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DISCRETE COXETER GROUPS

GYE-SEON LEE AND LUDOVIC MARQUIS

ABSTRACT. Coxeter groups are a special class of groups generated by involutions. They play important roles in the various areas of mathematics. This survey particularly focuses on how one uses Coxeter groups to construct interesting examples of discrete subgroups of Lie groups.

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1. INTRODUCTION

It is a fundamental problem in geometry and topology to understand discrete subgroups of Lie groups G. For example, when G is the isometry group $\mathrm{Isom}(\mathbb{H}^d)$ of the hyperbolic d-space $\mathbb{H}^d$, the study of discrete subgroups of $\mathrm{Isom}(\mathbb{H}^d)$ is closely related to that of complete hyperbolic d-manifolds. More precisely, there is a one-to-one correspondence between torsion-free discrete subgroups Γ of $\mathrm{Isom}(\mathbb{H}^d)$ and complete hyperbolic d-manifolds $\mathbb{H}^d/\Gamma$.

Convex cocompact subgroups of rank-one Lie groups G are specially an important class of discrete subgroups of G. In particular, given a finitely generated group Γ, the space of representations $\rho : \Gamma \to G$ whose image is convex cocompact is *open* in the representation space $\mathrm{Hom}(\Gamma, G)$, i.e., the space of all representations $\rho : \Gamma \to G$. So, if ρ is convex cocompact and is not isolated, then all the nearby representations of ρ are again convex cocompact, and hence discrete.

Recently, new notions of representations were introduced to generalize convex cocompact subgroups of rank-one Lie groups: Anosov representations in real semisimple Lie groups (see [Lab06, GW12]) and convex cocompact subgroups in real projective spaces (see [DGK17a]). Such representations also have the property of openness. As new theories are developed, it is also important to have many examples to support them. From this perspective, the role of Coxeter groups is crucial.

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The aim of this survey is to illustrate how one can build interesting examples of discrete Coxeter groups.

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2. Coxeter groups

2.1. **What is a Coxeter group?** A *Coxeter matrix* $M = (m_{s,t})_{s,t\in S}$ on a finite set S is a symmetric matrix with entries $m_{s,t} \in \{1,2,\dots,m,\dots,\infty\}$ such that the diagonal entries $m_{s,s} = 1$ and off-diagonal entries $m_{s,t} \neq 1$. From any Coxeter matrix $M = (m_{s,t})_{s,t\in S}$, one may obtain the *Coxeter group* W of M given by generators and relations:

$$W = \left\langle s \in S \mid (st)^{m_{s,t}} = 1,\ \forall s,t \in S \right\rangle.$$

Here, $(st)^\infty = 1$ means that there is no relation between s and t. Since $m_{s,s} = 1$, each generator s is an involution, i.e., $s^2 = 1$. We shall use the notation W, W_S or $W_{S,M}$ for a Coxeter group, depending on what is important to stress. One should remember that a Coxeter group is a group with a preferred generating set, namely S. The *rank* of W_S is the cardinality $\#S$ of S.

The *Coxeter diagram* of the Coxeter group W_S is the labeled graph $\mathscr{G}_W$ such that:

(i) the set of nodes[1] of $\mathscr{G}_W$ is the set S;
(ii) two nodes $s,t \in S$ are connected by an edge $\overline{st}$ of $\mathscr{G}_W$ if $m_{s,t} \in \{3,4,\dots,\infty\}$;
(iii) the label of the edge $\overline{st}$ is $m_{s,t}$ if $m_{s,t} \in \{4,5,\dots,\infty\}$.

It is well-known that for any subset T of S, the subgroup of W_S generated by T is the Coxeter group $W_{T,M'}$ with generating set T and exponents $m'_{s,t} = m_{s,t}$ for every $s,t \in T$ (see [Bou68, Chap. IV, Th. 2]). Such a subgroup W_T is called a *standard subgroup* of W_S.

The connected components of the graph $\mathscr{G}_{W_S}$ are graphs of the form $\mathscr{G}_{W_{S_i}}$, $i \in I$, where the $(S_i)_{i\in I}$ form a partition of S. The standard subgroups W_{S_i} are called the *irreducible components* of W_S. Since $m_{s,t} = 2$ if and only if $st = ts$, we see that the group W_S is the direct product of the subgroups W_{S_i} for $i \in I$. A Coxeter group W_S is *irreducible* when the Coxeter diagram $\mathscr{G}_{W_S}$ is connected, i.e., $\#I = 1$. A subset T of S is said to be "*something*" if the Coxeter group W_T is "something". For example, the word "something" can be replaced by "irreducible", and so on. Two subsets $T,U \subset S$ are *orthogonal* if $m_{t,u} = 2$ for every $t \in T$ and every $u \in U$.

The *Cosine matrix* of $W_{S,M}$ is the $S \times S$ symmetric matrix C_W whose entries are:

$$\left(\mathsf{C}_W\right)_{s,t} = -2\cos\left(\frac{\pi}{m_{s,t}}\right) \quad \text{for every } s,t \in S$$

[1] A Coxeter group often comes with a Coxeter polytope in such a way that the nodes of the Coxeter diagram are in bijection with the facets of the Coxeter polytope. We shall use the word *node* of the Coxeter diagram rather than *vertex* to make a distinction between the vertices of the Coxeter polytope and the vertices of the Coxeter diagram.

An irreducible Coxeter group W is said to be *spherical* (resp. *affine*) when the Cosine matrix C_W is positive definite (resp. positive semi-definite but not definite).

Theorem 2.1 (Coxeter [Cox32, Cox34] and Margulis–Vinberg [MV00])**.** *Let W_S be an irreducible Coxeter group. Then exactly one of the following is true:*

(i) If W_S is spherical, then W_S is a finite group.
(ii) If W_S is affine, then $\#S \geqslant 2$ and W_S is virtually[2] $\mathbb{Z}^{\#S-1}$.
(iii) Otherwise, W_S is large*, i.e., there exists a surjective homomorphism of a finite index subgroup of W_S onto a free group on two generators.*

Remark 2.2. These three cases are clearly exclusive. Consequently, if an irreducible Coxeter group W_S is finite (resp. infinite and virtually abelian), then W_S is spherical (resp. affine).

Remark 2.3. The irreducible spherical and irreducible affine Coxeter groups were classified by Coxeter [Cox32, Cox34]; see also Witt [Wit41]. The complete list can be found in Table 1.

A Coxeter group (not necessarily irreducible) is *spherical* (resp. *affine*) when all its irreducible components are spherical (resp. affine).

