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# Semiclassical Ohsawa-Takegoshi extension theorem and asymptotics of the orthogonal Bergman kernel.

Siarhei Finski

**Abstract.** In this paper, we study the asymptotics of Ohsawa-Takegoshi extension operator and orthogonal Bergman projector associated with high tensor powers of a positive line bundle.

More precisely, for a fixed submanifold in a complex manifold, we consider the operator which associates to a given holomorphic section of a positive line bundle over the submanifold the holomorphic extension of it to the ambient manifold with the minimal  $L^2$ -norm. When the tensor power of the line bundle tends to infinity, we prove an exponential estimate for the Schwartz kernel of this extension operator, and show that it admits a full asymptotic expansion in powers of the line bundle. Similarly, we study the asymptotics of the orthogonal Bergman kernel associated to the projection onto the holomorphic sections orthogonal to those which vanish along the submanifold. All our results are stated in the setting of manifolds and embeddings of bounded geometry.

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# 1 Introduction

One of the main goals of this article is to give an asymptotic version of Ohsawa-Takegoshi extension theorem for powers of the twisting positive line bundle tending to infinity. Another goal is to establish the asymptotic expansion of the Schwartz kernel of the orthogonal Bergman projector onto the holomorphic sections orthogonal to those which vanish along a submanifold.

More precisely, we fix two (not necessarily compact) complex manifolds  $X, Y$ , of dimensions  $n$  and  $m$  respectively. We fix also a complex embedding  $\iota : Y \rightarrow X$ , a positive line bundle  $(L, h^L)$  over  $X$  and an arbitrary Hermitian vector bundle  $(F, h^F)$  over  $X$ . In particular, we assume that for the curvature  $R^L$  of the Chern connection on  $(L, h^L)$ , the closed real  $(1, 1)$ -differential form

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L \quad (1.1)$$

is positive. We denote by  $g^{TX}$  the Riemannian metric on  $X$  induced by  $\omega$  as follows

$$g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot), \quad (1.2)$$

where  $J : TX \rightarrow TX$  is the complex structure on  $X$ . We denote by  $g^{TY}$  the induced metric on  $Y$ .

We assume throughout the whole article that the triple  $(X, Y, g^{TX})$ , and the Hermitian vector bundles  $(L, h^L)$ ,  $(F, h^F)$ , are of bounded geometry in the sense of Definitions 2.8, 2.14.

This means that we assume uniform lower bounds  $r_X, r_Y > 0$  on the injectivity radii of  $X, Y$ , the existence of the geodesic tubular neighborhood of  $Y$  of uniform size  $r_\perp > 0$  in  $X$ , and some uniform bounds on related curvatures and the second fundamental form of the embedding.

Now, we fix some positive (with respect to the orientation given by the complex structure) volume forms  $dv_X, dv_Y$  on  $X$  and  $Y$ . For smooth sections  $f, f'$  of  $L^p \otimes F$  over  $X$ , we define the  $L^2$ -scalar product using the pointwise scalar product  $\langle \cdot, \cdot \rangle_h$ , induced by  $h^L$  and  $h^F$ , as follows

$$\langle f, f' \rangle_{L^2(X)} := \int_X \langle f(x), f'(x) \rangle_h dv_X(x). \quad (1.3)$$

Similarly, using  $dv_Y$ , we introduce the  $L^2$ -scalar product for sections of  $\iota^*(L^p \otimes F)$  over  $Y$ . We denote by  $H_{(2)}^0(X, L^p \otimes F)$  and  $H_{(2)}^0(Y, \iota^*(L^p \otimes F))$  the vector spaces of holomorphic sections of  $L^p \otimes F$  over  $X$  and  $Y$  respectively with bounded  $L^2$ -norm.

We assume that for the Riemannian volume forms  $dv_{g^{TX}}, dv_{g^{TY}}$  of  $(X, g^{TX}), (Y, g^{TY})$ , for any  $k \in \mathbb{N}$ , there is  $C_k > 0$ , such that over  $X$  and  $Y$ , the following bounds hold

$$\left\| \frac{dv_{g^{TX}}}{dv_X} \right\|_{\mathcal{C}^k} \leq C_k, \quad \left\| \frac{dv_X}{dv_{g^{TX}}} \right\|_{\mathcal{C}^k} \leq C_k, \quad \left\| \frac{dv_{g^{TY}}}{dv_Y} \right\|_{\mathcal{C}^k} \leq C_k, \quad \left\| \frac{dv_Y}{dv_{g^{TY}}} \right\|_{\mathcal{C}^k} \leq C_k. \quad (1.4)$$

By extending Ohsawa-Takegoshi theorem, [33], [32], [11, §13], in Theorem 4.4, we prove that there is  $p_0 \in \mathbb{N}$ , such that for any  $p \geq p_0$ ,  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , there is  $f \in H_{(2)}^0(X, L^p \otimes F)$ , satisfying  $f|_Y = g$ . Then, for the Bergman projector  $B_p^Y$ , given by the orthogonal projection from the space of  $L^2$ -sections  $L^2(Y, \iota^*(L^p \otimes F))$  to  $H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , we define the extension operator

$$E_p : L^2(Y, \iota^*(L^p \otimes F)) \rightarrow H_{(2)}^0(X, L^p \otimes F), \quad (1.5)$$

by putting  $E_p g = f$ , where  $f|_Y = B_p^Y g$ , and  $f$  has the minimal  $L^2$ -norm among those  $f' \in H_{(2)}^0(X, L^p \otimes F)$  satisfying  $f'|_Y = B_p^Y g$  (the set of such  $f'$  is closed and convex, hence  $f$  exists

and it is unique by Hilbert projection theorem). In particular, for  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , we have  $(E_p g)|_Y = g$ . Ohsawa-Takegoshi extension theorem means in this context precisely that the operator  $E_p$  is bounded. In this article, we find an explicit asymptotic expansion of  $E_p$ , as  $p \rightarrow \infty$ .

More precisely, we identify the normal bundle  $N$  of  $Y$  in  $X$  as an orthogonal complement of  $TY$  in  $TX$  (with respect to  $g^{TX}$ ) so that we have the following orthogonal decomposition

$$TX|_Y \rightarrow TY \oplus N. \quad (1.6)$$

We denote by  $g^N$  the metric on  $N$  induced by  $g^{TX}$ . Let  $P^N : TX|_Y \rightarrow N$ ,  $P^Y : TX|_Y \rightarrow TY$ , be the projections induced by (1.6). Clearly,  $\nabla^N := P^N \nabla^{TX}|_Y$  defines a connection on  $N$ .

For  $y \in Y$ ,  $Z_N \in N_y$ , let  $\mathbb{R} \ni t \mapsto \exp_y^X(tZ_N) \in X$  be the geodesic in  $X$  in direction  $Z_N$ . Bounded geometry condition means, in particular, that this map induces a diffeomorphism of  $r_\perp$ -neighborhood of the zero section in  $N$  with a tubular neighborhood  $U$  of  $Y$  in  $X$ . From now on, we use this identification implicitly. Of course,  $(y, 0)$ ,  $y \in Y$ , then corresponds to  $Y$ .

We denote by  $\pi : U \rightarrow Y$  the natural projection  $(y, Z_N) \mapsto y$ . Over  $U$ , we identify  $L, F$  to  $\pi^*(L|_Y), \pi^*(F|_Y)$  by the parallel transport with respect to the respective Chern connections along the geodesic  $[0, 1] \ni t \mapsto (y, tZ_N) \in X$ ,  $|Z_N| < r_\perp$ , and we define a function  $\kappa_N$  as follows

$$dv_X = \kappa_N dv_Y \wedge dv_N, \quad (1.7)$$

where  $dv_N$  is the relative Riemannian volume form on  $(N, g^N)$ . Of course, we have  $\kappa_N|_Y = 1$  if

$$dv_X = dv_{g^{TX}}, \quad dv_Y = dv_{g^{TY}}. \quad (1.8)$$

Using the above identification, we define the operator  $E_p^0 : L^2(Y, \iota^*(L^p \otimes F)) \rightarrow L^2(X, L^p \otimes F)$  as follows. For  $g \in L^2(Y, \iota^*(L^p \otimes F))$ , we let  $(E_p^0 g)(x) = 0$ ,  $x \notin U$ , and in  $U$ , we put

$$(E_p^0 g)(y, Z_N) = (B_p^Y g)(y) \exp\left(-p \frac{\pi}{2} |Z_N|^2\right) \rho\left(\frac{|Z_N|}{r_\perp}\right). \quad (1.9)$$

where the norm  $|Z_N|$ ,  $Z_N \in N$ , is taken with respect to  $g^N$ , and  $\rho : \mathbb{R}_+ \rightarrow [0, 1]$  satisfies

$$\rho(x) = \begin{cases} 1, & \text{for } x < \frac{1}{4}, \\ 0, & \text{for } x > \frac{1}{2}. \end{cases} \quad (1.10)$$

Now, for  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , the section  $E_p^0 g$  satisfies  $(E_p^0 g)|_Y = g$ , but  $E_p^0 g$  is no longer holomorphic, unless  $g = 0$ . Nevertheless, as we shall see,  $E_p^0 g$  can be used to approximate very well the holomorphic section  $E_p g$ . More precisely, we have the following result.

**Theorem 1.1.** *There are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1$ , we have*

$$\|E_p - E_p^0\| \leq \frac{C}{p^{\frac{n-m+1}{2}}}. \quad (1.11)$$

where  $\|\cdot\|$  is the operator norm. Also, as  $p \rightarrow \infty$ , we have

$$\|E_p^0\| \sim \frac{1}{p^{\frac{n-m}{2}}} \cdot \sup_{y \in Y} \kappa_N^{\frac{1}{2}}(y). \quad (1.12)$$

Moreover, under assumption (1.8), in (1.11), one can replace  $p^{\frac{n-m+1}{2}}$  by  $p^{\frac{n-m+2}{2}}$  if and only if  $Y$  is a totally geodesic submanifold of  $(X, g^{TX})$ , i.e. the second fundamental form, see (1.19), vanishes.

*Remark 1.2.* The boundness of  $\kappa_N$  follows from the bounded geometry condition, see Corollary 2.10. Hence, the right-hand side of (1.12) is finite.

**Corollary 1.3.** There is  $p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}^*$ , there is  $C > 0$ , such that for any  $p \geq p_1$ ,  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , we have

$$\|\nabla^k(E_p g)\|_{L^2(X)} \leq \frac{C}{p^{\frac{n-m-k}{2}}} \cdot \|g\|_{L^2(Y)}, \quad (1.13)$$

where the connection  $\nabla$  is induced by the Chern and Levi-Civita connections.

*Remark 1.4.* The estimate of type (1.13) was lacking in Demailly's approach to the invariance of plurigenera for Kähler families, see [14, (4.19)]. Remark that in [14], the manifold  $X$  is an open strictly pseudoconvex domain  $U$ , given by the neighborhood of the diagonal of certain product manifold (hence,  $X$  is never compact), and the manifold  $Y$  is the (compact) diagonal. Our theorem applies to this setting because the Bergman metric on any strictly pseudoconvex domain has bounded geometry, see Klembeck [25, Theorem 1 and p. 279] and Greene-Krantz [22, p. 8].

Both Theorem 1.1 and Corollary 1.3 appear as almost direct consequences of more precise results about the asymptotics of the Schwartz kernel  $E_p(x, y) \in (L^p \otimes F)_x \otimes (L^p \otimes F)_y^*$ ,  $x \in X$ ,  $y \in Y$ , of  $E_p$  with respect to  $dv_Y$ . To state them, recall that  $E_p(x, y)$  is defined so that for any  $g \in L^2(Y, \iota^*(L^p \otimes F))$ ,  $x \in X$ , we have

$$(E_p g)(x) = \int_Y E_p(x, y) \cdot g(y) dv_Y(y). \quad (1.14)$$

**Theorem 1.5.** There are  $c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1$ ,  $x \in X$ ,  $y \in Y$ , the following estimate holds

$$\left| E_p(x, y) \right|_{\mathcal{C}^k(X \times Y)} \leq C p^{m + \frac{k}{2}} \exp(-c\sqrt{p} \text{dist}(x, y)), \quad (1.15)$$

where the pointwise  $\mathcal{C}^k$ -norm of an element from  $\mathcal{C}^\infty(X \times Y, (L^p \otimes F) \boxtimes (L^p \otimes F)^*)$  at a point  $(x, y) \in X \times Y$  is the sum of the norms induced by  $h^L, h^F$  and  $g^{TX}$ , evaluated at  $(x, y)$ , of the derivatives up to order  $k$  with respect to the connection induced by the Chern connections on  $L, F$  and the Levi-Civita connection on  $TX$ .

Theorem 1.5 implies that to understand fully the asymptotics of the Schwartz kernel of the extension operator, it suffices to do so in a neighborhood of a fixed point  $(y_0, y_0) \in Y \times Y$  in  $X \times Y$ . Our next result shows that after a reparametrization, given by a homothety with factor  $\sqrt{p}$  in certain coordinates around  $(y_0, y_0)$ , the kernel of this extension operator admits a complete asymptotic expansion in integer powers of  $\sqrt{p}$ , as  $p \rightarrow \infty$ . To state it, we fix some notation.

We fix a point  $y_0 \in Y$  and an orthonormal frame  $(e_1, \dots, e_{2m})$  (resp.  $(e_{2m+1}, \dots, e_{2n})$ ) in  $(T_{y_0}Y, g_{y_0}^{TY})$  (resp. in  $(N_{y_0}, g_{y_0}^N)$ ), such that for  $i = 1, \dots, n$ , the following identity is satisfied

$$J e_{2i-1} = e_{2i}. \quad (1.16)$$

For  $Z = (Z_Y, Z_N)$ ,  $Z_Y \in \mathbb{R}^{2m}$ ,  $Z_N \in \mathbb{R}^{2(n-m)}$ ,  $Z_Y = (Z_1, \dots, Z_{2m})$ ,  $Z_N = (Z_{2m+1}, \dots, Z_{2n})$ ,  $|Z_Y| < r_Y$ ,  $|Z_N| < r_\perp$ , we define a coordinate system  $\psi_{y_0} : B_0^{\mathbb{R}^{2m}}(r_Y) \times B_0^{\mathbb{R}^{2(n-m)}}(r_\perp) \rightarrow X$  by

$$\psi_{y_0}(Z_Y, Z_N) := \exp_{\exp_{y_0}^X(Z_Y)}(Z_N(Z_Y)), \quad (1.17)$$

where we identified  $Z_Y, Z_N$  to elements in  $T_{y_0}Y, N_{y_0}$ , using the fixed frames  $(e_1, \dots, e_{2m})$  and  $(e_{2m+1}, \dots, e_{2n})$ ,  $Z_N(Z_Y) \in N_{\exp_{y_0}^Y(Z_Y)}$  is the parallel transport of  $Z_N$  along the geodesic  $\exp_{y_0}^Y(tZ_Y)$ ,  $t = [0, 1]$ , with respect to the connection  $\nabla^N$  on  $N$ , and  $B_0^{\mathbb{R}^k}(\epsilon)$ ,  $\epsilon > 0$  means the euclidean ball of radius  $\epsilon$  around  $0 \in \mathbb{R}^k$ . The coordinates  $\psi_{y_0}$  are called the *Fermi coordinates* at  $y_0$ . Define the functions  $\kappa_{X,y_0} : B_0^{\mathbb{R}^{2m}}(r_Y) \times B_0^{\mathbb{R}^{2(n-m)}}(r_\perp) \rightarrow \mathbb{R}$ ,  $\kappa_{Y,y_0} : B_0^{\mathbb{R}^{2m}}(r_Y) \rightarrow \mathbb{R}$ , by

$$\begin{aligned} (\psi_{y_0}^* dv_X)(Z) &= \kappa_{X,y_0}(Z) dZ_1 \wedge \cdots \wedge dZ_{2n}, \\ (\psi_{y_0}^* dv_Y)(Z_Y) &= \kappa_{Y,y_0}(Z_Y) dZ_1 \wedge \cdots \wedge dZ_{2m}. \end{aligned} \quad (1.18)$$

Clearly, under assumptions (1.8), we have  $\kappa_{X,y_0}(0) = \kappa_{Y,y_0}(0) = 1$ . Once the point  $y_0$  is fixed, we usually omit it from the subscript of the  $\kappa$ -functions.

We define the *second fundamental form*  $A \in \mathcal{C}^\infty(Y, T^*Y \otimes \text{End}(TX|_Y))$  by

$$A := \nabla^{TX}|_Y - \nabla^{TY} \oplus \nabla^N. \quad (1.19)$$

Trivially,  $A$  takes values in skew-symmetric endomorphisms of  $TX|_Y$ , interchanging  $TY$  and  $N$ .

We fix an orthonormal frame  $f_1, \dots, f_r \in F_{y_0}$ , and define  $\tilde{f}_1, \dots, \tilde{f}_r$  by the parallel transport of  $f_1, \dots, f_r$  with respect to the Chern connection  $\nabla^F$  of  $(F, h^F)$ , done first along the path  $\psi(tZ_Y, 0)$ ,  $t \in [0, 1]$ , and then along the path  $\psi(Z_Y, tZ_N)$ ,  $t \in [0, 1]$ ,  $Z_Y \in \mathbb{R}^{2m}$ ,  $Z_N \in \mathbb{R}^{2(n-m)}$ ,  $|Z_Y| < r_Y$ ,  $|Z_N| < r_\perp$ . Similarly, we trivialize  $L$  in the neighborhood of  $y_0$ . Those frames allow us to interpret  $E_p(x, y)$  as an element of  $\text{End}(F_{y_0})$  for  $x \in X, y \in Y$  in a  $\min(r_\perp, r_X)$ -neighborhood of  $y_0$ .

We also define the function  $\mathcal{E}_{n,m}$  over  $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$  as follows

$$\mathcal{E}_{n,m}(Z, Z'_Y) = \exp\left(-\frac{\pi}{2} \sum_{i=1}^m (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i) - \frac{\pi}{2} \sum_{i=m+1}^n |z_i|^2\right), \quad (1.20)$$

where  $Z = (Z_Y, Z_N)$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $Z_N \in \mathbb{R}^{2(n-m)}$  and  $z_i, z'_i$  are given by  $z_i = Z_{2i-1} + \sqrt{-1}Z_{2i}$ ,  $z'_j = Z'_{2j-1} + \sqrt{-1}Z'_{2j}$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . As we show in Section 3.2,  $\mathcal{E}_{n,m}(Z, Z'_Y)$  is the Schwartz kernel of the extension operator for some model situation. The following theorem shows that in the general case, the extension operator is comparable to this model one.

**Theorem 1.6.** *For any  $r \in \mathbb{N}$ ,  $y_0 \in Y$ , there are  $J_r^E(Z, Z'_Y) \in \text{End}(F_{y_0})$  polynomials in  $Z \in \mathbb{R}^{2n}$ ,  $Z'_Y \in \mathbb{R}^{2m}$ , with the same parity as  $r$  and  $\deg J_r^E \leq 3r$ , whose coefficients are polynomials in  $\omega, R^{TX}, A, R^F, (dv_X/dv_{g^{TX}})^{\pm \frac{1}{2n}}, (dv_Y/dv_{g^{TY}})^{\pm \frac{1}{2n}}$ , and their derivatives of order  $\leq 2r$ , all evaluated at  $y_0$ , such that for the functions  $F_r^E := J_r^E \cdot \mathcal{E}_{n,m}$  over  $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$ , the following holds.*

*There are  $\epsilon, c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $y_0 \in Y$ ,  $p \geq p_1$ ,  $Z = (Z_Y, Z_N)$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $Z_N \in \mathbb{R}^{2(n-m)}$ ,  $|Z|, |Z'_Y| \leq \epsilon$ ,  $\alpha \in \mathbb{N}^{2n}$ ,  $\alpha' \in \mathbb{N}^{2m}$ ,  $|\alpha| + |\alpha'| \leq l$ , for  $Q_{k,l,l'}^1 := 6(16(n+2)(k+1) + l') + 2l$ , the following bound holds*

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z_Y^{\alpha'}} \left( \frac{1}{p^m} E_p(\psi_{y_0}(Z), \psi_{y_0}(Z'_Y)) - \sum_{r=0}^k p^{-\frac{r}{2}} F_r^E(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_X^{-\frac{1}{2}}(Z) \kappa_Y^{-\frac{1}{2}}(Z'_Y) \right) \right|_{\mathcal{C}^{l'}(Y)} \\ & \leq Cp^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \right)^{Q_{k,l,l'}^1} \exp\left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|)\right), \end{aligned} \quad (1.21)$$

where the  $\mathcal{C}^{l'}$ -norm is taken with respect to  $y_0$ . Also, the following identity holds

$$J_0^E(Z, Z'_Y) = \text{Id}_{F_{y_0}} \cdot \kappa_N^{\frac{1}{2}}(y_0). \quad (1.22)$$

Moreover, under assumption (1.8), we have

$$J_1^E(Z, Z'_Y) = \text{Id}_{F_{y_0}} \cdot \pi g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)), \quad (1.23)$$

where we implicitly identified  $Z \in \mathbb{R}^{2n}$  to an element in  $T_{y_0}X$  as  $Z := \sum Z_i \cdot e_i$ , similar notations have been used for  $Z_Y, Z_N, Z'_Y$ , and  $z_Y, z_N, z'_Y$  are the induced complex coordinates.

*Remark 1.7.* When  $Y$  is a point, we recover the asymptotic expansion of peak sections.

The operator  $E_p$  is very much related to the *orthogonal Bergman projector*. The last operator is defined as the orthogonal projector onto the holomorphic sections of  $L^p \otimes F$  over  $X$ , which are orthogonal to those vanishing along  $Y$ . To prove Theorems 1.5, 1.6, and out of independent interest, we establish analogous results for this projector. More precisely, consider the vector space

$$H_{(2)}^{0,0}(X, L^p \otimes F) := \left\{ f \in H_{(2)}^0(X, L^p \otimes F) : f|_Y = 0 \right\}. \quad (1.24)$$

An easy verification shows that (1.24) is a closed subspace. Let  $H_{(2)}^{0,\perp}(X, L^p \otimes F)$  be the orthogonal complement of  $H_{(2)}^{0,0}(X, L^p \otimes F)$  in  $H_{(2)}^0(X, L^p \otimes F)$  with respect to the  $L^2$ -scalar product.

Denote by  $B_p^\perp, B_p^0, B_p^X$  the orthogonal projection from  $L^2(X, L^p \otimes F)$  to  $H_{(2)}^{0,\perp}(X, L^p \otimes F), H_{(2)}^{0,0}(X, L^p \otimes F)$  and  $H_{(2)}^0(X, L^p \otimes F)$  respectively. Similarly to (1.14), we denote by  $B_p^\perp(x_1, x_2), B_p^0(x_1, x_2), B_p^X(x_1, x_2)$  the Schwartz kernels of  $B_p^\perp, B_p^0, B_p^X$  with respect to  $dv_X$ .

**Theorem 1.8.** *There are  $c > 0, p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1, x_1, x_2 \in X$ , the following estimate holds*

$$\left| B_p^\perp(x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \leq Cp^{n+\frac{k}{2}} \exp\left(-c\sqrt{p}(\text{dist}(x_1, x_2) + \text{dist}(x_1, Y) + \text{dist}(x_2, Y))\right), \quad (1.25)$$

where the norm  $\mathcal{C}^k$  is interpreted in the same way as in Theorem 1.5.

Hence, for the asymptotics of the Schwartz kernel of  $B_p^\perp$ , it is only left to study it in the neighborhood of a fixed point  $(y_0, y_0) \in Y \times Y$  in  $X \times X$ . To state our result in this direction, we define the function  $\mathcal{P}_{n,m}^\perp$  over  $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$  as follows

$$\mathcal{P}_{n,m}^\perp(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_{i=1}^n (|z_i|^2 + |z'_i|^2) + \pi \sum_{i=1}^m z_i \bar{z}'_i\right), \quad (1.26)$$

where  $Z, Z' \in \mathbb{R}^{2n}$ , and  $z_i, z'_i$  are given by  $z_i = Z_{2i-1} + \sqrt{-1}Z_{2i}, z'_i = Z'_{2i-1} + \sqrt{-1}Z'_{2i}$ , for  $i = 1, \dots, n$ . As we show in Section 3.2,  $\mathcal{P}_{n,m}^\perp(Z, Z')$  is the Schwartz kernel of the orthogonal Bergman projector for some model situation. Again, the following theorem shows that the general situation is comparable to this model one. We use the same trivialization of  $F$  and  $L$  in the neighborhood of  $y_0$  as in Theorem 1.6, and interpret  $B_p^\perp(x_1, x_2)$  as an element of  $\text{End}(F_{y_0})$ , for  $x_1, x_2 \in X$  in a  $\min(r_\perp, r_X)$ -neighborhood of  $y_0$ .

**Theorem 1.9.** *For any  $r \in \mathbb{N}, y_0 \in Y$ , there are polynomials  $J_r^\perp(Z, Z') \in \text{End}(F_{y_0}), Z, Z' \in \mathbb{R}^{2n}$ , with the same properties as in Theorem 1.6, such that for  $F_r^\perp := J_r^\perp \cdot \mathcal{P}_{n,m}^\perp$ , the following holds.*

There are  $\epsilon, c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $y_0 \in Y$ ,  $p \geq p_1$ ,  $Z = (Z_Y, Z_N)$ ,  $Z' = (Z'_Y, Z'_N)$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $Z_N, Z'_N \in \mathbb{R}^{2(n-m)}$ ,  $|Z|, |Z'| \leq \epsilon$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , for  $Q_{k,l,l'}^2 := 3(8(n+2)(k+1) + l') + l$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \frac{1}{p^n} B_p^\perp(\psi_{y_0}(Z), \psi_{y_0}(Z')) - \sum_{r=0}^k p^{-\frac{r}{2}} F_r^\perp(\sqrt{p}Z, \sqrt{p}Z') \kappa_X^{-\frac{1}{2}}(Z) \kappa_X^{-\frac{1}{2}}(Z') \right) \right|_{\mathcal{C}^{l'}(Y)} \leq Cp^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^{Q_{k,l,l'}^2} \exp \left( -c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|) \right), \quad (1.27)$$

where the  $\mathcal{C}^{l'}$ -norm is taken with respect to  $y_0$ . Also, we have

$$J_0^\perp(Z, Z') = \text{Id}_{F_{y_0}}. \quad (1.28)$$

Moreover, under the assumptions (1.8) and notations as in (1.23), we have

$$J_1^\perp(Z, Z') = \text{Id}_{F_{y_0}} \cdot \pi g(z_N + \bar{z}'_N, A(Z_Y - Z'_Y)(Z_Y - Z'_Y)). \quad (1.29)$$

*Remark 1.10.* We present an algorithmic way to compute the polynomials  $J_r^E, J_r^\perp$ .

Let us finally say few words about the tools we use in this article. The proofs of Theorems 1.5, 1.8 rely on the exponential estimate for the Bergman kernel, cf. Ma-Marinescu [30], and the refinement – in our asymptotic setting – of the Ohsawa-Takegoshi extension theorem, see Theorem 4.1, the proof of which is inspired by Bismut-Lebeau [3] and Demailly [10].

The proofs of Theorems 1.6, 1.9 rely on some techniques from spectral analysis, inspired by [3]; on the existence of the so-called uniform Stein atlases (see Definition 2.33) on Kähler manifolds of bounded geometry, which we establish using Hörmander's  $L^2$ -estimates; on the asymptotic expansion of the Bergman kernel due to Dai-Liu-Ma [9] and some technical results about the algebras of operators with Taylor-type expansion of the Schwartz kernel, which are inspired by the work of Ma-Marinescu [29], cf. [28, §7]. The general strategy for dealing with semi-classical limits here is inspired by Bismut [2] and Bismut-Vasserot [4].

The technical novelty of this paper, compared to [9], is that, unlike  $B_p^X$ , the operator  $B_p^\perp$  is *not* the spectral projector associated to the Kodaira Laplacian. This breaks apart most of the techniques used in [9], [28], as, for example, the relation with the heat kernel is no longer available. To remedy this, instead of Laplacian, we construct an ‘‘ad hoc’’ operator, see (5.2), based on the restriction map and  $B_p^X$ , so that  $B_p^\perp$  is the spectral projector associated to this new operator.

A similar idea of using spectral theory of operators other than Laplacian in the study of Bergman kernel has been used in symplectic reduction setting by Ma-Zhang [31]. In their manuscript, authors use the deformation of Laplacian by the Casimir operator, coming from Hamiltonian action of a compact Lie group. This deformation, differently from our setting, is a differential operator itself. This makes technical details of our article different, as here the spectral theory is applied to an operator, which is no longer local, and actually has a smooth Schwartz kernel.

We note that Theorems 1.8, 1.9, can be reformulated in terms of the so-called *logarithmic Bergman kernel*, which correspond to the Schwartz kernel of  $B_p^0$ . This is due to the obvious relation  $B_p^X = B_p^0 + B_p^\perp$ , and the fact that the Schwartz kernel of  $B_p^X$  is already well-understood by the results of Tian [38], Catlin [6], Zelditch [40], Dai-Liu-Ma [9] and Ma-Marinescu [30].

We, finally, mention that a problem, related to the one from this article, but in realms of non-Archimedean geometry, has been considered by Chen-Moriwaki [7] and Fang [18].



This paper is organized as follows. In Section 2, we recall the definitions of manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry. We compare Fermi coordinates to geodesic coordinates. We then introduce and study the notion of uniform Stein atlases. In Section 3, we prove that the set of operators over manifolds of bounded geometry, admitting certain bounds on the Schwartz kernels, forms an algebra under the composition. In Section 4, we establish a spectral bound for the restriction operator. Finally, in Section 5, by using all the above results, we establish the results announced in this section.

**Notations.** We use notations  $X, Y$  for complex manifolds and  $M, H$  for real manifolds. The complex (resp. real) dimensions of  $X, Y$  (resp.  $M, H$ ) are denoted here by  $n, m$ . An operator  $\iota$  always means an embedding  $\iota : Y \rightarrow X$  (resp.  $\iota : H \rightarrow M$ ). We denote by  $\text{Res}_Y$  (resp.  $\text{Res}_H$ ) the operator of restriction of sections of a certain vector bundle over  $X$  to  $Y$  (resp.  $M$  to  $H$ ). A variable  $p \in \mathbb{N}$  always denotes the power of a line bundle, and  $t \in \mathbb{R}$  is related to it by

$$t = \frac{1}{\sqrt{p}}. \quad (1.30)$$

For a Riemannian manifold  $(M, g^{TM})$ , we denote the Levi-Civita connection by  $\nabla^{TM}$ , and by  $dv_{g^{TM}}$  the Riemannian volume form. For a closed subset  $W \subset M$ ,  $r \geq 0$ , let  $B_W^M(r)$  be the ball of radius  $r$  around  $W$ . For a Hermitian vector bundle  $(E, h^E)$ , note  $B_r(E) := \{Z \in E : |Z|_{h^E} < r\}$ .

For a fixed volume form  $dv_M$  on  $M$ , we denote by  $L^2(dv_M, h^E)$  the space of  $L^2$ -sections of  $E$  with respect to  $dv_M$  and  $h^E$ . When  $dv_M = dv_{g^{TM}}$ , we also use the notation  $L^2(g^{TM}, h^E)$ . When there is no confusion about the data, we also use the simplified notation  $L^2(M)$  or even  $L^2$ .

For  $n \in \mathbb{N}^*$ , we denote by  $dv_{\mathbb{C}^n}$  the standard volume form on  $\mathbb{C}^n$ . We view  $\mathbb{C}^m$  (resp.  $\mathbb{R}^{2m}$ ) embedded in  $\mathbb{C}^n$  (resp.  $\mathbb{R}^{2n}$ ) by the first  $m$  coordinates. For  $Z \in \mathbb{R}^k$ , we denote by  $Z_l$ ,  $l = 1, \dots, k$ , the coordinates of  $Z$ . If  $Z \in \mathbb{R}^{2n}$ , we denote by  $z_i$ ,  $i = 1, \dots, n$  the induced complex coordinates  $z_i = Z_{2i-1} + \sqrt{-1}Z_{2i}$ . We frequently use the decomposition  $Z = (Z_Y, Z_N)$ , where  $Z_Y = (Z_1, \dots, Z_{2m})$  and  $Z_N = (Z_{2m+1}, \dots, Z_{2n})$ . For a fixed frame  $(e_1, \dots, e_{2n})$  in  $T_y X$ ,  $y \in Y$ , we implicitly identify  $Z$  (resp.  $Z_Y, Z_N$ ) to an element in  $T_y X$  (resp.  $T_y Y, N_y$ ) by

$$Z = \sum_{i=1}^{2n} Z_i e_i, \quad Z_Y = \sum_{i=1}^{2m} Z_i e_i, \quad Z_N = \sum_{i=2m+1}^{2n} Z_i e_i. \quad (1.31)$$

If the frame  $(e_1, \dots, e_{2n})$  satisfies (1.16), we denote  $\frac{\partial}{\partial z_i} := \frac{1}{2}(e_{2i-1} - \sqrt{-1}e_{2i})$ ,  $\frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}(e_{2i-1} + \sqrt{-1}e_{2i})$ , and identify  $z, \bar{z}$  to vectors in  $T_y X \otimes_{\mathbb{R}} \mathbb{C}$  as follows

$$z = \sum_{i=1}^n z_i \cdot \frac{\partial}{\partial z_i}, \quad \bar{z} = \sum_{i=1}^n \bar{z}_i \cdot \frac{\partial}{\partial \bar{z}_i}. \quad (1.32)$$

Clearly, in this identification, we have  $Z = z + \bar{z}$ ,  $(\text{Id} - \sqrt{-1}J)Z = 2z$ , and  $(\text{Id} + \sqrt{-1}J)Z = 2\bar{z}$ . We define  $z_Y, \bar{z}_Y \in T_y Y \otimes_{\mathbb{R}} \mathbb{C}$ ,  $z_N, \bar{z}_N \in N_y \otimes_{\mathbb{R}} \mathbb{C}$  in a similar way.

For  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ ,  $B = (B_1, \dots, B_k) \in \mathbb{C}^k$ , we write by

$$|\alpha| = \sum_{i=1}^k \alpha_i, \quad \alpha! = \prod_{i=1}^k (\alpha_i)!, \quad B^\alpha = \prod_{i=1}^k B_i^{\alpha_i}. \quad (1.33)$$

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## 2 Manifolds of bounded geometry and uniform Stein atlases

We recall the definitions of manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry and study some of their properties. Then, we introduce and study uniform Stein atlases.

Manifolds of bounded geometry are certain complete manifolds for which some uniform boundedness conditions on the curvature and injectivity radii are assumed. The reason for considering those type of manifolds in this article is twofold.

First, they appear naturally already in the study of compact manifolds. This is due to the fact that our main philosophy here is to reduce all the statements from general manifolds to  $\mathbb{C}^n$ . As we would like our theory to work with  $\mathbb{C}^n$  as well, which is no longer compact, the setting of compact manifolds is not an appropriate one for our needs. It turns out that the class of manifolds of bounded geometry is wide enough to contain both compact manifolds and  $\mathbb{C}^n$ , and restrictive enough to satisfy some essential estimates in our approach.

Second, the full force of Ohsawa-Takegoshi extension theorem comes from the fact that it can be applied to local problems over general weakly pseudoconvex domains. Many geometric applications of the extension theorem, as [14] (which also partially motivated the current article), are formulated for non-compact manifolds, see Remark 1.4. As we would like our theorems to be useful in those contexts as well, we need to abandon the compactness assumption.

This section is organized as follows. In Section 2.1, we recall the definitions of manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry. In Section 2.2, we compare the geodesic and Fermi coordinates. In Section 2.3, we calculate the holonomy of the vector bundle along the paths, adapted to the two coordinate systems. In Section 2.4, we give a formula for the complex structure in Fermi coordinates and study quasi-plurisubharmonicity of some functions on manifolds of bounded geometry. Finally, in Section 2.5, we introduce and study uniform Stein atlases, and prove that any Kähler manifold of bounded geometry admits uniform Stein atlas.

### 2.1 Coordinate-free and coordinate-wise descriptions of bounded geometry

In this section, we recall the definitions of manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry. There are mainly two ways to define objects of bounded geometry. Either one uses some bounds on the curvature (and related objects) – this is the coordinate-free description – or one uses charts constructed by geodesics and bounds the relevant structures (as the metric tensor) and its derivatives in these coordinates – the coordinate-wise description. The equivalence of these two perspectives is established in the works of Eichhorn [16], Schick [35] and Große-Schneider [23]. In this section, we recall those statements precisely.

**Definition 2.1.** We say that a Riemannian manifold  $(M, g^{TM})$  is of bounded geometry if the following two conditions are satisfied.

- (i) The injectivity radius of  $(M, g^{TM})$  is bounded below by a positive constant  $r_M$ .
- (ii) Every covariant derivative of the Riemann curvature tensor  $R^{TM}$  of  $M$  is bounded, i.e. for any  $k \in \mathbb{N}$ , there is a constant  $C_k > 0$  such that for any  $l = 0, \dots, k$ , we have

$$|(\nabla^{TM})^l R^{TM}| \leq C_k, \quad (2.1)$$

where  $\nabla^{TM}$  is the connection induced by the Levi-Civita connection, and the pointwise norm is taken with respect to  $g^{TM}$ .

*Remark 2.2.* The condition (i) from Definition 2.1 implies that  $(M, g^{TM})$  is complete.

