

# DISCRETE COXETER GROUPS

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#### **DISCRETE COXETER GROUPS**

#### GYE-SEON LEE AND LUDOVIC MARQUIS

ABSTRACT. Coxeter groups are a special class of groups generated by involutions. They play important roles in the various areas of mathematics. This survey particularly focuses on how one use Coxeter groups to construct interesting examples of discrete subgroups of Lie group.

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### 1. INTRODUCTION

It is a fundamental problem in geometry and topology to understand discrete subgroups of Lie groups G. For example, when G is the isometry group  $\operatorname{Isom}(\mathbb{H}^d)$  of the hyperbolic d-space  $\mathbb{H}^d$ , the study of discrete subgroups of  $\operatorname{Isom}(\mathbb{H}^d)$  is closely related to that of complete hyperbolic d-manifolds. More precisely, there is a one-to-one correspondence between torsion-free discrete subgroups  $\Gamma$  of  $\operatorname{Isom}(\mathbb{H}^d)$  and complete hyperbolic d-manifolds  $\mathbb{H}^d/\Gamma$ .

Convex cocompact subgroups of rank-one Lie groups G are specially an important class of discrete subgroups of G. In particular, given a finitely generated group  $\Gamma$ , the space of representations  $\rho: \Gamma \to G$  whose image is convex cocompact is *open* in the representation space Hom( $\Gamma, G$ ), i.e. the space of all the representations  $\rho: \Gamma \to G$ . So, if  $\rho$  is convex cocompact and is not isolated, then all the nearby representations of  $\rho$  are again convex cocompact, and hence discrete.

Recently, new notions of representations are introduced to generalize convex cocompact subgroups of rank-one Lie groups: Anosov representations in real semisimple Lie groups (see [Lab06, GW12]) and convex cocompact subgroups in real projective space (see [DGK17a]). Such representations also have the property of openness. As new theories are developed, it is also important to have many examples to support them. From this perspective, the role of Coxeter groups is crucial.

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*Key words and phrases.* Coxeter groups, discrete subgroups of Lie groups, reflection groups, convex cocompact subgroups, Anosov representations.

The aim of this survey is to illustrate how one can build interesting examples of discrete Coxeter groups.

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# 2. COXETER GROUPS

2.1. What is a Coxeter group? A Coxeter matrix  $M = (m_{s,t})_{s,t\in S}$  on a finite set S is a symmetric matrix with entries  $m_{s,t} \in \{1, 2, ..., m, ..., \infty\}$  such that the diagonal entries  $m_{s,s} = 1$  and off-diagonal entries  $m_{s,t} \neq 1$ . From any Coxeter matrix  $M = (m_{s,t})_{s,t\in S}$ , one may obtain the Coxeter group W of M given by generators and relations:

$$W = \left\langle s \in S \mid (st)^{m_{s,t}} = 1, \ \forall s, t \in S \right\rangle$$

It is implicit that  $(st)^{\infty} = 1$  means no relation between *s* and *t*. Since  $m_{s,s} = 1$ , each generator *s* is an involution, i.e.  $s^2 = 1$ . We shall use the notation *W*,  $W_S$  or  $W_{S,M}$  for the Coxeter group, depending on what is important to stress. One should remember that a Coxeter group is a group with a preferred generating set, namely *S*. The *rank* of  $W_S$  is the cardinality #*S* of *S*.

The *Coxeter diagram* of the Coxeter group  $W_S$  is the labeled graph  $\mathscr{G}_W$  such that:

- (i) the set of nodes<sup>1</sup> of  $\mathscr{G}_W$  is the set *S*;
- (ii) two nodes  $s, t \in S$  are connected by an edge  $\overline{st}$  of  $\mathscr{G}_W$  if  $m_{s,t} \in \{3, 4, \dots, \infty\}$ ;
- (iii) the label of the edge *st* is  $m_{s,t}$  if  $m_{s,t} \in \{4, 5, \dots, \infty\}$ .

It is well-known that for any subset T of S, the subgroup of  $W_S$  generated by T is the Coxeter group  $W_{T,M'}$  with generating set T and exponents  $m'_{s,t} = m_{s,t}$  for every  $s, t \in T$  (see [Bou68, Chap. IV, Th. 2]). Such a subgroup  $W_T$  is called a *standard subgroup* of  $W_S$ .

The connected components of the graph  $\mathscr{G}_{W_S}$  are graphs of the form  $\mathscr{G}_{W_{S_i}}$ ,  $i \in I$ , where the  $(S_i)_{i \in I}$  form a partition of S. The standard subgroups  $W_{S_i}$  are called the *irreducible components* of  $W_S$ . Since  $m_{s,t} = 2$  if and only if st = ts, we see that the group  $W_S$  is the direct product of the subgroups  $W_{S_i}$  for  $i \in I$ . A Coxeter group  $W_S$  is *irreducible* when the Coxeter diagram  $\mathscr{G}_{W_S}$  is connected, i.e. #I = 1. A subset T of S is said to be "something" if the Coxeter group  $W_T$  is "something". For example, the word "something" can be replaced by "irreducible", and so on. Two subsets  $T, U \subset S$  are *orthogonal* if  $m_{t,u} = 2$  for every  $t \in T$  and every  $u \in U$ .

The Cosine matrix of  $W_{S,M}$  is the  $S \times S$  symmetric matrix  $C_W$  whose entries are:

$$(C_W)_{s,t} = -2\cos\left(\frac{\pi}{m_{s,t}}\right)$$
 for every  $s, t \in S$ 

An irreducible Coxeter group W is said to be *spherical* (resp. *affine*) when the Cosine matrix  $C_W$  is positive definite (resp. positive semi-definite but not definite).

**Theorem 2.1** (Coxeter [Cox32, Cox34] and Margulis-Vinberg [MV00]). Let  $W_S$  be an irreducible Coxeter group. Then exactly one of the following is true:

<sup>&</sup>lt;sup>1</sup>A Coxeter group often comes with a Coxeter polytope in such a way that the nodes of the Coxeter diagram are in bijection with the facets of the Coxeter polytope. We shall use the word *node* of the Coxeter diagram rather than *vertex* to make distinction between the vertices of the Coxeter polytope and the vertices of the Coxeter diagram.

- (i) If  $W_S$  is spherical, then  $W_S$  is a finite group.
- (ii) If  $W_S$  is affine, then  $\#S \ge 2$  and  $W_S$  is virtually<sup>2</sup>  $\mathbb{Z}^{\#S-1}$ .
- (iii) Otherwise,  $W_S$  is large, i.e. there exists a surjective homomorphism of a finite index subgroup of  $W_S$  onto a free group on two generators.

*Remark* 2.2. These three cases are clearly exclusive. Consequently, if an irreducible Coxeter group  $W_S$  is finite (resp. infinite and virtually abelian), then  $W_S$  is spherical (resp. affine).

*Remark* 2.3. The irreducible spherical and irreducible affine Coxeter groups are classified by Coxeter [Cox32, Cox34]; see also Witt [Wit41]. The complete list can be found in Table 1.

A Coxeter group (not necessarily irreducible) is *spherical* (resp. *affine*) when all its irreducible components are *spherical* (resp. *affine*).

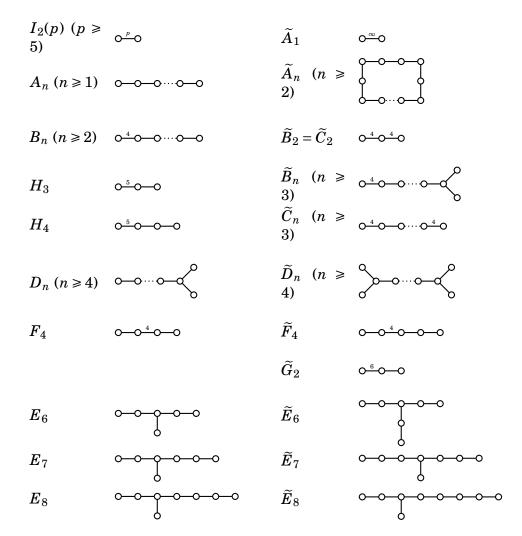


TABLE 1. The irreducible spherical Coxeter diagrams on the left and irreducible affine Coxeter diagrams on the right.

<sup>&</sup>lt;sup>2</sup>A group *G* is *virtually "something"* if there is a finite index subgroup  $H \leq G$  such that *H* is "something".

