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# Qualitative Integrals and Desintegrals – Towards a Logical View

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**Abstract.** This paper presents several variants of Sugeno integral, and in particular the idea of (qualitative) desintegrals, a dual of integrals. When evaluating an item, desintegrals are maximal if no defects at all are present, while integrals are maximal if all advantages are sufficiently present. This idea leads to a bipolar representation of preferences, by means of a pair made of an integral and a desintegral, whose possibilistic logic counterparts are outlined (in the case where criteria are binary).

**Keywords:** Sugeno integral, possibilistic logic, bipolar representation.

## 1 Introduction

In multi-criteria decision making, Sugeno integrals are commonly used as qualitative aggregation functions for evaluating objects on the basis of several criteria [10]. A Sugeno integral delivers a score between the minimum and the maximum of the partial ratings. The definition of Sugeno integral is based on a capacity (or fuzzy measure) which represents the importance of the sets of criteria. But the importance of the criteria can be exploited in different ways when aggregating partial evaluations. Especially, variants of Sugeno integral can be defined when the evaluation scale is taken as a Heyting algebra using an operator named the residuum.

When Sugeno integral is used, the criteria are considered positive: global evaluation increases with the partial ratings. If we consider negative criteria, then the global evaluation increases when the partial ratings decrease. In such a context it is possible to define other variants of Sugeno integral we call *desintegrals*. With these new kinds of negative aggregation functions, the better a criterion is satisfied the worse is the global evaluation. Besides, Sugeno integral can be encoded as a possibilistic logic base [8]. This paper partially extends this result to the desintegrals when the partial evaluations are binary values, which comes down to a logical encoding of monotonic set functions.

In order to illustrate our motivations, let us first present an example of how we can intuitively use a pair of specifications that can be represented by an integral and a desintegral, as we shall see later, for modeling preferences. We work in the framework of possibilistic logic with symbolic weights as in [11].

Let  $a, b, c$  and  $d$  be four properties and  $S$  be a scale, supposedly discrete and totally ordered with a greatest element denoted by 1 and a least element denoted by 0. To illustrate the “negative properties” side, associated with the idea of a desintegral, let us assume the following: if the properties  $a$  and  $b$  are satisfied then the global evaluation should remain below a certain level  $\theta < 1$  and if the property  $c$  is satisfied, the evaluation should be less than  $\lambda < 1$  with  $\lambda > \theta$  (where  $\lambda, \theta \in S$ ). For the “positive properties”, modeled by an integral, let us assume that if the property  $d$  is satisfied the global evaluation should be greater than  $\rho < 1$  and that if the properties  $a$  and  $d$  are satisfied the evaluation must be greater than  $\eta$  with  $\eta > \rho$  ( $\eta, \rho \in S$ ). As can be seen on the table in Figure 1, we obtain two symbolic distributions corresponding respectively to an upper bound of the evaluation  $x$  of the negative aspects and to a lower bound of the evaluation  $y$  of the positive aspects, in different situations. Note that if we impose a single evaluation rather than two, i.e;  $x = y$ , a consistency condition would be needed:  $\mu \leq \theta$ . But in the following, we consider the two types of evaluation separately.

a	b	c	d	negative aspects	positive aspects
1	1	1	1	$x \leq \theta$	$\eta \leq y$
1	1	1	0	$x \leq \theta$	$0 \leq y$
1	1	0	1	$x \leq \theta$	$\eta \leq y$
1	1	0	0	$x \leq \theta$	$0 \leq y$
1	0	1	1	$x \leq \lambda$	$\eta \leq y$
1	0	1	0	$x \leq \lambda$	$0 \leq y$
1	0	0	1	$x \leq 1$	$\eta \leq y$
1	0	0	0	$x \leq 1$	$0 \leq y$
0	1	1	1	$x \leq \lambda$	$\rho \leq y$
0	1	1	0	$x \leq \lambda$	$0 \leq y$
0	1	0	1	$x \leq 1$	$\rho \leq y$
0	1	0	0	$x \leq 1$	$0 \leq y$
0	0	1	1	$x \leq \lambda$	$\rho \leq y$
0	0	1	0	$x \leq \lambda$	$0 \leq y$
0	0	0	1	$x \leq 1$	$\rho \leq y$
0	0	0	0	$x \leq 1$	$0 \leq y$