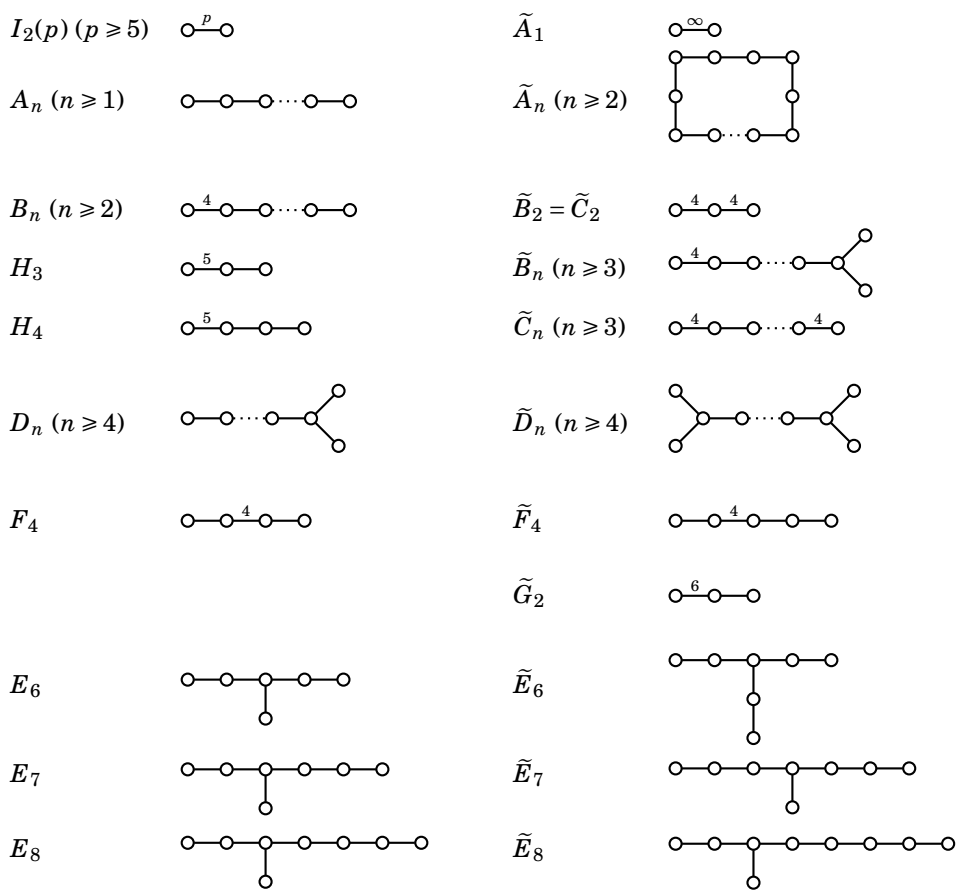

TABLE 1. The irreducible spherical Coxeter diagrams on the left and irreducible affine Coxeter diagrams on the right.

[2]A group G is *virtually "something"* if there is a finite index subgroup $H \leqslant G$ such that H is "something".

3. Hyperbolic reflection groups

3.1. **Hyperbolic polytopes.** Let $\mathbb{R}^{d,1}$ be the vector space $\mathbb{R}^{d+1}$ endowed with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ of signature[3] $(d,1)$, and let q be the associated quadratic form. A coordinate representation of q with respect to some basis of $\mathbb{R}^{d,1}$ is:

$$q(x) = x_1^2 + \cdots + x_d^2 - x_{d+1}^2.$$

A *hyperbolic d-space* $\mathbb{H}^d$ is a connected component of a hyperquadric:

$$\mathbb{H}^d = \{x \in \mathbb{R}^{d,1} \mid q(x) = -1 \text{ and } x_{d+1} > 0\}.$$

The isometry group of $\mathbb{H}^d$ is $\mathrm{O}^+_{d,1}(\mathbb{R})$, which consists of the elements of $\mathrm{O}_{d,1}(\mathbb{R})$ that preserve $\mathbb{H}^d$. We often work with the projective model of $\mathbb{H}^d$:

$$\mathbb{H}^d = \{x \in \mathbb{R}^{d,1} \mid q(x) < 0 \text{ and } x_{d+1} > 0\}/\mathbb{R}_+,$$

where the set $\mathbb{R}_+$ of positive scalars acts on $\mathbb{R}^{d,1} \smallsetminus \{0\}$ by multiplication. If we set

$$\mathbb{S}(\mathbb{R}^{d,1}) := (\mathbb{R}^{d,1} \smallsetminus \{0\})/\mathbb{R}_+,$$

then $\mathbb{H}^d$ is an open subset of the *projective sphere* $\mathbb{S}(\mathbb{R}^{d,1})$. The closure $\overline{\mathbb{H}^d}$ of $\mathbb{H}^d$ in $\mathbb{S}(\mathbb{R}^{d,1})$ is the *compactification* of $\mathbb{H}^d$.

A subset of $\mathbb{H}^d$ is a *hyperbolic d-polytope* if it is the intersection of a finite family of closed half-spaces of $\mathbb{H}^d$ and if it has non-empty interior. A *hyperbolic Coxeter d-polytope* (or simply a $\mathbb{H}^d$-Coxeter polytope) is a hyperbolic d-polytope P all its dihedral angles are sub-multiples of π. In other words, if two facets[4] s, t of P are adjacent,[5] then the dihedral angle $\theta(s,t)$ between s and t is equal to π/m for some integer $m \geqslant 2$. When two facets s,t are parallel, it is common to say that $\theta(s,t) = 0$. A hyperbolic Coxeter polytope is *right-angled* if all its dihedral angles are $\pi/2$.

Associated with a hyperbolic Coxeter polytope P is a Coxeter matrix $M = (m_{s,t})_{s,t\in S}$ on the set S of facets of P: if $s,t \in S$ are adjacent, then $m_{s,t} = \pi/\theta(s,t)$; otherwise, $m_{s,t} = \infty$. We denote by W_P the Coxeter group of the Coxeter matrix M, and call it the *Coxeter group of P*. If s is a facet of P, then σ_s denotes the reflection in the hyperplane containing s.

Theorem 3.1 (Poincaré [Poi83])**.** *Let P be a $\mathbb{H}^d$-Coxeter polytope, and W_P the Coxeter group of P. Then the homomorphism $\sigma : W_P \to \mathrm{Isom}(\mathbb{H}^d)$ defined by*

$$\sigma(s) = \sigma_s \quad \textit{for each } s \in S$$

is injective and the image $\Gamma_P := \sigma(W_P)$ is discrete. Moreover, P is a fundamental domain for the action of Γ_P on $\mathbb{H}^d$. In particular, if P has finite volume (resp. is compact), then Γ_P is a lattice (resp. uniform lattice) of $\mathrm{Isom}(\mathbb{H}^d)$.

A subgroup H of $\mathrm{Isom}(\mathbb{H}^d)$ is called a *hyperbolic reflection group* if $H = \Gamma_P$ for some $\mathbb{H}^d$-Coxeter polytope P. In this case, we call Γ_P the *reflection group of P*. Theorem 3.1 provides a nice way to construct discrete subgroups of $\mathrm{Isom}(\mathbb{H}^d)$, even lattices. We shall review in the next section the classification of $\mathbb{H}^d$-Coxeter polytopes of finite volume in small dimensions and their non-existence in large dimensions.

[3]The *signature* of a symmetric matrix B is the triple (p,q,r) of the positive, negative, and zero indices of inertia of B. In the case $r = 0$, we simply say that B is of signature (p,q).

[4]A face of P of codimension 1 (resp. 2) is called a *facet* (resp. *ridge*) of P.

[5]Two facets s and t of P are *adjacent* if $s \cap t$ is a ridge of P.

3.2. **Classical results in dimensions 2 and 3.**

3.2.1. *Dimension 2.* A necessary and sufficient condition for the existence of hyperbolic Coxeter 2-polytopes of finite volume follows immediately from:

Theorem 3.2. *Let $\theta_1,\dots,\theta_n$ be real numbers such that $0 \leqslant \theta_i < \pi$ for each $i = 1,\dots,n$. Then there exists a hyperbolic polygon of finite volume with dihedral angles $\theta_1,\dots,\theta_n$ if and only if $\sum_{i=1}^n \theta_i < (n-2)\pi$.*

3.2.2. *Dimension 3.* Hyperbolic Coxeter 3-polytopes of finite volume are well understood, notably thanks to the classification of hyperbolic 3-polytopes with dihedral angles $\leqslant \pi/2$ due to Andreev [And71a, And71b]. During his PhD, Roeder found and fixed a gap in the original proof of Andreev (see [RHD07]). Hodgson and Rivin [HR93] gave a characterization of hyperbolic 3-polytopes, which generalizes Andreev's theorem, in terms of a generalized Gauss map.

