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Geometric Models for (Temporally) Attributed Description Logics

Camille Bourgaux[1], Ana Ozaki[2], Jeff Z. Pan[3]

[1] DI ENS, ENS, CNRS, PSL University & Inria, Paris, France
[2] University of Bergen, Norway
[3] University of Edinburgh, United Kingdom

Abstract. In the search for knowledge graph embeddings that could capture ontological knowledge, geometric models of existential rules have been recently introduced. It has been shown that convex geometric regions capture the so-called quasi-chained rules. Attributed description logics (DL) have been defined to bridge the gap between DL languages and knowledge graphs, whose facts often come with various kinds of annotations that may need to be taken into account for reasoning. In particular, temporally attributed DLs are enriched by specific attributes whose semantics allows for some temporal reasoning. Considering that geometric models and (temporally) attributed DLs are promising tools designed for knowledge graphs, this paper investigates their compatibility, focusing on the attributed version of a Horn dialect of the DL-Lite family. We first adapt the definition of geometric models to attributed DLs and show that every satisfiable ontology has a convex geometric model. Our second contribution is a study of the impact of temporal attributes. We show that a temporally attributed DL may not have a convex geometric model in general but we can recover geometric satisfiability by imposing some restrictions on the use of the temporal attributes.

1 Introduction

Knowledge graph embeddings are popular latent representations of knowledge graphs (KG). In the search for KG embeddings that could capture ontological knowledge (i.e., schema of KG), geometric models of existential rules have been recently introduced [14]. Such models have several advantages. Notably, they ensure that facts which are valid in the embedding are logically consistent and deductively closed w.r.t. the ontology, and they can also be used to find plausible missing ontology rules. It has been shown that convex geometric regions capture the so-called quasi-chained rules, a fragment of first-order Horn logic. Attributed description logics (DL) have been defined to bridge the gap between DL ontology

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languages and KG, whose facts often come with various kinds of annotations that may need to be taken into account for reasoning. In particular, they were introduced as a formalism for dealing with the meta-knowledge present in KG, such as temporal validity, provenance, language, and others [17, 18, 8]. As time is of primary interest in KG, attributed DLs have been enriched with temporal attributes, whose semantics allows for some temporal reasoning over discrete time [23]. Considering that geometric models and (temporally) attributed DLs are promising tools designed for KG, this paper investigates their compatibility, focusing on the attributed version of a Horn dialect of the DL-Lite family.

Our contributions are as follows:

- We adapt the notion of geometric models for ((temporally) attributed) DLs; in particular, we use an arbitrary linear map to combine the individual geometric interpretations instead of restricting ourselves to vector concatenation, and define satisfaction of concept or role inclusions directly based on geometric inclusion relationship between the regions that interpret the concepts or roles.
- We show that every satisfiable attributed DL-Lite$^{\mathcal{H}}_{horn}$ ontology has a convex geometric model but there are satisfiable *temporally* attributed DL-Lite$^{\mathcal{H}}_{horn}$ ontologies without such a model.
- We exhibit restrictions on the use of temporal attributes that guarantee that temporally attributed DL-Lite$^{\mathcal{H}}_{horn}$ ontologies have a convex geometric model.

We define attributed DLs and geometric models in Section 2. Then, in Section 3, we study the relationship between satisfiability and the existence of convex geometric models in DL-Lite$^{\mathcal{H}}_{horn}$. We then extend our analysis for temporally attributed DL-Lite$^{\mathcal{H}}_{horn}$ in Section 4. In Section 5, we discuss related works and we conclude in Section 6. Omitted proofs are given in [9].

2 Geometric Models for Attributed Description Logics

In this section, we recall the framework of attributed DLs and define geometric models in this context.

2.1 Attributed DLs

We introduce attributed DLs by defining attributed DL-Lite [8], focusing on the DL-Lite$^{\mathcal{H}}_{horn}$ dialect. The notions presented here can be easily adapted to other attributed DLs, e.g. $\mathcal{EL}$, as in [17, 18]. Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be countably infinite and mutually disjoint sets of *concept*, *role*, and *individual names*. We assume that $\mathsf{N_I}$ is divided into two sets, called $\mathsf{N_i}$ and $\mathsf{N_a}$, and we refer to the elements in $\mathsf{N_a}$ as *annotation names*. We consider an additional set $\mathsf{N_U}$ of *set variables* and a set $\mathsf{N_V}$ of *object variables*. The set $\mathbf{S}$ of *specifiers* contains the following expressions:

- set variables $X \in \mathsf{N_U}$;
- *closed specifiers* $[a_1 : v_1, \ldots, a_n : v_n]$; and
- *open specifiers* $\lfloor a_1 : v_1, \ldots, a_n : v_n \rfloor$,

where $a_i \in \mathsf{N_a}$ and v_i is either an individual name in $\mathsf{N_a}$, an object variable in $\mathsf{N_V}$, or an expression of the form $X.a$, with X a set variable in $\mathsf{N_U}$ and a an individual name in $\mathsf{N_a}$. We use $X.a$ to refer to the (finite, possibly empty) set of all values of attribute a in an annotation set X. A *ground specifier* is a closed or open specifier built only over $\mathsf{N_a}$.

Syntax. A DL-Lite$^{\mathcal{H},@}_{horn}$ *concept* (resp. *role*) *assertion* is an expression $A(a)@S$ (resp. $R(a,b)@S$), with $A \in \mathsf{N_C}$ (resp. $R \in \mathsf{N_R}$), $a, b \in \mathsf{N_i}$, and $S \in \mathbf{S}$ a ground closed specifier. A DL-Lite$^{\mathcal{H},@}_{horn}$ *role inclusion* is an expression of the form:

$$X : S \quad (P \sqsubseteq Q), \tag{1}$$

where $S \in \mathbf{S}$ is a closed or open specifier, $X \in \mathsf{N_U}$ is a set variable, and P, Q are role expressions built according to the following syntax:

$$P \coloneqq R@S \mid R^{-}@S, \qquad Q \coloneqq P \mid \neg P \tag{2}$$

with $S \in \mathbf{S}$, $R \in \mathsf{N_R}$. A DL-Lite$^{\mathcal{H},@}_{horn}$ *concept inclusion* is of the form:

$$X_1 : S_1, \ldots, X_n : S_n \quad (\bigsqcap_{i=1}^{k} B_i \sqsubseteq C), \tag{3}$$

where $k, n \geq 1$, $S_1, \ldots, S_n \in \mathbf{S}$ are closed or open specifiers, $X_1, \ldots, X_n \in \mathsf{N_U}$ are set variables, and B_i, C are concept expressions built according to:

$$B \coloneqq A@S \mid \exists P, \quad C \coloneqq B \mid \bot, \tag{4}$$

where P is as in Equation (2), $A \in \mathsf{N_C}$ and $S \in \mathbf{S}$. Role expressions of the form P are called *roles* and concept expression of the form B are *basic concepts*. We further require that all object variables are *safe*, that is, if they occur on the right side of a concept/role inclusion or in a specifier associated with a set variable occurring on the right side then they must also occur on the left side of the inclusion (or in a specifier associated with a set variable occurring on the left).

