{\sup}(\Lambda, Q)$  are reached for some special matrices  $M$ . More precisely, we will show that:

- The infimum in  $T_{\inf}(\Lambda, Q)$  is reached for  $M = 0$ . In particular, this shows that  $T_{\inf}(\Lambda, M, Q) \geq T_{\inf}(\Lambda, 0, Q)$  for any  $M$ .
- The supremum in  $T_{\sup}(\Lambda, Q)$  is reached for  $M = 0$  if the condition (13) holds.

- If the condition (13) fails, then the supremum in  $T_{\text{sup}}(\Lambda, Q)$  is reached for the matrix  $M$  whose entries are all equal to zero, except for

$$m_{m+i,m}(x) = \frac{\lambda_{m+i}(x) - \lambda_m(x)}{-\lambda_m(x)} \ell^{i, \rho_0+1}, \quad \forall i \in \{\rho_0 + 1, \dots, p\},$$

where  $L^{-1} = (\ell^{ij})_{1 \leq i, j \leq p}$  and  $L$  is any matrix  $L$  coming from the *LCU* decomposition of  $Q$ .

**Remark 1.14.** For a subset  $E \subset L^\infty(0, 1)^{n \times n}$ , let us define

$$\begin{aligned} T_{\text{inf}}(\Lambda, Q)|_E &= \inf \{T_{\text{inf}}(\Lambda, M, Q) \mid M \in E\}, \\ T_{\text{sup}}(\Lambda, Q)|_E &= \sup \{T_{\text{inf}}(\Lambda, M, Q) \mid M \in E\}. \end{aligned}$$

Then, it follows from Remark 1.13 that we also have

$$\begin{aligned} T_{\text{inf}}(\Lambda, Q)|_E &= T_{\text{inf}}(\Lambda, Q), \\ T_{\text{sup}}(\Lambda, Q)|_E &= T_{\text{sup}}(\Lambda, Q), \end{aligned}$$

for every subset  $E$  that contains the matrix  $M = 0$  and the matrix  $M$  described in the third item of this remark.

For instance, if the speeds are constant, then  $T_{\text{inf}}(\Lambda, Q)$  and  $T_{\text{sup}}(\Lambda, Q)$  are also the lower and upper bounds of the minimal null control times among all the constant matrices  $M$ .

**Example 1.15.** Consider  $Q_1 \in \mathbb{R}^{4 \times 3}$  and  $Q_2 \in \mathbb{R}^{4 \times 4}$  of Example 1.7. For these matrices, the previous results of the literature [CN19, CN21a, HO21a] cannot be applied:  $Q_1, Q_2 \notin \mathcal{B}$  and  $\text{rank } Q_1, Q_2 = 3 < p = 4$ . We have (recall (5))

$$\begin{aligned} T_{\text{inf}}(\Lambda, Q_1) &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda), \quad T_7(\Lambda) + T_1(\Lambda), \quad T_4(\Lambda), \quad T_3(\Lambda)\} \\ &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda)\}, \end{aligned}$$

$$\begin{aligned} T_{\text{sup}}(\Lambda, Q_1) &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda), \quad T_6(\Lambda) + T_3(\Lambda)\} \\ &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda)\} \\ &= T_{\text{inf}}(\Lambda, Q_1), \end{aligned}$$

and

$$\begin{aligned} T_{\text{inf}}(\Lambda, Q_2) &= \max \{T_5(\Lambda) + T_1(\Lambda), \quad T_6(\Lambda) + T_2(\Lambda), \quad T_8(\Lambda) + T_3(\Lambda), \quad T_5(\Lambda), \quad T_4(\Lambda)\} \\ &= \max \{T_5(\Lambda) + T_1(\Lambda), \quad T_6(\Lambda) + T_2(\Lambda), \quad T_8(\Lambda) + T_3(\Lambda), \quad T_4(\Lambda)\}, \end{aligned}$$

$$T_{\text{sup}}(\Lambda, Q_2) = \max \{T_5(\Lambda) + T_1(\Lambda), \quad T_6(\Lambda) + T_2(\Lambda), \quad T_7(\Lambda) + T_4(\Lambda)\}.$$

**Remark 1.16.** If during the computations of the indices  $(r_k, c_k)$  we arrive at the last column, that is if we have

$$c_{k_0} = m, \tag{14}$$

for some  $k_0 \in \{1, \dots, p\}$ , then there is no need to find the next indices to be able to compute  $T_{\text{inf}}(\Lambda, Q)$  and  $T_{\text{sup}}(\Lambda, Q)$  since we know that the corresponding times will not be taken into account (because of (5)). For instance, for the matrix  $Q_1$  of Example 1.7 we can stop after the very first step (6) since it gives  $c_2 = 3$ , there is no need to go on and compute  $U_2$ .

**Remark 1.17.** Theorem 1.11 and its corollary generalize all the results of the literature that we are aware of on the null controllability of systems of the form (1) (except for the special case  $n = 2$ , which has been completely solved in [CVKB13, HO21b]):

- When the matrix  $Q$  is full row rank, that is,

$$\text{rank } Q = p, \quad (15)$$

exact and null controllability are equivalent properties for the system (1) (see e.g. [HO21a, Remark 1.3]) and it has been shown in [HO21a, Theorem 4.1] that  $T_{\text{inf}}(\Lambda, M, Q)$  is independent of  $M$  in that situation. Under the rank condition (15), it is clear that  $\rho_0 = p$  and the condition (13) is thus satisfied. It then follows from Corollary 1.12 that  $T_{\text{inf}}(\Lambda, M, Q)$  is independent of  $M$ . Therefore, our result encompasses the one of [HO21a].

- When  $m \leq p$  and  $Q \in \mathcal{B}$  (defined in (7)), it has been established in [CN21a, Theorem 1] that

$$T_{\text{sup}}(\Lambda, Q) \leq T_{[\text{CN}]}(\Lambda),$$

where we recall that  $T_{[\text{CN}]}(\Lambda)$  is given by (9). In that case, we see that  $r_k = c_k = k$  for every  $k \in \{1, \dots, m-1\}$  and either  $\rho_0 = m-1$  or  $\rho_0 = m$ . In all cases, we can check that

$$\max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\} = T_{[\text{CN}]}(\Lambda).$$

Therefore, item (ii) of Theorem 1.11 generalizes [CN21a, Theorem 1], which corresponded only to the inequality “ $\leq$ ” and only valid for matrices  $Q \in \mathcal{B}$ , but excluded for instance the matrices presented in Example 1.7.

- In fact, when  $\rho_0 = m$  in the previous point, the minimal null control time does not depend on  $M$ . More generally, if the condition (14) holds for some  $k_0 \leq \rho_0$ , then the condition (13) is satisfied (because of (5)) and it follows from Corollary 1.12 that

$$T_{\text{inf}}(\Lambda, M, Q) = \max_{k \in \{1, \dots, k_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda).$$

For instance, this condition is satisfied when the matrix  $Q$  has the block decomposition

$$Q = \begin{pmatrix} Q' \\ Q'' \end{pmatrix}, \quad \text{rank } Q' = m,$$

where  $Q' \in \mathbb{R}^{m \times m}$  and  $Q'' \in \mathbb{R}^{(p-m) \times m}$ .

## 1.5 Equivalent systems

The proof of our main result will first consist in transforming our initial system (1) into “equivalent” systems (from a controllability point of view) which have a simpler coupling structure. Let us make this notion of equivalent systems precise here. We will introduce it for a slightly broader class of systems than (1) because of the nature of the transformations that we will use in the sequel, this will be clear from Section 3. All the systems of this paper will have the following form:

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) + \Lambda(x) \frac{\partial y}{\partial x}(t, x) = M(x)y(t, x) + G(x)y_-(t, 0), \\ y_-(t, 1) = u(t), \quad y_+(t, 0) = Qy_-(t, 0), \\ y(0, x) = y^0(x), \end{cases} \quad (16)$$

where  $M \in L^\infty(0, 1)^{n \times n}$  and  $Q \in \mathbb{R}^{p \times m}$  as before, and  $G \in L^\infty(0, 1)^{n \times m}$ . Therefore, (16) is similar to (1) but it has the extra term with  $G$ . This system is well posed and the notions of controllability are similarly defined (see Section 2 below).

In what follows, we will refer to a system of the general form (16) as

$$(\Lambda, M, Q, G).$$

When a system does not contain a parameter ( $M$  or  $G$ ) we will use the notation  $-$  rather than writing 0, for instance we will use  $(\Lambda, M, Q, -)$  when the system does not contain  $G$ . The minimal null control time of the system  $(\Lambda, M, Q, G)$  will be denoted by  $T_{\text{inf}}(\Lambda, M, Q, G)$  (for consistency, we will keep using the notation  $T_{\text{inf}}(\Lambda, M, Q)$  rather than  $T_{\text{inf}}(\Lambda, M, Q, -)$ ).

Let us now give the precise definition of what we mean by equivalent systems in this work:

**Definition 1.18.** We say that two systems  $(\Lambda, M_1, Q_1, G_1)$  and  $(\Lambda, M_2, Q_2, G_2)$  are equivalent, and we write

$$(\Lambda, M_1, Q_1, G_1) \sim (\Lambda, M_2, Q_2, G_2),$$

if there exists an invertible bounded linear transformation

$$L : L^2(0, 1)^n \longrightarrow L^2(0, 1)^n,$$

such that, for every  $T > 0$ , the induced map  $\tilde{L} : C^0([0, T]; L^2(0, 1)^n) \longrightarrow C^0([0, T]; L^2(0, 1)^n)$  defined by  $(\tilde{L}y)(t) = L(y(t))$  for every  $t \in [0, T]$  satisfies

$$\tilde{L}(S_1) = S_2,$$

where  $S_i$  ( $i = 1, 2$ ) denotes the space of all the solutions  $y$  to the system  $(\Lambda, M_i, Q_i, G_i)$  in  $(0, T)$ .

It is not difficult to check that  $\sim$  is an equivalence relation and that two equivalent systems share the same controllability properties:

**Proposition 1.19.** *Let  $(\Lambda, M_1, Q_1, G_1) \sim (\Lambda, M_2, Q_2, G_2)$  be two equivalent systems. Then, for every  $T > 0$ , the system  $(\Lambda, M_1, Q_1, G_1)$  is null controllable in time  $T$  if, and only if, the system  $(\Lambda, M_2, Q_2, G_2)$  is null controllable in time  $T$ .*

In particular, two equivalent systems have the same minimal null control time. However, the converse is not true in general, an example has been detailed in Appendix A.

**Remark 1.20.** Let us emphasize that the notion of equivalent systems that we introduced here does not care how the control from one system is obtained from the control of the other system. It is different from the notion of (feedback) equivalence introduced in the seminal work [Bru70] in finite dimension, which was designed to transfer the stabilization properties of one system to another and thus required a more specific link between the two systems.

## 1.6 Outline of the proof

Since the proof of our main result involves many transformations, let us give a quick overview of the main steps before going into detail:

- 1) **Reduction of  $M$ .** First of all, we show in Section 3 that

$$(\Lambda, M, Q, -) \sim (\Lambda, -, Q, G^0),$$

for some  $G^0$ . It is nothing but a fundamental result of [HVDMK19, HDMVK16] that we rephrase here with the notion of equivalent systems. Consequently, we only have to focus on systems of the form  $(\Lambda, -, Q, G)$  in the sequel, which have the advantage of having a simpler coupling structure.

- 2) **Reduction of  $Q$ .** In Section 4, we show that the boundary coupling matrix  $Q$  can always be assumed in canonical form (Definition 1.3):

$$(\Lambda, -, Q, G^0) \sim (\Lambda, -, Q^0, G^1),$$

for some  $G^1$ . This is an important step that greatly simplifies the coupling structure of the system.

- 3) **Characterization of  $T_{\inf}(\Lambda, Q)$ .** The previous step allows us to characterize in Section 5 the smallest value of the minimal null control time. More precisely, we first establish that

$$\inf \{ T_{\inf}(\Lambda, -, Q^0, G^1) \mid G^1 \in L^\infty(0, 1)^{n \times m} \}$$

is equal to the quantity on the right-hand side of the equality (10). This is done by using a similar argument to the one in [HO21b]. We then show how to deduce the corresponding result for the initial system  $(\Lambda, M, Q, -)$ , thus proving the first part of our main result.

- 4) **Reduction of  $G_{+-}$ .** In view of the proof of the second part of our main result, we first show in Section 6 how to use the canonical form of  $Q^0$  to prove that

$$(\Lambda, -, Q^0, G^1) \sim (\Lambda, -, Q^0, G^2), \quad G^2 = \begin{pmatrix} G_{--}^1 \\ G_{+-}^2 \end{pmatrix},$$

for some  $G_{+-}^2$  which has the following structure:

$$g_{m+i, c_k}^2 = 0, \quad \forall k \in \{1, \dots, \rho\}, \forall i \geq r_k. \quad (17)$$

- 5) **Removal of  $G_{--}$ .** In Section 7 we then prove that the coupling term  $G_{--}$  has no influence on the minimal null control time:

$$T_{\inf}(\Lambda, -, Q^0, G^2) = T_{\inf}(\Lambda, -, Q^0, G^3), \quad G^3 = \begin{pmatrix} 0 \\ G_{+-}^2 \end{pmatrix}.$$

Unlike all the other steps, the proof is not based on the construction of a suitable transformation, it is based on a general compactness-uniqueness method adapted to the null controllability property and inspired from the previous works [CN21a, DO18].

- 6) **Characterization of  $T_{\sup}(\Lambda, Q)$ .** Finally, in Section 8, we characterize the largest value of the minimal null control time. More precisely, we first show that

$$\sup \left\{ T_{\inf}(\Lambda, -, Q^0, G^3) \mid G^3 = \begin{pmatrix} 0 \\ G_{+-}^2 \end{pmatrix} \text{ and } G_{+-}^2 \text{ satisfies (17)} \right\}$$

is equal to the quantity on the right-hand side of the equality (11). We then show how to deduce the corresponding result for the initial system  $(\Lambda, M, Q, -)$ , thus proving the second part of our main result.

**Remark 1.21.** All the steps described above are constructive, except for the one invoking a compactness-uniqueness argument. It would be interesting to be able to replace this step by a constructive approach (if possible).

## 2 Notations and solution along the characteristics

Before proceeding to the proof of our main result, we introduce in this section some notations and recall some results concerning the well-posedness of the non standard systems of the form (16).

## 2.1 The characteristics

We start with the characteristic curves associated with the system (16).

- First of all, throughout this paper it is convenient to extend  $\lambda_1, \dots, \lambda_n$  to functions of  $\mathbb{R}$  (still denoted by the same) such that  $\lambda_1, \dots, \lambda_n \in C^{0,1}(\mathbb{R})$  and

$$\lambda_1(x) < \dots < \lambda_m(x) \leq -\varepsilon < 0 < \varepsilon \leq \lambda_{m+1}(x) < \dots < \lambda_{m+p}(x), \quad \forall x \in \mathbb{R}, \quad (18)$$

for some  $\varepsilon > 0$  small enough. Since all the results of the present paper depend only on the values of  $\lambda_1, \dots, \lambda_n$  in  $[0, 1]$ , they do not depend on such an extension.

In what follows,  $i \in \{1, \dots, n\}$  is fixed.

- Let  $\chi_i$  be the flow associated with  $\lambda_i$ , i.e. for every  $(t, x) \in \mathbb{R} \times \mathbb{R}$ , the function  $s \mapsto \chi_i(s; t, x)$  is the solution to the ordinary differential equation (ODE)

$$\begin{cases} \frac{\partial \chi_i}{\partial s}(s; t, x) = \lambda_i(\chi_i(s; t, x)), & \forall s \in \mathbb{R}, \\ \chi_i(t; t, x) = x. \end{cases} \quad (19)$$

The existence and uniqueness of a (global) solution to the ODE (19) follows from the (global) Cauchy-Lipschitz theorem (see e.g. [Har02, Theorem II.1.1]). The uniqueness also yields the important group property

$$\chi_i(\sigma; s, \chi_i(s; t, x)) = \chi_i(\sigma; t, x), \quad \forall \sigma, s \in \mathbb{R}. \quad (20)$$

- Let us now introduce the entry and exit times  $s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x) \in \mathbb{R}$  of the flow  $\chi_i(\cdot; t, x)$  inside the domain  $[0, 1]$ , i.e. the respective unique solutions to

$$\begin{cases} \chi_i(s_i^{\text{in}}(t, x); t, x) = 1, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 0, & \text{if } i \in \{1, \dots, m\}, \\ \chi_i(s_i^{\text{in}}(t, x); t, x) = 0, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 1, & \text{if } i \in \{m+1, \dots, n\}. \end{cases}$$

Their existence and uniqueness are guaranteed by the condition (18).