To express properly Andreev's theorem, one needs some definitions. Two compact polytopes $\mathscr{P},\mathscr{P}'$ of the Euclidean space $\mathbb{R}^d$ are *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. A combinatorial equivalence class is called a *combinatorial polytope*. Note that if a hyperbolic polytope $P \subset \mathbb{H}^d$ is of finite volume, then the closure $\overline{P}$ of P in $\overline{\mathbb{H}^d}$ is combinatorially equivalent to a compact polytope of $\mathbb{R}^d$. A *labeled polytope* is a combinatorial polytope $\mathscr{P}$ with a labeling θ, that is, a function from the ridges of $\mathscr{P}$ to $(0,\pi/2]$. A hyperbolic polytope P of finite volume *realizes* a labeled polytope $\mathscr{P}$ if there exists a combinatorial equivalence ϕ between the faces of $\overline{P}$ and $\mathscr{P}$ such that the dihedral angle at the ridge e of $\overline{P}$ is the label $\theta(\phi(e))$ of $\mathscr{P}$.

Let $\mathscr{P}$ be a labeled 3-polytope with labeling θ. A *k-circuit* of $\mathscr{P}$ is a sequence of distinct facets $s_1,\dots,s_k$ such that $e_i := s_i \cap s_{i+1}$ (indices are modulo k) is an edge of $\mathscr{P}$. A k-circuit is *prismatic* if all the (closed) edges e_i are disjoint. The *angle sum* of a k-circuit is the real number $\sum_{i=1}^k \theta(e_i)$. A k-circuit is *spherical* (resp. *Euclidean*, resp. *hyperbolic*) if its angle sum is bigger than (resp. equal to, resp. less than) $(k-2)\pi$. A vertex v of $\mathscr{P}$ is *spherical* (resp. *Euclidean*) if the circuit consisting of the facets that contain v is spherical (resp. Euclidean). The *graph $W_{\mathscr{P}}$ of $\mathscr{P}$* is the graph whose nodes are the facets of $\mathscr{P}$ and such that two nodes s,t are connected if and only if $s \cap t$ is not an edge, or $s \cap t$ is an edge and $\theta(s \cap t) < \pi/2$.

Theorem 3.3 (Andreev [And71a, And71b]; see also Roeder–Hubbard–Dunbar [RHD07])**.** *Let $\mathscr{P}$ be a labeled 3-polytope whose underlying polytope is not a tetrahedron. Then there exists a compact (resp. finite volume) hyperbolic 3-polytope P that realizes $\mathscr{P}$ if and only if:*

(i) all the vertices of $\mathscr{P}$ are spherical (resp. spherical or Euclidean);
(ii) all the prismatic 3- and 4-circuits of $\mathscr{P}$ are hyperbolic;
(iii) the graph $W_{\mathscr{P}}$ of $\mathscr{P}$ is connected.

In that case, the polytope P is unique up to an isometry of $\mathbb{H}^3$.

Remark 3.4. A careful reader probably wonders what happen when the underlying polytope of $\mathscr{P}$ is a tetrahedron. This case is explained in [Roe06]. The list of all Coxeter tetrahedra of finite volume can be found in Tables 3 and 4.

Remark 3.5. The condition on the connectivity of $W_{\mathscr{P}}$ is often expressed by inequalities. One may notice that if the conditions (i) and (ii) are satisfied, then the disconnectedness of $W_{\mathscr{P}}$ implies that (see [RHD07, Prop. 1.5]):

- $\mathscr{P}$ is a right triangular prism, i.e., all the labels between the base facets and the joining facets are $\pi/2$, or
- $\mathscr{P}$ is a quadrilateral pyramid with Euclidean apex such that the labels of two opposite edges in the base are $\pi/2$.

3.3. **The Gram matrix of a hyperbolic polytope.** A hyperbolic d-polytope P is of the form

$$P = \cap_{i=1}^{N} H_i^-,$$

where H_i^- is a closed half-space of $\mathbb{H}^d$ whose boundary is a hyperplane H_i. Each H_i^- corresponds to a unique vector u_i with the property that:

$$H_i^- = \mathbb{H}^d \cap \{x \in \mathbb{R}^{d,1} \mid \langle x, u_i \rangle \leqslant 0\} \quad \text{and} \quad \langle u_i, u_i \rangle = 1.$$

The *Gram matrix* $G = (g_{i,j})$ of P is a symmetric $N \times N$ matrix with entries $g_{i,j} = \langle u_i, u_j \rangle$. A square matrix B is *reducible* if B is the direct sum of smaller square matrices B_1 and B_2 (after a reordering of the indices), i.e., $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$. Otherwise, B is *irreducible*. Every square matrix B shall be the direct sum of irreducible submatrices, each of which we call a *component* of B. The next theorem tells us that the hyperbolic polytopes are determined by the Gram matrices.

Theorem 3.6 (Vinberg [Vin85, Th. 2.1])**.** *Let $G = (g_{i,j})$ be an irreducible symmetric $N \times N$ matrix of signature $(d, 1, N-d-1)$ such that the diagonal entries $g_{i,i} = 1$ and off-diagonal entries $g_{i,j}$ are $\leqslant 0$. Then there exists a hyperbolic d-polytope P whose Gram matrix is G, and the polytope P is unique up to an isometry of $\mathbb{H}^d$.*

Remark 3.7. A detailed analysis of the Gram matrix can reveal the combinatorial structure of the hyperbolic polytope P (see [Vin85, Th. 3.1 & 3.2]) and whether P is compact or of finite volume (see [Vin85, Th. 4.1]).

3.4. **Lannér and quasi-Lannér Coxeter groups.** A Coxeter group W_S is *Lannér* (resp. *quasi-Lannér*) if $\det(\mathsf{C}_W) < 0$ and if for every proper subset $T \subset S$, the Coxeter group W_T is spherical (resp. spherical or irreducible affine). If P is a compact (resp. finite volume) Coxeter simplex, then the Coxeter group W_P is Lannér (resp. quasi-Lannér). Conversely, if W is a Lannér (resp. quasi-Lannér) Coxeter group of rank $d+1$, then its Cosine matrix C_W has signature $(d, 1)$ and there exists a compact (resp. finite volume) $\mathbb{H}^d$-Coxeter polytope whose Gram matrix is $\frac{1}{2}\mathsf{C}_W$.

In short, Lannér (resp. quasi-Lannér) Coxeter groups correspond to compact (resp. finite-volume) $\mathbb{H}^d$-Coxeter simplices. Such Coxeter simplices exist only in small dimensions.

Theorem 3.8 (Lannér [Lan50], Koszul [Kos67] and Chein [Che69])**.** *If W_S is a Lannér (resp. quasi-Lannér) Coxeter group, then $\#S \leqslant 5$ (resp. $\#S \leqslant 10$). Table 2 indicates the number of Lannér (resp. quasi-Lannér) Coxeter groups of a given rank. The list of Lannér and quasi-Lannér Coxeter groups of rank $\geqslant 4$ can be found in* [Che69].

The non-existence of hyperbolic Coxeter simplices in large dimensions is in fact the first side of a more general non-existence theorem in Section 3.5.

Remark 3.9. Tables 3 and 4 give us the list of all the Lannér or quasi-Lannér Coxeter groups of rank 4, which correspond to compact or finite volume hyperbolic tetrahedra, respectively.