A DL-Lite$^{\mathcal{H},@}_{horn}$ *ontology* is a set of DL-Lite$^{\mathcal{H},@}_{horn}$ assertions, role and concept inclusions. We say that an inclusion is *positive* if it does not contain negation or $\bot$. Also, we say that a DL-Lite$^{\mathcal{H},@}_{horn}$ ontology is *ground* if it does not contain variables. To simplify notation, we omit the specifier $\lfloor\rfloor$ (meaning "any annotation set") in role or concept expressions. In this sense, any DL-Lite$^{\mathcal{H}}_{horn}$ axiom is also a DL-Lite$^{\mathcal{H},@}_{horn}$ axiom. Moreover, we omit prefixes of the form $X : \lfloor\ \rfloor$, which state that there is no restriction on X.

Semantics. An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}_{\mathsf{i}}, \Delta^{\mathcal{I}}_{\mathsf{a}}, \cdot^{\mathcal{I}})$ of an attributed DL consists of a non-empty domain $\Delta^{\mathcal{I}}_{\mathsf{i}}$ of individuals, a non-empty domain $\Delta^{\mathcal{I}}_{\mathsf{a}}$ of annotations, and a function $\cdot^{\mathcal{I}}$. Individual names $a \in \mathsf{N_i}$ are interpreted as elements $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}_{\mathsf{i}}$ and individual names $a \in \mathsf{N_a}$ are interpreted as elements $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}_{\mathsf{a}}$. To interpret annotation sets, we use the set $\Phi^{\mathcal{I}} \coloneqq \{\Sigma \subseteq \Delta^{\mathcal{I}}_{\mathsf{a}} \times \Delta^{\mathcal{I}}_{\mathsf{a}} \mid \Sigma \text{ is finite}\}$ of all finite binary relations over $\Delta^{\mathcal{I}}_{\mathsf{a}}$. Each concept name $A \in \mathsf{N_C}$ is interpreted

as a set $A^{\mathcal{I}} \subseteq \Delta_{\mathsf{i}}^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ of elements with annotations, and each role name $R \in \mathsf{N_R}$ is interpreted as a set $R^{\mathcal{I}} \subseteq \Delta_{\mathsf{i}}^{\mathcal{I}} \times \Delta_{\mathsf{i}}^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ of pairs of elements with annotations. Each element (pair of elements) may appear with multiple different annotations. $\mathcal{I}$ *satisfies* a concept assertion $A(a)@[a_1\colon v_1, \ldots, a_n\colon v_n]$ if $(a^{\mathcal{I}}, \{(a_1^{\mathcal{I}}, v_1^{\mathcal{I}}), \ldots, (a_n^{\mathcal{I}}, v_n^{\mathcal{I}})\}) \in A^{\mathcal{I}}$. Role assertions are interpreted analogously. Expressions with free set or object variables are interpreted using variable assignments $\mathcal{Z}$ mapping object variables $x \in \mathsf{N_V}$ to elements $\mathcal{Z}(x) \in \Delta_{\mathsf{a}}^{\mathcal{I}}$ and set variables $X \in \mathsf{N_U}$ to finite binary relations $\mathcal{Z}(X) \in \Phi^{\mathcal{I}}$. For convenience, we also extend variable assignments to individual names, setting $\mathcal{Z}(a) = a^{\mathcal{I}}$ for every $a \in \mathsf{N_a}$. A specifier $S \in \mathbf{S}$ is interpreted as a set $S^{\mathcal{I},\mathcal{Z}} \subseteq \Phi^{\mathcal{I}}$ of matching annotation sets. We set $X^{\mathcal{I},\mathcal{Z}} := \{\mathcal{Z}(X)\}$ for variables $X \in \mathsf{N_U}$. The semantics of closed specifiers is defined as:

- $[a\colon v]^{\mathcal{I},\mathcal{Z}} := \{\{(a^{\mathcal{I}}, \mathcal{Z}(v))\}\}$ where $v \in \mathsf{N_a} \cup \mathsf{N_V}$;
- $[a\colon X.b]^{\mathcal{I},\mathcal{Z}} := \{\{(a^{\mathcal{I}}, \delta) \mid (b^{\mathcal{I}}, \delta) \in \mathcal{Z}(X)\}\}$;
- $[a_1\colon v_1, \ldots, a_n\colon v_n]^{\mathcal{I},\mathcal{Z}} := \{\bigcup_{i=1}^{n} F_i\}$ where $\{F_i\} = [a_i\colon v_i]^{\mathcal{I},\mathcal{Z}}$ for all $1 \leq i \leq n$.

$S^{\mathcal{I},\mathcal{Z}}$ therefore is a singleton set for variables and closed specifiers. For open specifiers, however, we define $\lfloor a_1\colon v_1, \ldots, a_n\colon v_n \rfloor^{\mathcal{I},\mathcal{Z}}$ to be the set:

$$\{F \subseteq \Phi^{\mathcal{I}} \mid F \supseteq G \text{ for } \{G\} = [a_1\colon v_1, \ldots, a_n\colon v_n]^{\mathcal{I},\mathcal{Z}}\}.$$

Now given $A \in \mathsf{N_C}$, $R \in \mathsf{N_R}$, and $S \in \mathbf{S}$, we define:

$$(A@S)^{\mathcal{I},\mathcal{Z}} := \{\delta \mid (\delta, F) \in A^{\mathcal{I}} \text{ for some } F \in S^{\mathcal{I},\mathcal{Z}}\},$$
$$(R@S)^{\mathcal{I},\mathcal{Z}} := \{(\delta, \epsilon) \mid (\delta, \epsilon, F) \in R^{\mathcal{I}} \text{ for some } F \in S^{\mathcal{I},\mathcal{Z}}\}.$$