- Since  $\lambda_i$  does not depend on time, we have an explicit formula for the inverse function  $\theta \mapsto \chi_i^{-1}(\theta; t, x)$ . Indeed, it solves

$$\begin{cases} \frac{\partial(\chi_i^{-1})}{\partial \theta}(\theta; t, x) = \frac{1}{\frac{\partial \chi_i}{\partial s}(\chi_i^{-1}(\theta; t, x); t, x)} = \frac{1}{\lambda_i(\theta)}, & \forall \theta \in \mathbb{R}, \\ \chi_i^{-1}(x; t, x) = t, \end{cases}$$

which gives

$$\chi_i^{-1}(\theta; t, x) = t + \int_x^\theta \frac{1}{\lambda_i(\xi)} d\xi.$$

It follows that

$$\begin{cases} s_i^{\text{in}}(t, x) = t - \int_x^1 \frac{1}{-\lambda_i(\xi)} d\xi, & s_i^{\text{out}}(t, x) = t + \int_0^x \frac{1}{-\lambda_i(\xi)} d\xi, & \text{if } i \in \{1, \dots, m\}, \\ s_i^{\text{in}}(t, x) = t - \int_0^x \frac{1}{\lambda_i(\xi)} d\xi, & s_i^{\text{out}}(t, x) = t + \int_x^1 \frac{1}{\lambda_i(\xi)} d\xi, & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (21)$$

- We have the following monotonic properties:

$$\begin{cases} \frac{\partial s_i^{\text{in}}}{\partial t} > 0, & \frac{\partial s_i^{\text{in}}}{\partial x} > 0, & \frac{\partial s_i^{\text{out}}}{\partial t} > 0, & \frac{\partial s_i^{\text{out}}}{\partial x} > 0, & \text{if } i \in \{1, \dots, m\}, \\ \frac{\partial s_i^{\text{in}}}{\partial t} > 0, & \frac{\partial s_i^{\text{in}}}{\partial x} < 0, & \frac{\partial s_i^{\text{out}}}{\partial t} > 0, & \frac{\partial s_i^{\text{out}}}{\partial x} < 0, & \text{if } i \in \{m+1, \dots, n\}, \end{cases} \quad (22)$$

and the following inverse formula, valid for every  $s, t \in \mathbb{R}$ :

$$\begin{cases} s < s_i^{\text{out}}(t, 1) & \iff & s_i^{\text{in}}(s, 0) < t, & \text{if } i \in \{1, \dots, m\}, \\ s < s_i^{\text{out}}(t, 0) & \iff & s_i^{\text{in}}(s, 1) < t, & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (23)$$

- Note as well that (recall (4))

$$T_i(\Lambda) = \begin{cases} s_i^{\text{out}}(0, 1) & \text{if } i \in \{1, \dots, m\}, \\ s_i^{\text{out}}(0, 0) & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (24)$$

- Finally, we introduce the non negative and increasing function  $\phi_i \in C^{1,1}(\mathbb{R})$  defined by

$$\phi_i(x) = \begin{cases} \int_0^x \frac{1}{-\lambda_i(\xi)} d\xi & \text{if } i \in \{1, \dots, m\}, \\ \int_0^x \frac{1}{\lambda_i(\xi)} d\xi & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (25)$$

Note that it is a bijection from  $[0, 1]$  to  $[0, T_i(\Lambda)]$ .

## 2.2 Solution along the characteristics

Let us now introduce the notion of solution for systems of the form (16). To this end, we have to restrict our discussion to the domain where the system evolves, i.e. on  $(0, T) \times (0, 1)$ ,  $T > 0$  being fixed. For every  $(t, x) \in (0, T) \times (0, 1)$ , we have

$$(s, \chi_i(s; t, x)) \in (0, t) \times (0, 1), \quad \forall s \in (\bar{s}_i^{\text{in}}(t, x), t),$$

where we introduced

$$\bar{s}_i^{\text{in}}(t, x) = \max \{0, s_i^{\text{in}}(t, x)\} < t.$$

We now proceed to formal computations in order to introduce the notion of solution for non smooth functions  $y$ . Writing the  $i$ -th equation of the system (16) along the characteristic  $\chi_i(s; t, x)$  for  $s \in [\bar{s}_i^{\text{in}}(t, x), t]$ , and using the chain rules yields the ODE

$$\begin{cases} \frac{d}{ds} y_i(s, \chi_i(s; t, x)) = \sum_{j=1}^n m_{ij}(\chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) + \sum_{j=1}^m g_{ij}(\chi_i(s; t, x)) y_j(s, 0), \\ y_i(\bar{s}_i^{\text{in}}(t, x), \chi_i(\bar{s}_i^{\text{in}}(t, x); t, x)) = b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x), \end{cases} \quad (26)$$

where the initial condition  $b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x)$  is given by the appropriate boundary or initial conditions in (16):

- for  $i \in \{1, \dots, m\}$ ,

$$b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x) = \begin{cases} u_i(s_i^{\text{in}}(t, x)) & \text{if } s_i^{\text{in}}(t, x) > 0, \\ y_i^0(\chi_i(0; t, x)) & \text{if } s_i^{\text{in}}(t, x) < 0, \end{cases} \quad (27)$$

- for  $i \in \{m+1, \dots, n\}$ ,

$$b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x) = \begin{cases} \sum_{j=1}^m q_{i-m,j} y_j(s_i^{\text{in}}(t, x), 0) & \text{if } s_i^{\text{in}}(t, x) > 0, \\ y_i^0(\chi_i(0; t, x)) & \text{if } s_i^{\text{in}}(t, x) < 0. \end{cases} \quad (28)$$

Integrating the ODE (26) over  $s \in [\bar{s}_i^{\text{in}}(t, x), t]$ , we obtain the following system of integral equations:

$$y_i(t, x) = b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x) + \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t, x)}^t m_{ij}(\chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds + \sum_{j=1}^m \int_{\bar{s}_i^{\text{in}}(t, x)}^t g_{ij}(\chi_i(s; t, x)) y_j(s, 0) ds. \quad (29)$$

This leads to the following notion of solution called “solution along the characteristics”:

**Definition 2.1.** Let  $T > 0$ ,  $y^0 \in L^2(0, 1)^n$  and  $u \in L^2(0, T)^m$  be fixed. We say that a function  $y : (0, T) \times (0, 1) \rightarrow \mathbb{R}^n$  is a solution to the system (16) in  $(0, T)$  if

$$y \in C^0([0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(0, T)^n),$$

and if the integral equation (29) is satisfied for every  $i \in \{1, \dots, n\}$  and for a.e.  $(t, x) \in (0, T) \times (0, 1)$ .

Using the Banach fixed-point theorem and suitable estimates, we can establish that the system (16) is globally well posed in this sense:

**Theorem 2.2.** For every  $T > 0$ ,  $y^0 \in L^2(0, 1)^n$  and  $u \in L^2(0, T)^m$ , there exists a unique solution  $y \in C^0([0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(0, T)^n)$  to the system (16) in  $(0, T)$ . Moreover, we have

$$\|y\|_{C^0([0, T]; L^2(0, 1)^n)} + \|y\|_{C^0([0, 1]; L^2(0, T)^n)} \leq C \left( \|y^0\|_{L^2(0, 1)^n} + \|u\|_{L^2(0, T)^m} \right), \quad (30)$$

for some  $C > 0$  that does not depend on  $y^0$  nor  $u$ .

For a proof of this result, we refer for instance to [CHOS21, Appendix A.2] (see also [CN19, Lemma 3.2] in the  $L^\infty$  setting).

### 3 Backstepping transformation

In this section, we use a Volterra transformation of the second kind to transform our initial system (1) into a system with a simpler coupling structure, this is the so-called backstepping method for partial differential equations. The content of this section is quite standard by now (yet, formulated differently here), see for instance [HVDKM19, Section 2.2] (or [CN19, Section 2]).

### 3.1 Removal of the diagonal terms

First of all, we perform a simple preliminary transformation in order to remove the diagonal terms in  $M$ . This is only a technical step, which is nevertheless necessary in view of the existence of the transformation that we will use in the next section, see Remark 3.3 below. For convenience, we introduce the set

$$\mathcal{M} = \{M \in L^\infty(0, 1)^{n \times n} \mid m_{ii} = 0, \quad \forall i \in \{1, \dots, n\}\}.$$

**Proposition 3.1.** *There exists a map  $\Psi : L^\infty(0, 1)^{n \times n} \rightarrow \mathcal{M}$  such that, for every  $M \in L^\infty(0, 1)^{n \times n}$ , we have*

$$(\Lambda, M, Q, -) \sim (\Lambda, \Psi(M), Q, -).$$

*Proof.* • We are going to use the spatial transformation

$$\tilde{y}(t, x) = E(x)y(t, x),$$

where  $E = \text{diag}(e_1, \dots, e_n) \in W^{1, \infty}(0, 1)^{n \times n}$  is the diagonal matrix whose entries are

$$e_i(x) = \exp\left(-\int_0^x \frac{m_{ii}(\xi)}{\lambda_i(\xi)} d\xi\right).$$

Clearly, this transformation is invertible on  $L^2(0, 1)^n$ .

- Assume now that  $y$  is a solution to the system  $(\Lambda, M, Q, -)$  for some  $y^0$  and  $u$  and let us show that  $\tilde{y}$  is then a solution to the system  $(\Lambda, \Psi(M), Q, -)$  for some  $\tilde{y}^0$  and  $\tilde{u}$ , where  $\Psi(M)$  will be determined below. We do it formally but this can be rigorously justified.

- The initial data is obviously  $\tilde{y}^0(x) = E(x)y^0(x)$ .
- The boundary condition at  $x = 0$  is clearly satisfied since  $\tilde{y}(t, 0) = y(t, 0)$ .
- Looking at the boundary condition at  $x = 1$ , the control  $\tilde{u}$  is

$$\tilde{u}(t) = \tilde{y}_-(t, 1) = E_{--}(1)y_-(t, 1).$$

- Using the equation satisfied by  $y$  and the fact that  $\Lambda$  and  $E$  commute, a computation shows that

$$\frac{\partial \tilde{y}}{\partial t}(t, x) + \Lambda(x) \frac{\partial \tilde{y}}{\partial x}(t, x) = \left(E(x)M(x) + \Lambda(x) \frac{\partial E}{\partial x}(x)\right) y(t, x).$$

Consequently,  $\tilde{y}$  satisfies the desired equation if we take

$$(\Psi(M))(x) = \left(E(x)M(x) + \Lambda(x) \frac{\partial E}{\partial x}(x)\right) E(x)^{-1}.$$

Now that  $\Psi$  is clearly identified, similar computations show that, conversely, if  $\tilde{y}$  is a solution to the system  $(\Lambda, \Psi(M), Q, -)$  for some  $\tilde{y}^0$  and  $\tilde{u}$ , then  $y$  is a solution to the system  $(\Lambda, M, Q, -)$  for some  $y^0$  and  $u$ .

- Finally, it is clear that  $\Psi(M) \in \mathcal{M}$  by construction. □

### 3.2 Backstepping transformation

We now recall an important result from [HVDKM19] and [HDMVK16] that we present here using the notion of equivalent system. To this end, we introduce the set

$$\mathcal{F} = \{A \in L^\infty(0, 1)^{n \times n} \mid A_{-+} = A_{+-} = 0\}.$$

**Theorem 3.2.** *For every  $A \in \mathcal{F}$ , there exists a map  $\Gamma_A : \mathcal{M} \rightarrow L^\infty(0, 1)^{n \times m}$  such that, for every  $M \in \mathcal{M}$ , we have*

$$(\Lambda, M, Q, -) \sim (\Lambda, -, Q, \Gamma_A(M)).$$

*Proof.* • We are going to use the spatial transformation

$$\tilde{y}(t, x) = y(t, x) - \int_0^x K(x, \xi) y(t, \xi) d\xi,$$

where  $K \in L^\infty(\mathcal{T})^{n \times n}$  and  $\mathcal{T}$  is the triangle

$$\mathcal{T} = \{(x, \xi) \in (0, 1) \times (0, 1) \mid x > \xi\}.$$

This transformation is always invertible on  $L^2(0, 1)^n$  since it is a Volterra transformation of the second kind (see e.g. [Hoc73, Theorem 2.5]).

- Assume now that  $y$  is a solution to the system  $(\Lambda, M, Q, -)$  for some  $y^0$  and  $u$  and let us show that  $\tilde{y}$  is then a solution to the system  $(\Lambda, -, Q, \Gamma_A(M))$  for some  $\tilde{y}^0$  and  $\tilde{u}$ , where  $\Gamma_A(M)$  will be determined below. We do it formally but this can be rigorously justified.

- The initial data is obviously  $\tilde{y}^0(x) = y^0(x) - \int_0^x K(x, \xi) y^0(\xi) d\xi$ .
- The boundary condition at  $x = 0$  is clearly satisfied since  $\tilde{y}(t, 0) = y(t, 0)$ .
- Looking at the boundary condition at  $x = 1$ , the control  $\tilde{u}$  is

$$\tilde{u}(t) = \tilde{y}_-(t, 1) = y_-(t, 1) - \int_0^1 H(\xi) y(t, \xi) d\xi, \quad (31)$$

where  $H(\xi) = (K_{--}(1, \xi) \quad K_{-+}(1, \xi))$ .

- Using the equation satisfied by  $y$ , integrating by parts, and using the boundary condition satisfied by  $y$  at  $x = 0$ , we have

$$\begin{aligned} \frac{\partial \tilde{y}}{\partial t}(t, x) + \Lambda(x) \frac{\partial \tilde{y}}{\partial x}(t, x) = & \\ & - \int_0^x \left( \Lambda(x) \frac{\partial K}{\partial x}(x, \xi) + \frac{\partial K}{\partial \xi}(x, \xi) \Lambda(\xi) + K(x, \xi) \left( \frac{\partial \Lambda}{\partial \xi}(\xi) + M(\xi) \right) \right) y(t, \xi) d\xi \\ & + (M(x) + K(x, x) \Lambda(x) - \Lambda(x) K(x, x)) y(t, x) \\ & - K(x, 0) \Lambda(0) \begin{pmatrix} \text{Id}_{\mathbb{R}^{m \times m}} \\ Q \end{pmatrix} y_-(t, 0). \end{aligned}$$

Consequently,  $\tilde{y}$  satisfies the desired equation if we take

$$(\Gamma_A(M))(x) = -K(x, 0) \Lambda(0) \begin{pmatrix} \text{Id}_{\mathbb{R}^{m \times m}} \\ Q \end{pmatrix}, \quad (32)$$

and provided that the kernel  $K$  satisfies the so-called kernel equations:

$$\begin{cases} \Lambda(x) \frac{\partial K}{\partial x}(x, \xi) + \frac{\partial K}{\partial \xi}(x, \xi) \Lambda(\xi) + K(x, \xi) \left( \frac{\partial \Lambda}{\partial \xi}(\xi) + M(\xi) \right) = 0, \\ \Lambda(x) K(x, x) - K(x, x) \Lambda(x) = M(x). \end{cases} \quad (33)$$

The existence of a solution to these equations will be discussed next.