Dimension $d = \#S - 1$	*♯ of quasi-Lannér not Lannér*	*♯ of Lannér Coxeter groups*
2	∞	∞
3	*23*	*9*
4	*9*	*5*
5	*12*	*0*
6	*3*	*0*
7	*4*	*0*
8	*4*	*0*
9	*3*	*0*

TABLE 2. The numbers of quasi-Lannér or Lannér Coxeter groups

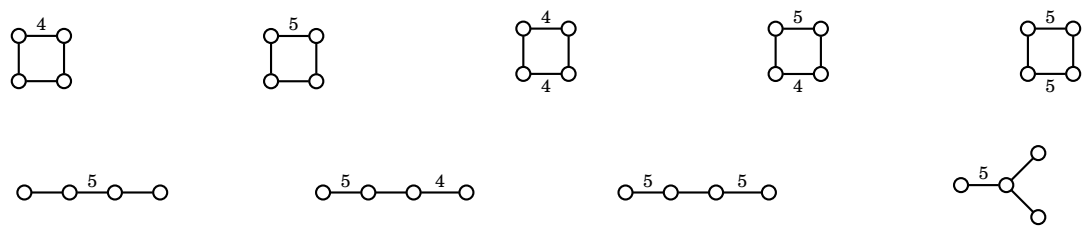

TABLE 3. The nine compact hyperbolic tetrahedra

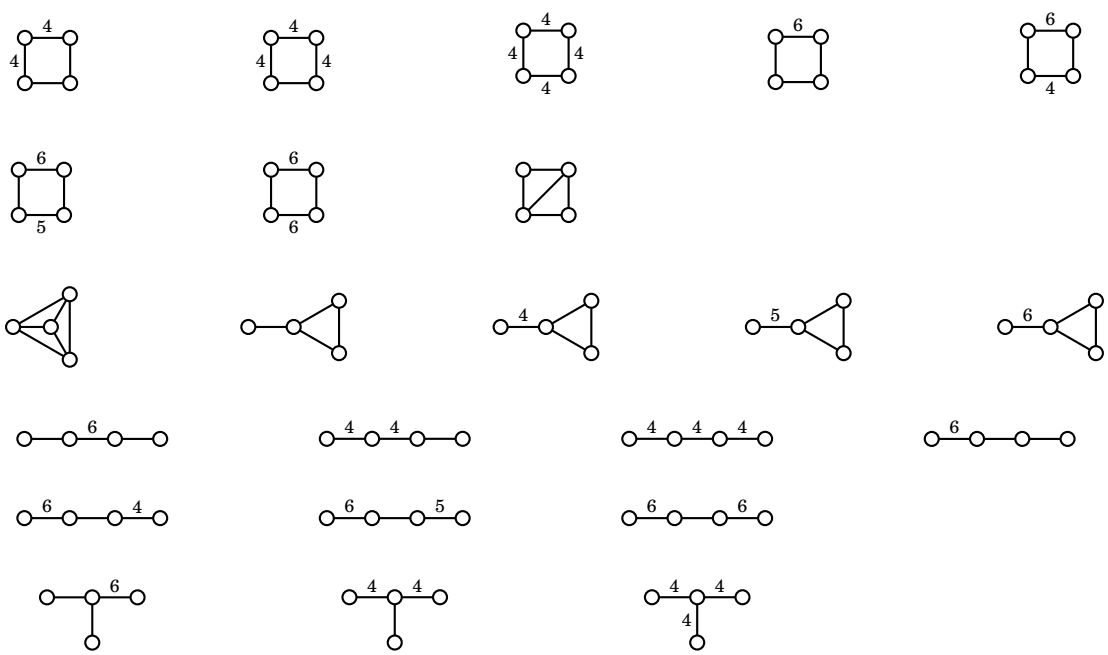

TABLE 4. The twenty-three hyperbolic tetrahedra of finite volume which are not compact

3.5. **Absence in large dimension.**

Theorem 3.10 (Vinberg [Vin84] and Prokhorov [Pro86])**.** *If Γ_P is a discrete reflection group of* $\mathrm{Isom}(\mathbb{H}^d)$ *with compact (resp. finite volume) fundamental domain P, then $d \leqslant 29$ (resp. $d \leqslant 995$).*

The proof of Vinberg (resp. Prokhorov) uses Nikulin's inequality for simple polytopes (resp. edge-simple polytopes) established in [Nik81] (resp. [Kho86]). The upper bounds in the right-angled case are better:

Theorem 3.11 (Potyagailo–Vinberg [PV05] and Dufour [Duf10])**.** *If Γ_P is a discrete reflection group of* $\mathrm{Isom}(\mathbb{H}^d)$ *with compact (resp. finite volume) right-angled fundamental domain P, then $d \leqslant 4$ (resp. $d \leqslant 12$).*

Except for compact right-angled polytopes, the upper bounds are far from being sharp.

- There exists a compact right-angled hyperbolic 4-polytope: 120-cell.
- Examples of finite volume right-angled d-polytopes are known in dimension $d \leqslant 8$ (see [PV05]).
- Examples of compact d-polytopes are known in dimension $d \leqslant 8$ (see [Bug84, Bug92] for $d = 7, 8$).
- Examples of finite volume d-polytopes are known in dimension $d \leqslant 21$ and $d \neq 20$ (see [All06] and the references therein for $d \leqslant 19$ and [Bor87] for $d = 21$).

Remark 3.12. Moussong observed that the argument of Vinberg [Vin84] may be extended to show that if a Coxeter group W is word hyperbolic and the nerve of W is a generalized homology $(d-1)$-sphere, then $d \leqslant 29$ (see [Dav08, Prop. 12.6.7]). Here, the *nerve* of a Coxeter group W_S is the poset of all nonempty spherical subsets of S partially ordered by inclusion, which is an abstract simplicial complex, and a *generalized homology d-sphere* is a homology d-manifold with the same homology as the d-sphere.

3.6. **Hyperbolic Coxeter polytopes with few facets.** The complete classification of compact or finite-volume hyperbolic polytopes is not an easy task. Only compact d-polytopes with N facets ($N \leqslant d+3$) and finite-volume d-polytopes with N facets ($N \leqslant d+2$) were classified. For more details, we refer the reader to the web page maintained by Felikson and Tumarkin:

https://www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html

This webpage contains all the known examples of hyperbolic Coxeter polytopes of dimension $\geqslant 4$.

3.7. **Convex cocompact hyperbolic reflection groups.** A subgroup Γ of $\mathrm{Isom}(\mathbb{H}^d)$ is *convex cocompact* if there exists a Γ-invariant convex subset $\mathscr{C}$ of $\mathbb{H}^d$ such that Γ acts properly discontinuously on $\mathscr{C}$ with compact quotient. Using the following theorems, one can easily check when a reflection group Γ_P of $\mathrm{Isom}(\mathbb{H}^d)$ is convex cocompact.

Theorem 3.13 (Desgroseilliers–Haglund [DH13, Th. 4.12])**.** *Let P be an $\mathbb{H}^d$-Coxeter polytope and let Γ_P be its reflection group. Then Γ_P is convex cocompact if and only if*

(i) Γ_P is word-hyperbolic, and
(ii) P has no pair of asymptotic facets.

Remark 3.14. The condition (ii) may be replaced by (ii') there is no pair of facets s, t of P such that $g_{s,t} = -1$, where $G = (g_{s,t})$ is the Gram matrix of P.

Theorem 3.15 (Moussong's hyperbolicity criterion [Mou88])**.** *Let W_S be a Coxeter group. Then W_S is word hyperbolic if and only if S does not contain two orthogonal non-spherical subsets, nor any affine subset of rank $\geqslant 3$.*

Remark 3.16. Desgroseilliers and Haglund [DH13, Th. 1.1] found a class of Coxeter groups which can be realized as a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^d)$, which is not a reflection group, and they conjectured that there exists a word-hyperbolic Coxeter group which admits a convex cocompact representation into $\mathrm{Isom}(\mathbb{H}^d)$ but which cannot be realized as a convex cocompact *reflection* group of $\mathrm{Isom}(\mathbb{H}^n)$ for any $n \in \mathbb{N}$.