### 3. Hyperbolic reflection groups

3.1. **Hyperbolic polytopes.** Let  $\mathbb{R}^{d,1}$  be the vector space  $\mathbb{R}^{d+1}$  endowed with a non-degenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  of signature<sup>3</sup> (*d*, 1), and let *q* be the associated quadratic form. A coordinate representation of *q* with respect to some basis of  $\mathbb{R}^{d,1}$  is:

$$q(x) = x_1^2 + \dots + x_d^2 - x_{d+1}^2.$$

A hyperbolic *d*-space  $\mathbb{H}^d$  is a connected component of a hyperquadric:

$$\mathbb{H}^{d} = \{ x \in \mathbb{R}^{d,1} \mid q(x) = -1 \text{ and } x_{d+1} > 0 \}.$$

The isometry group of  $\mathbb{H}^d$  is  $O_{d,1}^+(\mathbb{R})$ , which consists of the elements of  $O_{d,1}(\mathbb{R})$  that preserve  $\mathbb{H}^d$ . We often work with the projective model of  $\mathbb{H}^d$ :

$$\mathbb{H}^{d} = \{x \in \mathbb{R}^{d,1} \mid q(x) < 0 \text{ and } x_{d+1} > 0\} / \mathbb{R}_{+}$$

where the set  $\mathbb{R}_+$  of positive scalars acts on  $\mathbb{R}^{d,1} \setminus \{0\}$  by multiplication. If we set

$$\mathbb{S}(\mathbb{R}^{d,1}) := (\mathbb{R}^{d,1} \setminus \{0\})/\mathbb{R}_+,$$

then  $\mathbb{H}^d$  is an open subset of the *projective sphere*  $\mathbb{S}(\mathbb{R}^{d,1})$ . The closure  $\overline{\mathbb{H}^d}$  of  $\mathbb{H}^d$  in  $\mathbb{S}(\mathbb{R}^{d,1})$  is the *compactification* of  $\mathbb{H}^d$ .

A subset of  $\mathbb{H}^d$  is a hyperbolic *d*-polytope if it is the intersection of a finite family of closed half-spaces of  $\mathbb{H}^d$  and it has non-empty interior. A hyperbolic Coxeter *d*-polytope (or simply a  $\mathbb{H}^d$ -Coxeter polytope) is a hyperbolic *d*-polytope *P* all of whose dihedral angles are submultiples of  $\pi$ . In other words, if two facets<sup>4</sup> s, t of *P* are adjacent,<sup>5</sup> then the dihedral angle  $\theta(s,t)$  between s and t equals  $\pi/m$  for some integer  $m \ge 2$ . When two facets s,t are parallel, it is common to say that  $\theta(s,t) = 0$ . A hyperbolic Coxeter polytope is right-angled if all its dihedral angles are  $\pi/2$ .

Associated with a hyperbolic Coxeter polytope P is a Coxeter matrix  $M = (m_{s,t})_{s,t \in S}$  on the set S of facets of P: if  $s, t \in S$  are adjacent, then  $m_{s,t} = \pi/\theta(s,t)$ ; otherwise,  $m_{s,t} = \infty$ . We denote by  $W_P$  the Coxeter group of the Coxeter matrix M, and call it the *Coxeter group of* P. If s is a facet of P, then  $\sigma_s$  denotes the reflection in the hyperplane containing s.

**Theorem 3.1** (Poincaré [Poi83]). Let P be a  $\mathbb{H}^d$ -Coxeter polytope, and  $W_P$  the Coxeter group of P. Then the homomorphism  $\sigma: W_P \to \text{Isom}(\mathbb{H}^d)$  defined by

$$\sigma(s) = \sigma_s \quad for \ each \ s \in S$$

is injective and the image  $\Gamma_P := \sigma(W_P)$  is discrete. Moreover, P is a fundamental domain for the action of  $\Gamma_P$  on  $\mathbb{H}^d$ . In particular, if P has finite volume (resp. is compact), then  $\Gamma_P$  is a lattice (resp. uniform lattice) of Isom( $\mathbb{H}^d$ ).

A subgroup H of Isom( $\mathbb{H}^d$ ) is called a *hyperbolic reflection group* if  $H = \Gamma_P$  for some  $\mathbb{H}^d$ -Coxeter polytope P. In this case, we call  $\Gamma_P$  the *reflection group of* P. Theorem 3.1 provides a nice way to construct discrete subgroups of Isom( $\mathbb{H}^d$ ), even lattices. We shall review in the next section the classification of  $\mathbb{H}^d$ -Coxeter polytopes of finite volume in small dimension and their non-existence in large dimension.

<sup>&</sup>lt;sup>3</sup>The *signature* of a symmetric matrix *B* is the triple (p,q,r) of the positive, negative, and zero indices of inertia of *B*. In the case r = 0, we simply say that *B* is of signature (p,q).

<sup>&</sup>lt;sup>4</sup>A face of *P* of codimension 1 (resp. 2) is called a *facet* (resp. *ridge*) of *P*.

<sup>&</sup>lt;sup>5</sup>Two facets *s* and *t* of *P* are *adjacent* if  $s \cap t$  is a *ridge* of *P*.

## 3.2. Classical results in dimension 2 and 3.

3.2.1. *Dimension 2*. A necessary and sufficient condition for the existence of hyperbolic Coxeter 2-polytopes of finite volume follows immediately from:

**Theorem 3.2.** Let  $\theta_1, \ldots, \theta_n$  be real numbers such that  $0 \le \theta_i < \pi$  for each  $i = 1, \ldots, n$ . Then there exists a hyperbolic polygon of finite volume with dihedral angles  $\theta_1, \ldots, \theta_n$  if and only if  $\sum_{i=1}^n \theta_i < (n-2)\pi$ .

3.2.2. Dimension 3. Hyperbolic Coxeter 3-polytopes of finite volume are well understood, notably thanks to the classification of hyperbolic 3-polytopes with dihedral angle  $\leq \pi/2$  due to Andreev [And71a, And71b]. During his PhD, Roeder found and fixed a gap in the original proof of Andreev (see [RHD07]). Hodgson and Rivin [HR93] gave a characterization of hyperbolic 3-polytopes, which generalizes Andreev's theorem, in terms of a generalized Gauss map.

To express properly Andreev's theorem, one needs some definitions. Two compact polytopes  $\mathscr{P}, \mathscr{P}'$  of the Euclidean space  $\mathbb{R}^d$  are *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. A combinatorial equivalence class is called a *combinatorial polytope*. Note that if a hyperbolic polytope  $P \subset \mathbb{H}^d$  is of finite volume, then the closure  $\overline{P}$  of P in  $\overline{\mathbb{H}^d}$  is combinatorially equivalent to a compact polytope of  $\mathbb{R}^d$ . A *labeled polytope* is a combinatorial polytope  $\mathscr{P}$  with a labeling  $\theta$ , which is a function from the ridges of  $\mathscr{P}$  to  $(0, \pi/2]$ . A hyperbolic polytope P of finite volume *realizes* a labeled polytope  $\mathscr{P}$  if there exists a combinatorial equivalence  $\phi$  between the faces of  $\overline{P}$  and  $\mathscr{P}$  such that the dihedral angle at the ridge e of  $\overline{P}$  is the label  $\theta(\phi(e))$  of  $\mathscr{P}$ .

Let  $\mathscr{P}$  be a labeled 3-polytope with labeling  $\theta$ . A *k*-circuit of  $\mathscr{P}$  is a sequence of distinct facets  $s_1, \ldots, s_k$  such that  $e_i := s_i \cap s_{i+1}$  (indices are modulo k) is an edge of  $\mathscr{P}$ . A *k*-circuit is *prismatic* if all the (closed) edges  $e_i$  are disjoint. The *angle sum* of a *k*-circuit is the real number  $\sum_{i=1}^{k} \theta(e_i)$ . A *k*-circuit is *spherical* (resp. *Euclidean*, resp. *hyperbolic*) if its angle sum is bigger than (resp. equal to, resp. less than)  $(k-2)\pi$ . A vertex v of  $\mathscr{P}$  is *spherical* (resp. *Euclidean*) if the circuit consisting of the facets that contain v is spherical (resp. Euclidean). The graph  $W_{\mathscr{P}}$  of  $\mathscr{P}$  is the graph whose nodes are the facets of  $\mathscr{P}$  and such that two nodes s, t are connected if and only if  $s \cap t$  is not an edge, or  $s \cap t$  is an edge and  $\theta(s \cap t) < \pi/2$ .

**Theorem 3.3** (Andreev [And71a, And71b]; see also [RHD07]). Let  $\mathcal{P}$  be a labeled 3-polytope whose underlying polytope is not a tetrahedron. Then there exists a compact (resp. finite volume) hyperbolic 3-polytope P that realizes  $\mathcal{P}$  if and only if:

- (i) all the vertices of  $\mathcal{P}$  are spherical (resp. spherical or Euclidean);
- (ii) all the prismatic 3- and 4-circuits of  $\mathcal{P}$  are hyperbolic;
- (iii) the graph  $W_{\mathscr{P}}$  of  $\mathscr{P}$  is connected.

In that case, the polytope P is unique up to isometry of  $\mathbb{H}^3$ .

*Remark* 3.4. The careful reader probably wonder what happen when the underlying polytope of  $\mathscr{P}$  is a tetrahedron. This case is explained in [Roe06]. And the list of all Coxeter tetrahedra of finite volume can be found in Tables 3 and 4.