Now that  $\Gamma_A$  is clearly identified, similar computations show that, conversely, if  $\tilde{y}$  is a solution to the system  $(\Lambda, -, Q, \Gamma_A(M))$  for some  $\tilde{y}^0$  and  $\tilde{u}$ , then  $y$  is a solution to the system  $(\Lambda, M, Q, -)$  for some  $y^0$  and  $u$ .  $\square$

**Remark 3.3.** If we write the second condition of (33) component-wise:

$$(\lambda_i(x) - \lambda_j(x)) k_{ij}(x, x) = m_{ij}(x), \quad (34)$$

then we see that for  $i = j$  we shall necessarily have  $m_{ii} = 0$ . Therefore, it is necessary that  $M \in \mathcal{M}$  (otherwise the equation (34), and thus the kernel equations (33), have no solution). This explains why we had to perform a preliminary transformation in Section 3.1 to reduce the general case to this one.

From [HDMVK16, Section VI], we know that the kernel equations (33) have a solution (see also [HVDMK19, Remark A.2] to see how to deal with space-varying speeds). More precisely, we can extract the following result:

**Theorem 3.4.** *For every  $A \in \mathcal{F}$ , for every  $M \in \mathcal{M}$ , there exists a unique solution  $K \in L^\infty(\mathcal{T})^{n \times n}$  to the kernel equations (33) with:*

- For every  $i, j \in \{1, \dots, m\}$ :

$$\begin{aligned} k_{ij}(1, \xi) &= a_{ij}(\xi), & \text{if } i > j, \\ k_{ij}(x, 0) &= a_{ij}(x), & \text{if } i \leq j. \end{aligned} \quad (35)$$

- For every  $i, j \in \{m+1, \dots, n\}$ :

$$\begin{aligned} k_{ij}(x, 0) &= a_{ij}(x), & \text{if } i \geq j, \\ k_{ij}(1, \xi) &= a_{ij}(\xi), & \text{if } i < j. \end{aligned} \quad (36)$$

Moreover, we have the following additional regularities:

$$\begin{aligned} K &\in C^0((0, 1]; L^2(0, x)^{n \times n}), & K(x, \cdot) &\in L^\infty(0, x)^{n \times n}, & \forall x \in (0, 1]. \\ K &\in C^0([0, 1]; L^2(\xi, 1)^{n \times n}), & K(\cdot, \xi) &\in L^\infty(\xi, 1)^{n \times n}, & \forall \xi \in [0, 1]. \end{aligned}$$

As before, the notion of solution is to be understood in the sense of solution along the characteristics. By  $K \in C^0((0, 1]; L^2(0, x)^{n \times n})$  we mean that  $\|K(x_n, \cdot) - K(x, \cdot)\|_{L^2(0, \min\{x_n, x\})^{n \times n}} \rightarrow 0$  as  $x_n \rightarrow x$ , for every  $x \in (0, 1]$ , with a similar definition for  $K \in C^0([0, 1]; L^2(\xi, 1)^{n \times n})$ . Despite not mentioned in the literature, these important regularities can be deduced from the system of integral equations satisfied by the kernel. In particular, it shows that  $H$  and  $\Gamma_A(M)$  defined in (31) and (32) have the following regularities:

$$H \in L^\infty(0, 1)^{m \times n}, \quad \Gamma_A(M) \in L^\infty(0, 1)^{n \times m}.$$

**Remark 3.5.** The set  $\mathcal{F}$  corresponds to the set of boundary conditions that are free to choose for the kernel equations. The freedom for the boundary condition (35) was already used in the works [HDM15, HDMVK16, HVDKM19] in order to give to  $(\Gamma_A(M))_{--}$  a structure of strictly lower triangular matrix. However, in the present paper this will not be used and it is the other boundary condition (36) that will turn out to be essential (see Section 6 below).

## 4 Reduction of the boundary coupling matrix

In this section we perform some transformations to show that we can always assume that the boundary coupling matrix  $Q$  is in canonical form. More precisely, we prove the following result:

**Proposition 4.1.** *For every invertible upper triangular matrix  $U \in \mathbb{R}^{m \times m}$  and every invertible lower triangular matrix  $L \in \mathbb{R}^{p \times p}$ , there exists a map  $\Theta : L^\infty(0, 1)^{n \times m} \rightarrow L^\infty(0, 1)^{n \times m}$  such that, for every  $G \in L^\infty(0, 1)^{n \times m}$ , we have*

$$(\Lambda, -, Q, G) \sim (\Lambda, -, LQU, \Theta(G)).$$

*Proof.* • For any  $i, j \in \{1, \dots, n\}$ , we denote by  $\zeta_{ij}$  the solution to the ODE

$$\begin{cases} \frac{d}{ds} \zeta_{ij}(s) = \frac{\lambda_j(\zeta_{ij}(s))}{\lambda_i(s)}, & \forall s \in \mathbb{R}, \\ \zeta_{ij}(0) = 0. \end{cases}$$

- We first prove that, for every invertible upper triangular matrix  $U \in \mathbb{R}^{m \times m}$ , there exists a map  $\Theta_{--} : L^\infty(0, 1)^{m \times m} \rightarrow L^\infty(0, 1)^{m \times m}$  such that, for every  $G \in L^\infty(0, 1)^{n \times m}$ , we have

$$(\Lambda, -, Q, G) \sim \left( \Lambda, -, QU, \begin{pmatrix} \Theta_{--}(G_{--}) \\ G_{+-}U \end{pmatrix} \right).$$

To this end, we are going to use the spatial transformation

$$\tilde{y}_i(t, x) = \begin{cases} \sum_{k=i}^m u^{ik} y_k(t, \zeta_{ik}(x)) & \text{for } i \in \{1, \dots, m\}, \\ y_i(t, x) & \text{for } i \in \{m+1, \dots, n\}, \end{cases} \quad (37)$$

where  $U^{-1} = (u^{ik})_{1 \leq i, k \leq m}$ . Let us first show that this transformation is well defined and invertible. We can check that, for  $i \leq k \leq m$ , we have (recall (25))

$$\zeta_{ik}(x) = \phi_k^{-1}(\phi_i(x)). \quad (38)$$

In particular, for such indices,  $\zeta_{ik}$  is a  $C^1$ -diffeomorphism from  $(0, 1)$  to a subset of  $(0, 1)$  and thus the transformation (37) is well defined on  $L^2(0, 1)^n$ . Besides, using the property  $\zeta_{kj}(\zeta_{ik}(x)) = \zeta_{ij}(x)$  for  $i \leq k \leq j$ , we can check that its inverse is given by

$$y_k(t, x) = \begin{cases} \sum_{j=k}^m u_{kj} \tilde{y}_j(t, \zeta_{kj}(x)) & \text{for } k \in \{1, \dots, m\}, \\ \tilde{y}_k(t, x) & \text{for } k \in \{m+1, \dots, n\}. \end{cases}$$

- Assume now that  $y$  is a solution to the system  $(\Lambda, -, Q, G)$  for some  $y^0$  and  $u$  and let us show that  $\tilde{y}$  is then a solution to the system  $(\Lambda, -, QU, \tilde{G})$  for some  $\tilde{y}^0$  and  $\tilde{u}$ , where

$$\tilde{G} = \begin{pmatrix} \Theta_{--}(G_{--}) \\ G_{+-}U \end{pmatrix},$$

and where  $\Theta_{--}(G_{--})$  will be determined below. Once again, we do it formally but this can be rigorously justified.

- The initial data is obviously

$$\tilde{y}_i^0(x) = \begin{cases} \sum_{k=i}^m u^{ik} y_k^0(\zeta_{ik}(x)) & \text{for } i \in \{1, \dots, m\}, \\ y_i^0(x) & \text{for } i \in \{m+1, \dots, n\}. \end{cases}$$

- The boundary condition at  $x = 0$  is clearly satisfied since  $\tilde{y}_+ = y_+$  and  $\tilde{y}_-(t, 0) = U^{-1}y_-(t, 0)$ .
- Looking at the boundary condition at  $x = 1$ , the control  $\tilde{u}$  is

$$\tilde{u}_i(t) = \tilde{y}_i(t, 1) = \sum_{k=i}^m u^{ik} y_k(t, \zeta_{ik}(1)), \quad \forall i \in \{1, \dots, m\}.$$

- It is clear that  $\tilde{y}_+ = y_+$  satisfies the desired equation. Let us now fix  $i \in \{1, \dots, m\}$ . A computation shows that

$$\begin{aligned} \frac{\partial \tilde{y}_i}{\partial t}(t, x) + \lambda_i(x) \frac{\partial \tilde{y}_i}{\partial x}(t, x) - \sum_{j=1}^m \tilde{g}_{ij}(x) \tilde{y}_j(t, 0) = \\ \sum_{k=i}^m u^{ik} \left( -\lambda_k(\zeta_{ik}(x)) + \lambda_i(x) \frac{\partial \zeta_{ik}}{\partial x}(x) \right) \frac{\partial y_k}{\partial x}(t, \zeta_{ik}(x)) \\ + \sum_{\ell=1}^m \left( \sum_{k=i}^m u^{ik} g_{k\ell}(\zeta_{ik}(x)) - \sum_{j=1}^{\ell} \tilde{g}_{ij}(x) u^{j\ell} \right) y_\ell(t, 0). \end{aligned}$$

Consequently,  $\tilde{y}_i$  satisfies the desired equation, provided that

$$\sum_{k=i}^m u^{ik} g_{k\ell}(\zeta_{ik}(x)) - \sum_{j=1}^{\ell} \tilde{g}_{ij}(x) u^{j\ell} = 0, \quad \forall \ell \in \{1, \dots, m\}.$$

This uniquely determines  $\tilde{g}_{ij}$  for  $i, j \in \{1, \dots, m\}$  (and thus  $\Theta_{--}$ ):

$$\tilde{g}_{ij}(x) = \sum_{\ell=1}^j \left( \sum_{k=i}^m u^{ik} g_{k\ell}(\zeta_{ik}(x)) \right) u_{\ell j}.$$

Now that  $\Theta_{--}$  is clearly identified, similar computations show that, conversely, if  $\tilde{y}$  is a solution to the system  $(\Lambda, -, QU, \tilde{G})$  for some  $\tilde{y}^0$  and  $\tilde{u}$ , then  $y$  is a solution to the system  $(\Lambda, -, Q, G)$  for some  $y^0$  and  $u$ .

- Similarly, we can prove that, for every invertible lower triangular matrix  $L \in \mathbb{R}^{p \times p}$ , there exists a map  $\Theta_{+-} : L^\infty(0, 1)^{p \times m} \rightarrow L^\infty(0, 1)^{p \times m}$  such that, for every  $G \in L^\infty(0, 1)^{n \times m}$ , we have

$$(\Lambda, -, Q, G) \sim \left( \Lambda, -, LQ, \begin{pmatrix} G_{--} \\ \Theta_{+-}(G_{+-}) \end{pmatrix} \right).$$

This can be done using the spatial transformation

$$\tilde{y}_i(t, x) = \begin{cases} y_i(t, x) & \text{for } i \in \{1, \dots, m\}, \\ \sum_{k=m+1}^i \ell_{i-m, k-m} y_k(t, \zeta_{ik}(x)) & \text{for } i \in \{m+1, \dots, n\}, \end{cases}$$

((38) is still valid for the indices considered) where  $L = (\ell_{ij})_{1 \leq i, j \leq p}$  and taking

$$\tilde{g}_{ij}(x) = \sum_{k=m+1}^i \ell_{i-m, k-m} g_{kj}(\zeta_{ik}(x)),$$

for  $i \in \{m+1, \dots, n\}$  and  $j \in \{1, \dots, m\}$ , where  $\tilde{G}$  denotes the matrix

$$\tilde{G} = \begin{pmatrix} G_{--} \\ \Theta_{+-}(G_{+-}) \end{pmatrix}.$$

□

## 5 Smallest value of the minimal null control time

Thanks to the result of previous section it is from now on sufficient to consider boundary coupling matrices which are in canonical form. This is a big step forward, which already allows us to characterize the smallest value of the minimal null control time.

### 5.1 Characterization for systems $(\Lambda, -, Q, G)$

We start with systems of the form  $(\Lambda, -, Q, G)$ , we will discuss in the next section how to deduce the corresponding result for the initial system  $(\Lambda, M, Q, -)$ .

**Theorem 5.1.** *Let  $Q^0 \in \mathbb{R}^{p \times m}$  be in canonical form,  $G \in L^\infty(0, 1)^{n \times m}$  and  $T > 0$  be fixed.*

(i) *If the system  $(\Lambda, -, Q^0, G)$  is null controllable in time  $T$ , then necessarily*

$$T \geq \max \left\{ \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+1}(\Lambda), \quad T_m(\Lambda) \right\}. \quad (39)$$

(ii) *If  $T$  satisfies the condition (39), then the system  $(\Lambda, -, Q^0, -)$  (i.e. with  $G = 0$ ) is null controllable in time  $T$  with control  $u = 0$ .*

As for Theorem 1.11, we use the convention that the undefined quantities are simply not taken into account, which means that the condition (39) is reduced to  $T \geq \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}$  when  $\rho = 0$  (i.e. when  $Q^0 = 0$ ).

This result shows in particular that the smallest value that  $T_{\inf}(\Lambda, -, Q^0, G)$  can take with respect to  $G \in L^\infty(0, 1)^{n \times m}$  is equal to the quantity on the right-hand side of the inequality in (39). This can be extended to arbitrary boundary coupling matrices thanks to Proposition 4.1.

*Proof of Theorem 5.1.* We use the ideas of the proof of [HO21b, Lemma 3.3].

1) We first show that it is necessary that

$$T \geq \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}.$$

We point out that for this first step there is no need to assume that  $Q^0$  is in canonical form. Assume that  $T < \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}$ . Then, there exists  $i \in \{1, \dots, n\}$  such that  $T < T_i(\Lambda)$ . Let  $\omega_i$  be the open subset defined by

$$\omega_i = \{x \in (0, 1) \mid s_i^{\text{in}}(T, x) < 0\}. \quad (40)$$

Then, we have (see (24), (23) and (22))

$$T < T_i(\Lambda) \iff \omega_i \neq \emptyset.$$

For  $x \in \omega_i$ , the null controllability condition  $y_i(T, x) = 0$  is equivalent to (see (29))

$$0 = y_i^0(\chi_i(0; T, x)) + \int_0^T \sum_{j=1}^m g_{ij}(\chi_i(s; T, x)) y_j(s, 0) ds.$$

Since  $y_i^0 \in L^2(0, 1)$  is arbitrary and  $x \in \omega_i \mapsto \chi_i(0; T, x)$  is a  $C^1$ -diffeomorphism (its inverse is given by  $\xi \mapsto \chi_i(T; 0, \xi)$  thanks to (20)), this shows that the bounded linear operator  $K : L^2(0, T)^m \rightarrow L^2(\omega_i)$  defined by

$$(Kh)(x) = - \int_0^T \sum_{j=1}^m g_{ij}(\chi_i(s; T, x)) h_j(s) ds,$$

is surjective. This is impossible since its range is clearly a subset of  $L^\infty(\omega_i)$ , which is a proper subset of  $L^2(\omega_i)$ .