4. Projective reflection groups

4.1. **Tits–Vinberg's Theorem.** Let V be a vector space over $\mathbb{R}$, and let $\mathbb{S}(V)$ be the projective sphere. We denote by $\mathrm{SL}^{\pm}(V)$ the group of automorphism of $\mathbb{S}(V)$, i.e.,

$$\mathrm{SL}^{\pm}(V) = \{g \in \mathrm{GL}(V) \mid \det(g) = \pm 1\}.$$

We denote by $\hat{\mathbb{S}}$ the natural projection of $V \smallsetminus \{0\}$ to $\mathbb{S}(V)$, and let $\mathbb{S}(W) := \hat{\mathbb{S}}(W \smallsetminus \{0\})$ for any subset W of V. The complement of a projective hyperplane in $\mathbb{S}(V)$ consists of two connected components, each of which we call an *affine chart* of $\mathbb{S}(V)$. A *cone* is a subset of V which is invariant under multiplication by positive scalars. A subset $\mathscr{C}$ of $\mathbb{S}(V)$ is *convex* if there exists a convex cone U of V such that $\mathscr{C} = \mathbb{S}(V)$, *properly convex* if it is convex and its closure lies in some affine chart, and *strictly convex* if in addition its boundary does not contain any nontrivial projective line segment. Hyperbolic spaces are special examples of strictly convex open subsets of $\mathbb{S}(V)$.

A *projective polytope* is a properly convex subset P of $\mathbb{S}(V)$ such that P has a non-empty interior and $P = \cap_{i=1}^{N} \mathbb{S}(\{x \in V \mid \alpha_i(x) \leqslant 0\})$, where α_i, $i = 1, \dots, N$, are linear forms on V. We always assume that P has N facets, i.e., to define P, we need all the N linear forms $(\alpha_i)_{i=1}^{N}$. A *projective reflection* is an element of $\mathrm{SL}^{\pm}(V)$ of order 2 which is the identity on a hyperplane. Every projective reflection σ can be written as:

$$\sigma = \mathrm{Id} - \alpha \otimes v, \quad \text{i.e.,} \quad \sigma(x) = x - \alpha(x)v \quad \forall x \in V,$$

where α is a linear form on V and v is a vector of V such that $\alpha(v) = 2$.

Let P be a projective polytope and S the set of facets of P. A *reflection in a facet* $s \in S$ is a projective reflection σ_s which fixes each point of s. A *pre-mirror polytope* is a projective polytope P together with one reflection σ_s in each facet s of P. So, one may choose $\sigma_s = \mathrm{Id} - \alpha_s \otimes v_s$ with $\alpha_s(v_s) = 2$ such that $P = \cap_{s \in S} \mathbb{S}(\{x \in V \mid \alpha_s(x) \leqslant 0\})$. Note that the pairs (α_s, v_s) are uniquely determined only up to multiplication by a positive real number.

If P is a pre-mirror polytope, then Γ_P denotes the group generated by the reflections in the facets of P. We say that Γ_P is a *projective reflection group* if for any $\gamma \in \Gamma_P$,

$$\gamma(\mathring{P}) \cap \mathring{P} \neq \varnothing \quad \Rightarrow \quad \gamma = \mathrm{Id},$$

where $\mathring{P}$ denotes the interior of P.

In the next paragraph, we introduce a relevant tool to formulate Proposition 4.2 and Theorem 4.3 which express necessary and sufficient conditions for Γ_P of a pre-mirror polytope P to be a projective reflection group. A key notion is that of Cartan matrix of mirror polytope which generalizes the twice of the Gram matrix of hyperbolic polytope.

Definition 4.1. A *Cartan matrix* on a set S is a $S \times S$ matrix $\mathscr{A}_S = (a_{s,t})_{s,t\in S}$ which satisfies the conditions: (i) $a_{s,s} = 2$, $\forall s \in S$; (ii) $a_{s,t} \leqslant 0$, $\forall s \neq t \in S$; (iii) $a_{s,t} = 0 \Leftrightarrow a_{t,s} = 0$, $\forall s \neq t \in S$.

A *mirror polytope* is a pre-mirror polytope P such that the matrix $\mathscr{A}_P := (\alpha_s(v_t))_{s,t\in S}$ is a Cartan matrix. In this case, we call $\mathscr{A}_P$ the *Cartan matrix of* P. A Cartan matrix $\mathscr{A}$ on S is of *Coxeter type* if for any $s \neq t \in S$,

$$a_{s,t}a_{t,s} < 4 \quad \Rightarrow \quad \frac{\pi}{\arccos\left(\frac{1}{2}\sqrt{a_{s,t}a_{t,s}}\right)} \in \mathbb{N}.$$

A *projective Coxeter polytope* is a mirror polytope P whose Cartan matrix $\mathscr{A}_P$ is of Coxeter type. For each pair of adjacent facets s,t of P, the *dihedral angle* of the ridge $s \cap t$ is said to be $\pi/m_{s,t}$ if $a_{s,t}a_{t,s} = 4\cos^2(\pi/m_{s,t})$.

Proposition 4.2 (Vinberg [Vin71, Prop. 17])**.** *Let P be a pre-mirror polytope. If the group Γ_P is a projective reflection group, then P is a projective Coxeter polytope.*

A Cartan matrix $\mathscr{A}_S$ and a Coxeter group W_S are *compatible* when:

(i) $\forall s,t \in S, \quad m_{s,t} = 2 \Leftrightarrow a_{s,t} = 0$;
(ii) $\forall s,t \in S, \quad m_{s,t} < \infty \Leftrightarrow a_{s,t}a_{t,s} = 4\cos^2(\pi/m_{s,t})$;
(iii) $\forall s,t \in S, \quad m_{s,t} = \infty \Leftrightarrow a_{s,t}a_{t,s} \geqslant 4$.

It is clear that there is at most one Coxeter group compatible with a given Cartan matrix, and that a Cartan matrix $\mathscr{A}_S$ is compatible with some Coxeter group W_S if and only if $\mathscr{A}_S$ is of Coxeter type. If P is a projective Coxeter polytope, then W_P denotes the unique Coxeter group compatible with $\mathscr{A}_P$. The following is a generalization of Theorem 3.1 to the projective setting.

Theorem 4.3 (Bourbaki [Bou68, Chap. V] and Vinberg [Vin71, Th. 2])**.** [6] *Let P be a projective Coxeter polytope of $\mathbb{S}(V)$ with Coxeter group W_P, and let Γ_P be the group generated by the projective reflections $(\sigma_s)_{s\in S}$ in the facets of P. Then the following hold:*

(i) the homomorphism $\sigma : W_P \to \mathrm{SL}^{\pm}(V)$ defined by $\sigma(s) = \sigma_s$ is an isomorphism onto Γ_P;
(ii) the group Γ_P is a discrete projective reflection group;
(iii) the union $\mathscr{C}_P$ of the Γ_P-translates of P is a convex subset of $\mathbb{S}(V)$;
(iv) if Ω_P is the interior of $\mathscr{C}_P$, then Γ_P acts properly discontinuously on Ω_P.

4.2. **From Cartan matrices to mirror polytopes.** Given a Cartan matrix $\mathscr{A}_S$, there is a simple process to build a canonical mirror polytope $\Delta_{\mathscr{A}}$ such that $\mathscr{A}_{\Delta_{\mathscr{A}}} = \mathscr{A}_S$. In this construction, $\Delta_{\mathscr{A}}$ will be a simplex of dimension $\#S - 1$.

Let $V = \mathbb{R}^S$. We denote by $(e_s)_{s\in S}$ the canonical basis of V and $(e_s^*)_{s\in S}$ its dual basis. We set $\alpha_s := e_s^*$ and $v_s := \sum_t \mathscr{A}_{t,s}\, e_t$, i.e., v_s is the s-column vector of $\mathscr{A}_S$. Hence, by taking Δ to be (the projectivization of) the negative quadrant in $\mathbb{S}(V)$ and $\sigma_s = \mathrm{Id} - \alpha_s \otimes v_s$ to be the reflection in the facet $\Delta \cap \mathbb{S}(\mathrm{Ker}\,\alpha_s)$, we obtain a mirror polytope $\Delta_{\mathscr{A}}$ whose underlying polytope is a simplex of dimension $\#S - 1$. We call $\Delta_{\mathscr{A}}$ the *mirror simplex associated with* $\mathscr{A}_S$.