Let us fix  $x_0 \in M$  and an orthonormal frame  $(e_1, \dots, e_n)$  of  $(T_{x_0}M, g_{y_0}^{TM})$ . We identify  $\mathbb{R}^n$  to  $T_{x_0}M$  implicitly as in (1.31). We introduce the map  $\phi_{x_0} : \mathbb{R}^n \rightarrow M, x_0 \in M$ , as follows

$$\phi_{x_0}(Z) := \exp_{x_0}^M(Z). \quad (2.2)$$

As a general rule, whenever the point  $x_0$  is implicit, we drop it out from the subscript.

**Proposition 2.3** ([16, Theorem A]). Assume that for some  $k \in \mathbb{N}$ , the bound (2.1) holds. Then for any  $r_0 > 0$ , there is  $D_k > 0$ , such that for any  $x_0 \in X, l = 0, \dots, k$ , we have

$$\|g_{ij}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r_0))} \leq D_k, \quad \|g^{ij}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r_0))} \leq D_k, \quad (2.3)$$

where  $g_{ij}, i, j = 1, \dots, n$ , are the coefficients of the metric tensor  $\phi_{x_0}^* g^{TM}$ , and  $g^{ij}$  are the coefficients of the inverse tensor. Moreover,  $D_k$  depends only on  $k, C_k$  from (2.1) and  $r_0$ .

*Remark 2.4.* In [35, Proposition 3.3] it was also proved that the transition maps between two different geodesic coordinates can be bounded in a similar way. This gives us *the coordinate-wise approach* through geodesic coordinates to manifolds of bounded geometry.

Now, let us fix two metrics  $g_1^{TM}, g_2^{TM}$  on a manifold  $M$ . Assume that the injectivity radius of  $(M, g_1^{TM})$  is bounded below by a positive constant  $r_{M,1}$  (in particular,  $(M, g_1^{TM})$  is complete). Assume, moreover, that there is a section  $T \in \mathcal{C}(M, \text{End}(TM))$  such that

$$g_1^{TM}(u, v) = g_2^{TM}(Tu, v), \quad u, v \in TM, \quad (2.4)$$

and for any  $l = 0, \dots, k$ , the following bounds are satisfied

$$|(\nabla_1^{TM})^l T| \leq C_k, \quad |(\nabla_1^{TM})^l T^{-1}| \leq C_k, \quad (2.5)$$

where  $\nabla_1^{TM}$  is the Levi-Civita connection associated to  $g_1^{TM}$ . From (2.5), we conclude that for any  $x, y \in M$ , we have

$$\frac{1}{\sqrt{C_0}} \text{dist}_{M,1}(x, y) \leq \text{dist}_{M,2}(x, y) \leq \sqrt{C_0} \text{dist}_{M,1}(x, y), \quad (2.6)$$

where  $\text{dist}_{M,i}(x, y), i = 1, 2$ , is the distance on  $M$  induced by  $g_i^{TM}$ . From (2.6) and Hopf–Rinow theorem, we see that  $(M, g_2^{TM})$  is also complete. We denote by  $\phi_{x_0}^1 : \mathbb{R}^n \rightarrow M, \phi_{x_0}^2 : \mathbb{R}^n \rightarrow M$  the maps, constructed as in (2.2), but for  $(M, g_1^{TM})$  and  $(M, g_2^{TM})$  respectively. From (2.6), we see that for  $r' := \frac{r_{M,1}}{\sqrt{C_0}}$ , we have

$$\phi_{x_0}^2(B_0^{\mathbb{R}^n}(r')) \subset \phi_{x_0}^1(B_0^{\mathbb{R}^n}(r_{M,1})). \quad (2.7)$$

We denote  $h'_{x_0} := (\phi_{x_0}^1)^{-1} \circ \phi_{x_0}^2$ , and consider it as a map  $h'_{x_0} : B_0^{\mathbb{R}^n}(r') \rightarrow \mathbb{R}^n$ . By definition,

$$\phi_{x_0}^2 = \phi_{x_0}^1 \circ h'_{x_0}. \quad (2.8)$$

**Proposition 2.5.** Assume that for some  $k \in \mathbb{N}$ , the bounds (2.1) and (2.5) hold. Then there is  $D_k > 0$ , which depends only on  $k, C_k$  from (2.1) and (2.5), such that for any  $x_0 \in X, l = 0, \dots, k - 1$ ,

$$\|h'_{x_0}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r'))} \leq D_k. \quad (2.9)$$

*Proof.* Let us consider the metric  $g_{2,0}^{TM} := (\phi_{x_0}^1)^* g_2^{TM}$  over  $\mathbb{R}^n$ . By Proposition 2.3 and (2.5), we deduce that for the metric tensor  $g_{ij,0}$ , associated to  $g_{2,0}^{TM}$  on  $\mathbb{R}^n$ , as well as for its inverse  $g_0^{ij}$ , there is  $E_k > 0$ , which depends only on  $k, r_0, C_k$ , such that

$$\|g_{ij,0}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r_{M,1}))} \leq E_k, \quad \|g_0^{ij}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r_{M,1}))} \leq E_k. \quad (2.10)$$

Remark that the map  $h'_{x_0}$  coincides with the geodesic coordinates constructed on  $(\mathbb{R}^n, g_{2,0}^{TM})$ , centered at the point  $0 \in \mathbb{R}^n$ . It is now only left to use [35, Lemma 3.8].  $\square$

**Corollary 2.6.** Assume that there is  $C_3 > 0$ , such that the bounds (2.1) and (2.5) hold for  $k = 3$ . Then there is  $r_0 > 0$ , which depends only on  $C_3$  and  $r_{M,1}$ , such that  $h'_{x_0}|_{B_0^{\mathbb{R}^n}(r_0)}$  is a diffeomorphism onto its image. In particular, the injectivity radius of  $(M, g_2^{TM})$  is bounded below by  $r_0$ .

*Proof.* By the properties of the exponential mapping, we have  $dh'_{x_0}(0) = \text{Id}$ . Now, as from Proposition 2.5, the second derivative of  $h'_{x_0}$  is uniformly bounded over  $B_0^{\mathbb{R}^n}(r')$ , we conclude that there is  $r_0 > 0$ , which depends only on  $C_3$  and  $r_{M,1}$ , such that

$$\|dh'_{x_0} - \text{Id}\|_{\mathcal{C}^0(B_0^{\mathbb{R}^n}(r_0))} \leq \frac{1}{2}. \quad (2.11)$$

In particular, for any  $x_1, x_2 \in B_0^{\mathbb{R}^n}(r_0)$ , we have

$$\left| h'_{x_0}(x_1) - h'_{x_0}(x_2) - (x_1 - x_2) \right| \leq \frac{|x_1 - x_2|}{2}. \quad (2.12)$$

Hence,  $h'_{x_0}$  is injective on  $B_0^{\mathbb{R}^n}(r_0)$ . Since  $dh'_{x_0}$  is also invertible over  $B_0^{\mathbb{R}^n}(r_0)$ , by (2.11), the inverse function theorem implies that  $h'_{x_0}|_{B_0^{\mathbb{R}^n}(r_0)}$  is a diffeomorphism onto its image.  $\square$

**Proposition 2.7.** Assume  $(M, g^{TM})$  is of bounded geometry. Then for any  $0 < \epsilon < \frac{r_M}{4}$ , there are  $x_i \in M, i \in \mathbb{N}$ , such that  $B_{x_i}^M(\epsilon)$  cover  $M$  with a uniform finite overlap (independent of  $\epsilon$ ).

*Proof.* We essentially follow [23, Remark 4.6b)]. First of all, from (2.3), we see that there are  $c, C > 0$ , such that for any  $0 < \epsilon < r_M, x_0 \in M$ , the following holds

$$c\epsilon^n < \text{Vol}(B_{x_0}^M(\epsilon)) < C\epsilon^n. \quad (2.13)$$

Now, for a fixed  $0 < \epsilon < \frac{r_M}{4}$ , let us fix a maximal set of points  $x_i, i \in \mathbb{N}$ , such that the balls  $B_{x_i}^M(\frac{\epsilon}{2})$  are disjoint. Then the balls  $B_{x_i}^M(\epsilon)$  form a cover of  $M$ . We would like to prove that this cover has a uniform finite overlap, which is independent of  $\epsilon$ . For this, assume that for a fixed  $i \in \mathbb{N}$  and a subset  $\{y_1, \dots, y_l\}, l \in \mathbb{N}$  of  $\{x_i, i \in \mathbb{N}\}$ , the balls  $B_{y_j}^M(\epsilon)$  intersect  $B_{x_i}^M(\epsilon), j = 1, \dots, l, i \in \mathbb{N}$ . Then it is easy to see that  $B_{x_i}^M(4\epsilon)$  contains the union of  $B_{y_j}^M(\frac{\epsilon}{2})$ . As this is a disjoint union, we conclude from (2.13) that  $l \leq 8^n \frac{C}{c}$ , which finishes the proof.  $\square$

Now, let  $(H, g^{TH})$  be an embedded submanifold of  $(M, g^{TM})$ ,  $g^{TH} := g^{TM}|_H$ . We identify the normal bundle  $N$  of  $H$  in  $M$  as an orthogonal complement of  $TH$  in  $TM$  as in (1.6). We denote by  $g^N$  the metric on  $N$  induced by  $g^{TM}$ , and define  $\nabla^N$  as in (1.19). We denote by  $A$  the second fundamental form of the embedding of  $H$  in  $M$ , defined as in (1.19).

**Definition 2.8.** We say that the triple  $(M, H, g^{TM})$  is of bounded geometry if the following conditions are fulfilled.

- (i) The manifold  $(M, g^{TM})$  is of bounded geometry.
- (ii) The injectivity radius of  $(H, g^{TH})$  is bounded below by a positive constant  $r_H$ .
- (iii) There is a collar around  $H$  (a tubular neighborhood of fixed radius), i.e. there is  $r_\perp > 0$  such that for any  $x, y \in H$ , the normal geodesic balls  $B_{r_\perp}^\perp(x), B_{r_\perp}^\perp(y)$ , obtained by the application of the exponential mapping to vectors, orthogonal to  $H$ , of norm, bounded by  $r_\perp$ , are disjoint.
- (iv) Every covariant derivative of  $A$  of  $M$  is bounded, i.e. for all  $k \in \mathbb{N}$ , there is a constant  $C_k > 0$  such that for any  $l = 0, \dots, k$ , we have

$$|\nabla^l A| \leq C_k, \quad (2.14)$$

where  $\nabla$  is the connection induced by the Levi-Civita connection  $\nabla^{TH}$  and  $\nabla^N$ , and the pointwise norm is taken with respect to  $g^{TM}$ .

Clearly, condition (iii) from Definition 2.8 means that the map  $H \times B_{r_\perp}(N) \rightarrow M, (y, Z_N) \mapsto \exp_y^M(Z_N), y \in H, Z_N \in N_y, |Z_N| < r_\perp$ , is a diffeomorphism onto a tubular neighborhood  $U := B_H^M(r_\perp)$  of  $H$  in  $M$ . We define the projection

$$\pi : U \rightarrow H, \quad \exp_y^M(Z_N) \mapsto y. \quad (2.15)$$

We now fix a point  $y_0 \in H$  and an orthonormal frame  $(e_1, \dots, e_m)$  (resp.  $(e_{m+1}, \dots, e_n)$ ) in  $(T_{y_0}H, g^{TH})$  (resp. in  $(N, g^N)$ ). We define the *Fermi coordinates*,  $\psi_{y_0}$ , at  $y_0$  as in (1.17). Again, as both manifolds  $(M, g^{TM}), (H, g^{TH})$ , are complete, one can extend the domain of  $\psi_{x_0}$  to  $\mathbb{R}^n$ .

**Proposition 2.9** ([35, Lemma 3.9], [23, Theorem 4.9]). Assume that for some  $k \in \mathbb{N}$ , the bounds (2.1), (2.14) hold. Then for any  $r_0 > 0$ , there is  $D_k > 0$ , such that for any  $y_0 \in X, l = 0, \dots, k$ ,

$$\|g_{ij}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r_0))} \leq D_k, \quad \|g^{ij}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r_0))} \leq D_k. \quad (2.16)$$

where  $g_{ij}, i, j = 1, \dots, n$ , are the coefficients of the metric tensor  $\psi_{x_0}^* g^{TM}$ , and  $g^{ij}$  are the coefficients of the inverse tensor. Moreover,  $D_k$  depends only on  $k, r_0$  and  $C_k$  from (2.1), (2.14).

**Corollary 2.10.** Assume that for some  $k = 0$ , the bounds (2.1), (2.14) hold. Then there are  $c, C > 0$ , such that over  $B_H^M(r_\perp)$ , we have  $c \leq \kappa_N \leq C$ , where  $\kappa_N$  is defined as in (1.7).

Define the constant  $R > 0$  as follows

$$R := \min \left\{ \frac{r_M}{2}, \frac{r_H}{4}, \frac{r_\perp}{4} \right\}. \quad (2.17)$$

**Proposition 2.11.** Assume  $(M, H, g^{TM})$  is of bounded geometry. Then for any  $0 < \epsilon < \frac{R}{4}$ , there is a set of points  $y_i \in Y, i \in \mathbb{N}$ , such that  $B_{y_i}^H(\epsilon)$  have a uniform finite overlap (independent of  $\epsilon$ ) and cover  $B_H^M(\frac{\epsilon}{2})$ .

*Proof.* The proof is identical to the proof of Proposition 2.7 with only one change: instead of (2.10), one should use (2.16).  $\square$

Now, assume that the two metrics  $g_1^{TM}, g_2^{TM}$  on  $M$  are as in (2.4). For  $y_0 \in H$ , we denote by  $\psi_{y_0}^1 : \mathbb{R}^n \rightarrow M$ ,  $\psi_{y_0}^2 : \mathbb{R}^n \rightarrow M$  the maps, constructed as in (1.17), but for  $(M, g_1^{TM})$  and  $(M, g_2^{TM})$  respectively. We denote by  $r_{N,i}, r_{\perp,i}$  the lower bounds on the injectivity radii and the geodesic tubular neighborhood for  $(N, g_i^{TN})$ . From (2.6), we see that for  $r' := \frac{\min\{r_{N,1}, r_{\perp,1}\}}{\sqrt{C_0}}$ , for  $C_0$  as in (2.5), we may consider a map  $h''_{y_0} : B_0^{\mathbb{R}^n}(r') \rightarrow \mathbb{R}^n$ , which satisfies

$$\psi_{y_0}^2 = \psi_{y_0}^1 \circ h''_{y_0}. \quad (2.18)$$

**Proposition 2.12.** Assume that for some  $k \in \mathbb{N}$ , the bounds (2.1), (2.5) and (2.14) hold. Then there is  $D_k > 0$ , which depends only on  $k, C_k$  from (2.1), (2.5), (2.14) and  $r_{N,1}, r_{\perp,1}$ , such that for any  $y_0 \in X, l = 0, \dots, k-1$ , we have

$$\|h''_{y_0}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r'))} \leq D_k. \quad (2.19)$$

*Proof.* The proof is identical to the proof of Proposition 2.5.  $\square$

There is a diffeomorphism  $h_{y_0} : B_0^{\mathbb{R}^n}(R) \rightarrow \mathbb{R}^n$ , such that the following holds

$$\psi_{y_0} = \phi_{y_0} \circ h_{y_0}. \quad (2.20)$$

By repeating the proof of Proposition 2.5, we deduce the following.

**Proposition 2.13** ([23, Lemma 4.7]). Assume that for some  $k \in \mathbb{N}$ , the bounds (2.1), (2.5) and (2.14) hold. Then there is  $D_k > 0$ , which depends only on  $k, R, C_k$  from (2.1), (2.5) and (2.14), such that for any  $y_0 \in X, l = 0, \dots, k-1$ , we have

$$\|h_{y_0}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(R))} \leq D_k, \quad \|(h_{y_0})^{-1}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(R))} \leq D_k. \quad (2.21)$$

All in all, this gives the *coordinate-wise approach* through Fermi coordinates to triples of bounded geometry. Remark also that from Proposition 2.9, it follows that if  $(M, H, g^{TM})$  is of bounded geometry, then  $(H, g^{TH})$  is of bounded geometry, cf. [23, Lemma 4.5].

**Definition 2.14.** Let  $(E, \nabla^E, h^E)$  be a Hermitian vector bundle with a fixed Hermitian connection over a manifold  $(M, g^{TM})$  of bounded geometry. We say that  $(E, \nabla^E, h^E)$  is of bounded geometry if for any  $k \in \mathbb{N}$ , there is a constant  $C_k > 0$  such that for any  $l = 0, \dots, k$ , we have

$$|\nabla^l R^E| < C_k, \quad (2.22)$$

where  $\nabla$  is the connection induced by the Levi-Civita connection  $\nabla^{TM}$  and  $\nabla^E$ , and the pointwise norm is taken with respect to  $g^{TM}$ .

If  $(E, h^E)$  is a Hermitian vector bundle over a *complex manifold*, we say that it is of bounded geometry if  $(E, \nabla^E, h^E)$  is of bounded geometry for the Chern connection  $\nabla^E$  on  $(E, h^E)$ .

Let us now give the coordinate-wise description for vector bundles of bounded geometry. Let us first construct a trivialization of vector bundle  $(E, \nabla^E)$ ,  $\text{rk}(E) = r$ , as follows. We fix a point  $x_0 \in M$  and an orthonormal frame  $f_1, \dots, f_r \in E_{x_0}$ . Let  $\tilde{f}'_1, \dots, \tilde{f}'_r$  be a frame of  $E$  over  $B_{x_0}^M(r_M)$ , obtained by the parallel transport of  $f_1, \dots, f_r$  along the curve  $\phi_{x_0}(tZ)$ ,  $t \in [0, 1]$ ,  $Z \in T_{x_0}M$ ,  $|Z| < r_M$ . Clearly, as  $\nabla^E$  is Hermitian,  $\tilde{f}'_1, \dots, \tilde{f}'_r$  is an orthonormal frame over  $B_{x_0}^M(r_M)$ . We denote by  $\Gamma^{E'}$  the connection form of  $(E, \nabla^E)$  with respect to this frame.

**Proposition 2.15** ([16, Theorem B]). Assume that for some  $k \in \mathbb{N}^*$ , the bounds (2.1), (2.22) hold. Then there is  $D_k > 0$ , such that for any  $x_0 \in X$ ,  $l = 0, \dots, k-1$ , we have

$$\|\Gamma^{E'}\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(r_M))} \leq D_k. \quad (2.23)$$

Moreover,  $D_k$  depends only on  $k$ ,  $C_k$  from (2.1), (2.22) and  $r_M$ .

Now, let us consider another trivialization of  $(E, \nabla^E)$ . We place ourselves in a setting where  $(M, H, g^{TM})$  is a triple of bounded geometry. We fix a point  $y_0 \in H$  and an orthonormal frame  $f_1, \dots, f_r \in E_{y_0}$ . We define  $\tilde{f}_1, \dots, \tilde{f}_r$  by the parallel transport as it was done before Theorem 1.6. We denote by  $\Gamma^E$  the connection form of  $(E, \nabla^E)$  with respect to this frame.

**Proposition 2.16** ([23, Lemma 5.13]). Assume that for some  $k \in \mathbb{N}^*$ , the bounds (2.1), (2.14), (2.22) hold. Then there is  $D_k > 0$ , such that for any  $y_0 \in Y$ ,  $l = 0, \dots, k-1$ , we have

$$\|\Gamma^E\|_{\mathcal{C}^l(B_0^{\mathbb{R}^n}(R))} \leq D_k. \quad (2.24)$$

Moreover,  $D_k$  depends only on  $k$ ,  $C_k$  from (2.1), (2.14), (2.22) and  $R$ .

Let  $\xi_E$  be the function, defined in  $B_{y_0}^M(R)$ , with values in  $\text{End}(\mathbb{C}^r)$ , such that

$$(\tilde{f}_1, \dots, \tilde{f}_r) = \exp(\xi_E) \cdot (\tilde{f}'_1, \dots, \tilde{f}'_r), \quad (2.25)$$

where we view  $(\tilde{f}_1, \dots, \tilde{f}_r)$  and  $(\tilde{f}'_1, \dots, \tilde{f}'_r)$  as  $r \times 1$  matrices.

**Proposition 2.17.** Assume that for some  $k \in \mathbb{N}^*$ , the bounds (2.1), (2.14), (2.22) hold. Then there is  $D_k > 0$ , such that for any  $y_0 \in Y$ ,  $l = 0, \dots, k-1$ , we have

$$\|\xi_E\|_{\mathcal{C}^l(B_{y_0}^M(R))} \leq D_k. \quad (2.26)$$

Moreover,  $D_k$  depends only on  $k$ ,  $C_k$  from (2.1), (2.14), (2.22) and  $R$ .

*Proof.* The proof is analogous to the proof of Proposition 2.5. □

## 2.2 Diffeomorphism between Fermi and geodesic coordinates

In this section, we study the Taylor expansion of the diffeomorphism comparing geodesic and Fermi coordinates.

We conserve the notations from Section 2.1. We place ourselves in the setting of a triple  $(M, H, g^{TM})$  of bounded geometry. We define  $A \in \mathcal{C}^\infty(H, T^*H \otimes \text{End}(TM|_H))$  as in (1.19). We define an auxiliary form  $B \in \mathcal{C}^\infty(Y, \text{Sym}^2(T^*X|_Y) \otimes TX|_Y)$  in the notation (1.19) by

$$B(u) := B(u, u) := \frac{1}{2}A(P^H u)P^H u + A(P^H u)P^N u, \quad u \in TM|_H. \quad (2.27)$$

We fix a point  $y_0 \in H$  and an orthonormal frame  $(e_1, \dots, e_m)$  (resp.  $(e_{m+1}, \dots, e_n)$ ) in  $(T_{y_0}H, g^{TH})$  (resp. in  $(N, g^N)$ ). Recall that in (1.17) and (2.2), we defined two coordinate systems  $\psi_{y_0}, \phi_{y_0}$  in a neighborhood of  $y_0$ , and in (2.20), we defined a diffeomorphism  $h_{y_0} : B_0^{\mathbb{R}^n}(R) \rightarrow \mathbb{R}^n$ , for  $R$  from (2.17). We drop out  $y_0$  from the subscripts from now on. The main goal of this section is to study the Taylor expansion of  $h_{y_0}$  at  $0 \in \mathbb{R}^{2n}$ .

**Proposition 2.18.** The diffeomorphism  $h$  has the following Taylor expansion

$$h(Z) = Z + B(Z) + O(|Z|^3). \quad (2.28)$$

Moreover, the coefficients of order  $r$ ,  $r \in \mathbb{N}$ , in above Taylor expansion can be expressed in terms of  $R^{TM}$ ,  $A$ , and their derivatives up to order  $r - 2$  with respect to  $\nabla^{TM}$ , evaluated at  $y_0$ .

Before presenting the proof of Proposition 2.18, let us prove some auxiliary results.

**Lemma 2.19.** Let  $\gamma(t)$  be a geodesic in  $(M, g^{TM})$ , and let  $v := \gamma'(t)$ . Then for any  $k \in \mathbb{N}$ , the following identity between operators on smooth functions on  $M$  holds

$$v^{\otimes k} \cdot (\nabla^{TM})^{\otimes k} = \left( \frac{\partial}{\partial v} \right)^k, \quad (2.29)$$

where we view  $(\nabla^{TM})^{\otimes k}$  as an operator  $(\nabla^{TM})^{\otimes k} : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M, (T^*M)^{\otimes k})$ , and  $\cdot$  on the left-hand side of (2.29) means the contraction.

*Proof.* The proof is a consequence of the simple fact that  $\nabla_v^{TM} v = 0$  and the formula

$$(\nabla^{TM} \alpha)(X_1, \dots, X_{k+1}) = X_1 \alpha(X_2, \dots, X_{k+1}) - \sum_{i=2}^{k+1} \alpha(X_2, \dots, \nabla_{X_1}^{TM} X_i, \dots, X_{k+1}), \quad (2.30)$$

where  $\alpha$  is a  $k$ -form and  $X_1, \dots, X_{k+1}$  are some vector fields.  $\square$

**Corollary 2.20.** Let  $u$  be a smooth function on  $M$ . Then on the level of formal Taylor expansions, the following identity holds

$$u(\exp_{y_0}^M(Z)) = \sum_{k=0}^{\infty} \frac{1}{k!} Z^{\otimes k} \cdot (\nabla^{TM})^k u(y_0). \quad (2.31)$$

*Proof.* We have the following identity

$$u(\exp_{y_0}^M(Z)) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dt^k} u(\exp_{y_0}^M(tZ)). \quad (2.32)$$

The proof now is a direct consequence of Lemma 2.19.  $\square$

Both Lemma 2.19 and Corollary 2.20 appeared in Gavrilov [20, §2]. Let us now define the connection  $\nabla^a$  on  $TM|_H$  as follows

$$\nabla^a = \nabla^{TH} \oplus \nabla^N, \quad (2.33)$$

where we used the notation as in (1.19).

**Lemma 2.21.** Let  $\gamma(t)$  be a geodesic in  $(H, g^{TH})$ , and let  $v := \gamma'(t)$ . Let  $Z_1(t), \dots, Z_l(t) \in N$ ,  $l \in \mathbb{N}$  be vector fields along  $\gamma(t)$ , which are parallel along  $\gamma(t)$  with respect to  $\nabla^N$ . Then for any  $k \in \mathbb{N}$ , the following identity between operators on smooth functions on  $M$  holds

$$\begin{aligned} & \left( v^{\otimes k} \cdot (\nabla^{TH})^{\otimes k} \right) \circ \text{Res}_H \circ \left( (Z_1 \otimes \dots \otimes Z_l) \cdot (\nabla^{TM})^{\otimes l} \right) \\ &= \text{Res}_H \circ \left( (v^{\otimes k} \otimes Z_1 \otimes \dots \otimes Z_l) \cdot (\nabla^a)^{\otimes k} (\nabla^{TM})^{\otimes l} \right), \end{aligned} \quad (2.34)$$

where the compositions of the connections are interpreted in the same way as in Lemma 2.19.



*Proof.* The proof is a direct consequence of  $\nabla_v^a Z_i = \nabla_v^N Z_i = 0$ ,  $\nabla_v^{TH} v = 0$  and (2.30).  $\square$

*Proof of Proposition 2.18.* The main idea is to fix a smooth function  $u$  on  $M$  and to use Lemmas 2.19, 2.21 to get two different expressions for the Taylor expansion of  $u(\psi(Z))$ . By comparing the two expansions, we will get the Taylor expansion (2.28).

In one way, since  $\psi(Z_H, Z_N) = \phi(h(Z_H, Z_N))$ , we may apply Lemma 2.19 to get

$$u(\psi(Z_H, Z_N)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( h(Z_H, Z_N)^{\otimes k} \cdot (\nabla^{TM})^k u \right) (y_0). \quad (2.35)$$

In another way, let us first apply Lemma 2.19 to get

$$u(\psi(Z_H, Z_N)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( Z_N(Z_H)^{\otimes k} \cdot (\nabla^{TM})^k u \right) (\psi(Z_H, 0)), \quad (2.36)$$

where  $Z_N(Z_H)$  is the parallel transport of  $Z_N$  along the curve  $\exp_{y_0}^H(tZ_H)$ ,  $t \in [0, 1]$ , with respect to  $\nabla^N$ . We now apply Lemma 2.19 again, but this time on  $H$ , to get

$$\begin{aligned} & \left( Z_N(Z_H)^{\otimes k} \cdot (\nabla^{TM})^{\otimes k} u \right) (\psi(Z_H, 0)) \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} Z_H^{\otimes l} \cdot (\nabla^{TH})^{\otimes l} \left( Z_N(Z_H)^{\otimes k} \cdot (\nabla^{TM})^{\otimes k} u \right) (y_0). \end{aligned} \quad (2.37)$$

Finally, let us apply Lemma 2.21, (2.36) and (2.37), to get

$$u(\psi(Z_H, Z_N)) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{l!} \frac{1}{k!} \left( (Z_H^{\otimes l} \otimes Z_N^{\otimes k}) \cdot (\nabla^a)^{\otimes l} (\nabla^{TM})^{\otimes k} u \right) (y_0). \quad (2.38)$$

Now, we denote by  $h^{[r]}(Z)$ ,  $r \in \mathbb{N}$ , the homogeneous polynomial in  $(Z)$  of degree  $r$  such that the Taylor expansion of  $h(Z)$  is given by  $\sum_{r=0}^{\infty} h^{[r]}(Z)$ . Let us now take the homogeneous parts of (2.35) and compare them with (2.38). The comparison of the first degree gives us

$$(h^{[1]}(Z)u)(y_0) = (Zu)(y_0). \quad (2.39)$$

By comparing now the second degree, we get

$$\begin{aligned} & \left( h^{[2]}(Z_H, Z_N) \cdot \nabla^{TM} \right) u(y_0) + \frac{1}{2} \left( h^{[1]}(Z_H, Z_N)^{\otimes 2} \cdot (\nabla^{TM})^{\otimes 2} \right) u(y_0) \\ &= \left( \frac{1}{2} Z_H^{\otimes 2} \cdot (\nabla^a)^{\otimes 2} + Z_H \otimes Z_N \cdot \nabla^a \nabla^{TM} + \frac{1}{2} Z_N^{\otimes 2} \cdot (\nabla^{TM})^{\otimes 2} \right) u(y_0). \end{aligned} \quad (2.40)$$

From (1.19), (2.30), (2.40) and the fact that  $\nabla^{TM}$  has no torsion, we deduce

$$h^{[2]}(Z_H, Z_N) = \frac{1}{2} A_{y_0}(Z_H) Z_H + A_{y_0}(Z_H) Z_N. \quad (2.41)$$

From (2.39) and (2.41), we deduce (2.28).

Now, the coefficients of  $h^{[r]}(Z_H, Z_N)$  can be expressed in terms of  $A$ ,  $R^{TM}$ , and their derivatives up to order  $r-2$  because of (2.35), (2.38) and the fact that the antisymmetrization of  $(\nabla^{TM})^{\otimes 2}$  coincides with  $R^{TM}$ .  $\square$

### 2.3 Holonomy along the paths adapted to the two coordinate systems

This section is devoted to the comparison of two different trivializations of vector bundles, done in a neighborhood of submanifold. One uses the parallel transport adapted to the Fermi coordinate system, another one uses the geodesic coordinates.

We conserve the notations and assumptions from Section 2.2. Let  $(E, \nabla^E, h^E)$  be a Hermitian vector bundle of bounded geometry and rank  $r$  over  $(M, g^{TM})$ . We fix  $y_0 \in H$  and an orthonormal frame  $f_1, \dots, f_r \in (E_{y_0}, h_{y_0}^E)$ . Recall that in Section 2.1, over  $B_{y_0}^M(R)$ , we defined two orthonormal frames,  $\tilde{f}'_1, \dots, \tilde{f}'_r, \tilde{f}_1, \dots, \tilde{f}_r$ , of  $(E, h^E)$  using the parallel transports and the matrix function  $\xi_E$  in (2.25), which relates them.

**Proposition 2.22.** The following asymptotics holds

$$\xi_E(\psi(Z)) = O(|Z|^2). \quad (2.42)$$

If, moreover, we assume that  $(E, \nabla^E, h^E) := (L, \nabla^L, h^L)$  is a line bundle, and that there is a skew-adjoint endomorphism  $Q$  of  $TM$ , which is parallel with respect to  $\nabla^{TM}$  (i.e.  $\nabla^{TM}Q = 0$ ), which commutes with  $A$ , the restriction of which to  $H$  respects the decomposition (1.6), and such that for the curvature  $R^L$  of  $\nabla^L$ , and for any  $u, v \in TM$ , we have

$$\frac{\sqrt{-1}}{2\pi} R^L(u, v) = g^{TM}(Qu, v), \quad (2.43)$$

then the following more precise bound can be proven

$$\xi_L(\psi(Z)) = -\frac{1}{6} R_{y_0}^L(Z, B(Z)) + O(|Z|^4). \quad (2.44)$$

Moreover, the coefficients of order  $r$ ,  $r \in \mathbb{N}$ , in above Taylor expansions can be expressed in terms of  $R^{TM}$ ,  $R^E$ ,  $R^L$ ,  $A$ , and their derivatives up to order  $r - 2$  with respect to  $\nabla^{TM}$ ,  $\nabla^E$ ,  $\nabla^L$ , at  $y_0$ .

*Remark 2.23.* Assume  $(M, g^{TM})$  is endowed with a complex structure  $J$ , and  $g^{TM}$  is invariant under the action of it. Assume, moreover, that (2.43) holds for  $Q := J$  as in (1.2). Then all the requirements are satisfied for  $Q := J$ . This is because the invariance of  $g^{TM}$  under the action of  $J$ , and the fact that  $H$  is a complex submanifold, imply that the restrictions of  $J$  to  $H$  respects the decomposition (1.6). Also (2.43) implies that  $(M, J, g^{TM})$  is Kähler, which means that  $J$  is parallel with respect to  $\nabla^{TM}$ , cf. [28, Theorem 1.2.8]. Finally, by the definition of  $A$  and the fact that  $J|_H$  preserves  $TH$  and  $N$ , we see that  $J$  commutes with  $A$ . Proposition 2.22 will only be applied in this particular situation.

The proof is given in the end of this section. Before, let us state some auxiliary results.

**Lemma 2.24** ([28, Lemma 1.2.3] or [1, p. 38]). Let  $\tilde{t}'_i, i = 1, \dots, n$  be the vector fields, constructed by the parallel transport of  $e_i$  with respect to  $\nabla^{TM}$  along the curve  $\phi(tZ)$ ,  $t \in [0, 1]$ ,  $Z \in T_{y_0}M$ ,  $|Z| < r_M$ . Then

$$\frac{\partial \phi}{\partial Z_i} = \tilde{t}'_i + \sum_{j=1}^n O(|Z|^2) \tilde{t}'_j, \quad (2.45)$$

Moreover, the coefficients of order  $r$ ,  $r \in \mathbb{N}$ , of the Taylor expansion (2.45) can be expressed in terms of  $R^{TM}$ , and their derivatives up to order  $r - 2$  with respect to  $\nabla^{TM}$ , evaluated at  $y_0$ .

We denote  $e_H, e'_H \in TM$  (resp.  $e_N, e'_N$ ) the vector fields  $\frac{\partial\psi}{\partial Z_i}$  for  $i = 1, \dots, m$  (resp.  $i = m + 1, \dots, n$ ).

**Lemma 2.25.** For any endomorphism  $T$  of  $TM$ , which is parallel with respect to  $\nabla^{TM}$ , which commutes with  $A$ , the restriction of which to  $H$  respects the decomposition (1.6), we have

$$\begin{aligned} g_{\psi(Z_H, Z_N)}^{TM}(Te_N, e'_N) &= g_{y_0}^{TM}(Te_N, e'_N) + O(|Z|^2), \\ g_{\psi(Z_H, Z_N)}^{TM}(Te_N, e_H) &= O(|Z|^2). \\ g_{\psi(Z_H, Z_N)}^{TM}(Te_H, e'_H) &= g_{y_0}^{TM}(Te_H, e'_H) + g_{y_0}^{TM}(TA(e_H)Z_N, e'_H) \\ &\quad + g_{y_0}^{TM}(Te_H, A(e'_H)Z_N) + O(|Z|^2). \end{aligned} \tag{2.46}$$

Also, the following identity holds

$$g_{\psi(Z_H, 0)}^{TM}(Te_N, e'_N) = g_{y_0}^{TM}(Te_N, e'_N). \tag{2.47}$$

Moreover, the coefficients of order  $r$ ,  $r \in \mathbb{N}$ , in above Taylor expansions can be expressed in terms of  $R^{TM}$ ,  $A$ , and their derivatives up to order  $r - 2$  with respect to  $\nabla^{TM}$ ,  $\nabla^E$ ,  $\nabla^L$ , evaluated at  $y_0$ , and endomorphism  $T_{y_0}$ .

*Remark 2.26.* For  $N = M$  and  $T = \text{Id}$ , the result follows from [1, Proposition 1.28].

*Proof.* First of all, since  $T$  is parallel with respect to  $\nabla^{TM}$ , we see that

$$\frac{\partial}{\partial e_H} g_{\psi(Z_H, 0)}^{TM}(Te_N, e'_N) = g_{\psi(Z_H, 0)}^{TM}(T\nabla_{e_H}^{TM} e_N, e'_N) + g_{\psi(Z_H, 0)}^{TM}(Te_N, \nabla_{e_H}^{TM} e'_N). \tag{2.48}$$

However, by the definition of  $\psi$ , see (1.17), the restrictions of  $e_N, e'_N$  to  $H$  are parallel with respect to  $\nabla^N$ . By this, the fact that the form  $A$  exchanges  $N$  and  $TH$ , and the endomorphism  $T$  preserves the decomposition (1.6), we deduce that

$$\frac{\partial}{\partial e_H} g_{\psi(Z_H, 0)}^{TM}(Te_N, e'_N) = 0, \tag{2.49}$$

which readily implies (2.47).