2) Suppose now that  $\rho \neq 0$  (otherwise we are done) and that  $T$  is such that

$$\max \{T_{m+1}(\Lambda), T_m(\Lambda)\} \leq T < T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda),$$

where  $k_0 \in \{1, \dots, \rho\}$  is any index such that

$$T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda) = \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda).$$

We have seen in the previous step that the condition  $T \geq \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}$  means that all the subsets  $\omega_i$  defined in (40) are empty. In particular (recall also (22)),

$$s_{m+r_{k_0}}^{\text{in}}(T, x) > 0, \quad \forall x \in (0, 1).$$

Therefore, the null controllability condition  $y_{m+r_{k_0}}(T, x) = 0$  is equivalent to (see (29) and recall that  $Q^0$  is in canonical form)

$$0 = y_{c_{k_0}}(s_{m+r_{k_0}}^{\text{in}}(T, x), 0) + \int_{s_{m+r_{k_0}}^{\text{in}}(T, x)}^T \sum_{j=1}^m g_{m+r_{k_0}, j}(\chi_{m+r_{k_0}}(s; T, x)) y_j(s, 0) ds. \quad (41)$$

Let us now introduce the open subset  $\tilde{\omega}$  defined by

$$\tilde{\omega} = \left\{ x \in (0, 1) \mid s_{c_{k_0}}^{\text{in}}(s_{m+r_{k_0}}^{\text{in}}(T, x), 0) < 0 \right\}.$$

Using that  $T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda) = s_{m+r_{k_0}}^{\text{out}}(s_{c_{k_0}}^{\text{out}}(0,1),0)$  (see (24) and (21)), we can show by a similar reasoning as in the first step that

$$T < T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda) \iff \tilde{\omega} \neq \emptyset.$$

For  $x \in \tilde{\omega}$ , the identity (41) becomes (see (29))

$$\begin{aligned} 0 &= y_{c_{k_0}}^0(\chi_{c_{k_0}}(0; s_{m+r_{k_0}}^{\text{in}}(T,x),0)) \\ &+ \sum_{j=1}^m \int_{\bar{s}_{c_{k_0}}^{\text{in}}(s_{m+r_{k_0}}^{\text{in}}(T,x),0)}^{s_{m+r_{k_0}}^{\text{in}}(T,x)} g_{c_{k_0}j}(\chi_{c_{k_0}}(s; s_{m+r_{k_0}}^{\text{in}}(T,x),0)) y_j(s,0) ds \\ &+ \int_{s_{m+r_{k_0}}^{\text{in}}(T,x)}^T \sum_{j=1}^m g_{m+r_{k_0},j}(\chi_{m+r_{k_0}}(s;T,x)) y_j(s,0) ds. \end{aligned}$$

This leads to a contradiction by using the same argument as at the end of the first step.

- 3) Finally, it is not difficult to see from (29) that, when  $G = 0$ , the control  $u = 0$  brings the solution of the system  $(\Lambda, -, Q^0)$  to zero in any time  $T$  satisfying (39). □

## 5.2 Proof of the first part of Theorem 1.11

Let us now show how the previous results yield the desired characterization of the smallest minimal null control time for the initial system  $(\Lambda, M, Q, -)$ .

*Proof of item (i) of Theorem 1.11.* • Let  $M \in L^\infty(0,1)^{n \times n}$  and  $Q \in \mathbb{R}^{p \times m}$  be fixed. Let  $T > 0$  be such that the system  $(\Lambda, M, Q, -)$  is null controllable in time  $T$ .

- By Proposition 3.1 and Theorem 3.2, there exists  $G \in L^\infty(0,1)^{n \times m}$  such that the system  $(\Lambda, -, Q, G)$  is null controllable in time  $T$ .
- From Proposition 4.1, there exists  $\tilde{G} \in L^\infty(0,1)^{n \times m}$  such that the system  $(\Lambda, -, Q^0, \tilde{G})$  is null controllable in time  $T$ , where  $Q^0$  is the canonical form of  $Q$ .
- By item (i) of Theorem 5.1 we obtain that  $T$  has to satisfy the condition (39).

This establishes the following lower bound:

$$T_{\text{inf}}(\Lambda, M, Q) \geq \max \left\{ \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+1}(\Lambda), \quad T_m(\Lambda) \right\},$$

valid for every  $M \in L^\infty(0,1)^{n \times n}$ .

- This lower bound is reached for  $M = 0$ , this follows from Theorem 5.1 and Proposition 4.1 (using that  $\Theta(0) = 0$ ). □

### 5.3 Comments on the case $M = 0$

Let us conclude this section with some interesting remarks on the case  $M = 0$ . For  $M = 0$ , we can combine Theorem 5.1 with Proposition 4.1 (with  $G = 0$ , in which case their proofs are greatly simplified) to obtain a completely different proof of [Wec82, Theorems 1 and 2]. Our proof has several advantages. Firstly, we directly obtain a more explicit expression of the minimal null control time (see e.g. [HO21a, Remark 1.15]). On the other hand, we do not need to use the so-called duality and we are able to obtain an explicit control. More precisely, we can extract the following result from item (ii) of Theorem 5.1 and the proof of Proposition 4.1:

**Proposition 5.2.** *Let  $Q \in \mathbb{R}^{p \times m}$  and  $T$  satisfy (39). Then, the system  $(\Lambda, -, Q, -)$  is finite-time stabilizable with settling time  $T$ , with the following explicit feedback law:*

$$u_i(t) = - \sum_{k=i+1}^m u^{ik} y_k(t, \zeta_{ik}(1)), \quad i \in \{1, \dots, m\}, \quad (42)$$

where  $U^{-1} = (u^{ik})_{1 \leq i, k \leq m}$  and  $U$  is any matrix  $U$  coming from the LCU decomposition of  $Q$ .

We recall that the previous statement simply means that, if we replace the  $i$ -th component of  $u$  by the right-hand side of the formula (42) in the system (1) (with  $M = 0$ ), then the corresponding solution satisfies  $y(T, \cdot) = 0$  for every  $y^0 \in L^2(0, 1)^n$ . We also recall that systems with such boundary conditions are well posed (see e.g. [CN19, Section 3] in the  $L^\infty$  setting).

A similar result was obtained in the proof of [CN19, Proposition 1.6] when  $Q \in \mathcal{B}$  (defined in (7)), our result generalizes it to arbitrary  $Q \in \mathbb{R}^{p \times m}$ . Let us illustrate with an example that the feedback law that we have obtained (42) is also the same as in this reference when  $Q \in \mathcal{B}$ .

**Example 5.3.** *Let us consider the  $6 \times 6$  system used as example in [CN19, p. 1155]: we take  $p = m = 3$ , the negative speeds are*

$$\lambda_1 = -4 < \lambda_2 = -2 < \lambda_3 = -1 < 0,$$

*the positive speeds are arbitrary (subject to (3)), and we take the boundary coupling matrix*

$$Q = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ a & b & c \end{pmatrix},$$

*where  $a, b, c \in \mathbb{R}$  are arbitrary numbers.*

*Then, the Gaussian elimination easily shows that  $Q \in \mathcal{B}$  and*

$$U = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

*The feedback law is thus given by*

$$\begin{cases} y_1(t, 1) = y_2\left(t, \frac{1}{2}\right) + y_3\left(t, \frac{1}{4}\right), \\ y_2(t, 1) = -3y_3\left(t, \frac{1}{2}\right), \\ y_3(t, 1) = 0. \end{cases}$$

**Remark 5.4.** Let us also add that another advantage of not using the duality is that it can be useful to deal with other functional settings (e.g.  $C^1$ , provided that the inequality in (39) is strict).

## 6 Reduction to a canonical system

We are now left with the proof of the second part of Theorem 1.11, which is more difficult and require more work.

In this section, we will show how to use the canonical structure of the boundary coupling matrix to remove some coupling terms in the matrix  $G_{+-}$ . For any  $Q \in \mathbb{R}^{p \times m}$ , we introduce the set

$$\mathcal{C}(Q) = \{G_{+-} \in L^\infty(0, 1)^{p \times m} \mid g_{m+i, c_k} = 0, \quad \forall k \in \{1, \dots, \rho\}, \forall i \geq r_k\}$$

$$(\mathcal{C}(0) = L^\infty(0, 1)^{p \times m}).$$

The goal of this section is to prove the following result:

**Proposition 6.1.** *Assume that  $Q^0 \in \mathbb{R}^{p \times m}$  is in canonical form. Then, there exists a map  $\Upsilon : L^\infty(0, 1)^{p \times m} \rightarrow \mathcal{C}(Q^0)$  such that, for every  $G \in L^\infty(0, 1)^{n \times m}$ , we have*

$$(\Lambda, -, Q^0, G) \sim \left( \Lambda, -, Q^0, \left( \Upsilon(G_{+-}) \right) \right). \quad (43)$$

*Proof.* We assume that  $\rho \neq 0$  since otherwise there is nothing to prove. Reproducing the proof of Theorem 3.2 with the kernel

$$K = \begin{pmatrix} 0 & 0 \\ 0 & K_{++} \end{pmatrix},$$

we see that we have (43) if we take

$$(\Upsilon(G_{+-}))(x) = G_{+-}(x) - K_{++}(x, 0)\Lambda_{++}(0)Q^0 - \int_0^x K_{++}(x, \xi)G_{+-}(\xi) d\xi,$$

and provided that  $K_{++}$  satisfies

$$\begin{cases} \Lambda_{++}(x) \frac{\partial K_{++}}{\partial x}(x, \xi) + \frac{\partial K_{++}}{\partial \xi}(x, \xi)\Lambda_{++}(\xi) + K_{++}(x, \xi) \frac{\partial \Lambda_{++}}{\partial \xi}(\xi) = 0, \\ \Lambda_{++}(x)K_{++}(x, x) - K_{++}(x, x)\Lambda_{++}(x) = 0. \end{cases}$$

This is an uncoupled system with many solutions (as we already know from Theorem 3.4). Let us find a particular one that guarantees that  $\Upsilon(G_{+-}) \in \mathcal{C}(Q^0)$ . Let  $i, j \in \{m+1, \dots, n\}$  be fixed. The equation for  $k_{ij}$  is simply

$$\begin{cases} \lambda_i(x) \frac{\partial k_{ij}}{\partial x}(x, \xi) + \frac{\partial k_{ij}}{\partial \xi}(x, \xi)\lambda_j(\xi) + k_{ij}(x, \xi) \frac{\partial \lambda_j}{\partial \xi}(\xi) = 0, \\ k_{ij}(x, x) = 0, \quad \text{if } i \neq j. \end{cases} \quad (44)$$

Let  $s \mapsto \zeta_{ij}(s; x, \xi)$  be the associated characteristic passing through  $(x, \xi)$ :

$$\begin{cases} \frac{\partial \zeta_{ij}}{\partial s}(s; x, \xi) = \frac{\lambda_j(\zeta_{ij}(s; x, \xi))}{\lambda_i(s)}, \\ \zeta_{ij}(x; x, \xi) = \xi. \end{cases}$$

The solutions to (44) are explicit:

- If  $i \geq j$ , then there exists a unique solution to (44) which satisfies  $k_{ij}(x, 0) = a_{ij}(x)$  ( $a_{ij} \in L^\infty(0, 1)$  is arbitrary) and it is given by

$$k_{ij}(x, \xi) = \begin{cases} a_{ij}(s_{ij}^{\text{in}}(x, \xi)) \frac{\lambda_j(0)}{\lambda_j(\xi)} & \text{if } \xi < \zeta_{ij}(x; 0, 0), \\ 0 & \text{if } \xi > \zeta_{ij}(x; 0, 0), \end{cases}$$

where  $s_{ij}^{\text{in}}(x, \xi) \in (0, x)$  is the unique solution to

$$\zeta_{ij}(s_{ij}^{\text{in}}(x, \xi); x, \xi) = 0.$$

- If  $i < j$ , then there exists a unique solution to (44) which satisfies  $k_{ij}(1, \xi) = a_{ij}(\xi)$  ( $a_{ij} \in L^\infty(0, 1)$  is arbitrary) and it is given by

$$k_{ij}(x, \xi) = \begin{cases} a_{ij}(\zeta_{ij}(1; x, \xi)) \frac{\lambda_j(\zeta_{ij}(1; x, \xi))}{\lambda_j(\xi)} & \text{if } \xi < \zeta_{ij}(x; 1, 1), \\ 0 & \text{if } \xi > \zeta_{ij}(x; 1, 1). \end{cases}$$

We choose  $a_{ij} = 0$  for  $i < j$ , so that  $k_{ij} = 0$  for such indices. Let us now fix the remaining  $a_{ij}$  to ensure that  $\Upsilon(G_{+-}) \in \mathcal{C}(Q^0)$ . To this end, we fix  $i \in \{m+1, \dots, n\}$  such that  $E_i \neq \emptyset$ , where

$$E_i = \{\alpha \in \{1, \dots, \rho\} \mid m + r_\alpha \leq i\}.$$

The  $(i, c_\alpha)$ -th entry of  $\Upsilon(G_{+-})$  is equal to zero if, and only if,

$$0 = g_{ic_\alpha}(x) - \sum_{\ell=m+1}^n k_{i\ell}(x, 0) \lambda_\ell(0) q_{\ell-m, c_\alpha}^0 - \int_0^x \sum_{\ell=m+1}^n k_{i\ell}(x, \xi) g_{\ell, c_\alpha}(\xi) d\xi.$$

Using the explicit formulas for  $k_{ij}$  and the assumption that  $Q^0$  is in canonical form, for  $\alpha \in E_i$  this identity is equivalent to

$$0 = g_{ic_\alpha}(x) - a_{i, m+r_\alpha}(x) \lambda_{m+r_\alpha}(0) - \sum_{\ell=m+1}^i \int_0^{\zeta_{i\ell}(x; 0, 0)} a_{i\ell}(s_{i\ell}^{\text{in}}(x, \xi)) \frac{\lambda_\ell(0)}{\lambda_\ell(\xi)} g_{\ell, c_\alpha}(\xi) d\xi.$$

Using the change of variable  $\theta \mapsto \xi = \zeta_{i\ell}(x; \theta, 0)$  with the property

$$s_{i\ell}^{\text{in}}(x, \zeta_{i\ell}(x; \theta, 0)) = s_{i\ell}^{\text{in}}(\theta, 0) = \theta,$$

and isolating the terms for  $\ell = m + r_\beta$  with  $\beta \in E_i$ , this gives the following system of Volterra equations of the second kind:

$$a_{i, m+r_\alpha}(x) \lambda_{m+r_\alpha}(0) + \sum_{\beta \in E_i} \int_0^x a_{i, m+r_\beta}(\theta) h_{i, \alpha, m+r_\beta}(x, \theta) d\theta = f_{i\alpha}(x), \quad \alpha \in E_i, \quad (45)$$

with  $L^\infty$  kernel

$$h_{i\alpha\ell}(x, \theta) = \frac{\lambda_\ell(0)}{\lambda_\ell(\zeta_{i\ell}(x; \theta, 0))} g_{\ell, c_\alpha}(\zeta_{i\ell}(x; \theta, 0)) \frac{\partial \zeta_{i\ell}}{\partial \theta}(x; \theta, 0),$$

and  $L^\infty$  right-hand side

$$f_{i\alpha}(x) = g_{ic_\alpha}(x) - \sum_{\ell \in F_i} \int_0^x a_{i\ell}(\theta) h_{i\alpha\ell}(x, \theta) d\theta,$$

$$F_i = \{\ell \in \{m+1, \dots, i\} \mid \ell \notin \{m+r_1, \dots, m+r_\rho\} \text{ or } \ell \in \{m+r_\beta \mid \beta \notin E_i\}\}.$$

Setting

$$a_{i\ell} = 0, \quad \forall \ell \in F_i,$$

we have  $f_{i\alpha} = g_{ic_\alpha}$  and, once  $f_{i\alpha}$  is known, the remaining values  $a_{i, m+r_\alpha}$ ,  $\alpha \in E_i$ , are uniquely determined by solving the system (45). □

**Remark 6.2.** Let us point out that it is also possible to transform the matrix  $G_{--}$  into a strictly lower triangular matrix by using the kernel

$$K = \begin{pmatrix} K_{--} & 0 \\ 0 & 0 \end{pmatrix},$$

and by appropriately choosing some boundary conditions for  $K_{--}$  (this is the same proof as above with  $Q^0 = \text{Id}$ ). In summary, whatever  $Q \in \mathbb{R}^{p \times m}$  and  $G \in L^\infty(0, 1)^{n \times m}$  are, we have shown that we can always find some transformations so that we are reduced to the case where:

- $Q$  is in canonical form.
- $G_{--}$  is strictly lower triangular.
- $G_{+-} \in \mathcal{C}(Q)$ .

Let us add that, in general, it is not possible to remove more terms by using some transformations (e.g. the backstepping method). In other words, there is in general no simpler equivalent system. An example has been detailed in Appendix A. In this sense, systems  $(\Lambda, -, Q, G)$  with the above structure could be called “in canonical form”.