*Remark* 3.5. The condition on the connectivity of  $W_{\mathscr{P}}$  is often expressed by inequalities. One may notice that if the conditions (*i*) and (*ii*) are satisfied, then the disconnectedness of  $W_{\mathscr{P}}$  implies that (see [RHD07, Prop 1.5]):

- $\mathscr{P}$  is a right triangular prism, i.e. all the labels between the base facets and the joining facets are  $\pi/2$ , or
- $\mathscr{P}$  is a quadrilateral pyramid with Euclidean apex such that the labels of two opposite edges in the base are  $\pi/2$ .

#### 3.3. The Gram matrix of hyperbolic polytope. A hyperbolic *d*-polytope *P* is of the form

$$P = \cap_{i=1}^N H_i^-,$$

where  $H_i^-$  is a closed half-space of  $\mathbb{H}^d$  whose boundary is a hyperplane  $H_i$ . Each  $H_i^-$  corresponds to a unique vector  $u_i$  with the property that:

$$H_i^- = \mathbb{H}^d \cap \{x \in \mathbb{R}^{d,1} \mid \langle x, u_i \rangle \leq 0\} \text{ and } \langle u_i, u_i \rangle = 1.$$

The Gram matrix  $G = (g_{i,j})$  of P is a symmetric  $N \times N$  matrix with entries  $g_{i,j} = \langle u_i, u_j \rangle$ . A square matrix B is *reducible* if B is the direct sum of smaller square matrices  $B_1$  and  $B_2$  (after a reordering of the indices), i.e.  $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ . Otherwise, B is *irreducible*. Every square matrix B shall be the direct sum of irreducible submatrices, each of which we call a *component* of B. The next theorem tells us that the hyperbolic polytopes are determined by the Gram matrices.

**Theorem 3.6** ([Vin85, Th. 2.1]). Let  $G = (g_{i,j})$  be an irreducible symmetric  $N \times N$  matrix of signature (d, 1, N - d - 1) such that the diagonal entries  $g_{i,i} = 1$  and off-diagonal entries  $g_{i,j} \leq 0$ . Then there exists a hyperbolic d-polytope P whose Gram matrix is G, and the polytope P is unique up to isometry of  $\mathbb{H}^d$ .

*Remark* 3.7. A detailed analysis of the Gram matrix can reveal the combinatorial structure of the hyperbolic polytope P (see [Vin85, Th. 3.1 & 3.2]) and whether P is compact or of finite volume (see [Vin85, Th. 4.1]).

3.4. Lannér and quasi-Lannér Coxeter groups. A Coxeter group  $W_S$  is Lannér (resp. quasi-Lannér) if det $(C_W) < 0$  and for every proper subset  $T \subset S$ , the Coxeter group  $W_T$  is spherical (resp. spherical or irreducible affine). If P is a compact (resp. finite volume) Coxeter simplex, then the Coxeter group  $W_P$  is Lannér (resp. quasi-Lannér). Conversely, if W is a Lannér (resp. quasi-Lannér) Coxeter group of rank d + 1, then its Cosine matrix  $C_W$  has signature (d, 1) and there exists a compact (resp. finite volume)  $\mathbb{H}^d$ -Coxeter polytope whose Gram matrix is  $\frac{1}{2}C_W$ .

In short, Lannér (resp. quasi-Lannér) Coxeter groups correspond to compact (resp. finite-volume)  $\mathbb{H}^d$ -Coxeter simplices. Such Coxeter simplices exist only in small dimension.

**Theorem 3.8** (Lannér [Lan50], Koszul [Kos67] and Chein [Che69]). If  $W_S$  is a Lannér (resp. quasi-Lannér) Coxeter group, then  $\#S \leq 5$  (resp.  $\#S \leq 10$ ). Table 2 indicates the number of Lannér (resp. quasi-Lannér) of a given rank. The list of Lannér and quasi-Lannér Coxeter groups of rank  $\geq 4$  can be found in [Che69].

The non-existence of hyperbolic Coxeter simplices in large dimension is in fact the first side of a more general non-existence theorem in Section 3.5.

*Remark* 3.9. Tables 3 and 4 give us the list of all the Lannér or quasi-Lannér Coxeter groups of rank 4, which correspond to compact or finite volume hyperbolic tetrahedra, respectively.

#### DISCRETE COXETER GROUPS

$\begin{array}{c} Dimension \\ d=\#S-1 \end{array}$	♯ of quasi-Lannér not Lannér	♯ of Lannér Coxeter groups
2	$\infty$	$\infty$
3	23	9
4	9	5
5	12	0
6	3	0
7	4	0
8	4	0
9	3	0

TABLE 2. The numbers of quasi-Lannér or Lannér Coxeter groups

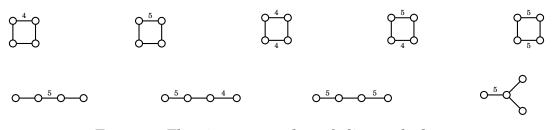


TABLE 3. The nine compact hyperbolic tetrahedra

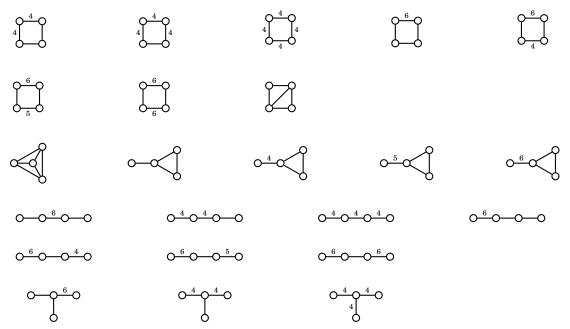


TABLE 4. The twenty-three hyperbolic tetrahedra of finite volume which are not compact

# 3.5. Absence in large dimension.

**Theorem 3.10** (Vinberg [Vin84] and Prokhorov [Pro86]). If  $\Gamma_P$  is a discrete reflection group of Isom( $\mathbb{H}^d$ ) with compact (resp. finite volume) fundamental domain P, then  $d \leq 29$  (resp.  $d \leq 995$ ).

The proof of Vinberg (resp. Prokhorov) used Nikulin's inequality for simple polytope (resp. edge-simple polytope) established in [Nik81] (resp. [Kho86]). The upper bounds in the right-angled case are better:

**Theorem 3.11** (Potyagailo-Vinberg [PV05] and Dufour [Duf10]). If  $\Gamma_P$  is a discrete reflection group of Isom( $\mathbb{H}^d$ ) with compact (resp. finite volume) right-angled fundamental domain P, then  $d \leq 4$  (resp.  $d \leq 12$ ).

Except for compact right-angled polytopes, the upper bounds are far from being sharp.

- There exists a compact right-angled hyperbolic 4-polytope: 120-cell.
- Examples of finite volume right-angled *d*-polytopes are known in dimension  $d \le 8$  (see [PV05]).
- Examples of compact *d*-polytope are known in dimension  $d \le 8$  (see [Bug84, Bug92] for d = 7, 8).
- Examples of finite volume *d*-polytope are known in dimension  $d \le 21$  and  $d \ne 20$  (see [All06] and reference therein for  $d \le 19$  and [Bor87] for d = 21).

*Remark* 3.12. Moussong observed that the argument of Vinberg [Vin84] may extend to show that if a Coxeter group W is word hyperbolic and the nerve of W is a generalized homology (d-1)-sphere, then  $d \leq 29$  (see [Dav08, Prop. 12.6.7]). Here, the *nerve* of a Coxeter group  $W_S$  is the poset of all nonempty spherical subset of S partially ordered by inclusion, which is an abstract simplicial complex, and a *generalized homology d-sphere* is a homology *d*-manifold with the same homology as the *d*-sphere.

3.6. **Hyperbolic Coxeter polytopes with few facets.** The complete classification of all compact or finite-volume hyperbolic polytopes is not an easy task. Only compact *d*-polytopes with *N* facets ( $N \le d+3$ ) and finite-volume *d*-polytopes with *N* facets ( $N \le d+2$ ) were classified. For more details, we refer the reader to the web page maintained by Felikson and Tumarkin:

https://www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html This webpage contains all the known examples of hyperbolic Coxeter polytopes of dimension  $\ge 4$ .

3.7. Convex cocompact hyperbolic reflection groups. A subgroup  $\Gamma$  of Isom( $\mathbb{H}^d$ ) is *convex cocompact* if there exists a  $\Gamma$ -invariant convex subset  $\mathscr{C}$  of  $\mathbb{H}^d$  such that  $\Gamma$  acts properly discontinuously on  $\mathscr{C}$  with compact quotient. Using the following theorems, one can easily check when a reflection group  $\Gamma_P$  of Isom( $\mathbb{H}^d$ ) is convex cocompact.

**Theorem 3.13** ([DH13, Th. 4.12]). Let P be a  $\mathbb{H}^d$ -Coxeter polytope and let  $\Gamma_P$  be its reflection group. Then  $\Gamma_P$  is convex cocompact if and only if

- (i)  $\Gamma_P$  is word-hyperbolic, and
- (ii) P has no pair of asymptotic facets.

*Remark* 3.14. The condition (*ii*) may be replaced by (*ii'*) there is no pair of facets s, t of P such that  $g_{s,t} = -1$ , where  $G = (g_{s,t})$  is the Gram matrix of P.