Recall that the vector fields  $\tilde{t}'_i, i = 1, \dots, n$ , were introduced in Lemma 2.24. As  $T$  is parallel,  $T\tilde{t}'_i$  is equal to the parallel transport of  $Te_i$ , along the same curve as the one used in the definition of  $\tilde{t}'_i$ . From this and (2.45), we deduce that

$$g_{\psi(Z_H, Z_N)}^{TM}\left(T\frac{\partial\phi}{\partial Z_i}, \frac{\partial\phi}{\partial Z_j}\right) = g_{y_0}^{TM}(Te_i, e_j) + O(|Z|^2). \tag{2.50}$$

Now, we see that Proposition 2.18, along with the fact that  $A$  exchanges  $N$  with  $TH$ , implies that for  $i = 1, \dots, m$  and  $j = m + 1, \dots, n$ , we have

$$\begin{aligned} \frac{\partial\psi}{\partial Z_j} &= \frac{\partial\phi}{\partial Z_j} + \sum_{l=1}^m \frac{\partial\phi}{\partial Z_l} g_{y_0}^{TM}(A(Z_H)e_j, e_l) + O(|Z|^2), \\ \frac{\partial\psi}{\partial Z_i} &= \frac{\partial\phi}{\partial Z_i} + \sum_{l=1}^m \frac{\partial\phi}{\partial Z_l} g_{y_0}^{TM}(A(e_i)Z_N, e_l) \\ &\quad + \sum_{l=m+1}^n \frac{\partial\phi}{\partial Z_l} g_{y_0}^{TM}(A(e_i)Z_H, e_l) + O(|Z|^2). \end{aligned} \tag{2.51}$$

From (2.50) and the first equation from (2.51), we establish the first equation in (2.46) and the remark after it. From (2.50) and (2.51), we also deduce that

$$g_{\psi(Z_H, Z_N)}^{TM}(Te_N, e_H) = g_{y_0}^{TM}(Te_N, e_H) + g_{y_0}^{TM}(TA(Z_H)e_N, e_H) + g_{y_0}^{TM}(Te_N, A(e_H)Z_H) + O(|Z|^2). \quad (2.52)$$

Now, since both  $\nabla^{TM}$  and  $\nabla^{TH}$  have no torsion, we have

$$A(u)v = A(v)u, \quad \text{for any } u, v \in TH. \quad (2.53)$$

From (2.53) and the fact that  $A$  is skew-adjoint and commutes with  $T$ , we deduce the second part of (2.46). The third part of (2.46) follows directly from (2.50) and (2.51).  $\square$

Recall that the projection  $\pi$  and the identification of  $E$  to  $\pi^*(E|_H)$  in a tubular neighborhood of  $H$  were defined before (1.9). We define the 1-form  $\Gamma^E$  with values in  $\text{End}(\pi^*(E|_H))$  as follows

$$\Gamma^E = \nabla^E - \pi^*(\nabla^E|_H), \quad (2.54)$$

where we implicitly used the above isomorphism. We introduce similar notations for  $L$ .

**Lemma 2.27.** Under the same assumptions as in Proposition 2.22, the following holds

$$\begin{aligned} \Gamma_{\psi(Z)}^E(e_N) &= \frac{1}{2}R_{\pi(\psi(Z))}^E(Z_N, e_N) + O(|Z|^2), \quad \Gamma_{\psi(Z)}^E(e_H) = R_{\pi(\psi(Z))}^E(Z_N, e_H) + O(|Z|^2), \\ \Gamma_{\psi(Z)}^L(e_N) &= \frac{1}{2}R_{\pi(\psi(Z))}^L(Z_N, e_N) + O(|Z|^3), \quad \Gamma_{\psi(Z)}^L(e_H) = O(|Z|^3). \end{aligned} \quad (2.55)$$

Moreover, the coefficients of order  $r$ ,  $r \in \mathbb{N}$ , in above Taylor expansions can be expressed in terms of  $R^{TM}$ ,  $R^L$ ,  $R^E$ ,  $A$ , and their derivatives up to order  $r - 2$  with respect to  $\nabla^{TM}$ ,  $\nabla^L$ ,  $\nabla^E$ ,  $\nabla^a$ , evaluated at  $y_0$ .

*Proof.* We follow closely the idea from [1, Proposition 1.18], cf. [28, Lemma 1.2.4], which corresponds to  $H = \{y_0\}$ . First of all, by (2.54), we have

$$(\pi^*(\nabla^E|_H)\Gamma^E) + \Gamma^E \wedge \Gamma^E + \pi^*(R^E|_H) = R^E. \quad (2.56)$$

We denote by  $\mathcal{R} \in \mathcal{C}^\infty(N, TN)$  is the tautological section of  $TN$ . As we identified a neighborhood  $U$  of  $H$  in  $M$  with a neighborhood of the zero section in  $N$ , we may look at  $\mathcal{R}$  as at the vector in the tangent space of  $U$ . If we write it in Fermi coordinates, we get

$$\mathcal{R} = \sum_{i=m+1}^n Z_i \frac{\partial}{\partial Z_i}. \quad (2.57)$$

By our choice of the trivialization,  $\iota_{\mathcal{R}}\Gamma^E = 0$ . Also,  $\iota_{\mathcal{R}}(\pi^*R^E|_H) = 0$ . Hence, by (2.56), we get

$$L_{\mathcal{R}}\Gamma^E = \left[ \iota_{\mathcal{R}}, \pi^*(\nabla^E|_H) \right] \Gamma^E = \iota_{\mathcal{R}}R^E. \quad (2.58)$$

From (2.57), for  $i = 1, \dots, m$ , and  $j = m + 1, \dots, n$ , we also obtain the following identities

$$L_{\mathcal{R}}dZ_i = 0, \quad L_{\mathcal{R}}dZ_j = dZ_j. \quad (2.59)$$

Using (2.59) and expanding both sides of (2.58) in Taylor series at  $Z_N = 0$ , we get

$$\begin{aligned} \sum_{\alpha} (|\alpha| + 1) (\partial^{\alpha} \Gamma^E)_{\psi(Z_H, 0)}(e_N) \frac{Z_N^{\alpha}}{\alpha!} &= \sum_{\alpha} (\partial^{\alpha} R^E)_{\psi(Z_H, 0)}(\mathcal{R}, e_N) \frac{Z_N^{\alpha}}{\alpha!}, \\ \sum_{\alpha} |\alpha| (\partial^{\alpha} \Gamma^E)_{\psi(Z_H, 0)}(e_H) \frac{Z_N^{\alpha}}{\alpha!} &= \sum_{\alpha} (\partial^{\alpha} R^E)_{\psi(Z_H, 0)}(\mathcal{R}, e_H) \frac{Z_N^{\alpha}}{\alpha!}. \end{aligned} \quad (2.60)$$

From (2.60), we deduce the first two equations of Lemma 2.27. From the fact that  $N$  is orthogonal to  $TH$ , (2.43) and the fact that the restriction of  $Q$  to  $H$  respects the decomposition (1.6), we see that  $R_{\psi(Z_H, Z_N)}^L(e_N, e_H) = O(|Z|)$ . The last two equations of Lemma 2.27 follow directly from Lemma 2.25, (2.43) and (2.60), applied for  $E := L$ .  $\square$

*Proof of Proposition 2.22.* We denote by  $\nabla$  the standard covariant derivative in  $E$ , defined locally in the frame  $\tilde{f}_1, \dots, \tilde{f}_r$ . We define the 1-form  $\Gamma_0^E$  with values in  $\text{End}(E_{y_0})$  as follows

$$\Gamma_0^E := \nabla^E - \nabla. \quad (2.61)$$

By (2.54), we have the following identity

$$\Gamma_{0, \psi(Z_H, Z_N)}^E = \pi^*(\Gamma_{0, \psi(Z_H, 0)}^E) + \Gamma_{\psi(Z_H, Z_N)}^E, \quad (2.62)$$

where we implicitly identified  $\Gamma_{\psi(Z_H, Z_N)}^E$  with a 1-form with values in  $\text{End}(E_{y_0})$  using our choice of the frame. By (2.43) and Lemma 2.27, applied once for the second summand in the right-hand side of (2.62), and once for  $M := H$ ,  $H := y_0$ , to treat the first summand, we see that for  $e := \frac{\partial \psi}{\partial Z_i}$ ,  $i = 1, \dots, n$ , we have

$$\begin{aligned} \Gamma_{0, \psi(Z)}^L(e) &= \frac{1}{2} \left( R_{y_0}^L(Z_H, e) + R_{\psi(Z_H, 0)}^L(Z_N, e) \right) + O(|Z|^3), \\ \Gamma_{0, \psi(Z)}^E(e) &= O(|Z|). \end{aligned} \quad (2.63)$$

Now, by (2.47) and (2.63), we conclude that

$$\Gamma_{0, \psi(Z)}^L(e) = \frac{1}{2} R_{y_0}^L(Z, e) + O(|Z|^3), \quad (2.64)$$

Now, we denote by  $\gamma(t)$  the geodesic  $\phi(th(Z))$ ,  $t \in [0, 1]$ . By the definition of  $h$ , we have  $\gamma(1) = \psi(Z)$ . From Proposition 2.18, we easily see

$$\psi^{-1}(\gamma(t)) = h^{-1}(th(Z)) = tZ + (t - t^2)B(Z) + O(|Z|^3). \quad (2.65)$$

Directly from the definition of  $\xi_E$ , the fact that  $\nabla_{\gamma'(t)}^E \tilde{f}_i^t = 0$ , (2.25), and the usual law of derivation of the exponential, we have

$$\int_0^1 \exp((1-s)\xi_E(\gamma(t))) \cdot \left( \frac{\partial}{\partial t} \xi_E(\gamma(t)) \right) \cdot \exp((s-1)\xi_E(\gamma(t))) ds = \Gamma_{0, \gamma(t)}^E(\gamma'(t)). \quad (2.66)$$

From (2.66) and the last equation from (2.63), we deduce (2.42). The statement about the coefficients of the Taylor expansions readily follows from (2.66) and the respective statement from Proposition 2.22.

Now, it is only left to establish (2.44). Since now  $E := L$  is a line bundle, (2.66) simplifies to

$$\frac{\partial}{\partial t} \xi_L(\gamma(t)) = \Gamma_{0,\gamma(t)}^L(\gamma'(t)). \quad (2.67)$$

From the identities

$$\int_0^1 t(1-2t)dt = -\frac{1}{6}, \quad \int_0^1 (t-t^2)dt = \frac{1}{6}, \quad (2.68)$$

we deduce the following asymptotics

$$\int_0^1 R_{y_0}^L \left( \psi^{-1}(\gamma(t)), \frac{\partial \psi^{-1}(\gamma(t))}{\partial t} \right) dt = -\frac{1}{3} R_{y_0}^L(Z, B(Z)) + O(|Z|^4). \quad (2.69)$$

From (2.64), (2.67) and (2.69), we deduce the result. The statement about the form of the coefficients follows from the respective statements in Proposition 2.18 and Lemma 2.27.  $\square$

## 2.4 Complex structure in Fermi coordinates and quasi-plurisubharmonicity

The main goal of this section is to give for Kähler manifolds of bounded geometry an approximate formula for the complex structure in Fermi coordinates. As a consequence, we establish quasi-plurisubharmonicity of several functions, which is crucial in certain  $L^2$ -estimates.

More precisely, assume  $(X, Y, g^{TX})$  is a Kähler triple of bounded geometry. We fix a point  $y_0 \in Y$  and an orthonormal frame  $(e_1, \dots, e_{2m})$  (resp.  $(e_{2m+1}, \dots, e_{2n})$ ) in  $(T_{y_0}Y, g_{y_0}^{TY})$  (resp. in  $(N_{y_0}, g_{y_0}^N)$ ), satisfying (1.16). Recall that in (1.17) and (2.2), we defined two coordinate systems  $\psi_{y_0}, \phi_{y_0}$  in a neighborhood of  $y_0$ . We denote by  $J$  the complex structure of  $X$ , and by  $\mathbf{J} = (\mathbf{J}_{ij})_{i,j=1}^{2n}$  its coordinates with respect to the basis  $\frac{\partial \psi}{\partial Z_i}, i = 1, \dots, 2n$ . It is a function, defined in  $B_0^{\mathbb{R}^{2n}}(R)$ , where  $R$  is as in (2.17), with matrix values of size  $2n \times 2n$ . We write  $\mathbf{J}$  in a block form  $\begin{pmatrix} \mathbf{J}_0 & \mathbf{J}_1 \\ \mathbf{J}_2 & \mathbf{J}_3 \end{pmatrix}$ , where  $\mathbf{J}_0$  has size  $2m \times 2m$ . The first result of this section goes as follows.

**Lemma 2.28.** The matrix  $\mathbf{J}$  has the following Taylor expansion

$$\mathbf{J} = J^0 + \begin{pmatrix} J^1 & 0 \\ 0 & 0 \end{pmatrix} + O(|Z|^2) \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + O(|Z||Z_N|) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.70)$$

where  $J^0$  is the diagonal block matrix with blocks  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , and  $J^1 = (J_{ij}^1)_{i,j=1}^{2m}$  is given by

$$J_{ij}^1 = -2g_{y_0}^{TM}(A(Je_i)Z_N, e_j), \quad (2.71)$$

for  $A$  defined in (1.19). Moreover, the constant in the  $O$ -terms is uniform on  $y_0 \in Y$ , and depends only on  $C_1$  from the bounds (2.1), (2.14) for  $k = 1$ .

*Proof.* The proof is essentially based on Lemma 2.24 and (2.51). Recall that the vector fields  $\tilde{t}'_i, i = 1, \dots, 2n$ , were constructed in Lemma 2.24 by the parallel transport of  $e_i$  with respect to  $\nabla^{TX}$  along the curve  $\phi(tZ), t \in [0, 1], Z \in T_{y_0}X$ . As  $(X, g^{TX})$  is Kähler, by [28, Theorem 1.2.8],  $J$  is parallel with respect to  $\nabla^{TX}$ . This means that by (1.16), for  $i = 1, \dots, n$ , we have

$$J\tilde{t}'_{2i-1} = \tilde{t}'_{2i}. \quad (2.72)$$

From (2.45) and (2.72), we conclude that

$$J \frac{\partial \phi}{\partial Z_{2i-1}} = \frac{\partial \phi}{\partial Z_{2i}} + O(|Z|^2). \quad (2.73)$$

Remark that (2.73) already implies Lemma 2.28 for  $Y := \{x_0\}$ . Now let us establish the general case. By the fact that  $A$  takes values in skew-adjoint matrices and commutes with  $J$ , the  $J$ -invariance of  $g^{TX}$  and (2.53), we deduce that for  $i = 1, \dots, 2m, j = 2m+1, \dots, 2n, l = 1, \dots, 2n$ , for  $Z_Y \in \mathbb{R}^{2m}, Z_N \in \mathbb{R}^{2(n-m)}$ , we have

$$\begin{aligned} g_{y_0}^{TX}(A(Z_Y)e_j, e_l) &= g_{y_0}^{TX}(A(Z_Y)Je_j, Je_l), \\ g_{y_0}^{TX}(A(e_i)Z_Y, e_l) &= g_{y_0}^{TX}(A(Je_i)Z_Y, Je_l), \\ g_{y_0}^{TX}(A(e_i)Z_N, e_l) &= -g_{y_0}^{TX}(A(Je_i)Z_N, Je_l), \end{aligned} \quad (2.74)$$

By (2.51), (2.73) and (2.74), we deduce

$$\mathbf{J} = J^0 + \begin{pmatrix} J^1 & 0 \\ 0 & 0 \end{pmatrix} + O(|Z|^2) \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (2.75)$$

where the  $O$ -term is uniform in the same sense as in (2.70). By (2.75), it is only left to prove that in the Taylor expansion of  $\mathbf{J}_3$ , the  $O(|Z|^2)$ -term can be replaced by the  $O(|Z||Z_N|)$ -term. But recall that for  $j = 2m+1, \dots, 2n$ ,  $\frac{\partial \psi}{\partial Z_j}|_Y$  are given by the parallel transport along  $\nabla^N$ . Hence, similarly to (2.72), we conclude that for  $j = m+1, \dots, n$ , we have

$$J \frac{\partial \psi}{\partial Z_{2j-1}}|_Y = \frac{\partial \psi}{\partial Z_{2j}}|_Y. \quad (2.76)$$

Thus, in the Taylor expansion of  $\mathbf{J}_3$ , only the terms containing  $|Z_N|$  appear.  $\square$

Now, recall that a function  $f : X \rightarrow [-\infty, +\infty[$  on a complex Hermitian manifold  $(X, \omega)$  is called quasi-plurisubharmonic if it is upper-semicontinuous and there is a constant  $C \in \mathbb{R}$ , such that the following inequality holds in the distributional sense

$$\sqrt{-1} \partial \bar{\partial} f \geq -C\omega. \quad (2.77)$$

We denote by  $\text{PSH}(X, C\omega)$  the set of quasi-plurisubharmonic functions  $f$ , verifying (2.77).

Recall that by standard properties of the plurisubharmonic functions, cf. [12, Theorem I.5.6], for any convex function  $\chi : \mathbb{R} \rightarrow \mathbb{R}$ , verifying  $0 \leq \chi' \leq 1$ , we have

$$\chi \circ f \in \text{PSH}(X, C\omega) \quad \text{for any } f \in \text{PSH}(X, C\omega). \quad (2.78)$$

Later on, on several occasions, we will need the following lemma.

**Lemma 2.29** ([24, Proposition 4.2]). Suppose that  $J$  is an almost complex structure on an open subset  $\Omega \subset \mathbb{R}^{2n}$ . Let  $v$  be a constant coefficient vector field on  $\Omega$  (i.e.  $v = \sum v_i \frac{\partial}{\partial Z_i}$  where  $v_i$  are constants). Then for any smooth function  $f : \Omega \rightarrow \mathbb{R}$ , the following identity holds

$$\begin{aligned} \partial \bar{\partial} f \left( v - \sqrt{-1}Jv, v + \sqrt{-1}Jv \right) &= (D^2 f)(v, v) + (D^2 f)(Jv, Jv) \\ &\quad + (Df) \left( D_{Jv} J(v) - D_v J(Jv) \right), \end{aligned} \quad (2.79)$$

where  $D^2 f$  is the standard double derivative of  $f$ , and  $D_u J$  is the derivative in the direction  $u \in \mathbb{R}^{2n}$ , of the almost complex structure, written in the standard coordinates of  $\mathbb{R}^{2n}$ .

For any  $x_0 \in X$ , we consider the function  $\alpha_{x_0} : B_{x_0}^X(r_X) \rightarrow \mathbb{R}$ , given by

$$\alpha_{x_0}(x) := \text{dist}_X(x_0, x)^2. \quad (2.80)$$

**Proposition 2.30.** Assume that a Kähler manifold  $(X, g^{TX})$  is of bounded geometry. Then there is  $r_0 > 0$ , such that for any  $x_0 \in X$ ,  $\alpha_{x_0} \in \text{PSH}(B_{x_0}^X(r_0), -\frac{1}{2}\omega)$ . Moreover, the constant  $r_0$  depends only on the constant  $C_1$  from the bound (2.1) and  $r_X$ .

*Proof.* We fix a point  $x_0 \in X$ , consider geodesic coordinates  $\phi_{x_0}$ , and introduce the vector field

$$v = \sum_{i=1}^{2n} a_i \frac{\partial \phi}{\partial Z_i}, \quad (2.81)$$

where  $a_i \in \mathbb{C}$ ,  $i = 1, \dots, 2n$  are certain constants, verifying  $\sum |a_i|^2 = 1$ . It is enough to verify that there is  $r_0 > 0$ , such that for any choice of  $a_i$ , over  $B_{x_0}^X(r_0)$ , we have

$$\partial \bar{\partial} \alpha_{x_0} \left( v - \sqrt{-1} Jv, v + \sqrt{-1} Jv \right) \geq 1. \quad (2.82)$$

Clearly, in  $\phi_{x_0}$ -coordinates,  $\alpha_{x_0}$  writes as

$$\alpha_{x_0}(Z) = \sum_{i=1}^{2n} |Z_i|^2. \quad (2.83)$$

Let us now use (2.79) and (2.83) to deduce that over  $B_{x_0}^X(r_X)$ , we have

$$\partial \bar{\partial} \alpha_{x_0} \left( v - \sqrt{-1} Jv, v + \sqrt{-1} Jv \right) = 2 + O(|Z|), \quad (2.84)$$

where the  $O$ -term can be bounded uniformly on the constant  $C_1$  from the bound (2.1). The statement follows directly from (2.84).  $\square$

Assume that a Kähler triple  $(X, Y, g^{TX})$  is of bounded geometry. Let us consider  $\delta_Y : X \setminus Y \rightarrow \mathbb{R}$ ,  $\alpha_Y : X \rightarrow \mathbb{R}$ , defined as

$$\begin{aligned} \delta_Y(x) &:= \log \left( \text{dist}_X(x, Y) \right) \cdot \rho \left( \frac{\text{dist}_X(x, Y)}{r_\perp} \right), \\ \alpha_Y(x) &:= \text{dist}_X(x, Y)^2 \cdot \rho \left( \frac{\text{dist}_X(x, Y)}{r_\perp} \right) + \left( 1 - \rho \left( \frac{\text{dist}_X(x, Y)}{r_\perp} \right) \right). \end{aligned} \quad (2.85)$$

**Theorem 2.31.** There is  $C > 0$ , such that  $\delta_Y \in \text{PSH}(X, C\omega)$ ,  $\alpha_Y \in \text{PSH}(X, C\omega)$ ,  $-\alpha_Y \in \text{PSH}(X, C\omega)$ . Moreover,  $C$  depends only on the constant  $C_1$  from the bounds (2.1), (2.14) and  $r_\perp$ .

*Proof.* The proofs for  $\delta_Y$ ,  $\alpha_Y$ ,  $-\alpha_Y$  are similar, so we concentrate on  $\delta_Y$ , which is slightly more difficult than the other two. From Lemma 2.28 and bounded geometry condition, we see that  $\sqrt{-1} \partial \bar{\partial} \delta_Y$  is uniformly bounded away from in  $B_Y^X(\frac{r_\perp}{4})$ . So it is enough to study this quantity only on  $B_Y^X(\frac{r_\perp}{4})$ . We fix a point  $y_0 \in Y$ , consider Fermi coordinates  $\psi_{y_0}$ , and introduce the vector field

$$v = \sum_{i=1}^{2m} a_i \frac{\partial \psi}{\partial Z_i} + \sum_{j=2m+1}^{2n} b_j \frac{\partial \psi}{\partial Z_j}, \quad (2.86)$$



where  $a_i, b_j \in \mathbb{C}$ ,  $i = 1, \dots, 2m$ ,  $j = 2m+1, \dots, 2n$ , are certain constants,  $\sum |a_i|^2 + \sum |b_j|^2 = 1$ . It is enough to verify that there is a constant  $C > 0$ , as described in the statement of the proposition we're proving, such that for any choice of  $a_i, b_j$  above, over  $\{\psi_{y_0}(Z_N) : |Z_N| < \frac{r_\perp}{4}\}$ , we have

$$\partial\bar{\partial}\delta_Y(v - \sqrt{-1}Jv, v + \sqrt{-1}Jv) \geq -C. \quad (2.87)$$

Let us now calculate each term on the right-hand side of (2.79) up to negligible terms. First of all, remark that in Fermi coordinates, over  $B_Y^X(\frac{r_\perp}{4})$ ,  $\delta_Y$  has particularly simple form

$$\delta_Y(Z) = \log |Z_N| = \frac{1}{2} \log \left( \sum_{j=2m+1}^{2n} |Z_j|^2 \right). \quad (2.88)$$

Hence, by (2.86), we deduce

$$(D^2\delta_Y)(v, v) = \frac{1}{|Z_N|^4} \left( \left( \sum_{j=2m+1}^{2n} |b_j|^2 \right) |Z_N|^2 - 2 \left( \sum_{j=2m+1}^{2n} b_j Z_j \right)^2 \right). \quad (2.89)$$

A similar calculation, using Lemma 2.28, gives that over  $\{\psi_{y_0}(Z_N) : |Z_N| < \frac{r_\perp}{4}\}$ , we have

$$(D^2\delta_Y)(Jv, Jv) = \frac{1}{|Z_N|^4} \left( \left( \sum_{j=2m+1}^{2n} |b_j|^2 \right) |Z_N|^2 - 2 \left( \sum_{j=m+1}^n (b_{2j-1} Z_{2j} - b_{2j} Z_{2j-1}) \right)^2 \right) + O(1), \quad (2.90)$$

where the  $O$ -term can be bounded uniformly in terms of the constant  $C_1$  from the bounds (2.1), (2.14) and  $r_\perp$ . From Lemma 2.28, we also see that

$$D\mathbf{J}_2, D\mathbf{J}_3 = O(|Z|), \quad (2.91)$$

where the  $O$ -term can be bounded uniformly in terms of the constant  $C_1$  from the bounds (2.1), (2.14). From (2.91), we see that the last two terms in (2.79) are negligible. From this (2.79), (2.89) and (2.90), we see that to deduce (2.87), it is now only left to prove the bound

$$\left( \sum_{j=2m+1}^{2n} |b_j|^2 \right) |Z_N|^2 \geq \left( \sum_{j=m+1}^n (b_{2j-1} Z_{2j} - b_{2j} Z_{2j-1}) \right)^2 + \left( \sum_{j=2m+1}^{2n} b_j Z_j \right)^2. \quad (2.92)$$

But (2.92) follows easily from the fact that the vectors  $(Z_{2m+2}, -Z_{2m+1}, \dots, Z_{2n}, -Z_{2n-1})$  and  $(Z_{2m+1}, \dots, Z_{2n})$  are orthogonal.  $\square$

## 2.5 Existence of uniform Stein atlases and related extension theorems

The main goal of this section is to introduce and study submanifolds with uniform Stein atlases. In particular, we prove that any Kähler manifold of bounded geometry admits uniform Stein atlas and we make a relation between the existence of uniform Stein atlas and the existence of holomorphic coordinates, defined on a sufficiently big geodesic ball.

Before this, recall that by Cauchy formula, the  $L^2$ -bound on holomorphic functions on the complex plane implies  $\mathcal{C}^k$ -bounds for any  $k \in \mathbb{N}$ . As the following lemma shows, an analogous statement holds for general Kähler manifolds of bounded geometry.

**Lemma 2.32.** Assume that a Kähler manifold  $(X, g^{TX})$  and a Hermitian vector bundle  $(E, h^E)$  over it have bounded geometry. Then for any  $k \in \mathbb{N}$ ,  $r_1 > r_0 > 0$ ,  $r_1 < \frac{r_X}{2}$ , there is  $C > 0$ , such that for any  $x_0 \in X$ , for any  $f \in H_{(2)}^0(B_{x_0}^X(r_1), E)$ , we have

$$\|f\|_{\mathcal{C}^k(B_{x_0}^X(r_0))} \leq C \|f\|_{L^2(B_{x_0}^X(r_1))}. \quad (2.93)$$

Moreover, the constant  $C$  depends only on  $r_0, r_1$  and the constants  $C_{k+n+4}$  from (2.1) and (2.22).

*Proof.* This statement follows from Ma-Marinescu [30, Lemma 2 and the remark before], applied for  $(L, h^L)$  trivial. Here we give an alternative proof for completeness.

Let us recall that Lichnerowicz formula, cf. [28, Theorem 1.4.7], expresses the Kodaira Laplacian  $\square^E$  of  $(E, h^E)$  through the Bochner Laplacian  $\Delta^E$ , cf. [28, (1.3.19)], associated to the Chern connection  $\nabla^E$  and the Levi-Civita connection  $\nabla^{TX}$ , as follows

$$2\square^E = \Delta^E + \frac{r^X}{4} + \left[ \Lambda_\omega, R^E + \frac{1}{2} \text{Tr}[R^{T^{1,0}X}] \right], \quad (2.94)$$

where  $r^X$  is the scalar curvature, and  $\Lambda_\omega$  is the contraction with the Kähler form  $\omega$  of  $(X, g^{TX})$ .

Clearly, for any  $k \in \mathbb{N}^*$ , the  $\mathcal{C}^k$ -norm of the coefficients of the operator  $\Delta^E$ , written in geodesic coordinates  $\phi_{x_0}$  and in the local trivialization of  $(E, h^E)$  as in Proposition 2.15, are uniformly bounded by the  $\mathcal{C}^{k+1}$ -bounds on  $\Gamma^{VE}$  from Proposition 2.15 and the Christoffel symbols  $\Gamma^{TX}$ . By Propositions 2.3, 2.15, those bounds, on their turn, depend only on the constants  $C_{k+2}$  from (2.1) and (2.22). By this, elliptic estimates, cf. [17, Theorem 6.3.5], and Sobolev embedding theorem, we conclude that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , which only depends on  $r_0, r_1$  and the constant  $C_{k+n+4}$  from Propositions 2.3, 2.15, such that for any  $g \in \mathcal{C}^\infty(X, E)$ , we have

$$\|g\|_{\mathcal{C}^k(B_{x_0}^X(r_0))} \leq C \sum_{i=0}^{k+n+1} \|(\square^E)^i g\|_{L^2(B_{x_0}^X(r_1))}. \quad (2.95)$$

We conclude by (2.95), applied for  $g := f$ , and the trivial fact  $\square^E f = 0$ .  $\square$

Recall that a complex manifold is said to be *Stein* if it is holomorphically convex, its global holomorphic functions separate points and give local coordinates at every point, cf. [12, Definition 1.6.16]. One can see, cf. [12, Theorem 1.6.18], that any Stein manifold is strongly pseudoconvex, meaning that it carries a smooth strictly plurisubharmonic exhaustion function. The famous result of Grauert from [21, Theorem 2], states that the converse for relatively compact subdomains of complex manifolds holds as well. In this section, we will use this perspective through strictly plurisubharmonic exhaustions on Stein manifolds.

**Definition 2.33.** We say that a Hermitian manifold  $(X, g^{TX})$  admits uniform Stein atlas if there are  $r_1 > r_0 > 0$ ,  $c, C > 0$ , such that for any  $x_0 \in X$ , there is a Stein neighborhood  $\Omega$  of  $x_0$ , such that  $B_{x_0}^X(r_0) \subset \Omega \subset B_{x_0}^X(r_1)$ , and a smooth strictly plurisubharmonic exhaustion function  $\delta$  on  $\Omega$ , verifying the following bounds

$$\begin{aligned} \delta > 0 \quad \text{over } \Omega, \quad \delta < C \quad \text{over } B_{x_0}^X(r_0), \\ \sqrt{-1} \partial \bar{\partial} \delta > c\omega, \end{aligned} \quad (2.96)$$

where  $\omega$  is the Hermitian form associated to  $g^{TX}$ .

**Theorem 2.34.** *Any Kähler manifold of bounded geometry  $(X, g^{TX})$  admits uniform Stein atlas. Moreover,  $r_1 > r_0 > 0$ ,  $c, C > 0$  from Definition 2.33 depend only on the constant  $C_1$  from the bound (2.1) and  $r_X$ .*

*Proof.* Recall that  $\alpha_{x_0}$  was defined in (2.80). By Proposition 2.30, there is a uniform constant  $r_1 > 0$ , which depends only on the constant  $C_1$  from the bound (2.1) and  $r_X$ , such that  $\sqrt{-1}\partial\bar{\partial}\alpha_{x_0} > \frac{1}{2}\omega$  over  $B_{x_0}^X(r_1)$ . This means, in particular, that  $\Omega := B_{x_0}^X(r_1)$  is strongly pseudoconvex. We now take  $r_0 := \frac{r_1}{2}$ , and let  $\delta := \frac{1}{r_1^2 - \alpha_{x_0}}$ . By (2.78), we conclude that the requirements of Definition 2.33 are satisfied for the above choice and  $c := \frac{1}{2r_1^2}$ ,  $C := \frac{2}{r_1^2}$ .  $\square$

*Remark 2.35.* An easy modification of the proof of Proposition 2.30 shows that the conclusion of Theorem 2.34 continues to hold even if one replaces the Kähler assumption by a weaker one that for any  $k \in \mathbb{N}$ , there is  $C_k > 0$ , such that for the complex structure  $J$ , we have  $|\nabla^k J| < C_k$ .

**Theorem 2.36.** *For any Kähler manifold of bounded geometry  $(X, g^{TX})$ , there are  $r_c, C > 0$ , such that for any  $x_0 \in X$ , there are holomorphic coordinates  $\chi := (h_1, \dots, h_n) : B_{x_0}^X(r_c) \rightarrow \mathbb{C}^n$ , verifying  $h_i(x_0) = 0$ ,  $|h_i| < C$ ,  $i = 1, \dots, n$ , and such that  $dh_i(x_0)$  form an orthonormal frame of  $(T^{(1,0)*}X, g^{T*X})$ . Moreover,  $r_c, C > 0$ , depend only on  $C_{n+7}$  from the bound (2.1) and  $r_X$ .*

*Remark 2.37.* From Lemma 2.32, all higher derivatives of  $h_i$  (with respect to  $\nabla^{TX}$ ) can be bounded uniformly on  $X$ . Hence, there is  $r_c^1 > 0$ , which depends only on  $C_{n+7}$  from the bound (2.1) and  $r_X$ , such that  $B_0^{\mathbb{C}^n}(r_c^1) \subset \text{Im } \chi$ .

The above observation with Proposition 2.3 imply that  $\chi$  are coordinates of bounded geometry in the sense of Cheng-Yau [8, Definition 1.1]. The existence of such coordinates on Kähler manifold of bounded geometry was proved by Wu-Yau [39, Theorem 9] by different methods.

*Proof.* The main idea of the proof is to first construct smooth functions  $h'_i$ ,  $i = 1, \dots, n$ , verifying the requirement from Theorem 2.36 about  $dh_i(x_0)$  and  $\bar{\partial}h'_i(x_0) = 0$ . Then by using Hörmander's  $L^2$ -estimates, Lemma 2.28 and Theorem 2.31, to show that one can perturb  $h'_i$  to  $h_i$  in such a way that  $h_i$  become holomorphic, with the same Taylor expansion of the first order at  $x_0$  as  $h'_i$ . Then from the  $L^2$ -estimate and Lemma 2.32, we get the  $\mathcal{C}^k$ -bounds on  $h_i$  for any given  $k$ , and from this, we conclude that in a uniform neighborhood,  $h_1, \dots, h_n$  are actually holomorphic coordinates, and not just any functions.

Let us now describe the proof in details. Let  $r_0, r_1 > 0$ ,  $\Omega$ ,  $\delta$  be as in Definition 2.33. First, we define  $h'_i$  in geodesic coordinates  $\phi_{x_0}$  by

$$h'_i := z_i \rho\left(\frac{|Z|}{r_0}\right), \quad (2.97)$$

where  $\rho$  is a bump function as in (1.10). By (2.3), there is  $C > 0$ , which depends only on  $r_0$  and  $C_0$  from (2.1), such that

$$\int_{\Omega} |h'_i|^2 dv_{g^{TX}} \leq C. \quad (2.98)$$

Directly from Lemma 2.28, applied for  $Y := \{x_0\}$ , we see that there is  $C > 0$ , which depends only on  $r_0$  and  $C_1$  from (2.1), such that over  $B_{x_0}^X(r_0)$ , we have

$$|\bar{\partial}h'_i| \leq C|Z|^2. \quad (2.99)$$

From the fact that  $\text{supp } \bar{\partial}h'_i \subset B_{x_0}^X(r_0)$ , the first estimates from (2.96), the coarea formula and (2.99), we conclude that for any  $c > 0$ , there is a uniform constant  $C$ , which depends only on  $r_0$ ,  $C$  from the first line in (2.96), and  $C_1$  from (2.1), such that for  $\delta_{\{x_0\}}$ , from Theorem 2.31,  $\Omega' := \Omega \setminus \{x_0\}$ , we have

$$\int_{\Omega'} |\bar{\partial}h'_i|^2 \exp(-c\delta) \cdot \exp(-(2n+1)\delta_{\{x_0\}}) dv_{g^{TX}} \leq C. \quad (2.100)$$

Now, as  $(X, g^{TX})$  is of bounded geometry, there is  $C > 0$ , such that the first Chern form of the canonical line bundle  $K_X$ , endowed with the metric  $h^{K_X}$ , induced by  $g^{TX}$ , satisfies

$$C\omega > c_1(K_X, h^{K_X}) > -C\omega, \quad (2.101)$$

moreover,  $C$  depends only on  $C_0$  from (2.1). Hence, by Theorem 2.31, we see that there is  $c > 0$ , which depends only on  $C_1$  from (2.1) and the constant  $c$  from the second line of (2.96), such that for  $E := K_X^{-1}$ ,  $h^E := (h^{K_X})^{-1} \cdot \exp(-c\delta) \cdot \exp(-(2n+1)\delta_{\{x_0\}})$ , the curvature  $R^E$  of  $(E, h^E)$  over  $\Omega$  satisfies

$$\sqrt{-1}R^E \geq \omega. \quad (2.102)$$

By (2.100), (2.102) and the result [10, Théorème 0.2], which states that  $\Omega'$  admits a complete Kähler metric, we may apply [12, Theorem VIII.6.1] for  $X := \Omega'$ ,  $g := \bar{\partial}h'_i$  and  $(E, h^E)$  as above to deduce that there are functions  $f_i$  over  $\Omega'$ , such that we have  $\bar{\partial}f_i = \bar{\partial}h'_i$ , and

$$\begin{aligned} \int_{\Omega'} |f_i|^2 \exp(-c\delta) \cdot \exp(-(2n+1)\delta_{\{x_0\}}) dv_{g^{TX}} \\ \leq \int_{\Omega'} |\bar{\partial}h'_i|^2 \exp(-c\delta) \cdot \exp(-(2n+1)\delta_{\{x_0\}}) dv_{g^{TX}}. \end{aligned} \quad (2.103)$$

We now put  $h_i := h'_i - f_i$ . Over  $\Omega'$ , we trivially have  $\bar{\partial}h_i = 0$ . Also, by (2.96), (2.98), (2.100) and (2.103), we get that

$$\int_{B_{x_0}^X(r_0)} |h_i|^2 dv_{g^{TX}} \leq C, \quad (2.104)$$

where  $C$  depends only on  $r_0$ ,  $C$  from the first line in (2.96), and  $C_1$  from (2.1). By the standard regularity result, [10, Lemme 6.9],  $h_i$  extends smoothly over  $x_0$ , and the equation  $\bar{\partial}h_i = 0$  holds on  $\Omega$ . From this, (2.100), (2.103) and the non-integrability of  $|Z| \exp(-(2n+1)\delta_{\{x_0\}})$  and  $\exp(-(2n+1)\delta_{\{x_0\}})$  near  $x_0$ , we deduce that  $f_i(x_0) = 0$  and  $df_i(x_0) = 0$ . Hence

$$h_i(x_0) = h'_i(x_0), \quad dh_i(x_0) = dh'_i(x_0). \quad (2.105)$$

As a consequence,  $\partial h_i(x_0)$  form an orthonormal basis of  $(T^{(1,0)*}X, g^{T^*X})$ .