Further DL expressions are defined as: $(R^{-}@S)^{\mathcal{I},\mathcal{Z}} := \{(\gamma, \delta) \mid (\delta, \gamma) \in (R@S)^{\mathcal{I},\mathcal{Z}}\}$, $\neg P^{\mathcal{I},\mathcal{Z}} := (\Delta_{\mathsf{i}}^{\mathcal{I}} \times \Delta_{\mathsf{i}}^{\mathcal{I}}) \backslash P^{\mathcal{I},\mathcal{Z}}$, $\exists P^{\mathcal{I},\mathcal{Z}} := \{\delta \mid \text{there is } (\delta, \epsilon) \in P^{\mathcal{I},\mathcal{Z}}\}$, $(B_1 \sqcap B_2)^{\mathcal{I},\mathcal{Z}} := B_1^{\mathcal{I},\mathcal{Z}} \cap B_2^{\mathcal{I},\mathcal{Z}}$, $\bot^{\mathcal{I},\mathcal{Z}} := \emptyset$. $\mathcal{I}$ *satisfies* a concept inclusion of the form (3) if, for all variable assignments $\mathcal{Z}$ that satisfy $\mathcal{Z}(X_i) \in S_i^{\mathcal{I},\mathcal{Z}}$ for all $1 \leq i \leq n$, we have $(\bigsqcap_{i=1}^{k} B_i)^{\mathcal{I},\mathcal{Z}} \subseteq C^{\mathcal{I},\mathcal{Z}}$. Satisfaction of role inclusions is defined analogously. An interpretation $\mathcal{I}$ satisfies an ontology $\mathcal{O}$, or is a *model* of $\mathcal{O}$, if it satisfies all of its axioms. As usual, $\models$ denotes the induced logical entailment relation.

For ground specifiers $\{S, T\} \subseteq \mathbf{S}$, we write $S \Rightarrow T$ if T is an open specifier, and the set of attribute-value pairs $a : b$ in S is a superset of the set of attribute-value pairs in T.

2.2 Geometric Models

We now define the geometric interpretations of attributed relations. Let m be an integer and $f : \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^{2 \cdot m}$ be a fixed but arbitrary linear map satisfying the following:

(i) the restriction of f to $\mathbb{R}^m \times \{0\}^m$ is injective;
(ii) the restriction of f to $\{0\}^m \times \mathbb{R}^m$ is injective;
(iii) $f(\mathbb{R}^m \times \{0\}^m) \cap f(\{0\}^m \times \mathbb{R}^m) = \{0^{2 \cdot m}\}$;

where 0^m denotes the vector $(0, \ldots, 0)$ with m zeros. Intuitively, individuals will be interpreted as vectors from $\mathbb{R}^m$ and f will be used to combine two vectors to interpret pairs of individuals.

Definition 1 (Geometric Interpretation). *An m-dimensional f-geometric interpretation η of $(\mathsf{N_C}, \mathsf{N_R}, \mathsf{N_i}, \mathsf{N_a})$ assigns*

- *to each $A \in \mathsf{N_C}$ and ground $S \in \mathbf{S}$ a region $\eta(A@S) \subseteq \mathbb{R}^m$,*
- *to each $R \in \mathsf{N_R}$ and ground $S \in \mathbf{S}$ a region $\eta(R@S) \subseteq \mathbb{R}^{2 \cdot m}$, and*
- *to each $a \in \mathsf{N_i}$ a vector $\eta(a) \in \mathbb{R}^m$.*

Moreover, for all $\{S, T\} \subseteq \mathbf{S}$ and $E \in \mathsf{N_C} \cup \mathsf{N_R}$, if $S \Rightarrow T$ then $\eta(E@S) \subseteq \eta(E@T)$. We say that η is convex *if, for every $E \in \mathsf{N_C} \cup \mathsf{N_R}$, every ground $S \in \mathbf{S}$, every $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \eta(E@S)$, and every $\lambda \in [0, 1]$, if $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \eta(E@S)$ then $(1 - \lambda)\boldsymbol{v}_1 + \lambda \boldsymbol{v}_2 \in \eta(E@S)$.*

The interpretation of ground complex concept or role expressions is as follows. Assume all specifiers occurring in expressions below are ground, $R \in \mathsf{N_R}$, P is a role, and B, B_i are basic concepts. Then,

- $\eta(R^-@S) := \{f(\delta, \delta') \mid f(\delta', \delta) \in \eta(R@S)\}$,
- $\eta(\neg P) := \mathbb{R}^{2 \cdot m} \setminus \eta(P)$,
- $\eta(\exists P) := \{\delta \mid \exists \delta', f(\delta, \delta') \in \eta(P)\}$,
- $\eta(\bigsqcap_{i=1}^n B_i) := \bigcap_{i=1}^n \eta(B_i)$,
- $\eta(\bot) := \emptyset$.

We may omit "of $(\mathsf{N_C}, \mathsf{N_R}, \mathsf{N_i}, \mathsf{N_a})$" when we speak about m-dimensional f-geometric interpretations. The interest of geometric interpretations is that concept and role assertions translate into membership in geometric regions and ground concept or role inclusions translate into geometric inclusions.

Definition 2 (Satisfaction of Ground Axioms). *An m-dimensional f-geometric interpretation η satisfies*

- *a concept assertion $A(a)@S$, denoted $\eta \models A(a)@S$, if $\eta(a) \in \eta(A@S)$;*
- *a role assertion $R(a, b)@S$, denoted $\eta \models R(a, b)@S$, if $f(\eta(a), \eta(b)) \in \eta(R@S)$;*
- *a ground role inclusion $P \sqsubseteq Q$, denoted $\eta \models P \sqsubseteq Q$, if $\eta(P) \subseteq \eta(Q)$;*
- *a ground concept inclusion $\bigsqcap_{i=1}^n B_i \sqsubseteq C$, denoted $\eta \models \bigsqcap_{i=1}^n B_i \sqsubseteq C$, if we have that $\eta(\bigsqcap_{i=1}^n B_i) \subseteq \eta(C)$.*

We are ready for the first theorem which establishes that our more general notion of geometric models still has the same properties of the geometric models originally proposed [14].