**Theorem 3.15** (Moussong's hyperbolicity criterion [Mou88]). Let  $W_S$  be a Coxeter group. Then  $W_S$  is word hyperbolic if and only if S does not contain two orthogonal non-spherical subsets, nor any affine subset of rank  $\geq 3$ .

*Remark* 3.16. Desgroseilliers and Haglund [DH13, Th. 1.1] found a class of Coxeter groups which can be realized as convex cocompact subgroup of  $\text{Isom}(\mathbb{H}^d)$ , which is not a reflection group, and they conjectured that there exists a word-hyperbolic Coxeter group which admits a convex cocompact representation into  $\text{Isom}(\mathbb{H}^d)$  but which cannot be realized as convex cocompact *reflection* group of  $\text{Isom}(\mathbb{H}^n)$  for any  $n \in \mathbb{N}$ .

#### 4. PROJECTIVE REFLECTION GROUPS

4.1. **Tits-Vinberg's Theorem.** Let *V* be a vector space over  $\mathbb{R}$ , and let  $\mathbb{S}(V)$  be the projective sphere. We denote by  $\mathrm{SL}^{\pm}(V)$  the group of automorphism of  $\mathbb{S}(V)$ , i.e.

$$\mathrm{SL}^{\pm}(V) = \{g \in \mathrm{GL}(V) \mid \det(g) = \pm 1\}.$$

We denote by  $\hat{S}$  the natural projection of  $V \setminus \{0\}$  to S(V), and let  $S(W) := \hat{S}(W \setminus \{0\})$  for any subset W of V. The complement of a projective hyperplane in S(V) consists of two connected components, each of which we call an *affine chart* of S(V). A *cone* is a subset of V which is invariant under multiplication by positive scalars. A subset  $\mathscr{C}$  of S(V) is *convex* if there exists a convex cone U of V such that  $\mathscr{C} = S(V)$ , *properly convex* if it is convex and its closure lies in some affine chart, and *strictly convex* if in addition its boundary does not contain any nontrivial projective line segment. Hyperbolic spaces are special examples of strictly convex open subsets of S(V).

A projective polytope is a properly convex subset P of  $\mathbb{S}(V)$  such that P has a non-empty interior and  $P = \bigcap_{i=1}^{N} \mathbb{S}(\{x \in V \mid \alpha_i(x) \leq 0\})$ , where  $\alpha_i$ , i = 1, ..., N, are linear forms on V. We always assume that P has N facets, i.e. to define P, we need all the N linear forms  $(\alpha_i)_{i=1}^N$ . A projective reflection is an element of  $SL^{\pm}(V)$  of order 2 which is the identity on a hyperplane. Every projective reflection  $\sigma$  can be written as:

$$\sigma = \operatorname{Id} - \alpha \otimes v$$
, i.e.  $\sigma(x) = x - \alpha(x)v \quad \forall x \in V$ ,

where  $\alpha$  is a linear form on *V* and *v* is a vector of *V* such that  $\alpha(v) = 2$ .

Let *P* be a projective polytope and *S* the set of facets of *P*. A reflection in a facet  $s \in S$  is a projective reflection  $\sigma_s$  which fixes each point of *s*. A pre-mirror polytope is a projective polytope *P* together with one reflection  $\sigma_s$  in each facet *s* of *P*. So, one may choose  $\sigma_s =$  $\mathrm{Id} - \alpha_s \otimes v_s$  with  $\alpha_s(v_s) = 2$  such that  $P = \bigcap_{s \in S} \mathbb{S}(\{x \in V \mid \alpha_s(x) \leq 0\})$ . Note that the couples  $(\alpha_s, v_s)$  are uniquely determined only up to multiplication by a positive real numbers.

If *P* is a pre-mirror polytope, then  $\Gamma_P$  denotes the group generated by the reflections in the facets of *P*. We say that  $\Gamma_P$  is a *projective reflection group* if for any  $\gamma \in \Gamma_P$ ,

$$\gamma(\check{P})\cap\check{P}\neq\varnothing\quad\Rightarrow\quad\gamma=\mathrm{Id},$$

where  $\mathring{P}$  denotes the interior of P.

In the next paragraph, we introduce a relevant tool to formulate Proposition 4.2 and Theorem 4.3 which express necessary and sufficient conditions for  $\Gamma_P$  of a pre-mirror polytope P to be a projective reflection group. A key is the notion of Cartan matrix of mirror polytope which generalizes the twice of the Gram matrix of hyperbolic polytope. **Definition 4.1.** A *Cartan matrix* on a set *S* is a  $S \times S$  matrix  $\mathscr{A}_S = (a_{s,t})_{s,t \in S}$  which satisfies the conditions: (*i*)  $a_{s,s} = 2$ ,  $\forall s \in S$ ; (*ii*)  $a_{s,t} \leq 0$ ,  $\forall s \neq t \in S$ ; (*iii*)  $a_{s,t} = 0 \Leftrightarrow a_{t,s} = 0$ ,  $\forall s \neq t \in S$ .

A *mirror polytope* is a pre-mirror polytope *P* such that the matrix  $\mathscr{A}_P := (\alpha_s(v_t))_{s,t\in S}$  is a Cartan matrix. In this case, we call  $\mathscr{A}_P$  the *Cartan matrix of P*. A Cartan matrix  $\mathscr{A}$  on *S* is of *type Coxeter* if for any  $s \neq t \in S$ ,

$$a_{s,t}a_{t,s} < 4 \quad \Rightarrow \quad \frac{\pi}{\arccos\left(\frac{1}{2}\sqrt{a_{s,t}a_{t,s}}\right)} \in \mathbb{N}.$$

A projective Coxeter polytope is a mirror polytope P whose Cartan matrix  $\mathscr{A}_P$  is of type Coxeter. For each pair of adjacent facets s,t of P, the *dihedral angle* of the ridge  $s \cap t$  is said to be  $\pi/m_{s,t}$  if  $a_{s,t}a_{t,s} = 4\cos^2(\pi/m_{s,t})$ .

**Proposition 4.2** ([Vin71, Prop. 17]). Let P be a pre-mirror polytope. If the group  $\Gamma_P$  is a projective reflection group, then P is a projective Coxeter polytope.

A Cartan matrix  $\mathscr{A}_S$  and a Coxeter group  $W_S$  are *compatible* when:

(i)  $\forall s, t \in S$ ,  $m_{s,t} = 2 \Leftrightarrow a_{s,t} = 0$ ; (ii)  $\forall s, t \in S$ ,  $m_{s,t} < \infty \Leftrightarrow a_{s,t}a_{t,s} = 4\cos^2(\pi/m_{s,t})$ ; (iii)  $\forall s, t \in S$ ,  $m_{s,t} = \infty \Leftrightarrow a_{s,t}a_{t,s} \ge 4$ .

It is clear that there is at most one Coxeter group compatible with a given Cartan matrix, and that a Cartan matrix  $\mathscr{A}_S$  is compatible with some Coxeter group  $W_S$  if and only if  $\mathscr{A}_S$  is of Coxeter type. If P is a projective Coxeter polytope, then  $W_P$  denotes the unique Coxeter group compatible with  $\mathscr{A}_P$ . The following is a generalization of Theorem 3.1 to the projective setting.

**Theorem 4.3** ([Bou68, Chap. V] and [Vin71, Th. 2]). <sup>6</sup> Let P be a projective Coxeter polytope of S(V) with Coxeter group  $W_P$ , and let  $\Gamma_P$  be the group generated by the projective reflections  $(\sigma_s)_{s \in S}$  in the facets of P. Then the following hold:

- (i) the homomorphism  $\sigma: W_P \to SL^{\pm}(V)$  defined by  $\sigma(s) = \sigma_s$  is an isomorphism onto  $\Gamma_P$ ;
- (ii) the group  $\Gamma_P$  is a discrete projective reflection group;
- (iii) the union  $\mathscr{C}_P$  of the  $\Gamma_P$ -translates of P is a convex subset of  $\mathbb{S}(V)$ ;
- (iv) if  $\Omega_P$  is the interior of  $\mathscr{C}_P$ , then  $\Gamma_P$  acts properly discontinuously on  $\Omega_P$ .

4.2. From Cartan matrix to mirror polytope. Given a Cartan matrix  $\mathscr{A}_S$ , there is a simple process to build a canonical mirror polytope  $\Delta_{\mathscr{A}}$  such that  $\mathscr{A}_{\Delta_{\mathscr{A}}} = \mathscr{A}_S$ . In this construction,  $\Delta_{\mathscr{A}}$  will be a simplex of dimension #S - 1.