## 7 Reduction by compactness-uniqueness

In this section, we show that, even though we can not in general fully remove  $G_{--}$  by using some transformations (Remark 6.2), nevertheless, the two systems share the same minimal null control time:

**Theorem 7.1.** *Let  $Q \in \mathbb{R}^{p \times m}$  be fixed. For every  $G \in L^\infty(0, 1)^{n \times m}$ , we have*

$$T_{\text{inf}}(\Lambda, -, Q, G) = T_{\text{inf}}\left(\Lambda, -, Q, \begin{pmatrix} 0 \\ G_{+-} \end{pmatrix}\right). \quad (46)$$

**Remark 7.2.** Let us emphasize once again that it is impossible to prove Theorem 7.1 by using some transformations to pass from one system to the other (e.g. backstepping). In other words, these two systems are in general not equivalent (in the sense of Definition 1.18). Therefore, a different method is necessary to prove Theorem 7.1. We will do it thanks to a compactness-uniqueness method adapted to the null controllability property.

### 7.1 A compactness-uniqueness method for the null controllability

We will present here a general compactness-uniqueness method adapted to the null controllability property. We will see in the next section how to use it in order to obtain Theorem 7.1.

First of all, let us briefly recall some basic facts about abstract linear control systems. All along this section,  $H$  and  $U$  are two complex Hilbert spaces,  $A : D(A) \subset H \rightarrow H$  is the generator of a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  on  $H$  and  $B \in \mathcal{L}(U, D(A^*))'$ . Here and in what follows,  $E'$  denotes the anti-dual of the complex space  $E$ , that is the complex (Banach) space of all continuous conjugate linear forms. We will use the convention that an inner product of a complex Hilbert space is conjugate linear in its second argument. One of the reason why we have to consider complex (and not real) spaces is because we will use below a condition involving the spectral elements of the operator  $A$ , we will explain how to deal with real Banach spaces in practice at the end of this section in Remark 7.8.

Let us now consider the evolution problem associated with the pair  $(A, B)$ , i.e.

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t), & t \in (0, T), \\ y(0) = y^0, \end{cases} \quad (47)$$

where  $T > 0$ ,  $y(t)$  is the state at time  $t$ ,  $y^0$  is the initial data and  $u(t)$  is the control at time  $t$ .

Let us recall a standard procedure to define a notion of solution in  $H$  to (47) for non smooth functions. We formally multiply (47) by a smooth function  $z$ , integrate over an arbitrary time interval  $(0, \tau) \subset (0, T)$ , perform an integration by parts and use the adjoints to obtain the identity

$$\langle y(\tau), z(\tau) \rangle_H - \langle y^0, z(0) \rangle_H + \int_0^\tau \left\langle y(t), -\frac{d}{dt}z(t) - A^*z(t) \right\rangle_H dt = \int_0^\tau \langle u(t), B^*z(t) \rangle_U dt.$$

Particularizing this identity for the solution  $z$  to the so-called adjoint system

$$\begin{cases} -\frac{d}{dt}z(t) = A^*z(t), & t \in (0, \tau), \\ z(\tau) = z^1, \end{cases} \quad (48)$$

i.e.  $z(t) = S(\tau - t)^*z^1$ , where  $z^1$  is arbitrary, this leads to the following notion of solution in  $H$ :

**Definition 7.3.** Let  $T > 0$ ,  $y^0 \in H$  and  $u \in L^2(0, T; U)$  be fixed. We say that a function  $y : [0, T] \rightarrow H$  is a solution to (47) if  $y \in C^0([0, T]; H)$  and

$$\langle y(\tau), z^1 \rangle_H - \langle y^0, z(0) \rangle_H = \int_0^\tau \langle u(t), B^*z(t) \rangle_U dt, \quad (49)$$

for every  $\tau \in (0, T]$  and  $z^1 \in D(A^*)$ , where  $z \in C^0([0, \tau]; D(A^*))$  is the solution to the adjoint system (48).

For the system (47) to be well posed in this sense, the space  $H$  has to satisfy some properties.

**Definition 7.4.** We say that  $H$  is an admissible subspace for the system  $(A, B)$  if the following regularity property holds:

$$\forall \tau > 0, \exists C > 0, \int_0^\tau \|B^*z(t)\|_U^2 dt \leq C \|z^1\|_H^2, \quad \forall z^1 \in D(A^*), \quad (50)$$

where  $z \in C^0([0, \tau]; D(A^*))$  is the solution to the adjoint system (48).

We recall that, thanks to basic semigroup properties, it is equivalent to prove (50) for one single  $\tau > 0$ .

If  $H$  is an admissible subspace for  $(A, B)$ , then the map

$$z^1 \in D(A^*) \mapsto \int_0^\tau \langle u(t), B^*z(t) \rangle_U dt,$$

can be extended to a continuous conjugate linear form on  $H$ . Thus, we have a natural definition for the map  $\tau \in [0, T] \mapsto y(\tau) \in H$  through the formula (49). It can be proved that this map is also continuous and that it depends continuously on  $y^0$  and  $u$  on compact time intervals (see e.g. [Cor07, Theorem 2.37]). This establishes the so-called well-posedness of the abstract control system (47) in  $H$ .

Now that we have a notion of continuous solution for the system (47) in the space  $H$ , we can speak of its controllability properties in  $H$ .

**Definition 7.5.** We say that the system (47) is null controllable in time  $T$  if, for every  $y^0 \in H$ , there exists  $u \in L^2(0, T; U)$  such that the corresponding solution  $y \in C^0([0, T]; H)$  to the system (47) satisfies

$$y(T) = 0.$$

It is also well known that controllability has a dual concept named observability. We have the following characterization (see e.g. [Cor07, Theorem 2.44]):

**Theorem 7.6.** *Let  $T > 0$  be fixed. The system  $(A, B)$  is null controllable in time  $T$  if, and only if, there exists  $C > 0$  such that, for every  $z^1 \in D(A^*)$ ,*

$$\|z(0)\|_H^2 \leq C \int_0^T \|B^* z(t)\|_U^2 dt,$$

where  $z \in C^0([0, T]; D(A^*))$  is the solution to the adjoint system (48) (with  $\tau = T$ ).

After these basic reminders, we can now clearly introduce the general compactness-uniqueness result on which the proof of Theorem 7.1 will rely on.

**Theorem 7.7.** *Let  $H$  and  $U$  be two complex Hilbert spaces. Let  $A : D(A) \subset H \rightarrow H$  be the generator of a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  on  $H$  and let  $B \in \mathcal{L}(U, D(A^*)')$ . We assume that  $H$  is an admissible subspace for  $(A, B)$  and that  $(A, B)$  satisfies the so-called Fattorini-Hautus test, namely:*

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \quad (51)$$

Assume in addition that there exists  $T_0 > 0$  such that, for every  $T > T_0$ , the following two properties hold:

- (i) *There exist two complex Banach spaces  $E_1, E_2$ , a compact operator  $P : E_1 \rightarrow E_2$ , a linear operator  $L : D(A^*) \rightarrow E_1$  and  $C > 0$  such that, for every  $z^1 \in D(A^*)$ ,*

$$\|z(0)\|_H^2 \leq C \left( \int_0^T \|B^* z(t)\|_U^2 dt + \|PLz^1\|_{E_2}^2 \right), \quad (52)$$

$$\|Lz^1\|_{E_1}^2 \leq C \left( \|z(0)\|_H^2 + \int_0^T \|B^* z(t)\|_U^2 dt \right), \quad (53)$$

where  $z \in C^0([0, T]; D(A^*))$  is the solution to the adjoint system (48) (with  $\tau = T$ ).

- (ii) *For every  $0 < t_1 < t_2 < T - T_0$ , there exists  $C > 0$  such that, for every  $z^1 \in D(A^*)$ ,*

$$\|z(t_2)\|_H^2 \leq C \left( \|z(t_1)\|_H^2 + \int_{t_1}^{t_2} \|B^* z(t)\|_U^2 dt \right), \quad (54)$$

where  $z \in C^0([0, T]; D(A^*))$  is the solution to the adjoint system (48) (with  $\tau = T$ ).

Then, the system  $(A, B)$  is null controllable in time  $T$  for every  $T > T_0$ .

The proof of this result is postponed to Appendix B for the sake of the presentation. It is based on arguments developed in the proofs of [CN21a, Theorem 2] and [DO18, Lemma 2.6] (see also the references therein). Let us just mention at this point that, in general, the compactness-uniqueness method is designed for the exact controllability property. It is only thanks to the property (54) that we are able to consider the null controllability property here.

**Remark 7.8.** In most applications we encounter real systems, that is  $H$  and  $U$  are real Banach spaces. To apply what precedes, we have to consider their so-called complexifications as well as the complexifications of the operators  $A$  and  $B$ . By splitting the complex system (i.e. the system corresponding to these complexifications) into real and imaginary parts, it is not difficult to check that the real system is controllable if, and only if, so is the complex system.

## 7.2 Proof of Theorem 7.1

Let us now show how to use the general result Theorem 7.7 in order to obtain Theorem 7.1. We only prove the inequality “ $\leq$ ” in (46) (which is the most important one), the other inequality can be established similarly. Let then  $T_0 > 0$  be such that

$$\left( \Lambda, -, Q, \begin{pmatrix} 0 \\ G_{+-} \end{pmatrix} \right) \quad (55)$$

is null controllable in time  $T_0$  and let us show that necessarily  $T_{\text{inf}}(\Lambda, -, Q, G) \leq T_0$ . This will follow from Theorem 7.7 once we will have checked that the system  $(\Lambda, -, Q, G)$  satisfies all the assumptions of this result.

First of all, we have to recast the system  $(\Lambda, -, Q, G)$  as an abstract evolution system of the form (47). This is quite standard. To identify what are the operators  $A$  and  $B$  (in fact, we first find  $A^*$  and  $B^*$ ), we repeat the procedure that led to Definition 7.3 on the system (16) (with  $M = 0$ ), where taking the adjoints is replaced by an integration by parts in space. This gives the following.

- The state and control spaces are

$$H = L^2(0, 1)^n (= L^2(0, 1; \mathbb{C}^n)), \quad U = \mathbb{C}^m.$$

They are equipped with their usual inner products.

- The unbounded linear operator  $A : D(A) \subset H \rightarrow H$  is defined for every  $y \in D(A)$  by

$$(Ay)(x) = -\Lambda(x) \frac{\partial y}{\partial x}(x) + G(x)y_-(0), \quad x \in (0, 1),$$

with domain

$$D(A) = \left\{ y \in H^1(0, 1)^n \mid y_-(1) = 0, \quad y_+(0) = Qy_-(0) \right\}.$$

- It is clear that  $D(A)$  is dense in  $H$  since it contains  $C_c^\infty(0, 1)^n$ . A computation shows that

$$D(A^*) = \left\{ z \in H^1(0, 1)^n \mid z_-(0) = R^* z_+(0) + \int_0^1 K(\xi)^* z(\xi) d\xi, \quad z_+(1) = 0 \right\},$$

where  $R \in \mathbb{R}^{p \times m}$  and  $K \in L^\infty(0, 1)^{n \times m}$  are defined by

$$R = -\Lambda_{++}(0)Q\Lambda_{--}(0)^{-1}, \quad K(\xi) = -G(\xi)\Lambda_{--}(0)^{-1},$$

and we have, for every  $z \in D(A^*)$ ,

$$(A^*z)(x) = \Lambda(x) \frac{\partial z}{\partial x}(x) + \frac{\partial \Lambda}{\partial x}(x)z(x), \quad x \in (0, 1).$$

- The control operator  $B \in \mathcal{L}(U, D(A^*)')$  is given for every  $u \in U$  and  $z \in D(A^*)$  by

$$\langle Bu, z \rangle_{D(A^*)', D(A^*)} = \langle u, -\Lambda_{--}(1)z_{-}(1) \rangle_{\mathbb{C}^m}.$$

Note that  $B$  is well defined since  $Bu$  is continuous on  $H^1(0, 1)^n$  (by the trace theorem  $H^1(0, 1)^n \hookrightarrow C^0([0, 1]^n)$ ) and since the graph norm  $\|\cdot\|_{D(A^*)}$  and  $\|\cdot\|_{H^1(0, 1)^n}$  are equivalent norms on  $D(A^*)$ .

- Finally, the adjoint  $B^* \in \mathcal{L}(D(A^*), U)$  is given for every  $z \in D(A^*)$  by

$$B^*z = -\Lambda_{--}(1)z_{-}(1).$$

We can prove that  $A$  is closed and that both  $A, A^*$  are quasi-dissipative, so that  $A$  generates a  $C_0$ -semigroup by a well-known corollary of Lumer-Phillips theorem.

Since the other properties to check depend on the adjoint system, it is convenient to write it explicitly:

$$\begin{cases} \frac{\partial z}{\partial t}(t, x) + \Lambda(x) \frac{\partial z}{\partial x}(t, x) = -\frac{\partial \Lambda}{\partial x}(x)z(t, x), \\ z_{-}(t, 0) = R^*z_{+}(t, 0) + \int_0^1 K(\xi)^*z(t, \xi) d\xi, \quad z_{+}(t, 1) = 0, \\ z(T, x) = z^1(x). \end{cases}$$

Using the method of characteristics it is easy to prove the estimate (50) for  $\tau \leq T_1(\Lambda)$ , which shows that  $H$  is an admissible subspace for  $(A, B)$ .

Therefore, the abstract control system is well posed in  $H$ . To rigorously justify that this pair  $(A, B)$  is “the” abstract form of  $(\Lambda, -, Q, G)$  we have to reason in terms of notions of solution:

**Proposition 7.9.** *The solution to system  $(\Lambda, -, Q, G)$  in the sense of Definition 2.1 coincides with the solution to abstract system (47) in the sense of Definition 7.3 corresponding to the pair  $(A, B)$  introduced above.*

*Proof.* We argue by approximation. Let  $y^0 \in L^2(0, 1)^n$ ,  $u \in L^2(0, T)^m$  be fixed and let  $y$  be the corresponding solution to system  $(\Lambda, -, Q, G)$  in the sense of Definition 2.1.

- We take two approximations  $(y^{0,k})_k \subset H_0^1(0, 1)^n$  and  $(u^k)_k \subset H_0^1(0, T)^m$  such that

$$y^{0,k} \rightarrow y^0 \text{ in } L^2(0, 1)^n, \quad u^k \rightarrow u \text{ in } L^2(0, T)^m. \quad (56)$$

Let  $y^k$  be the solution corresponding to  $y^{0,k}$  and  $u^k$  in the sense of Definition 2.1. Since  $y^{0,k}$  and  $u^k$  obviously satisfy the  $C^0$  compatibility conditions

$$y_-^{0,k}(1) = u^k(0), \quad y_+^{0,k}(0) = Qy_-^{0,k}(0),$$

we can prove that

$$y^k \in C^0([0, T]; H^1(0, 1)^n) \cap C^0([0, 1]; H^1(0, T)^n)$$

(for instance, by adapting the fixed point approach of [CHOS21, Appendix A.2] in the above space – the regularity  $G \in L^\infty(0, 1)^{n \times m}$  is enough after a suitable change of variable). In particular,  $y^k \in H^1((0, T) \times (0, 1)^n)$  and it satisfies (16) almost everywhere (with  $M = 0$ ,  $y = y^k$ ,  $u = u^k$  and  $y^0 = y^{0,k}$ ).

- Repeating the procedure that led to Definition 7.3 we easily check that  $y^k$  is the solution to abstract system (47) in the sense of Definition 7.3, i.e. it satisfies identity (49) (with  $y = y^k$ ,  $y^0 = y^{0,k}$  and  $u = u^k$ ). Using (56) and

$$y^k \rightarrow y \text{ in } C^0([0, T]; L^2(0, 1)^n),$$

(this follows from (30) and (56)), we can pass to the limit  $k \rightarrow +\infty$  in this identity to obtain that  $y$  is the solution to abstract system (47) in the sense of Definition 7.3.  $\square$

We will now check that our pair  $(A, B)$  satisfies the assumptions of Theorem 7.7.

- The Fattorini-Hautus test (51) is easy to check. Indeed, if  $\lambda \in \mathbb{C}$  and  $z \in D(A^*)$  are such that  $A^*z = \lambda z$  and  $B^*z = 0$ , then in particular  $z \in H^1(0, 1)^n$  solves the system of linear ODEs

$$\begin{cases} \frac{\partial z}{\partial x}(x) = \Lambda(x)^{-1} \left( -\frac{\partial \Lambda}{\partial x}(x) + \lambda \text{Id}_{\mathbb{R}^n \times n} \right) z(x), & x \in (0, 1), \\ z(1) = 0, \end{cases}$$

so that  $z = 0$  by uniqueness.