Theorem 3. *Let η be an m-dimensional f-geometric interpretation. For every linear map f' satisfying (i)-(iii), the m-dimensional f'-geometric interpretation η' defined as:*

- *$\eta'(a) := \eta(a)$, for all $a \in \mathsf{N_i}$;*
- *$\eta'(A@S) := \eta(A@S)$, for all $A \in \mathsf{N_C}$ and ground $S \in \mathbf{S}$; and*

– $\eta'(R@S) := \{f'(\delta, \delta') \mid f(\delta, \delta') \in \eta(R@S)\}$, *for all* $R \in \mathsf{N_R}$ *and ground* $S \in \mathbf{S}$*;*

is such that $\eta \models \alpha$ *iff* $\eta' \models \alpha$*, for all ground axioms* α*.*

Proof (Sketch). This result follows from the fact that there is an isomorphism between the regions in η and η'.

To define when a geometric interpretation is a model of a (possibly not ground) DL-Lite$^{\mathcal{H},@}_{horn}$ ontology, we need to define when such an interpretation satisfies non-ground concept or role inclusions. To do so, we use a standard interpretation built from the geometric interpretation. Given a ground specifier S and an annotation name $\star$, we define $F^\star_S = \{(a, b) \mid a{:}\,b \text{ occurs in } S\} \cup \{(\star, \star) \mid \text{if } S \text{ is open}\}$. Given an m-dimensional f-geometric interpretation η, a subset D_a of $\mathsf{N_a}$ and an annotation name $\star \in \mathsf{N_a} \setminus D_\mathsf{a}$, we define an interpretation $\mathcal{I}(\eta, D^\star_\mathsf{a})$ as follows.

– $\Delta_\mathsf{i}^{\mathcal{I}(\eta, D^\star_\mathsf{a})} = \mathbb{R}^m$ and, for all $a \in \mathsf{N_i}$, $a^{\mathcal{I}(\eta, D^\star_\mathsf{a})} = \eta(a)$;
– $\Delta_\mathsf{a}^{\mathcal{I}(\eta, D^\star_\mathsf{a})} = \mathsf{N_a}$ and for all $a \in \mathsf{N_a}$, $a^{\mathcal{I}(\eta, D^\star_\mathsf{a})} = a$;
– $A^{\mathcal{I}(\eta, D^\star_\mathsf{a})} = \{(\delta, F^\star_S) \mid \delta \in \eta(A@S), S \text{ built on } D_\mathsf{a}\}$, for all $A \in \mathsf{N_C}$;
– $R^{\mathcal{I}(\eta, D^\star_\mathsf{a})} = \{(\delta, \epsilon, F^\star_S) \mid f(\delta, \epsilon) \in \eta(R@S), S \text{ built on } D_\mathsf{a}\}$, for all $R \in \mathsf{N_R}$.

The next proposition shows that η and $\mathcal{I}(\eta, D^\star_\mathsf{a})$ satisfy the same ground axioms built using only annotation names from D_a. It follows in particular that η satisfies all ground axioms of an ontology $\mathcal{O}$ iff $\mathcal{I}(\eta, \mathsf{N}^\star_\mathcal{O})$ does, where $\mathsf{N}_\mathcal{O}$ is the set of annotation names from $\mathsf{N_a}$ that occur in $\mathcal{O}$.

Theorem 4. *Let* η *be an* m*-dimensional* f*-geometric interpretation. Let* α *be a ground axiom, i.e.,* α *is either a concept/role assertion or a ground concept/role inclusion. Let* D_α *be the set of annotation names that occur in* α *and let* D *be a subset of* $\mathsf{N_a}$ *such that* $D_\alpha \subseteq D$*. Let* $\star$ *be an annotation name that does not occur in* D*. Then, the following holds:* $\eta \models \alpha$ *iff* $\mathcal{I}(\eta, D^\star) \models \alpha$*.*

Proof (Sketch). The proof relies heavily on the definition of $F^\star_S$ and the requirement that $\eta(E@S) \subseteq \eta(E@T)$ when $S \Rightarrow T$ in Definition 1.

We are now ready to define geometric models of DL-Lite$^{\mathcal{H},@}_{horn}$ ontologies.

Definition 5 (Geometric Model). *Let* $\mathcal{O}$ *be a DL-Lite*$^{\mathcal{H},@}_{horn}$ *ontology, and let* $\mathsf{N}_\mathcal{O}$ *be the set of annotation names from* $\mathsf{N_a}$ *that occur in* $\mathcal{O}$ *and* $\star$ *an annotation name that does not occur in* $\mathcal{O}$*. An* m*-dimensional* f*-geometric interpretation* η *is a model of* $\mathcal{O}$ *if* $\mathcal{I}(\eta, \mathsf{N}^\star_\mathcal{O})$ *is a model of* $\mathcal{O}$*.*

3 Satisfiability and Convex Geometric Models

We start by recalling some definitions and results on geometric interpretations of an ontology containing existential rules [14]. An existential rule is an expression of the form $B_1 \wedge \cdots \wedge B_n \rightarrow \exists X_1, \ldots, X_j.H$ where $n \geq 0$, the B_i's and H are atoms built from sets of predicates, constants and variables, and the X_i's are variables.

A negative constraint is a rule whose head is $\bot$. An existential rule or negative constraint is *quasi-chained* if for all $1 \leq i \leq n$, $|(\mathsf{vars}(B_1) \cup \cdots \cup \mathsf{vars}(B_{i-1})) \cap \mathsf{vars}(B_i)| \leq 1$, where $\mathsf{vars}(B)$ denotes the variables that occur in B. It is easy to see that a DL-Lite$^{\mathcal{H}}_{horn}$ ontology without negative role inclusions can be translated into a quasi-chained ontology. Negative role inclusions are not quasi-chained: their translation to rules is indeed of the form $P_1(x, y) \wedge P_2(x, y) \rightarrow \bot$ where the body atoms share two variables. A (standard) model $\mathcal{M}$ of an existential rules ontology $\mathcal{K}$ is a set of facts that contains all facts from $\mathcal{K}$ and satisfies all existential rules from $\mathcal{K}$. In this setting, for every fact α, $\alpha \in \mathcal{M}$ iff $\mathcal{M} \models \alpha$.