Let  $V = \mathbb{R}^S$ . We denote by  $(e_s)_{s \in S}$  the canonical basis of V and  $(e_s^*)_{s \in S}$  its dual basis. We set  $\alpha_s := e_s^*$  and  $v_s := \sum_t \mathscr{A}_{t,s} e_t$ , i.e.  $v_s$  is the *s*-column vector of  $\mathscr{A}_S$ . Hence, by taking  $\Delta$  to be the (projectivization of) negative quadrant in  $\mathbb{S}(V)$  and  $\sigma_s = \mathrm{Id} - \alpha_s \otimes v_s$  to be the reflection in the facet  $\Delta \cap \mathbb{S}(\mathrm{Ker} \, \alpha_s)$ , we obtain a mirror polytope  $\Delta_{\mathscr{A}}$  whose underlying polytope is a simplex of dimension #S - 1. We call  $\Delta_{\mathscr{A}}$  the *mirror simplex associated to*  $\mathscr{A}_S$ .

In the case where  $W_S$  is any Coxeter group and  $\mathscr{A}_S = C_W$ , the mirror simplex associated to  $\mathscr{A}_S$  is called the *Tits simplex associated to*  $W_S$  and denoted by  $\Delta_W$ . The corresponding

<sup>&</sup>lt;sup>6</sup>Theorem 4.3 was proved by Tits for  $\Delta_W$ , which we define in Section 4.2, and by Vinberg for the general case.

representation  $\sigma: W_S \to S(\mathbb{R}^S)$  is dual to Tits geometric representation described in [Bou68].

For example, if W is a spherical (resp. irreducible affine), then  $\Delta_W$  gives rise to the classical tiling of  $\mathbb{S}(V)$  (resp. of an affine chart) with  $\Gamma_{\Delta_W}$  in the isometry group of sphere (resp. Euclidean space). If W is Lannér (resp. quasi-Lannér), then  $\Omega_{\Delta_W}$  is the projective model of the hyperbolic space and  $\Delta_W$  (resp.  $\Delta_W \cap \Omega_{\Delta_W}$ ) is the hyperbolic Coxeter polytope whose Coxeter group is W.

An irreducible Cartan matrix  $\mathscr{A}$  has a simple eigenvalue  $\lambda_{\mathscr{A}}$  which corresponds to an eigenvector with positive entries and has the smallest modulus among the eigenvalues of  $\mathscr{A}$ , by the Perron-Frobenius theorem. We say that  $\mathscr{A}$  is of *positive*, *zero* or *negative type* when  $\lambda_{\mathscr{A}}$  is positive, zero or negative, respectively. For example, the Gram matrix of a hyperbolic polytope of finite volume is always of negative type. Now, the following is a generalization of Theorem 3.6 to the projective setting.

**Theorem 4.4** ([Vin71, Cor. 1]). Let  $\mathscr{A}$  be a Cartan matrix of size  $N \times N$ . Assume that  $\mathscr{A}$  is irreducible, of negative type and of rank d + 1. Then there exists a mirror d-polytope P with N facets unique up to automorphism of  $\mathbb{S}(\mathbb{R}^{d+1})$  such that  $\mathscr{A}_P = \mathscr{A}$ .

*Remark* 4.5. Theorem 4.4 is not explicitly stated in [Vin71, Cor. 1] for non-Coxeter polytopes, but may be proved from [Vin71, Prop. 13 & 15].

4.3. **Anosov reflection groups.** Anosov representations are discrete representations of word-hyperbolic groups into semisimple Lie group with good dynamical properties. They have received a lot of attention and have been much studied recently (see e.g. [Lab06, GW12] for the definition of Anosov representation). But examples of Anosov representations of word hyperbolic groups, which are more complicated than free groups and surface groups, into Lie group of higher rank are less known. The following theorem, which generalizes Theorem 3.13, tells us that any infinite, word hyperbolic, irreducible Coxeter groups admit Anosov representations.

**Theorem 4.6** ([DGK<sup>+</sup>21, Cor. 1.18]). Let P be a projective Coxeter polytope of S(V) with Coxeter group  $W_S$ . Suppose that  $W_S$  is word-hyperbolic. Then the following are equivalent:

- the representation  $\sigma: W_S \to SL^{\pm}(V)$  defined by  $\sigma(s) = \sigma_s$  is  $P_1$ -Anosov (i.e. Anosov with respect to the stabilizer of a line in V);
- $\mathcal{A}_{s,t}\mathcal{A}_{t,s} > 4$  for all  $s \neq t$  with  $m_{s,t} = \infty$ .

*Remark* 4.7. Anosov reflection groups in O(p,q) can be used to give a new proof of Theorem 3.15 (Moussong's hyperbolicity criterion); see [DGK17b, LM19].

**Theorem 4.8** ([LM19, Th. A]). In dimension d = 4, ..., 8, there exists a projective Coxeter polytope of  $S(\mathbb{R}^{d+2})$  with Coxeter group  $W_S$  such that:

- the group  $W_S$  is word-hyperbolic and its boundary is a (d-1)-sphere;
- the image of the representation  $\sigma: W_S \to SL^{\pm}(\mathbb{R}^{d+2})$  defined by  $\sigma(s) = \sigma_s$  lies in  $O_{d,2}(\mathbb{R})$ ;
- the representation  $\sigma: W_S \to O_{d,2}(\mathbb{R})$  is *P*-Anosov, where *P* is the stabilizer of an isotropic line;
- the group  $W_S$  is not quasi-isometric to  $\mathbb{H}^4$ .

4.4. **Convex cocompact projective reflection groups.** An infinite discrete subgroup  $\Gamma$  of  $SL^{\pm}(V)$  is *convex cocompact in* S(V) if it acts properly discontinuously on some properly convex open subset  $\Omega$  of S(V) and cocompactly on a nonempty  $\Gamma$ -invariant closed convex subset  $\mathscr{C}$  of  $\Omega$  whose closure in S(V) contains all accumulation points of all possible  $\Gamma$ -orbits  $\Gamma \cdot y$  with  $y \in \Omega$ .

The notion of Convex cocompactness in S(V) introduced in [DGK17a], in some sense, generalizes that of Anosov representation, but it does not require that the group  $\Gamma$  is word hyperbolic. There is also a simple characterization of convex cocompactness for projective reflection groups:

**Theorem 4.9** ([DGK<sup>+</sup>21, Th. 1.3]). Let P be a projective Coxeter polytope of S(V) with infinite irreducible Coxeter group  $W_S$ , and  $\sigma: W_S \to SL^{\pm}(V)$  the representation defined by  $\sigma(s) = \sigma_s$ . If  $\sigma(W_S)$  is convex cocompact in S(V), then  $W_S$  satisfies the following two conditions:

- (i) S does not contain two orthogonal non-spherical subsets;
- (ii) if S contains an irreducible affine subset T of rank  $\geq 3$ , then  $W_T$  is of type  $\tilde{A}_k$  where k = #T 1.

**Theorem 4.10** ([DGK<sup>+</sup>21, Th. 1.8]). Let *P* be a projective Coxeter polytope of S(V) with infinite irreducible Coxeter group  $W_S$ ,  $\mathscr{A}_S = (\mathscr{A}_{s,t})_{s,t\in S}$  the Cartan matrix of *P*, and  $\sigma : W_S \to SL^{\pm}(V)$  the representation defined by  $\sigma(s) = \sigma_s$ . If  $W_S$  satisfies the conditions (i) and (ii) of Theorem 4.9, then the following are equivalent:

- $\sigma(W)$  is convex cocompact in S(V);
- for any irreducible standard subgroup  $W_T$  of  $W_S$  with  $\emptyset \neq T \subset S$ , the Cartan submatrix  $\mathscr{A}_T := (\mathscr{A}_{s,t})_{s,t \in T}$  is not zero type;
- det( $\mathscr{A}_T$ )  $\neq 0$  for all  $T \subset S$  with  $W_T$  of type  $A_k$ ,  $k \ge 1$ .

As a result, any infinite, irreducible Coxeter group  $W_S$  satisfying the conditions (*i*) and (*ii*) of Theorem 4.9 admits projective reflection groups, which are convex cocompact in  $S(\mathbb{R}^N)$  with N = #S (see [DGK<sup>+</sup>21, Th. 1.3]).

4.5. Divisible and quasi-divisible domains. Every properly convex open subset  $\Omega$  of  $\mathbb{S}(V)$  admits a Hilbert metric  $d_{\Omega}$  on  $\Omega$  so that the group Aut( $\Omega$ ) of automorphisms of  $\mathbb{S}(V)$  preserving  $\Omega$  acts on  $\Omega$  by isometries for  $d_{\Omega}$ . A properly convex domain  $\Omega$  is *divisible* (resp. *quasi-divisible*) by  $\Gamma$  if there exists a discrete subgroup  $\Gamma$  of Aut( $\Omega$ ) such that  $\Omega/\Gamma$  is compact (resp. of finite volume with respect to the Hausdorff measure induced by  $d_{\Omega}$ ). (see e.g. [Mar14] for more details for the Hilbert metric and the Hausdorff measure).

In general, it is difficult to construct divisible or quasi-divisible domains with various properties. But, in small dimension, one can use *perfect* or *quasi-perfect* projective Coxeter polytopes to build such domains. We first introduce the definition of 2-perfect polytopes, which is slightly more general than that of perfect or quasi-perfect polytopes, and Section 5 illustrates some interesting examples of divisible domains.

