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1 Equivalent one-dimensional first-order linear hyperbolic
2 systems and range of the minimal null control time with
3 respect to the internal coupling matrix

4 Long Hu* Guillaume Olive†

5 September 16, 2021

6 **Abstract**

7 In this paper, we are interested in the minimal null control time of one-dimensional
8 first-order linear hyperbolic systems by one-sided boundary controls. Our main result is an
9 explicit characterization of the smallest and largest values that this minimal null control time
10 can take with respect to the internal coupling matrix. In particular, we obtain a complete
11 description of the situations where the minimal null control time is invariant with respect
12 to all the possible choices of internal coupling matrices. The proof relies on the notion of
13 equivalent systems, in particular the backstepping method, a canonical LU -decomposition
14 for boundary coupling matrices and a compactness-uniqueness method adapted to the null
15 controllability property.

16 **Keywords.** Hyperbolic systems, Boundary controllability, Minimal null control time, Equiv-
17 alent systems, Backstepping method, LU -decomposition, Compactness-uniqueness method

18 **2010 Mathematics Subject Classification.** 35L40, 93B05

19 **1 Introduction and main result**

20 **1.1 Problem description**

21 In this article we are interested in the null controllability properties of the following class of
22 one-dimensional first-order linear hyperbolic systems, which appears for instance in linearized
23 Saint-Venant equations and many other physical models of balance laws (see e.g. [BC16, Chapter
24 1] and many references therein):

25
$$\begin{cases} \frac{\partial y}{\partial t}(t, x) + \Lambda(x) \frac{\partial y}{\partial x}(t, x) = M(x)y(t, x), \\ y_-(t, 1) = u(t), \quad y_+(t, 0) = Qy_-(t, 0), \\ y(0, x) = y^0(x). \end{cases} \quad (1)$$

26 In (1), $t > 0$, $x \in (0, 1)$, $y(t, \cdot)$ is the state at time t , y^0 is the initial data and $u(t)$ is the
27 control at time t . We denote by $n \geq 2$ the total number of equations of the system. The matrix

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28 $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ is assumed to be diagonal:

$$29 \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n), \quad (2)$$

30 with $m \geq 1$ negative speeds and $p \geq 1$ positive speeds ($m + p = n$) such that:

$$31 \quad \lambda_1(x) < \dots < \lambda_m(x) < 0 < \lambda_{m+1}(x) < \dots < \lambda_{m+p}(x), \quad \forall x \in [0, 1]. \quad (3)$$

32 Finally, the matrix $M \in L^\infty(0, 1)^{n \times n}$ couples the equations of the system inside the domain and
 33 the constant matrix $Q \in \mathbb{R}^{p \times m}$ couples the equations of the system on the boundary $x = 0$.

All along this paper, for a vector (or vector-valued function) $v \in \mathbb{R}^n$ and a matrix (or matrix-valued function) $A \in \mathbb{R}^{n \times n}$, we use the notation

$$v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix}, \quad A = \begin{pmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{pmatrix},$$

34 where $v_- \in \mathbb{R}^m$, $v_+ \in \mathbb{R}^p$ and $A_{--} \in \mathbb{R}^{m \times m}$, $A_{-+} \in \mathbb{R}^{m \times p}$, $A_{+-} \in \mathbb{R}^{p \times m}$, $A_{++} \in \mathbb{R}^{p \times p}$.

We recall that the system (1) is well posed in $(0, T)$ for every $T > 0$: for every $y^0 \in L^2(0, 1)^n$ and $u \in L^2(0, T)^m$, there exists a unique solution

$$y \in C^0([0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(0, T)^n)$$

35 to the system (1). By solution we mean “solution along the characteristics”, this will be detailed
 36 in Section 2 below.

37 The regularity $C^0([0, T]; L^2(0, 1)^n)$ of the solution allows us to consider control problems in
 38 the space $L^2(0, 1)^n$:

39 **Definition 1.1.** Let $T > 0$. We say that the system (1) is:

- 40 • **exactly controllable in time T** if, for every $y^0, y^1 \in L^2(0, 1)^n$, there exists $u \in L^2(0, T)^m$
 41 such that the corresponding solution y to the system (1) in $(0, T)$ satisfies $y(T, \cdot) = y^1$.
- 42 • **null controllable in time T** if the previous property holds at least for $y^1 = 0$.

43 Clearly, exact controllability implies null controllability, but the converse is not true in general.

44 These notions also depend on the time T and, since controllability in time T_1 implies controllability
 45 in time T_2 for every $T_2 \geq T_1$, it is natural to try to find the smallest possible control time,
 46 the so-called “minimal control time”. This problem was recently completely solved in [HO21]
 47 for the notion of exact controllability and we will investigate here what happens for the null
 48 controllability.

Definition 1.2. For any Λ, M and Q as above, we denote by $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$ the
 minimal null control time of the system (1), that is

$$T_{\text{inf}}(\Lambda, M, Q) = \inf \{T > 0 \mid \text{the system (1) is null controllable in time } T\}.$$

49 The time $T_{\text{inf}}(\Lambda, M, Q)$ is named “minimal” null control time according to the current liter-
 50 ature, despite it is not always a minimal element of the set. We keep this naming here, but we
 51 use the notation with the “inf” to avoid eventual confusions. The time $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$
 52 is thus the unique time that satisfies the following two properties:

- 53 • If $T > T_{\text{inf}}(\Lambda, M, Q)$, then the system (1) is null controllable in time T .
- 54 • If $T < T_{\text{inf}}(\Lambda, M, Q)$, then the system (1) is not null controllable in time T .

Finally, let us introduce the elementary times $T_1(\Lambda), \dots, T_n(\Lambda)$ defined by

$$T_i(\Lambda) = \begin{cases} \int_0^1 \frac{1}{-\lambda_i(\xi)} d\xi & \text{if } i \in \{1, \dots, m\}, \\ \int_0^1 \frac{1}{\lambda_i(\xi)} d\xi & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (4)$$

For the rest of this article it is important to keep in mind that the assumption (3) implies the following order relation among $T_i(\Lambda)$:

$$\begin{cases} T_1(\Lambda) \leq \dots \leq T_m(\Lambda), \\ T_{m+p}(\Lambda) \leq \dots \leq T_{m+1}(\Lambda). \end{cases} \quad (5)$$

1.2 The LCU decomposition

An important feature of the present article is that no assumption will be required on the boundary coupling matrices Q . To be able to handle such a general case and state our main result we introduce a notion of canonical form.

Definition 1.3. We say that a matrix $Q^0 \in \mathbb{R}^{p \times m}$ is in canonical form if either $Q^0 = 0$ or there exist an integer $\rho \geq 1$, row indices $r_1, \dots, r_\rho \in \{1, \dots, p\}$ with $r_1 < \dots < r_\rho$ and distinct column indices $c_1, \dots, c_\rho \in \{1, \dots, m\}$ such that

$$\begin{cases} q_{ij}^0 = 1 & \text{if } (i, j) \in \{(r_1, c_1), \dots, (r_\rho, c_\rho)\}, \\ q_{ij}^0 = 0 & \text{otherwise.} \end{cases}$$

For $Q^0 = 0$ we set $\rho = 0$ for convenience.

Note that we necessarily have $\rho = \text{rank } Q^0$.

Example 1.4. *The matrices*

$$Q_1^0 = \begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ \boxed{1} & 0 & 0 \end{pmatrix}, \quad Q_2^0 = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{pmatrix}.$$

are both in canonical form, with

$$\begin{aligned} \text{for } Q_1^0: & \quad (r_1, c_1) = (1, 2), \quad (r_2, c_2) = (2, 3), \quad (r_3, c_3) = (4, 1), \\ \text{for } Q_2^0: & \quad (r_1, c_1) = (1, 1), \quad (r_2, c_2) = (2, 2), \quad (r_3, c_3) = (4, 3). \end{aligned}$$

Using the Gaussian elimination we can transform any matrix into a canonical form, this is what we will call in this article the “LCU decomposition” (for Lower–Canonical–Upper decomposition). More precisely, we have

Proposition 1.5. *For every $Q \in \mathbb{R}^{p \times m}$, there exists a unique $Q^0 \in \mathbb{R}^{p \times m}$ such that the following two properties hold:*

- (i) *There exists an upper triangular matrix $U \in \mathbb{R}^{m \times m}$ with only ones on its diagonal and there exists an invertible lower triangular matrix $L \in \mathbb{R}^{p \times p}$ such that*

$$LQU = Q^0.$$

71 (ii) Q^0 is in canonical form.

72 We call Q^0 the canonical form of Q .

73 We mention that, because of possible zero rows or columns of Q^0 , the matrices L and U are
74 in general not unique.

75 With this proposition, we can extend the definition of the indices $(r_1, c_1), \dots, (r_\rho, c_\rho)$ to any
76 nonzero matrix:

77 **Definition 1.6.** For any nonzero matrix $Q \in \mathbb{R}^{p \times m}$, we denote by $(r_1, c_1), \dots, (r_\rho, c_\rho)$ the
78 positions of the nonzero entries of its canonical form ($r_1 < \dots < r_\rho$).

79 As previously mentioned, the existence of the LCU decomposition is a direct consequence of
80 the Gaussian elimination, where the matrix U corresponds to rightward column substitutions,
81 whereas the matrix L corresponds to downward row substitutions and then normalization to 1 of
82 the remaining nonzero entries. Let us present some examples which will make this point clearer.

Example 1.7. We illustrate how to find the decomposition of Proposition 1.5 in practice. Con-
sider

$$Q_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 5 \\ 0 & 1 & 2 \\ 4 & -4 & 4 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 5 & -1 & 8 \\ 0 & 1 & 1 & 1 \\ -1 & 3 & 6 & 4 \end{pmatrix}.$$

83 Let us deal with Q_1 first. We look at the first row, we take the first nonzero entry as pivot. We
84 remove the entries to the right on the same row by doing the column substitution $C_3 \leftarrow C_3 - 2C_2$,
85 which gives

$$86 \quad Q_1 U_1 = Q_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ 4 & -4 & 12 \end{pmatrix}. \quad (6)$$

We now look at the next row and take as new pivot the first nonzero entry that is not in the
column of the previous pivot, that is, not in C_2 . Since there is no entry to the right of this new
pivot, there is nothing to do and we move to the next row. Since this next row has no nonzero
element which is not in C_2, C_3 , we move again to the next and last row. We take as new pivot
the first nonzero entry that is not in C_2 or C_3 and we remove the entries to the right on the
same row by doing the column substitutions $C_2 \leftarrow C_2 + C_1$ and $C_3 \leftarrow C_3 - 3C_1$, which gives

$$Q_1 U_1 U_2 = Q_1 U_1 \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

87 Working then on the rows with downward substitutions only (starting with the first row) and
88 finally normalizing to 1 the remaining nonzero entries, we see that Q_1 becomes Q_1^0 of Example
89 1.4. Similarly, it can be checked that the canonical form of Q_2 is in fact Q_2^0 of Example 1.4.

90 **Remark 1.8.** Observe that we only need to compute the matrix U in order to find the indices
91 $(r_1, c_1), \dots, (r_\rho, c_\rho)$.

92 The uniqueness of the LCU decomposition is less straightforward and we refer for instance
93 to the arguments in the proof of [DJM06, Theorem 1] or to [HO21, Appendix A] for a proof.

94 **Remark 1.9.** In the Gaussian elimination process described above, we absolutely do not want to
 95 perform any permutation of the rows. This is because we have ordered the speeds of our system
 96 in a particular way (recall (3)). The fact that we use right multiplication by upper triangular
 97 matrices and left multiplication by lower triangular matrices is also dictated by this choice of
 98 order (for instance in [HO21] the speeds were ordered differently and right multiplication by
 99 lower triangular matrices was considered instead).

100 1.3 Literature

Boundary controllability of one-dimensional first-order hyperbolic systems has been widely investigated since the late 1960s. Two pioneering works are [Rus67] and the celebrated survey paper [Rus78], in which the author established the null controllability of the system (1) in any time $T \geq T_{[\text{Rus}]}(\Lambda)$, where $T_{[\text{Rus}]}(\Lambda)$ is the sum of the largest transport times from opposite directions, that is,

$$T_{[\text{Rus}]}(\Lambda) = T_{m+1}(\Lambda) + T_m(\Lambda).$$

101 An important feature of this result is that it is valid whatever are the internal and boundary
 102 coupling matrices M and Q . In other words, the time $T_{[\text{Rus}]}(\Lambda)$ gives an upper bound for the
 103 minimal null control time $T_{\text{inf}}(\Lambda, M, Q)$ with respect to these matrices.

104 In general, no better time can be expected. More precisely, it is easy to see that there exist
 105 matrices M and Q such that $T_{\text{inf}}(\Lambda, M, Q) = T_{[\text{Rus}]}(\Lambda)$ (simply take $M = 0$ and Q the matrix
 106 whose entries are all equal to zero except for $q_{1,m} = 1$). However, for most of the matrices M and
 107 Q , this upper bound is too large. Indeed, by just slightly restricting the class of such matrices
 108 (in particular, for Q), it is possible to have a strictly better upper bound than $T_{[\text{Rus}]}(\Lambda)$.

109 This fact was already observed in [Rus78], where the author tried to find the minimal null
 110 control time in the particular case of conservation laws ($M = 0$), rightly by looking more closely
 111 at the properties of the boundary coupling matrix. He could not solve this problem though and
 112 he left it as an open problem ([Rus78, Remark p. 656]). This was eventually solved few years
 113 later in [Wec82], where the author gave an explicit expression of the minimal null control time
 114 in terms of some indices related to Q .

115 Concerning systems of balance laws ($M \neq 0$), finding the minimal null control time for
 116 arbitrary M and Q is still an open challenging problem. Recently, there has been a resurgence on
 117 the characterization of such a time. A first result in this direction has been obtained in [CN19]
 118 with an improvement of the upper bound $T_{[\text{Rus}]}(\Lambda)$ for a certain class of boundary coupling
 119 matrices Q . More precisely, they considered the class \mathcal{B} defined by

$$120 \quad \mathcal{B} = \{Q \in \mathbb{R}^{p \times m} \mid (8) \text{ is satisfied for every } i \in \{1, \dots, \min\{p, m-1\}\}\}, \quad (7)$$

121 where the condition (8) is:

$$122 \quad \text{the } i \times i \text{ matrix formed from the first } i \text{ rows and the first } i \text{ columns of } Q \text{ is invertible,} \quad (8)$$

123 (it is understood that the set \mathcal{B} is empty when $m = 1$). For this class of boundary coupling
 124 matrices, the authors then showed that the upper bound $T_{[\text{Rus}]}(\Lambda)$ can be reduced to the time
 125 $T_{[\text{CN}]}(\Lambda)$ defined by

$$126 \quad T_{[\text{CN}]}(\Lambda) = \begin{cases} \max \left\{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_k(\Lambda), T_m(\Lambda) \right\} & \text{if } m \geq p, \\ \max_{k \in \{1, \dots, m\}} T_{m+k}(\Lambda) + T_k(\Lambda) & \text{if } m < p. \end{cases} \quad (9)$$

127 This was first done for some generic internal coupling matrices or under rather stringent con-
 128 ditions ([CN19, Theorem 1.1 and 1.5]) but the same authors were then able to remove these
 129 restrictions in [CN21a, Theorem 1 and 3].

On the other hand, when the boundary coupling matrix Q is full row rank, the problem of finding the minimal null control time, and not only an upper bound, has also been recently completely solved in [HO21]. More precisely, it is proved in [HO21, Theorem 1.12 and Remark 1.3] that

$$\text{rank } Q = p \implies T_{\text{inf}}(\Lambda, M, Q) = \max \left\{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), T_m(\Lambda) \right\}.$$

130 We see in this case that the minimal null control time has the remarkable property to be in-
 131 dependent of the internal coupling matrix M . In particular, this is the same time as the one
 132 found for conservation laws in [Wec82], yet with a more explicit expression. For $m > p$, this
 133 generalizes the aforementioned results of [CN19, CN21a] in two ways: firstly, this is a result for
 134 arbitrary full row rank boundary coupling matrices (not only for $Q \in \mathcal{B}$) and, secondly, this
 135 obviously establishes that no better time can be obtained (even for $Q \in \mathcal{B}$ this is not proved in
 136 [CN19, CN21a]). We mention this because the results of the present paper will share these two
 137 features.

138 For the special case of 2×2 systems, the minimal null control time has also been found in
 139 [CVKB13] when the boundary coupling matrix (which is then a scalar) is not zero and in [HO20]
 140 when the boundary coupling is reduced to zero. Notably, in the second situation, the minimal
 141 null control time depends on the behavior of the internal coupling matrix M ([HO20, Theorem
 142 1.5]).

143 Finally, we would like to mention the related works [CHOS21, CN21b, AKM21] concerning
 144 time-dependent systems and [Li10, LR10, Hu15, CN20a, CN20b] for quasilinear systems.

145 As we have discussed, finding what exactly is the minimal null control time turns out to be a
 146 difficult task. Instead, in this article we propose to look for the smallest and largest values that
 147 the minimal null control time $T_{\text{inf}}(\Lambda, M, Q)$ can take with respect to the internal coupling matrix
 148 M . Our main result is an explicit and easy-to-compute formula for both of these times. We will
 149 also completely characterize all the parameters Λ and Q for which $T_{\text{inf}}(\Lambda, M, Q)$ is invariant with
 150 respect to all $M \in L^\infty(0, 1)^{n \times n}$. We will show that our results generalize all the known works
 151 that have been previously quoted. In the course of the proof we will obtain some new results
 152 even for conservation laws ($M = 0$), notably with an explicit feedback law stabilizing the system
 153 in the minimal time.

154 Our proof relies on the notion of equivalent systems, in particular the backstepping method
 155 with the results of [HDMVK16, HVDMK19], the introduction of a canonical LU -decomposition
 156 for boundary coupling matrix Q in the same spirit as in [HO21], as well as a compactness-
 157 uniqueness method adapted to the null controllability inspired from the works [CN21a, DO18].

158 1.4 Main result and comments

159 As we have seen in the previous section, to explicitly characterize $T_{\text{inf}}(\Lambda, M, Q)$ for arbitrary M
 160 and Q is still a challenging open problem. Instead, we propose to find the smallest and largest
 161 values that it can take with respect to the internal coupling matrix M .