In the case where W_S is any Coxeter group and $\mathscr{A}_S = \mathsf{C}_W$, the mirror simplex associated with $\mathscr{A}_S$ is called the *Tits simplex associated with* W_S and denoted by Δ_W. The corresponding

[6] Theorem 4.3 was proved by Tits for Δ_W, which we define in Section 4.2, and by Vinberg for the general case.

representation $\sigma : W_S \to \mathbb{S}(\mathbb{R}^S)$ is dual to Tits geometric representation described in [Bou68].

For example, if W is spherical (resp. irreducible affine), then Δ_W gives rise to the classical tiling of $\mathbb{S}(V)$ (resp. of an affine chart) with Γ_{Δ_W} in the isometry group of the sphere (resp. Euclidean space). If W is Lannér (resp. quasi-Lannér), then Ω_{Δ_W} is the projective model of the hyperbolic space and Δ_W (resp. $\Delta_W \cap \Omega_{\Delta_W}$) is the hyperbolic Coxeter polytope whose Coxeter group is W.

By the Perron–Frobenius theorem, an irreducible Cartan matrix $\mathscr{A}$ has a simple eigenvalue $\lambda_{\mathscr{A}}$ which corresponds to an eigenvector with positive entries and has the smallest modulus among the eigenvalues of $\mathscr{A}$. We say that $\mathscr{A}$ is of *positive*, *zero* or *negative type* when $\lambda_{\mathscr{A}}$ is positive, zero or negative, respectively. For example, the Gram matrix of a hyperbolic polytope of finite volume is always of negative type. Now, the following is a generalization of Theorem 3.6 to the projective setting.

Theorem 4.4 (Vinberg [Vin71, Cor. 1])**.** *Let $\mathscr{A}$ be a Cartan matrix of size $N \times N$. Assume that $\mathscr{A}$ is irreducible, of negative type and of rank $d+1$. Then there exists a unique mirror d-polytope P, up to automorphism of $\mathbb{S}(\mathbb{R}^{d+1})$, such that $\mathscr{A}_P = \mathscr{A}$.*

Remark 4.5. Theorem 4.4 is not explicitly stated in [Vin71, Cor. 1] for non-Coxeter polytopes, but may be proved from [Vin71, Prop. 13 & 15].

4.3. **Anosov reflection groups.** Anosov representations are discrete representations of word-hyperbolic groups into semisimple Lie group with good dynamical properties. They have received a lot of attention and have been much studied recently (see e.g. [Lab06, GW12] for the definition of Anosov representation). But examples of Anosov representations of word hyperbolic groups, which are more complicated than free groups and surface groups, into Lie group of higher rank are less known. The following theorem, which generalizes Theorem 3.13, tells us that any infinite, word hyperbolic, irreducible Coxeter group admits Anosov representations.

Theorem 4.6 ([DGK+21, Cor. 1.18])**.** *Let P be a projective Coxeter polytope of $\mathbb{S}(V)$ with Coxeter group W_S. Suppose that W_S is word-hyperbolic. Then the following are equivalent:*

- *the representation $\sigma : W_S \to \mathrm{SL}^{\pm}(V)$ defined by $\sigma(s) = \sigma_s$ is P_1-Anosov (i.e., Anosov with respect to the stabilizer of a line in V);*
- *$\mathscr{A}_{s,t}\mathscr{A}_{t,s} > 4$ for all $s \neq t$ with $m_{s,t} = \infty$.*

Remark 4.7. Anosov reflection groups in $\mathrm{O}(p,q)$ can be used to give a new proof of Theorem 3.15 (Moussong's hyperbolicity criterion); see [DGK17b, LM19].

Theorem 4.8 ([LM19, Th. A])**.** *In dimension $d = 4, \dots, 8$, there exists a projective Coxeter polytope of $\mathbb{S}(\mathbb{R}^{d+2})$ with Coxeter group W_S such that:*

- *the group W_S is word-hyperbolic and its boundary is a $(d-1)$-sphere;*
- *the image of the representation $\sigma : W_S \to \mathrm{SL}^{\pm}(\mathbb{R}^{d+2})$ defined by $\sigma(s) = \sigma_s$ lies in $\mathrm{O}_{d,2}(\mathbb{R})$;*
- *the representation $\sigma : W_S \to \mathrm{O}_{d,2}(\mathbb{R})$ is P-Anosov, where P is the stabilizer of an isotropic line;*
- *the group W_S is not quasi-isometric to $\mathbb{H}^d$.*

4.4. **Convex cocompact projective reflection groups.** An infinite discrete subgroup Γ of $\mathrm{SL}^{\pm}(V)$ is *convex cocompact in* $\mathbb{S}(V)$ if it acts properly discontinuously on some properly convex open subset Ω of $\mathbb{S}(V)$ and cocompactly on a nonempty Γ-invariant closed convex subset $\mathscr{C}$ of Ω whose closure in $\mathbb{S}(V)$ contains all accumulation points of all possible Γ-orbits $\Gamma \cdot y$ with $y \in \Omega$.

The notion of Convex cocompactness in $\mathbb{S}(V)$ introduced in [DGK17a], in some sense, generalizes that of Anosov representation, but it does not require that the group Γ is word hyperbolic. There is also a simple characterization of convex cocompactness for projective reflection groups:

Theorem 4.9 ([DGK+21, Th. 1.3])**.** *Let P be a projective Coxeter polytope of $\mathbb{S}(V)$ with infinite irreducible Coxeter group W_S, and $\sigma : W_S \to \mathrm{SL}^{\pm}(V)$ the representation defined by $\sigma(s) = \sigma_s$. If $\sigma(W_S)$ is convex cocompact in $\mathbb{S}(V)$, then W_S satisfies the following two conditions:*

(i) S does not contain two orthogonal non-spherical subsets;
(ii) if S contains an irreducible affine subset T of rank $\geqslant 3$, then W_T is of type $\widetilde{A}_k$ where $k = \#T - 1$.

Theorem 4.10 ([DGK+21, Th. 1.8])**.** *Let P be a projective Coxeter polytope of $\mathbb{S}(V)$ with infinite irreducible Coxeter group W_S, $\mathscr{A}_S = (\mathscr{A}_{s,t})_{s,t\in S}$ the Cartan matrix of P, and $\sigma : W_S \to \mathrm{SL}^{\pm}(V)$ the representation defined by $\sigma(s) = \sigma_s$. If W_S satisfies the conditions (i) and (ii) of Theorem 4.9, then the following are equivalent:*

- *$\sigma(W)$ is convex cocompact in $\mathbb{S}(V)$;*
- *for any irreducible standard subgroup W_T of W_S with $\varnothing \neq T \subset S$, the Cartan submatrix $\mathscr{A}_T := (\mathscr{A}_{s,t})_{s,t\in T}$ is not of zero type;*
- $\det(\mathscr{A}_T) \neq 0$ *for all $T \subset S$ with W_T of type $\widetilde{A}_k$, $k \geqslant 1$.*

As a result, any infinite, irreducible Coxeter group W_S satisfying the conditions (i) and (ii) of Theorem 4.9 admits projective reflection groups, which are convex cocompact in $\mathbb{S}(\mathbb{R}^N)$ with $N = \#S$ (see [DGK+21, Th. 1.3]).