Below,  $C$  denotes a positive number that may change from line to line but that never depends on  $z^1$  or  $t$ .

- The inequality (54) is also not difficult to check. Indeed, for  $0 < t_1 < t_2 < T - T_0$ , using the method of characteristics, we have

$$\|z_-(t_2, \cdot)\|_{L^2(0,1)^m}^2 \leq C \left( \|z_-(t_1, \cdot)\|_{L^2(0,1)^m}^2 + \int_{t_1}^{t_2} \|z_-(t, 1)\|_{\mathbb{C}^m}^2 dt \right),$$

and, provided that  $T_0 \geq T_{m+1}(\Lambda)$  and using that  $z_+(\cdot, 1) = 0$ , we also have

$$\|z_+(t_2, \cdot)\|_{L^2(0,1)^p}^2 \leq C \|z_+(t_1, \cdot)\|_{L^2(0,1)^p}^2.$$

We recall that, since the system (55) is null controllable in time  $T_0$  by assumption, we necessarily have  $T_0 \geq T_{m+1}(\Lambda)$  (see the first step of the proof of Theorem 5.1).

- Let us now investigate the estimate (52). Let  $T > T_0$ . We will prove that there exists  $H \in L^\infty((0, T) \times (0, T))^{m \times m}$  such that, for every  $z^1 \in L^2(0, 1)^n$ ,

$$\|z(0, \cdot)\|_{L^2(0,1)^n}^2 \leq C \left( \int_0^T \|z_-(t, 1)\|_{\mathbb{C}^m}^2 dt + \int_0^T \left\| \int_0^T H(t, s) z_-(s, 0) ds \right\|_{\mathbb{C}^m}^2 dt \right). \quad (57)$$

Let us first make some preliminary observations. We denote by  $\zeta$  the solution to the adjoint system of (55) in  $(0, T)$  with final data  $z^1$ , and we set

$$\theta = z - \zeta. \quad (58)$$

Clearly, it satisfies

$$\begin{cases} \frac{\partial \theta}{\partial t}(t, x) + \Lambda(x) \frac{\partial \theta}{\partial x}(t, x) = -\frac{\partial \Lambda}{\partial x}(x) \theta(t, x), \\ \theta_-(t, 0) = R^* \theta_+(t, 0) + \int_0^1 K_{+-}(\xi)^* \theta_+(t, \xi) d\xi + \int_0^1 K_{--}(\xi)^* z_-(t, \xi) d\xi, & \theta_+(t, 1) = 0, \\ \theta(T, x) = 0. \end{cases}$$

Using the method of characteristics, we immediately see that

$$\theta_+ = 0. \quad (59)$$

Consequently,  $\theta_-$  solves

$$\begin{cases} \frac{\partial \theta_-}{\partial t}(t, x) + \Lambda_{--}(x) \frac{\partial \theta_-}{\partial x}(t, x) = -\frac{\partial \Lambda_{--}}{\partial x}(x) \theta_-(t, x), \\ \theta_-(t, 0) = \int_0^1 K_{--}(\xi)^* z_-(t, \xi) d\xi, \\ \theta_-(T, x) = 0. \end{cases}$$

Since  $T > T_0 \geq T_m(\Lambda)$ , using the method of characteristics, it is not difficult to see that, for  $t \in (0, T)$ , we have

$$\begin{aligned} \|\theta_-(t, 0)\|_{\mathbb{C}^m}^2 &= \left\| \int_0^1 K_{--}(\xi)^* z_-(t, \xi) d\xi \right\|_{\mathbb{C}^m}^2 \\ &\leq C \left( \left\| \int_t^T H(t, s) z_-(s, 0) ds \right\|_{\mathbb{C}^m}^2 + \int_0^t \|z_-(s, 1)\|_{\mathbb{C}^m}^2 ds \right), \end{aligned} \quad (60)$$

for some  $H \in L^\infty((0, T) \times (0, T))^{m \times m}$  independent of  $z^1$ . Let us now prove the desired estimate (57). Since by assumption the system (55) is null controllable in time  $T_0$ , and thus in time  $T > T_0$ , the solution  $\zeta$  to its adjoint system satisfies (see Theorem 7.6)

$$\|\zeta(0, \cdot)\|_{L^2(0,1)^n}^2 \leq C \int_0^T \|\zeta_-(t, 1)\|_{\mathbb{C}^m}^2 dt.$$

Recalling (58) and (59), it follows that

$$\begin{aligned} \|z(0, \cdot)\|_{L^2(0,1)^n}^2 &\leq 2 \|\theta(0, \cdot)\|_{L^2(0,1)^n}^2 + 2 \|\zeta(0, \cdot)\|_{L^2(0,1)^n}^2 \\ &= 2 \|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + 2 \|\zeta(0, \cdot)\|_{L^2(0,1)^n}^2 \\ &\leq 2 \|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + C \int_0^T \|\zeta_-(t, 1)\|_{\mathbb{C}^m}^2 dt \\ &\leq 2 \|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + 2C \int_0^T \|\theta_-(t, 1)\|_{\mathbb{C}^m}^2 dt + 2C \int_0^T \|z_-(t, 1)\|_{\mathbb{C}^m}^2 dt. \end{aligned}$$

On the other hand, using the method of characteristics and the condition  $\theta_-(T, \cdot) = 0$ , we have

$$\|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + \int_0^T \|\theta_-(t, 1)\|_{\mathbb{C}^m}^2 dt \leq C \int_0^T \|\theta_-(t, 0)\|_{\mathbb{C}^m}^2 dt.$$

Combined with (60) this leads to the desired estimate (57).

- The estimate (57) suggests to consider the linear operators

$$P : L^2(0, T)^m \longrightarrow L^2(0, T)^m, \quad L : D(A^*) \longrightarrow L^2(0, T)^m,$$

defined by

$$(Pv)(t) = \int_0^T H(t, s)v(s) ds, \quad (Lz^1)(s) = z_-(s, 0).$$

From the previous point, (52) is fulfilled. It is also well-known that operators of the form of  $P$  are compact. Finally, we easily check with the method of characteristics that  $L$  satisfies the remaining estimate (53). This concludes the proof of Theorem 7.1.

## 8 Largest value of the minimal null control time

In this last section we will finally prove the second part of Theorem 1.11.

### 8.1 Characterization for systems $(\Lambda, -, Q, G)$

We start with systems of the form  $(\Lambda, -, Q, G)$ , we will deal with the initial system  $(\Lambda, M, Q, -)$  in the next section.

**Theorem 8.1.** *Let  $Q^0 \in \mathbb{R}^{p \times m}$  be in canonical form and let  $G \in L^\infty(0, 1)^{n \times m}$  with  $G_{--} = 0$  and  $G_{+-} \in \mathcal{C}(Q^0)$ .*

(i) *The system  $(\Lambda, -, Q^0, G)$  is null controllable in time  $T$  for every*

$$T \geq \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\}, \quad (61)$$

where we recall that  $\rho_0$  is defined in the statement of Theorem 1.11.

(ii) *Assume that the condition (13) fails and let  $G \in \mathbb{R}^{n \times m}$  be the constant matrix whose entries are all equal to zero except for*

$$g_{m+\rho_0+1, m} = 1.$$

*If the corresponding system  $(\Lambda, -, Q^0, G)$  is null controllable in time  $T$ , then  $T$  has to satisfy the condition (61).*

As for Theorem 1.11, we use the convention that the undefined quantities are simply not taken into account, which more precisely gives:

- If  $\rho_0 = 0$ , then the condition (61) is  $T \geq T_{m+1}(\Lambda) + T_m(\Lambda)$ .
- If  $\rho_0 = p$ , then the condition (61) is  $T \geq \max \{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_m(\Lambda) \}$ .

In the second part of the statement we only discussed the case when (13) fails since otherwise the time on the right-hand side of the inequality in (61) coincides with the time on the right-hand side of the inequality in (39) and it follows from item (i) of Theorem 5.1 that item (i) of Theorem 8.1 then becomes a necessary and sufficient condition.

This result shows in particular that the largest value that  $T_{\inf}(\Lambda, -, Q^0, G)$  can take with respect to  $G \in L^\infty(0, 1)^{n \times m}$  when  $G_{--} = 0$  and  $G_{+-} \in \mathcal{C}(Q^0)$  is equal to the quantity on the right-hand side of the inequality in (61). This can be extended to arbitrary boundary coupling matrices and arbitrary  $G \in L^\infty(0, 1)^{n \times m}$  thanks to Proposition 4.1, Proposition 6.1 and Theorem 7.1.

*Proof of Theorem 8.1.* 1) We begin with the proof of the first item. Let first  $i \in \{1, \dots, m\}$  be fixed. Since  $T \geq T_i(\Lambda)$ , which means that  $s_i^{\text{in}}(T, x) > 0$  for every  $x \in (0, 1)$  as we have seen in the first step of the proof of Theorem 5.1, and since  $G_{--} = 0$  by assumption, the null controllability condition  $y_i(T, \cdot) = 0$  is equivalent to (see (29) and (27))

$$u_i(s_i^{\text{in}}(T, \cdot)) = 0 \quad \text{in } (0, 1).$$

Since  $i \leq m$ , the map  $x \mapsto s_i^{\text{in}}(T, x)$  is non decreasing (see (22)) with  $s_i^{\text{in}}(T, 1) = T$ . Thus, the previous condition is also equivalent to

$$u_i = 0 \quad \text{in } (s_i^{\text{in}}(T, 0), T). \quad (62)$$

- 2) Let us now consider  $i \in \{m+1, \dots, n\}$ . Since  $T \geq T_i(\Lambda)$ , the null controllability condition  $y_i(T, x) = 0$  is equivalent to (see (29) and (28))

$$a_i(x) + b_i(x) = 0,$$

where

$$a_i(x) = \sum_{\substack{j=1 \\ j \notin \{c_1, \dots, c_{\rho_0}\}}}^m \left( q_{i-m, j}^0 y_j(s_i^{\text{in}}(T, x), 0) + \int_{s_i^{\text{in}}(T, x)}^T g_{ij}(\chi_i(s; T, x)) y_j(s, 0) ds \right),$$

and

$$b_i(x) = \sum_{k=1}^{\rho_0} \left( q_{i-m, c_k}^0 y_{c_k}(s_i^{\text{in}}(T, x), 0) + \int_{s_i^{\text{in}}(T, x)}^T g_{ic_k}(\chi_i(s; T, x)) y_{c_k}(s, 0) ds \right).$$

- We first consider the case  $i \geq m + \rho_0 + 1$  (which happens only if  $\rho_0 < p$ ). Clearly, we have  $b_i = 0$  in that situation since  $Q^0$  is in canonical form,  $G_{+-} \in \mathcal{C}(Q^0)$  and (12). Let us show that we can choose  $u_j$  for  $j \notin \{c_1, \dots, c_{\rho_0}\}$  so that  $a_i = 0$  as well. Since  $x \mapsto s_i^{\text{in}}(T, x)$  is non increasing for  $i \geq m + 1$  (recall (22)), it is sufficient to choose it such that

$$y_j(\cdot, 0) = 0 \quad \text{in } (s_i^{\text{in}}(T, 1), T). \quad (63)$$

Since  $T \geq T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda)$  by assumption, we have in particular  $T \geq T_i(\Lambda) + T_j(\Lambda)$  for the indices  $i, j$  considered (recall (5)). This condition can be written as  $T \geq s_i^{\text{out}}(s_j^{\text{out}}(0, 1), 0)$  (see (21) and (24)) or, equivalently (see (23)),

$$s_j^{\text{in}}(s_i^{\text{in}}(T, 1), 0) \geq 0.$$

Since  $s \mapsto s_j^{\text{in}}(s, 0)$  is increasing (see (22)), this is equivalent to

$$s_j^{\text{in}}(s, 0) > 0, \quad \forall s \in (s_i^{\text{in}}(T, 1), T).$$

As a result, we see that (63) holds if, and only if, (see (29), (27) and recall that  $G_{--} = 0$ )

$$u_j(s_j^{\text{in}}(\cdot, 0)) = 0 \quad \text{in } (s_i^{\text{in}}(T, 1), T).$$

Using again that  $s \mapsto s_j^{\text{in}}(s, 0)$  is increasing, this means that

$$u_j = 0 \quad \text{in } (s_j^{\text{in}}(s_i^{\text{in}}(T, 1), 0), s_j^{\text{in}}(T, 0)).$$

Observe that this is compatible with (62) since these two intervals are disjoint.

- Let us now consider the case  $i \leq m + \rho_0$  (which happens only if  $\rho_0 \neq 0$ ). Since  $Q^0$  is in canonical form,  $G_{+-} \in \mathcal{C}(Q^0)$  and (12), we see that  $a_i(x) + b_i(x) = 0$  is equivalent to

$$a_i(x) + y_{c_{i-m}}(s_i^{\text{in}}(T, x), 0) + \sum_{k=i-m+1}^{\rho_0} \int_{s_i^{\text{in}}(T, x)}^T g_{ic_k}(\chi_i(s; T, x)) y_{c_k}(s, 0) ds = 0. \quad (64)$$

Let us show that we can choose  $u_{c_1}, \dots, u_{c_{\rho_0}}$  so that this identity is satisfied. By assumption, we have  $T \geq T_i(\Lambda) + T_{c_{i-m}}(\Lambda)$  for every  $i \in \{m+1, \dots, m + \rho_0\}$ . As in the previous point we can check that this condition can be written as

$$s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, 1), 0) \geq 0.$$

Since  $x \mapsto s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0)$  is decreasing (see (22)), this is equivalent to

$$s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0) > 0, \quad \forall x \in (0, 1).$$

As a result, we see that (64) holds if, and only if, (see (29), (27) and recall that  $G_{--} = 0$ )

$$u_{c_{i-m}} \left( s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0) \right) = -a_i(x) - \sum_{k=i-m+1}^{\rho_0} \int_{s_i^{\text{in}}(T, x)}^T g_{ic_k}(\chi_i(s; T, x)) y_{c_k}(s, 0) ds.$$

Since  $a_i$  is known (it only concerns  $u_j$  for  $j \notin \{c_1, \dots, c_{\rho_0}\}$ ), we see by induction (starting with  $i = m + \rho_0$ ) that this formula determines the values of  $u_{c_{i-m}}$  in the interval

$$\left( s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, 1), 0), \quad s_{c_{i-m}}^{\text{in}}(T, 0) \right)$$

(the map  $x \mapsto s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0)$  is non increasing and  $s_i^{\text{in}}(T, 0) = T$ ). Observe once again that this is compatible with (62) since these two intervals are disjoint.

This concludes the proof of the first item (i) of Theorem 8.1.