Definition 1.10. We define

$$\begin{aligned} T_{\text{inf}}(\Lambda, Q) &= \inf \left\{ T_{\text{inf}}(\Lambda, M, Q) \mid M \in L^\infty(0, 1)^{n \times n} \right\}, \\ T_{\text{sup}}(\Lambda, Q) &= \sup \left\{ T_{\text{inf}}(\Lambda, M, Q) \mid M \in L^\infty(0, 1)^{n \times n} \right\}. \end{aligned}$$

162 The main result of the present paper is the following explicit characterization of these two
 163 quantities:

164 **Theorem 1.11.** *Let $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ satisfy (2)-(3) and let $Q \in \mathbb{R}^{p \times m}$ be fixed.*

165 (i) *We have*

$$166 \quad T_{\inf}(\Lambda, Q) = \max \left\{ \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+1}(\Lambda), \quad T_m(\Lambda) \right\}, \quad (10)$$

167 where we recall that the indices (r_k, c_k) are defined in Definition 1.6.

168 (ii) *We have*

$$169 \quad T_{\sup}(\Lambda, Q) = \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\}, \quad (11)$$

where ρ_0 is the largest integer $i \in \{1, \dots, p\}$ such that

the $i \times m$ matrix formed from the first i rows of Q has rank i ,

170 with $\rho_0 = 0$ if the first row of Q is equal to zero.

171 In the statement of Theorem 1.11, we use the convention that the undefined quantities are
 172 simply not taken into account, which more precisely gives:

- 173 • If $\rho = 0$, then $T_{\inf}(\Lambda, 0) = \max \{T_{m+1}(\Lambda), \quad T_m(\Lambda)\}$.
- 174 • If $\rho_0 = 0$, then $T_{\sup}(\Lambda, Q) = T_{m+1}(\Lambda) + T_m(\Lambda)$.
- 175 • If $\rho_0 = p$, then $T_{\sup}(\Lambda, Q) = \max \{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_m(\Lambda) \}$.

176 An equivalent definition of ρ_0 is (when the first row of Q is not equal to zero)

$$177 \quad \rho_0 = \max \{i \in \{1, \dots, p\} \mid r_k = k, \quad \forall k \in \{1, \dots, i\}\}. \quad (12)$$

178 We emphasize that ρ_0 is defined for any Q , it is not a condition like $Q \in \mathcal{B}$.

179 By investigating the possibilities of equality $T_{\inf}(\Lambda, Q) = T_{\sup}(\Lambda, Q)$, we obtain the following
 180 important consequence of Theorem 1.11:

181 **Corollary 1.12.** *Let $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ satisfy (2)-(3) and let $Q \in \mathbb{R}^{p \times m}$ be fixed. The map
 182 $M \mapsto T_{\inf}(\Lambda, M, Q)$ is constant on $L^\infty(0, 1)^{n \times n}$ if, and only if, Λ and Q satisfy*

$$183 \quad \rho_0 = p \quad \text{or} \quad \left(0 < \rho_0 < p \quad \text{and} \quad \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda) \geq T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right). \quad (13)$$

184 **Remark 1.13.** In the proof of Theorem 1.11, we will show in fact that the infimum in (10) and
 185 the supremum in (11) are reached for some special matrices M . More precisely, we will show
 186 that:

- 187 • The infimum in (10) is reached for $M = 0$.
- 188 • The supremum in (11) is reached for $M = 0$ if the condition (13) holds.

- If the condition (13) fails, then the supremum in (11) is reached for the matrix M whose entries are all equal to zero, except for

$$m_{m+i,m}(x) = \frac{\lambda_{m+i}(x) - \lambda_m(x)}{-\lambda_m(x)} \ell^{i,\rho_0+1}, \quad \forall i \in \{\rho_0 + 1, \dots, p\},$$

189 where $L^{-1} = (\ell^{ij})_{1 \leq i,j \leq p}$ and L is any matrix L coming from the *LCU* decomposition of
190 Q .

Example 1.14. For $Q_1 \in \mathbb{R}^{4 \times 3}$ and $Q_2 \in \mathbb{R}^{4 \times 4}$ of Example 1.7 we have (recall (5))

$$\begin{aligned} T_{\inf}(\Lambda, Q_1) &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda), \quad T_7(\Lambda) + T_1(\Lambda), \quad T_4(\Lambda), \quad T_3(\Lambda)\} \\ &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda)\}, \end{aligned}$$

$$\begin{aligned} T_{\sup}(\Lambda, Q_1) &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda), \quad T_6(\Lambda) + T_3(\Lambda)\} \\ &= \max \{T_4(\Lambda) + T_2(\Lambda), \quad T_5(\Lambda) + T_3(\Lambda)\} \\ &= T_{\inf}(\Lambda, Q_1), \end{aligned}$$

and

$$\begin{aligned} T_{\inf}(\Lambda, Q_2) &= \max \{T_5(\Lambda) + T_1(\Lambda), \quad T_6(\Lambda) + T_2(\Lambda), \quad T_8(\Lambda) + T_3(\Lambda), \quad T_5(\Lambda), \quad T_4(\Lambda)\} \\ &= \max \{T_5(\Lambda) + T_1(\Lambda), \quad T_6(\Lambda) + T_2(\Lambda), \quad T_8(\Lambda) + T_3(\Lambda), \quad T_4(\Lambda)\}, \end{aligned}$$

$$T_{\sup}(\Lambda, Q_2) = \max \{T_5(\Lambda) + T_1(\Lambda), \quad T_6(\Lambda) + T_2(\Lambda), \quad T_7(\Lambda) + T_4(\Lambda)\}.$$

191 **Remark 1.15.** If during the computations of the indices (r_k, c_k) we arrive at the last column,
192 that is if we have

$$193 \quad c_{k_0} = m, \quad (14)$$

194 for some $k_0 \in \{1, \dots, p\}$, then there is no need to find the next indices to be able to compute
195 $T_{\inf}(\Lambda, Q)$ and $T_{\sup}(\Lambda, Q)$ since we know that the corresponding times will not be taken into
196 account (because of (5)). For instance, for the matrix Q_1 of Example 1.7 we can stop after the
197 very first step (6) since it gives $c_2 = 3$, there is no need to go on and compute U_2 .

198 **Remark 1.16.** Theorem 1.11 and its corollary generalize all the results of the literature that
199 we are aware of on the null controllability of systems of the form (1) (except for the special case
200 $n = 2$, which has been completely solved in [CVKB13, HO20]):

- When the matrix Q is full row rank, that is,

$$202 \quad \text{rank } Q = p, \quad (15)$$

203 exact and null controllability are equivalent properties for the system (1) (see e.g. [HO21,
204 Remark 1.3]) and it has been shown in [HO21, Theorem 4.1] that $T_{\inf}(\Lambda, M, Q)$ is inde-
205 pendent of M in that situation. Under the rank condition (15), it is clear that $\rho_0 = p$ and
206 the condition (13) is thus satisfied. It then follows from Corollary 1.12 that $T_{\inf}(\Lambda, M, Q)$
207 is independent of M . Therefore, our result encompasses the one of [HO21].

- When $m \leq p$ and $Q \in \mathcal{B}$ (defined in (7)), it has been established in [CN21a, Theorem 1] that

$$T_{\sup}(\Lambda, Q) \leq T_{[\text{CN}]}(\Lambda),$$

where we recall that $T_{[\text{CN}]}(\Lambda)$ is given by (9). In that case, we see that $r_k = c_k = k$ for every $k \in \{1, \dots, m-1\}$ and either $\rho_0 = m-1$ or $\rho_0 = m$. In all cases, we can check that

$$\max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\} = T_{[\text{CN}]}(\Lambda).$$

208 Therefore, item (ii) of Theorem 1.11 generalizes [CN21a, Theorem 1], which corresponded
 209 only to the inequality “ \leq ” and only valid for matrices $Q \in \mathcal{B}$, but excluded for instance
 210 the matrices presented in Example 1.7. We mention that, since the speeds are ordered, we
 211 cannot simply renumber the unknowns so that, after this transformation, the new matrix
 212 Q belongs to \mathcal{B} .

- In fact, when $\rho_0 = m$ in the previous point, the minimal null control time does not depend on M . More generally, if the condition (14) holds for some $k_0 \leq \rho_0$, then the condition (13) is satisfied (because of (5)) and it follows from Corollary 1.12 that

$$T_{\text{inf}}(\Lambda, M, Q) = \max_{k \in \{1, \dots, k_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda).$$

For instance, this condition is satisfied when the matrix Q has the block decomposition

$$Q = \begin{pmatrix} Q' \\ Q'' \end{pmatrix}, \quad \text{rank } Q' = m,$$

213 where $Q' \in \mathbb{R}^{m \times m}$ and $Q'' \in \mathbb{R}^{(p-m) \times m}$.

214 1.5 Equivalent systems

215 The proof of our main result will first consist in transforming our initial system (1) into “equiva-
 216 lent” systems (from a controllability point of view) which have a simpler coupling structure. Let
 217 us make this notion of equivalent systems precise here. We will introduce it for a slightly broader
 218 class of systems than (1) because of the nature of the transformations that we will use in the
 219 sequel, this will be clear from Section 3. All the systems of this paper will have the following
 220 form:

$$221 \begin{cases} \frac{\partial y}{\partial t}(t, x) + \Lambda(x) \frac{\partial y}{\partial x}(t, x) = M(x)y(t, x) + G(x)y_-(t, 0), \\ y_-(t, 1) = u(t), \quad y_+(t, 0) = Qy_-(t, 0), \\ y(0, x) = y^0(x), \end{cases} \quad (16)$$

222 where $M \in L^\infty(0, 1)^{n \times n}$ and $Q \in \mathbb{R}^{p \times m}$ as before, and $G \in L^\infty(0, 1)^{n \times m}$. Therefore, (16) is
 223 similar to (1) but it has the extra term with G . This system is well posed and the notions of
 224 controllability are similarly defined (see Section 2 below).

In what follows, we will refer to a system of the general form (16) as

$$(\Lambda, M, Q, G).$$

225 When a system does not contain a parameter (M or G) we will use the notation – rather than
 226 writing 0, for instance we will use $(\Lambda, M, Q, -)$ when the system does not contain G . The minimal
 227 null control time of the system (Λ, M, Q, G) will be denoted by $T_{\text{inf}}(\Lambda, M, Q, G)$ (for consistency,
 228 we will keep using the notation $T_{\text{inf}}(\Lambda, M, Q)$ rather than $T_{\text{inf}}(\Lambda, M, Q, -)$).

229 Let us now give the precise definition of what we mean by equivalent systems in this work:

Definition 1.17. We say that two systems (Λ, M_1, Q_1, G_1) and (Λ, M_2, Q_2, G_2) are equivalent, and we write

$$(\Lambda, M_1, Q_1, G_1) \sim (\Lambda, M_2, Q_2, G_2),$$

if there exists an invertible bounded linear transformation

$$L : L^2(0, 1)^n \longrightarrow L^2(0, 1)^n,$$

such that, for every $T > 0$, the induced map $\tilde{L} : C^0([0, T]; L^2(0, 1)^n) \longrightarrow C^0([0, T]; L^2(0, 1)^n)$ defined by $(\tilde{L}y)(t) = L(y(t))$ for every $t \in [0, T]$ satisfies

$$\tilde{L}(S_1) = S_2,$$

230 where S_i ($i = 1, 2$) denotes the space of all the solutions y to the system (Λ, M_i, Q_i, G_i) in $(0, T)$.

231 It is not difficult to check that \sim is an equivalence relation and that two equivalent systems
232 share the same controllability properties:

233 **Proposition 1.18.** *Let $(\Lambda, M_1, Q_1, G_1) \sim (\Lambda, M_2, Q_2, G_2)$ be two equivalent systems. Then, for
234 every $T > 0$, the system (Λ, M_1, Q_1, G_1) is null controllable in time T if, and only if, the system
235 (Λ, M_2, Q_2, G_2) is null controllable in time T .*

236 In particular, two equivalent systems have the same minimal null control time. However, the
237 converse is not true in general, an example has been detailed in Appendix A.

238 **Remark 1.19.** Let us emphasize that the notion of equivalent systems that we introduced here
239 does not care how the control from one system is obtained from the control of the other system.
240 It is different from the notion of (feedback) equivalence introduced in the seminal work [Bru70]
241 in finite dimension, which was designed to transfer the stabilization properties of one system to
242 another and thus required a more specific link between the two systems.

243 1.6 Outline of the proof

244 Since the proof of our main result involves many transformations, let us give a quick overview of
245 the main steps before going into detail:

- 1) First of all, we show in Section 3 that

$$(\Lambda, M, Q, -) \sim (\Lambda, -, Q, G),$$

246 for some G . It is nothing but a fundamental result of [HVDMK19, HDMVK16] that we
247 rephrase here with the notion of equivalent systems. Consequently, we only have to focus
248 on systems of the form $(\Lambda, -, Q, G)$ in the sequel, which have the advantage of having a
249 simpler coupling structure.

- 2) In Section 4, we show that the boundary coupling matrix Q can always be assumed in canonical form (Definition 1.3):

$$(\Lambda, -, Q, G) \sim (\Lambda, -, Q^0, \tilde{G}),$$

250 for some \tilde{G} . This is an important step that greatly simplifies the coupling structure of the
251 system.

- 3) Notably, this allows us to characterize in Section 5 the smallest value of the minimal null control time. More precisely, we first establish that

$$\inf \left\{ T_{\text{inf}}(\Lambda, -, Q^0, \tilde{G}) \mid \tilde{G} \in L^\infty(0, 1)^{n \times m} \right\}$$

252 is equal to the quantity on the right-hand side of the equality (10). This is done by using
 253 a similar argument to the one in [HO20]. We then show how to deduce the corresponding
 254 result for the initial system $(\Lambda, M, Q, -)$, thus proving the first part of our main result.

- 4) In view of the proof of the second part of our main result, we first show in Section 6 how to use the canonical form of Q^0 to prove that

$$(\Lambda, -, Q^0, \tilde{G}) \sim \left(\Lambda, -, Q^0, \begin{pmatrix} \tilde{G}_{--} \\ \hat{G}_{+-} \end{pmatrix} \right),$$

255 for some \hat{G}_{+-} which has the following structure:

$$256 \quad \hat{g}_{m+i, c_k} = 0, \quad \forall k \in \{1, \dots, \rho\}, \forall i \geq r_k. \quad (17)$$

- 5) In Section 7 we then prove that the coupling term \tilde{G}_{--} has no influence on the minimal null control time:

$$T_{\text{inf}} \left(\Lambda, -, Q^0, \begin{pmatrix} \tilde{G}_{--} \\ \hat{G}_{+-} \end{pmatrix} \right) = T_{\text{inf}} \left(\Lambda, -, Q^0, \begin{pmatrix} 0 \\ \hat{G}_{+-} \end{pmatrix} \right).$$

257 Unlike all the other steps, the proof is not based on the construction of a suitable trans-
 258 formation, it is based on a general compactness-uniqueness method adapted to the null
 259 controllability property and inspired from the previous works [CN21a, DO18].

- 6) Finally, in Section 8, we characterize the largest value of the minimal null control time. More precisely, we first show that

$$\sup \left\{ T_{\text{inf}} \left(\Lambda, -, Q^0, \begin{pmatrix} 0 \\ \hat{G}_{+-} \end{pmatrix} \right) \mid \hat{G}_{+-} \text{ satisfies (17)} \right\}$$

260 is equal to the quantity on the right-hand side of the equality (11). We then show how
 261 to deduce the corresponding result for the initial system $(\Lambda, M, Q, -)$, thus proving the
 262 second part of our main result.

263 **Remark 1.20.** All the steps described above are constructive, except for the one invoking a
 264 compactness-uniqueness argument. It would be interesting to be able to replace this step by a
 265 constructive approach (if possible).

266 2 Notations and solution along the characteristics

267 Before proceeding to the proof of our main result, we introduce in this section some notations
 268 and recall some results concerning the well-posedness of the non standard systems of the form
 269 (16).

2.1 The characteristics

We start with the characteristic curves associated with the system (16).

- First of all, throughout this paper it is convenient to extend $\lambda_1, \dots, \lambda_n$ to functions of \mathbb{R} (still denoted by the same) such that $\lambda_1, \dots, \lambda_n \in C^{0,1}(\mathbb{R})$ and

$$\lambda_1(x) < \dots < \lambda_m(x) \leq -\varepsilon < 0 < \varepsilon \leq \lambda_{m+1}(x) < \dots < \lambda_{m+p}(x), \quad \forall x \in \mathbb{R}, \quad (18)$$

for some $\varepsilon > 0$ small enough. Since all the results of the present paper depend only on the values of $\lambda_1, \dots, \lambda_n$ in $[0, 1]$, they do not depend on such an extension.

In what follows, $i \in \{1, \dots, n\}$ is fixed.

- Let χ_i be the flow associated with λ_i , i.e. for every $(t, x) \in \mathbb{R} \times \mathbb{R}$, the function $s \mapsto \chi_i(s; t, x)$ is the solution to the ordinary differential equation (ODE)

$$\begin{cases} \frac{\partial \chi_i}{\partial s}(s; t, x) = \lambda_i(\chi_i(s; t, x)), & \forall s \in \mathbb{R}, \\ \chi_i(t; t, x) = x. \end{cases} \quad (19)$$

The existence and uniqueness of a (global) solution to the ODE (19) follows from the (global) Cauchy-Lipschitz theorem (see e.g. [Har02, Theorem II.1.1]). The uniqueness also yields the important group property

$$\chi_i(\sigma; s, \chi_i(s; t, x)) = \chi_i(\sigma; t, x), \quad \forall \sigma, s \in \mathbb{R}. \quad (20)$$

- Let us now introduce the entry and exit times $s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x) \in \mathbb{R}$ of the flow $\chi_i(\cdot; t, x)$ inside the domain $[0, 1]$, i.e. the respective unique solutions to

$$\begin{cases} \chi_i(s_i^{\text{in}}(t, x); t, x) = 1, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 0, & \text{if } i \in \{1, \dots, m\}, \\ \chi_i(s_i^{\text{in}}(t, x); t, x) = 0, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 1, & \text{if } i \in \{m+1, \dots, n\}. \end{cases}$$

Their existence and uniqueness are guaranteed by the condition (18).