From Lemma 2.32 and (2.104), we conclude that there is a constant  $C > 0$ , which depends only on  $r_0, r_1$  and the constant  $C_{n+7}$  from (2.1), such that

$$\|h_i\|_{\mathcal{C}^3(B_{x_0}^X(r_0))} \leq C. \quad (2.106)$$

By repeating the proof of Corollary 2.6, we conclude that there is  $r_c > 0$ , which depend only on  $C_{n+7}$  from the bound (2.1) and  $r_X$ , such that  $(h_1, \dots, h_n)$  form holomorphic coordinates over  $B_{x_0}^X(r_c)$ . This finishes the proof.  $\square$

**Lemma 2.38.** Let  $(E, h^E)$  be a Hermitian vector bundle of bounded geometry over a Kähler manifold of bounded geometry  $(X, g^{TX})$ . Then there are  $r_0, C > 0$ , such that for any  $x_0 \in X$ ,  $e \in E_{x_0}$ ,  $|e| = 1$ , there exists  $f \in H^0(B_{x_0}^X(r_0), E)$ ,  $f(x_0) = e$ , verifying

$$\|f\|_{L^2(B_{x_0}^X(r_0))} \leq C. \quad (2.107)$$

Moreover, the constant  $C$  depends only on  $r_0, r_1, c, C$  from (2.96) and the constant  $C_0$  from (2.1) and (2.22).

*Proof.* The proof can be deduced from Hörmander's  $L^2$ -estimates, Lemmas 2.27, 2.28 and Theorem 2.31 in exactly the same way as in the proof of Theorem 2.36. Let us describe a more direct approach relying on Ohsawa-Takegoshi extension theorem.

Let  $\Omega, \delta, r_0$  be as in Definition 2.33 By Theorem 2.31 and (2.101), there is a constant  $c > 0$ , which depends only on the constant  $c$  from (2.96) and the constant  $C_1$  from (2.1) and (2.22), such that for  $E := E \otimes K_X^{-1}$ ,  $h^E := h^E \otimes (h^{K_X})^{-1} \cdot \exp(-c\delta)$ , the curvature  $R^E$  of  $(E, h^E)$  satisfies

$$\sqrt{-1}R^E - c_1(K_X, h^{K_X}) \cdot \text{Id}_E + 2an\sqrt{-1}\partial\bar{\partial}\delta_{\{x_0\}} \cdot \text{Id}_E \geq 0, \quad \text{for any } a \in [1, 2], \quad (2.108)$$

From Ohsawa-Takegoshi extension theorem in the form of [13, Theorem 2.8], applied for  $X := \Omega$ ,  $\psi := 2n\delta_{\{x_0\}}$ , we see that there is a section  $f$ , such that  $f(x_0) = e$  and

$$\int_{\Omega} |f|^2 \exp(-c\delta) \frac{\exp(-2n\delta_{\{x_0\}})}{1 + 2n\delta_{x_0}^2} dv_{g^{TX}} \leq 17|e|^2 \exp(-c\delta(x_0)). \quad (2.109)$$

Since for any  $C > 0$ , the function  $u \mapsto \frac{\exp(-2nu)}{1+2nu^2}$ ,  $u < C$ , is uniformly bounded from below by a universal constant  $c > 0$ , we deduce from (2.109) that

$$\int_{\Omega} |f|^2 \exp(-c\delta) dv_{g^{TX}} \leq \frac{17}{c} |e|^2 \exp(-c\delta(x_0)). \quad (2.110)$$

We now take  $r_0$  as in Definition 2.33, and the result follows from (2.96) and (2.109).  $\square$

**Corollary 2.39.** Assume we are in the setting of Lemma 2.38. Then there are  $r_0, C > 0$ , such that for any  $x_0 \in X$ , there is a local holomorphic frame  $(f_1, \dots, f_r) \in H^0(B_{x_0}^X(r_0), E)$  of  $E$ , such that

$$\|f_i\|_{L^2(B_{x_0}^X(r_0))} \leq C, \quad (2.111)$$

and  $(f_1(x_0), \dots, f_r(x_0))$  give an orthonormal frame of  $(E_{x_0}, h_{x_0}^E)$ . Moreover, the constant  $C$  depends only on  $r_0$ , the constants  $c, C$  from (2.96) and the constant  $C_{n+7}$  from (2.1) and (2.22).

*Proof.* The statement follows Lemma 2.38 in the same way as Corollary 2.6 follows from Proposition 2.5.  $\square$

**Theorem 2.40.** Assume that the Kähler triple  $(X, Y, g^{TX})$  is of bounded geometry. Then there are  $r_c, C > 0$ , such that for any  $y_0 \in Y$ , there are holomorphic coordinates  $\chi := (h_1, \dots, h_n) : B_{y_0}^X(r_c) \rightarrow \mathbb{C}$ ,  $h_i(x_0) = 0$ ,  $|h_i| < C$ ,  $i = 1, \dots, n$ ,  $h_{m+1}|_Y, \dots, h_n|_Y = 0$ , and such that  $\partial h_i(x_0)$  form an orthonormal frame of  $(T^{(1,0)*}X, g^{T^*X})$ . Moreover,  $r_c, C$  depend only on  $r_X$  and the constant  $C_{n+7}$  from (2.1) and (2.14).

*Remark 2.41.* As in Remark 2.37, we see that there is  $r_c^1 > 0$ , which depends only on  $C_{n+7}$  from the bounds (2.1), (2.14),  $r_Y, r_X, r_\perp$ , such that  $B_0^{\mathbb{C}^n}(r_c^1) \subset \text{Im } \chi$ .

*Proof.* The proof is very similar to the proof of Theorem 2.36, so we only highlight the main steps. For  $i = 1, \dots, m$ , we define  $h_i$  in the same way as in Theorem 2.36. For  $i = m + 1, \dots, n$ , we define  $h'_i$  in Fermi coordinates  $\psi_{x_0}$  by

$$h'_i := z_i \rho\left(\frac{|Z_N|}{r_0}\right), \quad (2.112)$$

where the bump function  $\rho$  is as in (1.10). As in (2.99), directly from Lemma 2.28, we see that there is  $C > 0$ , which depends only on  $r_0$  and  $C_1$  from (2.1) and (2.14), such that over  $B_{x_0}^X(r_0)$ , for  $i = m + 1, \dots, n$ , we have

$$|\bar{\partial}h'_i| \leq C|Z||Z_N|. \quad (2.113)$$

By Theorem 2.31, we see that there is  $c > 0$ , which depends only on  $C_1$  from (2.1) and (2.14), and the constant  $c$  from the second line of (2.96), such that the curvature  $R^E$  of  $(E, h^E)$ ,  $E := K_X^{-1}$ ,  $h^E := (h^{K_X})^{-1} \cdot \exp(-c\delta) \cdot \exp(-(2m+1)\delta_{\{x_0\}}) \cdot \exp(-2(n-m)\delta_Y)$ , for  $\delta_{\{x_0\}}, \delta_Y$ , defined in Theorem 2.31, satisfies (2.102). As in (2.103), we construct  $f_i$  over  $\Omega' := \Omega \setminus Y$ . By using the same argument as after (2.104), we conclude that  $f_i$  extends over  $Y$ , and by the fact that  $\exp(-2(n-m)\delta_Y)$  is non-integrable near  $Y$ , and  $|Z| \exp(-(2m+1)\delta_{\{x_0\}}) \cdot \exp(-2(n-m)\delta_Y)$  is non-integrable near  $y_0$ , we conclude that  $f_i|_Y = 0$ , and (2.105) holds for  $i = m + 1, \dots, n$ . From this point, the proof is a word-to-word repetition of the proof of Theorem 2.36.  $\square$

### 3 Operator algebras on manifolds of bounded geometry

In this section, we prove that the set of operators over manifolds of bounded geometry, admitting certain bounds on the Schwartz kernels, forms an algebra under the composition. This will later play a crucial role in our approach to the main statements of this article.

This section is organized as follows. In Section 3.1, we show that the set of operators with an exponential decay of the Schwartz kernel forms an algebra on manifolds of bounded geometry. Then in Section 3.2, we give explicit formulas for the Schwartz kernels of the orthogonal Bergman projector and the extension operator on the pair  $(\mathbb{C}^n, \mathbb{C}^m)$ . We also recall the composition rules for operators with related kernels. Finally, in Section 3.3, we prove that the set of operators, whose Schwartz kernel admits Taylor-type expansion, forms an algebra under the composition.

#### 3.1 Algebra of operators with an exponential decay of the Schwartz kernel

In this section, by relying on Bishop-Gromov inequality, we show that the set of operators with an exponential decay forms an algebra on manifolds of bounded geometry.

Let  $(M, g^{TM})$  be a complete manifold with a positive bound on the injectivity radius  $r_M$ . We fix an arbitrary sequence of Hermitian vector bundles  $(E_p, \nabla^{E_p}, h^{E_p})$ ,  $p \in \mathbb{N}^*$ , on  $M$ , endowed with a fixed connection. We fix  $s > 0$  and assume that the Ricci curvature  $\text{Ric}_{g^{TM}}$  of  $(M, g^{TM})$  satisfies the bound

$$\text{Ric}_{g^{TM}} \geq -(n-1)s. \quad (3.1)$$

Clearly, (3.1) is satisfied for some  $s > 0$  once  $(M, g^{TM})$  is of bounded geometry.

Let us fix a volume form  $dv_M$  on  $M$ , such that the condition, analogous to (1.4), is satisfied with respect to the metric  $g^{TM}$ . Let us fix  $q \in \mathbb{N}^*$ , and a sequence of operators  $A_p^1, \dots, A_p^q$ ,  $p \in \mathbb{N}^*$ , acting on  $\mathcal{C}^\infty(M, E_p)$  by the convolutions with smooth kernels  $A_p^1(x_1, x_2), \dots, A_p^q(x_1, x_2) \in E_{p,x_1} \otimes E_{p,x_2}^*$ ,  $x_1, x_2 \in X$ , with respect to the volume form  $dv_M$ . We assume that for any  $k \in \mathbb{N}$ , there are  $c, C_h > 0$ ,  $h = 1, \dots, q$ , such that for any  $x_1, x_2 \in X$ , we have

$$\left| A_p^h(x_1, x_2) \right|_{\mathcal{C}^k(M \times M)} \leq C_h \cdot p^{\frac{n+k}{2}} \cdot \exp\left(-c\sqrt{p} \cdot \text{dist}(x_1, x_2)\right), \quad (3.2)$$

and there is  $h$  among  $1, \dots, q$ , such that even stronger bound holds

$$\left| A_p^h(x_1, x_2) \right|_{\mathcal{C}^k(M \times M)} \leq C_h \cdot p^{\frac{n+k}{2}} \cdot \exp\left(-c\sqrt{p} \cdot (\text{dist}(x_1, x_2) + \text{dist}(x_1, W) + \text{dist}(x_2, W))\right), \quad (3.3)$$

where  $W$  is a closed subset in  $M$ , and the  $\mathcal{C}^k$ -norm is taken in the sense of Theorem 1.5.

**Lemma 3.1.** There is  $C_0 > 0$ , which depends only on  $s, r_M, n$  and the constant  $c$  from (3.2) and (3.3), such that for any  $p \in \mathbb{N}^*$ ,  $\sqrt{p} > 4(n-1)\frac{\sqrt{s}}{c}$ , the following operator is well-defined

$$D_p := A_p^1 \circ \dots \circ A_p^q, \quad (3.4)$$

and for  $C := \prod_{h=1}^q C_h$ , the following bound holds

$$\left| D_p(x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \leq C_0^q C p^{\frac{n+k}{2}} \cdot \exp\left(-\frac{c}{8}\sqrt{p} \cdot (\text{dist}(x_1, x_2) + \text{dist}(x_1, W) + \text{dist}(x_2, W))\right). \quad (3.5)$$

In the proof of Lemma 3.1 and elsewhere, the following proposition plays a crucial role.

**Proposition 3.2.** There is a constant  $C' > 0$ , which depends only on  $n, s, r_M$ , such that for any  $x_0 \in M$ ,  $l > 2(n-1)\sqrt{s}$ , the following bound holds

$$\int_M \exp(-l \text{dist}(x_0, x)) dv_{g^{TM}}(x) < \frac{C'}{l^n}. \quad (3.6)$$

*Proof.* The main idea of our proof is to use Bishop-Gromov inequality to bound the volumes of balls and spheres in our manifold. We denote by  $v(n, s, r)$  the volume of a ball of radius  $r$  in the space form of constant curvature  $-s$ ,  $s > 0$ , cf. [34, p. 69]. From [34, pp. 69, 269], we see that

$$v(n, s, r) = \text{Vol}(S^{n-1}) \cdot \int_0^r \left( \frac{\sinh(\sqrt{s}r)}{\sqrt{s}} \right)^{n-1} dr, \quad (3.7)$$

where  $\text{Vol}(S^{n-1})$  is the volume of the unit sphere  $S^{n-1}$ , endowed with the standard Riemannian metric. From (3.7), we see that for any  $s > 0$ , there is  $C > 0$ , such that for any  $r > 0$ , we have

$$v(n, s, r) \leq C \exp((n-1)\sqrt{s}r). \quad (3.8)$$

Bishop-Gromov inequality [34, Lemma 9.36] states that the following function is non-increasing

$$\mathbb{R} \ni r \mapsto \frac{\text{Vol}(B_{x_0}^M(r))}{v(n, s, r)}. \quad (3.9)$$

Moreover, the limit of (3.9), as  $r \rightarrow 0$ , is equal to 1. In particular, for any  $r \geq 0$ , we have

$$\text{Vol}(B_{x_0}^M(r)) \leq v(n, s, r). \quad (3.10)$$

We now decompose the integral in (3.2) into two parts: over  $B_{x_0}^M(r_M)$  and over its complement,  $V$ . For the second part, we have the following bound

$$\int_V \exp(-l \text{dist}(x_0, x)) dv_{g^{TM}}(x) \leq \sum_{i=1}^{\infty} \exp(-ilr_M) \text{Vol}\left(B_{x_0}^M((i+1)r_M) \setminus B_{x_0}^M(ir_M)\right). \quad (3.11)$$

However, from (3.8) and (3.10), we see that

$$\text{Vol}\left(B_{x_0}^M((i+1)r_M) \setminus B_{x_0}^M(ir_M)\right) \leq C \exp((i+1)(n-1)\sqrt{s}r_M). \quad (3.12)$$

By combining (3.11) and (3.12), for  $l > 2(n-1)\sqrt{s}$ , we easily get the bound

$$\int_V \exp(-l \text{dist}(x_0, x)) dv_{g^{TM}}(x) \leq C \exp\left(-\frac{lr_M}{2}\right). \quad (3.13)$$

It is now only left to bound the integral over  $B_{x_0}^M(r_M)$ . Since the distance function  $x \mapsto \text{dist}(x_0, x)$  is smooth for  $x \in B_{x_0}^M(r_M)$ , and the norm of its gradient is equal to 1, the smooth coarea formula gives us for  $r < r_M$  the following identity

$$\text{Vol}(B_{x_0}^M(r)) = \int_0^r \text{Vol}(S_{x_0}^M(r')) dr', \quad (3.14)$$

where  $S_{x_0}^M(r)$  is the volume of the  $r$ -sphere around  $x_0$ . By differentiating the function (3.9), and using (3.14), we see that (3.9) is equivalent to the following bound

$$\frac{\text{Vol}(S_{x_0}^M(r))}{\text{Vol}(B_{x_0}^M(r))} \leq \frac{t(n, s, r)}{v(n, s, r)}, \quad (3.15)$$

where  $t(n, s, r) = \frac{d}{dr}v(n, s, r)$ . From (3.7), (3.10) and (3.15), we deduce

$$\text{Vol}(S_{x_0}^M(r)) \leq \text{Vol}(S^{n-1}) \cdot \left(\frac{\sinh(\sqrt{s}r)}{\sqrt{s}}\right)^{n-1}. \quad (3.16)$$

From (3.16), we see that there is  $C > 0$ , which depends only on  $s$  and  $r_M$ , such that

$$\int_0^{r_M} \exp(-lr) \text{Vol}(S_{x_0}^M(r)) dr \leq \frac{C}{ln}. \quad (3.17)$$

Now, from the smooth coarea formula as in (3.14), we have

$$\int_{B_{x_0}^M(r_M)} \exp(-l \text{dist}(x_0, x)) dv_{g^{TM}}(x) = \int_0^{r_M} \exp(-lr) \text{Vol}(S_{x_0}^M(r)) dr. \quad (3.18)$$

From (3.13), (3.17) and (3.18), we conclude.  $\square$



*Proof of Lemma 3.1.* For simplicity of the presentation, we only present the proof for  $k = 0$ , as the general case is treated in an analogous way. We trivially have the following identity

$$D_p(x_1, x_2) = \int_{M^{\times(q-1)}} A_p^1(x_1, z_1) \cdot A_p^2(z_1, z_2) \cdot \dots \cdot A_p^q(z_{q-1}, x_2) dv_M(z_1) \cdot \dots \cdot dv_M(z_{q-1}). \quad (3.19)$$

Now, the triangle inequality readily implies

$$\begin{aligned} \text{dist}(x_1, z_1) + \text{dist}(z_1, z_2) + \dots + \text{dist}(z_{q-1}, x_2) &\geq \text{dist}(x_1, x_2), \\ \text{dist}(x_1, z_1) + \dots + \text{dist}(z_{q-1}, x_2) + \text{dist}(z_i, W) &\geq \text{dist}(x_1, W). \end{aligned} \quad (3.20)$$

Then for  $C$  as in (3.5), by (3.2), (3.3), (3.19) and (3.20), the following bound holds

$$\begin{aligned} |D_p(x_1, x_2)| &\leq Cp^n \cdot \exp\left(-\frac{c}{4}\sqrt{p} \cdot (\text{dist}(x_1, W) + \text{dist}(x_1, x_2))\right) \cdot \\ &\cdot \int_M p^n \exp\left(-\frac{c}{2}\sqrt{p} \cdot \text{dist}(z_{q-1}, x_2)\right) \cdot \int_M p^n \exp\left(-\frac{c}{2}\sqrt{p} \cdot \text{dist}(z_{q-2}, z_{q-1})\right) \cdot \dots \\ &\cdot \int_M p^n \exp\left(-\frac{c}{2}\sqrt{p} \cdot \text{dist}(z_1, z_2)\right) dv_M(z_1) \cdot \dots \cdot dv_M(z_{q-1}). \end{aligned} \quad (3.21)$$

By triangle inequality, we also have

$$\text{dist}(x_1, W) + \text{dist}(x_1, x_2) \geq \text{dist}(x_2, W). \quad (3.22)$$

By applying Proposition 3.2 and (1.4), for the integrals over  $z_1, \dots, z_{q-1}$  from (3.21), and (3.22), we get (3.5) for  $k = 0$  from (3.21) for  $C_0 := C'$  from Proposition 3.2.  $\square$

### 3.2 Model operators on the complex vector space and the kernel calculus

In this section, we consider the model situation of the complex vector space, for which an explicit formula for the Schwartz kernels of the orthogonal Bergman projector and the extension operator can be given. We then use those explicit formulas to give a description for the compositions of the operators, the Schwartz kernels of which can be expressed using the above kernels. This section (as well as the next one) is motivated in many ways by the works of Ma-Marinescu [29], [28].

Endow  $\mathbb{C}^n$  with the standard Riemannian metric and consider a trivialized complex line bundle  $L_0$  on  $\mathbb{C}^n$ . We endow  $L_0$  with the Hermitian metric  $h^{L_0}$ , given by

$$\|1\|_{h^{L_0}}(Z) = \exp\left(-\frac{\pi}{2}|Z|^2\right), \quad (3.23)$$

where  $Z$  is the natural real coordinate on  $\mathbb{C}^n$ , and 1 is the trivializing section of  $L_0$ . An easy verification shows that (3.23) implies that (1.2) holds in our setting. Recall that [28, §4.1.6] shows that the Kodaira Laplacian  $\mathcal{L}$  on  $\mathcal{C}^\infty(X, L_0 \otimes F_0)$  is given by

$$\mathcal{L} = \sum_{i=1}^n b_i b_i^+, \quad (3.24)$$

where  $b_i, b_i^+$  are *creation* and *annihilation* operators, defined as

$$b_i = -2\frac{\partial}{\partial z_i} + \pi\bar{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial \bar{z}_i} + \pi z_i. \quad (3.25)$$

A classical calculation, cf. [28, Theorem 4.1.20], shows that the orthonormal basis with respect to the induced  $L^2$ -norm of  $\ker \mathcal{L}$  is given by the functions

$$\left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} z^\beta \exp\left(-\frac{\pi}{2}|Z|^2\right), \quad \beta \in \mathbb{N}^{\times n}. \quad (3.26)$$

In particular, [28, (4.1.84)], the Bergman kernel  $\mathcal{P}_n$  of  $\mathbb{C}^n$  is given by

$$\mathcal{P}_n(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_{i=1}^n (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i)\right), \quad \text{for } Z, Z' \in \mathbb{C}^n. \quad (3.27)$$

Also, we see easily that the Schwartz kernel of the orthogonal Bergman kernel, corresponding to the projection onto holomorphic sections orthogonal to those which vanish along  $\mathbb{C}^m$ , is given by

$$\mathcal{P}_{n,m}^\perp(Z, Z') = \sum_{\beta \in \mathbb{N}^{\times m}} \left(\frac{\pi^{|\beta|}}{\beta!}\right) z^\beta \bar{z}'^\beta \exp\left(-\frac{\pi}{2}|Z|^2\right). \quad (3.28)$$

By simplifying the above expression, using (3.27), we see that  $\mathcal{P}_{n,m}^\perp(Z, Z')$  corresponds precisely to the quantity, defined in (1.26).

Let us calculate the  $L^2$ -extension operator  $\mathcal{E}_{n,m}$ , extending each element from  $(\ker \mathcal{L})|_Y$  to an element from  $\ker \mathcal{L}$  with the minimal  $L^2$ -norm. From (3.26), we easily see that for  $Z_Y \in \mathbb{C}^m$ ,  $Z_N \in \mathbb{C}^{n-m}$  and  $g \in (\ker \mathcal{L})|_Y$ , we have

$$(\mathcal{E}_{n,m}g)(Z_Y, Z_N) = g(Z_Y) \exp\left(-\frac{\pi}{2}|Z_N|^2\right). \quad (3.29)$$

We extend  $\mathcal{E}_{n,m}$  to the whole  $L^2$ -space by  $g \mapsto (\mathcal{E}_{n,m} \circ \mathcal{P}_m)g$ . From (3.29), we see that the kernel of  $\mathcal{E}_{n,m}$  corresponds precisely to the quantity, defined in (1.20).

Now, a lot of calculations in this article will have something to do with compositions of operators having Schwartz kernels, given by the product of polynomials with the above kernels. For that reason, the following lemma will be of utmost importance in what follows.

**Lemma 3.3.** For any polynomials  $A_1(Z, Z')$ ,  $A_2(Z, Z')$ ,  $Z, Z' \in \mathbb{R}^{2n}$ , there is a polynomial  $A_3 := \mathcal{K}_{n,m}[A_1, A_2]$ , the coefficients of which are polynomials of the coefficients of  $A_1, A_2$ , such that

$$(A_1 \cdot \mathcal{P}_{n,m}^\perp) \circ (A_2 \cdot \mathcal{P}_{n,m}^\perp) = A_3 \cdot \mathcal{P}_{n,m}^\perp. \quad (3.30)$$

Moreover,  $\deg A_3 \leq \deg A_1 + \deg A_2$ . Also, if both polynomials  $A_1, A_2$  are even or odd (resp. one is even, another is odd), then the polynomial  $A_3$  is even (resp. odd).

Similarly, there is a polynomial  $A'_3 := \mathcal{K}'_{n,m}[A_1, A_2]$  with the same properties as  $A_3$ , such that

$$(A_1 \cdot \mathcal{P}_n) \circ (A_2 \cdot \mathcal{P}_{n,m}^\perp) = A'_3 \cdot \mathcal{P}_{n,m}^\perp. \quad (3.31)$$

Finally, for any polynomials  $A_1(Z, Z')$ ,  $A_2(Z_Y, Z'_Y)$ , where  $Z, Z' \in \mathbb{R}^{2n}$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ , there is a polynomial  $A''_3 := \mathcal{K}''_{n,m}[A_1, A_2]$  in  $(Z, Z'_Y)$ , with the same properties as  $A_3$ , such that

$$(A_1 \cdot \mathcal{P}_{n,m}^\perp) \circ \mathcal{E}_{n,m} \circ (A_2 \cdot \mathcal{P}_m) = A''_3 \cdot \mathcal{E}_{n,m}. \quad (3.32)$$

*Proof.* First of all, since  $\mathcal{P}_{n,n}^\perp = \mathcal{P}_n$ , for  $n = m$ , (3.30) was proved in [28, Lemma 7.1.1, (7.1.6)] by the use of so-called kernel calculus. Let us now show that the general case of (3.30) can be reduced to this special one. For this, remark that by (1.26), we have

$$\begin{aligned} \mathcal{P}_{n,m}^\perp(Z, Z') &= \mathcal{P}_{n,m}^\perp(Z, Z'_Y) \cdot \exp\left(-\frac{\pi}{2}|Z'_N|^2\right), & \mathcal{P}_{n,m}^\perp(Z, Z'_Y) &= \mathcal{P}_n(Z, Z'_Y), \\ \mathcal{P}_{n,m}^\perp(Z, Z') &= \mathcal{P}_n(Z_Y, Z') \cdot \exp\left(-\frac{\pi}{2}|Z_N|^2\right), & \mathcal{P}_{n,m}^\perp(Z_Y, Z') &= \mathcal{P}_n(Z_Y, Z'). \end{aligned} \quad (3.33)$$

We decompose the polynomials  $A_1, A_2$  as follows

$$A_1(Z, Z') = \sum_{\alpha} Z_N^\alpha \cdot A_1^\alpha(Z_Y, Z'), \quad A_2(Z, Z') = \sum_{\alpha'} A_2^{\alpha'}(Z, Z'_Y) Z_N^{\alpha'}, \quad (3.34)$$

where  $\alpha, \alpha' \in \mathbb{N}^{2(n-m)}$ ,  $|\alpha| \leq \deg A_1$ ,  $|\alpha'| \leq \deg A_2$ . Now, by (3.33) and (3.34), we have

$$\begin{aligned} \left( (A_1 \cdot \mathcal{P}_{n,m}^\perp) \circ (A_2 \cdot \mathcal{P}_{n,m}^\perp) \right)(Z, Z') &= \exp\left(-\frac{\pi}{2}(|Z_N|^2 + |Z'_N|^2)\right) \\ &\cdot \sum_{\alpha} \sum_{\alpha'} Z_N^\alpha Z_N^{\alpha'} \cdot \left( (A_1^\alpha \cdot \mathcal{P}_n) \circ (A_2^{\alpha'} \cdot \mathcal{P}_n) \right)(Z_Y, Z'_Y). \end{aligned} \quad (3.35)$$

From (3.33) and (3.35), we conclude

$$\mathcal{K}_{n,m}[A_1, A_2] = \sum_{\alpha} \sum_{\alpha'} Z_N^\alpha Z_N^{\alpha'} \mathcal{K}_{n,n}[A_1^\alpha, A_2^{\alpha'}]. \quad (3.36)$$

Now, (3.30) follows from (3.36) and the fact that (3.30) holds for  $n = m$ .

Along the same lines, we obtain

$$\begin{aligned} \left( (A_1 \cdot \mathcal{P}_n) \circ (A_2 \cdot \mathcal{P}_{n,m}^\perp) \right)(Z, Z') &= \exp\left(-\frac{\pi}{2}|Z'_N|^2\right) \\ &\cdot \sum_{\alpha'} Z_N^{\alpha'} \cdot \left( (A_1 \cdot \mathcal{P}_n) \circ (A_2^{\alpha'} \cdot \mathcal{P}_n) \right)(Z, Z'_Y). \end{aligned} \quad (3.37)$$

From (3.33) and (3.37), we conclude

$$\mathcal{K}'_{n,m}[A_1, A_2](Z, Z') = \sum_{\alpha'} Z_N^{\alpha'} \cdot \mathcal{K}_{n,n}[A_1, A_2^{\alpha'}](Z, Z'_Y). \quad (3.38)$$

Now, (3.31) follows from (3.38) and the fact that (3.30) holds for  $n = m$ .

Now, by (3.29), we first write

$$\mathcal{E}_{n,m}(Z, Z'_Y) = \exp\left(-\frac{\pi}{2}|Z_N|^2\right) \cdot \mathcal{P}_m(Z_Y, Z'_Y). \quad (3.39)$$

From (3.30) for  $n := m$ , (3.33) and (3.39), we obtain

$$\left( \mathcal{E}_{n,m} \circ (A_2 \cdot \mathcal{P}_m) \right)(Z, Z'_Y) = \mathcal{K}_{m,m}[1, A_2](Z_Y, Z'_Y) \cdot \mathcal{P}_{n,m}^\perp(Z, Z'_Y). \quad (3.40)$$

From (3.30) and (3.40), we deduce

$$\mathcal{K}''_{n,m}[A_1, A_2](Z, Z'_Y) = \mathcal{K}_{n,m} \left[ A_1, \mathcal{K}_{m,m}[1, A_2] \right](Z, Z'_Y). \quad (3.41)$$

Now, (3.32) follows from (3.41) and the fact that (3.30) holds.  $\square$

*Remark 3.4.* Directly from the definitions, we have

$$\begin{aligned}\mathcal{K}_{n,m}[A_1 \cdot P(Z'), A_2] &= \mathcal{K}_{n,m}[A_1, P(Z) \cdot A_2], \\ \mathcal{K}_{n,m}[A_1, A_2 \cdot P(Z')] &= \mathcal{K}_{n,m}[A_1, A_2] \cdot P(Z').\end{aligned}\quad (3.42)$$

Using the kernel calculus from [28, §7.1] or the explicit calculations, one can verify, cf. [28, (7.1.10)], that for  $i, j \leq m$ , the following holds

$$\begin{aligned}\mathcal{K}_{n,m}[1, z_i z_j] &= z_i z_j, & \mathcal{K}_{n,m}[1, z_i \bar{z}_j] &= \frac{1}{\pi} \delta_{ij} + z_i \bar{z}'_j, \\ \mathcal{K}_{n,m}[1, \bar{z}_i \bar{z}_j] &= \bar{z}'_i \bar{z}'_j, \\ \mathcal{K}_{n,m}[1, P_i(Z) z_i] &= \mathcal{K}_{n,m}[1, P_i(Z)] z_i, & \mathcal{K}_{n,m}[1, P_i(Z) \bar{z}_i] &= \mathcal{K}_{n,m}[1, P_i(Z)] \bar{z}'_i,\end{aligned}\quad (3.43)$$

where  $P_i(Z)$  doesn't depend on  $z_i$  and  $\bar{z}_i$ . Finally, for any  $k = 2m+1, \dots, 2n$ , from the calculation  $\int_{\mathbb{R}} Z \exp(-\pi|Z|^2) dZ = 0$ , we get

$$\mathcal{K}_{n,m}\left[A_1(Z_Y, Z'_Y), Z_k \cdot A_2(Z_Y, Z'_Y)\right] = 0. \quad (3.44)$$

### 3.3 Algebra of operators with Taylor-type expansion of the Schwartz kernel

The main goal of this section is to prove that the set of operators, acting on the sections of a trivial vector bundle over  $\mathbb{C}^n$  by the convolution with Schwartz kernel admitting Taylor-type expansion and exponential decay away from  $\mathbb{C}^m \subset \mathbb{C}^n$ , forms an algebra under composition. We also extend this result for operators on general triples of bounded geometry.

Let us now state precisely our results. The proofs will be given in the end of this section. We fix  $q \in \mathbb{N}$ ,  $q \geq 2$ , and operators  $\mathcal{G}_t, \mathcal{A}_t^1, \dots, \mathcal{A}_t^q$ ,  $t \in [0, 1]$ , acting on the sections of the trivial vector bundle  $\mathbb{C}^{r_0}$  over  $\mathbb{C}^n$  by the convolutions with smooth kernels  $\mathcal{G}_t(Z, Z'), \mathcal{A}_t^1(Z, Z'), \dots, \mathcal{A}_t^q(Z, Z') \in \text{End}(\mathbb{C}^{r_0})$  with respect to the volume form  $dv_{\mathbb{C}^n}$  on  $\mathbb{C}^n$ . We assume that there are  $c_0, q_1 > 0$ , such that for any  $l \in \mathbb{N}$ , there are  $C > 0$ ,  $Q_{h,1} \geq 0$ ,  $h = 1, \dots, q$ , such that for any  $t \in [0, 1]$ ,  $Z, Z' \in \mathbb{R}^{2n}$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , we have

$$\begin{aligned}\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{A}_t^h(Z, Z') \right| &\leq C \left(1 + |Z| + |Z'|\right)^{Q_{h,1}+q_1 l} \exp\left(-c_0(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right), \\ \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{G}_t(Z, Z') \right| &\leq C \left(1 + |Z| + |Z'|\right)^{Q_{1,1}+q_1 l} \exp\left(-c_0|Z - Z'|\right).\end{aligned}\quad (3.45)$$

**Lemma 3.5.** The operators  $\mathcal{D}_t := \mathcal{A}_t^1 \circ \dots \circ \mathcal{A}_t^q$ ,  $\mathcal{D}'_t := \mathcal{G}_t \circ \mathcal{A}_t^2 \circ \dots \circ \mathcal{A}_t^q$  are well-defined and have smooth Schwartz kernels  $\mathcal{D}_t(Z, Z')$ ,  $\mathcal{D}'_t(Z, Z')$  with respect to  $dv_{\mathbb{C}^n}$ . Moreover, for any  $l \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $t \in [0, 1]$ ,  $Z, Z' \in \mathbb{R}^{2n}$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , we have

$$\begin{aligned}\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{R}_t(Z, Z') \right| &\leq C \left(1 + |Z| + |Z'|\right)^{Q_{1,1}+\dots+Q_{q,1}+q_1 l} \\ &\quad \cdot \exp\left(-\frac{c_0}{8}(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right),\end{aligned}\quad (3.46)$$

where  $\mathcal{R}_t$  designates either  $\mathcal{D}_t$  or  $\mathcal{D}'_t$ .

Now, assume, in addition to (3.45), that for any  $r \in \mathbb{N}$ ,  $h \in 1, \dots, q$ , there are  $\mathcal{J}_r^h(Z, Z') \in \text{End}(\mathbb{C}^{r_0})$  polynomials in  $Z, Z' \in \mathbb{R}^{2n}$ , such that  $\mathcal{F}_r^h := \mathcal{J}_r^h \cdot \mathcal{P}_{n,m}^\perp$  (resp.  $\mathcal{F}'_r := \mathcal{J}_r^1 \cdot \mathcal{P}_n$ ) appear as coefficients of Taylor-type expansion for  $\mathcal{A}_t^h$  (resp.  $\mathcal{G}_t$ ). More precisely, we assume that there are  $\epsilon_0, c_1, q_1, q_2 > 0$ , such that for any  $k, l \in \mathbb{N}$ ,  $h = 1, \dots, q$ , there are  $C > 0, Q_{h,2} \geq 0$ , such that for any  $t \in [0, 1], Z, Z' \in \mathbb{R}^{2n}, |Z|, |Z'| \leq \frac{\epsilon_0}{t}, \alpha, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{A}_t^h(Z, Z') - \sum_{r=0}^k t^r \mathcal{F}_r^h(Z, Z') \right) \right| \leq C t^{k+1} \left( 1 + |Z| + |Z'| \right)^{Q_{h,2} + q_1 l + q_2 k} \exp \left( -c_1 (|Z_Y - Z'_Y| + |Z_N| + |Z'_N|) \right). \quad (3.47)$$

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{G}_t(Z, Z') - \sum_{r=0}^k t^r \mathcal{F}'_r(Z, Z') \right) \right| \leq C t^{k+1} \left( 1 + |Z| + |Z'| \right)^{Q_{1,2} + q_1 l + q_2 k} \exp \left( -c_1 |Z - Z'| \right). \quad (3.48)$$

Define polynomials  $\mathcal{J}_{r,0}(Z, Z') \in \text{End}(\mathbb{C}^r)$  (resp.  $\mathcal{J}'_{r,0}(Z, Z') \in \text{End}(\mathbb{C}^r)$ ),  $r \in \mathbb{N}$ , in  $Z, Z' \in \mathbb{R}^{2n}$ , as follows

$$\begin{aligned} \mathcal{J}_{r,0} &= \sum_{\lambda} \mathcal{K}_{n,m} \left[ \mathcal{J}_{\lambda_1}^1, \mathcal{K}_{n,m} \left[ \mathcal{J}_{\lambda_2}^2, \dots, \mathcal{K}_{n,m} \left[ \mathcal{J}_{\lambda_{q-1}}^{q-1}, \mathcal{J}_{\lambda_q}^q \right] \dots \right] \right], \\ \mathcal{J}'_{r,0} &= \sum_{\lambda} \mathcal{K}'_{n,m} \left[ \mathcal{J}_{\lambda_1}^1, \mathcal{K}_{n,m} \left[ \mathcal{J}_{\lambda_2}^2, \dots, \mathcal{K}_{n,m} \left[ \mathcal{J}_{\lambda_{q-1}}^{q-1}, \mathcal{J}_{\lambda_q}^q \right] \dots \right] \right], \end{aligned} \quad (3.49)$$

where  $\lambda$  runs over all partitions  $(\lambda_1, \dots, \lambda_q)$  of  $r$  by natural numbers  $\lambda_i$ . We let  $\mathcal{F}_{r,0} := \mathcal{J}_{r,0} \cdot \mathcal{P}_{n,m}^\perp$  (resp.  $\mathcal{F}'_{r,0} := \mathcal{J}'_{r,0} \cdot \mathcal{P}_{n,m}^\perp$ ).