Given a set $\mathcal{R}$ of relation names and a set X of constants and labelled nulls, a m-dimensional geometric interpretation η of $(\mathcal{R}, X)$ assigns to each k-ary relation R from $\mathcal{R}$ a region $\eta(R) \subseteq \mathbb{R}^{k.m}$ and to each object o from X a vector $\eta(o) \in \mathbb{R}^m$. Tuples of individuals are interpreted using vectors concatenation, which plays the role of the linear map f we use to interpret a pair of individuals: for every $R \in \mathcal{R}$ and $o_1, \ldots, o_k \in X$, $\eta \models R(o_1, \ldots, o_k)$ if $\eta(o_1) \oplus \cdots \oplus \eta(o_k) \in \eta(R)$. The authors define

$$\Phi(\eta) = \{R(o_1, \ldots, o_k) \mid R \in \mathcal{R}, o_1, \ldots, o_k \in X, \eta \models R(o_1, \ldots, o_k)\}.$$

Proposition 3 in [14] states that if $\mathcal{K}$ is a quasi-chained ontology and $\mathcal{M}$ is a finite model of $\mathcal{K}$, then $\mathcal{K}$ has a convex geometric model η such that $\Phi(\eta) = \mathcal{M}$. This transfers to the DL setting as follows. Let $\mathcal{O}$ be a quasi-chained DL ontology and $\mathcal{J}$ be a finite model of $\mathcal{O}$ such that $\Delta^{\mathcal{J}} = \mathsf{N_i}$, and $a^{\mathcal{J}} = a$ for every $a \in \mathsf{N_i}$ (which implies that for every concept A, if $\delta \in A^{\mathcal{J}}$, then $\mathcal{J} \models A(\delta)$, and similarly for roles). Then $\mathcal{O}$ has a convex geometric model η such that

$$\{E(\boldsymbol{t}) \mid \eta \models E(\boldsymbol{t})\} = \{E(\boldsymbol{t}) \mid \mathcal{J} \models E(\boldsymbol{t})\},$$

where $E \in \mathsf{N_C} \cup \mathsf{N_R}$ and $\boldsymbol{t}$ is a tuple with the arity of E. In the following, we may similarly write $E(\boldsymbol{t})@S$ to refer to an atom of the form $A(a)@S$ or $R(a, b)@S$.

Theorem 6. *Let $\mathcal{O}$ be a satisfiable DL-Lite$^{\mathcal{H},@}_{horn}$ ontology without negative role inclusions. $\mathcal{O}$ has a convex geometric model.*

Proof (Sketch). We use grounding to translate $\mathcal{O}$ into an equisatisfiable DL-Lite$^{\mathcal{H}}_{horn}$ ontology. Since $\mathcal{O}$ does not contain negative role inclusion, the obtained DL-Lite$^{\mathcal{H}}_{horn}$ ontology is quasi-chained. We can thus apply the result by Gutiérrez-Basulto and Schockaert [14] to get a convex m-dimensional f-geometric model with f being vector concatenation. This geometric model is used to construct a geometric model of $\mathcal{O}$.

4 Adding Time

In this section, we discuss the ability of convex geometric models to capture temporally attributed DLs. We show that we need to restrict the expressivity of the temporal ontology to get a convex geometric model.

We introduce temporally attributed DLs by defining temporally attributed DL-Lite$^{\mathcal{H}}_{horn}$, called DL-Lite$^{\mathcal{H},\mathbb{T},@}_{horn}$, as in [23]. The description logic DL-Lite$^{\mathcal{H},\mathbb{T},@}_{horn}$ is defined as a multi-sorted version of DL-Lite$^{\mathcal{H},@}_{horn}$, where time points and intervals are seen as datatypes. *Time points* are elements of $\mathsf{N_T}$, and *time intervals* are elements of $\mathsf{N_T^2}$. These sets and the set of *(abstract) individual names* $\mathsf{N_I}$ are mutually disjoint. Time points are represented in a discrete manner by natural numbers, and we assume that elements of $\mathsf{N_T}$ ($\mathsf{N_T^2}$) are (pairs of) numbers. A pair of numbers k, ℓ in $\mathsf{N_T^2}$ is denoted $[k, \ell]$.

The annotation names: $\mathsf{time}, \mathsf{before}, \mathsf{after}, \mathsf{until}, \mathsf{since}, \mathsf{during}, \mathsf{between} \in \mathsf{N_a}$ are called *temporal attributes* and have their own semantics. Basically, time is used to mark a point in time and before and after refer to *some* point in the past and in the future, respectively. The temporal attributes until and since refer to *all* points in the past and all point in the future (e.g. since 2020 the KR conference became an annual event). Finally, during is an interval which represents a period of time (it refers to all points in the interval) and $\mathsf{between}$ is an interval of uncertainty for when an event happened. The *value type* of $\mathsf{time}, \mathsf{before}, \mathsf{after}, \mathsf{until}, \mathsf{since}$ is $\mathsf{N_T}$, while the value type of $\mathsf{during}, \mathsf{between}$ is $\mathsf{N_T^2}$. We write $\mathsf{valtype}(a)$ to refer to the value type of the annotation name a. Object variables are now taken from pairwise disjoint sets $\mathsf{Var}(\mathsf{N_a})$, $\mathsf{Var}(\mathsf{N_T})$, and $\mathsf{Var}(\mathsf{N_T^2})$.

Annotation set specifiers are defined as in Section 2.1 with the difference that for each $a_i \in \mathsf{N_a}$ and each v_i in attribute value pair $a_i : v_i$ we require compatibility between the value type of its attribute, that is:

- $v_i \in \mathsf{valtype}(a_i) \cup \mathsf{Var}(\mathsf{valtype}(a_i))$, or
- $v_i = [v, w]$ with $\mathsf{valtype}(a_i) = \mathsf{N_T^2}$ and v, w in $\mathsf{N_T} \cup \mathsf{Var}(\mathsf{N_T})$, or
- $v_i = X.b$ with $X \in \mathsf{N_U}$, $b \in \mathsf{N_a}$, and $\mathsf{valtype}(a_i) = \mathsf{valtype}(b)$.