- Since λ_i does not depend on time, we have an explicit formula for the inverse function $\theta \mapsto \chi_i^{-1}(\theta; t, x)$. Indeed, it solves

$$\begin{cases} \frac{\partial(\chi_i^{-1})}{\partial \theta}(\theta; t, x) = \frac{1}{\frac{\partial \chi_i}{\partial s}(\chi_i^{-1}(\theta; t, x); t, x)} = \frac{1}{\lambda_i(\theta)}, & \forall \theta \in \mathbb{R}, \\ \chi_i^{-1}(x; t, x) = t, \end{cases}$$

which gives

$$\chi_i^{-1}(\theta; t, x) = t + \int_x^\theta \frac{1}{\lambda_i(\xi)} d\xi.$$

It follows that

$$\begin{cases} s_i^{\text{in}}(t, x) = t - \int_x^1 \frac{1}{-\lambda_i(\xi)} d\xi, & s_i^{\text{out}}(t, x) = t + \int_0^x \frac{1}{-\lambda_i(\xi)} d\xi, & \text{if } i \in \{1, \dots, m\}, \\ s_i^{\text{in}}(t, x) = t - \int_0^x \frac{1}{\lambda_i(\xi)} d\xi, & s_i^{\text{out}}(t, x) = t + \int_x^1 \frac{1}{\lambda_i(\xi)} d\xi, & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (21)$$

288 • We have the following monotonic properties:

$$289 \quad \begin{cases} \frac{\partial s_i^{\text{in}}}{\partial t} > 0, & \frac{\partial s_i^{\text{in}}}{\partial x} > 0, & \frac{\partial s_i^{\text{out}}}{\partial t} > 0, & \frac{\partial s_i^{\text{out}}}{\partial x} > 0, & \text{if } i \in \{1, \dots, m\}, \\ \frac{\partial s_i^{\text{in}}}{\partial t} > 0, & \frac{\partial s_i^{\text{in}}}{\partial x} < 0, & \frac{\partial s_i^{\text{out}}}{\partial t} > 0, & \frac{\partial s_i^{\text{out}}}{\partial x} < 0, & \text{if } i \in \{m+1, \dots, n\}, \end{cases} \quad (22)$$

290 and the following inverse formula, valid for every $s, t \in \mathbb{R}$:

$$291 \quad \begin{cases} s < s_i^{\text{out}}(t, 1) & \iff & s_i^{\text{in}}(s, 0) < t, & \text{if } i \in \{1, \dots, m\}, \\ s < s_i^{\text{out}}(t, 0) & \iff & s_i^{\text{in}}(s, 1) < t, & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (23)$$

292 • Note as well that (recall (4))

$$293 \quad T_i(\Lambda) = \begin{cases} s_i^{\text{out}}(0, 1) & \text{if } i \in \{1, \dots, m\}, \\ s_i^{\text{out}}(0, 0) & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (24)$$

294 • Finally, we introduce the non negative and increasing function $\phi_i \in C^{1,1}(\mathbb{R})$ defined by

$$295 \quad \phi_i(x) = \begin{cases} \int_0^x \frac{1}{-\lambda_i(\xi)} d\xi & \text{if } i \in \{1, \dots, m\}, \\ \int_0^x \frac{1}{\lambda_i(\xi)} d\xi & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (25)$$

296 Note that it is a bijection from $[0, 1]$ to $[0, T_i(\Lambda)]$.

297 2.2 Solution along the characteristics

Let us now introduce the notion of solution for systems of the form (16). To this end, we have to restrict our discussion to the domain where the system evolves, i.e. on $(0, T) \times (0, 1)$, $T > 0$ being fixed. For every $(t, x) \in (0, T) \times (0, 1)$, we have

$$(s, \chi_i(s; t, x)) \in (0, t) \times (0, 1), \quad \forall s \in (\bar{s}_i^{\text{in}}(t, x), t),$$

where we introduced

$$\bar{s}_i^{\text{in}}(t, x) = \max \{0, s_i^{\text{in}}(t, x)\} < t.$$

298 We now proceed to formal computations in order to introduce the notion of solution for
299 non smooth functions y . Writing the i -th equation of the system (16) along the characteristic
300 $\chi_i(s; t, x)$ for $s \in [\bar{s}_i^{\text{in}}(t, x), t]$, and using the chain rules yields the ODE

$$301 \quad \begin{cases} \frac{d}{ds} y_i(s, \chi_i(s; t, x)) = \sum_{j=1}^n m_{ij}(\chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) + \sum_{j=1}^m g_{ij}(\chi_i(s; t, x)) y_j(s, 0), \\ y_i(\bar{s}_i^{\text{in}}(t, x), \chi_i(\bar{s}_i^{\text{in}}(t, x); t, x)) = b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x), \end{cases} \quad (26)$$

302 where the initial condition $b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x)$ is given by the appropriate boundary or initial
303 conditions in (16):

304 • for $i \in \{1, \dots, m\}$,

$$305 \quad b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x) = \begin{cases} u_i(s_i^{\text{in}}(t, x)) & \text{if } s_i^{\text{in}}(t, x) > 0, \\ y_i^0(\chi_i(0; t, x)) & \text{if } s_i^{\text{in}}(t, x) < 0, \end{cases} \quad (27)$$

306 • for $i \in \{m+1, \dots, n\}$,

$$307 \quad b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x) = \begin{cases} \sum_{j=1}^m q_{i-m,j} y_j(s_i^{\text{in}}(t, x), 0) & \text{if } s_i^{\text{in}}(t, x) > 0, \\ y_i^0(\chi_i(0; t, x)) & \text{if } s_i^{\text{in}}(t, x) < 0. \end{cases} \quad (28)$$

Integrating the ODE (26) over $s \in [\bar{s}_i^{\text{in}}(t, x), t]$, we obtain the following system of integral equations:

$$y_i(t, x) = b_i(y_i^0, u_i, y_-(\cdot, 0))(t, x) + \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t, x)}^t m_{ij}(\chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds \\ + \sum_{j=1}^m \int_{\bar{s}_i^{\text{in}}(t, x)}^t g_{ij}(\chi_i(s; t, x)) y_j(s, 0) ds. \quad (29)$$

308 This leads to the following notion of solution called “solution along the characteristics”:

Definition 2.1. Let $T > 0$, $y^0 \in L^2(0, 1)^n$ and $u \in L^2(0, T)^m$ be fixed. We say that a function $y : (0, T) \times (0, 1) \rightarrow \mathbb{R}^n$ is a solution to the system (16) in $(0, T)$ if

$$y \in C^0([0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(0, T)^n),$$

309 and if the integral equation (29) is satisfied for every $i \in \{1, \dots, n\}$ and for a.e. $(t, x) \in (0, T) \times$
310 $(0, 1)$.

311 Using the Banach fixed-point theorem and suitable estimates, we can establish that the system
312 (16) is globally well posed in this sense:

313 **Theorem 2.2.** For every $T > 0$, $y^0 \in L^2(0, 1)^n$ and $u \in L^2(0, T)^m$, there exists a unique
314 solution $y \in C^0([0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(0, T)^n)$ to the system (16) in $(0, T)$. Moreover,
315 we have

$$316 \quad \|y\|_{C^0([0, T]; L^2(0, 1)^n)} + \|y\|_{C^0([0, 1]; L^2(0, T)^n)} \leq C \left(\|y^0\|_{L^2(0, 1)^n} + \|u\|_{L^2(0, T)^m} \right), \quad (30)$$

317 for some $C > 0$ that does not depend on y^0 nor u .

318 For a proof of this result, we refer for instance to [CHOS21, Appendix A.2] (see also [CN19,
319 Lemma 3.2] in the L^∞ setting).

320 3 Backstepping transformation

321 In this section, we use a Volterra transformation of the second kind to transform our initial
322 system (1) into a system with a simpler coupling structure, this is the so-called backstepping
323 method for partial differential equations. The content of this section is quite standard by now
324 (yet, formulated differently here), see for instance [HVDKM19, Section 2.2] (or [CN19, Section
325 2]).

3.1 Removal of the diagonal terms

First of all, we perform a simple preliminary transformation in order to remove the diagonal terms in M . This is only a technical step, which is nevertheless necessary in view of the existence of the transformation that we will use in the next section, see Remark 3.3 below. For convenience, we introduce the set

$$\mathcal{M} = \{M \in L^\infty(0, 1)^{n \times n} \mid m_{ii} = 0, \quad \forall i \in \{1, \dots, n\}\}.$$

Proposition 3.1. *There exists a map $\Psi : L^\infty(0, 1)^{n \times n} \rightarrow \mathcal{M}$ such that, for every $M \in L^\infty(0, 1)^{n \times n}$, we have*

$$(\Lambda, M, Q, -) \sim (\Lambda, \Psi(M), Q, -).$$

Proof. • We are going to use the spatial transformation

$$\tilde{y}(t, x) = E(x)y(t, x),$$

where $E = \text{diag}(e_1, \dots, e_n) \in W^{1, \infty}(0, 1)^{n \times n}$ is the diagonal matrix whose entries are

$$e_i(x) = \exp\left(-\int_0^x \frac{m_{ii}(\xi)}{\lambda_i(\xi)} d\xi\right).$$

Clearly, this transformation is invertible on $L^2(0, 1)^n$.

• Assume now that y is a solution to the system $(\Lambda, M, Q, -)$ for some y^0 and u and let us show that \tilde{y} is then a solution to the system $(\Lambda, \Psi(M), Q, -)$ for some \tilde{y}^0 and \tilde{u} , where $\Psi(M)$ will be determined below. We do it formally but this can be rigorously justified.

– The initial data is obviously $\tilde{y}^0(x) = E(x)y^0(x)$.

– The boundary condition at $x = 0$ is clearly satisfied since $\tilde{y}(t, 0) = y(t, 0)$.

– Looking at the boundary condition at $x = 1$, the control \tilde{u} is

$$\tilde{u}(t) = \tilde{y}_-(t, 1) = E_{--}(1)y_-(t, 1).$$

– Using the equation satisfied by y and the fact that Λ and E commute, a computation shows that

$$\frac{\partial \tilde{y}}{\partial t}(t, x) + \Lambda(x) \frac{\partial \tilde{y}}{\partial x}(t, x) = \left(E(x)M(x) + \Lambda(x) \frac{\partial E}{\partial x}(x)\right) y(t, x).$$

Consequently, \tilde{y} satisfies the desired equation if we take

$$(\Psi(M))(x) = \left(E(x)M(x) + \Lambda(x) \frac{\partial E}{\partial x}(x)\right) E(x)^{-1}.$$

Now that Ψ is clearly identified, similar computations show that, conversely, if \tilde{y} is a solution to the system $(\Lambda, \Psi(M), Q, -)$ for some \tilde{y}^0 and \tilde{u} , then y is a solution to the system $(\Lambda, M, Q, -)$ for some y^0 and u .

• Finally, it is clear that $\Psi(M) \in \mathcal{M}$ by construction. □

3.2 Backstepping transformation

We now recall an important result from [HVDMK19] and [HDMVK16] that we present here using the notion of equivalent system. To this end, we introduce the set

$$\mathcal{F} = \{A \in L^\infty(0, 1)^{n \times n} \mid A_{-+} = A_{+-} = 0\}.$$

Theorem 3.2. *For every $A \in \mathcal{F}$, there exists a map $\Gamma_A : \mathcal{M} \rightarrow L^\infty(0, 1)^{n \times m}$ such that, for every $M \in \mathcal{M}$, we have*

$$(\Lambda, M, Q, -) \sim (\Lambda, -, Q, \Gamma_A(M)).$$

Proof. • We are going to use the spatial transformation

$$\tilde{y}(t, x) = y(t, x) - \int_0^x K(x, \xi) y(t, \xi) d\xi,$$

where $K \in L^\infty(\mathcal{T})^{n \times n}$ and \mathcal{T} is the triangle

$$\mathcal{T} = \{(x, \xi) \in (0, 1) \times (0, 1) \mid x > \xi\}.$$

This transformation is always invertible on $L^2(0, 1)^n$ since it is a Volterra transformation of the second kind (see e.g. [Hoc73, Theorem 2.5]).

• Assume now that y is a solution to the system $(\Lambda, M, Q, -)$ for some y^0 and u and let us show that \tilde{y} is then a solution to the system $(\Lambda, -, Q, \Gamma_A(M))$ for some \tilde{y}^0 and \tilde{u} , where $\Gamma_A(M)$ will be determined below. We do it formally but this can be rigorously justified.

- The initial data is obviously $\tilde{y}^0(x) = y^0(x) - \int_0^x K(x, \xi) y^0(\xi) d\xi$.
- The boundary condition at $x = 0$ is clearly satisfied since $\tilde{y}(t, 0) = y(t, 0)$.
- Looking at the boundary condition at $x = 1$, the control \tilde{u} is

$$\tilde{u}(t) = \tilde{y}_-(t, 1) = y_-(t, 1) - \int_0^1 H(\xi) y(t, \xi) d\xi, \quad (31)$$

where $H(\xi) = (K_{--}(1, \xi) \quad K_{-+}(1, \xi))$.

- Using the equation satisfied by y , integrating by parts, and using the boundary condition satisfied by y at $x = 0$, we have

$$\begin{aligned} \frac{\partial \tilde{y}}{\partial t}(t, x) + \Lambda(x) \frac{\partial \tilde{y}}{\partial x}(t, x) = & \\ & - \int_0^x \left(\Lambda(x) \frac{\partial K}{\partial x}(x, \xi) + \frac{\partial K}{\partial \xi}(x, \xi) \Lambda(\xi) + K(x, \xi) \left(\frac{\partial \Lambda}{\partial \xi}(\xi) + M(\xi) \right) \right) y(t, \xi) d\xi \\ & + (M(t, x) + K(x, x) \Lambda(x) - \Lambda(x) K(x, x)) y(t, x) \\ & - K(x, 0) \Lambda(0) \begin{pmatrix} \text{Id}_{\mathbb{R}^{m \times m}} \\ Q \end{pmatrix} y_-(t, 0). \end{aligned}$$

Consequently, \tilde{y} satisfies the desired equation if we take

$$(\Gamma_A(M))(x) = -K(x, 0) \Lambda(0) \begin{pmatrix} \text{Id}_{\mathbb{R}^{m \times m}} \\ Q \end{pmatrix}, \quad (32)$$

351 and provided that the kernel K satisfies the so-called kernel equations:

$$352 \quad \begin{cases} \Lambda(x) \frac{\partial K}{\partial x}(x, \xi) + \frac{\partial K}{\partial \xi}(x, \xi) \Lambda(\xi) + K(x, \xi) \left(\frac{\partial \Lambda}{\partial \xi}(\xi) + M(\xi) \right) = 0, \\ \Lambda(x) K(x, x) - K(x, x) \Lambda(x) = M(x). \end{cases} \quad (33)$$

353 The existence of a solution to these equations will be discussed next.

354 Now that Γ_A is clearly identified, similar computations show that, conversely, if \tilde{y} is a
355 solution to the system $(\Lambda, -, Q, \Gamma_A(M))$ for some \tilde{y}^0 and \tilde{u} , then y is a solution to the
356 system $(\Lambda, M, Q, -)$ for some y^0 and u .

357 □

358 **Remark 3.3.** If we write the second condition of (33) component-wise:

$$359 \quad (\lambda_i(x) - \lambda_j(x)) k_{ij}(x, x) = m_{ij}(x), \quad (34)$$

360 then we see that for $i = j$ we shall necessarily have $m_{ii} = 0$. Therefore, it is necessary that
361 $M \in \mathcal{M}$ (otherwise the equation (34), and thus the kernel equations (33), have no solution).
362 This explains why we had to perform a preliminary transformation in Section 3.1 to reduce the
363 general case to this one.

364 From [HDMVK16, Section VI], we know that the kernel equations (33) have a solution (see
365 also [HVDMK19, Remark A.2] to see how to deal with space-varying speeds). More precisely,
366 we can extract the following result:

367 **Theorem 3.4.** *For every $A \in \mathcal{F}$, for every $M \in \mathcal{M}$, there exists a unique solution $K \in$
368 $L^\infty(\mathcal{T})^{n \times n}$ to the kernel equations (33) with:*

369 • For every $i, j \in \{1, \dots, m\}$:

$$370 \quad \begin{aligned} k_{ij}(1, \xi) &= a_{ij}(\xi), & \text{if } i > j, \\ k_{ij}(x, 0) &= a_{ij}(x), & \text{if } i \leq j. \end{aligned} \quad (35)$$

371 • For every $i, j \in \{m+1, \dots, n\}$:

$$372 \quad \begin{aligned} k_{ij}(x, 0) &= a_{ij}(x), & \text{if } i \geq j, \\ k_{ij}(1, \xi) &= a_{ij}(\xi), & \text{if } i < j. \end{aligned} \quad (36)$$

Moreover, we have the following additional regularities:

$$\begin{aligned} K &\in C^0((0, 1]; L^2(0, x)^{n \times n}), & K(x, \cdot) &\in L^\infty(0, x)^{n \times n}, & \forall x \in (0, 1]. \\ K &\in C^0([0, 1]; L^2(\xi, 1)^{n \times n}), & K(\cdot, \xi) &\in L^\infty(\xi, 1)^{n \times n}, & \forall \xi \in [0, 1]. \end{aligned}$$

As before, the notion of solution is to be understood in the sense of solution along the characteristics. By $K \in C^0((0, 1]; L^2(0, x)^{n \times n})$ we mean that $\|K(x_n, \cdot) - K(x, \cdot)\|_{L^2(0, \min\{x_n, x\})^{n \times n}} \rightarrow 0$ as $x_n \rightarrow x$, for every $x \in (0, 1]$, with a similar definition for $K \in C^0([0, 1]; L^2(\xi, 1)^{n \times n})$. Despite not mentioned in the literature, these important regularities can be deduced from the system of integral equations satisfied by the kernel. In particular, it shows that H and $\Gamma_A(M)$ defined in (31) and (32) have the following regularities:

$$H \in L^\infty(0, 1)^{m \times n}, \quad \Gamma_A(M) \in L^\infty(0, 1)^{n \times m}.$$

373 **Remark 3.5.** The set \mathcal{F} corresponds to the set of boundary conditions that are free to choose
374 for the kernel equations. The freedom for the boundary condition (35) was already used in the
375 works [HDM15, HDMVK16, HVDKM19] in order to give to $(\Gamma_A(M))_{--}$ a structure of strictly
376 lower triangular matrix. However, in the present paper this will not be used and it is the other
377 boundary condition (36) that will turn out to be essential (see Section 6 below).