**Lemma 3.6.** In the notations of (3.45), (3.47), for any  $k, l \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $t \in [0, 1], Z, Z' \in \mathbb{R}^{2n}, |Z|, |Z'| \leq \frac{\epsilon_0}{2t}, \alpha, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l$ , and for  $Q := \max\{\sum_{h=1}^q Q_{h,1}, \sum_{h=1}^q Q_{h,2}\}$ ,  $c_2 := \min\{c_0, c_1\}$ , the following estimate holds

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{R}_t(Z, Z') - \sum_{r=0}^k t^r \mathcal{G}_{r,0}(Z, Z') \right) \right| \leq C t^{k+1} \left( 1 + |Z| + |Z'| \right)^{Q + q_1 l + q_2 k} \exp \left( -\frac{c_2}{8} (|Z_Y - Z'_Y| + |Z_N| + |Z'_N|) \right), \quad (3.50)$$

where  $\mathcal{R}_t$  (resp.  $\mathcal{G}_{r,0}$ ) designates either  $\mathcal{D}_t$  or  $\mathcal{D}'_t$  (resp. either  $\mathcal{F}_{r,0}$  or  $\mathcal{F}'_{r,0}$ ).

Now, let us formulate similar results for general manifolds. More precisely, let  $(X, Y, g^{TX})$  be a triple of bounded geometry. Let  $dv_X$  be a volume form over  $X$ , satisfying (1.4). We fix a Hermitian line (resp. vector) bundle  $(L, h^L)$  (resp.  $(F, h^F)$ ) on  $X$ .

Let us fix  $q \in \mathbb{N}^*$ , and a sequence of operators  $A_p^1, \dots, A_p^q, p \in \mathbb{N}^*$ , acting on  $\mathcal{C}^\infty(X, L^p \otimes F)$  by the convolutions with smooth kernels  $A_p^1(x_1, x_2), \dots, A_p^q(x_1, x_2) \in (L^p \otimes F)_{x_1} \otimes (L^p \otimes F)_{x_2}^*$  with respect to the volume form  $dv_X$ . We assume that there are  $c_0 > 0, p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $h = 1, \dots, q, x_1, x_2 \in X, p \geq p_1$ , we have

$$\left| A_p^h(x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \leq C p^{n + \frac{k}{2}} \exp \left( -c_0 \sqrt{p} \text{dist}(x_1, x_2) \right), \quad (3.51)$$

and there is  $h$  among  $1, \dots, q$ , such that even stronger bound holds

$$\begin{aligned} & |A_p^h(x_1, x_2)|_{\mathcal{C}^k(X \times X)} \\ & \leq Cp^{n+\frac{k}{2}} \exp\left(-c_0\sqrt{p}(\text{dist}(x_1, x_2) + \text{dist}(x_1, Y) + \text{dist}(x_2, Y))\right). \end{aligned} \quad (3.52)$$

We fix  $y_0 \in Y$  and trivialize  $(L, h^L)$  (resp.  $(F, h^F)$ ) in a neighborhood of  $y_0$  using Fermi coordinates and parallel transport with respect to  $\nabla^L$  (resp.  $\nabla^F$ ) as we did before Theorem 1.6. Assume that for any  $h = 1, \dots, q$ ,  $r \in \mathbb{N}$ , there are  $J_{r,h}^\perp(Z, Z') \in \text{End}(F_{y_0})$  polynomials in  $Z, Z' \in \mathbb{R}^{2n}$ , such that for  $F_{r,h}^\perp := J_{r,h}^\perp \cdot \mathcal{P}_{n,m}^\perp$ , the following holds.

There are  $\epsilon_0, c_1, q_1, q_2 > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k, l \in \mathbb{N}$ ,  $h = 1, \dots, q$ , there are  $C > 0$ ,  $Q_{h,3} \geq 0$ , such that for any  $p \geq p_1$ ,  $Z = (Z_Y, Z_N)$ ,  $Z' = (Z'_Y, Z'_N)$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $Z_N, Z'_N \in \mathbb{R}^{2(n-m)}$ ,  $|Z|, |Z'| \leq \epsilon_0$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , the following bound holds

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \frac{1}{p^n} A_p^h(\psi_{y_0}(Z), \psi_{y_0}(Z')) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,h}^\perp(\sqrt{p}Z, \sqrt{p}Z') \kappa_X^{-\frac{1}{2}}(Z) \kappa_X^{-\frac{1}{2}}(Z') \right) \right| \\ & \leq Cp^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^{Q_{h,3}+q_1l+q_2k} \\ & \quad \cdot \exp\left(-c_1\sqrt{p}(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right). \end{aligned} \quad (3.53)$$

Define polynomials  $J_{r,D}^\perp(Z, Z') \in \text{End}(F_{y_0})$ ,  $r \in \mathbb{N}$ , in  $Z, Z' \in \mathbb{R}^{2n}$ , as follows

$$J_{r,D}^\perp = \sum_{\lambda} \mathcal{K}_{n,m} \left[ J_{\lambda_1,1}^\perp, \mathcal{K}_{n,m} \left[ J_{\lambda_2,2}^\perp, \dots, \mathcal{K}_{n,m} \left[ J_{\lambda_{q-1},q-1}^\perp, J_{\lambda_q,q}^\perp \right] \dots \right] \right], \quad (3.54)$$

where  $\lambda$  runs over all partitions  $(\lambda_1, \dots, \lambda_q)$  of  $r$  by natural numbers  $\lambda_i$ , and let  $F_{r,D}^\perp := J_{r,D}^\perp \cdot \mathcal{P}_{n,m}^\perp$ .

**Lemma 3.7.** In the notations of (3.51), (3.52), (3.53), for any  $k, l \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1$ ,  $Z = (Z_Y, Z_N)$ ,  $Z' = (Z'_Y, Z'_N)$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $Z_N, Z'_N \in \mathbb{R}^{2(n-m)}$ ,  $|Z|, |Z'| \leq \frac{\epsilon_0}{2}$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , for  $D_p$  as in (3.4),  $Q := \sum_{h=1}^q Q_{h,3}$ ,  $c_2 = \min\{c_0, c_1\}$ , we have

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \frac{1}{p^n} D_p(\psi_{y_0}(Z), \psi_{y_0}(Z')) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,D}^\perp(\sqrt{p}Z, \sqrt{p}Z') \kappa_X^{-\frac{1}{2}}(Z) \kappa_X^{-\frac{1}{2}}(Z') \right) \right| \\ & \leq Cp^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^{Q+q_1l+q_2k} \\ & \quad \cdot \exp\left(-\frac{c_2}{8}\sqrt{p}(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right). \end{aligned} \quad (3.55)$$

Let  $dv_Y$  be a volume form over  $Y$ , satisfying (1.4). We fix an operator  $C_p$ ,  $p \in \mathbb{N}^*$ , acting on  $\mathcal{C}^\infty(Y, \iota^*(L^p \otimes F))$  by the convolution with smooth kernel  $C_p(y_1, y_2)$ ,  $y_1, y_2 \in Y$ , with respect to the volume form  $dv_Y$ . Assume there is  $c_2 > 0$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that

$$|C_p(y_1, y_2)|_{\mathcal{C}^k(Y \times Y)} \leq Cp^{m+\frac{k}{2}} \exp\left(-c_2\sqrt{p}\text{dist}(y_1, y_2)\right). \quad (3.56)$$

Assume that for a fixed  $y_0 \in Y$ , for any  $r \in \mathbb{N}$ , there are  $J_{r,C}(Z_Y, Z'_Y) \in \text{End}(F_{y_0})$  polynomials in  $Z_Y, Z'_Y \in \mathbb{R}^m$ , such that for  $F_{r,C} := J_{r,C} \cdot \mathcal{P}_m$ , the following condition holds. There are

$\epsilon_0, c_3, q_1, q_2 > 0, p_1 \in \mathbb{N}^*$ , such that for  $k, l \in \mathbb{N}$ , there are  $C > 0, Q_C \geq 0$ , such that for  $p \geq p_1, Z_Y, Z'_Y \in \mathbb{R}^{2m}, |Z_Y|, |Z'_Y| \leq \epsilon_0, \alpha, \alpha' \in \mathbb{N}^{2m}, |\alpha| + |\alpha'| \leq l$ , the following bound holds

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^\alpha \partial Z'_Y{}^{\alpha'}} \left( \frac{1}{p^m} C_p(\psi_{y_0}(Z_Y), \psi_{y_0}(Z'_Y)) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,C}(\sqrt{p}Z_Y, \sqrt{p}Z'_Y) \kappa_Y^{-\frac{1}{2}}(Z_Y) \kappa_Y^{-\frac{1}{2}}(Z'_Y) \right) \right| \leq C p^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z'_Y| \right)^{Q_C+q_1l+q_2k} \exp\left(-c_3\sqrt{p}|Z_Y - Z'_Y|\right). \quad (3.57)$$

Recall that  $\kappa_N$  and  $E_p^0$  were defined in (1.7) and (1.9) respectively. We denote

$$D_p := A_p^1 \circ (\kappa_N^{-\frac{1}{2}} \cdot E_p^0) \circ C_p. \quad (3.58)$$

Define polynomials  $J_{r,D}^E(Z, Z') \in \text{End}(F_{y_0}), r \in \mathbb{N}$ , in  $Z \in \mathbb{R}^{2n}, Z' \in \mathbb{R}^{2m}$ , by

$$J_{r,D}^E = \sum_{r_0=0}^r \mathcal{K}''_{n,m}[J_{r_0,1}^\perp, J_{r-r_0,C}], \quad (3.59)$$

and let  $F_{r,D}^E := J_{r,D}^E \cdot \mathcal{E}_{n,m}$ .

**Lemma 3.8.** In the notations of (3.51), (3.53), (3.56), (3.57), for any  $k, l \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1, Z = (Z_Y, Z_N), Z_Y, Z'_Y \in \mathbb{R}^{2m}, Z_N \in \mathbb{R}^{2(n-m)}, |Z|, |Z'_Y| \leq \frac{\epsilon_0}{2}, \alpha \in \mathbb{N}^{2n}, \alpha' \in \mathbb{N}^{2m}, |\alpha| + |\alpha'| \leq l, Q := Q_{1,3} + Q_C, c_4 := \min\{c_0, c_1, c_2, c_3\}$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'_Y{}^{\alpha'}} \left( \frac{1}{p^m} D_p(\psi_{y_0}(Z), \psi_{y_0}(Z'_Y)) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,D}^E(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_X^{-\frac{1}{2}}(Z) \kappa_Y^{-\frac{1}{2}}(Z'_Y) \right) \right| \leq C p^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \right)^{Q+q_1l+q_2k} \exp\left(-\frac{c_4}{8}\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|)\right). \quad (3.60)$$

*Proof.* The proof is completely analogous to the proof of Lemma 3.1, done for  $W := \mathbb{C}^m$ , with only one change. Instead of Proposition 3.2, one has to use that for any  $c > 0, Q \geq 0$ , there is a constant  $C$ , such that for any  $A \in \mathbb{C}^n$ , we have

$$\int_{\mathbb{C}^n} |Z|^Q \exp(-c|Z - A|) dZ_1 \wedge \cdots \wedge dZ_{2n} \leq C(1 + |A|^Q), \quad (3.61)$$

which can be shown easily by the change of variables  $Z \mapsto Z + A$ .  $\square$

*Proof of Lemma 3.6.* To simplify the presentation, we restrict ourselves to the case  $l = 0$ , as the general case is treated in an analogous way. We present first the proof that the asymptotic expansion (3.50) holds for  $k = 0$ . Analogously to (3.19), we have

$$\mathcal{D}_t(Z, Z') = \int_{(\mathbb{R}^{2n})^{\times(q-1)}} \mathcal{A}_t^1(Z, Z_1) \mathcal{A}_t^2(Z_1, Z_2) \cdots \mathcal{A}_t^q(Z_{q-1}, Z') \cdot dv_{\mathbb{R}^{2n}}(Z_1) \cdots dv_{\mathbb{R}^{2n}}(Z_{q-1}) \quad (3.62)$$

We decompose the integral (3.62) into a sum of two integrals. The first one is over the set  $Q_t := B_0^{\mathbb{R}^{2n}}(\frac{\epsilon_0}{t})^{\times(q-1)}$ , and the second one is over its complement  $Q_t^c$ . Let us bound the contribution in

(3.62) coming from the integral on  $Q_t^c$ . Similarly to (3.20), for any  $Z, Z' \in \mathbb{R}^{2n}$ ,  $|Z|, |Z'| \leq \frac{\epsilon_0}{2t}$ , and any  $(Z_1, \dots, Z_{s-1}) \in Q_t^c$ , we have

$$\text{dist}(Z, Z_1) + \text{dist}(Z_1, Z_2) + \dots + \text{dist}(Z_{s-1}, Z') \geq \frac{\epsilon_0}{2t}. \quad (3.63)$$

By (3.45), similarly to the proof of Lemma 3.1 and (3.61), there is  $C > 0$ , such that

$$\left| \int_{Q_t^c} \mathcal{A}_t^1(Z, Z_1) \cdot \mathcal{A}_t^2(Z_1, Z_2) \cdot \dots \cdot \mathcal{A}_t^q(Z_{q-1}, Z') dv_{\mathbb{C}^n}(Z_1) \cdot \dots \cdot dv_{\mathbb{C}^n}(Z_{q-1}) \right| \leq C(1 + |Z| + |Z'|)^{\sum_{h=1}^q Q_{h,1}} \exp\left(-\frac{c}{8}(|Z_Y - Z'_Y| + |Z_N| + |Z'_N| + \frac{\epsilon_0}{2t})\right). \quad (3.64)$$

From the fact that for any  $c, \epsilon > 0$ ,  $k \in \mathbb{N}$ , there is a constant  $C > 0$ , such that  $\exp(-c\epsilon/t) < Ct^k$ , for any  $t \in [0, 1]$ , implies that the right-hand side of, (3.64) is majorated by the right-hand side of (3.47), and hence, the contribution of an integral over  $Q_t^c$  is negligible.

Now, let  $A_1, A_2, A_3$  be as in Lemma 3.3. By (3.64), we see that for any  $\epsilon > 0$ , there is  $C > 0$ , which depends only on  $k, A_1, A_2$ , such that for  $Z, Z' \in \mathbb{R}^{2n}$ ,  $|Z|, |Z'| \leq \frac{\epsilon}{2t}$ , we have

$$\left| (A_3 \cdot \mathcal{P}_{n,m}^\perp)(Z, Z') - \int_{|Z_1| < \frac{\epsilon}{t}} (A_1 \cdot \mathcal{P}_{n,m}^\perp)(Z, Z_1) \cdot (A_2 \cdot \mathcal{P}_{n,m}^\perp)(Z_1, Z') dv_{\mathbb{C}^n}(Z_1) \right| \leq C(1 + |Z| + |Z'|)^{\deg A_1 + \deg A_2} \exp\left(-\frac{c}{8}(|Z_Y - Z'_Y| + |Z_N| + |Z'_N| + \frac{\epsilon}{2t})\right). \quad (3.65)$$

Of course, an estimate as in (3.65), holds for any number of polynomials. From (3.64) and (3.65), we deduce the first part of Lemma 3.6 for  $k = 0$ .

Now, to extend the argument for general  $k \in \mathbb{N}$ . For any  $k' \in \mathbb{N}$ , in the notations of (3.47), we introduce

$$\mathcal{B}_t^{l,k'}(Z, Z') := \sum_{r=0}^{k'} t^r \mathcal{F}_r^l(Z, Z'). \quad (3.66)$$

Then we decompose  $\mathcal{D}_t$  into a sum of elements  $\mathcal{C}_t^1 \circ \dots \circ \mathcal{C}_t^q$ , where each  $\mathcal{C}_t^l$ ,  $l = 1, \dots, q$ , is either equal to  $\mathcal{A}_t^l - \mathcal{B}_t^{l,k'}$  or to  $\mathcal{B}_t^{l,k'}$ , and the sum of  $k'$  for all the multiplicands adds up to  $k$ . For the term, which consists of the compositions of  $\mathcal{B}_t^l$ , we apply (3.64) and (3.65) to conclude that it has an asymptotic expansion of the form (3.50) up to the error term of the form as in the right-hand side of (3.64). From (3.65) and the definition of  $\mathcal{K}_{n,m}[\cdot, \cdot]$  from Lemma 3.3, the coefficients of this asymptotic expansion are given by polynomials  $\mathcal{J}_{r,0}$ , defined in (3.49).

For other terms, we decompose the integral into two parts: over  $Q_t$  and over  $Q_t^c$ . To bound the contribution from  $Q_t$ , we use (3.47) and Lemma 3.5. To bound the contribution from  $Q_t^c$ , we use (3.45) and proceed as in (3.64). By combining the two bounds, we see that they contribute no more than the right-hand side of (3.47). This finishes the proof of the first part.

The proof of the second part proceeds along the same lines with only one change: instead of the first part of Lemma 3.3, one should use the second part.  $\square$

*Proof of Lemma 3.7.* The proof is very similar to the proof of Lemma 3.6, so we only highlight the main steps. We fix  $Z, Z' \in \mathbb{R}^{2n}$ ,  $|Z|, |Z'| < \frac{\epsilon_0}{2}$ , and let  $x = \psi_{y_0}(Z)$ ,  $x' = \psi_{y_0}(Z')$ . We decompose the integral in the formula (3.19), into two parts. The first one is over the set  $Q := B_{y_0}^X(\epsilon_0)^{\times(q-1)}$ ,



and the second one is over its complement  $Q^c$ . Similarly to (3.64), but relying on Proposition 3.2, we deduce the bound

$$\left| \int_{Q^c} A_p^1(x, x_1) \cdot A_p^2(x_1, x_2) \cdot \dots \cdot A_p^q(x_{q-1}, x') dv_X(x_1) \cdot \dots \cdot dv_X(x_{q-1}) \right| \leq Cp^{n+\frac{k}{2}} \cdot \exp\left(-\frac{c}{8}\sqrt{p} \cdot (\text{dist}(x, x') + \text{dist}(x, Y) + \text{dist}(x', Y) + \frac{\epsilon_0}{2})\right), \quad (3.67)$$

But since for any  $c, \epsilon > 0, k \in \mathbb{N}$ , there is  $C > 0$ , such that  $\exp(-c\epsilon\sqrt{p}) < Cp^{-k}$ , for any  $p \in \mathbb{N}^*$ , the right-hand side of (3.67), is majorated by the right-hand side of (3.55), and hence negligible. Now, to deal with the integration over  $Q$ , we pass to  $\psi_{y_0}$ -coordinates. The change of the variables introduces the  $\kappa_X$  factor for every volume form. It will be canceled with the two  $\kappa_X^{-1/2}$  factors, which appear in the asymptotic expansion (3.53). Once two factors are canceled, and we make a change of the variables  $Z \mapsto \frac{Z}{\sqrt{p}}$ , we reduce the problem to the estimates of the form (3.65). The proof is now finished exactly as in the proof of Lemma 3.6.  $\square$

*Proof of Lemma 3.8.* The proof is analogous to the proof of Lemma 3.7 with only one change: instead of relying on (3.30) in the estimate (3.65), one has to rely on (3.32). The reason why the factor  $\kappa_N^{-\frac{1}{2}}$  appears in (3.58) is due to the identity

$$\kappa_X(Z) = \kappa_N(\psi_{y_0}(Z)) \cdot \kappa_Y(Z_Y), \quad (3.68)$$

which implies that the term  $\kappa_X$ , appearing after the passage to  $\psi_{y_0}$ -coordinates, disappear with the terms  $\kappa_X^{-\frac{1}{2}}$  and  $\kappa_Y^{-\frac{1}{2}}$ , which appear in the Taylor-type expansions of  $A_p$  and  $C_p$ .  $\square$

## 4 Spectral bound for the restriction operator

The main goal of this section is to prove a spectral bound for the restriction operator.

More precisely, we conserve the notation from Section 1 and assume that the triple  $(X, Y, g^{TX})$  is of bounded geometry. Consider the restriction operator  $\text{Res}_p : H_{(2)}^{0,\perp}(X, L^p \otimes F) \rightarrow \mathcal{C}^\infty(Y, \iota^*(L^p \otimes F))$ , defined as  $f \mapsto f|_Y$ . Directly from Propositions 2.9, 2.11 and Lemma 2.32, we see that the restriction to  $Y$  of an element from  $H_{(2)}^{0,\perp}(X, L^p \otimes F)$  lies in  $H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ . Hence, we may regard  $\text{Res}_p$  as an operator

$$\text{Res}_p : H_{(2)}^{0,\perp}(X, L^p \otimes F) \rightarrow H_{(2)}^0(Y, \iota^*(L^p \otimes F)). \quad (4.1)$$

The main result of this section goes as follows.

**Theorem 4.1.** *There are  $c, C > 0, p_1 \in \mathbb{N}^*$  such that for any  $p \geq p_1$ , we have*

$$cp^{\frac{n-m}{2}} \leq \|\text{Res}_p\| \leq Cp^{\frac{n-m}{2}}, \quad (4.2)$$

where  $\|\cdot\|$  is the operator norm with respect to the  $L^2$ -scalar products (1.3) on  $X$  and  $Y$ .

*Remark 4.2.* a) For compact manifolds, a similar statement appeared in Sun [37, Theorem 3.3] for  $(F, h^F)$  trivial. However, it seems that only the lower bound was discussed there. Also we, unfortunately, didn't understand the argument of the author in his proof of the lower bound. In particular, the passage from the second equation before the announcement of [37, Theorem 3.3] to the first equation appears rather mysterious for us. Our proof is different.

b) A more precise version of (4.2), containing the asymptotics of  $\|\text{Res}_p\|$ , will be given in [19].

This section is organized as follows. In Section 4.1, we calculate the first two terms of the Taylor expansion of the  $\bar{\partial}$ -operator in a shrinking tubular neighborhood of  $Y$ . This will play a crucial role in our proof of the lower bound of Theorem 4.1 in Section 4.2. In Section 4.3, we establish the upper bound in Theorem 4.1.

## 4.1 Taylor expansion of the holomorphic differential near submanifold

The main goal of this section is to calculate the first two terms of the Taylor expansion of  $\bar{\partial}^{L^p \otimes F}$ -operator, considered in a shrinking neighborhood of  $Y$  of size  $\frac{1}{\sqrt{p}}$ , as  $p \rightarrow \infty$ . Our result is motivated by the Taylor expansions of the associated Dirac operator due to Bismut-Lebeau [3, Theorem 8.18], which corresponds to trivial  $(L, h^L)$  in our setting, and the Taylor expansion due to Dai-Liu-Ma [9, Theorem 4.6], which corresponds to  $Y$  equal to a point in our setting.

More precisely, as in Section 1, we consider a triple  $(X, Y, g^{TX})$  of bounded geometry. By means of the exponential map as in (1.7), we identify a neighborhood of the zero section  $B_{r_\perp}(N)$  in the normal bundle  $N$ , to a neighborhood  $U := B_Y^X(r_\perp)$  of  $Y$  in  $X$ .

Recall that the projection  $\pi : U \rightarrow Y$  and the identifications of  $L, F$  to  $\pi^*(L|_Y), \pi^*(F|_Y)$  in  $B_Y^X(r_\perp)$  were defined before (1.9). We similarly identify  $TX$  to  $\pi^*(TX|_Y)$  over  $B_Y^X(r_\perp)$  using the parallel transport with respect to the Levi-Civita connection  $\nabla^{TX}$ . Remark that since  $g^{TX}$  is Kähler by (1.2), the decomposition  $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$  is preserved by  $\nabla^{TX}$ , cf. [28, Theorem 1.2.8]. In other words, the identification of  $TX$  with  $\pi^*(TX|_Y)$  induces the identifications

$$\tau : \pi^*(T^{(1,0)}X|_Y) \rightarrow T^{(1,0)}X|_U, \quad \tau : \pi^*(T^{(0,1)}X|_Y) \rightarrow T^{(0,1)}X|_U. \quad (4.3)$$

We define the 1-forms  $\Gamma^F, \Gamma^L$  with values in  $\text{End}(\pi^*(F|_Y)), \text{End}(\pi^*(L|_Y))$  as in (2.54) using the above isomorphisms. Recall also that the connection  $\nabla^N$  on  $N$  was introduced before (1.19). The connection  $\nabla^N$  induces the splitting

$$TN = N \oplus T^H N \quad (4.4)$$

of the tangent space of the total space of  $N$ . Here  $T^H N$  is the horizontal part of  $N$  with respect to the connection  $N$ . If  $U \in TY$ , we denote by  $U^H \in T^H N$  the horizontal lift of  $U$  in  $T^H N$ .

For  $\epsilon > 0$ , we denote by  $\mathbb{E}(\epsilon)$  (resp.  $\mathbb{E}$ ) the set of smooth sections of  $\pi^*(L^p|_Y \otimes F|_Y)$  on  $B_\epsilon(N)$  (resp. on the total space of  $N$ ). We also denote by  $\mathbb{E}^{(0,1)}(\epsilon)$  (resp.  $\mathbb{E}^{(0,1)}$ ) the set of smooth sections of  $\pi^*(T^{*(0,1)}X|_Y) \otimes \pi^*(L^p|_Y \otimes F|_Y)$  on  $B_\epsilon(N)$  (resp. on the total space of  $N$ ).

Clearly, the above isomorphisms allow us to see  $\bar{\partial}^{L^p \otimes F}$  as an operator

$$\bar{\partial}^{L^p \otimes F} : \mathbb{E}(r_\perp) \rightarrow \mathbb{E}^{(0,1)}(r_\perp). \quad (4.5)$$

We fix a point  $y_0 \in Y$  and an orthonormal frame  $(e_1, \dots, e_{2m})$  (resp.  $(e_{2m+1}, \dots, e_{2n})$ ) in  $(T_{y_0}Y, g^{TY})$  (resp. in  $(N_{y_0}, g_{y_0}^N)$ ) such that (1.16) is satisfied. Using the exponential coordinates on  $Y$  and the parallel transport of  $(e_{2m+1}, \dots, e_{2n})$  along the geodesics on  $Y$ , similarly to Fermi coordinates  $\psi_{y_0}$  in (1.17), we introduce complex coordinates  $z_1, \dots, z_m$  on  $Y$  and  $z_{m+1}, \dots, z_n$  on  $N$ . Using those coordinates, we define the operators

$$\bar{\partial}_H^{L^p \otimes F}, \mathcal{L}_N^{L^p \otimes F} : \mathbb{E} \rightarrow \mathbb{E}^{(0,1)}, \quad (4.6)$$

by prescribing their action at a point  $(y_0, Z_N)$ ,  $Z_N \in \mathbb{R}^{2(n-m)}$ , as follows

$$\bar{\partial}_H^{L^p \otimes F} = \sum_{i=1}^m d\bar{z}_i|_{y_0} \cdot \left( \frac{\partial}{\partial \bar{z}_i} \Big|_{y_0} \right)^H, \quad \mathcal{L}_N^{L^p \otimes F} = \sum_{i=m+1}^n d\bar{z}_i|_{y_0} \cdot \left( \frac{\partial}{\partial \bar{z}_i} + \frac{\pi z_i}{2} \right). \quad (4.7)$$

The first differentiation in (4.7) is well-defined because  $\pi_* \left( \frac{\partial}{\partial \bar{z}_i} \Big|_{y_0} \right)^H = \frac{\partial}{\partial \bar{z}_i} \Big|_{y_0}$  is of type  $(0, 1)$ , and the second derivation is well-defined because the vector bundles are trivialized along fibers of  $\pi$ .

We use notation (1.30), and for any  $\epsilon > 0$  define the rescaling operator  $F_t : \mathbb{E}(\epsilon) \rightarrow \mathbb{E}(\frac{\epsilon}{t})$  for  $f \in \mathbb{E}(\epsilon)$  as follows

$$(F_t f)(y, Z_N) := f(y, tZ_N), \quad (y, Z_N) \in B_\epsilon(N). \quad (4.8)$$

The operator  $F_t : \mathbb{E}^{(0,1)}(\epsilon) \rightarrow \mathbb{E}^{(0,1)}(\frac{\epsilon}{t})$  is defined in an analogous way.

**Theorem 4.3.** *As  $p \rightarrow \infty$ , we have*

$$F_t \circ \bar{\partial}^{L^p \otimes F} \circ F_t^{-1} = \frac{1}{t} \mathcal{L}_N^{L^p \otimes F} + \bar{\partial}_H^{L^p \otimes F} + O(t|Z_N|^2 \bar{\partial}_N + t|Z_N| \bar{\partial}_H + t|Z_N|), \quad (4.9)$$

where  $O(t|Z_N|^2 \bar{\partial}_N + t|Z_N| \bar{\partial}_H + t|Z_N|)$  is an operator of the form  $\sum_{i=1}^m a_i(t, y, Z_N) \cdot d\bar{z}_i|_{y_0} \cdot \left( \frac{\partial}{\partial \bar{z}_i} \Big|_{y_0} \right)^H + \sum_{j=m+1}^n b_j(t, y, Z_N) \cdot d\bar{z}_j|_{y_0} \cdot \frac{\partial}{\partial \bar{z}_j} + c(t, y, Z_N)$ , such that there is a constant  $C > 0$ , for which  $|a_i(t, y, Z_N)| \leq Ct|Z_N|^2$ ,  $|b_j(t, y, Z_N)| \leq Ct|Z_N|$ ,  $|c(t, y, Z_N)| \leq Ct|Z_N|$  holds for any  $y \in Y$ ,  $|Z_N| < r_\perp$ ,  $i = 1, \dots, m$ , and  $j = m+1, \dots, n$ .

*Proof.* To simplify the notations, we denote  $\frac{\tau \partial}{\partial \bar{z}_i} := \tau \left( \frac{\partial}{\partial \bar{z}_i} \Big|_{y_0} \right)$  and  $\frac{\tau \partial}{\partial Z_i} := \tau \left( \frac{\partial}{\partial Z_i} \Big|_{y_0} \right)$ . By (4.3), the action of the operator  $\bar{\partial}^{L^p \otimes F}$ , viewed as in (4.5), at a point  $(y_0, Z_N)$ ,  $|Z_N| < r_\perp$ , is given by

$$\bar{\partial}^{L^p \otimes F} = \sum_{i=1}^n dz_i|_{y_0} \cdot \frac{\tau \partial}{\partial \bar{z}_i} + \sum_{i=1}^n dz_i|_{y_0} \cdot \left( p\Gamma_{(y_0, Z_N)}^L \left( \frac{\tau \partial}{\partial \bar{z}_i} \right) + \Gamma_{(y_0, Z_N)}^F \left( \frac{\tau \partial}{\partial \bar{z}_i} \right) \right). \quad (4.10)$$

A calculation from Bismut-Lebeau [3, p. 94-96] shows

$$F_t \circ \left( \sum_{i=1}^n dz_i|_{y_0} \cdot \frac{\tau \partial}{\partial \bar{z}_i} \right) \circ F_t^{-1} = \frac{1}{t} \left( \sum_{i=m+1}^n d\bar{z}_i|_{y_0} \cdot \frac{\partial}{\partial \bar{z}_i} \right) + \bar{\partial}_H^{L^p \otimes F} - \dim Y \cdot \nu^*(y) \wedge + O(t|Z_N|^2 \bar{\partial}_N + t|Z_N| \bar{\partial}_H), \quad (4.11)$$

where the  $O$ -term is interpreted as in (4.9), and  $\nu \in \mathcal{C}^\infty(Y, N)$  is the dual of the mean curvature of  $\iota$ , defined as follows

$$\nu := \frac{1}{2m} \sum_{i=1}^{2m} A(e_i) e_i, \quad (4.12)$$

where the sum runs over an orthonormal basis of  $(TY, g^{TY})$ . However, as in our setting the manifolds are Kähler (hence,  $J$  commutes with  $A$ ), we see by (2.53) that for  $i = 1, \dots, m$ , we have

$$A(e_{2i-1}) e_{2i-1} = A(Je_{2i}) Je_{2i} = JA(Je_{2i}) e_{2i} = JA(e_{2i}) Je_{2i} = -A(e_{2i}) e_{2i}, \quad (4.13)$$

for the basis  $e_1, \dots, e_{2m}$ , as in (1.16). Hence, we see that

$$\nu = 0, \quad (4.14)$$

which, for compact manifolds, is a consequence of the well-known fact that every complex submanifold of a Kähler manifold is volume minimizing in its homology class, cf. [12, III.(1.25)].

Remark that in [3], no assumption of bounded geometry was made, and (4.11) was stated locally in  $Y$ . However, in the proof of Bismut-Lebeau, the  $O$ -term comes from the expansion of the Christoffel symbols and the connection forms, and hence by Propositions 2.9, 2.16, it can be bounded uniformly in  $Y$ .

Now, from Lemma 2.24 and (2.51), we see that for  $i = 1, \dots, 2m$ ,  $j = 2m + 1, \dots, 2n$ , at the point  $(y_0, Z_N)$ , we have

$$\begin{aligned} \frac{\tau \partial}{\partial Z_j} &= \frac{\partial \psi}{\partial Z_j} + O(|Z_N|^2), \\ \frac{\tau \partial}{\partial Z_i} &= \frac{\partial \psi}{\partial Z_i} + \sum_{l=1}^{2m} \frac{\partial \psi}{\partial Z_l} g_{y_0}^{TM}(A(e_i)Z_N, e_l) + O(|Z_N|^2). \end{aligned} \quad (4.15)$$

From Lemma 2.27 and (4.15), we deduce

$$\begin{aligned} \Gamma_{(y_0, Z_N)}^F \left( \frac{\tau \partial}{\partial \bar{z}_i} \right) &= O(|Z_N|), \quad \Gamma_{(y_0, Z_N)}^F \left( \frac{\tau \partial}{\partial \bar{z}_j} \right) = O(|Z_N|), \\ \Gamma_{(y_0, Z_N)}^L \left( \frac{\tau \partial}{\partial \bar{z}_i} \right) &= O(|Z_N|^3), \quad \Gamma_{(y_0, Z_N)}^L \left( \frac{\tau \partial}{\partial \bar{z}_j} \right) = \frac{1}{2} R_{y_0}^L \left( Z_N, \frac{\partial}{\partial \bar{z}_i} \right) + O(|Z_N|^3). \end{aligned} \quad (4.16)$$

Now, from (4.10), (4.11), (4.14) and (4.16), we get

$$\begin{aligned} F_t \circ \bar{\partial}^{L^p \otimes F} \circ F_t^{-1} &= \frac{1}{t} \sum_{i=m+1}^n d\bar{z}_i \cdot \left( \frac{\partial}{\partial \bar{z}_i} + \frac{1}{2} R_{y_0}^L \left( Z_N, \frac{\partial}{\partial \bar{z}_i} \right) \right) + \bar{\partial}_H^{L^p \otimes F} \\ &\quad + O(t|Z_N|^2 \bar{\partial}_N + t|Z_N| \bar{\partial}_H + t|Z_N|). \end{aligned} \quad (4.17)$$

Finally, by (1.1), we have  $R_{y_0}^L \left( Z_N, \frac{\partial}{\partial \bar{z}_i} \right) = \pi z_i$ . By this and (4.17), we conclude.  $\square$

## 4.2 Extension theorem, a proof of the lower bound in Theorem 4.1

In this section, we establish the lower bound in Theorem 4.1. For this, in the notations from Section 4.1, we show that the following refined version of Ohsawa-Takegoshi extension theorem holds.

**Theorem 4.4.** *There are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1$  and  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , there is  $f \in H_{(2)}^0(X, L^p \otimes F)$ , such that  $f|_Y = g$  and*

$$\|f\|_{L^2(X)} \leq \frac{C}{p^{\frac{n-m}{2}}} \|g\|_{L^2(Y)}. \quad (4.18)$$

*Proof of the lower bound in Theorem 4.1.* Let  $C$  be as in (4.18). As  $E_p$  is defined as the minimal extension with respect to the  $L^2$ -norm, for any  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , by Theorem 4.4, we have

$$\|E_p g\|_{L^2(X)} \leq \frac{C}{p^{\frac{n-m}{2}}} \|g\|_{L^2(Y)}. \quad (4.19)$$

This means exactly that the lower bound in Theorem 4.1 holds for  $c := C^{-1}$ .  $\square$

By the above, to settle the main goal of this section, we need to establish Theorem 4.4. The main idea of the proof of Theorem 4.4 is to pass through the general framework of the proof of Ohsawa-Takegoshi extension theorem, cf. [11, §11]. We choose a smooth extension of  $g$  over  $X$ , and then obtain the holomorphic extension by modifying the smooth one using a solution of  $\bar{\partial}$ -equation with singular weight, which forces the solution to annihilate along  $Y$ .