Let *P* be a projective Coxeter polytope of  $\mathbb{S}(V)$  and *S* the set of facets of *P*. Given a vertex v of *P*, we denote by  $S_v$  the set of facets that contain v. For any  $s \in S_v$ , the projective reflection  $\sigma_s$  induces a projective reflection  $\overline{\sigma}_s$  of the projective space  $\mathbb{S}(V/\langle v \rangle)$ , where  $\langle v \rangle$  is the subspace spanned by v and  $V/\langle v \rangle$  is the quotient vector space. The projection of *P* to  $\mathbb{S}(V/\langle v \rangle)$  with the reflections  $(\overline{\sigma}_s)_{s \in S_v}$  defines a projective Coxeter polytope  $P_v$  of  $\mathbb{S}(V/\langle v \rangle)$ , called the *link of P at v*.

**Definition 4.11.** A projective Coxeter *d*-polytope *P* is *elliptic* (resp. *parabolic*, resp. *loxo-dromic*) when each component of  $\mathscr{A}_P$  is of positive type (resp. zero type, resp. negative type) and the rank of  $\mathscr{A}_P$  is d + 1 (resp. *d*, resp. d + 1).

*Remark* 4.12. If *P* is elliptic, then  $W_P$  is a spherical Coxeter group and *P* is the Tits simplex associated to  $W_P$ . If *P* is parabolic, then  $W_P$  is an affine Coxeter group, *P* is the Cartesian product of the Tits simplices associated to the irreducible components of  $W_P$ , and  $\Omega_P$  is an affine chart of  $\mathbb{S}(V)$ .

A projective Coxeter polytope P is *perfect* (resp. *quasi-perfect*, resp. 2-*perfect*) when all its vertex link are elliptic (resp. elliptic or parabolic, resp. perfect). For example, quasi-perfect Coxeter polytopes should be 2-perfect.

*Remark* 4.13. A perfect Coxeter polytope is either elliptic, parabolic or irreducible loxodromic, by [Vin71, Prop. 26].

Let *P* be an irreducible loxodromic Coxeter polytope and  $\Gamma_P$  the projective reflection group of *P*. Then  $\Omega_P$  is a properly convex domain, hence it admits a Hilbert metric  $d_{\Omega_P}$ . A projective Coxeter polytope *P* is perfect if and only if the action of  $\Gamma_P$  on  $\Omega_P$  is cocompact, by [Vin71, Th. 2]. So, in this case, the domain  $\Omega_P$  is divisible by  $\Gamma_P$ .

The action of  $\Gamma_P$  on  $\Omega_P$  is said to be of *finite covolume* if  $P \cap \Omega_P$  has finite volume with respect the Hausdorff measure  $\mu_{\Omega_P}$  induced by  $d_{\Omega_P}$ , and of *geometrically finite* if  $\mu_{\Omega_P}(P \cap \mathscr{C}(\Lambda_{\Omega_P})) < \infty$ , where  $\Lambda_P$  is the limit set of  $\Gamma_P$  and  $\mathscr{C}(\Lambda_P)$  is the convex hull of  $\Lambda_P$  of  $\Omega_P$  (see [Mar17] for more details).

**Theorem 4.14** ([Mar17, Th. A]). Let P be an irreducible, loxodromic, 2-perfect Coxeter polytope of S(V). Then the action of  $\Gamma_P$  on  $\Omega_P$  is always geometrically finite, and

- $\Gamma_P$  is of finite covolume if and only if *P* is quasi-perfect;
- $\Gamma_P$  is convex cocompact in S(V) if and only if all the vertex links of P are elliptic or loxodromic.

4.6. **Cocompact action of Coxeter groups.** There are many examples of discrete Coxeter subgroups of  $SL^{\pm}(V)$  other than projective reflection groups. However, if a Coxeter group  $\Gamma$  divides a properly convex domain, then  $\Gamma = \Gamma_P$  for some projective Coxeter polytope *P*:

**Theorem 4.15** ([Dav08, Prop. 10.9.7] and [CD00]; see [LM19, Lem. 5.4]). Let W be a Coxeter group, and let  $\rho: W \to SL^{\pm}(V)$  be a faithful representation. Suppose that there exists a divisible domain  $\Omega$  by  $\rho(W)$ . Then the following hold:

- (i) for each  $s \in S$ , the image  $\rho(s)$  of s is a projective reflection of  $\mathbb{S}(V)$ ;
- (ii)  $\rho(W)$  is a projective reflection group generated by  $(\rho(s))_{s \in S}$ .

*Remark* 4.16. It is an open question whether Theorem 4.15 still holds when the word "divisible" is replaced by "quasi-divisible".

### 5. EXAMPLES OF PROJECTIVE REFLECTION GROUPS

The construction of projective reflection groups had led to several existence theorems in convex projective geometry.

A properly convex domain  $\Omega$  of  $\mathbb{S}(V)$  is *decomposable* if a cone of V lifting  $\Omega$  is a non-trivial direct product of two smaller cones, and *homogeneous* if the group Aut( $\Omega$ ) acts transitively on  $\Omega$ . Since the homogeneous quasi-divisible domains are well-understood by [Vin63, Koe99],

only inhomogeneous ones are of interest to us. So, all properly convex domains in this section are assumed to be inhomogeneous and indecomposable.

5.1. **Kac-Vinberg's example.** The first example of divisible 2-domain which is not a hyperbolic plane was found by Kac and Vinberg [KV67]. They used perfect projective Coxeter triangles P with the Cartan matrix  $\mathscr{A}_P = (\mathscr{A}_{i,j})_{i,j=1,2,3}$  such that (*i*) each entry  $\mathscr{A}_{i,j}$  is a negative integer, (*ii*) det( $\mathscr{A}_P$ ) < 0, and (*iii*)  $\mathscr{A}_{1,2}\mathscr{A}_{2,3}\mathscr{A}_{3,1} \neq \mathscr{A}_{1,3}\mathscr{A}_{3,2}\mathscr{A}_{2,1}$ . Here, the condition (*i*) implies that the projective reflection group  $\Gamma_P$  of P is a subgroup of SL(3,  $\mathbb{Z}$ ), (*ii*) implies that  $\mathscr{A}_P$  is of negative type, and finally (*iii*) implies that  $\Omega_P$  is not a hyperbolic plane (see Figure 1).



FIGURE 1. Triangles with dihedral angles  $\pi/3$ ,  $\pi/3$  and  $\pi/6$  on the left, with dihedral angles  $\pi/3$ ,  $\pi/4$  and  $\pi/6$  on the center, and with dihedral angles  $\pi/6$ ,  $\pi/6$  and  $\pi/6$  on the right.

*Remark* 5.1. Let  $\hat{\Gamma}_P$  be any finite-index torsion-free subgroup of  $\Gamma_P$ . Then  $\hat{\Gamma}_P$  is an infinite index subgroup of SL(3,  $\mathbb{Z}$ ) and is Zariski dense in SL(3,  $\mathbb{R}$ ). In other words,  $\hat{\Gamma}_P$  is a *thin* surface group (see [KLLR19] for an introduction to thin groups).

5.2. **Benoist's examples and more.** The first known examples of divisible *d*-domains  $\Omega$  which are not strictly convex were introduced by Benoist [Ben06a] in dimension d = 3, ..., 7 (see Figure 2). In such examples, the discrete group  $\Gamma$  which divides  $\Omega$  is relatively hyperbolic with respect to virtual  $\mathbb{Z}^{d-1}$ . Later, different examples of non-strictly convex divisible *d*-domains were found in [CLM20] in dimension d = 4, ..., 8, and the group  $\Gamma$  dividing such *d*-domain is relatively hyperbolic with respect to a collection of virtually free abelian subgroup of rank < d - 1. Except in dimension 3 (see [BDL]), all the known examples were built from projective reflection groups.



FIGURE 2. A collection of properly embedded triangles in the non-strictly convex divisible 3-domains is colored. Each triangle is preserved by a subgroup of  $\Gamma$ , which is virtually  $\mathbb{Z}^2$ .

*Remark* 5.2. A generalization of Thurston's hyperbolic Dehn filling theorem to the projective setting led to the examples in [CLM20].

In [Ben06b], Benoist found the first example of word-hyperbolic group  $\Gamma$ , not quasi-isometric to the hyperbolic space, that divides a properly convex 4-domain  $\Omega$ , again using projective reflection groups. Since  $\Gamma$  is word hyperbolic,  $\Omega$  should be strictly convex by [Ben04]. Shortly after, Kapovich [Kap07] found examples in any dimension  $d \ge 4$ , using Gromov-Thurston manifolds [GT87].