378 4 Reduction of the boundary coupling matrix

379 In this section we perform some transformations to show that we can always assume that the
380 boundary coupling matrix Q is in canonical form. More precisely, we prove the following result:

Proposition 4.1. *For every invertible upper triangular matrix $U \in \mathbb{R}^{m \times m}$ and every invertible lower triangular matrix $L \in \mathbb{R}^{p \times p}$, there exists a map $\Theta : L^\infty(0, 1)^{n \times m} \rightarrow L^\infty(0, 1)^{n \times m}$ such that, for every $G \in L^\infty(0, 1)^{n \times m}$, we have*

$$(\Lambda, -, Q, G) \sim (\Lambda, -, LQU, \Theta(G)).$$

Proof. • For any $i, j \in \{1, \dots, n\}$, we denote by ζ_{ij} the solution to the ODE

$$\begin{cases} \frac{d}{ds} \zeta_{ij}(s) = \frac{\lambda_j(\zeta_{ij}(s))}{\lambda_i(s)}, & \forall s \in \mathbb{R}, \\ \zeta_{ij}(0) = 0. \end{cases}$$

- We first prove that, for every invertible upper triangular matrix $U \in \mathbb{R}^{m \times m}$, there exists a map $\Theta_{--} : L^\infty(0, 1)^{m \times m} \rightarrow L^\infty(0, 1)^{m \times m}$ such that, for every $G \in L^\infty(0, 1)^{n \times m}$, we have

$$(\Lambda, -, Q, G) \sim \left(\Lambda, -, QU, \begin{pmatrix} \Theta_{--}(G_{--}) \\ G_{+-}U \end{pmatrix} \right).$$

381 To this end, we are going to use the spatial transformation

$$382 \quad \tilde{y}_i(t, x) = \begin{cases} \sum_{k=i}^m u^{ik} y_k(t, \zeta_{ik}(x)) & \text{for } i \in \{1, \dots, m\}, \\ y_i(t, x) & \text{for } i \in \{m+1, \dots, n\}, \end{cases} \quad (37)$$

383 where $U^{-1} = (u^{ik})_{1 \leq i, k \leq m}$. Let us first show that this transformation is well defined and
384 invertible. We can check that, for $i \leq k \leq m$, we have (recall (25))

$$385 \quad \zeta_{ik}(x) = \phi_k^{-1}(\phi_i(x)). \quad (38)$$

In particular, for such indices, ζ_{ik} is a C^1 -diffeomorphism from $(0, 1)$ to a subset of $(0, 1)$ and thus the transformation (37) is well defined on $L^2(0, 1)^n$. Besides, using the property $\zeta_{kj}(\zeta_{ik}(x)) = \zeta_{ij}(x)$ for $i \leq k \leq j$, we can check that its inverse is given by

$$y_k(t, x) = \begin{cases} \sum_{j=k}^m u_{kj} \tilde{y}_j(t, \zeta_{kj}(x)) & \text{for } k \in \{1, \dots, m\}, \\ \tilde{y}_k(t, x) & \text{for } k \in \{m+1, \dots, n\}. \end{cases}$$

- Assume now that y is a solution to the system $(\Lambda, -, Q, G)$ for some y^0 and u and let us show that \tilde{y} is then a solution to the system $(\Lambda, -, QU, \tilde{G})$ for some \tilde{y}^0 and \tilde{u} , where

$$\tilde{G} = \begin{pmatrix} \Theta_{--}(G_{--}) \\ G_{+-}U \end{pmatrix},$$

386 and where $\Theta_{--}(G_{--})$ will be determined below. Once again, we do it formally but this
387 can be rigorously justified.

- The initial data is obviously

$$\tilde{y}_i^0(x) = \begin{cases} \sum_{k=i}^m u^{ik} y_k^0(\zeta_{ik}(x)) & \text{for } i \in \{1, \dots, m\}, \\ y_i^0(x) & \text{for } i \in \{m+1, \dots, n\}. \end{cases}$$

388 – The boundary condition at $x = 0$ is clearly satisfied since $\tilde{y}_+ = y_+$ and $\tilde{y}_-(t, 0) =$
389 $U^{-1}y_-(t, 0)$.

- Looking at the boundary condition at $x = 1$, the control \tilde{u} is

$$\tilde{u}_i(t) = \tilde{y}_i(t, 1) = \sum_{k=i}^m u^{ik} y_k(t, \zeta_{ik}(1)), \quad \forall i \in \{1, \dots, m\}.$$

- It is clear that $\tilde{y}_+ = y_+$ satisfies the desired equation. Let us now fix $i \in \{1, \dots, m\}$. A computation shows that

$$\begin{aligned} \frac{\partial \tilde{y}_i}{\partial t}(t, x) + \lambda_i(x) \frac{\partial \tilde{y}_i}{\partial x}(t, x) - \sum_{j=1}^m \tilde{g}_{ij}(x) \tilde{y}_j(t, 0) = \\ \sum_{k=i}^m u^{ik} \left(-\lambda_k(\zeta_{ik}(x)) + \lambda_i(x) \frac{\partial \zeta_{ik}}{\partial x}(x) \right) \frac{\partial y}{\partial x}(t, \zeta_{ik}(x)) \\ + \sum_{\ell=1}^m \left(\sum_{k=i}^m u^{ik} g_{k\ell}(\zeta_{ik}(x)) - \sum_{j=1}^{\ell} \tilde{g}_{ij}(x) u^{j\ell} \right) y_\ell(t, 0). \end{aligned}$$

Consequently, \tilde{y}_i satisfies the desired equation, provided that

$$\sum_{k=i}^m u^{ik} g_{k\ell}(\zeta_{ik}(x)) - \sum_{j=1}^{\ell} \tilde{g}_{ij}(x) u^{j\ell} = 0, \quad \forall \ell \in \{1, \dots, m\}.$$

This uniquely determines \tilde{g}_{ij} for $i, j \in \{1, \dots, m\}$ (and thus Θ_{--}):

$$\tilde{g}_{ij}(x) = \sum_{\ell=1}^j \left(\sum_{k=i}^m u^{ik} g_{k\ell}(\zeta_{ik}(x)) \right) u_{\ell j}.$$

390 Now that Θ_{--} is clearly identified, similar computations show that, conversely, if \tilde{y} is a
391 solution to the system $(\Lambda, -, QU, \tilde{G})$ for some \tilde{y}^0 and \tilde{u} , then y is a solution to the system
392 $(\Lambda, -, Q, G)$ for some y^0 and u .

- Similarly, we can prove that, for every invertible lower triangular matrix $L \in \mathbb{R}^{p \times p}$, there exists a map $\Theta_{+-} : L^\infty(0, 1)^{p \times m} \rightarrow L^\infty(0, 1)^{p \times m}$ such that, for every $G \in L^\infty(0, 1)^{n \times m}$, we have

$$(\Lambda, -, Q, G) \sim \left(\Lambda, -, LQ, \begin{pmatrix} G_{--} \\ \Theta_{+-}(G_{+-}) \end{pmatrix} \right).$$

This can be done using the spatial transformation

$$\tilde{y}_i(t, x) = \begin{cases} y_i(t, x) & \text{for } i \in \{1, \dots, m\}, \\ \sum_{k=m+1}^i \ell_{i-m, k-m} y_k(t, \zeta_{ik}(x)) & \text{for } i \in \{m+1, \dots, n\}, \end{cases}$$

((38) is still valid for the indices considered) where $L = (\ell_{ij})_{1 \leq i, j \leq p}$ and taking

$$\tilde{g}_{ij}(x) = \sum_{k=m+1}^i \ell_{i-m, k-m} g_{kj}(\zeta_{ik}(x)),$$

for $i \in \{m+1, \dots, n\}$ and $j \in \{1, \dots, m\}$, where \tilde{G} denotes the matrix

$$\tilde{G} = \begin{pmatrix} G_{--} \\ \Theta_{+-}(G_{+-}) \end{pmatrix}.$$

393

□

394 5 Smallest value of the minimal null control time

395 Thanks to the result of previous section it is from now on sufficient to consider boundary coupling
396 matrices which are in canonical form. This is a big step forward, which already allows us to
397 characterize the smallest value of the minimal null control time.

398 5.1 Characterization for systems $(\Lambda, -, Q, G)$

399 We start with systems of the form $(\Lambda, -, Q, G)$, we will discuss in the next section how to deduce
400 the corresponding result for the initial system $(\Lambda, M, Q, -)$.

401 **Theorem 5.1.** *Let $Q^0 \in \mathbb{R}^{p \times m}$ be in canonical form, $G \in L^\infty(0, 1)^{n \times m}$ and $T > 0$ be fixed.*

402 (i) *If the system $(\Lambda, -, Q^0, G)$ is null controllable in time T , then necessarily*

$$403 \quad T \geq \max \left\{ \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+1}(\Lambda), \quad T_m(\Lambda) \right\}. \quad (39)$$

404 (ii) *If T satisfies the condition (39), then the system $(\Lambda, -, Q^0, -)$ (i.e. with $G = 0$) is null
405 controllable in time T with control $u = 0$.*

406 As for Theorem 1.11, we use the convention that the undefined quantities are simply not taken
407 into account, which means that the condition (39) is reduced to $T \geq \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}$
408 when $\rho = 0$ (i.e. when $Q^0 = 0$).

409 This result shows in particular that the smallest value that $T_{\inf}(\Lambda, -, Q^0, G)$ can take with
410 respect to $G \in L^\infty(0, 1)^{n \times m}$ is equal to the quantity on the right-hand side of the inequality in
411 (39). This can be extended to arbitrary boundary coupling matrices thanks to Proposition 4.1.

412 *Proof of Theorem 5.1.* We use the ideas of the proof of [HO20, Lemma 3.3].

1) We first show that it is necessary that

$$T \geq \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}.$$

413 We point out that for this first step there is no need to assume that Q^0 is in canonical
414 form. Assume that $T < \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}$. Then, there exists $i \in \{1, \dots, n\}$ such
415 that $T < T_i(\Lambda)$. Let ω_i be the open subset defined by

$$416 \quad \omega_i = \{x \in (0, 1) \mid s_i^{\text{in}}(T, x) < 0\}. \quad (40)$$

Then, we have (see (24), (23) and (22))

$$T < T_i(\Lambda) \iff \omega_i \neq \emptyset.$$

For $x \in \omega_i$, the null controllability condition $y_i(T, x) = 0$ is equivalent to (see (29))

$$0 = y_i^0(\chi_i(0; T, x)) + \int_0^T \sum_{j=1}^m g_{ij}(\chi_i(s; T, x)) y_j(s, 0) ds.$$

Since $y_i^0 \in L^2(0, 1)$ is arbitrary and $x \in \omega_i \mapsto \chi_i(0; T, x)$ is a C^1 -diffeomorphism (its
inverse is given by $\xi \mapsto \chi_i(T; 0, \xi)$ thanks to (20)), this shows that the bounded linear
operator $K : L^2(0, T)^m \rightarrow L^2(\omega_i)$ defined by

$$(Kh)(x) = - \int_0^T \sum_{j=1}^m g_{ij}(\chi_i(s; T, x)) h_j(s) ds,$$

417 is surjective. This is impossible since its range is clearly a subset of $L^\infty(\omega_i)$, which is a
418 proper subset of $L^2(\omega_i)$.

2) Suppose now that $\rho \neq 0$ (otherwise we are done) and that T is such that

$$\max \{T_{m+1}(\Lambda), T_m(\Lambda)\} \leq T < T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda),$$

where $k_0 \in \{1, \dots, \rho\}$ is any index such that

$$T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda) = \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda).$$

We have seen in the previous step that the condition $T \geq \max \{T_{m+1}(\Lambda), T_m(\Lambda)\}$ means
that all the subsets ω_i defined in (40) are empty. In particular (recall also (22)),

$$s_{m+r_{k_0}}^{\text{in}}(T, x) > 0, \quad \forall x \in (0, 1).$$

419 Therefore, the null controllability condition $y_{m+r_{k_0}}(T, x) = 0$ is equivalent to (see (29) and
420 recall that Q^0 is in canonical form)

$$421 \quad 0 = y_{c_{k_0}}(s_{m+r_{k_0}}^{\text{in}}(T, x), 0) + \int_{s_{m+r_{k_0}}^{\text{in}}(T, x)}^T \sum_{j=1}^m g_{m+r_{k_0}, j}(\chi_{m+r_{k_0}}(s; T, x)) y_j(s, 0) ds. \quad (41)$$

Let us now introduce the open subset $\tilde{\omega}$ defined by

$$\tilde{\omega} = \left\{ x \in (0, 1) \mid s_{c_{k_0}}^{\text{in}}(s_{m+r_{k_0}}^{\text{in}}(T, x), 0) < 0 \right\}.$$

Using that $T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda) = s_{m+r_{k_0}}^{\text{out}}(s_{c_{k_0}}^{\text{out}}(0,1),0)$ (see (24) and (21)), we can show by a similar reasoning as in the first step that

$$T < T_{m+r_{k_0}}(\Lambda) + T_{c_{k_0}}(\Lambda) \iff \tilde{\omega} \neq \emptyset.$$

For $x \in \tilde{\omega}$, the identity (41) becomes (see (29))

$$\begin{aligned} 0 &= y_{c_{k_0}}^0(\chi_{c_{k_0}}(0; s_{m+r_{k_0}}^{\text{in}}(T,x),0)) \\ &\quad + \sum_{j=1}^m \int_{\bar{s}_{c_{k_0}}^{\text{in}}(s_{m+r_{k_0}}^{\text{in}}(T,x),0)}^{s_{m+r_{k_0}}^{\text{in}}(T,x)} g_{c_{k_0}j}(\chi_{c_{k_0}}(s; s_{m+r_{k_0}}^{\text{in}}(T,x),0)) y_j(s,0) ds \\ &\quad + \int_{s_{m+r_{k_0}}^{\text{in}}(T,x)}^T \sum_{j=1}^m g_{m+r_{k_0},j}(\chi_{m+r_{k_0}}(s;T,x)) y_j(s,0) ds. \end{aligned}$$

4.2.2 This leads to a contradiction by using the same argument as at the end of the first step.

4.2.3 3) Finally, it is not difficult to see from (29) that, when $G = 0$, the control $u = 0$ brings the
4.2.4 solution of the system $(\Lambda, -, Q^0)$ to zero in any time T satisfying (39).

4.2.5 □

4.2.6 5.2 Proof of the first part of Theorem 1.11

4.2.7 Let us now show how the previous results yield the desired characterization of the smallest
4.2.8 minimal null control time for the initial system $(\Lambda, M, Q, -)$.

4.2.9 *Proof of item (i) of Theorem 1.11.* • Let $M \in L^\infty(0,1)^{n \times n}$ and $Q \in \mathbb{R}^{p \times m}$ be fixed. Let
4.3.0 $T > 0$ be such that the system $(\Lambda, M, Q, -)$ is null controllable in time T .

4.3.1 – By Proposition 3.1 and Theorem 3.2, there exists $G \in L^\infty(0,1)^{n \times m}$ such that the
4.3.2 system $(\Lambda, -, Q, G)$ is null controllable in time T .

4.3.3 – From Proposition 4.1, there exists $\tilde{G} \in L^\infty(0,1)^{n \times m}$ such that the system $(\Lambda, -, Q^0, \tilde{G})$
4.3.4 is null controllable in time T , where Q^0 is the canonical form of Q .

4.3.5 – By item (i) of Theorem 5.1 we obtain that T has to satisfy the condition (39).

This establishes the following lower bound:

$$T_{\text{inf}}(\Lambda, M, Q) \geq \max \left\{ \max_{k \in \{1, \dots, \rho\}} T_{m+r_k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+1}(\Lambda), \quad T_m(\Lambda) \right\},$$

4.3.6 valid for every $M \in L^\infty(0,1)^{n \times n}$.

4.3.7 • This lower bound is reached for $M = 0$, this follows from Theorem 5.1 and Proposition 4.1
4.3.8 (using that $\Theta(0) = 0$).

4.3.9 □

440 **5.3 Comments on the case $M = 0$**

441 Let us conclude this section with some interesting remarks on the case $M = 0$. For $M = 0$, we
 442 can combine Theorem 5.1 with Proposition 4.1 (with $G = 0$, in which case their proofs are greatly
 443 simplified) to obtain a completely different proof of [Wec82, Theorems 1 and 2]. Our proof has
 444 several advantages. Firstly, we directly obtain a more explicit expression of the minimal null
 445 control time (see e.g. [HO21, Remark 1.15]). On the other hand, we do not need to use the
 446 so-called duality and we are able to obtain an explicit control. More precisely, we can extract
 447 the following result from item (ii) of Theorem 5.1 and the proof of Proposition 4.1:

448 **Proposition 5.2.** *Let $Q \in \mathbb{R}^{p \times m}$ and T satisfy (39). Then, the system $(\Lambda, -, Q, -)$ is finite-time
 449 stabilizable with settling time T , with the following explicit feedback law:*

$$450 \quad u_i(t) = - \sum_{k=i+1}^m u^{ik} y_k(t, \zeta_{ik}(1)), \quad i \in \{1, \dots, m\}, \quad (42)$$

451 where $U^{-1} = (u^{ik})_{1 \leq i, k \leq m}$ and U is any matrix U coming from the LCU decomposition of Q .

452 We recall that the previous statement simply means that, if we replace the i -th component
 453 of u by the right-hand side of the formula (42) in the system (1) (with $M = 0$), then the
 454 corresponding solution satisfies $y(T, \cdot) = 0$ for every $y^0 \in L^2(0, 1)^n$. We also recall that systems
 455 with such boundary conditions are well posed (see e.g. [CN19, Section 3] in the L^∞ setting).

456 A similar result was obtained in the proof of [CN19, Proposition 1.6] when $Q \in \mathcal{B}$ (defined
 457 in (7)), our result generalizes it to arbitrary $Q \in \mathbb{R}^{p \times m}$. Let us illustrate with an example that
 458 the feedback law that we have obtained (42) is also the same as in this reference when $Q \in \mathcal{B}$.

Example 5.3. *Let us consider the 6×6 system used as example in [CN19, p. 1155]: we take
 $p = m = 3$, the negative speeds are*

$$\lambda_1 = -4 < \lambda_2 = -2 < \lambda_3 = -1 < 0,$$

the positive speeds are arbitrary (subject to (3)), and we take the boundary coupling matrix

$$Q = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ a & b & c \end{pmatrix},$$

459 where $a, b, c \in \mathbb{R}$ are arbitrary numbers.

Then, the Gaussian elimination easily shows that $Q \in \mathcal{B}$ and

$$U = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The feedback law is thus given by

$$\begin{cases} y_1(t, 1) = y_2\left(t, \frac{1}{2}\right) + y_3\left(t, \frac{1}{4}\right), \\ y_2(t, 1) = -3y_3\left(t, \frac{1}{2}\right), \\ y_3(t, 1) = 0. \end{cases}$$

460 **Remark 5.4.** Let us also add that another advantage of not using the duality is that it can
 461 be useful to deal with other functional settings (e.g. C^1 , provided that the inequality in (39) is
 462 strict).