The novelty here is that instead of choosing an arbitrary smooth extension, as it is done in [11, §11], we choose a specific one, given by the operator (1.9). The incentive for doing so comes from the fact that we would like to get some estimates, related to this extension, which would be uniform in  $p$ , and this doesn't seem doable for an arbitrary choice of the extension. Choosing a specific extension allows us to use Theorem 4.3 to prove the uniform versions of some  $L^2$ -estimates, which are indispensable in the proof. For those estimates and after, we need the following proposition.

**Proposition 4.5.** For any  $k \in \mathbb{N}$ , there are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1$  and  $f \in H_{(2)}^0(X, L^p \otimes F)$ , we have

$$\|\nabla^k f\|_{L^2(X)} \leq Cp^{\frac{k}{2}} \|f\|_{L^2(X)}, \quad (4.20)$$

where  $\nabla$  is the covariant derivative with respect to the induced Chern and Levi-Civita connections.

For the proof of Proposition 4.5 and in many places later, the following result will be crucial.

**Theorem 4.6.** For any  $k \in \mathbb{N}$ , there are  $c, C > 0$ ,  $p_1 \in \mathbb{N}^*$  such that for  $p \geq p_1$ , we have

$$\left| B_p^X(x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \leq Cp^{n+\frac{k}{2}} \cdot \exp(-c\sqrt{p} \cdot \text{dist}(x_1, x_2)), \quad (4.21)$$

where  $\mathcal{C}^k$ -norm here is interpreted as in Theorem 1.5.

*Proof of Theorem 4.6.* For compact manifolds, Theorem 4.6 was implicit in Dai-Liu-Ma [9, Theorem 4.18]. For manifolds of bounded geometry it was proved by Ma-Marinescu in [30, Theorem 1]. For the summary of numerous previous works on the off-diagonal estimates of the Bergman kernel as in (4.21), refer to [30, p. 1328].  $\square$

*Proof of Proposition 4.5.* For compact manifolds, the statement is very classical. For manifolds of bounded geometry, the proof can be obtained by a slight modification of the argument in [30, Lemma 2]. We present here an alternative proof, based on Theorem 4.6. First of all, by Theorem 4.6 and Proposition 3.2, there are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$  such that for any  $x_0 \in X$ , for  $p \geq p_1$ , we have

$$\int_X |(\nabla^k B_p^X)(x_0, x)| dv_X(x) \leq Cp^{\frac{k}{2}}, \quad \int_X |(\nabla^k B_p^X)(x, x_0)| dv_X(x) \leq Cp^{\frac{k}{2}}. \quad (4.22)$$

Also, since  $f = B_p^X f$ ,  $\nabla^k f$  admits the integral representation

$$\nabla^k f(x_1) = \int_X (\nabla^k B_p^X)(x_1, x_2) \cdot f(x_2) dv_X(x_2). \quad (4.23)$$

We deduce (4.20) directly from (4.22), (4.23) and Young's inequality for integral operators, cf. [36, Theorem 0.3.1] applied for  $p, q = 2$ ,  $r = 1$  in the notations of [36].  $\square$

*Proof of Theorem 4.4.* Recall that the operator  $E_p^0$  was defined in (1.9). We would like to verify that for any  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ ,  $y_0 \in Y$ , we have

$$\alpha(y_0) = 0, \quad \text{where} \quad \alpha := \bar{\partial}^{L^p \otimes F}(E_p^0 g). \quad (4.24)$$

Indeed, let us work in a neighborhood  $V := B_Y^X(\frac{r_\perp}{4})$  of  $Y$  in  $X$ . Recall that  $t \in \mathbb{R}_+$  and  $F_t$  were defined in (4.8). Then in the notations of (1.9), on  $V$ , we have

$$E_p^0 g = F_t^{-1} \tilde{g}, \quad \tilde{g}(y, Z_N) = g(y) \exp\left(-\frac{\pi}{2}|Z_N|^2\right). \quad (4.25)$$

Recall that  $\bar{\partial}_H^{L^p \otimes F}$ ,  $\mathcal{L}_N^{L^p \otimes F}$  were defined in (4.7). A trivial calculation shows that on  $V$ , we have

$$\mathcal{L}_N^{L^p \otimes F} \tilde{g} = 0. \quad (4.26)$$

Also, since  $\nabla^N$  preserves  $g^N$ , in the notations of (4.7), similarly to [3, (8.97)], we have

$$\left(\left(\frac{\partial}{\partial \bar{z}_i}\right) \Big|_{y_0}\right)^H \tilde{g}(y_0, Z_N) = \left(\frac{\partial}{\partial \bar{z}_i} g\right)(y_0) \exp\left(-\frac{\pi}{2}|Z_N|^2\right). \quad (4.27)$$

As a consequence of (4.27) and the fact that  $g$  is holomorphic, we obtain

$$\bar{\partial}_H^{L^p \otimes F} \tilde{g} = 0. \quad (4.28)$$

From (4.25), (4.26), (4.28) and the fact that all the residue terms in Theorem 4.3 contain  $|Z_N|$ , we deduce (4.24).

Now, using the  $L^2$ -estimates, let us construct a holomorphic perturbation of  $E_p^0 g$ , satisfying the assumptions of Theorem 4.4. Recall that  $\delta_Y : X \setminus Y \rightarrow \mathbb{R}$ ,  $\alpha_Y : X \rightarrow \mathbb{R}$ , were defined in (2.85). For  $\epsilon > 0$ , let us now define the weight  $\delta_p : X \setminus Y \rightarrow \mathbb{R}$  as follows

$$\delta_p := 2(n-m)\delta_Y - \epsilon p \alpha_Y. \quad (4.29)$$

By taking  $\epsilon$  small, by Theorem 2.31, we see that there exists  $p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1$ , over  $X$ , the following inequality holds in the distributional sense

$$p\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \delta_p > \frac{p}{2}\omega. \quad (4.30)$$

Let us fix  $\epsilon$  small enough, such that for any  $|Z_N| < r_\perp$ , we have

$$\frac{\pi}{2}|Z_N|^2 - \epsilon \alpha_Y(y, Z_N) \geq \frac{\pi}{4}|Z_N|^2. \quad (4.31)$$

We will now prove that there are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1$ ,  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$  and  $\alpha$  defined in (4.24), we have

$$\int_{X \setminus Y} |\alpha|^2 e^{-\delta_p} dv_X \leq C \|g\|_{L^2(Y)}^2. \quad (4.32)$$

By Corollary 2.10, Lemmas 2.28, 2.29, (1.9) and (4.31), we see that there are  $c, C > 0, p_1 \in \mathbb{N}^*$  such for any  $p \geq p_1$ , we have

$$\int_{X \setminus B_Y^X(\frac{r_\perp}{4})} |\alpha|^2 e^{-\delta_p} dv_X \leq C \exp(-cp) \left( \|g\|_{L^2(Y)}^2 + \|\nabla g\|_{L^2(Y)}^2 \right). \quad (4.33)$$

Now, as  $\alpha$  has support in  $B_Y^X(\frac{r_\perp}{2})$ , it is enough to work in  $(y, Z_N)$ ,  $y \in Y, Z_N \in N_y$  coordinates. To estimate the integral over  $B_Y^X(\frac{r_\perp}{4})$ , we use (4.25) and make the change of variables by  $F_t$  to get

$$\begin{aligned} & \int_{B_Y^X(\frac{r_\perp}{4})} |\alpha|^2 e^{-\delta_p} dv_Y \wedge dv_N \\ &= \int_{B_{\frac{r_\perp}{4t}}(N)} \left| (F_t \circ \bar{\partial}^{L^p \otimes F} \circ F_t^{-1}) \tilde{g} \right|^2 (y, tZ_N) \frac{e^{\epsilon p \alpha_Y(y, tZ_N)}}{|Z_N|^{2(n-m)}} dv_Y \wedge dv_N. \end{aligned} \quad (4.34)$$

By using Corollary 2.10, (4.26), (4.27), (4.28), (4.31), (4.34), and the fact that for  $j = 1, 2$ ,

$$\int_{\mathbb{R}^{2(n-m)}} \frac{|Z_N|^j \exp(-\frac{\pi}{4}|Z_N|^2) dv_{\mathbb{R}^{2(n-m)}}(Z_N)}{|Z_N|^{2(n-m)}} < +\infty, \quad (4.35)$$

we conclude that there are  $C > 0, p_1 \in \mathbb{N}$ , such that for any  $p \geq p_1, g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$  and  $\alpha$  as in (4.24), we have

$$\int_{B_Y^X(\frac{r_\perp}{4})} |\alpha|^2 e^{-\delta_p} dv_X \leq Ct \left( \|g\|_{L^2(Y)}^2 + \|\nabla g\|_{L^2(Y)}^2 \right). \quad (4.36)$$

From Proposition 4.5 and (4.36), we deduce (4.32).

By [10, Theorem 1.5],  $X \setminus Y$  is a complete Kähler manifold. Hence, by (4.30), we may resolve the  $\bar{\partial}$ -equation on  $X \setminus Y$ , see [11, Proposition 13.4]. From this, the trivial fact that  $\bar{\partial}^{L^p \otimes F} \alpha = 0$  and (4.32), we see that there are  $C > 0, p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1, g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , there is a section  $f_0 \in \mathcal{C}^\infty(X \setminus Y, L^p \otimes F)$ , such that

$$\bar{\partial}^{L^p \otimes F} f_0 = \alpha, \quad \int_{X \setminus Y} |f_0|^2 e^{-\delta_p} dv_X \leq \frac{C}{p} \|g\|_{L^2(Y)}^2. \quad (4.37)$$

Let us prove that  $f := E_p^0 g - f_0$  verifies the assumptions of Theorem 4.4.

From (4.37), we see that over  $X \setminus Y$ ,  $\bar{\partial}^{L^p \otimes F} f = 0$ . Also, by (4.37), we easily get that  $f \in L^2(X)$ . By the standard regularity result, [10, Lemme 6.9],  $f$  extends smoothly and the equation  $\bar{\partial}^{L^p \otimes F} f = 0$  holds on  $X$ . In particular,  $f_0$  extends smoothly as well. However, since  $\exp(-2(n-m)\delta_Y)$  is not integrable, the  $L^2$ -bound (4.37) implies that  $f_0|_Y = 0$ . Hence, we conclude that  $f|_Y = g$ . It is only left to verify that  $f$  satisfies the needed  $L^2$ -bound (4.18).

An easy calculation, using Corollary 2.10, shows that there are  $c, C > 0$ , such that we have

$$\frac{c}{p^{\frac{n-m}{2}}} \|g\|_{L^2(Y)} \leq \|E_p^0 g\|_{L^2(X)} \leq \frac{C}{p^{\frac{n-m}{2}}} \|g\|_{L^2(Y)}. \quad (4.38)$$

Let us now prove the following bound

$$\int_X |f_0|^2 e^{-\delta_p} dv_X \geq Cp^{n-m} \int_X |f_0|^2 dv_X. \quad (4.39)$$

This will be clearly enough for our needs, as from the  $L^2$ -bound in (4.37), (4.38) and (4.39), we would deduce the  $L^2$ -bound (4.18).

First of all, since  $\alpha_Y \geq \min\{\frac{1}{2}(r_\perp)^2, \frac{1}{2}\}$  on  $X \setminus B_Y^X(r_\perp)$ , there are  $c, C > 0$ , such that

$$\int_{X \setminus B_Y^X(r_\perp)} |f_0|^2 e^{-\delta p} dv_X \geq C \exp(\epsilon c p) \int_{X \setminus B_Y^X(r_\perp)} |f_0|^2 dv_X \quad (4.40)$$

It is now only left to give the lower bound for the integrand on the left-hand side of (4.39), where the integration is done over  $B_Y^X(r_\perp)$ . But remark that from (2.85), over  $B_Y^X(r_\perp)$ , there is  $C > 0$ , such that for any  $p \in \mathbb{N}^*$ , we have  $e^{-\delta p} \geq Cp^{n-m}$ . From this, we deduce

$$\int_{B_Y^X(r_\perp)} |f_0|^2 e^{-\delta p} dv_X \geq Cp^{n-m} \int_{B_Y^X(r_\perp)} |f_0|^2 dv_X. \quad (4.41)$$

From (4.40) and (4.41), we obtain (4.39).  $\square$

*Remark 4.7.* Our proof shows that there is  $E_p^1 : H_{(2)}^0(Y, \iota^*(L^p \otimes F)) \rightarrow H_{(2)}^0(X, L^p \otimes F)$ , verifying  $(E_p^1 g)|_Y = g$  for  $g \in H_{(2)}^0(Y, \iota^*(L^p \otimes F))$ , and such that (1.11) holds for  $E_p^1$  instead of  $E_p^0$ .

### 4.3 Asymptotic trace theorem, a proof of the upper bound in Theorem 4.1

The main goal of this section is to give a proof of the upper bound in Theorem 4.1. The main idea of our proof is to apply the trace theorem from the theory of Sobolev spaces in a blown-up neighborhood of  $y_0 \in Y$ .

To describe it precisely, recall that for a fixed point  $y_0 \in Y$ , the Fermi coordinates  $\psi_{y_0}$  were defined in (1.17). We identify  $L$  (resp.  $F$ ) to  $L_{y_0}$  (resp.  $F_{y_0}$ ) in a neighborhood  $U := B_{y_0}^X(R)$ , for  $R$  as in (2.17), by using the parallel transport in the same way as it was done before Theorem 1.6.

We fix a unitary vector in  $(L_{y_0}, h^{L_{y_0}})$ , and trivialize  $(L, h^L)$  over  $U$  using it and the above parallel transport. This allows us to see the Kodaira Laplacians  $\square^{L^p \otimes F}$  on  $L^p \otimes F$  as operators  $L_{p, y_0}$ , acting on a fixed space  $\mathcal{C}^\infty(U, F_{y_0})$ .

We use notation (1.30), and define  $S_t : \mathcal{C}^\infty(U, F_{y_0}) \rightarrow \mathcal{C}^\infty(B_{y_0}^X(tR), F_{y_0})$  as follows

$$S_t f(Z) := f\left(\frac{Z}{t}\right). \quad (4.42)$$

Let  $\nabla$  be the ordinary differentiation operator on  $T_{y_0}X$ . Over  $\mathcal{C}^\infty(U, F_{y_0})$ , we can define the following operators

$$\begin{aligned} \nabla_{0, \cdot} &:= \nabla + \frac{1}{2} R_{y_0}^L(Z, \cdot), \\ \mathcal{L} &:= - \sum_{i=1}^{2n} (\nabla_{0, e_i})^2 - 4n\pi, \\ \mathcal{L}_t &:= t^2 S_t^{-1} L_{p, y_0} S_t. \end{aligned} \quad (4.43)$$

According to [28, §4.1.6],  $\mathcal{L}$  coincides with the operator defined in (3.24). The following statement, essentially parallel to [28, Lemma 1.6.6] (which corresponds to  $Y = \{y_0\}$  in our notation), is crucial in our proof of the upper bound in Theorem 4.1.



**Lemma 4.8.** When  $t \rightarrow 0$ , the following asymptotics holds

$$\mathcal{L}_t = \mathcal{L} + O(t\nabla^2), \quad (4.44)$$

where  $O(t\nabla^2)$  means a second order differential operator with coefficients bounded for any  $k \in \mathbb{N}$  in  $\mathcal{C}^k$ -norm by  $C_k t$  for a certain constant  $C_k > 0$ , which is uniform on  $y_0 \in Y$ , and depends only on the constant  $C_{k+1}$  from (2.1), (2.14) and (2.22).

*Proof.* The proof proceeds through the use of Lichnerowicz formula exactly in the same way as in the proof of [28, Lemma 1.6.6]. Of course, the function  $\kappa$  from [28, Lemma 1.6.6] doesn't appear in our considerations because it doesn't influence the highest order of the asymptotics of  $\mathcal{L}_t$ . The fact that we do not work in geodesic coordinates, but in Fermi coordinates, doesn't pose any problems as well, since according to Proposition 2.18, the derivative at 0 of the diffeomorphism which relates the two coordinate systems is equal to the identity, and, thus, doesn't influence the highest order of the asymptotics of  $\mathcal{L}_t$ . The uniform boundness in  $O$ -term follows Propositions 2.3, 2.9 and 2.16.  $\square$

*Proof of the upper bound in Theorem 4.1.* We would like to prove that there are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1$ ,  $y_0 \in Y$  and  $f_p \in H^0(X, L^p \otimes F)$ , the following bound holds

$$\|f_p|_Y\|_{L^2(B_{y_0}^Y(t))} \leq Cp^{\frac{n-m}{2}} \|f_p\|_{L^2(B_{y_0}^X(4t))}. \quad (4.45)$$

Clearly, once (4.45) is established, the upper bound in Theorem 4.1 readily follows after a summation over different  $y_i \in Y$ ,  $i \in \mathbb{N}$ , chosen as in Proposition 2.11.

Now, for  $\epsilon > 0$ , we denote by  $L_{X,0}^2(\epsilon)$ ,  $L_{Y,0}^2(\epsilon)$  the  $L^2$ -spaces on  $B_0^{\mathbb{R}^{2n}}(\epsilon)$ ,  $B_0^{\mathbb{R}^{2m}}(\epsilon)$  of sections of  $(F_{y_0}, h_{y_0}^F)$ , associated with the metrics  $g_{y_0}^{TX}$  and  $g_{y_0}^{TY}$  using the fixed choice of the orthonormal frames in  $T_{y_0}Y$ ,  $N_{y_0}$ . A similar notation is used for the corresponding Sobolev space  $H_{X,0}^k(\epsilon)$ ,  $k \in \mathbb{N}$ . By Proposition 2.3, there is a constant  $C > 1$ , which depends only on  $C_0$  from (2.3), applied for  $X$  and  $Y$ , such that for any  $y_0 \in Y$ ,  $\epsilon < \min\{r_X, r_Y\}$ ,  $h \in \mathcal{C}^\infty(X, L^p \otimes F)$ , we have

$$\begin{aligned} C^{-1} \|h\|_{L_{X,0}^2(\epsilon)} &\leq \|h\|_{L^2(B_{y_0}^X(\epsilon))} \leq C \|h\|_{L_{X,0}^2(\epsilon)}, \\ C^{-1} \|h|_Y\|_{L_{Y,0}^2(\epsilon)} &\leq \|h|_Y\|_{L^2(B_{y_0}^Y(\epsilon))} \leq C \|h|_Y\|_{L_{Y,0}^2(\epsilon)}, \end{aligned} \quad (4.46)$$

where we identified  $h$  as an element of  $\mathcal{C}^\infty(B_{y_0}^X(\epsilon), F_{y_0})$  as in the beginning of this section.

Let us denote  $f_t := S_t^{-1}(f_p|_U)$ , where by the above trivialization we view  $f_p$  as an element of  $\mathcal{C}^\infty(U, F_{y_0})$ . Then by the change of variables, we have

$$\|f_p|_Y\|_{L_{Y,0}^2(t\epsilon)} = p^{-\frac{m}{2}} \|f_t|_Y\|_{L_{Y,0}^2(\epsilon)}, \quad \|f_p\|_{L_{X,0}^2(4t\epsilon)} = p^{-\frac{n}{2}} \|f_t\|_{L_{X,0}^2(4\epsilon)}. \quad (4.47)$$

Now, the operator  $\mathcal{L}$  is elliptic. From elliptic estimates, cf. [17, Theorem 6.3.5], and Lemma 4.8, there is a constant  $C > 0$  (which depends only on  $\epsilon$  and on the constant  $C_{k+n-m+2}$  from (2.1), (2.14) and (2.22)), such that for any  $h \in \mathcal{C}^\infty(U, F_{x_0})$ ,

$$\|h\|_{H_{X,0}^{n-m}(\epsilon/4)} \leq C \sum_{i=0}^{n-m} \|\mathcal{L}_t h\|_{L_{X,0}^2(\epsilon/2)}. \quad (4.48)$$

As in our coordinates, the points  $(Z_Y, 0)$ ,  $Z_Y \in \mathbb{R}^{2m}$ ,  $|Z_Y| < r_Y$  correspond to a neighborhood of  $y_0$  in  $Y$ , we may apply the trace theorem from the theory of Sobolev spaces, cf. [17, Theorem 5.5.1], to see that there is a constant  $C > 0$ , which depends only on  $r_0$ , such that for any  $h \in \mathcal{C}^\infty(V, F_{x_0})$ , we have

$$\|h|_Y\|_{L^2_0(r_0/2)} \leq C \|h\|_{H^m_{X,0}(r_0)}. \quad (4.49)$$

We note that by (4.43) and the fact that  $f_p \in H^0_{(2)}(X, L^p \otimes F)$ , we have  $\mathcal{L}_t f_t = 0$ . From this, (4.46), (4.47) and (4.48), (4.49), applied for  $h := f_t$ , we finally deduce (4.45).  $\square$

## 5 Asymptotic expansions of the two kernels

The main goal of this section is to study the asymptotics of the Schwartz kernels of the orthogonal Bergman projector and the extension operator. This section is organized as follows. In Section 5.1, we establish Theorem 1.8. We also show that despite the global nature of the orthogonal Bergman kernel, the asymptotic expansion of it depends only locally on the geometry of the problem. In Section 5.2, we prove that after certain reparametrization given by a homothety in Fermi coordinates, the Bergman kernel admits a complete asymptotic expansion. In Section 5.3, we establish Theorem 1.9. Finally, in Section 5.4, we prove all the other results announced in Section 1.

### 5.1 Exponential decay and localization of the orthogonal Bergman kernel

The first main goal of this section is to prove that the orthogonal Bergman kernel has exponential off-diagonal decay, i.e. to establish Theorem 1.8. The second main goal is to establish that the asymptotic expansion of the orthogonal Bergman kernel is localized.

The main difficulty here is that the projection  $B_p^\perp$  is not the spectral projection associated to the Laplacian. Hence, the methods, developed by Dai-Liu-Ma in [9], cf. [28, Proposition 4.1.5], based on the finite propagation speed property for the wave equation, cannot be applied.

One can remedy this by extending a slightly more technical approach through the  $L^2$ -estimates with varying weights, similar to what has been done by Lindholm [27, Proposition 9].

We follow here an alternative approach. The main idea of our proof of Theorem 1.8 is to construct the ‘‘approximate projection’’  $B_p^{\perp,a}$  onto the  $H^{0,\perp}_{(2)}(X, L^p \otimes F)$  (the precise meaning of this is given after (5.5)), such that the Schwartz kernel of it verifies the estimate analogous to (1.25). Then by means of the spectral theory, essentially relying on Theorem 4.1, we relate  $B_p^{\perp,a}$  and  $B_p^\perp$ , and show that the above estimate persists through this relation. An exponential estimate of the Bergman kernel, see Theorem 4.6, plays a crucial role in our approach. The advantage of this approach is that it introduces some of the techniques, which will later play a crucial role in our proof of Theorem 1.9, where the  $L^2$ -estimates, it seems, can not be used to provide a direct proof.

For the proof, let us first introduce some preliminary results. We use the notations and assumptions from Theorem 1.8. We will need the following famous statement.

**Proposition 5.1.** For any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \in \mathbb{N}$ ,  $f \in H^0_{(2)}(X, L^p \otimes F)$  and  $x \in X$ , we have

$$|\nabla^k f(x)| \leq Cp^{(n+k)/2} \|f\|_{L^2(X)}. \quad (5.1)$$

*Proof.* For compact manifolds, the statement is very classical. For manifolds of bounded geometry, it was proved by Ma-Marinescu [30, Lemma 2].  $\square$

*Proof of Theorem 1.8.* Let us construct the “approximate projection”  $B_p^{\perp,a}$  onto  $H_{(2)}^{0,\perp}(X, L^p \otimes F)$ . For this, define  $A_p := E_p^0 \circ \text{Res}_Y \circ B_p^X$ , where  $E_p^0$  was defined in (1.9). Then we define  $B_p^{\perp,a} := L^2(X, L^p \otimes F) \rightarrow L^2(X, L^p \otimes F)$  as follows

$$B_p^{\perp,a} := A_p^* \circ A_p. \quad (5.2)$$

Clearly, by Theorem 4.6, Lemma 3.1 and (1.9), we see that there are  $c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1$  and  $x_1, x_2 \in X$ , we have

$$|B_p^{\perp,a}(x_1, x_2)|_{\mathcal{C}^k(X \times X)} \leq Cp^n \cdot \exp\left(-c\sqrt{p} \cdot (\text{dist}(x_1, x_2) + \text{dist}(x_1, Y) + \text{dist}(x_2, Y))\right). \quad (5.3)$$

Also, for any  $f \in L^2(X, L^p \otimes F)$ , we have  $\langle B_p^{\perp,a}f, f \rangle_{L^2} = \langle A_p f, A_p f \rangle_{L^2}$ . Hence in the notations of (1.24), we have

$$\ker B_p^{\perp,a} = H_{(2)}^{0,0}(X, L^p \otimes F) + \ker B_p^X. \quad (5.4)$$

By Theorem 4.1 and (4.38), the operators  $B_p^{\perp,a}$ ,  $p \in \mathbb{N}$ , possess a *uniform spectral gap*, i.e. there are  $a, b > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $p \geq p_1$ , we have

$$\text{Spec}(B_p^{\perp,a}) \subset \{0\} \cup [a, b]. \quad (5.5)$$

The properties (5.4) and (5.5) justify the name “approximate projection”, as  $B_p^{\perp}$  is the only self-adjoint operator on  $H_{(2)}^0(X, L^p \otimes F)$ , satisfying (5.4) and (5.5) for  $a, b = 1$ .

We will now show that the estimate (1.25) follows formally from (5.3), (5.4) and (5.5). To do so, from (5.4) and (5.5), we see that for  $\epsilon := 1 - \frac{a}{2b}$ , and for any  $r \in \mathbb{N}^*$ , the following bound holds

$$\left\| \left( B_p^X - \frac{B_p^{\perp,a}}{2b} \right)^r - B_p^X + B_p^{\perp} \right\| \leq \epsilon^r. \quad (5.6)$$

Since the operator under the norm of (5.6) vanishes on the orthogonal complement of  $H_{(2)}^0(X, L^p \otimes F)$  and takes values inside of  $H_{(2)}^0(X, L^p \otimes F)$ , Proposition 5.1 and (5.6) then imply that for any  $k \in \mathbb{N}$ , there are  $C > 0$ ,  $l \in \mathbb{N}$ , such that for any  $p \geq p_1$ ,  $r \in \mathbb{N}$ , we have

$$\left\| \left( B_p^X - \frac{B_p^{\perp,a}}{2b} \right)^r - B_p^X + B_p^{\perp} \right\|_{\mathcal{C}^k(X \times X)} \leq Cp^{n+k/2} \epsilon^r. \quad (5.7)$$

Now, using the trivial identity  $(B_p^X)^r = B_p^X$ , we expand  $(B_p^X - \frac{B_p^{\perp,a}}{2b})^r - B_p^X$  into  $2^r - 1$  summands, such that each summand contains  $B_p^{\perp,a}$  as a multiple. By Theorem 4.6 and (5.3), we may apply Lemma 3.1 for  $W := Y$ , to bound each of those summands. Hence, there are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $x_1, x_2 \in X$ ,  $p \geq p_1$ ,  $r \in \mathbb{N}$ , we have

$$\begin{aligned} & \left| \left( \left( B_p^X - \frac{B_p^{\perp,a}}{2b} \right)^r - B_p^X \right)(x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \\ & \leq C^r p^{n+k/2} \cdot \exp\left(-\frac{c}{8}\sqrt{p} \cdot (\text{dist}(x_1, x_2) + \text{dist}(x_1, Y) + \text{dist}(x_2, Y))\right). \end{aligned} \quad (5.8)$$

Now, we fix  $x_1, x_2 \in X$ , and let

$$r := \left\lceil \frac{c}{16 \log(\max(C, 2))} \sqrt{p} \cdot (\text{dist}(x_1, x_2) + \text{dist}(x_1, Y) + \text{dist}(x_2, Y)) \right\rceil, \quad (5.9)$$

where  $\lceil \cdot \rceil$  is the ceil function. Then in the right-hand side of (5.8), for this choice of  $r$ , the power of  $C$  becomes negligible with respect to the last multiplicand. Then (1.25) holds by (5.7), (5.8) and the inequality  $\epsilon < 1$ .  $\square$

Now, in the second part of this section, we show that despite the global nature of the orthogonal Bergman kernel, the asymptotic expansion of it depends only on the local geometry of the problem.

More precisely, we fix  $X, Y, (L, h^L), (F, h^F), dv_X, dv_Y$  as in Section 1. We denote by  $X', Y', (L', h^{L'}), (F', h^{F'}), dv_{X'}, dv_{Y'}$  some other manifold, submanifold, etc. We define  $g^{TX}, g^{TX'}$  as in (1.2). We fix  $y_0 \in Y, y'_0 \in Y'$  and assume that there is a holomorphic diffeomorphism between  $U := B_{y_0}^X(r_0)$  and  $U' := B_{y'_0}^{X'}(r_0)$  for  $r_0 < r_X, r_{Y'}$ , such that it induces a diffeomorphism of  $Y \cap U$  to  $Y' \cap U'$  and sends  $dv_X, dv_Y$  to  $dv_{X'}, dv_{Y'}$  respectively. We also assume that the diffeomorphism can be extended to holomorphic isometries between  $(L, h^L), (F, h^F)$  and  $(L', h^{L'}), (F', h^{F'})$ . In particular, it is a local isometry between  $(X, g^{TX}, y_0)$  and  $(X', g^{TX'}, y'_0)$ .

We denote by  $B_p^{X'}, B_p^{\perp}$  the Bergman projector and the orthogonal Bergman projector associated with  $X', (L', h^{L'}), (F', h^{F'}), dv_{X'}$ . For  $x_1, x_2 \in X$ , we denote by  $B_p^{X'}(x_1, x_2), B_p^{\perp}(x_1, x_2)$  the Schwartz kernels of those operators with respect to  $dv_{X'}$ .

**Theorem 5.2.** *There is  $p_1 \in \mathbb{N}^*$ , such that for any  $r_0 > 0$  as above, there is  $c > 0$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for  $V := B_{y_0}^X(r_0/2)$  and any  $p \geq p_1$ , we have*

$$\|B_p^{\perp} - B_p^{\perp}\|_{\mathcal{C}^k(V \times V)} \leq C \exp(-c\sqrt{p}). \quad (5.10)$$

The proof of Theorem 5.2 is based on the following proposition.

**Proposition 5.3.** *There is  $p_1 \in \mathbb{N}^*$ , such that for any  $r_0 > 0$  as above, there is  $c > 0$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for  $V := B_{y_0}^X(r_0/2)$  and any  $p \geq p_1$ , we have*

$$\|B_p^X - B_p^{X'}\|_{\mathcal{C}^k(V \times V)} \leq C \exp(-c\sqrt{p}). \quad (5.11)$$

*Proof.* As our proof here is an easy modification of the proof of [30, Theorem 1], we only briefly explain the main necessary modifications. First, we denote by  $\square_p$  (resp.  $\square'_p$ ) the Kodaira Laplacians on  $\mathcal{C}^\infty(X, L^p \otimes F)$  (resp.  $\mathcal{C}^\infty(X', L^p \otimes F')$ ). We would like to prove that there are constants  $a, c_1, c_2 > 0$  such that for any  $k \in \mathbb{N}, u_0 > 0$ , there is  $C > 0$ , such that for any  $u \geq u_0, p \in \mathbb{N}, x_1, x_2 \in V := B_{y_0}^X(r_0/2)$ , the following estimates hold

$$\left| \left( \exp\left(-\frac{u}{p}\square_p\right) - \exp\left(-\frac{u}{p}\square'_p\right) \right)(x_1, x_2) \right|_{\mathcal{C}^k(V \times V)} \leq C p^{n+k/2} \cdot \exp\left(c_1 u - \frac{ap}{u} \cdot (\text{dist}(x_1, x_2) + 1)\right). \quad (5.12)$$

$$\left| \left( \square_p \exp\left(-\frac{u}{p}\square_p\right) - \square'_p \exp\left(-\frac{u}{p}\square'_p\right) \right)(x_1, x_2) \right|_{\mathcal{C}^k(V \times V)} \leq C p^{n+1+k/2} \cdot \exp\left(-c_2 u - \frac{ap}{u} \cdot (\text{dist}(x_1, x_2) + 1)\right). \quad (5.13)$$

The proof is a direct modification of the proof of [30, (3.1), (3.2)], which states that

$$\left| \exp\left(-\frac{u}{p}\square_p\right)(x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \leq C p^{n+k/2} \cdot \exp\left(c_1 u - \frac{ap}{u} \cdot \text{dist}(x_1, x_2)\right), \quad (5.14)$$

$$\left| \square_p \exp \left( -\frac{u}{p} \square_p \right) (x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \leq Cp^{n+1+k/2} \cdot \exp \left( -c_2 u - \frac{ap}{u} \cdot \text{dist}(x_1, x_2) \right). \quad (5.15)$$

More precisely, first of all, (5.12), (5.13) are consequences of (5.14) and (5.15) for  $\text{dist}(x_1, x_2) > \frac{r_0}{2}$ . Now, in the case  $\text{dist}(x_1, x_2) < \frac{r_0}{2}$ , the proof of (5.12), (5.13) is the same as the proof of (5.14), (5.15) in [28, proof of Theorem 4] with only one change. In the notations of [28, proof of Theorem 4], for  $h$  instead of  $\sqrt{p} \text{dist}(x_1, x_2) / \epsilon$ , one should take  $\sqrt{p} \frac{r_0}{2\epsilon}$ . Then, again in the notations of [28, proof of Theorem 4], due to finite propagation speed of solutions of hyperbolic equations, cf. [30, (3.9)], and the fact that the Laplacians  $\square_p$  and  $\square'_p$  coincide in  $U$ , the difference  $(H_{u,h}(\frac{1}{\sqrt{p}} \square_p) - H_{u,h}(\frac{1}{\sqrt{p}} \square'_p))(x_1, \cdot)$  vanishes.

Once (5.12) and (5.13) are established, the proof proceeds in exactly the same way with one final modification: the estimates of the quantities associated to  $\square_p$  should be replaced by the estimates of the difference of the same quantities associated to  $\square_p$  and  $\square'_p$ . Now, the estimate (5.11) follows from (5.12), (5.13) and the following identity, cf. [30, (3.14)],

$$\begin{aligned} B_p^X - B_p^{X'} &= \left( \exp \left( -\frac{u}{p} \square_p \right) - \exp \left( -\frac{u}{p} \square'_p \right) \right) \\ &\quad + \int_u^{+\infty} \left( \frac{1}{p} \square'_p \exp \left( -\frac{u_1}{p} \square'_p \right) - \frac{1}{p} \square_p \exp \left( -\frac{u_1}{p} \square_p \right) \right) du_1, \end{aligned} \quad (5.16)$$

by using exactly the same estimates as in [30, proof of Theorem 1].  $\square$

*Proof of Theorem 5.2.* The proof is an easy modification of the proof of Theorem 1.8, so we only present the main idea. We use the notation from the proof of Theorem 1.8. We denote by  $B_p^{\perp, a}$  the operator, constructed as  $B_p^{\perp, a}$  in (5.2), but for  $X', Y', (L', h^{L'}), (F', h^{F'}), dv_{X'}, dv_{Y'}$ . We fix a smooth cut-off function  $\rho_0 : U \rightarrow [0, 1]$  (resp.  $\rho_1$ ), equal to 1 on  $V$  (resp. on  $B_{y_0}^X(\frac{3}{4}r_0)$ ), and 0 on  $\partial B_{y_0}^X(\frac{2}{3}r_0)$  (resp. on  $\partial U$ ). Since  $U$  and  $U'$  are identified by a fixed diffeomorphism, we may regard  $\rho_0, \rho_1$  as functions, defined on  $X$  or  $X'$  by extending them by zero. For  $r \in \mathbb{N}$ , we consider the difference

$$D_p := \rho_0 \left( \left( B_p^X - \frac{B_p^{\perp, a}}{2b} \right)^r - B_p^X \right) \rho_0 - \rho_0 \left( \left( B_p^{X'} - \frac{B_p^{\perp, a}}{2b} \right)^r - B_p^{X'} \right) \rho_0. \quad (5.17)$$

Once the brackets in (5.17) are opened, one can replace each multiplicand  $A$  in the resulting expression by the sum  $\rho_1 A \rho_1 + (1 - \rho_1) A \rho_1 + (1 - \rho_1) A (1 - \rho_1) + (1 - \rho_1) A (1 - \rho_1)$ . We denote by  $R_p$  the sum of the terms, which contain at least one  $(1 - \rho_1)$ -term. Clearly, by Lemma 3.1, there is  $p_1 \in \mathbb{N}^*$ , such that for  $p \geq p_1$ ,  $W := \partial B_{y_0}^X(\frac{5}{7}r_0)$ , for any  $r_0 > 0$ , there are  $c, C > 0$ , such that

$$\|R_p\|_{\mathcal{C}^k(V \times V)} \leq C^r p^{n+k/2} \cdot \exp(-c\sqrt{p}). \quad (5.18)$$

As  $\text{supp } \rho_1 \subset U$ , using the diffeomorphism between  $U$  and  $U'$ , we may interpret all the operators  $\rho_1 B_p^{\perp, a} \rho_1, \rho_1 (B_p^{\perp, a}) \rho_1, \rho_1 B_p^X \rho_1, \rho_1 B_p^{X'} \rho_1$  as operators, acting over the same space,  $X$ . Let us now study the terms, where only  $\rho_1$  appear. For those terms, it is easy to see that one can rearrange the summands so that the terms in (5.17) with  $\rho_1$  can be expressed as a sum of  $(2^r - 1)2^r$  elements, each of which would contain as a multiplicand either  $\rho_1 (B_p^{\perp, a} - B_p^{\perp, a}) \rho_1$ , or  $\rho_1 (B_p^X - B_p^{X'}) \rho_1$  and one of  $\rho_1 (B_p^{\perp, a}) \rho_1$  or  $\rho_1 (B_p^{\perp, a}) \rho_1$ . Then by Theorem 4.6, Proposition 5.3, (5.3) and Lemma 3.1 applied for  $W := \partial U$ , we see that there are  $c, C > 0$ , such that for any  $p \geq p_1$ , we get the bound

$$\|D_p - R_p\|_{\mathcal{C}^k(V \times V)} \leq C^r p^{n+k/2} \cdot \exp(-c\sqrt{p}). \quad (5.19)$$

By summing up (5.18) and (5.19), we finally deduce that for any  $p \geq p_1$ , we have

$$\|D_p\|_{\mathcal{C}^k(V \times V)} \leq C^r p^{n+k/2} \cdot \exp(-c\sqrt{p}). \quad (5.20)$$

Now, by taking a sum of (5.7) and the analogous estimate for  $X', Y', (L', h^{L'}), (F', h^{F'}), dv_{X'}, dv_{Y'}$ , we get that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1, r \in \mathbb{N}$ , we have

$$\|D_p + B_p^\perp - B_p'^\perp\|_{\mathcal{C}^k(V \times V)} \leq Cp^{n+k/2} \epsilon^r. \quad (5.21)$$

We now adjust  $r$  as follows

$$r := \left\lceil \frac{c}{4 \log(\max(C, 2))} \sqrt{p} \right\rceil. \quad (5.22)$$

Then the contribution of  $C^r$  in (5.20) gets eliminated by the last multiplicand in the right-hand side of (5.20). The proof is now finished by (5.20) and (5.21).  $\square$

## 5.2 Bergman kernel asymptotics in Fermi coordinates

The main goal of this section is to prove that after a reparametrization given by a homothety with factor  $\sqrt{p}$  in Fermi coordinates, the Bergman kernel admits a complete asymptotic expansion in powers of  $\sqrt{p}$ , as  $p \rightarrow \infty$ . The proof relies on the analogous result of Dai-Liu-Ma [9], stated in geodesic coordinates, and the calculations from Sections 2.2, 2.3.