- 3) Let us now prove item (ii) of Theorem 8.1. Assume that the condition (13) fails, let  $G$  be the constant matrix introduced in the statement, and assume that the corresponding system  $(\Lambda, -, Q^0, G)$  is null controllable in time  $T$ . Since (13) fails, the condition (61) is simply

$$T \geq T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda). \quad (65)$$

Since the system  $(\Lambda, -, Q^0, G)$  is null controllable in time  $T$  by assumption, the following  $2 \times 2$  subsystem also has to be null controllable in time  $T$ :

$$\begin{cases} \frac{\partial y_m}{\partial t}(t, x) + \lambda_m(x) \frac{\partial y_m}{\partial x}(t, x) = 0, \\ \frac{\partial y_{m+\rho_0+1}}{\partial t}(t, x) + \lambda_{m+\rho_0+1}(x) \frac{\partial y_{m+\rho_0+1}}{\partial x}(t, x) = y_m(t, 0), \\ y_m(t, 1) = u_m(t), \quad y_{m+\rho_0+1}(t, 0) = q_{\rho_0+1, m}^0 y_m(t, 0). \end{cases}$$

Let us show that, whether  $q_{\rho_0+1, m}^0 = 1$  or  $q_{\rho_0+1, m}^0 = 0$ , we necessarily have (65). If  $q_{\rho_0+1, m}^0 = 1$ , then this follows from item (i) of Theorem 5.1. Let us then consider the case  $q_{\rho_0+1, m}^0 = 0$ . As before, it is clearly necessary that  $T \geq T_{m+\rho_0+1}(\Lambda)$  and, under this condition, the null controllability condition  $y_{m+\rho_0+1}(T, x) = 0$  becomes equivalent to (see (29) and (28))

$$\int_{s_{m+\rho_0+1}^{\text{in}}(T, x)}^T y_m(s, 0) ds = 0.$$

Using the change of variable  $\xi \mapsto s = s_{m+\rho_0+1}^{\text{in}}(T, \xi)$ , this holds if, and only if,

$$\int_0^x y_m(s_{m+\rho_0+1}^{\text{in}}(T, \xi), 0) \frac{\partial s_{m+\rho_0+1}^{\text{in}}(T, \xi)}{\partial \xi} d\xi = 0.$$

Taking the derivative with respect to  $x$ , this is also equivalent to

$$y_m(s_{m+\rho_0+1}^{\text{in}}(T, \cdot), 0) = 0 \quad \text{in } (0, 1).$$

It is now not difficult to see that we can choose  $u_m$  such that this condition holds if, and only if, we have (65). □

## 8.2 Proof of the second part of Theorem 1.11

Let us now show how to combine all the previous results in order to obtain the desired characterization of the largest minimal null control time for the initial system  $(\Lambda, M, Q, -)$ .

*Proof of item (ii) of Theorem 1.11.* Let  $Q \in \mathbb{R}^{p \times m}$  be fixed.

- 1) • By item (i) of Theorem 8.1, we have

$$T_{\inf}(\Lambda, -, Q^0, G) \leq \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\},$$

for every  $G \in L^\infty(0, 1)^{n \times m}$  with  $G_{--} = 0$  and  $G_{+-} \in \mathcal{C}(Q^0)$ , where  $Q^0$  is the canonical form of  $Q$ .

- By Theorem 7.1, this inequality remains true for every  $G \in L^\infty(0, 1)^{n \times m}$  with  $G_{+-} \in \mathcal{C}(Q^0)$ .
- By Proposition 6.1, this inequality remains true for every  $G \in L^\infty(0, 1)^{n \times m}$ .
- By Proposition 4.1, this inequality remains true by changing  $Q^0$  into  $Q$ .
- By Proposition 3.1 and Theorem 3.2, this inequality remains true for the system  $(\Lambda, M, Q, -)$  for any  $M \in L^\infty(0, 1)^{n \times n}$ .

In summary, we have established the following upper bound:

$$T_{\inf}(\Lambda, M, Q) \leq \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\},$$

valid for every  $M \in L^\infty(0, 1)^{n \times n}$ .

- 2) Let us now show that this upper bound is reached for some special  $M$ . If the condition (13) is satisfied, then this upper bound coincides with the lower bound, and we know that this latter is reached for  $M = 0$  (Remark 1.13). Let us now assume that the condition (13) is not satisfied.

- Then, we know from Theorem 8.1 that this upper bound is the minimal null control time of the system  $(\Lambda, -, Q^0, G)$  for the constant matrix  $G \in \mathbb{R}^{n \times m}$  whose entries are all equal to zero except for

$$g_{m+\rho_0+1, m} = 1.$$

Let us now find the corresponding matrix  $M$ .

- We decompose  $Q$  in its canonical form:  $LQU = Q^0$ . By Proposition 4.1, this upper bound is the minimal null control time of the system  $(\Lambda, -, Q, \widehat{G})$  for any  $\widehat{G}$  such that  $\Theta(\widehat{G}) = G$ . Now, it follows from the proof of Proposition 4.1 that, for constant matrices,  $\Theta$  is simply given by

$$\Theta(\widehat{G}) = \begin{pmatrix} U^{-1}\widehat{G}_{--}U \\ L\widehat{G}_{+-}U \end{pmatrix}, \quad \forall \widehat{G} \in \mathbb{R}^{n \times m}.$$

Therefore,  $\widehat{G}$  is the matrix whose entries are all equal to zero except for

$$\widehat{g}_{m+i, m} = \ell^{i, \rho_0+1}, \quad \forall i \in \{\rho_0 + 1, \dots, p\},$$

where  $L^{-1} = (\ell^{ij})_{1 \leq i, j \leq p}$ .

- By Theorem 3.2, this upper bound is the minimal null control time of the system  $(\Lambda, M, Q, -)$  for any  $M \in \mathcal{M}$  such that  $\Gamma_A(M) = \widehat{G}$  for some  $A \in \mathcal{F}$ . Let us determine  $A$  and  $M$  such that this identity holds. By definition of  $\Gamma_A(M)$  (see (32)), this is equivalent to

$$\begin{cases} 0 = -K_{--}(x, 0)\Lambda_{--}(0) - K_{-+}(x, 0)\Lambda_{++}(0)Q, \\ \widehat{G}_{+-} = -K_{+-}(x, 0)\Lambda_{--}(0) - K_{++}(x, 0)\Lambda_{++}(0)Q, \end{cases}$$

where  $K$  is the solution to the kernel equations (33) with additional boundary conditions (35)-(36) provided by  $A$ . Let us rewrite these kernel equations by blocks:

$$\begin{cases} \Lambda_{--}(x) \frac{\partial K_{--}}{\partial x}(x, \xi) + \frac{\partial K_{--}}{\partial \xi}(x, \xi) \Lambda_{--}(\xi) \\ + K_{--}(x, \xi) \left( \frac{\partial \Lambda_{--}}{\partial \xi}(\xi) + M_{--}(\xi) \right) + K_{-+}(x, \xi) M_{+-}(\xi) = 0, \\ \Lambda_{--}(x) K_{--}(x, x) - K_{--}(x, x) \Lambda_{--}(x) = M_{--}(x). \end{cases}$$

$$\begin{cases} \Lambda_{--}(x) \frac{\partial K_{-+}}{\partial x}(x, \xi) + \frac{\partial K_{-+}}{\partial \xi}(x, \xi) \Lambda_{++}(\xi) \\ + K_{--}(x, \xi) M_{-+}(\xi) + K_{-+}(x, \xi) \left( \frac{\partial \Lambda_{++}}{\partial \xi}(\xi) + M_{++}(\xi) \right) = 0, \\ \Lambda_{--}(x) K_{-+}(x, x) - K_{-+}(x, x) \Lambda_{++}(x) = M_{-+}(x). \end{cases}$$

$$\begin{cases} \Lambda_{++}(x) \frac{\partial K_{+-}}{\partial x}(x, \xi) + \frac{\partial K_{+-}}{\partial \xi}(x, \xi) \Lambda_{--}(\xi) \\ + K_{+-}(x, \xi) \left( \frac{\partial \Lambda_{--}}{\partial \xi}(\xi) + M_{--}(\xi) \right) + K_{++}(x, \xi) M_{+-}(\xi) = 0, \\ \Lambda_{++}(x) K_{+-}(x, x) - K_{+-}(x, x) \Lambda_{--}(x) = M_{+-}(x). \end{cases}$$

$$\begin{cases} \Lambda_{++}(x) \frac{\partial K_{++}}{\partial x}(x, \xi) + \frac{\partial K_{++}}{\partial \xi}(x, \xi) \Lambda_{++}(\xi) \\ + K_{+-}(x, \xi) M_{-+}(\xi) + K_{++}(x, \xi) \left( \frac{\partial \Lambda_{++}}{\partial \xi}(\xi) + M_{++}(\xi) \right) = 0, \\ \Lambda_{++}(x) K_{++}(x, x) - K_{++}(x, x) \Lambda_{++}(x) = M_{++}(x). \end{cases}$$

Note that the subsystems satisfied by  $(K_{--}, K_{-+})$  and  $(K_{+-}, K_{++})$  are not coupled. By uniqueness of the solution to these equations (see Theorem 3.4), we see that

$$\begin{aligned} M_{-+} = 0 &\implies K_{-+} = 0, \\ M_{++} = A_{++} = 0, \quad M_{-+} = 0 &\implies K_{++} = 0, \\ M_{--} = A_{--} = 0, \quad K_{-+} = 0 &\implies K_{--} = 0. \end{aligned}$$

Therefore, it only remains to determine  $M_{+-}$  such that

$$K_{+-}(x, 0) = -\widehat{G}_{+-} \Lambda_{--}(0)^{-1}.$$

Let  $i \in \{m+1, \dots, n\}$  and  $j \in \{1, \dots, m\}$  be fixed. The equation for  $k_{ij}$  is now simply

$$\begin{cases} \lambda_i(x) \frac{\partial k_{ij}}{\partial x}(x, \xi) + \frac{\partial k_{ij}}{\partial \xi}(x, \xi) \lambda_j(\xi) + k_{ij}(x, \xi) \frac{\partial \lambda_j}{\partial \xi}(\xi) = 0, \\ k_{ij}(x, x) = \frac{m_{ij}(x)}{\lambda_i(x) - \lambda_j(x)}. \end{cases} \quad (66)$$

Let  $s \mapsto \zeta_{ij}(s; x, \xi)$  be the associated characteristic passing through  $(x, \xi)$ :

$$\begin{cases} \frac{\partial \zeta_{ij}}{\partial s}(s; x, \xi) = \frac{\lambda_j(\zeta_{ij}(s; x, \xi))}{\lambda_i(s)}, \\ \zeta_{ij}(x; x, \xi) = \xi. \end{cases}$$

The solution to (66) is explicit:

$$k_{ij}(x, \xi) = \frac{m_{ij}(s_{ij}^{\text{in}}(x, \xi))}{\lambda_i(s_{ij}^{\text{in}}(x, \xi)) - \lambda_j(s_{ij}^{\text{in}}(x, \xi))} \frac{\lambda_j(s_{ij}^{\text{in}}(x, \xi))}{\lambda_j(\xi)},$$

where  $s_{ij}^{\text{in}}(x, \xi) \in (0, x)$  is the unique solution to

$$\zeta_{ij}(s_{ij}^{\text{in}}(x, \xi); x, \xi) = s_{ij}^{\text{in}}(x, \xi).$$

Thus, the desired condition  $k_{ij}(\cdot, 0) = -\hat{g}_{ij}/\lambda_j(0)$  is equivalent to

$$m_{ij}(s_{ij}^{\text{in}}(x, 0)) = \frac{\lambda_i(s_{ij}^{\text{in}}(x, 0)) - \lambda_j(s_{ij}^{\text{in}}(x, 0))}{-\lambda_j(s_{ij}^{\text{in}}(x, 0))} \hat{g}_{ij}, \quad x \in (0, 1).$$

□

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## A An example of non equivalent hyperbolic systems

In this appendix we present an explicit example of hyperbolic systems which are not equivalent in the sense of Definition 1.18. This example is important to illustrate that, in general, it is not possible to obtain a simpler system than the one we obtained in the present article if we only use invertible transformations (see Remark 6.2). It also motivates the use of the compactness-uniqueness method to establish the important result Theorem 7.1. We refer to [CN19, Section 4.3] for a close but different example.

We consider the following simple  $3 \times 3$  systems with constant coefficients:

$$\begin{cases} \frac{\partial y_1}{\partial t}(t, x) - \frac{\partial y_1}{\partial x}(t, x) = 0, \\ \frac{\partial y_2}{\partial t}(t, x) - \frac{1}{2} \frac{\partial y_2}{\partial x}(t, x) = ay_1(t, 0), \\ \frac{\partial y_3}{\partial t}(t, x) + \frac{\partial y_3}{\partial x}(t, x) = by_2(t, 0), \end{cases} \quad (67)$$

where  $a, b \in \mathbb{R}$  are some parameters, and with boundary conditions

$$\begin{cases} y_1(t, 1) = u_1(t), \\ y_2(t, 1) = u_2(t), \end{cases} \quad y_3(t, 0) = y_1(t, 0). \quad (68)$$

We are in the case  $m = 2, p = 1$  and the matrices  $\Lambda, G$  and  $Q$  are

$$\Lambda = \left( \begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ \hline 0 & 0 & 1 \end{array} \right), \quad G = G_{ab} = \begin{pmatrix} 0 & 0 \\ a & 0 \\ 0 & b \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Note as well that we are in an ideal configuration:

- $Q$  is in canonical form.
- $(G_{ab})_{--}$  is strictly lower triangular.
- $(G_{ab})_{+-} \in \mathcal{C}(Q)$ .

Clearly,  $\rho = \rho_0 = 1$  and  $(r_1, c_1) = (1, 1)$ . It follows from the results of the present article (actually, a direct proof is also possible) that the minimal null control time of the system (67)-(68) is

$$T_{\inf}(\Lambda, -, Q, G_{ab}) = 2, \quad \forall a, b \in \mathbb{R}.$$

In particular, the system (67)-(68) is null controllable in time  $T$  for every  $T > 2$ . Let us now study the null controllability properties of this system in this critical time:

**Proposition A.1.** *The system (67)-(68) is null controllable in time  $T = 2$  if, and only if,*

$$ab \notin \Sigma = \left\{ -\left(\frac{\pi}{2} + k\pi\right)^2 \mid k \in \mathbb{N} \right\}. \quad (69)$$

**Remark A.2.** It follows from this result and Proposition 1.19 that

$$(\Lambda, -, Q, G_{ab}) \quad \text{is not equivalent to} \quad (\Lambda, -, Q, G_{cd}), \quad \text{if } ab \in \Sigma, cd \notin \Sigma.$$

In particular, it is not possible to transform the system  $(\Lambda, -, Q, G_{ab})$  into  $(\Lambda, -, Q, G_{0b})$  or  $(\Lambda, -, Q, G_{a0})$  when  $ab \in \Sigma$  (in other words, we cannot remove  $(G_{ab})_{--}$  nor  $(G_{ab})_{+-}$  in this case).