4.63 6 Reduction to a canonical system

4.64 We are now left with the proof of the second part of Theorem 1.11, which is more difficult and
4.65 require more work.

In this section, we will show how to use the canonical structure of the boundary coupling matrix to remove some coupling terms in the matrix G_{+-} . For any $Q \in \mathbb{R}^{p \times m}$, we introduce the set

$$4.66 \quad \mathcal{C}(Q) = \{G_{+-} \in L^\infty(0, 1)^{p \times m} \mid g_{m+i, c_k} = 0, \quad \forall k \in \{1, \dots, \rho\}, \forall i \geq r_k\}$$

$$4.67 \quad (\mathcal{C}(0) = L^\infty(0, 1)^{p \times m}).$$

The goal of this section is to prove the following result:

4.68 **Proposition 6.1.** *Assume that $Q^0 \in \mathbb{R}^{p \times m}$ is in canonical form. Then, there exists a map*
4.69 $\Upsilon : L^\infty(0, 1)^{p \times m} \rightarrow \mathcal{C}(Q^0)$ *such that, for every $G \in L^\infty(0, 1)^{n \times m}$, we have*

$$4.70 \quad (\Lambda, -, Q^0, G) \sim \left(\Lambda, -, Q^0, \left(\Upsilon(G_{+-}) \right) \right). \quad (43)$$

Proof. We assume that $\rho \neq 0$ since otherwise there is nothing to prove. Reproducing the proof of Theorem 3.2 with the kernel

$$K = \begin{pmatrix} 0 & 0 \\ 0 & K_{++} \end{pmatrix},$$

we see that we have (43) if we take

$$(\Upsilon(G_{+-}))(x) = G_{+-}(x) - K_{++}(x, 0)\Lambda_{++}(0)Q^0 - \int_0^x K_{++}(x, \xi)G_{+-}(\xi) d\xi,$$

and provided that K_{++} satisfies

$$\begin{cases} \Lambda_{++}(x) \frac{\partial K_{++}}{\partial x}(x, \xi) + \frac{\partial K_{++}}{\partial \xi}(x, \xi)\Lambda_{++}(\xi) + K_{++}(x, \xi) \frac{\partial \Lambda_{++}}{\partial \xi}(\xi) = 0, \\ \Lambda_{++}(x)K_{++}(x, x) - K_{++}(x, x)\Lambda_{++}(x) = 0. \end{cases}$$

4.71 This is an uncoupled system with many solutions (as we already know from Theorem 3.4). Let
4.72 us find a particular one that guarantees that $\Upsilon(G_{+-}) \in \mathcal{C}(Q^0)$. Let $i, j \in \{m+1, \dots, n\}$ be
4.73 fixed. The equation for k_{ij} is simply

$$4.74 \quad \begin{cases} \lambda_i(x) \frac{\partial k_{ij}}{\partial x}(x, \xi) + \frac{\partial k_{ij}}{\partial \xi}(x, \xi)\lambda_j(\xi) + k_{ij}(x, \xi) \frac{\partial \lambda_j}{\partial \xi}(\xi) = 0, \\ k_{ij}(x, x) = 0, \quad \text{if } i \neq j. \end{cases} \quad (44)$$

Let $s \mapsto \zeta_{ij}(s; x, \xi)$ be the associated characteristic passing through (x, ξ) :

$$\begin{cases} \frac{\partial \zeta_{ij}}{\partial s}(s; x, \xi) = \frac{\lambda_j(\zeta_{ij}(s; x, \xi))}{\lambda_i(s)}, \\ \zeta_{ij}(x; x, \xi) = \xi. \end{cases}$$

4.75 The solutions to (44) are explicit:

- If $i \geq j$, then there exists a unique solution to (44) which satisfies $k_{ij}(x, 0) = a_{ij}(x)$ ($a_{ij} \in L^\infty(0, 1)$ is arbitrary) and it is given by

$$k_{ij}(x, \xi) = \begin{cases} a_{ij}(s_{ij}^{\text{in}}(x, \xi)) \frac{\lambda_j(0)}{\lambda_j(\xi)} & \text{if } \xi < \zeta_{ij}(x; 0, 0), \\ 0 & \text{if } \xi > \zeta_{ij}(x; 0, 0), \end{cases}$$

where $s_{ij}^{\text{in}}(x, \xi) \in (0, x)$ is the unique solution to

$$\zeta_{ij}(s_{ij}^{\text{in}}(x, \xi); x, \xi) = 0.$$

- If $i < j$, then there exists a unique solution to (44) which satisfies $k_{ij}(1, \xi) = a_{ij}(\xi)$ ($a_{ij} \in L^\infty(0, 1)$ is arbitrary) and it is given by

$$k_{ij}(x, \xi) = \begin{cases} a_{ij}(\zeta_{ij}(1; x, \xi)) \frac{\lambda_j(\zeta_{ij}(1; x, \xi))}{\lambda_j(\xi)} & \text{if } \xi < \zeta_{ij}(x; 1, 1), \\ 0 & \text{if } \xi > \zeta_{ij}(x; 1, 1). \end{cases}$$

We choose $a_{ij} = 0$ for $i < j$, so that $k_{ij} = 0$ for such indices. Let us now fix the remaining a_{ij} to ensure that $\Upsilon(G_{+-}) \in \mathcal{C}(Q^0)$. To this end, we fix $i \in \{m+1, \dots, n\}$ such that $E_i \neq \emptyset$, where

$$E_i = \{\alpha \in \{1, \dots, \rho\} \mid m + r_\alpha \leq i\}.$$

The (i, c_α) -th entry of $\Upsilon(G_{+-})$ is equal to zero if, and only if,

$$0 = g_{ic_\alpha}(x) - \sum_{\ell=m+1}^n k_{i\ell}(x, 0) \lambda_\ell(0) q_{\ell-m, c_\alpha}^0 - \int_0^x \sum_{\ell=m+1}^n k_{i\ell}(x, \xi) g_{\ell, c_\alpha}(\xi) d\xi.$$

Using the explicit formulas for k_{ij} and the assumption that Q^0 is in canonical form, for $\alpha \in E_i$ this identity is equivalent to

$$0 = g_{ic_\alpha}(x) - a_{i, m+r_\alpha}(x) \lambda_{m+r_\alpha}(0) - \sum_{\ell=m+1}^i \int_0^{\zeta_{i\ell}(x; 0, 0)} a_{i\ell}(s_{i\ell}^{\text{in}}(x, \xi)) \frac{\lambda_\ell(0)}{\lambda_\ell(\xi)} g_{\ell, c_\alpha}(\xi) d\xi.$$

Using the change of variable $\theta \mapsto \xi = \zeta_{i\ell}(x; \theta, 0)$ with the property

$$s_{i\ell}^{\text{in}}(x, \zeta_{i\ell}(x; \theta, 0)) = s_{i\ell}^{\text{in}}(\theta, 0) = \theta,$$

and isolating the terms for $\ell = m + r_\beta$ with $\beta \in E_i$, this gives the following system of Volterra equations of the second kind:

$$a_{i, m+r_\alpha}(x) \lambda_{m+r_\alpha}(0) + \sum_{\beta \in E_i} \int_0^x a_{i, m+r_\beta}(\theta) h_{i, \alpha, m+r_\beta}(x, \theta) d\theta = f_{i\alpha}(x), \quad \alpha \in E_i, \quad (45)$$

with L^∞ kernel

$$h_{i\alpha\ell}(x, \theta) = \frac{\lambda_\ell(0)}{\lambda_\ell(\zeta_{i\ell}(x; \theta, 0))} g_{\ell, c_\alpha}(\zeta_{i\ell}(x; \theta, 0)) \frac{\partial \zeta_{i\ell}}{\partial \theta}(x; \theta, 0),$$

and L^∞ right-hand side

$$f_{i\alpha}(x) = g_{ic_\alpha}(x) - \sum_{\ell \in F_i} \int_0^x a_{i\ell}(\theta) h_{i\alpha\ell}(x, \theta) d\theta,$$

$$F_i = \{\ell \in \{m+1, \dots, i\} \mid \ell \notin \{m+r_1, \dots, m+r_\rho\} \text{ or } \ell \in \{m+r_\beta \mid \beta \notin E_i\}\}.$$

Setting

$$a_{i\ell} = 0, \quad \forall \ell \in F_i,$$

we have $f_{i\alpha} = g_{ic_\alpha}$ and, once $f_{i\alpha}$ is known, the remaining values $a_{i, m+r_\alpha}$, $\alpha \in E_i$, are uniquely determined by solving the system (45).

481

□

Remark 6.2. Let us point out that it is also possible to transform the matrix G_{--} into a strictly lower triangular matrix by using the kernel

$$K = \begin{pmatrix} K_{--} & 0 \\ 0 & 0 \end{pmatrix},$$

482 and by appropriately choosing some boundary conditions for K_{--} (this is the same proof as
483 above with $Q^0 = \text{Id}$). In summary, whatever $Q \in \mathbb{R}^{p \times m}$ and $G \in L^\infty(0, 1)^{n \times m}$ are, we have
484 shown that we can always find some transformations so that we are reduced to the case where:

- 485 • Q is in canonical form.
- 486 • G_{--} is strictly lower triangular.
- 487 • $G_{+-} \in \mathcal{C}(Q)$.

488 Let us add that, in general, it is not possible to remove more terms by using some transforma-
489 tions (e.g. the backstepping method). In other words, there is in general no simpler equivalent
490 system. An example has been detailed in Appendix A. In this sense, systems $(\Lambda, -, Q, G)$ with
491 the above structure could be called “in canonical form”.

492 7 Reduction by compactness-uniqueness

493 In this section, we show that, even though we can not in general fully remove G_{--} by using
494 some transformations (Remark 6.2), nevertheless, the two systems share the same minimal null
495 control time:

496 **Theorem 7.1.** *Let $Q \in \mathbb{R}^{p \times m}$ be fixed. For every $G \in L^\infty(0, 1)^{n \times m}$, we have*

$$497 \quad T_{\text{inf}}(\Lambda, -, Q, G) = T_{\text{inf}}\left(\Lambda, -, Q, \begin{pmatrix} 0 \\ G_{+-} \end{pmatrix}\right). \quad (46)$$

498 **Remark 7.2.** Let us emphasize once again that it is impossible to prove Theorem 7.1 by using
499 some transformations to pass from one system to the other (e.g. backstepping). In other words,
500 these two systems are in general not equivalent (in the sense of Definition 1.17). Therefore, a
501 different method is necessary to prove Theorem 7.1. We will do it thanks to a compactness-
502 uniqueness method adapted to the null controllability property.

503 7.1 A compactness-uniqueness method for the null controllability

504 We will present here a general compactness-uniqueness method adapted to the null controllability
505 property. We will see in the next section how to use it in order to obtain Theorem 7.1.

506 First of all, let us briefly recall some basic facts about abstract linear control systems. All
507 along this section, H and U are two complex Hilbert spaces, $A : D(A) \subset H \rightarrow H$ is the generator
508 of a C_0 -semigroup $(S(t))_{t \geq 0}$ on H and $B \in \mathcal{L}(U, D(A^*)')$. Here and in what follows, E' denotes
509 the anti-dual of the complex space E , that is the complex (Banach) space of all continuous
510 conjugate linear forms. We will use the convention that an inner product of a complex Hilbert
511 space is conjugate linear in its second argument. One of the reason why we have to consider
512 complex (and not real) spaces is because we will use below a condition involving the spectral
513 elements of the operator A , we will explain how to deal with real Banach spaces in practice at
514 the end of this section in Remark 7.8.

515 Let us now consider the evolution problem associated with the pair (A, B) , i.e.

$$516 \quad \begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t), & t \in (0, T), \\ y(0) = y^0, \end{cases} \quad (47)$$

517 where $T > 0$, $y(t)$ is the state at time t , y^0 is the initial data and $u(t)$ is the control at time t .

Let us recall a standard procedure to define a notion of solution in H to (47) for non smooth functions. We formally multiply (47) by a smooth function z , integrate over an arbitrary time interval $(0, \tau) \subset (0, T)$, perform an integration by parts and use the adjoints to obtain the identity

$$\langle y(\tau), z(\tau) \rangle_H - \langle y^0, z(0) \rangle_H + \int_0^\tau \left\langle y(t), -\frac{d}{dt}z(t) - A^*z(t) \right\rangle_H dt = \int_0^\tau \langle u(t), B^*z(t) \rangle_U dt.$$

518 Particularizing this identity for the solution z to the so-called adjoint system

$$519 \quad \begin{cases} -\frac{d}{dt}z(t) = A^*z(t), & t \in (0, \tau), \\ z(\tau) = z^1, \end{cases} \quad (48)$$

520 i.e. $z(t) = S(\tau - t)^*z^1$, where z^1 is arbitrary, this leads to the following notion of solution in H :

521 **Definition 7.3.** Let $T > 0$, $y^0 \in H$ and $u \in L^2(0, T; U)$ be fixed. We say that a function
522 $y : [0, T] \rightarrow H$ is a solution to (47) if $y \in C^0([0, T]; H)$ and

$$523 \quad \langle y(\tau), z^1 \rangle_H - \langle y^0, z(0) \rangle_H = \int_0^\tau \langle u(t), B^*z(t) \rangle_U dt, \quad (49)$$

524 for every $\tau \in (0, T]$ and $z^1 \in D(A^*)$, where $z \in C^0([0, \tau]; D(A^*))$ is the solution to the adjoint
525 system (48).

526 For the system (47) to be well posed in this sense, the space H has to satisfy some properties.

527 **Definition 7.4.** We say that H is an admissible subspace for the system (A, B) if the following
528 regularity property holds:

$$529 \quad \forall \tau > 0, \exists C > 0, \int_0^\tau \|B^*z(t)\|_U^2 dt \leq C \|z^1\|_H^2, \quad \forall z^1 \in D(A^*), \quad (50)$$

530 where $z \in C^0([0, \tau]; D(A^*))$ is the solution to the adjoint system (48).

531 We recall that, thanks to basic semigroup properties, it is equivalent to prove (50) for one
532 single $\tau > 0$.

If H is an admissible subspace for (A, B) , then the map

$$z^1 \in D(A^*) \longmapsto \int_0^\tau \langle u(t), B^*z(t) \rangle_U dt,$$

533 can be extended to a continuous conjugate linear form on H . Thus, we have a natural definition
534 for the map $\tau \in [0, T] \mapsto y(\tau) \in H$ through the formula (49). It can be proved that this map
535 is also continuous and that it depends continuously on y^0 and u on compact time intervals (see
536 e.g. [Cor07, Theorem 2.37]). This establishes the so-called well-posedness of the abstract control
537 system (47) in H .

538 Now that we have a notion of continuous solution for the system (47) in the space H , we can
539 speak of its controllability properties in H .

Definition 7.5. We say that the system (47) is null controllable in time T if, for every $y^0 \in H$, there exists $u \in L^2(0, T; U)$ such that the corresponding solution $y \in C^0([0, T]; H)$ to the system (47) satisfies

$$y(T) = 0.$$

540 It is also well known that controllability has a dual concept named observability. We have
541 the following characterization (see e.g. [Cor07, Theorem 2.44]):

Theorem 7.6. *Let $T > 0$ be fixed. The system (A, B) is null controllable in time T if, and only if, there exists $C > 0$ such that, for every $z^1 \in D(A^*)$,*

$$\|z(0)\|_H^2 \leq C \int_0^T \|B^* z(t)\|_U^2 dt,$$

542 where $z \in C^0([0, T]; D(A^*))$ is the solution to the adjoint system (48) (with $\tau = T$).

543 After these basic reminders, we can now clearly introduce the general compactness-uniqueness
544 result on which the proof of Theorem 7.1 will rely on.

545 **Theorem 7.7.** *Let H and U be two complex Hilbert spaces. Let $A : D(A) \subset H \rightarrow H$ be the
546 generator of a C_0 -semigroup $(S(t))_{t \geq 0}$ on H and let $B \in \mathcal{L}(U, D(A^*)')$. We assume that H is
547 an admissible subspace for (A, B) and that (A, B) satisfies the so-called Fattorini-Hautus test,
548 namely:*

$$549 \ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \quad (51)$$

550 Assume in addition that there exists $T_0 > 0$ such that, for every $T > T_0$, the following two
551 properties hold:

(i) *There exist two complex Banach spaces E_1, E_2 , a compact operator $P : E_1 \rightarrow E_2$, a linear operator $L : D(A^*) \rightarrow E_1$ and $C > 0$ such that, for every $z^1 \in D(A^*)$,*

$$\|z(0)\|_H^2 \leq C \left(\int_0^T \|B^* z(t)\|_U^2 dt + \|PLz^1\|_{E_2}^2 \right), \quad (52)$$

$$\|Lz^1\|_{E_1}^2 \leq C \left(\|z(0)\|_H^2 + \int_0^T \|B^* z(t)\|_U^2 dt \right), \quad (53)$$

552 where $z \in C^0([0, T]; D(A^*))$ is the solution to the adjoint system (48) (with $\tau = T$).

553 (ii) *For every $0 < t_1 < t_2 < T - T_0$, there exists $C > 0$ such that, for every $z^1 \in D(A^*)$,*

$$554 \|z(t_2)\|_H^2 \leq C \left(\|z(t_1)\|_H^2 + \int_{t_1}^{t_2} \|B^* z(t)\|_U^2 dt \right), \quad (54)$$

555 where $z \in C^0([0, T]; D(A^*))$ is the solution to the adjoint system (48) (with $\tau = T$).

556 Then, the system (A, B) is null controllable in time T for every $T > T_0$.

557 The proof of this result is postponed to Appendix B for the sake of the presentation. It is
558 based on arguments developed in the proofs of [CN21a, Theorem 2] and [DO18, Lemma 2.6] (see
559 also the references therein). Let us just mention at this point that, in general, the compactness-
560 uniqueness method is designed for the exact controllability property. It is only thanks to the
561 property (54) that we are able to consider the null controllability property here.

562 **Remark 7.8.** In most applications we encounter real systems, that is H and U are real Banach
563 spaces. To apply what precedes, we have to consider their so-called complexifications as well
564 as the complexifications of the operators A and B . By splitting the complex system (i.e. the
565 system corresponding to these complexifications) into real and imaginary parts, it is not difficult
566 to check that the real system is controllable if, and only if, so is the complex system.