We use notations from Section 1 and assume that  $(X, g^{TX})$  is of bounded geometry. Recall that  $A \in \mathcal{C}^\infty(Y, T^*Y \otimes \text{End}(TX|_Y)), R > 0, B \in \mathcal{C}^\infty(Y, \text{Sym}^2(T^*X|_Y) \otimes TX|_Y)$ , and  $\nu \in \mathcal{C}^\infty(Y, N)$  were defined in (1.19), (2.17), (2.27) and (4.12) respectively.

We fix a point  $y_0 \in Y$  and an orthonormal frame  $(e_1, \dots, e_{2m})$  (resp.  $(e_{2m+1}, \dots, e_{2n})$ ) in  $(T_{y_0}Y, g^{TY})$  (resp. in  $(N_{y_0}, g_{y_0}^N)$ ) as in (1.16). Recall that the Fermi coordinates  $\psi_{y_0}$  were defined in (1.17). Recall that the function  $\kappa_X$  in a neighborhood of  $y_0$ , was defined in (1.18). We fix an orthonormal frame  $(f_1, \dots, f_r)$  of  $(F_{y_0}, h_{y_0}^F)$  and define the orthonormal frame  $(\tilde{f}_1, \dots, \tilde{f}_r)$  of  $(F, h^F)$  in a neighborhood of  $y_0$ , as in Section 2.1. Similarly, we trivialize  $(L, h^L)$ . The choice of those frames allows us to interpret  $B_p^X(x_1, x_2)$  as an element of  $\text{End}(F_{y_0})$  for  $x_1, x_2 \in X$  in a neighborhood of  $y_0$ . Recall that  $\mathcal{P}_n$  was defined in (3.27).

**Proposition 5.4.** For any  $r \in \mathbb{N}$ , there are  $J_r(Z, Z') \in \text{End}(F_{y_0})$  polynomials in  $Z, Z' \in \mathbb{R}^{2n}$ , satisfying the same assumptions as polynomials from Theorem 1.9, such that for  $F_r := J_r \cdot \mathcal{P}_n$ , the following holds. There are  $\epsilon, c > 0$ , such that for any  $k, l, l' \in \mathbb{N}$ , there exists  $C > 0$ , such that for any  $y_0 \in Y, p \in \mathbb{N}^*, Z, Z' \in \mathbb{R}^{2n}, |Z|, |Z'| \leq \epsilon, \alpha, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l, Q_{k,l,l'}^3 := 3(n+k+l'+2) + l$ , the following bound holds

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \frac{1}{p^n} B_p^X(\psi_{y_0}(Z), \psi_{y_0}(Z')) - \sum_{r=0}^k p^{-\frac{r}{2}} F_r(\sqrt{p}Z, \sqrt{p}Z') \kappa_X^{-\frac{1}{2}}(Z) \kappa_X^{-\frac{1}{2}}(Z') \right) \right|_{\mathcal{C}^{l'}(Y)} \\ & \leq Cp^{-(k+1-l)/2} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^{Q_{k,l,l'}^3} \exp\left(-c\sqrt{p}|Z - Z'|\right), \end{aligned} \quad (5.23)$$

where the  $\mathcal{C}^{l'}$ -norm is taken with respect to  $y_0$ . Also, the following identity holds

$$J_0(Z, Z') = \text{Id}_{F_{y_0}}. \quad (5.24)$$

Moreover, under the assumptions (1.8), we have

$$J_1(Z, Z') = \text{Id}_{F_{y_0}} \cdot \pi \left( g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)) + g(\bar{z}'_N, A(z_Y - z'_Y)(z_Y - z'_Y)) \right). \quad (5.25)$$

Before describing the proof of Proposition 5.4, let us recall the relevant asymptotic expansion of Dai-Liu-Ma. Recall first that for  $x_0 \in X$ , the coordinates  $\phi_{x_0}$  in a neighborhood of  $x_0$  were defined in (2.2). Let us define the function  $\kappa'_X$  in a neighborhood of  $x_0$  by the following formula

$$(\phi_{x_0}^* dv_X)(Z) = \kappa'_X(Z) dZ_1 \wedge \cdots \wedge dZ_{2n}. \quad (5.26)$$

We use the orthonormal frame  $(\tilde{f}'_1, \dots, \tilde{f}'_r)$  of  $(F, h^F)$ , defined in (2.25) in a neighborhood of  $y_0$ . Similarly, we trivialize  $(L, h^L)$ . The choice of those frames allows us to interpret the Schwartz kernel of  $B_p^X$  as an element of  $\text{End}(F_{y_0})$  for  $x_1, x_2 \in X$  in a neighborhood of  $y_0$ . We denote this element here by  $B_p^{X'}(x_1, x_2)$  to distinguish with  $B_p^X(x_1, x_2)$  previously defined.

**Theorem 5.5** (Dai-Liu-Ma [9], cf. [28, Theorem 4.2.1]). *For any  $r \in \mathbb{N}$ ,  $x_0 \in X$ , there are  $J'_r(Z, Z') \in \text{End}(F_{x_0})$  polynomials in  $Z, Z' \in \mathbb{R}^{2n}$ , satisfying the same assumptions as polynomials from Theorem 1.9, such that for  $F'_r := J'_r \cdot \mathcal{P}_n$ , the following holds. There are  $\epsilon, c > 0$ , such that for any  $k, l, l' \in \mathbb{N}$ , there exists  $C > 0$ , such that for any  $x_0 \in X$ ,  $p \in \mathbb{N}^*$ ,  $Z, Z' \in \mathbb{R}^{2n}$ ,  $|Z|, |Z'| \leq \epsilon$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , and  $Q_{k,l,l'}^4 := 2(n + k + l' + 2) + l$ , we have*

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \frac{1}{p^n} B_p^{X'}(\phi_{x_0}(Z), \phi_{x_0}(Z')) - \sum_{r=0}^k p^{-\frac{r}{2}} F'_r(\sqrt{p}Z, \sqrt{p}Z') \kappa'_X{}^{-\frac{1}{2}}(Z) \kappa'_X{}^{-\frac{1}{2}}(Z') \right) \right|_{\mathcal{C}^{l'}(X)} \\ \leq C p^{-(k+1-l)/2} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^{Q_{k,l,l'}^4} \exp \left( -c\sqrt{p}|Z - Z'| \right), \quad (5.27)$$

where the  $\mathcal{C}^{l'}$ -norm is taken with respect to  $y_0$ . Also, the following identity holds

$$J'_0(Z, Z') = \text{Id}_{F_{y_0}}. \quad (5.28)$$

Moreover, under the assumptions (1.8), we have

$$J'_1(Z, Z') = 0. \quad (5.29)$$

*Proof.* In [9] and [28], Theorem 5.5 did not appear in exactly the same form as it appears here. Let us, nevertheless, explain that our formulations are equivalent. First, in [9], (5.27) was stated for compact manifolds. For manifolds of bounded geometry, the result is from [28, Theorems 4.1.21, 6.1.1, Remark 4.1.26 and Problem 6.1]. Second, in [9] and [28], the result is stated for geodesic coordinates associated to a metric  $g_0^{TX}$  verifying  $dv_{g_0^{TX}} = dv_X$ . Here, we prefer to state the expansion (5.27) for geodesic coordinates associated to  $g^{TX}$  because the Bergman kernel depends only on  $dv_X$  and not on  $g_0^{TX}$ , which is not natural to the problem.

To obtain (5.27) from the result of Dai-Liu-Ma, it is enough to apply their result for the metric  $g_0^{TX} := g^{TX} \cdot (dv_X/dv_{g^{TX}})^{\frac{1}{2n}}$ , which has bounded geometry by Proposition 2.5, Corollary 2.6 and (1.4). The formula for the leading term of the expansion, (5.28), is the same as in Dai-Liu-Ma by Proposition 2.12 and due to the fact that the derivative of the transition map between the geodesic coordinate systems associated to  $g_0^{TX}$  and  $g^{TX}$ , evaluated at the point  $0 \in \mathbb{R}^{2n}$ , is equal to identity. Finally, the formula (5.29) here coincides with the one from Dai-Liu-Ma because under the assumptions (1.8), our coordinates coincide.  $\square$

*Proof of Proposition 5.4.* Recall that the diffeomorphism  $h$  was defined in (2.20), and the functions  $\xi_L, \xi_F$  were defined in (2.25). Directly from the definitions, we obtain the following relation between the Schwartz kernels

$$B_p^X(\psi(Z), \psi(Z')) = \exp(-p\xi_L^* - \xi_F^*)(\psi(Z')) \cdot B_p^{X'}(\phi(h(Z)), \phi(h(Z'))) \cdot \exp(-p\xi_L - \xi_F)(\psi(Z)). \quad (5.30)$$

From (5.30), we see that to establish Proposition 5.4, it is necessary to study the Taylor expansions of each term in (5.27) for  $Z := h(Z), Z' := h(Z')$ .

From Theorem 5.5, (5.30), the fact that from Proposition 2.18,  $h(Z) = Z + O(|Z|^2)$ , the fact that from Proposition 2.22,  $\xi_F(\psi(Z)) = O(|Z|)$  and  $\xi_L(\psi(Z)) = O(|Z|^3)$ , the fact that the above  $O$ -terms are uniform by the coordinate description of bounded geometry, and the fact that the coefficients of the higher order Taylor expansions can be expressed in terms of  $R^{TX}, A, R^F, \omega$ , as described in Propositions 2.18, 2.22, we deduce (5.23) and (5.24).

To establish (5.25), let us place ourselves in coordinates as in (3.27). By (1.32), we see that

$$2 \sum_{i=1}^n z_i \bar{z}'_i = g_{y_0}^{TM} \left( (\text{Id} - \sqrt{-1}J)Z, (\text{Id} + \sqrt{-1}J)Z' \right). \quad (5.31)$$

From this, Proposition 2.18 and (1.32), we get

$$\begin{aligned} |h(Z)|^2 &= |Z|^2 + 2g_{y_0}^{TM}(B(Z), Z) + O(|Z|^4), \\ 2 \sum_{i=1}^n (z_i \bar{z}'_i)(h(Z), h(Z')) &= 2 \sum_{i=1}^n z_i \bar{z}'_i + 4g_{y_0}^{TM}(B(Z), \bar{z}') + 4g_{y_0}^{TM}(z, B(Z')) + O(|Z|^4). \end{aligned} \quad (5.32)$$

From (3.27) and (5.32), we conclude that

$$\begin{aligned} \mathcal{P}_n(\sqrt{p}h(Z), \sqrt{p}h(Z')) &= \mathcal{P}_n(\sqrt{p}Z, \sqrt{p}Z') \cdot \\ &\cdot \left( 1 - \frac{\pi}{\sqrt{p}} \left( g_{y_0}^{TM}(B(\sqrt{p}Z), \sqrt{p}Z) + g_{y_0}^{TM}(B(\sqrt{p}Z'), \sqrt{p}Z') \right. \right. \\ &\quad \left. \left. - 2g_{y_0}^{TM}(B(\sqrt{p}Z), \sqrt{p}\bar{z}') - 2g_{y_0}^{TM}(\sqrt{p}z, B(\sqrt{p}Z')) \right) + O\left(\frac{1}{p}|\sqrt{p}Z|^4\right) \right). \end{aligned} \quad (5.33)$$

Now, from Proposition 2.22 and (1.2), we get

$$\exp(-p\xi_L - \xi_F)(\psi(Z)) = 1 - \frac{\sqrt{-1}\pi}{3\sqrt{p}} g_{y_0}^{TM}(J\sqrt{p}Z, B(\sqrt{p}Z)) + O\left(|Z|^2 + \frac{1}{p}|\sqrt{p}Z|^4\right). \quad (5.34)$$

By Lemma 2.25, applied for  $M := X, H := Y$  and  $T := \text{Id}$ , we get

$$\kappa_X^{1/2}(Z) = 1 - mg_{y_0}^{TM}(\nu, Z) + O(|Z|^2), \quad (5.35)$$

where  $\nu \in \mathcal{C}^\infty(Y, N)$  was defined (4.12). We now apply (5.35) for  $X := X$  and  $Y := \{y_0\}$ , to get

$$\kappa_X'^{1/2}(Z) = 1 + O(|Z|^2). \quad (5.36)$$



From (4.14), (5.27), (5.30), (5.33), (5.34), (5.35), (5.36) and an easy calculation, we deduce

$$\begin{aligned} J_1(Z, Z') &= -\pi \text{Id}_{F_{y_0}} \cdot \left( g_{y_0}^{TM}(B(Z), Z) + g_{y_0}^{TM}(B(Z'), Z') \right. \\ &\quad \left. - 2 \left( g_{y_0}^{TM}(B(Z), \bar{z}') + g_{y_0}^{TM}(z, B(Z')) \right) \right. \\ &\quad \left. + \frac{\sqrt{-1}}{3} \left( g_{y_0}^{TM}(JZ, B(Z)) - g_{y_0}^{TM}(JZ', B(Z')) \right) \right). \end{aligned} \quad (5.37)$$

By using the fact that  $A$  commutes with  $J$  and takes its values in skew-adjoint matrices, we get

$$\begin{aligned} g_{y_0}^{TM}(Z, B(Z)) &= -\frac{1}{2} \left( g_{y_0}^{TM}(z_N, A(Z_Y)\bar{z}_Y) + g_{y_0}^{TM}(\bar{z}_N, A(Z_Y)z_Y) \right), \\ g_{y_0}^{TM}(JZ, B(Z)) &= \frac{3\sqrt{-1}}{2} \left( g_{y_0}^{TM}(z_N, A(Z_Y)\bar{z}_Y) - g_{y_0}^{TM}(\bar{z}_N, A(Z_Y)z_Y) \right). \end{aligned} \quad (5.38)$$

Now, we use (2.53) and the fact that  $A$  commutes with  $J$  to deduce

$$A(Z_Y)z_Y = A(z_Y)z_Y, \quad A(Z_Y)\bar{z}_Y = A(\bar{z}_Y)\bar{z}_Y. \quad (5.39)$$

From (2.53), (5.37), (5.38) and (5.39), we deduce (5.25).  $\square$

### 5.3 Orthogonal Bergman kernel asymptotics, a proof of Theorem 1.9

The main goal of this section is to prove Theorem 1.9. For this, we first use the localization property of the orthogonal Bergman kernel, established in Theorem 5.2, to reduce our problem to a special case  $X' := \mathbb{C}^n$ ,  $Y' := \mathbb{C}^m$  and trivial  $L', F'$  (but endowed with non trivial metrics). Then on the pair  $X', Y'$  we make a homothety and show that the renormalization of the relevant operators converges to the perturbations of the model operators from Section 3.2.

Let us first describe precisely the construction of  $X', Y', L', F'$  and the metrics on them. As we rely later on the complex structure of  $X', Y', L', F'$  in an essential way, the construction through geodesic coordinates, as in [28], wouldn't suffice for our needs. We rely here in our non-compact setting on the results of Section 2.5, and we propose an approach, which in the compact case essentially coincides with the approach of Dinh-Ma-Nguyen [15, after (2.23)].

We fix  $y_0 \in Y$ . Let  $r_c, r_c^1 > 0$  and  $\chi : B_{y_0}^X(r_c) \rightarrow \mathbb{C}^n$ ,  $U := B_{y_0}^X(r_c)$ ,  $V := \text{Im } \chi$ , as in Theorem 2.40 and Remark 2.41. We put  $X' := \mathbb{C}^n$ ,  $Y' := \mathbb{C}^m$ .

Let holomorphic frame  $\sigma$  of  $L$  over  $B_{y_0}^X(r_0)$ ,  $r_0 > 0$ , be constructed as in Lemma 2.38. Assume, for simplicity,  $4r_c < r_0$  (if this is not the case, put  $r_c := \frac{r_0}{8}$ ). Define the function  $\theta(Z)$  over  $V$  by

$$\exp(-2\theta(Z)) := h^L(\sigma, \sigma). \quad (5.40)$$

From Lemma 2.32 and (2.111), we see that there is  $C > 0$ , which depends only on  $r_c$  and the constant  $C_{n+6}$  from (2.1) and (2.22), such that

$$\|\theta\|_{\mathcal{C}^2(V)} \leq C. \quad (5.41)$$

Denote by  $\theta_0^{[1]}$  and  $\theta_0^{[2]}$  the first and the second order Taylor expansions of  $\theta_0 := \theta \circ \chi^{-1}$  at 0. For  $r_c^1 > \epsilon > 0$ , we now define  $\theta_\epsilon : \mathbb{C}^n \rightarrow \mathbb{R}$  as follows

$$\theta_\epsilon(Z) := \rho\left(\frac{|Z|}{\epsilon}\right)\theta_0(Z) + \left(1 - \rho\left(\frac{|Z|}{\epsilon}\right)\right) \cdot \left(\theta_0(0) + \theta_0^{[1]}(Z) + \theta_0^{[2]}(Z)\right), \quad (5.42)$$

where  $\rho$  is a bump function as in (1.10). Let  $h_\epsilon^{L'}$  be the metric on  $L' := X' \times \mathbb{C}$  defined by

$$h_\epsilon^{L'}(1, 1)(Z) := \exp(-2\theta_\epsilon(Z)). \quad (5.43)$$

Let  $R_\epsilon^{L'}$  be the curvature of the Chern connection  $\nabla_\epsilon^{L'}$  on  $(L', h_\epsilon^{L'})$ . By (1.2), (5.41) and Remark 2.37, it is easy to see that there is a constant  $\epsilon_0 > 0$ , such that for any  $\epsilon < \epsilon_0$ , we have

$$\inf \left\{ \sqrt{-1} R_{\epsilon, Z}^{L'}(u, Ju) / |u|^2 : u \in T_Z X', Z \in X' \right\} \geq \pi, \quad (5.44)$$

where  $|u|$  is the norm of  $u \in T_Z X'$ , calculated by a (trivial) identification of  $T_Z X'$  with  $T_0 X'$ . Moreover,  $\epsilon_0$  depends only on  $r_c$  and the constant  $C_{n+6}$  from (2.1) and (2.22). From now on, we fix such  $\epsilon_0$  and remove it from all subsequent subscripts. We assume, for simplicity, that  $4r_c^1 < \epsilon_0$ . From (5.44), we see, in particular, that  $(L', h^{L'})$  is positive. We denote by  $g^{TX'}$  the metric on  $X'$ , defined through  $(L', h^{L'})$  as in (1.2). Of course, by (5.42),  $\chi$  is then a local holomorphic isometry, defined over  $B_{y_0}^X(\frac{r_c}{2})$ , between  $(X, g^{TX})$  and  $(X', g^{TX'})$ .

Now, let  $r_0 > 0$  and a holomorphic frame  $(f_1, \dots, f_r)$  of  $F$  over  $B_{y_0}^X(r_0)$  be as in Corollary 2.39. Assume, for simplicity, that  $4r_c^1 < r_0$ . Define the function  $h_{ij}^F(x) \in \mathbb{C}$ ,  $x \in U$ , as follows  $h_{ij}^F := h^F(f_i, f_j)$ . Consider the functions  $h_{ij}^{F,0} : \mathbb{C}^n \rightarrow \mathbb{C}$ , defined by

$$h_{ij}^{F,0}(Z) := \rho\left(\frac{|Z|}{r_c^1}\right) h_{ij}^F(\chi^{-1}(Z)) + \delta_{ij} \left(1 - \rho\left(\frac{|Z|}{r_c^1}\right)\right). \quad (5.45)$$

Clearly,  $h_{ij}^{F,0}$  takes values in Hermitian matrices. Define the metric  $h^{F'}$  on  $F' := X' \times \mathbb{C}^r$  by

$$h^{F'}(1_i, 1_j)(Z) := h_{ij}^{F,0}(Z), \quad (5.46)$$

where  $1_l$ ,  $l \in 1, \dots, r$  is the constant vector in  $F'$ , given by  $(0, \dots, 0, 1, 0, \dots, 0)$ , where 1 is put in the  $l$ -th place. The pair  $(F', h^{F'})$  is a Hermitian vector bundle on  $X'$ .

Clearly, the triple  $(X', Y', g^{TX'})$  and the Hermitian vector bundles  $(L', h^{L'})$ ,  $(F', h^{F'})$  are of bounded geometry. Moreover, the corresponding constants  $C_k$  from (2.1), (2.14), (2.22), associated to  $(X', Y', g^{TX'})$ ,  $(L', h^{L'})$ ,  $(F', h^{F'})$ , can be bounded in terms of the corresponding constants  $C_{k+n+6}$  from (2.1), (2.14), (2.22), associated to  $(X, Y, g^{TX})$ ,  $(L, h^L)$ ,  $(F, h^F)$ .

We denote by  $g_0^{TX'}$ ,  $h_0^{F'}$  the restrictions of  $g^{TX'}$ ,  $h^{F'}$  to  $0 \in X'$ . We interpret  $g_0^{TX'}$  as metric on  $X'$  by using the standard trivialization of  $TX'$  coming from the linear structure. Similarly, we interpret  $h_0^{F'}$  as Hermitian metric on  $F'$  over  $X'$  by using the trivialization of  $F'$ .

As the expansion from Theorem 1.9 is stated in Fermi coordinates, we need an analogue of those coordinates on  $X'$ . For technical reasons, which will become clear after (5.54), we need a global diffeomorphism of  $X'$ , which coincides with Fermi coordinates in a small neighborhood of 0. For  $0 < \epsilon < \min\{r_c, R\}$ , where  $R > 0$  is as in (2.17), we define  $\psi_\epsilon(Z)$  for  $Z \in \mathbb{C}^n$  as follows

$$\psi_\epsilon(Z) = \psi(Z) \rho\left(\frac{|Z|}{\epsilon}\right) + Z \left(1 - \rho\left(\frac{|Z|}{\epsilon}\right)\right), \quad (5.47)$$

where  $\psi(Z) \in X'$  the Fermi coordinate, defined as in (1.17), but for the pair  $(X', g^{TX'})$ . From Propositions 2.9, 2.12, we see that there is  $\epsilon_2 > 0$  such that for  $\epsilon < \epsilon_2$ , the derivative of  $\psi_\epsilon(Z)$  is invertible for all  $Z \in \mathbb{C}^n$ , and  $|D\psi_\epsilon - \text{Id}| < \frac{1}{2}$ . Moreover, the constant  $\epsilon_2$  depends only on  $r_X, r_Y, r_\perp$  and  $C_0$  from (2.1), (2.5) and (2.14). For simplicity, we assume that  $4r_c^1 < \epsilon_2$ . We fix such

$\epsilon_2$  and denote  $\psi_{\epsilon_2}$  by  $\psi_0$  from now on. As  $\psi_0(Z)$  obviously satisfies  $\psi_0(Z) \rightarrow \infty$ , as  $|Z| \rightarrow \infty$ , by the invertibility of  $D\psi_0$  and Hadamard's global inverse function theorem, cf. [26, Theorem 6.2.4],  $\psi_0$  is a diffeomorphism. Clearly, as Fermi coordinates preserve  $Y'$ , by (5.47),  $\psi_0|_{\mathbb{C}^m}$  is a diffeomorphism on  $\mathbb{C}^m$ .

We define the volume form  $dv_{X'}$  on  $X'$  as follows

$$dv_{X'} := \rho\left(\frac{|Z|}{r_c}\right)(\chi^{-1})^* dv_X + \left(1 - \rho\left(\frac{|Z|}{r_c}\right)\right) dZ_1 \wedge \cdots \wedge dZ_{2n}. \quad (5.48)$$

Similarly to (1.18), we define the function  $\kappa_{X'} : X' \rightarrow \mathbb{R}$  as follows

$$(\psi_0^* dv_{X'})(Z) = \kappa_{X'}(Z) dZ_1 \wedge \cdots \wedge dZ_{2n}. \quad (5.49)$$

Clearly, since  $\psi_0(Z) = Z$ , as  $|Z| \rightarrow \infty$ , we have  $\kappa_{X'}(Z) = 1$ , as  $|Z| \rightarrow \infty$ .

Now, let us fix  $e \in L'_0$  and  $f_1, \dots, f_r \in F'_0$ , and consider the orthonormal frames  $\tilde{e}$  and  $\tilde{f}_1, \dots, \tilde{f}_r$ , constructed as in Theorem 1.6, for  $L', F'$ , but instead of  $\psi$ , using  $\psi_0$ . As  $\psi_0$  is globally defined, those frames become also globally defined.

Let us now consider the Bergman projector  $B_p^{X'}$  (resp. the orthogonal Bergman projector  $B_p^{\perp'}$ ) associated to  $X', L', F'$ . This is a self-adjoint operator, acting on the  $L^2$ -space  $L^2(dv_{X'}, h^{L' \otimes p \otimes F'})$ . The above orthonormal frames allow us to see  $B_p^{X'}$  (resp.  $B_p^{\perp'}$ ) as a self-adjoint operator, acting on the  $L^2$ -space  $L^2(dv_{X'}, h_0^{F'})$ . We denote by  $B_p^{X'}(x_1, x_2)$ ,  $B_p^{\perp'}(x_1, x_2)$  the Schwartz kernels of those operators with respect to  $dv_{X'}$ . The following theorem essentially shows that it is enough to establish our main result of this section for  $X', L', F'$  instead of  $X, L, F$ .

**Lemma 5.6.** There are  $c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $y_0 \in Y$ ,  $p \geq p_1$ ,  $x_1, x_2 \in B_{y_0}^X(\frac{r_c}{4})$ , the following estimate holds

$$\left| B_p^{\perp'}(x_1, x_2) - B_p^{\perp'}(x_1, x_2) \right|_{\mathcal{C}^k(X \times X)} \leq C \exp(-c\sqrt{p}), \quad (5.50)$$

where we implicitly identified  $x_1, x_2$  to points in  $X'$  using  $\chi$ .

*Proof.* It follows from Theorem 5.2 and the fact that  $\chi$  is a holomorphic diffeomorphism, which extends to isometries between  $(L, h^L)$ ,  $(F, h^F)$  and  $(L', h^{L'})$ ,  $(F', h^{F'})$  over  $B_{y_0}^X(\frac{r_c}{2})$ .  $\square$

The advantage of passing from  $X, L, F$  to  $X', L', F'$  is twofold. First, since all the vector bundles are now trivialized, the operators  $B_p^{X'}$  and  $B_p^{\perp'}$  might be considered as operators, acting on the same space (independent of  $p$ ). Second, as  $X'$  is equal to  $\mathbb{C}^n$ , we can use the homothety on  $X'$  in our analysis. We will use both of those features in what follows.

We define  $t > 0$ ,  $S_t$  as in (1.30) and (4.42). Clearly, for any  $f, f' \in L^2(g_0^{TX'}, h_0^{F'})$ , we have

$$\langle S_t f, S_t f' \rangle_{L^2(g_0^{TX'})} = t^{2n} \langle f, f' \rangle_{L^2(g_0^{TX'})}. \quad (5.51)$$

We also consider a map  $U : L^2(g_0^{TX'}, h_0^{F'}) \rightarrow L^2(dv_{X'}, h_0^{F'})$ , defined as follows

$$(Uf)(Z) = \kappa_{X'}^{-1/2}(\psi_0^{-1}(Z)) \cdot f(\psi_0^{-1}(Z)). \quad (5.52)$$

An easy verification using (5.49) shows that  $U$  is well-defined and it is an isometry, i.e.

$$\langle Uf, Uf' \rangle_{L^2(dv_{X'})} = \langle f, f' \rangle_{L^2(g_0^{TX'})}. \quad (5.53)$$

We now consider another operators  $\mathcal{B}_t, \mathcal{B}_t^\perp$ , acting on  $L^2(g_0^{TX'}, h_0^{F'})$  as follows

$$\begin{aligned}\mathcal{B}_t &:= S_t^{-1} \circ U^{-1} \circ B_p^{X'} \circ U \circ S_t, \\ \mathcal{B}_t^\perp &:= S_t^{-1} \circ U^{-1} \circ B_p^{\perp'} \circ U \circ S_t.\end{aligned}\tag{5.54}$$

From (5.51) and (5.53), we see that  $\mathcal{B}_t, \mathcal{B}_t^\perp$  are self-adjoint.

**Lemma 5.7.** For any  $l \in \mathbb{N}$ , there exists  $C > 0$ , such that for any  $p \in \mathbb{N}^*$ ,  $f \in \text{Im}(\mathcal{B}_t)$ ,  $\alpha \in \mathbb{N}^{2n}$ ,  $|\alpha| \leq l$ , we have

$$\left| \frac{\partial^{|\alpha|}}{\partial Z^\alpha} f(Z) \right| \leq C \|f\|_{L^2(g_0^{TX'})}.\tag{5.55}$$

*Proof.* Clearly, by (5.54), we have

$$\text{Im } \mathcal{B}_t \subset (S_t^{-1} \circ U^{-1})(H_{(2)}^0(X', L^p \otimes F')).\tag{5.56}$$

We conclude by this, Proposition 5.1, (4.42) and (5.51).  $\square$

We denote by  $\mathcal{B}_t(Z, Z')$ ,  $\mathcal{B}_t^\perp(Z, Z')$  the Schwartz kernels of  $\mathcal{B}_t, \mathcal{B}_t^\perp$  with respect to the volume form  $dv_{g_0^{TX'}}$  on  $X'$ . An easy calculation shows that they are related to the Schwartz kernels  $B_p^{X'}(Z, Z')$ ,  $B_p^{\perp'}(Z, Z')$  of  $B_p^{X'}, B_p^{\perp'}$  with respect to  $dv_{X'}$  as follows

$$\begin{aligned}\mathcal{B}_t(Z, Z') &= t^{2n} B_p^{X'}(\psi_0(tZ), \psi_0(tZ')) \kappa_{X'}^{\frac{1}{2}}(tZ) \kappa_{X'}^{\frac{1}{2}}(tZ'), \\ \mathcal{B}_t^\perp(Z, Z') &= t^{2n} B_p^{\perp'}(\psi_0(tZ), \psi_0(tZ')) \kappa_{X'}^{\frac{1}{2}}(tZ) \kappa_{X'}^{\frac{1}{2}}(tZ'),\end{aligned}\tag{5.57}$$

We will now introduce the operator  $\mathcal{E}^0$ , sending the sections of  $F'$  over  $Y'$  to the sections of  $F'$  over  $X'$  by the following formula

$$(\mathcal{E}^0 f)(Z_Y, Z_N) = f(Z_Y) \exp\left(-\frac{\pi}{2}|Z_N|^2\right).\tag{5.58}$$

Denote by  $\mathcal{A}_t$  the operator, acting on  $L^2(g_0^{TX'}, h_0^{F'})$ , as follows

$$\mathcal{A}_t := \mathcal{E}^0 \circ \text{Res}_{Y'} \circ \mathcal{B}_t,\tag{5.59}$$

where  $\text{Res}_{Y'}$  is the restriction operator (which is well-defined in the considered composition as  $\mathcal{B}_t$  has smooth Schwartz kernel). We define the operator  $\mathcal{C}_t$  on  $L^2(g_0^{TX'}, h_0^{F'})$ , as follows

$$\mathcal{C}_t := \mathcal{A}_t^* \circ \mathcal{A}_t,\tag{5.60}$$

where  $\mathcal{A}_t^*$  is the adjoint of  $\mathcal{A}_t$ . The operator  $\mathcal{C}_t$  will be our main tool in the proof of Theorem 1.9. Remark its similarity with (5.2). Let us study some of its properties.

**Lemma 5.8.** The operator  $\mathcal{C}_t$  is self-adjoint, and there are constants  $a, b > 0$ , such that

$$\text{Spec}(\mathcal{C}_t) \subset \{0\} \cup [a, b],\tag{5.61}$$

for any  $t > 0$ . Moreover, we have the following

$$(\ker \mathcal{C}_t)^\perp = (S_t^{-1} \circ U)(H_{(2)}^{0,\perp}(X', L^p \otimes F')).\tag{5.62}$$

*Proof.* The proof is identical to (5.4) and (5.5).  $\square$

**Lemma 5.9.** The Schwartz kernel  $\mathcal{C}_t(Z, Z')$  of  $\mathcal{C}_t$  with respect to  $dv_{\mathbb{C}^n}$  satisfies the following bound. There are  $c > 0, p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1, Z, Z' \in \mathbb{R}^{2n}, \alpha, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{C}_t(Z, Z') \right|_{\mathcal{C}^{l'}(Y)} \leq C \exp\left(-c(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right), \quad (5.63)$$

where the  $\mathcal{C}^{l'}$ -norm is calculated with respect to  $y_0$ , which was fixed in the beginning of the section.

*Proof.* From Theorem 4.6, (5.57) and (5.59), we conclude that a bound like (5.63) holds for  $\mathcal{A}_t(Z, Z')$  instead of  $\mathcal{C}_t(Z, Z')$ . It is uniform in  $\mathcal{C}^{l'}$ -norm with respect to the choice of  $y_0 \in Y$  because the construction of  $X', Y', (L', h^{L'})$ , etc. depends smoothly on  $y_0$  and the Bergman kernel expansion depends also smoothly on  $y_0$ . We now conclude by Lemma 3.1 and (5.60).  $\square$

**Lemma 5.10.** For any  $r \in \mathbb{N}$ , there are  $\mathcal{J}'_r(Z, Z') \in \text{End}(F'_0)$  polynomials in  $Z, Z' \in \mathbb{R}^{2n}$  with the same properties as in Theorem 1.6, such that for  $\mathcal{F}'_r := \mathcal{J}'_r \cdot \mathcal{P}_{n,m}^\perp$ , the following holds.

There are  $\epsilon, c > 0, p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for  $p \geq p_1, Z, Z' \in T_{y_0}X, |Z|, |Z'| \leq \frac{\epsilon}{t}, \alpha, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{C}_t(Z, Z') - \sum_{r=0}^k t^r \mathcal{F}'_r(Z, Z') \right) \right|_{\mathcal{C}^{l'}(Y)} \leq C t^{k+1-m} \cdot \left(1 + |Z| + |Z'|\right)^{Q_{k,l,l'}^5} \exp\left(-c(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right), \quad (5.64)$$

where  $Q_{k,l,l'}^5 := 3(2n + k + l' + 4) + l$ . Also, the following identity holds

$$\mathcal{J}'_0(Z, Z') = \text{Id}_{F_{y_0}}. \quad (5.65)$$

Moreover, under the assumptions (1.8), we have

$$\mathcal{J}'_1(Z, Z') = \text{Id}_{F_{y_0}} \cdot \pi \left( g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)) + g(\bar{z}'_N, A(z_Y - z'_Y)(z_Y - z'_Y)) \right). \quad (5.66)$$

*Remark 5.11.* From (5.63), (5.64), (5.65) and the fact that  $\mathcal{P}_{n,m}^\perp$  is the orthogonal projector by the results of Section 3.2, we see that for  $a, b$  from Lemma 5.8, we have  $a \leq 1 \leq b$ .