A *time-sorted interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}_{\mathsf{i}}, \Delta^{\mathcal{I}}_{\mathsf{a}}, \cdot^{\mathcal{I}})$ is an interpretation with a domain $\Delta^{\mathcal{I}}_{\mathsf{a}}$ that is a disjoint union of $\Delta^{\mathcal{I}}_{A} \cup \Delta^{\mathcal{I}}_{T} \cup \Delta^{\mathcal{I}}_{2T}$, where $\Delta^{\mathcal{I}}_{A}$ is the *abstract domain of annotations*, $\Delta^{\mathcal{I}}_{T}$ (the *temporal domain*) is a finite or infinite interval, and $\Delta^{\mathcal{I}}_{2T} = \Delta^{\mathcal{I}}_{T} \times \Delta^{\mathcal{I}}_{T}$. We interpret individual names in $\mathsf{N_i}$ as elements in $\Delta^{\mathcal{I}}_{\mathsf{i}}$; annotation names in $\mathsf{N_a}$ as elements in $\Delta^{\mathcal{I}}_{A}$; time points $t \in \mathsf{N_T}$ as $t^{\mathcal{I}} \in \Delta^{\mathcal{I}}_{T}$; and intervals $[t, t'] \in \mathsf{N_T^2}$ as $[t, t']^{\mathcal{I}} = (t^{\mathcal{I}}, t'^{\mathcal{I}}) \in \Delta^{\mathcal{I}}_{2T}$. A pair $(\delta, \epsilon) \in \Delta^{\mathcal{I}}_{A} \times \Delta^{\mathcal{I}}_{\mathsf{a}}$ is *well-typed*, if:

1. $\delta = a^{\mathcal{I}}$ for an attribute 'a' of value type $\mathsf{N_T}$ and $\epsilon \in \Delta^{\mathcal{I}}_{T}$; or
2. $\delta = a^{\mathcal{I}}$ for an attribute 'a' of value type $\mathsf{N_T^2}$ and $\epsilon \in \Delta^{\mathcal{I}}_{2T}$; or
3. $\delta = a^{\mathcal{I}}$ for an attribute 'a' of value type $\mathsf{N_a}$ and $\epsilon \in \Delta^{\mathcal{I}}_{A}$.

Let $\Phi^{\mathcal{I}}$ be the set of all finite sets of well-typed pairs. The function $\cdot^{\mathcal{I}}$ maps concept names $A \in \mathsf{N_C}$ to $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}_{\mathsf{i}} \times \Phi^{\mathcal{I}}$ and role names $R \in \mathsf{N_R}$ to $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}_{\mathsf{i}} \times \Delta^{\mathcal{I}}_{\mathsf{i}} \times \Phi^{\mathcal{I}}$. The semantics of terms is given by variable assignments, which for a time-sorted interpretation $\mathcal{I}$ is defined as a function $\mathcal{Z}$ that maps

- set variables $X \in \mathsf{N_U}$ to finite binary relations $\mathcal{Z}(X) \in \Phi^{\mathcal{I}}$, and
- object variables $x \in \mathsf{Var}(\mathsf{N_I}) \cup \mathsf{Var}(\mathsf{N_T}) \cup \mathsf{Var}(\mathsf{N_T^2})$ to elements $\mathcal{Z}(x) \in \Delta^{\mathcal{I}}_{I} \cup \Delta^{\mathcal{I}}_{T} \cup \Delta^{\mathcal{I}}_{2T}$ (respecting their types).

For (set or object) variables x, we define $x^{\mathcal{I},\mathcal{Z}} := \mathcal{Z}(x)$, and for abstract individuals, time points, or time intervals a, we define $a^{\mathcal{I},\mathcal{Z}} := a^{\mathcal{I}}$. The semantics of specifiers is as in Section 2.1 with the difference that values can also be time points and intervals:

- $[a\colon v]^{\mathcal{I},\mathcal{Z}} := \{\{(a^{\mathcal{I}}, v^{\mathcal{I},\mathcal{Z}})\}\}$, with $v \in \mathsf{valtype}(a) \cup \mathsf{Var}(\mathsf{valtype}(a))$;
- $[a\colon [v,w]]^{\mathcal{I},\mathcal{Z}} := \{\{(a^{\mathcal{I}}, (v^{\mathcal{I},\mathcal{Z}}, w^{\mathcal{I},\mathcal{Z}}))\}\}$, with $\mathsf{valtype}(a) = \mathsf{N}_{\mathsf{T}}^2$, and $v, w \in \mathsf{N}_{\mathsf{T}} \cup \mathsf{Var}(\mathsf{N}_{\mathsf{T}})$.

We are now ready to formally define the semantics of temporal attributes.

Definition 7. *Consider a temporal domain $\Delta_T^{\mathcal{I}}$ and a domain $\Delta_{\mathsf{i}}^{\mathcal{I}}$ of individuals and a domain $\Delta_{\mathsf{a}}^{\mathcal{I}}$ of annotations, and let $(\mathcal{I}_i)_{i \in \Delta_T^{\mathcal{I}}}$ be a sequence of (non-temporal) interpretations with domains $\Delta_{\mathsf{i}}^{\mathcal{I}}$ and $\Delta_{\mathsf{a}}^{\mathcal{I}}$, such that, for all $a \in \mathsf{N}_{\mathsf{I}}$, we have $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$ for all $i, j \in \Delta_T^{\mathcal{I}}$. We define a* global interpretation *for $(\mathcal{I}_i)_{i \in \Delta_T^{\mathcal{I}}}$ as a time-sorted interpretation $\mathcal{I} = (\Delta_{\mathsf{i}}^{\mathcal{I}}, \Delta_{\mathsf{a}}^{\mathcal{I}}, \cdot^{\mathcal{I}})$ as follows. Let $a^{\mathcal{I}} = a^{\mathcal{I}_i}$ for all $a \in \mathsf{N}_{\mathsf{I}}$. For any finite set $F \in \Phi^{\mathcal{I}}$, let $F_I := F \cap (\Delta_A^{\mathcal{I}} \times \Delta_A^{\mathcal{I}})$ denote its abstract part without any temporal attributes. For any $A \in \mathsf{N}_{\mathsf{C}}$, $\delta \in \Delta_{\mathsf{i}}^{\mathcal{I}}$, and $F \in \Phi^{\mathcal{I}}$ with $F \setminus F_I \neq \emptyset$, we have $(\delta, F) \in A^{\mathcal{I}}$ if and only if $(\delta, F_I) \in A^{\mathcal{I}_i}$ for some $i \in \Delta_T^{\mathcal{I}}$, and the following conditions hold for all $(a^{\mathcal{I}}, x) \in F$:*

- *if $a = \mathsf{time}$, then $(\delta, F_I) \in A^{\mathcal{I}_x}$,*
- *if $a = \mathsf{before}$, then $(\delta, F_I) \in A^{\mathcal{I}_j}$ for some $j < x$,*
- *if $a = \mathsf{after}$, then $(\delta, F_I) \in A^{\mathcal{I}_j}$ for some $j > x$,*
- *if $a = \mathsf{until}$, then $(\delta, F_I) \in A^{\mathcal{I}_j}$ for all $j \leq x$,*
- *if $a = \mathsf{since}$, then $(\delta, F_I) \in A^{\mathcal{I}_j}$ for all $j \geq x$,*
- *if $a = \mathsf{between}$, then $(\delta, F_I) \in A^{\mathcal{I}_j}$ for some $j \in [x]$,*
- *if $a = \mathsf{during}$, then $(\delta, F_I) \in A^{\mathcal{I}_j}$ for all $j \in [x]$,*

where $[x]$ for an element $x \in \Delta_{2T}^{\mathcal{I}}$ denotes the finite interval represented by the pair of numbers x, and $j \in \Delta_T^{\mathcal{I}}$. For roles $R \in \mathsf{N}_{\mathsf{R}}$, we define $(\delta, \epsilon, F) \in R^{\mathcal{I}}$ analogously.