567 7.2 Proof of Theorem 7.1

568 Let us now show how to use the general result Theorem 7.7 in order to obtain Theorem 7.1. We
569 only prove the inequality “ \leq ” in (46) (which is the most important one), the other inequality
570 can be established similarly. Let then $T_0 > 0$ be such that

$$571 \quad \left(\Lambda, -, Q, \begin{pmatrix} 0 \\ G_{+-} \end{pmatrix} \right) \quad (55)$$

572 is null controllable in time T_0 and let us show that necessarily $T_{\text{inf}}(\Lambda, -, Q, G) \leq T_0$. This will
573 follow from Theorem 7.7 once we will have checked that the system $(\Lambda, -, Q, G)$ satisfies all the
574 assumptions of this result.

575 First of all, we have to recast the system $(\Lambda, -, Q, G)$ as an abstract evolution system of the
576 form (47). This is quite standard. To identify what are the operators A and B (in fact, we first
577 find A^* and B^*), we repeat the procedure that led to Definition 7.3 on the system (16) (with
578 $M = 0$), where taking the adjoints is replaced by an integration by parts in space. This gives
579 the following.

- The state and control spaces are

$$H = L^2(0, 1)^n (= L^2(0, 1; \mathbb{C}^n)), \quad U = \mathbb{C}^m.$$

580 They are equipped with their usual inner products.

- The unbounded linear operator $A : D(A) \subset H \rightarrow H$ is defined for every $y \in D(A)$ by

$$(Ay)(x) = -\Lambda(x) \frac{\partial y}{\partial x}(x) + G(x)y_-(0), \quad x \in (0, 1),$$

with domain

$$D(A) = \{y \in H^1(0, 1)^n \mid y_-(1) = 0, \quad y_+(0) = Qy_-(0)\}.$$

- It is clear that $D(A)$ is dense in H since it contains $C_c^\infty(0, 1)^n$. A computation shows that

$$D(A^*) = \left\{ z \in H^1(0, 1)^n \mid z_-(0) = R^* z_+(0) + \int_0^1 K(\xi)^* z(\xi) d\xi, \quad z_+(1) = 0 \right\},$$

where $R \in \mathbb{R}^{p \times m}$ and $K \in L^\infty(0, 1)^{n \times m}$ are defined by

$$R = -\Lambda_{++}(0)Q\Lambda_{--}(0)^{-1}, \quad K(\xi) = -G(\xi)\Lambda_{--}(0)^{-1},$$

and we have, for every $z \in D(A^*)$,

$$(A^*z)(x) = \Lambda(x) \frac{\partial z}{\partial x}(x) + \frac{\partial \Lambda}{\partial x}(x)z(x), \quad x \in (0, 1).$$

- The control operator $B \in \mathcal{L}(U, D(A^*)')$ is given for every $u \in U$ and $z \in D(A^*)$ by

$$\langle Bu, z \rangle_{D(A^*)', D(A^*)} = \langle u, -\Lambda_{--}(1)z_-(1) \rangle_{C^m}.$$

581 Note that B is well defined since Bu is continuous on $H^1(0, 1)^n$ (by the trace theorem
 582 $H^1(0, 1)^n \hookrightarrow C^0([0, 1]^n)$ and since the graph norm $\|\cdot\|_{D(A^*)}$ and $\|\cdot\|_{H^1(0, 1)^n}$ are equivalent
 583 norms on $D(A^*)$).

- Finally, the adjoint $B^* \in \mathcal{L}(D(A^*), U)$ is given for every $z \in D(A^*)$ by

$$B^*z = -\Lambda_{--}(1)z_-(1).$$

584 We can prove that A is closed and that both A, A^* are quasi-dissipative, so that A generates
 585 a C_0 -semigroup by a well-known corollary of Lumer-Phillips theorem.

Since the other properties to check depend on the adjoint system, it is convenient to write it explicitly:

$$\begin{cases} \frac{\partial z}{\partial t}(t, x) + \Lambda(x) \frac{\partial z}{\partial x}(t, x) = -\frac{\partial \Lambda}{\partial x}(x)z(t, x), \\ z_-(t, 0) = R^*z_+(t, 0) + \int_0^1 K(\xi)^*z(t, \xi) d\xi, \quad z_+(t, 1) = 0, \\ z(T, x) = z^1(x). \end{cases}$$

586 Using the method of characteristics it is easy to prove the estimate (50) for $\tau \leq T_1(\Lambda)$, which
 587 shows that H is an admissible subspace for (A, B) .

588 Therefore, the abstract control system is well posed in H . To rigorously justify that this pair
 589 (A, B) is “the” abstract form of $(\Lambda, -, Q, G)$ we have to reason in terms of notions of solution:

590 **Proposition 7.9.** *The solution to system $(\Lambda, -, Q, G)$ in the sense of Definition 2.1 coincides*
 591 *with the solution to abstract system (47) in the sense of Definition 7.3 corresponding to the pair*
 592 *(A, B) introduced above.*

593 *Proof.* We argue by approximation. Let $y^0 \in L^2(0, 1)^n$, $u \in L^2(0, T)^m$ be fixed and let y be the
 594 corresponding solution to system $(\Lambda, -, Q, G)$ in the sense of Definition 2.1.

- We take two approximations $(y^{0,k})_k \subset H_0^1(0, 1)^n$ and $(u^k)_k \subset H_0^1(0, T)^m$ such that

$$596 \quad y^{0,k} \rightarrow y^0 \text{ in } L^2(0, 1)^n, \quad u^k \rightarrow u \text{ in } L^2(0, T)^m. \quad (56)$$

Let y^k be the solution corresponding to $y^{0,k}$ and u^k in the sense of Definition 2.1. Since
 $y^{0,k}$ and u^k obviously satisfy the C^0 compatibility conditions

$$y_-^{0,k}(1) = u^k(0), \quad y_+^{0,k}(0) = Qy_-^{0,k}(0),$$

we can prove that

$$y^k \in C^0([0, T]; H^1(0, 1)^n) \cap C^0([0, 1]; H^1(0, T)^n)$$

597 (for instance, by adapting the fixed point approach of [CHOS21, Appendix A.2] in the above
 598 space – the regularity $G \in L^\infty(0, 1)^{n \times m}$ is enough after a suitable change of variable). In
 599 particular, $y^k \in H^1((0, T) \times (0, 1)^n)$ and it satisfies (16) almost everywhere (with $M = 0$,
 600 $y = y^k$, $u = u^k$ and $y^0 = y^{0,k}$).

- Repeating the procedure that led to Definition 7.3 we easily check that y^k is the solution to abstract system (47) in the sense of Definition 7.3, i.e. it satisfies identity (49) (with $y = y^k$, $y^0 = y^{0,k}$ and $u = u^k$). Using (56) and

$$y^k \rightarrow y \text{ in } C^0([0, T]; L^2(0, 1)^n),$$

601 (this follows from (30) and (56)), we can pass to the limit $k \rightarrow +\infty$ in this identity to
 602 obtain that y is the solution to abstract system (47) in the sense of Definition 7.3.

603 □

604 We will now check that our pair (A, B) satisfies the assumptions of Theorem 7.7.

- The Fattorini-Hautus test (51) is easy to check. Indeed, if $\lambda \in \mathbb{C}$ and $z \in D(A^*)$ are such that $A^*z = \lambda z$ and $B^*z = 0$, then in particular $z \in H^1(0, 1)^n$ solves the system of linear ODEs

$$\begin{cases} \frac{\partial z}{\partial x}(x) = \Lambda(x)^{-1} \left(-\frac{\partial \Lambda}{\partial x}(x) + \lambda \text{Id}_{\mathbb{R}^n \times n} \right) z(x), & x \in (0, 1), \\ z(1) = 0, \end{cases}$$

605 so that $z = 0$ by uniqueness.

606 Below, C denotes a positive number that may change from line to line but that never depends
 607 on z^1 or t .

- The inequality (54) is also not difficult to check. Indeed, for $0 < t_1 < t_2 < T - T_0$, using the method of characteristics, we have

$$\|z_-(t_2, \cdot)\|_{L^2(0,1)^m}^2 \leq C \left(\|z_-(t_1, \cdot)\|_{L^2(0,1)^m}^2 + \int_{t_1}^{t_2} \|z_-(t, 1)\|_{\mathbb{C}^m}^2 dt \right),$$

and, provided that $T_0 \geq T_{m+1}(\Lambda)$ and using that $z_+(\cdot, 1) = 0$, we also have

$$\|z_+(t_2, \cdot)\|_{L^2(0,1)^p}^2 \leq C \|z_+(t_1, \cdot)\|_{L^2(0,1)^p}^2.$$

608 We recall that, since the system (55) is null controllable in time T_0 by assumption, we
 609 necessarily have $T_0 \geq T_{m+1}(\Lambda)$ (see the first step of the proof of Theorem 5.1).

- Let us now investigate the estimate (52). Let $T > T_0$. We will prove that there exists
 610 $H \in L^\infty((0, T) \times (0, T))^{m \times m}$ such that, for every $z^1 \in L^2(0, 1)^n$,

612
$$\|z(0, \cdot)\|_{L^2(0,1)^n}^2 \leq C \left(\int_0^T \|z_-(t, 1)\|_{\mathbb{C}^m}^2 dt + \int_0^T \left\| \int_0^T H(t, s) z_-(s, 0) ds \right\|_{\mathbb{C}^m}^2 dt \right). \quad (57)$$

613 Let us first make some preliminary observations. We denote by ζ the solution to the adjoint
 614 system of (55) in $(0, T)$ with final data z^1 , and we set

615
$$\theta = z - \zeta. \quad (58)$$

Clearly, it satisfies

$$\begin{cases} \frac{\partial \theta}{\partial t}(t, x) + \Lambda(x) \frac{\partial \theta}{\partial x}(t, x) = -\frac{\partial \Lambda}{\partial x}(x) \theta(t, x), \\ \theta_-(t, 0) = R^* \theta_+(t, 0) + \int_0^1 K_{+-}(\xi)^* \theta_+(t, \xi) d\xi + \int_0^1 K_{--}(\xi)^* z_-(t, \xi) d\xi, & \theta_+(t, 1) = 0, \\ \theta(T, x) = 0. \end{cases}$$

616

Using the method of characteristics, we immediately see that

617

$$\theta_+ = 0. \quad (59)$$

Consequently, θ_- solves

$$\begin{cases} \frac{\partial \theta_-}{\partial t}(t, x) + \Lambda_{--}(x) \frac{\partial \theta_-}{\partial x}(t, x) = -\frac{\partial \Lambda_{--}}{\partial x}(x) \theta_-(t, x), \\ \theta_-(t, 0) = \int_0^1 K_{--}(\xi)^* z_-(t, \xi) d\xi, \\ \theta_-(T, x) = 0. \end{cases}$$

Since $T > T_0 \geq T_m(\Lambda)$, using the method of characteristics, it is not difficult to see that, for $t \in (0, T)$, we have

$$\begin{aligned} \|\theta_-(t, 0)\|_{\mathbb{C}^m}^2 &= \left\| \int_0^1 K_{--}(\xi)^* z_-(t, \xi) d\xi \right\|_{\mathbb{C}^m}^2 \\ &\leq C \left(\left\| \int_t^T H(t, s) z_-(s, 0) ds \right\|_{\mathbb{C}^m}^2 + \int_0^t \|z_-(s, 1)\|_{\mathbb{C}^m}^2 ds \right), \end{aligned} \quad (60)$$

for some $H \in L^\infty((0, T) \times (0, T))^{m \times m}$ independent of z^1 . Let us now prove the desired estimate (57). Since by assumption the system (55) is null controllable in time T_0 , and thus in time $T > T_0$, the solution ζ to its adjoint system satisfies (see Theorem 7.6)

$$\|\zeta(0, \cdot)\|_{L^2(0,1)^n}^2 \leq C \int_0^T \|\zeta_-(t, 1)\|_{\mathbb{C}^m}^2 dt.$$

Recalling (58) and (59), it follows that

$$\begin{aligned} \|z(0, \cdot)\|_{L^2(0,1)^n}^2 &\leq 2 \|\theta(0, \cdot)\|_{L^2(0,1)^n}^2 + 2 \|\zeta(0, \cdot)\|_{L^2(0,1)^n}^2 \\ &= 2 \|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + 2 \|\zeta(0, \cdot)\|_{L^2(0,1)^n}^2 \\ &\leq 2 \|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + C \int_0^T \|\zeta_-(t, 1)\|_{\mathbb{C}^m}^2 dt \\ &\leq 2 \|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + 2C \int_0^T \|\theta_-(t, 1)\|_{\mathbb{C}^m}^2 dt + 2C \int_0^T \|z_-(t, 1)\|_{\mathbb{C}^m}^2 dt. \end{aligned}$$

On the other hand, using the method of characteristics and the condition $\theta_-(T, \cdot) = 0$, we have

$$\|\theta_-(0, \cdot)\|_{L^2(0,1)^m}^2 + \int_0^T \|\theta_-(t, 1)\|_{\mathbb{C}^m}^2 dt \leq C \int_0^T \|\theta_-(t, 0)\|_{\mathbb{C}^m}^2 dt.$$

618

Combined with (60) this leads to the desired estimate (57).

- The estimate (57) suggests to consider the linear operators

$$P : L^2(0, T)^m \longrightarrow L^2(0, T)^m, \quad L : D(A^*) \longrightarrow L^2(0, T)^m,$$

defined by

$$(Pv)(t) = \int_0^T H(t, s)v(s) ds, \quad (Lz^1)(s) = z_-(s, 0).$$

619

From the previous point, (52) is fulfilled. It is also well-known that operators of the form of P are compact. Finally, we easily check with the method of characteristics that L satisfies the remaining estimate (53). This concludes the proof of Theorem 7.1.

620

621

622 8 Largest value of the minimal null control time

623 In this last section we will finally prove the second part of Theorem 1.11.

624 8.1 Characterization for systems $(\Lambda, -, Q, G)$

625 We start with systems of the form $(\Lambda, -, Q, G)$, we will deal with the initial system $(\Lambda, M, Q, -)$
626 in the next section.

627 **Theorem 8.1.** *Let $Q^0 \in \mathbb{R}^{p \times m}$ be in canonical form and let $G \in L^\infty(0, 1)^{n \times m}$ with $G_{--} = 0$
628 and $G_{+-} \in \mathcal{C}(Q^0)$.*

629 (i) *The system $(\Lambda, -, Q^0, G)$ is null controllable in time T for every*

$$630 \quad T \geq \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\}, \quad (61)$$

631 *where we recall that ρ_0 is defined in the statement of Theorem 1.11.*

(ii) *Assume that the condition (13) fails and let $G \in \mathbb{R}^{n \times m}$ be the constant matrix whose entries
are all equal to zero except for*

$$g_{m+\rho_0+1, m} = 1.$$

632 *If the corresponding system $(\Lambda, -, Q^0, G)$ is null controllable in time T , then T has to satisfy
633 the condition (61).*

634 As for Theorem 1.11, we use the convention that the undefined quantities are simply not
635 taken into account, which more precisely gives:

- 636 • If $\rho_0 = 0$, then the condition (61) is $T \geq T_{m+1}(\Lambda) + T_m(\Lambda)$.
- 637 • If $\rho_0 = p$, then the condition (61) is $T \geq \max \{ \max_{k \in \{1, \dots, p\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_m(\Lambda) \}$.

638 In the second part of the statement we only discussed the case when (13) fails since otherwise
639 the time on the right-hand side of the inequality in (61) coincides with the time on the right-hand
640 side of the inequality in (39) and it follows from item (i) of Theorem 5.1 that item (i) of Theorem
641 8.1 then becomes a necessary and sufficient condition.

642 This result shows in particular that the largest value that $T_{\inf}(\Lambda, -, Q^0, G)$ can take with
643 respect to $G \in L^\infty(0, 1)^{n \times m}$ when $G_{--} = 0$ and $G_{+-} \in \mathcal{C}(Q^0)$ is equal to the quantity on the
644 right-hand side of the inequality in (61). This can be extended to arbitrary boundary coupling
645 matrices and arbitrary $G \in L^\infty(0, 1)^{n \times m}$ thanks to Proposition 4.1, Proposition 6.1 and Theorem
646 7.1.