*Proof.* Recall that polynomials  $J_r(Z, Z')$  and  $\epsilon, c > 0$  were defined in Proposition 5.4. From Proposition 5.4, (3.33), (5.57) and (5.59), we conclude that for

$$\mathcal{J}_{r,0}(Z, Z') := J_r(Z_Y, Z'_Y), \quad (5.67)$$

we can define  $\mathcal{F}_{r,0} := \mathcal{J}_{r,0} \cdot \mathcal{P}_{n,m}^\perp$ , so that for  $|Z|, |Z'| < \frac{\epsilon}{t}$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{A}_t(Z, Z') - \sum_{r=0}^k t^r \mathcal{F}_{r,0}(Z, Z') \right) \right|_{\mathcal{C}^{l'}(Y)} \leq C t^{k+1-m}.$$

$$\cdot \left(1 + |Z| + |Z'|\right)^{Q_{k,l,l'}}^3 \exp\left(-c(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right). \quad (5.68)$$

From Lemma 3.6, (5.60) and (5.68), we conclude that there are polynomials  $\mathcal{J}'_r$  for which (5.64) holds for  $c := \frac{c}{8}$  and  $\epsilon := \frac{\epsilon}{4}$ . Moreover, in the notations of Lemma 3.3, we have

$$\mathcal{J}'_r(Z, Z') = \sum_{r'=0}^r \mathcal{K}_{n,m}[\overline{\mathcal{J}_{r_0,0}(Z', Z)}, \mathcal{J}_{r-r_0,0}(Z, Z')]. \quad (5.69)$$

From (5.24), (5.67) and (5.69), we deduce (5.65). From (3.43) and (5.25), we deduce

$$\mathcal{K}_{n,m}[\overline{\mathcal{J}_{0,0}(Z', Z)}, \mathcal{J}_{1,0}(Z, Z')] = \text{Id}_{F_{y_0}} \cdot \pi g(\bar{z}'_N, A(z_Y - z'_Y)(z_Y - z'_Y)). \quad (5.70)$$

Now, remark that the summands from the second equation in (5.69) are the adjoints of each other. From this and (5.70), we deduce (5.66).  $\square$

*Proof of Theorem 1.9.* Let us show that Theorem 1.9 follows from Lemmas 5.6, 5.7, 5.8, 5.9, 5.10. More precisely, from Lemma 5.6 and (5.57), we see that it is enough to establish that there are polynomials  $J_r^\perp$ ,  $r \in \mathbb{N}$  as in Theorem 1.9, and  $F_r^\perp$  as in Theorem 1.9 and  $\epsilon, c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there exists  $C > 0$ , such that for any  $p \geq p_1$ ,  $Z, Z' \in \mathbb{R}^{2n}$ ,  $|Z|, |Z'| \leq \frac{\epsilon}{l}$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , the following bound holds

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{B}_t^\perp(Z, Z') - \sum_{r=0}^k t^r F_r^\perp(Z, Z') \right) \right|_{\mathcal{G}^{l'}(Y)} \\ & \leq Ct^{k+1} \left(1 + |Z| + |Z'|\right)^{Q_{k,l,l'}^2} \exp\left(-c(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right). \end{aligned} \quad (5.71)$$

We will establish now that the following bound holds

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{B}_t^\perp(Z, Z') - \sum_{r=0}^{2k+1} t^r F_r^\perp(Z, Z') \right) \right|_{\mathcal{G}^{l'}(Y)} \leq Ct^{2k+2} \left(1 + |Z| + |Z'|\right)^{Q_{2k+1,l,l'}''}, \quad (5.72)$$

where  $Q_{k,l,l'}'' := 3(4(n+2)(k+1) + l') + l$ . To see that it would suffice, remark that from Theorem 1.8, the bound on the degrees of  $J_r^\perp$ ,  $r \in \mathbb{N}$ , and the fact that the polynomials  $J_r^\perp$  depend smoothly on  $y_0$ , we see that there are  $c, C > 0$ ,  $p_1 \in \mathbb{N}$ , such that for  $p \geq p_1$ ,  $Z, Z' \in \mathbb{R}^{2n}$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , the following bound holds

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{B}_t^\perp(Z, Z') - \sum_{r=0}^{2k+1} t^r F_r^\perp(Z, Z') \right) \right|_{\mathcal{G}^{l'}(Y)} \\ & \leq C \left(1 + |Z| + |Z'|\right)^{6k+l+l'+6} \exp\left(-c(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right). \end{aligned} \quad (5.73)$$

From (5.72), (5.73) and Cauchy inequality, we deduce (5.71). Hence, it is left to establish (5.72), on which we concentrate from now on.

From (5.54), we see that  $\mathcal{B}_t^\perp, \mathcal{B}_t$  are the only self-adjoint operators on  $L^2(g_0^{TX'}, h_0^{F'})$  such that

$$\begin{aligned} \text{Spec} \mathcal{B}_t^\perp & \subset \{0, 1\}, \quad (\ker \mathcal{B}_t^\perp)^\perp = (S_t^{-1} \circ U)(H_{(2)}^{0,\perp}(X', L^p \otimes F')), \\ \text{Spec} \mathcal{B}_t & \subset \{0, 1\}, \quad (\ker \mathcal{B}_t)^\perp = (S_t^{-1} \circ U)(H_{(2)}^0(X', L^p \otimes F')), \end{aligned} \quad (5.74)$$

From this and Lemma 5.8, we see that for  $a, b$  as in Lemma 5.8, we have

$$\mathcal{B}_t^\perp = \int_{\Omega} \frac{1}{\lambda - \mathcal{C}_t} d\lambda, \quad (5.75)$$

where  $\Omega \subset \mathbb{C}$  is a circle not containing  $\{0\}$  inside of it, but containing  $\{a, b\}$ . By Remark 5.11, we see that  $\Omega$  contains 1 inside of it.

Let us first show the existence of polynomials  $J_r^\perp(Z, Z') \in \text{End}(F_{y_0})$  as in Theorem 1.9 and provide an algorithmic way of constructing them. We denote by  $\mathcal{C}_0$  the operator, acting on  $L^2(g_0^{T_{X'}}, h_0^{F'})$  by the convolution with smooth kernel  $\mathcal{P}_{n,m}^\perp(Z, Z') \cdot \text{Id}_{F'_0}$ . By the results of Section 3.2, the operator  $\mathcal{C}_0$  is an orthogonal projection, hence we have

$$\frac{1}{\lambda - \mathcal{C}_0} = \frac{1 - \mathcal{C}_0}{\lambda} + \frac{\mathcal{C}_0}{\lambda - 1}. \quad (5.76)$$

Now, let us apply the resolvent formula

$$\frac{1}{\lambda - \mathcal{C}_t} - \frac{1}{\lambda - \mathcal{C}_0} = \frac{1}{\lambda - \mathcal{C}_t} (\mathcal{C}_t - \mathcal{C}_0) \frac{1}{\lambda - \mathcal{C}_0}. \quad (5.77)$$

By using (5.76) and (5.77) inductively, we see that for any  $k \in \mathbb{N}$ , we can represent

$$\frac{1}{\lambda - \mathcal{C}_t} = \frac{1}{\lambda - \mathcal{C}_t} A_k + B_k, \quad (5.78)$$

where the operators  $A_k$  (resp.  $B_k$ ) are the linear combinations with coefficients given by some universal rational functions (in  $\lambda$ ) of the operators of the form

$$B(\mathcal{C}_t - \mathcal{C}_0)^{k_1} \mathcal{C}_0 \cdots \mathcal{C}_0 (\mathcal{C}_t - \mathcal{C}_0)^{k_l} B', \quad (5.79)$$

where  $l \in \mathbb{N}$ ,  $k_1, \dots, k_l \in \mathbb{N}^*$ ,  $k_1 + \dots + k_l = k$ , (resp.  $k_1 + \dots + k_l \leq k$ ) and  $B, B'$  are either the identity operators or  $\mathcal{C}_0$ . Now, from Lemma 5.8 and (5.74), we deduce that

$$\frac{1}{\lambda - \mathcal{C}_t} = \mathcal{B}_t \frac{1}{\lambda - \mathcal{C}_t} + \frac{1 - \mathcal{B}_t}{\lambda}. \quad (5.80)$$

We now rewrite (5.78) using (5.80) as follows

$$\frac{1}{\lambda - \mathcal{C}_t} = \mathcal{B}_t \frac{1}{\lambda - \mathcal{C}_t} A_k + \frac{1 - \mathcal{B}_t}{\lambda} A_k + B_k. \quad (5.81)$$

Now, by Lemmas 3.5, 5.10, each of the terms in (5.79) admit Taylor-type expansions as in (5.64) for  $Z, Z' \in \mathbb{C}^n$ ,  $|Z|, |Z'| < \frac{\epsilon}{2t}$ , where  $\epsilon$  is as in Lemma 5.10. From this and Lemma 3.6, we deduce that there are polynomials, having the same properties as described in Theorem 1.9, which we denote by  $J_r^\perp(Z, Z') \in \text{End}(F_{y_0})$ , such that for  $F_r^\perp$ , defined as in Theorem 1.9, there are  $\epsilon, c > 0, p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for  $p \geq p_1, Z, Z' \in T_{y_0} X$ ,  $|Z|, |Z'| \leq \frac{\epsilon}{2t}, \alpha, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l$ , the following bound holds

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \left( \int_{\Omega} B_k d\lambda \right) (Z, Z') - \sum_{r=0}^k t^r F_r^\perp(Z, Z') \right) \right|_{\mathcal{C}^{l'}(Y)}$$

$$\leq Ct^{k+1} \left(1 + |Z| + |Z'|\right)^{Q''_{k,l,l'}}. \quad (5.82)$$

Also, since by Lemma 5.10, the first term of the Taylor-type expansion of  $\mathcal{C}_t - \mathcal{C}_0$  vanishes, and each term in  $A_k$  has exactly  $k$  multiplicands  $\mathcal{C}_t - \mathcal{C}_0$ , by Lemma 3.5, we conclude that there is  $c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there are  $C > 0$ , such that for  $p \geq p_1$ ,  $Z, Z' \in T_{y_0} X$ ,  $|Z|, |Z'| \leq \frac{\epsilon}{2t}$ ,  $\alpha, \alpha' \in \mathbb{N}^{2n}$ ,  $|\alpha| + |\alpha'| \leq l$ , the following bound holds

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \left( \int_{\Omega} \frac{1 - \mathcal{B}_t}{\lambda} A_k d\lambda \right) (Z, Z') \right) \right|_{\mathcal{C}^{\nu}(Y)} \leq Ct^k \left(1 + |Z| + |Z'|\right)^{Q''_{k,l,l'}}. \quad (5.83)$$

It is, hence, only left to show that the first summand on the right-hand side of (5.81) is bounded by the term in the right-hand side of (5.72). For this, let us fix  $Z' \in \mathbb{C}^n$ ,  $|Z'| < \frac{\epsilon}{2t}$ . From Lemma 3.5, similarly to (5.83), we deduce that there is  $C > 0$ , such that we have

$$\left| \frac{\partial^{|\alpha'|}}{\partial Z'^{\alpha'}} A_k(Z, Z') \right| \leq Ct^k (1 + |Z| + |Z'|)^{Q''_{k,l,l'}} \exp \left( -\frac{c}{8} (|Z_Y - Z'_Y| + |Z_N| + |Z'_N|) \right). \quad (5.84)$$

From (5.84), we readily deduce that

$$\left\| \frac{\partial^{|\alpha'|}}{\partial Z'^{\alpha'}} A_k(\cdot, Z') \right\|_{L^2(g_0^{T X'})} \leq Ct^k (1 + |Z'|)^{Q''_{k,l,l'}}. \quad (5.85)$$

However, by the choice of  $\Omega$ , there exists  $c > 0$ , such that for any  $\lambda \in \Omega$ , we have

$$\text{Spec} \left( \mathcal{B}_t \frac{1}{\lambda - \mathcal{C}_t} \right) \subset [0, c]. \quad (5.86)$$

From (5.85) and (5.86), we deduce

$$\left\| \frac{\partial^{|\alpha'|}}{\partial Z'^{\alpha'}} \left( \mathcal{B}_t \frac{1}{\lambda - \mathcal{C}_t} A_k \right) (\cdot, Z') \right\|_{L^2(g_0^{T X'})} \leq Ct^k (1 + |Z'|)^{Q''_{k,l,l'}}. \quad (5.87)$$

From Lemma 5.7 and (5.87), we deduce that for any  $Z' \in \mathbb{C}^n$ ,  $|Z'| < \frac{\epsilon}{2t}$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( \mathcal{B}_t \frac{1}{\lambda - \mathcal{C}_t} A_k \right) (Z, Z') \right| \leq Ct^k (1 + |Z'|)^{Q''_{k,l,l'}}. \quad (5.88)$$

From (5.75), (5.78), (5.82), (5.83) and (5.88), we deduce (5.72). Also, from (5.82), we deduce the general algorithm for the construction of the polynomials  $J_r^\perp$ .

Now it only left to prove (1.28) and (1.29). For this, let us find explicit formula for  $B_1$ , as the polynomials  $J_0^\perp, J_1^\perp$ , can be then read off from (5.82). For this, we apply once (5.80) and then twice (5.77) to get

$$\frac{1}{\lambda - \mathcal{C}_t} = \frac{1}{\lambda - \mathcal{C}_0} + \frac{1}{\lambda - \mathcal{C}_0} (\mathcal{C}_t - \mathcal{C}_0) \frac{1}{\lambda - \mathcal{C}_0} + \frac{1}{\lambda - \mathcal{C}_t} (\mathcal{C}_t - \mathcal{C}_0) \frac{1}{\lambda - \mathcal{C}_0} (\mathcal{C}_t - \mathcal{C}_0) \frac{1}{\lambda - \mathcal{C}_0}. \quad (5.89)$$

From (5.76) and (5.89), we deduce that

$$B_1 = \left( \frac{1 - \mathcal{C}_0}{\lambda} + \frac{\mathcal{C}_0}{\lambda - 1} \right) + \left( \frac{1 - \mathcal{C}_0}{\lambda} + \frac{\mathcal{C}_0}{\lambda - 1} \right) (\mathcal{C}_t - \mathcal{C}_0) \left( \frac{1 - \mathcal{C}_0}{\lambda} + \frac{\mathcal{C}_0}{\lambda - 1} \right). \quad (5.90)$$



Now, by Remark 5.11 and the choice of  $\Omega$ , we have the identities

$$\begin{aligned} \int_{\Omega} \frac{1}{\lambda} d\lambda &= 0, & \int_{\Omega} \frac{1}{\lambda-1} d\lambda &= 1, & \int_{\Omega} \frac{1}{\lambda^2} d\lambda &= 0, \\ \int_{\Omega} \frac{1}{(\lambda-1)^2} d\lambda &= 0, & \int_{\Omega} \frac{1}{\lambda(\lambda-1)} d\lambda &= 1. \end{aligned} \quad (5.91)$$

From (5.90) and (5.91), we deduce

$$\int_{\Omega} B_1 d\lambda = \mathcal{C}_0 + (1 - \mathcal{C}_0)(\mathcal{C}_t - \mathcal{C}_0)\mathcal{C}_0 + \mathcal{C}_0(\mathcal{C}_t - \mathcal{C}_0)(1 - \mathcal{C}_0). \quad (5.92)$$

Hence, by using the fact that  $\mathcal{C}_0^2 = \mathcal{C}_0$ , the fact that by Lemma 5.10, the first term of the Taylor-type expansion of  $\mathcal{C}_t - \mathcal{C}_0$  vanishes, and (5.92), we deduce (1.28).

Now, let us establish (1.29). By using the notations from Proposition 5.4, Remark 3.4 and (5.66), we deduce from (5.82) and (5.92) the following identity

$$J_1^{\perp}(Z, Z') = \mathcal{K}_{n,m}[\mathcal{J}'_1, \mathcal{J}'_0] - 2\mathcal{K}_{n,m}[\mathcal{J}'_0, \mathcal{K}_{n,m}[\mathcal{J}'_1, \mathcal{J}'_0]] + \mathcal{K}_{n,m}[\mathcal{J}'_0, \mathcal{J}'_1]. \quad (5.93)$$

We will now calculate each term in (5.93). From (3.43), (3.44) and (5.66), we deduce

$$\mathcal{K}_{n,m}[\mathcal{J}'_1, \mathcal{J}'_0](Z, Z') = \text{Id}_{F_{y_0}} \cdot \pi g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)). \quad (5.94)$$

From (3.44) and (5.94), we deduce

$$\mathcal{K}_{n,m}[\mathcal{J}'_0, \mathcal{K}_{n,m}[\mathcal{J}'_1, \mathcal{J}'_0]](Z, Z') = 0. \quad (5.95)$$

Now, (1.29) follows directly from (5.93), (5.94) and (5.95).  $\square$

## 5.4 Asymptotics of the extension operator, proofs of Theorems 1.1, 1.5, 1.6

The main goal of this section is to prove Theorems 1.1, 1.5, 1.6 and Corollary 1.3. The main idea is to find an algebraic expression for  $E_p$  in terms of  $B_p^{\perp}$ ,  $B_p^Y$  and  $\text{Res}_Y$  and then to get the needed results by transferring the statements from Theorems 1.8, 1.9. We conserve the notations from Section 1. Define the operators  $G_p : L^2(Y, \iota^*(L^p \otimes F)) \rightarrow L^2(Y, \iota^*(L^p \otimes F))$  and  $I_p : L^2(Y, \iota^*(L^p \otimes F)) \rightarrow L^2(X, L^p \otimes F)$  as follows

$$G_p := \text{Res}_Y \circ I_p - B_p^Y, \quad I_p := B_p^{\perp} \circ E_p^0 \circ B_p^Y. \quad (5.96)$$

**Lemma 5.12.** There are  $c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1$ ,  $y_1, y_2 \in Y$ , the following estimate holds

$$\left| G_p(y_1, y_2) \right|_{\mathcal{C}^k(Y \times Y)} \leq Cp^{m+\frac{k-1}{2}} \cdot \exp(-c\sqrt{p} \cdot \text{dist}(y_1, y_2)), \quad (5.97)$$

where the pointwise  $\mathcal{C}^k$ -norm is interpreted as in Theorem 1.5.

Also, for any  $r \in \mathbb{N}$ ,  $y_0 \in Y$ , there are  $J_{r,G}(Z_Y, Z'_Y) \in \text{End}(F_{y_0})$  polynomials in  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ , satisfying the same properties as in Theorem 1.6, such that for  $F_{r,G} := J_{r,G} \cdot \mathcal{P}_m$ , the

following holds. There are  $\epsilon, c > 0, p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $y_0 \in Y, p \geq p_1, Z_Y, Z'_Y \in \mathbb{R}^{2m}, |Z_Y|, |Z'_Y| \leq \epsilon, \alpha, \alpha' \in \mathbb{N}^{2m}, |\alpha| + |\alpha'| \leq l$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^\alpha \partial Z'_Y{}^{\alpha'}} \left( \frac{1}{p^m} G_p(\psi_{y_0}(Z_Y), \psi_{y_0}(Z'_Y)) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,G}(\sqrt{p}Z_Y, \sqrt{p}Z'_Y) \kappa_Y^{-\frac{1}{2}}(Z_Y) \kappa_Y^{-\frac{1}{2}}(Z'_Y) \right) \right|_{\mathcal{C}^{l'}(Y)} \leq Cp^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z'_Y| \right)^{2Q_{k,l,l'}^2} \exp \left( -c\sqrt{p}|Z_Y - Z'_Y| \right), \quad (5.98)$$

where  $Q_{k,l,l'}^2 > 0$  is defined in Theorem 1.9. Moreover, we have

$$J_{0,G} = 0. \quad (5.99)$$

And under assumption (1.8), we even have

$$J_{1,G} = 0, \quad (5.100)$$

so that  $p^{m+\frac{k-1}{2}}$  in (5.97) in this case can be replaced by  $p^{m-1+\frac{k}{2}}$ .

*Proof.* First of all, by Theorem 4.6 and Lemma 3.1, there are  $c > 0, p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for  $p \geq p_1, y_1, y_2 \in Y$ , the following estimate holds

$$\left| G_p(y_1, y_2) \right|_{\mathcal{C}^k(Y \times Y)} \leq Cp^{n+\frac{k}{2}} \cdot \exp \left( -c\sqrt{p} \cdot \text{dist}(y_1, y_2) \right). \quad (5.101)$$

Since for any fixed  $\epsilon > 0$ , (5.101) implies (5.97) for  $y_1, y_2 \in Y$  verifying  $\text{dist}(y_1, y_2) > \epsilon$  and a different choice of  $c, C$ , it is now enough to establish (5.98), (5.99) to get (5.97) for all  $y_1, y_2 \in Y$ . Let us concentrate on this now.

Let  $J'_{r,Y}(Z_Y, Z'_Y) \in \text{End}(F_{y_0})$  be the polynomials in  $Z_Y, Z'_Y \in \mathbb{R}^{2m}, r \in \mathbb{N}$ , given by Theorem 5.5, applied for  $X := Y$ . Recall that polynomials  $J_r^\perp(Z, Z') \in \text{End}(F_{y_0})$ , were defined in Theorem 1.9. We rewrite  $I_p$  in the following equivalent form

$$I_p := \left( B_p^\perp \cdot \kappa_N^{\frac{1}{2}} \right) \circ \left( \kappa_N^{-\frac{1}{2}} \cdot E_p^0 \right) \circ B_p^Y. \quad (5.102)$$

For  $k \in \mathbb{N}$ , let us now write the Taylor expansions of  $\kappa_N^{\frac{1}{2}}, \kappa_N^{-\frac{1}{2}}$  in a neighborhood of  $y_0$  as follows

$$\kappa_N^{\frac{1}{2}}(Z) = \sum_{i=0}^k \kappa_{N,i}^{\frac{1}{2}}(Z) + O(|Z|^{k+1}), \quad \kappa_N^{-\frac{1}{2}}(Z) = \sum_{i=0}^k \kappa_{N,i}^{-\frac{1}{2}}(Z) + O(|Z|^{k+1}), \quad (5.103)$$

where  $\kappa_{N,i}^{\frac{1}{2}}(Z), \kappa_{N,i}^{-\frac{1}{2}}(Z)$  are homogeneous polynomials of degree  $i$ . We denote now

$$J_r^{\perp, \kappa} := \sum_{r_0=0}^r J_{r_0}^\perp(Z, Z') \cdot \kappa_{N, r-r_0}^{\frac{1}{2}}(Z'). \quad (5.104)$$

By Theorems 1.8, 1.9, 4.6, 5.5 and Lemma 3.8, we conclude that for

$$J_{r,I}^E(Z, Z'_Y) = \sum_{r_0=0}^r \mathcal{K}''_{n,m}[J_{r_0}^{\perp, \kappa}, J'_{r-r_0,Y}] \quad (5.105)$$

and for  $F_{r,I}^E := J_{r,I}^E \cdot \mathcal{E}_{n,m}$ , the following holds. There are  $\epsilon, c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1$ ,  $y_0 \in Y$ ,  $Z = (Z_Y, Z_N)$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $Z_N \in \mathbb{R}^{2(n-m)}$ ,  $|Z|, |Z'_Y| \leq \epsilon$ ,  $\alpha \in \mathbb{N}^{2n}$ ,  $\alpha' \in \mathbb{N}^{2m}$ ,  $|\alpha| + |\alpha'| \leq l$ , we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z_Y^{\alpha'}} \left( \frac{1}{p^m} I_p(\psi_{y_0}(Z), \psi_{y_0}(Z'_Y)) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,I}^E(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_X^{-\frac{1}{2}}(Z) \kappa_Y^{-\frac{1}{2}}(Z'_Y) \right) \right|_{\mathcal{C}^l(Y)} \leq Cp^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \right)^{2Q_{k,l,l'}} \exp\left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|)\right). \quad (5.106)$$

From Theorem 5.5, (5.96) and (5.106), we conclude that (5.98) holds for

$$J_{r,G}(Z_Y, Z'_Y) := \sum_{r_0=0}^r \kappa_{N,r_0}^{-\frac{1}{2}}(Z_Y) \cdot J_{r-r_0,I}^E(Z_Y, Z'_Y) - J'_{r,Y}(Z_Y, Z'_Y). \quad (5.107)$$

From (1.28), (5.28) and (5.105), we conclude

$$J_{0,I}^E = \text{Id}_{F_{y_0}} \cdot \mathcal{K}''_{n,m}[1, 1] \cdot \kappa_N^{\frac{1}{2}}(y_0). \quad (5.108)$$

From (5.28), (5.107) and (5.108), we obtain (5.99).

Now, we assume (1.8). From (1.29), (3.41), (3.42), (3.44), (4.14), (5.29), (5.35), and (5.105), we get

$$J_{1,I}^E(Z, Z'_Y) = \text{Id}_{F_{y_0}} \cdot \pi g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)). \quad (5.109)$$

We obtain (5.100) from (5.107) and (5.109).  $\square$

**Lemma 5.13.** There are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for  $p \geq p_1$ , we have

$$\|G_p\| \leq \frac{C}{\sqrt{p}}. \quad (5.110)$$

Moreover, under assumption (1.8), in the above inequality, we can replace  $\sqrt{p}$  by  $p$ .

*Proof.* First of all, by Proposition 3.2 and Lemma 5.12, there are  $C > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $y_0 \in Y$ , for  $p \geq p_1$ , we have

$$\int_Y |G_p(y_0, y)| dv_Y(y) \leq \frac{C}{\sqrt{p}}, \quad \int_Y |G_p(y, y_0)| dv_Y(y) \leq \frac{C}{\sqrt{p}}. \quad (5.111)$$

The result now follows from (5.111) and Young's inequality for integral operators, cf. [36, Theorem 0.3.1] applied for  $p, q = 2$ ,  $r = 1$  in the notations of [36]. The second part is proved using (5.100) in exactly the same way, one only has to rely on the comment after (5.100).  $\square$

Lemma 5.13 now implies that there is  $p_1 \in \mathbb{N}^*$ , such that for  $p \geq p_1$ , the infinite sum

$$T_p := \sum_{i=1}^{\infty} (-1)^i G_p^i. \quad (5.112)$$

is well-defined as an operator on  $L^2(Y, \iota^*(L^p \otimes F))$ .

**Lemma 5.14.** The following identity holds

$$E_p = I_p + I_p \circ T_p. \quad (5.113)$$

*Proof.* Since  $B_p^\perp$  has values in  $H_{(2)}^{0,\perp}(X, L^p \otimes F)$ , and  $G_p$  vanishes on the kernel of  $B_p^Y$  by (5.96), it is enough to establish that

$$\text{Res}_Y \circ (I_p + I_p \circ T_p) = B_p^Y. \quad (5.114)$$

This identity follows from the observation that for any  $i \in \mathbb{N}^*$ , we have

$$\text{Res}_Y \circ B_p^\perp \circ E_p^0 \circ G_p^i = G_p^{i+1} + G_p^i, \quad (5.115)$$

which, on its turn, follows by an application of the right composition with  $G_p^i$  in (5.96), and the identity  $B_p^Y \circ G_p = G_p$ , following from the trivial fact that the restriction of a holomorphic section is holomorphic.  $\square$

*Proof of Theorem 1.5.* By Lemmas 3.1, 5.12, and the fact that for any  $C > 0$  there is  $p_1 \in \mathbb{N}^*$  such that for  $p \geq p_1$ , the sum  $\sum_{i=0}^{\infty} (\frac{C}{\sqrt{p}})^i$  converges, we deduce that there is  $c > 0$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for  $p \geq p_1$ , the following estimate for the Schwartz kernel holds

$$\left| T_p(y_1, y_2) \right|_{\mathcal{C}^k(Y \times Y)} \leq Cp^{m+\frac{k-1}{2}} \cdot \exp(-c\sqrt{p} \cdot \text{dist}(y_1, y_2)). \quad (5.116)$$

Moreover, under assumption (1.8), we can replace  $p^{m+\frac{k-1}{2}}$  by  $p^{m-1+\frac{k}{2}}$ . We conclude by Theorem 1.8, Lemmas 3.1, 5.14 and (5.116).  $\square$

*Proof of Theorem 1.6.* Let us fix  $k \in \mathbb{N}$  and establish Theorem 1.6 for it. By Lemmas 3.7, 5.12, we conclude that for any  $r \in \mathbb{N}$ ,  $y_0 \in Y$ , there are  $J_{r,T}(Z_Y, Z'_Y) \in \text{End}(F_{y_0})$  polynomials in  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ , satisfying the same properties as in Theorem 1.6, such that for  $F_{r,T} := J_{r,T} \cdot \mathcal{P}_m$ , the following holds. There are  $\epsilon, c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k, l, l' \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $y_0 \in Y$ ,  $p \geq p_1$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $|Z_Y|, |Z'_Y| \leq \epsilon$ ,  $\alpha, \alpha' \in \mathbb{N}^{2m}$ ,  $|\alpha| + |\alpha'| \leq l$ , the following bound holds

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^\alpha \partial Z'_Y{}^{\alpha'}} \left( \frac{1}{p^m} T_p(\psi_{y_0}(Z_Y), \psi_{y_0}(Z'_Y)) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,T}(\sqrt{p}Z_Y, \sqrt{p}Z'_Y) \kappa_Y^{-\frac{1}{2}}(Z_Y) \kappa_Y^{-\frac{1}{2}}(Z'_Y) \right) \right|_{\mathcal{C}^{l'}(Y)} \leq Cp^{-\frac{k+1-l}{2}} \left( 1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z'_Y| \right)^{Q''_{k,l,l'}} \exp(-c\sqrt{p}|Z_Y - Z'_Y|), \quad (5.117)$$

where  $Q'''_{k,l,l'} := 6(8(n+2)(2k+1) + l') + 2l$ . Moreover, we have

$$J_{0,T} = 0. \quad (5.118)$$

And under assumption (1.8), we even have

$$J_{1,T} = 0. \quad (5.119)$$

Now, by Theorems 1.8, 1.9, Lemma 3.8, (5.106) and (5.117), we deduce that the asymptotic expansion (1.21) holds for

$$J_r^E := J_{r,I}^E + \sum_{r_1=0}^r \mathcal{K}''_{n,m}[J_{r_1,I}^E, J_{r-r_1,T}]. \quad (5.120)$$

From (5.108) and (5.118), we deduce (1.22). Moreover, under assumption (1.8), from (5.109), (5.118) and (5.119), we deduce (1.23).  $\square$

*Proof of Theorem 1.1.* Let us denote  $K_p := E_p - E_p^0$ . From Theorem 1.5 and (1.9), we conclude that there are  $c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $p \geq p_1$ ,  $x \in X$ ,  $y \in Y$ , the following estimate holds

$$\left| K_p(x, y) \right|_{\mathcal{C}^k(X \times Y)} \leq Cp^{m+\frac{k}{2}} \exp(-c\sqrt{p}\text{dist}(x, y)). \quad (5.121)$$

Let us now denote by  $J'_{r,Y}(Z_Y, Z'_Y)$ ,  $r \in \mathbb{N}$ , the polynomials from Theorem 5.5, applied for  $X := Y$ . From Theorems 1.6, 5.5, we deduce that for polynomials

$$J_{r,K}^E(Z, Z'_Y) := J_r^E(Z, Z'_Y) - \sum_{r_0=0}^r \kappa_{N,r-r_0}^{\frac{1}{2}}(Z_Y) \cdot J'_{r_0,Y}(Z_Y, Z'_Y), \quad r \in \mathbb{N}, \quad (5.122)$$

and the functions  $F_{r,K}^E := J_{r,K}^E \cdot \mathcal{E}_{n,m}$  over  $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$ , the following holds. There are  $\epsilon, c > 0$ ,  $p_1 \in \mathbb{N}^*$ , such that for any  $k \in \mathbb{N}$ , there is  $C > 0$ , such that for any  $y_0 \in Y$ ,  $p \geq p_1$ ,  $Z = (Z_Y, Z_N)$ ,  $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ ,  $Z_N \in \mathbb{R}^{2(n-m)}$ ,  $|Z|, |Z'_Y| \leq \epsilon$ , we have

$$\begin{aligned} & \left| \frac{1}{p^m} K_p(\psi_{y_0}(Z), \psi_{y_0}(Z'_Y)) - \sum_{r=0}^k p^{-\frac{r}{2}} F_{r,K}^E(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_X^{-\frac{1}{2}}(Z) \kappa_Y^{-\frac{1}{2}}(Z'_Y) \right| \\ & \leq Cp^{-\frac{k+1}{2}} \left( 1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \right)^{Q_{k,0,0}^1} \exp\left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|)\right). \end{aligned} \quad (5.123)$$

From (1.22), (5.28) and (5.122), we deduce

$$J_{0,K}^E(Z, Z'_Y) = 0. \quad (5.124)$$

Moreover, under assumption (1.8), from (1.23), (5.29) and (5.122), we deduce

$$J_{1,K}^E(Z, Z'_Y) = J_1^E(Z, Z'_Y). \quad (5.125)$$

From (5.121), (5.123), (5.124) and the use of Young's inequality for integral operators as in Lemma 5.13, we deduce (1.11). In an analogous way, from (1.23), (5.121), (5.123), (5.124) and (5.125), under additional assumptions (1.8) and  $A = 0$ , we see that in (1.11), one can replace  $p^{\frac{n-m+1}{2}}$  by  $p^{\frac{n-m+2}{2}}$ .

Now, an easy calculation, using (1.7), shows that for any  $g \in L^2(Y, \iota^*(L^p \otimes F))$ , we have

$$\|E_p^0 g\|_{L^2(dv_X)} = \|f(p, y) \cdot B_p^Y g\|_{L^2(dv_Y)}, \quad (5.126)$$

where the function  $f : \mathbb{N} \times Y \rightarrow \mathbb{R}$  is defined as

$$f(p, y)^2 := \int_{\mathbb{R}^{2(n-m)}} \kappa_N(y, \sqrt{p}Z_N) \cdot \exp(-p\pi|Z_N|^2) \rho\left(\frac{|Z_N|}{r_\perp}\right)^2 dZ_{2m+1} \wedge \cdots \wedge dZ_{2n}. \quad (5.127)$$

By an easy calculation using Gaussian integral, there is  $c > 0$ , such that, as  $p \rightarrow \infty$ , we have

$$f(p, y)^2 = \frac{\kappa_N(y)}{p^{n-m}} + O\left(\frac{1}{p^{n-m+\frac{1}{2}}}\right). \quad (5.128)$$

From (1.11), (5.126), (5.127) and (5.128), we deduce (1.12).

Now, consider the Toeplitz operator  $T_{\kappa_N, p} : L^2(Y, \iota^*(L^p \otimes F)) \rightarrow L^2(Y, \iota^*(L^p \otimes F))$ , given by

$$T_{\kappa_N, p} g := B_p^Y (\kappa_N \cdot B_p^Y g). \quad (5.129)$$

Then, we clearly have

$$\langle T_{\kappa_N, p} g, g \rangle_{L^2(dv_Y)} = \langle \kappa_N \cdot B_p^Y g, B_p^Y g \rangle_{L^2(dv_Y)}. \quad (5.130)$$

Thus, by (5.126) and (5.128), we have

$$\|E_p^0 g\|_{L^2(dv_X)} = \frac{1}{p^{\frac{n-m}{2}}} \|T_{\kappa_N, p}\|^{\frac{1}{2}} + O\left(\frac{1}{p^{\frac{n-m+1}{2}}}\right). \quad (5.131)$$

Recall that for compact manifolds  $Y$ , Bordemann-Meinrenken-Schlichenmaier [5, Theorem 4.1] (for  $(F, h^F)$  trivial) and Ma-Marinescu [29, Theorem 3.19, (3.91)] (for any  $(F, h^F)$ ) established that there is  $C > 0$ , such that

$$\sup_{y \in Y} \kappa_N(y) - \frac{C}{\sqrt{p}} \leq \|T_{\kappa_N, p}\| \leq \sup_{y \in Y} \kappa_N(y). \quad (5.132)$$

The same proof shows that the upper bound of (5.132) continues to hold for non-compact manifolds as well. Recall that the lower bound of (5.132) was proved in [29] using the asymptotic expansion of the peak section, localized at the point where the supremum of  $\kappa_N$  is achieved. As the asymptotic expansion of the peak section is based on the asymptotic expansion of the Bergman kernel and the exponential bound on it, and both those results continue to hold on manifolds of bounded geometry by the results of [30], cf. Theorems 4.6, 5.5, we see that (5.132) continues to hold in full generality, see [28, §7.5], when the supremum of  $\kappa_N$  is achieved. Now, if the supremum is not achieved, then the same proof gives us that for any  $\epsilon > 0$ , there is  $p_1 \in \mathbb{N}^*$ , such that for  $p \geq p_1$ , we have

$$\sup_{y \in Y} \kappa_N(y) - \epsilon \leq \|T_{\kappa_N, p}\| \quad (5.133)$$

We deduce (1.12) by (5.131), (5.132) and (5.133).

Now it is only left to prove that if  $A \neq 0$ , then under additional assumption (1.8), one can not replace  $p^{\frac{n-m+1}{2}}$  by  $p^{\frac{n-m+2}{2}}$ . For this, remark that as long as  $A \neq 0$ , by (1.23), the operator, acting on  $\mathbb{C}^n$  with the kernel  $F_1^E(Z, Z'_Y)$ , has non-zero norm. Then, by the calculations, similar to (5.126), we see that the operator, acting on  $\mathbb{C}^n$  with the kernel  $F_1^E(\sqrt{p}Z, \sqrt{p}Z'_Y)$ , has norm of order  $\frac{1}{p^{\frac{n-m}{2}}}$ , as  $p \rightarrow \infty$ . We deduce from this and Theorem 1.6 that if  $A \neq 0$ , then under additional assumption (1.8), one can not replace  $p^{\frac{n-m+1}{2}}$  by  $p^{\frac{n-m+2}{2}}$ .  $\square$

*Proof of Corollary 1.3.* It is a direct consequence of Proposition 4.5 and (1.12).  $\square$

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