## 6. HITCHIN COMPONENT OF POLYGON GROUPS

Let *P* be a compact hyperbolic polygon with dihedral angles  $\pi/m_1, \ldots, \pi/m_k$ , and let *W* be the Coxeter group of *P*. The conjugacy classes of discrete and faithful representations of *W* to PGL(2,  $\mathbb{R}$ ) form a connected component  $\mathcal{T}$  of  $\chi(W, \text{PGL}(2, \mathbb{R})) := \text{Hom}(W, \text{PGL}(2, \mathbb{R}))/\text{PGL}(2, \mathbb{R})$ , i.e. the space of conjugacy classes of representations of *W* to PGL(2,  $\mathbb{R}$ ).

For any  $n \ge 2$ , there is a unique irreducible representation  $\kappa : PGL(2,\mathbb{R}) \to PGL(n,\mathbb{R})$  up to conjugation. This gives rise to an embedding:

$$\mathcal{T} \to \chi(W, \mathrm{PGL}(n, \mathbb{R})) := \mathrm{Hom}(W, \mathrm{PGL}(n, \mathbb{R}))/\mathrm{PGL}(n, \mathbb{R})$$

The image of this embedding is called the *Fuchsian locus* and the component of  $\chi(W, PGL(n, \mathbb{R}))$  containing the Fuchsian locus is the *Hitchin component* Hit( $W, PGL(n, \mathbb{R})$ ). A representation  $\rho : W \to PGL(n, \mathbb{R})$  is called a *Hitchin representation* if its  $PGL(n, \mathbb{R})$ -conjugacy class is an element of Hit( $W, PGL(n, \mathbb{R})$ ).

**Theorem 6.1** ([ALS18, Th. 1.1 & 1.2]). Let P be a compact hyperbolic polygon with dihedral angles  $\pi/m_1, \ldots, \pi/m_k$ , and W its Coxeter group. Then each Hitchin representation in Hit(W,PGL(n,  $\mathbb{R}$ )) is discrete and faithfull, and Hit(W,PGL(n,  $\mathbb{R}$ )) is an open cell of dimension

$$-(n^2-1)+\sum_{\ell=2}^n\sum_{i=1}^k \left\lfloor \ell\left(1-\frac{1}{m_i}\right) \right\rfloor,$$

where  $\lfloor x \rfloor$  denotes the biggest integer not bigger than x.

For example, the PGL( $2m, \mathbb{R}$ ) (resp. PGL( $2m + 1, \mathbb{R}$ )) Hitchin component of the Coxeter group associated to a right-angled hyperbolic *k*-gon ( $k \ge 5$ ) is an open cell of dimension (k - 4) $m^2 + 1$  (resp. (k - 4)( $m^2 + m$ )).

*Remark* 6.2. In the case of n = 2 (resp. n = 3), Theorem 6.1 was proved by Thurston [Thu] (resp. Choi-Goldman [CG05]).

*Remark* 6.3. Let  $\rho$  be any Hitchin representation in Hit(W, PGL(n,  $\mathbb{R}$ )). In the case  $n \ge 4$ , the image of each generator of W should be an involution but not a projective reflection, hence  $\rho$  is not a projective reflection group.

### 7. PROPERLY DISCONTINUOUS AFFINE GROUPS

7.1. **Auslander's conjecture and Milnor's question.** In the 60's, Auslander raised the following conjecture:

**Conjecture 7.1** (Auslander [Aus64]). Every discrete subgroup  $\Gamma$  of the affine group Aff( $\mathbb{R}^d$ ) which acts properly discontinuously and cocompactly on  $\mathbb{R}^d$  is virtually solvable.

In the 70's, Milnor asked if the Auslander's conjecture still holds without the condition that the action is cocompact:

**Question 7.2** (Milnor [Mil77]). Is every discrete subgroup  $\Gamma$  of Aff( $\mathbb{R}^d$ ) which acts properly discontinuously on  $\mathbb{R}^d$  virtually solvable?

In 1983, Fried and Goldman [FG83] showed that the Auslander's conjecture is true in dimension 3, and Margulis answered the Milnor's question negatively:

**Theorem 7.3** (Margulis [Mar83, Mar87]). There exists a properly discontinuous action of the free group on two generators on  $\mathbb{R}^3$ .

Even if some progress have been made over the years toward Auslander's conjecture (see e.g. [Tom16, AMS12, AMS20] for a proof assuming  $d \le 6$  and [GK84, Tom90, AMS10] for a proof assuming the linear part is contained in a particular class of semisimple Lie subgroups), Auslander's conjecture is still open.

Back to Milnor's question, the existence and property of properly discontinuous affine action of free groups on  $\mathbb{R}^n$  have been actively studied (see e.g. [Dru92, CDG16, DGK15, GLM09, AMS02, Smi16b, Smi16a] or the survey [DDGS20]).

7.2. **Properly discontinuous affine Coxeter groups.** Before the following theorem, properly discontinuous affine actions by non-virtually solvable non-free groups were unknown.

**Theorem 7.4** ([DGK20, Th. 1.1]). Any right-angled Coxeter group of rank k admits a proper affine actions on  $\mathbb{R}^{k(k-1)/2}$ .

*Remark* 7.5. The action preserves a bilinear form and in some particular cases, one can find much smaller affine space on which the Coxeter groups acts (see [DGK20, Prop. 1.6]).

#### REFERENCES

- [All06] Daniel Allcock. Infinitely many hyperbolic Coxeter groups through dimension 19. Geom. Topol., 10:737–758, 2006. 8
- [ALS18] Daniele Alessandrini, Gye-Seon Lee, and Florent Schaffhauser. Hitchin components for orbifolds. arXiv e-prints, page arXiv:1811.05366, November 2018. 15
- [AMS02] Herbert Abels, Gregory Margulis, and Gregory Soifer. On the zariski closure of the linear part of a properly discontinuous group of affine transformations. *Journal of Differential Geometry*, 60(2), feb 2002. 16
- [AMS10] Herbert Abels, Gregory Margulis, and Gregory Soifer. The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant. *Geometriae Dedicata*, 153(1):1–46, dec 2010. 16
- [AMS12] Herbert Abels, Gregory Margulis, and Gregory Soifer. The Auslander conjecture for dimension less than 7. *arXiv e-prints*, page arXiv:1211.2525, November 2012. 16
- [AMS20] Herbert Abels, Gregory Margulis, and Gregory Soifer. The Auslander conjecture for dimension less then 7. *arXiv e-prints*, page arXiv:2011.12788, November 2020. 16
- [And71a] Evgeny M. Andreev. On convex polyhedra in Lobachevskij spaces. Math. USSR, Sb., 10:413-440, 1971. 5
- [And71b] Evgeny M. Andreev. On convex polyhedra of finite volume in Lobachevskii space. *Math. USSR, Sb.*, 12:255–259, 1971. 5
- [Aus64] Louis Auslander. The structure of complete locally affine manifolds. *Topology*, 3:131–139, jul 1964. 15
- [BDL] Samuel A. Ballas, Jeffrey Danciger, and Gye-Seon Lee. Convex projective structures on nonhyperbolic three-manifolds. *Geom. Topol.*, (3):1593–1646. 14