Proof of Theorem 8.1. 1) We begin with the proof of the first item. Let first $i \in \{1, \dots, m\}$
be fixed. Since $T \geq T_i(\Lambda)$, which means that $s_i^{\text{in}}(T, x) > 0$ for every $x \in (0, 1)$ as we have
seen in the first step of the proof of Theorem 5.1, and since $G_{--} = 0$ by assumption, the
null controllability condition $y_i(T, \cdot) = 0$ is equivalent to (see (29) and (27))

$$u_i(s_i^{\text{in}}(T, \cdot)) = 0 \quad \text{in } (0, 1).$$

647 Since $i \leq m$, the map $x \mapsto s_i^{\text{in}}(T, x)$ is non decreasing (see (22)) with $s_i^{\text{in}}(T, 1) = T$. Thus,
648 the previous condition is also equivalent to

$$649 \quad u_i = 0 \quad \text{in } (s_i^{\text{in}}(T, 0), T). \quad (62)$$

2) Let us now consider $i \in \{m+1, \dots, n\}$. Since $T \geq T_i(\Lambda)$, the null controllability condition $y_i(T, x) = 0$ is equivalent to (see (29) and (28))

$$a_i(x) + b_i(x) = 0,$$

where

$$a_i(x) = \sum_{\substack{j=1 \\ j \notin \{c_1, \dots, c_{\rho_0}\}}}^m \left(q_{i-m, j}^0 y_j(s_i^{\text{in}}(T, x), 0) + \int_{s_i^{\text{in}}(T, x)}^T g_{ij}(\chi_i(s; T, x)) y_j(s, 0) ds \right),$$

and

$$b_i(x) = \sum_{k=1}^{\rho_0} \left(q_{i-m, c_k}^0 y_{c_k}(s_i^{\text{in}}(T, x), 0) + \int_{s_i^{\text{in}}(T, x)}^T g_{ic_k}(\chi_i(s; T, x)) y_{c_k}(s, 0) ds \right).$$

650 • We first consider the case $i \geq m + \rho_0 + 1$ (which happens only if $\rho_0 < p$). Clearly, we
 651 have $b_i = 0$ in that situation since Q^0 is in canonical form, $G_{+-} \in \mathcal{C}(Q^0)$ and (12).
 652 Let us show that we can choose u_j for $j \notin \{c_1, \dots, c_{\rho_0}\}$ so that $a_i = 0$ as well. Since
 653 $x \mapsto s_i^{\text{in}}(T, x)$ is non increasing for $i \geq m + 1$ (recall (22)), it is sufficient to choose
 654 it such that

$$y_j(\cdot, 0) = 0 \quad \text{in } (s_i^{\text{in}}(T, 1), T). \quad (63)$$

Since $T \geq T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda)$ by assumption, we have in particular $T \geq T_i(\Lambda) + T_j(\Lambda)$ for the indices i, j considered (recall (5)). This condition can be written as $T \geq s_i^{\text{out}}(s_j^{\text{out}}(0, 1), 0)$ (see (21) and (24)) or, equivalently (see (23)),

$$s_j^{\text{in}}(s_i^{\text{in}}(T, 1), 0) \geq 0.$$

Since $s \mapsto s_j^{\text{in}}(s, 0)$ is increasing (see (22)), this is equivalent to

$$s_j^{\text{in}}(s, 0) > 0, \quad \forall s \in (s_i^{\text{in}}(T, 1), T).$$

As a result, we see that (63) holds if, and only if, (see (29), (27) and recall that $G_{--} = 0$)

$$u_j(s_j^{\text{in}}(\cdot, 0)) = 0 \quad \text{in } (s_i^{\text{in}}(T, 1), T).$$

Using again that $s \mapsto s_j^{\text{in}}(s, 0)$ is increasing, this means that

$$u_j = 0 \quad \text{in } (s_j^{\text{in}}(s_i^{\text{in}}(T, 1), 0), s_j^{\text{in}}(T, 0)).$$

656 Observe that this is compatible with (62) since these two intervals are disjoint.

657 • Let us now consider the case $i \leq m + \rho_0$ (which happens only if $\rho_0 \neq 0$). Since Q^0 is
 658 in canonical form, $G_{+-} \in \mathcal{C}(Q^0)$ and (12), we see that $a_i(x) + b_i(x) = 0$ is equivalent
 659 to

$$660 \quad a_i(x) + y_{c_{i-m}}(s_i^{\text{in}}(T, x), 0) + \sum_{k=i-m+1}^{\rho_0} \int_{s_i^{\text{in}}(T, x)}^T g_{ic_k}(\chi_i(s; T, x)) y_{c_k}(s, 0) ds = 0. \quad (64)$$

Let us show that we can choose $u_{c_1}, \dots, u_{c_{\rho_0}}$ so that this identity is satisfied. By assumption, we have $T \geq T_i(\Lambda) + T_{c_{i-m}}(\Lambda)$ for every $i \in \{m+1, \dots, m + \rho_0\}$. As in the previous point we can check that this condition can be written as

$$s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, 1), 0) \geq 0.$$

Since $x \mapsto s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0)$ is decreasing (see (22)), this is equivalent to

$$s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0) > 0, \quad \forall x \in (0, 1).$$

As a result, we see that (64) holds if, and only if, (see (29), (27) and recall that $G_{--} = 0$)

$$u_{c_{i-m}} \left(s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0) \right) = -a_i(x) - \sum_{k=i-m+1}^{\rho_0} \int_{s_i^{\text{in}}(T, x)}^T g_{ic_k}(\chi_i(s; T, x)) y_{c_k}(s, 0) ds.$$

Since a_i is known (it only concerns u_j for $j \notin \{c_1, \dots, c_{\rho_0}\}$), we see by induction (starting with $i = m + \rho_0$) that this formula determines the values of $u_{c_{i-m}}$ in the interval

$$\left(s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, 1), 0), \quad s_{c_{i-m}}^{\text{in}}(T, 0) \right)$$

661 (the map $x \mapsto s_{c_{i-m}}^{\text{in}}(s_i^{\text{in}}(T, x), 0)$ is non increasing and $s_i^{\text{in}}(T, 0) = T$). Observe once
662 again that this is compatible with (62) since these two intervals are disjoint.

663 This concludes the proof of the first item (i) of Theorem 8.1.

664 3) Let us now prove item (ii) of Theorem 8.1. Assume that the condition (13) fails, let G
665 be the constant matrix introduced in the statement, and assume that the corresponding
666 system $(\Lambda, -, Q^0, G)$ is null controllable in time T . Since (13) fails, the condition (61) is
667 simply

$$T \geq T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda). \quad (65)$$

Since the system $(\Lambda, -, Q^0, G)$ is null controllable in time T by assumption, the following 2×2 subsystem also has to be null controllable in time T :

$$\begin{cases} \frac{\partial y_m}{\partial t}(t, x) + \lambda_m(x) \frac{\partial y_m}{\partial x}(t, x) = 0, \\ \frac{\partial y_{m+\rho_0+1}}{\partial t}(t, x) + \lambda_{m+\rho_0+1}(x) \frac{\partial y_{m+\rho_0+1}}{\partial x}(t, x) = y_m(t, 0), \\ y_m(t, 1) = u_m(t), \quad y_{m+\rho_0+1}(t, 0) = q_{\rho_0+1, m}^0 y_m(t, 0). \end{cases}$$

Let us show that, whether $q_{\rho_0+1, m}^0 = 1$ or $q_{\rho_0+1, m}^0 = 0$, we necessarily have (65). If $q_{\rho_0+1, m}^0 = 1$, then this follows from item (i) of Theorem 5.1. Let us then consider the case $q_{\rho_0+1, m}^0 = 0$. As before, it is clearly necessary that $T \geq T_{m+\rho_0+1}(\Lambda)$ and, under this condition, the null controllability condition $y_{m+\rho_0+1}(T, x) = 0$ becomes equivalent to (see (29) and (28))

$$\int_{s_{m+\rho_0+1}^{\text{in}}(T, x)}^T y_m(s, 0) ds = 0.$$

Using the change of variable $\xi \mapsto s = s_{m+\rho_0+1}^{\text{in}}(T, \xi)$, this holds if, and only if,

$$\int_0^x y_m(s_{m+\rho_0+1}^{\text{in}}(T, \xi), 0) \frac{\partial s_{m+\rho_0+1}^{\text{in}}(T, \xi)}{\partial \xi} d\xi = 0.$$

Taking the derivative with respect to x , this is also equivalent to

$$y_m(s_{m+\rho_0+1}^{\text{in}}(T, \cdot), 0) = 0 \quad \text{in } (0, 1).$$

669 It is now not difficult to see that we can choose u_m such that this condition holds if, and
670 only if, we have (65).

671 □

672 8.2 Proof of the second part of Theorem 1.11

673 Let us now show how to combine all the previous results in order to obtain the desired charac-
674 terization of the largest minimal null control time for the initial system $(\Lambda, M, Q, -)$.

675 *Proof of item (ii) of Theorem 1.11.* Let $Q \in \mathbb{R}^{p \times m}$ be fixed.

- 1) • By item (i) of Theorem 8.1, we have

$$T_{\inf}(\Lambda, -, Q^0, G) \leq \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\},$$

676 for every $G \in L^\infty(0, 1)^{n \times m}$ with $G_{--} = 0$ and $G_{+-} \in \mathcal{C}(Q^0)$, where Q^0 is the
677 canonical form of Q .

- 678 • By Theorem 7.1, this inequality remains true for every $G \in L^\infty(0, 1)^{n \times m}$ with $G_{+-} \in$
679 $\mathcal{C}(Q^0)$.
- 680 • By Proposition 6.1, this inequality remains true for every $G \in L^\infty(0, 1)^{n \times m}$.
- 681 • By Proposition 4.1, this inequality remains true by changing Q^0 into Q .
- 682 • By Proposition 3.1 and Theorem 3.2, this inequality remains true for the system
683 $(\Lambda, M, Q, -)$ for any $M \in L^\infty(0, 1)^{n \times n}$.

In summary, we have established the following upper bound:

$$T_{\inf}(\Lambda, M, Q) \leq \max \left\{ \max_{k \in \{1, \dots, \rho_0\}} T_{m+k}(\Lambda) + T_{c_k}(\Lambda), \quad T_{m+\rho_0+1}(\Lambda) + T_m(\Lambda) \right\},$$

684 valid for every $M \in L^\infty(0, 1)^{n \times n}$.

- 685 2) Let us now show that this upper bound is reached for some special M . If the condition
686 (13) is satisfied, then this upper bound coincides with the lower bound, and we know that
687 this latter is reached for $M = 0$ (Remark 1.13). Let us now assume that the condition (13)
688 is not satisfied.

- Then, we know from Theorem 8.1 that this upper bound is the minimal null control
time of the system $(\Lambda, -, Q^0, G)$ for the constant matrix $G \in \mathbb{R}^{n \times m}$ whose entries are
all equal to zero except for

$$g_{m+\rho_0+1, m} = 1.$$

689 Let us now find the corresponding matrix M .

- We decompose Q in its canonical form: $LQU = Q^0$. By Proposition 4.1, this upper
bound is the minimal null control time of the system $(\Lambda, -, Q, \widehat{G})$ for any \widehat{G} such
that $\Theta(\widehat{G}) = G$. Now, it follows from the proof of Proposition 4.1 that, for constant
matrices, Θ is simply given by

$$\Theta(\widehat{G}) = \begin{pmatrix} U^{-1}\widehat{G}_{--}U \\ L\widehat{G}_{+-}U \end{pmatrix}, \quad \forall \widehat{G} \in \mathbb{R}^{n \times m}.$$

Therefore, \widehat{G} is the matrix whose entries are all equal to zero except for

$$\widehat{g}_{m+i, m} = \ell^{i, \rho_0+1}, \quad \forall i \in \{\rho_0 + 1, \dots, p\},$$

690 where $L^{-1} = (\ell^{ij})_{1 \leq i, j \leq p}$.

- By Theorem 3.2, this upper bound is the minimal null control time of the system $(\Lambda, M, Q, -)$ for any $M \in \mathcal{M}$ such that $\Gamma_A(M) = \widehat{G}$ for some $A \in \mathcal{F}$. Let us determine A and M such that this identity holds. By definition of $\Gamma_A(M)$ (see (32)), this is equivalent to

$$\begin{cases} 0 = -K_{--}(x, 0)\Lambda_{--}(0) - K_{-+}(x, 0)\Lambda_{++}(0)Q, \\ \widehat{G}_{+-} = -K_{+-}(x, 0)\Lambda_{--}(0) - K_{++}(x, 0)\Lambda_{++}(0)Q, \end{cases}$$

where K is the solution to the kernel equations (33) with additional boundary conditions (35)-(36) provided by A . Let us rewrite these kernel equations by blocks:

$$\begin{cases} \Lambda_{--}(x) \frac{\partial K_{--}}{\partial x}(x, \xi) + \frac{\partial K_{--}}{\partial \xi}(x, \xi) \Lambda_{--}(\xi) \\ + K_{--}(x, \xi) \left(\frac{\partial \Lambda_{--}}{\partial \xi}(\xi) + M_{--}(\xi) \right) + K_{-+}(x, \xi) M_{+-}(\xi) = 0, \\ \Lambda_{--}(x) K_{--}(x, x) - K_{--}(x, x) \Lambda_{--}(x) = M_{--}(x). \end{cases}$$

$$\begin{cases} \Lambda_{--}(x) \frac{\partial K_{-+}}{\partial x}(x, \xi) + \frac{\partial K_{-+}}{\partial \xi}(x, \xi) \Lambda_{++}(\xi) \\ + K_{--}(x, \xi) M_{-+}(\xi) + K_{-+}(x, \xi) \left(\frac{\partial \Lambda_{++}}{\partial \xi}(\xi) + M_{++}(\xi) \right) = 0, \\ \Lambda_{--}(x) K_{-+}(x, x) - K_{-+}(x, x) \Lambda_{++}(x) = M_{-+}(x). \end{cases}$$

$$\begin{cases} \Lambda_{++}(x) \frac{\partial K_{+-}}{\partial x}(x, \xi) + \frac{\partial K_{+-}}{\partial \xi}(x, \xi) \Lambda_{--}(\xi) \\ + K_{+-}(x, \xi) \left(\frac{\partial \Lambda_{--}}{\partial \xi}(\xi) + M_{--}(\xi) \right) + K_{++}(x, \xi) M_{+-}(\xi) = 0, \\ \Lambda_{++}(x) K_{+-}(x, x) - K_{+-}(x, x) \Lambda_{--}(x) = M_{+-}(x). \end{cases}$$

$$\begin{cases} \Lambda_{++}(x) \frac{\partial K_{++}}{\partial x}(x, \xi) + \frac{\partial K_{++}}{\partial \xi}(x, \xi) \Lambda_{++}(\xi) \\ + K_{+-}(x, \xi) M_{-+}(\xi) + K_{++}(x, \xi) \left(\frac{\partial \Lambda_{++}}{\partial \xi}(\xi) + M_{++}(\xi) \right) = 0, \\ \Lambda_{++}(x) K_{++}(x, x) - K_{++}(x, x) \Lambda_{++}(x) = M_{++}(x). \end{cases}$$

Note that the subsystems satisfied by (K_{--}, K_{-+}) and (K_{+-}, K_{++}) are not coupled. By uniqueness of the solution to these equations (see Theorem 3.4), we see that

$$\begin{aligned} M_{-+} = 0 &\implies K_{-+} = 0, \\ M_{++} = A_{++} = 0, \quad M_{-+} = 0 &\implies K_{++} = 0, \\ M_{--} = A_{--} = 0, \quad K_{-+} = 0 &\implies K_{--} = 0. \end{aligned}$$

Therefore, it only remains to determine M_{+-} such that

$$K_{+-}(x, 0) = -\widehat{G}_{+-} \Lambda_{--}(0)^{-1}.$$

691
692

Let $i \in \{m+1, \dots, n\}$ and $j \in \{1, \dots, m\}$ be fixed. The equation for k_{ij} is now simply

693

$$\begin{cases} \lambda_i(x) \frac{\partial k_{ij}}{\partial x}(x, \xi) + \frac{\partial k_{ij}}{\partial \xi}(x, \xi) \lambda_j(\xi) + k_{ij}(x, \xi) \frac{\partial \lambda_j}{\partial \xi}(\xi) = 0, \\ k_{ij}(x, x) = \frac{m_{ij}(x)}{\lambda_i(x) - \lambda_j(x)}. \end{cases} \quad (66)$$

Let $s \mapsto \zeta_{ij}(s; x, \xi)$ be the associated characteristic passing through (x, ξ) :

$$\begin{cases} \frac{\partial \zeta_{ij}}{\partial s}(s; x, \xi) = \frac{\lambda_j(\zeta_{ij}(s; x, \xi))}{\lambda_i(s)}, \\ \zeta_{ij}(x; x, \xi) = \xi. \end{cases}$$

The solution to (66) is explicit:

$$k_{ij}(x, \xi) = \frac{m_{ij}(s_{ij}^{\text{in}}(x, \xi))}{\lambda_i(s_{ij}^{\text{in}}(x, \xi)) - \lambda_j(s_{ij}^{\text{in}}(x, \xi))} \frac{\lambda_j(s_{ij}^{\text{in}}(x, \xi))}{\lambda_j(\xi)},$$

where $s_{ij}^{\text{in}}(x, \xi) \in (0, x)$ is the unique solution to

$$\zeta_{ij}(s_{ij}^{\text{in}}(x, \xi); x, \xi) = s_{ij}^{\text{in}}(x, \xi).$$

Thus, the desired condition $k_{ij}(\cdot, 0) = -\hat{g}_{ij}/\lambda_j(0)$ is equivalent to

$$m_{ij}(s_{ij}^{\text{in}}(x, 0)) = \frac{\lambda_i(s_{ij}^{\text{in}}(x, 0)) - \lambda_j(s_{ij}^{\text{in}}(x, 0))}{-\lambda_j(s_{ij}^{\text{in}}(x, 0))} \hat{g}_{ij}, \quad x \in (0, 1).$$

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□

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701 A An example of non equivalent hyperbolic systems

702 In this appendix we present an explicit example of hyperbolic systems which are not equivalent
703 in the sense of Definition 1.17. This example is important to illustrate that, in general, it is not
704 possible to obtain a simpler system than the one we obtained in the present article if we only
705 use invertible transformations (see Remark 6.2). It also motivates the use of the compactness-
706 uniqueness method to establish the important result Theorem 7.1. We refer to [CN19, Section
707 4.3] for a close but different example.

708 We consider the following simple 3×3 systems with constant coefficients:

$$\begin{cases} \frac{\partial y_1}{\partial t}(t, x) - \frac{\partial y_1}{\partial x}(t, x) = 0, \\ \frac{\partial y_2}{\partial t}(t, x) - \frac{1}{2} \frac{\partial y_2}{\partial x}(t, x) = ay_1(t, 0), \\ \frac{\partial y_3}{\partial t}(t, x) + \frac{\partial y_3}{\partial x}(t, x) = by_2(t, 0), \end{cases} \quad (67)$$

710 where $a, b \in \mathbb{R}$ are some parameters, and with boundary conditions

$$\begin{cases} y_1(t, 1) = u_1(t), \\ y_2(t, 1) = u_2(t), \end{cases} \quad y_3(t, 0) = y_1(t, 0). \quad (68)$$

We are in the case $m = 2, p = 1$ and the matrices Λ, G and Q are

$$\Lambda = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{array} \right), \quad G = G_{ab} = \begin{pmatrix} 0 & 0 \\ a & 0 \\ 0 & b \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

712 Note as well that we are in an ideal configuration:

- 713 • Q is in canonical form.
- 714 • $(G_{ab})_{--}$ is strictly lower triangular.
- 715 • $(G_{ab})_{+-} \in \mathcal{C}(Q)$.

Clearly, $\rho = \rho_0 = 1$ and $(r_1, c_1) = (1, 1)$. It follows from the results of the present article (actually, a direct proof is also possible) that the minimal null control time of the system (67)-(68) is

$$T_{\inf}(\Lambda, -, Q, G_{ab}) = 2, \quad \forall a, b \in \mathbb{R}.$$

716 In particular, the system (67)-(68) is null controllable in time T for every $T > 2$. Let us now
717 study the null controllability properties of this system in this critical time:

718 **Proposition A.1.** *The system (67)-(68) is null controllable in time $T = 2$ if, and only if,*

$$719 \quad ab \notin \Sigma = \left\{ -\left(\frac{\pi}{2} + k\pi\right)^2 \mid k \in \mathbb{N} \right\}. \quad (69)$$

Remark A.2. It follows from this result and Proposition 1.18 that

$$(\Lambda, -, Q, G_{ab}) \quad \text{is not equivalent to} \quad (\Lambda, -, Q, G_{cd}), \quad \text{if } ab \in \Sigma, cd \notin \Sigma.$$

720 In particular, it is not possible to transform the system $(\Lambda, -, Q, G_{ab})$ into $(\Lambda, -, Q, G_{0b})$ or
721 $(\Lambda, -, Q, G_{a0})$ when $ab \in \Sigma$ (in other words, we cannot remove $(G_{ab})_{--}$ nor $(G_{ab})_{+-}$ in this
722 case).