Definition 8 (Temporal Geometric Interpretation). *A temporal m-dimensional f-geometric interpretation with temporal domain Δ_T is a sequence $(\eta_j)_{j \in \Delta_T}$ of m-dimensional f-geometric interpretations. An m-dimensional f-geometric interpretation η is* global *for $(\eta_j)_{j \in \Delta_T}$ and $D_{\mathsf{a}} \subseteq \mathsf{N}_{\mathsf{a}}$ if $\mathcal{I}(\eta, (D_{\mathsf{a}} \cup \mathsf{N}_{\mathsf{T}} \cup \mathsf{N}_{\mathsf{T}}^2)^{\star})$ is global for $(\mathcal{I}(\eta_j, D_{\mathsf{a}}^{\star}))_{j \in \Delta_T}$.*

Let $\mathsf{N}_{\mathcal{O}}$ denote the union of all elements in N_{a}, N_{T}, and $\mathsf{N}_{\mathsf{T}}^2$ occurring in $\mathcal{O}$.

Definition 9 (Geometric Model). *Let $\mathcal{O}$ be a DL-Lite$_{horn}^{\mathcal{H},\mathbb{T},@}$ ontology. An f-geometric interpretation η is an m-dimensional f-geometric model of $\mathcal{O}$ if it is global for a sequence $(\eta_j)_{j \in \Delta_T}$ of m-dimensional f-geometric interpretations and $\mathsf{N}_{\mathcal{O}}$, plus $\mathcal{I}(\eta, \mathsf{N}_{\mathcal{O}}^{\star})$ satisfies $\mathcal{O}$.*

Example 10 shows that even if temporal specifiers are only of the form time and during, convex geometric models may not exist for satisfiable DL-Lite$_{horn}^{\mathcal{H},\mathbb{T},@}$ ontologies.

Example 10. Let

$$\mathcal{O} = \{\exists R@[\mathsf{during}\colon [1,2]] \sqsubseteq A, \exists R@[\mathsf{time}\colon 1] \sqcap A \sqsubseteq \bot,$$
$$R(a,a)@[\mathsf{time}\colon 1], R(b,b)@[\mathsf{time}\colon 1], R(a,b)@[\mathsf{time}\colon 2], R(b,a)@[\mathsf{time}\colon 2]\}$$

and let η be a convex f-geometric model of $\mathcal{O}$. Let $\delta = 0.5\eta(a) + 0.5\eta(b)$. By the convexity of $\eta(R@[\mathsf{time}\colon 1])$ and $\eta(R@[\mathsf{time}\colon 2])$, we have that

$$f(\delta,\delta) \in \eta(R@[\mathsf{time}\colon 1]) \text{ and } f(\delta,\delta) \in \eta(R@[\mathsf{time}\colon 2]),$$

so $\mathcal{I}(\eta, \mathsf{N}^\star_{\mathcal{O}}) \models R(\delta,\delta)@[\mathsf{time}\colon 1]$ and $\mathcal{I}(\eta, \mathsf{N}^\star_{\mathcal{O}}) \models R(\delta,\delta)@[\mathsf{time}\colon 2]$. It follows that $\mathcal{I}(\eta, \mathsf{N}^\star_{\mathcal{O}}) \models R(\delta,\delta)@[\mathsf{during}\colon [1,2]]$. Since $\mathcal{I}(\eta, \mathsf{N}^\star_{\mathcal{O}}) \models R@[\mathsf{during}\colon [1,2]] \sqsubseteq A$, we also have that $\mathcal{I}(\eta, \mathsf{N}^\star_{\mathcal{O}}) \models A(\delta)$. Hence $\mathcal{I}(\eta, \mathsf{N}^\star_{\mathcal{O}}) \not\models \exists R@[\mathsf{time}\colon 1] \sqcap A \sqsubseteq \bot$. This means that $\mathcal{I}(\eta, \mathsf{N}^\star_{\mathcal{O}})$ is not a model of $\mathcal{O}$. ◁

To overcome this problem, we introduce a restriction on the specifiers allowed on roles. We introduce *atemporal specifiers*. An atemporal specifier is a specifier S that can only be interpreted as a set $S^{\mathcal{I},\mathcal{Z}} \subseteq \Phi^{\mathcal{I}}$ of matching annotation sets that do not contain any temporal attribute.

To show that convex geometric models can capture some DL-Lite$^{\mathcal{H},\mathbb{T},@}_{horn}$ ontologies, we use concept inclusions with conjunctions on the left-hand side, which can be expressed in DL-Lite$^{\mathcal{H}}_{horn}$. The following example shows that adding conjunction in role inclusions may however lead to satisfiable ontologies not having a convex model, even for plain DLs.

Example 11. Assume role conjunctions are allowed in the ontology. Let

$$\mathcal{O} = \{R_1 \sqcap R_2 \sqsubseteq R_3, \exists R_1 \sqsubseteq A, \exists R_3 \sqcap A \sqsubseteq \bot, R_1(a,a), R_1(b,b), R_2(a,b), R_2(b,a)\}$$

and let η be a convex f-geometric model of $\mathcal{O}$. Let $\delta = 0.5\eta(a) + 0.5\eta(b)$. By the convexity of $\eta(R_1)$ and $\eta(R_2)$, we have that

$$f(\delta,\delta) \in \eta(R_1) \text{ and } f(\delta,\delta) \in \eta(R_2), \text{ hence } f(\delta,\delta) \in \eta(R_1 \sqcap R_2).$$

Then since η is a model of $R_1 \sqcap R_2 \sqsubseteq R_3$, $f(\delta,\delta) \in \eta(R_3)$ so $\delta \in \eta(\exists R_3)$. Moreover, since η is a model of $\exists R_1 \sqsubseteq A$, and $\delta \in \eta(\exists R_1)$, $\delta \in \eta(A)$. Hence $\eta \not\models \exists R_3 \sqcap A \sqsubseteq \bot$ so η is not a model of $\mathcal{O}$. ◁

We now state the main result of this section, which states that, under certain conditions, satisfiable DL-Lite$^{\mathcal{H},\mathbb{T},@}_{horn}$ ontologies have a convex geometric model. The need of Condition (i) is already illustrated by Example 10 whereas Condition (ii) ensures that the underlying logic is Horn (that is, it does not have disjunctions which can be expressed with the temporal attributes before, after and between).