- [Ben04] Yves Benoist. Convexes divisibles. I. In Algebraic groups and arithmetic, pages 339–374. Tata Inst. Fund. Res., Mumbai, 2004. 15
- [Ben06a] Yves Benoist. Convexes divisibles. IV. Structure du bord en dimension 3. Invent. Math., 164(2):249–278, 2006. 14
- [Ben06b] Yves Benoist. Convexes hyperboliques et quasiisométries. Geom. Dedicata, 122:109-134, 2006. 15
- [Bor87] Richard Borcherds. Automorphism groups of Lorentzian lattices. J. Algebra, 111(1):133-153, 1987. 8
- [Bou68] Nicolas Bourbaki. Groupes et algèbres de Lie, Chapitre IV, V, VI. Hermann, 1968. 2, 10, 11
- [Bug84] Vadim O. Bugaenko. Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring  $\mathbb{Z}[(\sqrt{5}+1)/2]$ . Vestnik Moskov. Univ. Ser. I Mat. Mekh., (5):6–12, 1984. 8
- [Bug92] Vadim O. Bugaenko. Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices. In *Lie groups, their discrete subgroups, and invariant theory*, volume 8 of *Adv. Soviet Math.*, pages 33–55. Amer. Math. Soc., Providence, RI, 1992. 8
- [CD00] Ruth Charney and Michael Davis. When is a Coxeter system determined by its Coxeter group? J. London Math. Soc. (2), 61(2):441–461, 2000. 13
- [CDG16] Virginie Charette, Todd A. Drumm, and William M. Goldman. Proper affine deformations of the one-holed torus. Transform. Groups, 21(4):953-1002, 2016. 16
- [CG05] Suhyoung Choi and William M. Goldman. The deformation spaces of convex ℝP<sup>2</sup>-structures on 2orbifolds. Amer. J. Math., 127(5):1019–1102, 2005. 15
- [Che69] Michel Chein. Recherche des graphes des matrices de Coxeter hyperboliques d'ordre ≤ 10. Rev. Française Informat. Recherche Opérationnelle, 3(Ser. R-3):3–16, 1969. 6
- [CLM20] Suhyoung Choi, Gye-Seon Lee, and Ludovic Marquis. Convex projective generalized Dehn filling. Ann. Sci. Éc. Norm. Supér. (4), 53(1):217–266, 2020. 14, 15
- [Cox32] Harold Coxeter. The Polytopes with Regular-Prismatic Vertex Figures. Proc. London Math. Soc. (2), 34(2):126–189, 1932. 2, 3
- [Cox34] Harold Coxeter. Discrete groups generated by reflections. Ann. of Math. (2), 35(3):588–621, 1934. 2, 3
- [Dav08] Michael W. Davis. The geometry and topology of Coxeter groups, volume 32 of London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2008. 8, 13
- [DDGS20] Jeffrey Danciger, Todd A. Drumm, William M. Goldman, and Ilia Smilga. Proper actions of discrete groups of affine transformations. special volume Dynamics, Geometry, Number Theory: the Impact of Margulis on Modern Mathematics, to be published by the University of Chicago Press, page arXiv:2002.09520, February 2020. 16
- [DGK15] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Margulis spacetimes via the arc complex. Inventiones mathematicae, 204(1):133–193, aug 2015. 16
- [DGK17a] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Convex cocompact actions in real projective geometry. *arXiv e-prints*, page arXiv:1704.08711, April 2017. 1, 12
- [DGK17b] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Convex cocompactness in pseudoriemannian hyperbolic spaces. *Geometriae Dedicata*, 192(1):87–126, dec 2017. 11
- [DGK20] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Proper affine actions for right-angled coxeter groups. *Duke Mathematical Journal*, 169(12), sep 2020. 16
- [DGK<sup>+</sup>21] Jeffrey Danciger, François Guéritaud, Fanny Kassel, Gye-Seon Lee, and Ludovic Marquis. Convex cocompactness for Coxeter groups. *arXiv e-prints*, page arXiv:2102.02757, February 2021. 11, 12
- [DH13] Marc Desgroseilliers and Frédéric Haglund. On some convex cocompact groups in real hyperbolic space. *Geometry & Topology*, 17(4):2431–2484, aug 2013. 8, 9
- [Dru92] Todd A. Drumm. Fundamental polyhedra for margulis space-times. Topology, 31(4):677–683, oct 1992. 16
- [Duf10] Guillaume Dufour. Notes on right-angled Coxeter polyhedra in hyperbolic spaces. *Geom. Dedicata*, 147:277–282, 2010. 8
- [FG83] David Fried and William M Goldman. Three-dimensional affine crystallographic groups. Advances in Mathematics, 47(1):1–49, jan 1983. 16
- [GK84] William M. Goldman and Yoshinobu Kamishima. The fundamental group of a compact flat lorentz space form is virtually polycyclic. *Journal of Differential Geometry*, 19(1), jan 1984. 16
- [GLM09] William M. Goldman, François Labourie, and Gregory Margulis. Proper affine actions and geodesic flows of hyperbolic surfaces. *Annals of Mathematics*, 170(3):1051–1083, nov 2009. 16

- [GT87] Mikhail Gromov and William Thurston. Pinching constants for hyperbolic manifolds. *Invent. Math.*, 89(1):1–12, 1987. 15
- [GW12] Olivier Guichard and Anna Wienhard. Anosov representations: domains of discontinuity and applications. *Inventiones mathematicae*, 190(2):357–438, feb 2012. 1, 11
- [HR93] Craig D. Hodgson and Igor Rivin. A characterization of compact convex polyhedra in hyperbolic 3space. *Invent. Math.*, 111(1):77–111, 1993. 5
- [Kap07] Michael Kapovich. Convex projective structures on Gromov-Thurston manifolds. Geom. Topol., 11:1777-1830, 2007. 15
- [Kho86] Askold. G. Khovanskii. Hyperplane sections of polyhedra, toric varieties and discrete groups in Lobachevsky space. Funktsional. Anal. i Prilozhen., 20(1):50–61, 96, 1986. 8
- [KLLR19] Alex Kontorovich, Darren D. Long, Alexander Lubotzky, and Alan W. Reid. What is ... a thin group? Notices Amer. Math. Soc., 66(6):905–910, 2019. 14
- [Koe99] Max Koecher. The Minnesota notes on Jordan algebras and their applications, volume 1710 of Lecture Notes in Mathematics. Springer-Verlag, 1999. 13
- [Kos67] Jean-Louis Koszul. Lectures on hyperbolic Coxeter groups. University of Notre Dame, 1967. 6
- [KV67] Victor G. Kac and Érnest B. Vinberg. Quasi-homogeneous cones. Math. Notes, 1:231–235, 1967. 14
- [Lab06] François Labourie. Anosov flows, surface groups and curves in projective space. Invent. Math., 165(1):51-114, 2006. 1, 11
- [Lan50] Folke Lannér. On complexes with transitive groups of automorphisms. Comm. Sém., Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.], 11:71, 1950. 6
- [LM19] Gye-Seon Lee and Ludovic Marquis. Anti-de Sitter strictly GHC-regular groups which are not lattices. Trans. Am. Math. Soc., 372(1):153–186, 2019. 11, 13
- [Mar83] Gregory Margulis. Free completely discontinuous groups of affine transformations. *Dokl. Akad. Nauk* SSSR, 272(4):785–788, 1983. 16
- [Mar87] Gregory Margulis. Complete affine locally flat manifolds with a free fundamental group. *Journal of* Soviet Mathematics, 36(1):129–139, jan 1987. 16
- [Mar14] Ludovic Marquis. Around groups in Hilbert geometry. In *Handbook of Hilbert geometry*, volume 22 of *IRMA Lect. Math. Theor. Phys.*, pages 207–261. Eur. Math. Soc., Zürich, 2014. 12
- [Mar17] Ludovic Marquis. Coxeter group in Hilbert geometry. Groups Geom. Dyn., 11(3):819-877, 2017. 13
- [Mil77] John Milnor. On fundamental groups of complete affinely flat manifolds. Advances in Mathematics, 25(2):178–187, aug 1977. 16
- [Mou88] Gabor Moussong. *Hyperbolic Coxeter groups*. ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Ph.D.)– The Ohio State University. 9
- [MV00] Gregory Margulis and Érnest Vinberg. Some linear groups virtually having a free quotient. J. Lie Theory, 10(1):171–180, 2000. 2
- [Nik81] Viacheslav V. Nikulin. Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebro-geometric applications. In *Current problems in mathematics, Vol.* 18, pages 3–114. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981. 8
- [Poi83] Henri Poincaré. Mémoire: Les groupes kleinéens. Acta Mathematica, 3(0):49–92, 1883. 4
- [Pro86] M. N. Prokhorov. Absence of discrete groups of reflections with a noncompact fundamental polyhedron of finite volume in a Lobachevsky space of high dimension. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(2):413-424, 1986. 8
- [PV05] Leonid Potyagailo and Ernest Vinberg. On right-angled reflection groups in hyperbolic spaces. Comment. Math. Helv., 80(1):63-73, 2005. 8
- [RHD07] Roland K. W. Roeder, John H. Hubbard, and William D. Dunbar. Andreev's theorem on hyperbolic polyhedra. Ann. Inst. Fourier, 57(3):825–882, 2007. 5
- [Roe06] Roland K. W. Roeder. Compact hyperbolic tetrahedra with non-obtuse dihedral angles. Publ. Mat., Barc., 50(1):211–227, 2006. 5
- [Smi16a] Ilia Smilga. Proper affine actions: a sufficient criterion. to appear in Mathematische Annalen, page arXiv:1612.08942, December 2016. 16
- [Smi16b] Ilia Smilga. Proper affine actions on semisimple lie algebras. Annales de l'institut Fourier, 66(2):785– 831, 2016. 16
- [Thu] William Thurston. The Geometry and Topology of Three-Manifolds. Lecture notes. 15

- [Tom90] George Tomanov. The virtual solvability of the fundamental group of a generalized lorentz space form. *Journal of Differential Geometry*, 32(2), jan 1990. 16
- [Tom16] George Tomanov. Properly discontinuous group actions on affine homogeneous spaces. *Tr. Mat. Inst. Steklova*, 292(Algebra, Geometriya i Teoriya Chisel):268–279, 2016. 16
- [Vin63] Érnest B. Vinberg. The theory of homogeneous convex cones. Trudy Moskov. Mat. Obv č., 12:303–358, 1963. 13
- [Vin71] Érnest B. Vinberg. Discrete linear groups that are generated by reflections. Izv. Akad. Nauk SSSR Ser. Mat., 35:1072–1112, 1971. 10, 11, 13
- [Vin84] Érnest B. Vinberg. Absence of crystallographic groups of reflections in Lobachevskiĭ spaces of large dimension. Trudy Moskov. Mat. Obshch., 47:68–102, 246, 1984. 8
- [Vin85] Érnest B. Vinberg. Hyperbolic groups of reflections. Uspekhi Mat. Nauk, 40(1(241)):29–66, 255, 1985.
  6
- [Wit41] Ernst Witt. Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe. Abh. Math. Sem. Hansischen Univ., 14(1):289–322, 1941. 3

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