Proof of Proposition A.1. The solution to the system (67)-(68) is explicit (see Section 2):

$$y_1(t, x) = \begin{cases} u_1(t-1+x) & \text{if } t-1+x > 0, \\ y_1^0(t+x) & \text{if } t-1+x < 0, \end{cases}$$

$$y_2(t, x) = \begin{cases} u_2(t-2(1-x)) + a \int_{t-2(1-x)}^t y_1(s, 0) ds & \text{if } t-2(1-x) > 0, \\ y_2^0\left(\frac{t}{2}+x\right) + a \int_0^t y_1(s, 0) ds & \text{if } t-2(1-x) < 0, \end{cases}$$

$$y_3(t, x) = \begin{cases} y_1(t-x, 0) + b \int_{t-x}^t y_2(s, 0) ds & \text{if } t-x > 0, \\ y_3^0(-t+x) + b \int_0^t y_2(s, 0) ds & \text{if } t-x < 0. \end{cases}$$

Clearly, the null controllability condition $y_1(2, \cdot) = 0$ is satisfied if, and only if,

$$u_1 = 0 \quad \text{in } (1, 2).$$

Similarly, the null controllability condition $y_2(2, \cdot) = 0$ holds if, and only if,

$$u_2(t) = -a \int_t^2 y_1(s, 0) ds, \quad t \in (0, 2).$$

Thus, the control u_2 is uniquely determined once the values of the control u_1 in $(0, 1)$ are known. The remaining condition $y_3(2, x) = 0$ is equivalent to

$$y_1(2-x, 0) + b \int_{2-x}^2 y_2(s, 0) ds = 0,$$

and thus to

$$u_1(1-x) + b \int_{2-x}^2 y_2^0\left(\frac{s}{2}\right) ds + abx \int_0^1 y_1^0(\theta) d\theta + ab \int_{2-x}^2 \int_1^s u_1(\theta-1) d\theta ds = 0.$$

723 Using the change of variables $t = 1-x$ and $\sigma = \theta-1$, this is also equivalent to

$$724 \quad u_1(t) + ab \int_t^1 \int_0^s u_1(\sigma) d\sigma ds = f(t), \quad t \in (0, 1), \quad (70)$$

725 where we introduced the following function depending only on the initial data:

$$726 \quad f(t) = -b \int_{1+t}^2 y_2^0\left(\frac{s}{2}\right) ds - ab(1-t) \int_0^1 y_1^0(\theta) d\theta. \quad (71)$$

727 This identity can be rewritten as

$$728 \quad (\text{Id} - K) \begin{pmatrix} u_1 \\ \int_0^{\cdot} u_1(\sigma) d\sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad (72)$$

where $K : L^2(0, 1)^2 \rightarrow L^2(0, 1)^2$ is the operator defined by

$$\left(K \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) (t) = \begin{pmatrix} -ab \int_t^1 \beta(s) ds \\ \int_0^t \alpha(s) ds \end{pmatrix}.$$

729 Since K is compact, the Fredholm alternative says that (72) has a solution if, and only if,

$$730 \quad \begin{pmatrix} f \\ 0 \end{pmatrix} \in (\ker(\text{Id} - K^*))^\perp. \quad (73)$$

A simple computation shows that

$$\left(K^* \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \right) (s) = \begin{pmatrix} \int_s^1 \tilde{\beta}(t) dt \\ -ab \int_0^s \tilde{\alpha}(t) dt \end{pmatrix}.$$

It follows that $\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \in \ker(\text{Id} - K^*)$ if, and only if, $\tilde{\beta}(s) = -ab \int_0^s \tilde{\alpha}(t) dt$ and $\tilde{\alpha}$ solves the following second order linear ODE:

$$\begin{cases} \tilde{\alpha}''(s) - ab\tilde{\alpha}(s) = 0, \\ \tilde{\alpha}'(0) = 0, \\ \tilde{\alpha}(1) = 0. \end{cases}$$

We can check that this ODE has a nonzero solution if, and only if,

$$ab \in \Sigma,$$

731 where Σ is the set introduced in (69). It follows that we have two possibilities:

732 • If $ab \notin \Sigma$, then $\ker(\text{Id} - K^*) = \{0\}$ and (72) has a (unique) solution u_1 . This shows that
 733 the system (67)-(68) is null controllable in time $T = 2$.

• If $ab \in \Sigma$, then there exists a nonzero $\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \in \ker(\text{Id} - K^*)$. Necessarily, $\tilde{\alpha} \neq 0$ and thus

$$\exists f \in C_c^\infty(0, 1), \quad \langle \tilde{\alpha}, f \rangle_{L^2(0,1)} \neq 0.$$

734 It is clear that we can construct y_2^0 and y_1^0 that satisfy (71) for this f (take for instance
 735 $y_1^0 = 0$ and $y_2^0(x) = f'(2x - 1)/b$ for $x \in [1/2, 1]$ and $y_2^0(x) = 0$ otherwise, note that
 736 $b \neq 0$ in the case considered). For such a f , the condition (73) fails and thus there is no
 737 corresponding solution u_1 to (70), meaning that the system (67)-(68) is not null controllable
 738 in time $T = 2$.

739

□

740 **Remark A.3.** We have seen during the proof that, when $ab \notin \Sigma$, the control that brings the
 741 solution to zero in the critical time $T = 2$ is unique (it can also be written explicitly).

742 B Proof of the abstract compactness-uniqueness result

743 The goal of this appendix is to give a proof of Theorem 7.7. It is inspired from the proofs of
 744 [CN21a, Theorem 2] and [DO18, Lemma 2.6] (see also the references therein).

Here and in what follows, it will be more convenient to work with the expression $S(t)^*z^1$
 rather than $z(t) = S(T - t)^*z^1$. The corresponding assumptions (52), (53) and (54) become:

$$\|S(T)^*z^1\|_H^2 \leq C \left(\int_0^T \|B^*S(t)^*z^1\|_U^2 dt + \|PLz^1\|_{E_2}^2 \right), \quad (74)$$

$$\|Lz^1\|_{E_1}^2 \leq C \left(\|S(T)^*z^1\|_H^2 + \int_0^T \|B^*S(t)^*z^1\|_U^2 dt \right), \quad (75)$$

$$\|S(T - t_2)^*z^1\|_H^2 \leq C \left(\|S(T - t_1)^*z^1\|_H^2 + \int_{T-t_2}^{T-t_1} \|B^*S(t)^*z^1\|_U^2 dt \right). \quad (76)$$

745 1) Let $T > T_0$ be fixed. By duality (see Theorem 7.6), we have to prove that there exists
 746 $C > 0$ such that, for every $z^1 \in D(A^*)$,

$$747 \quad \|S(T)^*z^1\|_H^2 \leq C \int_0^T \|B^*S(t)^*z^1\|_U^2 dt. \quad (77)$$

We argue by contradiction and assume that the observability inequality (77) does not hold.
 Then, there exists a sequence $(z_n^1)_{n \geq 1} \subset D(A^*)$ such that, for every $n \geq 1$,

$$\|S(T)^*z_n^1\|_H^2 > n \int_0^T \|B^*S(t)^*z_n^1\|_U^2 dt.$$

In particular $S(T)^*z_n^1 \neq 0$ and we can normalize z_n^1 , still denoted by the same, in such a
 way that

$$\|S(T)^*z_n^1\|_H = 1, \quad \int_0^T \|B^*S(t)^*z_n^1\|_U^2 dt \xrightarrow{n \rightarrow +\infty} 0.$$

Using the estimate (75) we obtain that

$$(Lz_n^1)_{n \geq 1} \text{ is bounded in } E_1.$$

Since P is compact, we can extract a subsequence, still denoted by $(z_n^1)_{n \geq 1}$, such that

$$(PLz_n^1)_{n \geq 1} \text{ converges in } E_2.$$

Using now the estimate (74), we obtain that $(S(T)^*z_n^1)_{n \geq 1}$ is a Cauchy sequence in H , and thus converges: there exists $f \in H$ such that

$$S(T)^*z_n^1 \xrightarrow{n \rightarrow +\infty} f \text{ in } H.$$

748 Besides, $f \neq 0$ since $\|f\|_H = 1$. In other words, we have shown that

$$749 \quad N_T \neq \{0\}, \quad (78)$$

where N_τ is the subspace defined for every $\tau > 0$ by

$$N_\tau = \left\{ f \in H \mid \begin{array}{l} \exists (z_n^1)_{n \geq 1} \subset D(A^*), \quad S(\tau)^*z_n^1 \xrightarrow{n \rightarrow +\infty} f \text{ in } H, \\ B^*S(\cdot)^*z_n^1 \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^2(0, \tau; U) \end{array} \right\}.$$

750 Let us now study the properties of these subspaces.

751 2) First of all, it forms a non-increasing sequence of subspaces:

$$752 \quad N_{\tau_2} \subset N_{\tau_1}, \quad \forall \tau_2 \geq \tau_1 > 0. \quad (79)$$

753 Indeed, if $f \in N_{\tau_2}$ and $(z_n^1)_{n \geq 1} \subset D(A^*)$ denotes an associated sequence, then we easily
754 check that $f \in N_{\tau_1}$ by considering the sequence $(S(\tau_2 - \tau_1)^*z_n^1)_{n \geq 1}$.

755 3) Let us now show that

$$756 \quad \dim N_\tau < +\infty, \quad \forall \tau > T_0. \quad (80)$$

By Riesz theorem, it is equivalent to show that the closed unit ball of N_τ is compact. Let then $(f^k)_{k \geq 1} \subset N_\tau$ be such that $\|f^k\|_H \leq 1$ for every $k \geq 1$. Let $(z_n^{1,k})_{n \geq 1} \subset D(A^*)$ be an associated sequence. In particular, for every $k \geq 1$, there exists $n_k \geq 1$ such that, denoting by $w^{1,k} = z_{n_k}^{1,k}$, we have

$$\|S(\tau)^*w^{1,k} - f^k\|_H \leq \frac{1}{k}, \quad \|B^*S(\cdot)^*w^{1,k}\|_{L^2(0, \tau; U)} \leq \frac{1}{k}, \quad \forall k \geq 1.$$

757 Since $(f^k)_{k \geq 1}$ is bounded, so is $(S(\tau)^*w^{1,k})_{k \geq 1}$. Using the same reasoning as in Step 1),
758 we deduce from the estimates (75) and (74) that $(S(\tau)^*w^{1,k})_{k \geq 1}$ is a Cauchy sequence. It
759 follows that $(f^k)_{k \geq 1}$ is a Cauchy sequence as well, and thus converges.

760 4) The next step is to establish that

$$761 \quad N_\tau \subset D(A^*), \quad A^*(N_\tau) \subset N_{\tau - \varepsilon}, \quad \forall \tau \in (T_0, T), \quad \forall \varepsilon \in (0, \tau - T_0). \quad (81)$$

Let then $f \in N_\tau$. By definition, there exists a sequence $(z_n^1)_{n \geq 1} \subset D(A^*)$ such that

$$S(\tau)^*z_n^1 \xrightarrow{n \rightarrow +\infty} f \text{ in } H, \quad (82)$$

$$B^*S(\cdot)^*z_n^1 \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^2(0, \tau; U). \quad (83)$$

Using the estimate (76) with $t_1 = T - \tau$ and $t_2 = T - (\tau - \varepsilon)$, we see that

$$(S(\tau - \varepsilon)^* z_n^1)_{n \geq 1} \text{ is bounded in } H.$$

762 As before, it follows from the estimates (75) and (74) that there exists $g \in H$ such that

$$763 \quad S(\tau - \varepsilon)^* z_n^1 \xrightarrow{n \rightarrow +\infty} g \text{ in } H. \quad (84)$$

Noting (82), by uniqueness of the limit, we have

$$f = S(\varepsilon)^* g.$$

Let us now prove that $g \in D(A^*)$. By definition of the domain of the generator of a semigroup, we have to show that, for any sequence $t_n > 0$ with $t_n \rightarrow 0$ as $n \rightarrow +\infty$, the sequence

$$u_n = \frac{S(t_n)^* g - g}{t_n}$$

764 converges in H as $n \rightarrow +\infty$ and that its limit does not depend on the sequence $(t_n)_n$. Let
765 $n_0 \geq 1$ be large enough so that $t_n \leq \varepsilon$ for every $n \geq n_0$. From (84) and (83) we easily see
766 that

$$767 \quad S(t)^* g \in N_{\tau - \varepsilon}, \quad \forall t \in [0, \varepsilon]. \quad (85)$$

Thus,

$$u_n \in N_{\tau - \varepsilon}, \quad \forall n \geq n_0.$$

Let now $\mu \in \rho(A^*) \neq \emptyset$ be fixed and let us introduce the following norm on $N_{\tau - \varepsilon}$:

$$\|z\|_{-1} = \|(\mu - A^*)^{-1} z\|_H.$$

768 Since $(\mu - A^*)^{-1} g \in D(A^*)$, we have

$$769 \quad (\mu - A^*)^{-1} u_n = \frac{S(t_n)^* - \text{Id}}{t_n} (\mu - A^*)^{-1} g \xrightarrow{n \rightarrow +\infty} A^* (\mu - A^*)^{-1} g \quad \text{in } H. \quad (86)$$

Therefore, $(u_n)_{n \geq n_0}$ is a Cauchy sequence in $N_{\tau - \varepsilon}$ for the norm $\|\cdot\|_{-1}$. Since $N_{\tau - \varepsilon}$ is finite dimensional (reall (80)), all the norms are equivalent on $N_{\tau - \varepsilon}$. Thus, $(u_n)_{n \geq n_0}$ is a Cauchy sequence for the usual norm $\|\cdot\|_H$ as well and, as a result, converges for this norm. It is clear from (86) that its limit does not depend on the sequence $(t_n)_n$. This shows that $g \in D(A^*)$ and thus $f = S(\varepsilon)^* g \in D(A^*)$. In addition, we have

$$A^* f = A^* S(\varepsilon)^* g = \lim_{h \rightarrow 0^+} \frac{S(\varepsilon)^* g - S(\varepsilon - h)^* g}{h} \in N_{\tau - \varepsilon} \quad (\text{by (85)}).$$

770 This shows that $A^*(N_\tau) \subset N_{\tau - \varepsilon}$.

5) Let us now prove that

$$N_\tau \subset \ker B^*, \quad \forall \tau \in (T_0, T).$$

Let $\varepsilon \in (0, \tau - T_0)$ be arbitrary. We use the same notations as in the previous step. Since, by assumption, H is an admissible subspace for (A, B) (see Definition 7.4), the map $z^1 \in D(A^*) \mapsto B^* S(\varepsilon - \cdot)^* z^1 \in L^2(0, \varepsilon; U)$ can be extended to a bounded linear operator $\Psi \in \mathcal{L}(H, L^2(0, \varepsilon; U))$. From (84) and continuity of Ψ , we have

$$\Psi S(\tau - \varepsilon)^* z_n^1 \xrightarrow{n \rightarrow +\infty} \Psi g, \quad \text{in } L^2(0, \varepsilon; U).$$

Since $z_n^1 \in D(A^*)$, we have $(\Psi S(\tau - \varepsilon)^* z_n^1)(t) = B^* S(\tau - t) z_n^1$ for $t \in (0, \varepsilon)$. From (83) and uniqueness of the limit, we deduce that

$$\Psi g = 0.$$

Since $g \in D(A^*)$, we have $(\Psi g)(t) = B^* S(\varepsilon - t)^* g$ and the map $t \in [0, \varepsilon] \mapsto B^* S(\varepsilon - t)^* g$ is continuous. It follows that

$$B^* f = B^* S(\varepsilon)^* g = (\Psi g)(0) = 0.$$

771 6) Next, we observe that there exist $\tau \in (T_0, T)$ and $\varepsilon \in (0, \tau - T_0)$ such that

$$772 \quad N_\tau = N_{\tau - \varepsilon}. \quad (87)$$

Indeed, from (80) and (79), the sequence of integers $(\dim N_{T - (T - T_0)/k})_{k \geq 2}$ is non-increasing and thus stationary: there exists $k_0 \geq 2$ such that

$$\dim N_{T - (T - T_0)/k} = \dim N_{T - (T - T_0)/k_0}, \quad \forall k \geq k_0.$$

Denoting by $\delta = (T - T_0)/k_0 \in (0, T - T_0)$ and using (79), we then have

$$N_\tau = N_{T - \delta}, \quad \forall \tau \in [T - \delta, T].$$

773 The desired claim easily follows.

7) Consequently, for $\tau \in (T_0, T)$ and $\varepsilon \in (0, \tau - T_0)$ fixed such that (87) holds, the restriction of A^* to N_τ is a linear operator from the finite dimensional space N_τ into itself (recall (81)). Besides, $N_\tau \neq \{0\}$ since it contains N_T by (79) and we have (78). Therefore, this restriction has at least one eigenvalue (recall that H is a complex Hilbert space), i.e. there exist $\lambda \in \mathbb{C}$ and a nonzero $\phi \in N_\tau$ such that

$$A^* \phi = \lambda \phi.$$

774 Since $N_\tau \subset \ker B^*$, we also have $\phi \in \ker B^*$ and this is a contradiction with the Fattorini-
775 Hautus test (51).

776 This concludes the proof of Theorem 7.7.

Remark B.1. Let us stress that the end of our proof differs from the one in [CN21a, Section 2.2]. Indeed, in this reference, the conclusion of the proof relied on the fact that the semigroup is nilpotent, that is

$$\exists T > 0, \quad S(T)^* z^1 = 0, \quad \forall z^1 \in H.$$

777 This readily implies that the operator A^* has no eigenvalues and this is how the authors conclude
778 that $N_T = \{0\}$. On the other hand, in our proof above, we only made use of the Fattorini-
779 Hautus test (51) (which is trivially checked if the operator A^* has no eigenvalues). Besides, this
780 is optimal, in the sense that this test is always a necessary condition for the system (A, B) to be
781 null controllable in some time.

782 Finally, let us add that for the example of the hyperbolic system $(\Lambda, -, Q, G)$ the correspond-
783 ing adjoint semigroup is not always nilpotent. Notably, the strictly lower triangular structure of
784 G_{--} was used at the end of [CN21a, Section 2.2] to prove such a property.

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