Theorem 12. *Let $\mathcal{O}$ be a satisfiable DL-Lite$^{\mathcal{H},\mathbb{T},@}_{horn}$ ontology without negative role inclusions and such that (i) all specifiers attached to a role in $\mathcal{O}$ are atemporal, and (ii)* before, after *and* between *do not occur in $\mathcal{O}$. Then $\mathcal{O}$ has a convex geometric model.*

5 Related Work

Traditionally, most KG embedding models are time-unaware. These models embed both entities and relations in a low-dimensional latent space based on some regularities of target KG. They can be used as approximate reasoning methods [25, 24] for completing KG without using the schema. Typical embedding models include the translation based models, such as TransE [7] and bilinear models, such as ComplEx [29] and SimplE [16]. From the expressiveness perspective, TransE and DisMult have been shown to be not fully expressive; however, CompleEx and SimplE are fully expressive. Gutierrez-Basulto and Schockaert [14] use geometric models to study the compatibility between TBox/ontology and KG embeddings. They show that bilinear models (inc. ComplEx and SimplE) cannot strictly represent relation subsumption rules. Wiharja et al. [30] show that many well known KG embeddings based on KG completion methods are not impressive, when schema aware correctness is considered, despite good performance reported in silver standard based evaluations. Currently, more and more applications are involving dynamic KG, where knowledge in practice is time-variant and consists of sequences of observations. For example, in recommendation systems based on KG, new items and new user actions appear in real time. Accordingly, temporal KG embedding models incorporate time information into their node and relation representations. We next discuss how temporal information is taken into account in KG embeddings and how it has been used in combination with classical DLs.

Temporal Knowledge Graph Embeddings. Temporal KG embedding models can be seen as extensions of static KG embedding models. A basic approach is to collapse the dynamic graph into a static graph by aggregating the temporal observations over time [19]. This approach, however, may lose large amounts of information. An alternative approach is to give more weights to snapshots that are more recent [28]. Another alternative approach to aggregating temporal observations is to apply decomposition methods to dynamic graphs. The idea is to model a KG as an order 4 tensor and decompose it using CP or Tucker, or other decomposition methods to obtain entity, relation, and timestamp embeddings [12]. In addition to aggregation based approaches, there are approaches extending static KG embedding, such as TransE, by adding a timestamp embedding into the score function [15, 21]. Jiang et al. [15] only use such timestamps to maintain temporal order, while using Integer Linear Programming to encode the temporal consistency information as constraints. Ma et al. [21] extend several models (Tucker, RESCAL, HolE, ComplEx, DistMult) by adding a timestamp embedding to their score functions. These models may not work well when the number of timestamps is large. Furthermore, since they only learn embeddings for observed timestamps, they cannot generalize to unseen timestamps. Dasgupta et al. [11] fragments a temporally-scoped input KG into multiple static subgraphs with each subgraph corresponding to a timestamp. There are also approaches of applying random walk models for temporal KG. E.g., Bian et al [6] use metapath2vec to generate random walks on both the initial KG and the updated nodes and re-compute the embeddings for these nodes. These approaches mainly leverage

the temporal aspect of dynamic graphs to reduce the computations. However, they may fail at capturing the evolution and the temporal patterns of the nodes. Another natural choice for modeling temporal KG is by extending sequence models to graph data. E.g., García-Duran [13] extend TransE and DistMult by combining the relation and timestamp through a character LSTM, so as to learn representations for time-augmented KG facts that can be used in conjunction with existing scoring functions for link prediction. Ma et al. [21] argue that temporal KG embeddings could also be used as models for cognitive episodic memory (facts we remember and can recollect) and for semantic memory (current facts we know) can be generated from episodic memory by a marginalization operation.

Temporal Description Logics. In the DL literature, there are several approaches for representing and reasoning temporal information [20, 33, 5]. Schmiedel [27] was the first to propose an extension of the description logics (the $\mathcal{FLENR}^-$ DL in this case) with an interval-based temporal logic, with the temporal quantifier **at**, the existential and universal temporal quantifiers **sometime** and **alltime**. Artale and Franconi [2, 3] considered a class of interval-based temporal description logics by reducing the expressivity to keep the property of decidability of the logic proposed by Schmiedel [27]. Schild proposed $\mathcal{ALCT}$ [26], extending $\mathcal{ALC}$ with point-based modal temporal connectives from tense logic [10], including existential future ($\Diamond$), universal future ($\Box$), next instant ($\bigcirc$),until ($\mathcal{U}$), reflexive until (**U**). Wolter and Zakharyaschev studied the $\mathcal{ALC}^-_{\mathcal{M}}$ DL and showed that it is decidable in the class of linear, discrete and unbounded temporal structures [31]. They also showed that the $\mathcal{ALC}_{\mathcal{M}}$ DL (extending the $\mathcal{ALC}^-_{\mathcal{M}}$ DL with global roles) is undecidable [32]. Temporal operators can be used in a temporal ABox as well, allowing the use of next instant ($\bigcirc\varphi$) and until ($\varphi\mathcal{U}\psi$) with ABox assertions [4]. Ozaki et al. [22, 23] propose temporally attributed DLs, which allows the use of absolute temporal information in both TBoxes and ABoxes. They show that the satisfiability of ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ ontologies is ExpTime-complete, and that the satisfiability of ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ ontologies without the temporal attributes $\mathsf{between}$, before and after is PTime-complete.

6 Conclusion

We investigate how geometric models can (or cannot) be used to capture rules about annotated data expressed in the formalism of attributed DLs. We show that every satisfiable attributed DL-Lite$^{\mathcal{H}}_{horn}$ ontology has a convex geometric model and that this is also the case when allowing the use of temporal attributes under some restrictions. There is still a long way to make this result practical since we still require an embedding technique that would construct such a model. In this direction, we highlight the work of Abboud et al. [1], where relations are mapped to convex regions in the format of hyper-rectangles.

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