4.5. **Divisible and quasi-divisible domains.** Every properly convex open subset Ω of $\mathbb{S}(V)$ admits a Hilbert metric d_Ω on Ω so that the group $\mathrm{Aut}(\Omega)$ of automorphisms of $\mathbb{S}(V)$ preserving Ω acts on Ω by isometries for d_Ω. A properly convex domain Ω is *divisible* (resp. *quasi-divisible*) *by* Γ if there exists a discrete subgroup Γ of $\mathrm{Aut}(\Omega)$ such that Ω/Γ is compact (resp. of finite volume with respect to the Hausdorff measure induced by d_Ω). (see e.g. [Mar14] for more details for the Hilbert metric and the Hausdorff measure).

In general, it is difficult to construct divisible or quasi-divisible domains with various properties. But, in small dimension, one can use *perfect* or *quasi-perfect* projective Coxeter polytopes to build such domains. We first introduce the definition of 2-perfect polytopes, which is slightly more general than that of perfect or quasi-perfect polytopes, and in Section 5 we give some interesting examples of divisible domains.

Let P be a projective Coxeter polytope of $\mathbb{S}(V)$ and S the set of facets of P. Given a vertex v of P, we denote by S_v the set of facets that contain v. For any $s \in S_v$, the projective reflection σ_s induces a projective reflection $\overline{\sigma}_s$ of the projective space $\mathbb{S}(V/\langle v\rangle)$, where $\langle v\rangle$ is the subspace spanned by v and $V/\langle v\rangle$ is the quotient vector space. The projection of P to $\mathbb{S}(V/\langle v\rangle)$ with the reflections $(\overline{\sigma}_s)_{s\in S_v}$ define a projective Coxeter polytope P_v of $\mathbb{S}(V/\langle v\rangle)$, called the *link of P at v*.

Definition 4.11. A projective Coxeter d-polytope P is *elliptic* (resp. *parabolic*, resp. *loxodromic*) when each component of $\mathscr{A}_P$ is of positive type (resp. zero type, resp. negative type) and the rank of $\mathscr{A}_P$ is $d+1$ (resp. d, resp. $d+1$).

Remark 4.12. If P is elliptic, then W_P is a spherical Coxeter group and P is the Tits simplex associated with W_P. If P is parabolic, then W_P is an affine Coxeter group, P is the Cartesian product of the Tits simplices associated with the irreducible components of W_P, and Ω_P is an affine chart of $\mathbb{S}(V)$.

A projective Coxeter polytope P is *perfect* (resp. *quasi-perfect*, resp. *2-perfect*) when all its vertex links are elliptic (resp. elliptic or parabolic, resp. perfect). For example, quasi-perfect Coxeter polytopes should be 2-perfect.

Remark 4.13. By [Vin71, Prop. 26], a perfect Coxeter polytope is either elliptic, parabolic or irreducible loxodromic.

Let P be an irreducible loxodromic Coxeter polytope and Γ_P the projective reflection group of P. Then Ω_P is a properly convex domain, hence it admits a Hilbert metric d_{Ω_P}. By [Vin71, Th. 2], a projective Coxeter polytope P is perfect if and only if the action of Γ_P on Ω_P is cocompact. So, in this case, the domain Ω_P is divisible by Γ_P.

The action of Γ_P on Ω_P is said to be of *finite covolume* if $P \cap \Omega_P$ has finite volume with respect the Hausdorff measure μ_{Ω_P} induced by d_{Ω_P}, and *geometrically finite* if $\mu_{\Omega_P}(P \cap \mathscr{C}(\Lambda_{\Omega_P})) < \infty$, where Λ_P is the limit set of Γ_P and $\mathscr{C}(\Lambda_P)$ is the convex hull of Λ_P of Ω_P (see [Mar17] for more details).

Theorem 4.14 ([Mar17, Th. A])**.** *Let P be an irreducible, loxodromic, 2-perfect Coxeter polytope of $\mathbb{S}(V)$. Then the action of Γ_P on Ω_P is always geometrically finite, and*

- *Γ_P is of finite covolume if and only if P is quasi-perfect;*
- *Γ_P is convex cocompact in $\mathbb{S}(V)$ if and only if all the vertex links of P are elliptic or loxodromic.*

4.6. **Cocompact action of Coxeter groups.** There are many examples of discrete Coxeter subgroups of $\mathrm{SL}^{\pm}(V)$ other than projective reflection groups. However, if a Coxeter group Γ divides a properly convex domain, then $\Gamma = \Gamma_P$ for some projective Coxeter polytope P:

Theorem 4.15 (Davis [Dav08, Prop. 10.9.7] and Charney–Davis [CD00]; see [LM19, Lem. 5.4])**.** *Let W be a Coxeter group, and let $\rho : W \to \mathrm{SL}^{\pm}(V)$ be a faithful representation. Suppose that there exists a convex domain Ω divisible by $\rho(W)$. Then the following hold:*

(i) for each $s \in S$, the image $\rho(s)$ of s is a projective reflection of $\mathbb{S}(V)$;
(ii) $\rho(W)$ is a projective reflection group generated by $(\rho(s))_{s\in S}$.

Remark 4.16. It is an open question whether Theorem 4.15 still holds when the word "divisible" is replaced by "quasi-divisible".

5. Examples of projective reflection groups

The construction of projective reflection groups had led to several existence theorems in convex projective geometry.

A properly convex domain Ω of $\mathbb{S}(V)$ is *decomposable* if a cone of V lifting Ω is a non-trivial direct product of two smaller cones, and *homogeneous* if the group $\mathrm{Aut}(\Omega)$ acts transitively on Ω. Since the homogeneous quasi-divisible domains are well-understood by [Vin63, Koe99],

only inhomogeneous ones are of interest to us. So, all properly convex domains in this section are assumed to be inhomogeneous and indecomposable.

5.1. **Kac–Vinberg's example.** The first example of divisible 2-domain which is not a hyperbolic plane was found by Kac and Vinberg [KV67]. They used perfect projective Coxeter triangles P with Cartan matrix $\mathscr{A}_P = (\mathscr{A}_{i,j})_{i,j=1,2,3}$ such that (i) each entry $\mathscr{A}_{i,j}$ is an integer, (ii) $\det(\mathscr{A}_P) < 0$, and (iii) $\mathscr{A}_{1,2}\mathscr{A}_{2,3}\mathscr{A}_{3,1} \neq \mathscr{A}_{1,3}\mathscr{A}_{3,2}\mathscr{A}_{2,1}$. Here, the condition (i) implies that the projective reflection group Γ_P of P is a subgroup of $\mathrm{SL}(3,\mathbb{Z})$, (ii) implies that $\mathscr{A}_P$ is of negative type, and finally (iii) implies that Ω_P is not a hyperbolic plane (see Figure 1).

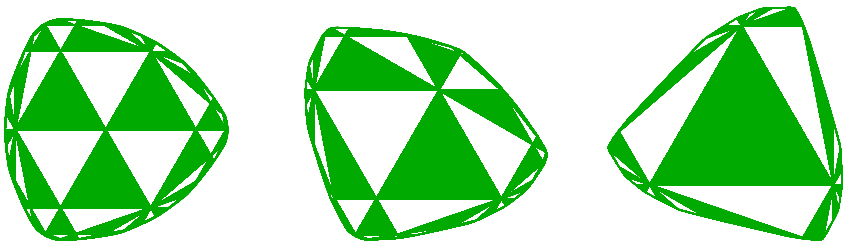

FIGURE 1. Triangles with dihedral angles $\pi/3$, $\pi/3$ and $\pi/6$ on the left, with dihedral angles $\pi/3$, $\pi/4$ and $\pi/6$ on the center, and with dihedral angles $\pi/6$, $\pi/6$ and $\pi/6$ on the right.

Remark 5.1. Let $\hat{\Gamma}_P$ be any finite-index torsion-free subgroup of Γ_P. Then $\hat{\Gamma}_P$ is an infinite index subgroup of $\mathrm{SL}(3,\mathbb{Z})$ and is Zariski dense in $\mathrm{SL}(3,\mathbb{R})$. In other words, $\hat{\Gamma}_P$ is a *thin* surface group (see [KLLR19] for an introduction to thin groups).