*Proof of Proposition A.1.* The solution to the system (67)-(68) is explicit (see Section 2):

$$y_1(t, x) = \begin{cases} u_1(t-1+x) & \text{if } t-1+x > 0, \\ y_1^0(t+x) & \text{if } t-1+x < 0, \end{cases}$$

$$y_2(t, x) = \begin{cases} u_2(t-2(1-x)) + a \int_{t-2(1-x)}^t y_1(s, 0) ds & \text{if } t-2(1-x) > 0, \\ y_2^0\left(\frac{t}{2}+x\right) + a \int_0^t y_1(s, 0) ds & \text{if } t-2(1-x) < 0, \end{cases}$$

$$y_3(t, x) = \begin{cases} y_1(t-x, 0) + b \int_{t-x}^t y_2(s, 0) ds & \text{if } t-x > 0, \\ y_3^0(-t+x) + b \int_0^t y_2(s, 0) ds & \text{if } t-x < 0. \end{cases}$$

Clearly, the null controllability condition  $y_1(2, \cdot) = 0$  is satisfied if, and only if,

$$u_1 = 0 \quad \text{in } (1, 2).$$

Similarly, the null controllability condition  $y_2(2, \cdot) = 0$  holds if, and only if,

$$u_2(t) = -a \int_t^2 y_1(s, 0) ds, \quad t \in (0, 2).$$

Thus, the control  $u_2$  is uniquely determined once the values of the control  $u_1$  in  $(0, 1)$  are known. The remaining condition  $y_3(2, x) = 0$  is equivalent to

$$y_1(2-x, 0) + b \int_{2-x}^2 y_2(s, 0) ds = 0,$$

and thus to

$$u_1(1-x) + b \int_{2-x}^2 y_2^0\left(\frac{s}{2}\right) ds + abx \int_0^1 y_1^0(\theta) d\theta + ab \int_{2-x}^2 \int_1^s u_1(\theta-1) d\theta ds = 0.$$

Using the change of variables  $t = 1-x$  and  $\sigma = \theta-1$ , this is also equivalent to

$$u_1(t) + ab \int_t^1 \int_0^s u_1(\sigma) d\sigma ds = f(t), \quad t \in (0, 1), \quad (70)$$

where we introduced the following function depending only on the initial data:

$$f(t) = -b \int_{1+t}^2 y_2^0\left(\frac{s}{2}\right) ds - ab(1-t) \int_0^1 y_1^0(\theta) d\theta. \quad (71)$$

This identity can be rewritten as

$$(\text{Id} - K) \begin{pmatrix} u_1 \\ \int_0^{\cdot} u_1(\sigma) d\sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad (72)$$

where  $K : L^2(0, 1)^2 \rightarrow L^2(0, 1)^2$  is the operator defined by

$$\left( K \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) (t) = \begin{pmatrix} -ab \int_t^1 \beta(s) ds \\ \int_0^t \alpha(s) ds \end{pmatrix}.$$

Since  $K$  is compact, the Fredholm alternative says that (72) has a solution if, and only if,

$$\begin{pmatrix} f \\ 0 \end{pmatrix} \in (\ker(\text{Id} - K^*))^\perp. \quad (73)$$

A simple computation shows that

$$\left( K^* \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \right) (s) = \begin{pmatrix} \int_s^1 \tilde{\beta}(t) dt \\ -ab \int_0^s \tilde{\alpha}(t) dt \end{pmatrix}.$$

It follows that  $\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \in \ker(\text{Id} - K^*)$  if, and only if,  $\tilde{\beta}(s) = -ab \int_0^s \tilde{\alpha}(t) dt$  and  $\tilde{\alpha}$  solves the following second order linear ODE:

$$\begin{cases} \tilde{\alpha}''(s) - ab\tilde{\alpha}(s) = 0, \\ \tilde{\alpha}'(0) = 0, \\ \tilde{\alpha}(1) = 0. \end{cases}$$

We can check that this ODE has a nonzero solution if, and only if,

$$ab \in \Sigma,$$

where  $\Sigma$  is the set introduced in (69). It follows that we have two possibilities:

- If  $ab \notin \Sigma$ , then  $\ker(\text{Id} - K^*) = \{0\}$  and (72) has a (unique) solution  $u_1$ . This shows that the system (67)-(68) is null controllable in time  $T = 2$ .
- If  $ab \in \Sigma$ , then there exists a nonzero  $\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \in \ker(\text{Id} - K^*)$ . Necessarily,  $\tilde{\alpha} \neq 0$  and thus

$$\exists f \in C_c^\infty(0, 1), \quad \langle \tilde{\alpha}, f \rangle_{L^2(0,1)} \neq 0.$$

It is clear that we can construct  $y_2^0$  and  $y_1^0$  that satisfy (71) for this  $f$  (take for instance  $y_1^0 = 0$  and  $y_2^0(x) = f'(2x - 1)/b$  for  $x \in [1/2, 1]$  and  $y_2^0(x) = 0$  otherwise, note that  $b \neq 0$  in the case considered). For such a  $f$ , the condition (73) fails and thus there is no corresponding solution  $u_1$  to (70), meaning that the system (67)-(68) is not null controllable in time  $T = 2$ .

□

**Remark A.3.** We have seen during the proof that, when  $ab \notin \Sigma$ , the control that brings the solution to zero in the critical time  $T = 2$  is unique (it can also be written explicitly).

## B Proof of the abstract compactness-uniqueness result

The goal of this appendix is to give a proof of Theorem 7.7. It is inspired from the proofs of [CN21a, Theorem 2] and [DO18, Lemma 2.6] (see also the references therein).

Here and in what follows, it will be more convenient to work with the expression  $S(t)^* z^1$  rather than  $z(t) = S(T - t)^* z^1$ . The corresponding assumptions (52), (53) and (54) become:

$$\|S(T)^* z^1\|_H^2 \leq C \left( \int_0^T \|B^* S(t)^* z^1\|_U^2 dt + \|PLz^1\|_{E_2}^2 \right), \quad (74)$$

$$\|Lz^1\|_{E_1}^2 \leq C \left( \|S(T)^* z^1\|_H^2 + \int_0^T \|B^* S(t)^* z^1\|_U^2 dt \right), \quad (75)$$

$$\|S(T - t_2)^* z^1\|_H^2 \leq C \left( \|S(T - t_1)^* z^1\|_H^2 + \int_{T-t_2}^{T-t_1} \|B^* S(t)^* z^1\|_U^2 dt \right). \quad (76)$$

- 1) Let  $T > T_0$  be fixed. By duality (see Theorem 7.6), we have to prove that there exists  $C > 0$  such that, for every  $z^1 \in D(A^*)$ ,

$$\|S(T)^* z^1\|_H^2 \leq C \int_0^T \|B^* S(t)^* z^1\|_U^2 dt. \quad (77)$$

We argue by contradiction and assume that the observability inequality (77) does not hold. Then, there exists a sequence  $(z_n^1)_{n \geq 1} \subset D(A^*)$  such that, for every  $n \geq 1$ ,

$$\|S(T)^* z_n^1\|_H^2 > n \int_0^T \|B^* S(t)^* z_n^1\|_U^2 dt.$$

In particular  $S(T)^* z_n^1 \neq 0$  and we can normalize  $z_n^1$ , still denoted by the same, in such a way that

$$\|S(T)^* z_n^1\|_H = 1, \quad \int_0^T \|B^* S(t)^* z_n^1\|_U^2 dt \xrightarrow{n \rightarrow +\infty} 0.$$

Using the estimate (75) we obtain that

$$(Lz_n^1)_{n \geq 1} \text{ is bounded in } E_1.$$

Since  $P$  is compact, we can extract a subsequence, still denoted by  $(z_n^1)_{n \geq 1}$ , such that

$$(PLz_n^1)_{n \geq 1} \text{ converges in } E_2.$$

Using now the estimate (74), we obtain that  $(S(T)^*z_n^1)_{n \geq 1}$  is a Cauchy sequence in  $H$ , and thus converges: there exists  $f \in H$  such that

$$S(T)^*z_n^1 \xrightarrow{n \rightarrow +\infty} f \text{ in } H.$$

Besides,  $f \neq 0$  since  $\|f\|_H = 1$ . In other words, we have shown that

$$N_T \neq \{0\}, \quad (78)$$

where  $N_\tau$  is the subspace defined for every  $\tau > 0$  by

$$N_\tau = \left\{ f \in H \mid \begin{array}{l} \exists (z_n^1)_{n \geq 1} \subset D(A^*), \\ S(\tau)^*z_n^1 \xrightarrow{n \rightarrow +\infty} f \text{ in } H, \\ B^*S(\cdot)^*z_n^1 \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^2(0, \tau; U) \end{array} \right\}.$$

Let us now study the properties of these subspaces.

- 2) First of all, it forms a non-increasing sequence of subspaces:

$$N_{\tau_2} \subset N_{\tau_1}, \quad \forall \tau_2 \geq \tau_1 > 0. \quad (79)$$

Indeed, if  $f \in N_{\tau_2}$  and  $(z_n^1)_{n \geq 1} \subset D(A^*)$  denotes an associated sequence, then we easily check that  $f \in N_{\tau_1}$  by considering the sequence  $(S(\tau_2 - \tau_1)^*z_n^1)_{n \geq 1}$ .

- 3) Let us now show that

$$\dim N_\tau < +\infty, \quad \forall \tau > T_0. \quad (80)$$

By Riesz theorem, it is equivalent to show that the closed unit ball of  $N_\tau$  is compact. Let then  $(f^k)_{k \geq 1} \subset N_\tau$  be such that  $\|f^k\|_H \leq 1$  for every  $k \geq 1$ . Let  $(z_n^{1,k})_{n \geq 1} \subset D(A^*)$  be an associated sequence. In particular, for every  $k \geq 1$ , there exists  $n_k \geq 1$  such that, denoting by  $w^{1,k} = z_{n_k}^{1,k}$ , we have

$$\|S(\tau)^*w^{1,k} - f^k\|_H \leq \frac{1}{k}, \quad \|B^*S(\cdot)^*w^{1,k}\|_{L^2(0, \tau; U)} \leq \frac{1}{k}, \quad \forall k \geq 1.$$

Since  $(f^k)_{k \geq 1}$  is bounded, so is  $(S(\tau)^*w^{1,k})_{k \geq 1}$ . Using the same reasoning as in Step 1), we deduce from the estimates (75) and (74) that  $(S(\tau)^*w^{1,k})_{k \geq 1}$  has a Cauchy subsequence. It follows that  $(f^k)_{k \geq 1}$  has a Cauchy subsequence as well, and thus has a converging subsequence.

- 4) The next step is to establish that

$$N_\tau \subset D(A^*), \quad A^*(N_\tau) \subset N_{\tau - \varepsilon}, \quad \forall \tau \in (T_0, T), \quad \forall \varepsilon \in (0, \tau - T_0). \quad (81)$$

Let then  $f \in N_\tau$ . By definition, there exists a sequence  $(z_n^1)_{n \geq 1} \subset D(A^*)$  such that

$$S(\tau)^*z_n^1 \xrightarrow{n \rightarrow +\infty} f \text{ in } H, \quad (82)$$

$$B^*S(\cdot)^*z_n^1 \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^2(0, \tau; U). \quad (83)$$

Using the estimate (76) with  $t_1 = T - \tau$  and  $t_2 = T - (\tau - \varepsilon)$ , we see that

$$(S(\tau - \varepsilon)^* z_n^1)_{n \geq 1} \text{ is bounded in } H.$$

As before, it follows from the estimates (75) and (74) that there exists  $g \in H$  such that

$$S(\tau - \varepsilon)^* z_n^1 \xrightarrow{n \rightarrow +\infty} g \text{ in } H. \quad (84)$$

Noting (82), by uniqueness of the limit, we have

$$f = S(\varepsilon)^* g.$$

Let us now prove that  $g \in D(A^*)$ . By definition of the domain of the generator of a semigroup, we have to show that, for any sequence  $t_n > 0$  with  $t_n \rightarrow 0$  as  $n \rightarrow +\infty$ , the sequence

$$u_n = \frac{S(t_n)^* g - g}{t_n}$$

converges in  $H$  as  $n \rightarrow +\infty$  and that its limit does not depend on the sequence  $(t_n)_n$ . Let  $n_0 \geq 1$  be large enough so that  $t_n \leq \varepsilon$  for every  $n \geq n_0$ . From (84) and (83) we easily see that

$$S(t)^* g \in N_{\tau - \varepsilon}, \quad \forall t \in [0, \varepsilon]. \quad (85)$$

Thus,

$$u_n \in N_{\tau - \varepsilon}, \quad \forall n \geq n_0.$$

Let now  $\mu \in \rho(A^*) \neq \emptyset$  be fixed and let us introduce the following norm on  $N_{\tau - \varepsilon}$ :

$$\|z\|_{-1} = \|(\mu - A^*)^{-1} z\|_H.$$

Since  $(\mu - A^*)^{-1} g \in D(A^*)$ , we have

$$(\mu - A^*)^{-1} u_n = \frac{S(t_n)^* - \text{Id}}{t_n} (\mu - A^*)^{-1} g \xrightarrow{n \rightarrow +\infty} A^* (\mu - A^*)^{-1} g \quad \text{in } H. \quad (86)$$

Therefore,  $(u_n)_{n \geq n_0}$  is a Cauchy sequence in  $N_{\tau - \varepsilon}$  for the norm  $\|\cdot\|_{-1}$ . Since  $N_{\tau - \varepsilon}$  is finite dimensional (reall (80)), all the norms are equivalent on  $N_{\tau - \varepsilon}$ . Thus,  $(u_n)_{n \geq n_0}$  is a Cauchy sequence for the usual norm  $\|\cdot\|_H$  as well and, as a result, converges for this norm. It is clear from (86) that its limit does not depend on the sequence  $(t_n)_n$ . This shows that  $g \in D(A^*)$  and thus  $f = S(\varepsilon)^* g \in D(A^*)$ . In addition, we have

$$A^* f = A^* S(\varepsilon)^* g = \lim_{h \rightarrow 0^+} \frac{S(\varepsilon)^* g - S(\varepsilon - h)^* g}{h} \in N_{\tau - \varepsilon} \quad (\text{by (85)}).$$

This shows that  $A^*(N_\tau) \subset N_{\tau - \varepsilon}$ .

5) Let us now prove that

$$N_\tau \subset \ker B^*, \quad \forall \tau \in (T_0, T).$$

Let  $\varepsilon \in (0, \tau - T_0)$  be arbitrary. We use the same notations as in the previous step. Since, by assumption,  $H$  is an admissible subspace for  $(A, B)$  (see Definition 7.4), the map  $z^1 \in D(A^*) \mapsto B^* S(\varepsilon - \cdot)^* z^1 \in L^2(0, \varepsilon; U)$  can be extended to a bounded linear operator  $\Psi \in \mathcal{L}(H, L^2(0, \varepsilon; U))$ . From (84) and continuity of  $\Psi$ , we have

$$\Psi S(\tau - \varepsilon)^* z_n^1 \xrightarrow{n \rightarrow +\infty} \Psi g, \quad \text{in } L^2(0, \varepsilon; U).$$

Since  $z_n^1 \in D(A^*)$ , we have  $(\Psi S(\tau - \varepsilon)^* z_n^1)(t) = B^* S(\tau - t) z_n^1$  for  $t \in (0, \varepsilon)$ . From (83) and uniqueness of the limit, we deduce that

$$\Psi g = 0.$$

Since  $g \in D(A^*)$ , we have  $(\Psi g)(t) = B^* S(\varepsilon - t)^* g$  and the map  $t \in [0, \varepsilon] \mapsto B^* S(\varepsilon - t)^* g$  is continuous. It follows that

$$B^* f = B^* S(\varepsilon)^* g = (\Psi g)(0) = 0.$$

6) Next, we observe that there exist  $\tau \in (T_0, T)$  and  $\varepsilon \in (0, \tau - T_0)$  such that

$$N_\tau = N_{\tau - \varepsilon}. \quad (87)$$

Indeed, from (80) and (79), the sequence of integers  $(\dim N_{T - (T - T_0)/k})_{k \geq 2}$  is non-increasing and thus stationary: there exists  $k_0 \geq 2$  such that

$$\dim N_{T - (T - T_0)/k} = \dim N_{T - (T - T_0)/k_0}, \quad \forall k \geq k_0.$$

Denoting by  $\delta = (T - T_0)/k_0 \in (0, T - T_0)$  and using (79), we then have

$$N_\tau = N_{T - \delta}, \quad \forall \tau \in [T - \delta, T).$$

The desired claim easily follows.

7) Consequently, for  $\tau \in (T_0, T)$  and  $\varepsilon \in (0, \tau - T_0)$  fixed such that (87) holds, the restriction of  $A^*$  to  $N_\tau$  is a linear operator from the finite dimensional space  $N_\tau$  into itself (recall (81)). Besides,  $N_\tau \neq \{0\}$  since it contains  $N_T$  by (79) and we have (78). Therefore, this restriction has at least one eigenvalue (recall that  $H$  is a complex Hilbert space), i.e. there exist  $\lambda \in \mathbb{C}$  and a nonzero  $\phi \in N_\tau$  such that

$$A^* \phi = \lambda \phi.$$

Since  $N_\tau \subset \ker B^*$ , we also have  $\phi \in \ker B^*$  and this is a contradiction with the Fattorini-Hautus test (51).

This concludes the proof of Theorem 7.7.

**Remark B.1.** Let us stress that the end of our proof differs from the one in [CN21a, Section 2.2]. Indeed, in this reference, the conclusion of the proof relied on the fact that the semigroup is nilpotent, that is

$$\exists T > 0, \quad S(T)^* z^1 = 0, \quad \forall z^1 \in H.$$

This readily implies that the operator  $A^*$  has no eigenvalues and this is how the authors conclude that  $N_T = \{0\}$ . On the other hand, in our proof above, we only made use of the Fattorini-Hautus test (51) (which is trivially checked if the operator  $A^*$  has no eigenvalues). Besides, this is optimal, in the sense that this test is always a necessary condition for the system  $(A, B)$  to be null controllable in some time.

Finally, let us add that for the example of the hyperbolic system  $(\Lambda, -, Q, G)$  the corresponding adjoint semigroup is not always nilpotent. Notably, the strictly lower triangular structure of  $G_{--}$  was used at the end of [CN21a, Section 2.2] to prove such a property.

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