5.2. **Benoist's examples and more.** The first known examples of divisible d-domains Ω which are not strictly convex were introduced by Benoist [Ben06a] in dimension $d = 3,\dots,7$ (see Figure 2). In such examples, the discrete group Γ which divides Ω is relatively hyperbolic with respect to virtual $\mathbb{Z}^{d-1}$. Later, different examples of non-strictly convex divisible d-domains were found in [CLM20] in dimension $d = 4,\dots,8$, and the group Γ dividing such d-domain is relatively hyperbolic with respect to a collection of virtually free abelian subgroup of rank $< d-1$. Except in dimension 3 (see [BDL]), all the known examples were built from projective reflection groups.

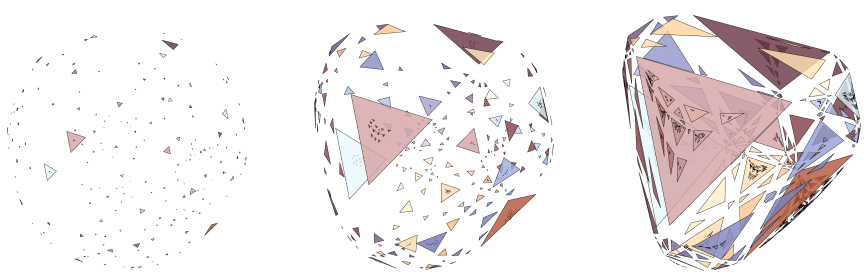

FIGURE 2. A collection of properly embedded triangles in the non-strictly convex divisible 3-domains is colored. Each triangle is preserved by a subgroup of Γ, which is virtually $\mathbb{Z}^2$.

Remark 5.2. A generalization of Thurston's hyperbolic Dehn filling theorem to the projective setting led to the examples in [CLM20].

In [Ben06b], Benoist found the first example of word-hyperbolic group Γ, not quasi-isometric to the hyperbolic space, that divides a properly convex 4-domain Ω, again using projective reflection groups. Since Γ is word hyperbolic, Ω should be strictly convex by [Ben04]. Shortly after, Kapovich [Kap07] found examples in any dimension $d \geqslant 4$, using Gromov–Thurston manifolds [GT87].

6. Hitchin component of polygon groups

Let P be a compact hyperbolic polygon with dihedral angles $\pi/m_1, \dots, \pi/m_k$, and let W be the Coxeter group of P. The conjugacy classes of discrete and faithful representations of W to $\mathrm{PGL}(2,\mathbb{R})$ form a connected component $\mathscr{T}$ of $\chi(W,\mathrm{PGL}(2,\mathbb{R})) := \mathrm{Hom}(W,\mathrm{PGL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$, i.e., the space of conjugacy classes of representations of W to $\mathrm{PGL}(2,\mathbb{R})$.

For any $n \geqslant 2$, there is a unique irreducible representation $\kappa : \mathrm{PGL}(2,\mathbb{R}) \to \mathrm{PGL}(n,\mathbb{R})$ up to conjugation. This gives rise to an embedding:

$$\mathscr{T} \to \chi(W,\mathrm{PGL}(n,\mathbb{R})) := \mathrm{Hom}(W,\mathrm{PGL}(n,\mathbb{R}))/\mathrm{PGL}(n,\mathbb{R})$$

The image of this embedding is called the *Fuchsian locus* and the component of $\chi(W,\mathrm{PGL}(n,\mathbb{R}))$ containing the Fuchsian locus is the *Hitchin component* $\mathrm{Hit}(W,\mathrm{PGL}(n,\mathbb{R}))$. A representation $\rho : W \to \mathrm{PGL}(n,\mathbb{R})$ is called a *Hitchin representation* if its $\mathrm{PGL}(n,\mathbb{R})$-conjugacy class is an element of $\mathrm{Hit}(W,\mathrm{PGL}(n,\mathbb{R}))$.

Theorem 6.1 ([ALS18, Th. 1.1 & 1.2])**.** *Let P be a compact hyperbolic polygon with dihedral angles $\pi/m_1, \dots, \pi/m_k$, and W its Coxeter group. Then each Hitchin representation in $\mathrm{Hit}(W,\mathrm{PGL}(n,\mathbb{R}))$ is discrete and faithfull, and $\mathrm{Hit}(W,\mathrm{PGL}(n,\mathbb{R}))$ is an open cell of dimension*

$$-(n^2-1) + \sum_{\ell=2}^{n} \sum_{i=1}^{k} \left\lfloor \ell \left(1 - \frac{1}{m_i}\right) \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the biggest integer not bigger than x.

For example, the $\mathrm{PGL}(2m,\mathbb{R})$ (resp. $\mathrm{PGL}(2m+1,\mathbb{R})$) Hitchin component of the Coxeter group associated with a right-angled hyperbolic k-gon ($k \geqslant 5$) is an open cell of dimension $(k-4)m^2+1$ (resp. $(k-4)(m^2+m)$).

Remark 6.2. In the case of $n = 2$ (resp. $n = 3$), Theorem 6.1 was proved by Thurston [Thu] (resp. Choi–Goldman [CG05]).

Remark 6.3. Let ρ be any Hitchin representation in $\mathrm{Hit}(W,\mathrm{PGL}(n,\mathbb{R}))$. In the case $n \geqslant 4$, the image of each generator of W should be an involution but not a projective reflection, hence ρ is not a projective reflection group.

7. Properly discontinuous affine groups

7.1. **Auslander's conjecture and Milnor's question.** In the 1960s, Auslander raised the following conjecture:

Conjecture 7.1 (Auslander [Aus64])**.** *Every discrete subgroup Γ of the affine group $\mathrm{Aff}(\mathbb{R}^d)$ which acts properly discontinuously and cocompactly on $\mathbb{R}^d$ is virtually solvable.*

In the 1970s, Milnor asked if Auslander's conjecture still holds without the condition that the action is cocompact:

Question 7.2 (Milnor [Mil77])**.** *Is every discrete subgroup* Γ *of* $\mathrm{Aff}(\mathbb{R}^d)$ *which acts properly discontinuously on* $\mathbb{R}^d$ *virtually solvable?*

In 1983, Fried and Goldman [FG83] showed that Auslander's conjecture is true in dimension 3, and Margulis answered Milnor's question negatively:

Theorem 7.3 (Margulis [Mar83, Mar87])**.** *There exists a properly discontinuous affine action of the free group on two generators on* $\mathbb{R}^3$.

Even if some progress have been made over the years towards Auslander's conjecture (see e.g. [Tom16, AMS20] for a proof assuming $d \leqslant 6$ and [GK84, Tom90, AMS10] for a proof assuming the linear part is contained in a particular class of semisimple Lie subgroups), Auslander's conjecture is still open.

Back to Milnor's question, the existence and property of properly discontinuous affine action of free groups on $\mathbb{R}^n$ have been actively studied (see e.g. [Dru92, CDG16, DGK15, GLM09, AMS02, Smi16b, Smi16a] or the survey [DDGS20]).

7.2. **Properly discontinuous affine Coxeter groups.** Before the following theorem, properly discontinuous affine actions by non-virtually solvable non-free groups were unknown.

Theorem 7.4 (Danciger–Guéritaud–Kassel [DGK20, Th. 1.1])**.** *Any right-angled Coxeter group of rank k admits a properly discontinuous affine action on* $\mathbb{R}^{k(k-1)/2}$.

Remark 7.5. The action preserves a bilinear form and in some particular cases, one can find much smaller affine space on which the Coxeter groups acts (see [DGK20, Prop. 1.6]).

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