**Fig. 1.** Symbolic distributions corresponding to the idea of desintegral (upper bound) and to the idea of integral (lower bound)

This article deals with the bipolar representation of preferences (see [4] for an overview on the representation of preferences) in a qualitative framework by extending the notion of Sugeno integral. Section 3 presents two types of integrals and two types of desintegrals, and establishes relations between them when the criteria are binary. Section 4 studies the logical counterpart of integrals and desintegrals in this case. Before concluding, Section 5 briefly discusses the idea of using a pair of integral / desintegral to describe acceptable objects. But first we present a reminder of possibilistic logic and we introduce the algebraic framework necessary for the evaluations.

## 2 Framework and Notations

This section provides a reminder on possibilistic logic, and introduces the algebraic framework required for the evaluations of objects using qualitative integrals and desintegrals whose definitions are presented in the next section.

### 2.1 Possibilistic Logic

Let  $B^N = \{(\varphi_j, \alpha_j) \mid j = 1, \dots, m\}$  be a possibilistic logic base where  $\varphi_j$  is a propositional logic formula and  $\alpha_j \in \mathcal{L} \subseteq [0, 1]$  is a priority level [7]. The logical conjunctions and disjunctions are denoted by  $\wedge$  and  $\vee$ . Each formula  $(\varphi_j, \alpha_j)$  means that  $N(\varphi_j) \geq \alpha_j$ , where  $N$  is a necessity measure, i.e., a set function satisfying the property  $N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$ . A necessity measure is associated to a possibility distribution  $\pi$  as follows:

$N(\varphi) = \min_{\omega \notin M(\varphi)} (1 - \pi(\omega)) = 1 - \Pi(\neg\varphi)$ , where  $\Pi$  is the possibility measure associated to  $N$  and  $M(\varphi)$  is the set of models induced by the underlying propositional language for which  $\varphi$  is true.

The base  $B^N$  is associated to the least informative possibility distribution induced by the constraints  $N(\varphi_j) \geq \alpha_j$ , namely,  $\pi_B^N(\omega) = \min_{j=1, \dots, m} \pi_{(\varphi_j, \alpha_j)}(\omega)$  on the set of interpretations, where  $\pi_{(\varphi_j, \alpha_j)}(\omega) = 1$  if  $\omega \in M(\varphi_j)$ , and  $\pi_{(\varphi_j, \alpha_j)}(\omega) = 1 - \alpha_j$  if  $\omega \notin M(\varphi_j)$ . An interpretation  $\omega$  is all the more possible as it does not violate any formula  $\varphi_j$  having a higher priority level  $\alpha_j$ . Hence, this possibility distribution is expressed as a min-max combination:

$$\pi_B^N(\omega) = \min_{j=1, \dots, m} \max(1 - \alpha_j, I_{M(\varphi_j)}(\omega))$$

where  $I_{M(\varphi_j)}$  is the characteristic function of  $M(\varphi_j)$ . So, if  $\omega \notin M(\varphi_j)$ ,  $\pi_B^N(\omega) \leq 1 - \alpha_j$ , and if  $\omega \in \bigcap_{j \in J} M(\neg\varphi_j)$ ,  $\pi_B^N(\omega) \leq \min_{j \in J} (1 - \alpha_j)$ . It is a description “from above” of  $\pi_B^N$ . A possibilistic base  $B^N$  can be transformed in a base where the formulas  $\varphi_i$  are clauses (without altering the distribution  $\pi_B^N$ ). We can still see  $B^N$  as a conjunction of weighted clauses, i.e., as an extension of the conjunctive normal form.

A dual representation in possibilistic logic is based on guaranteed possibility measures. Hence a logical formula is a pair  $[\psi, \beta]$ , interpreted as the constraint  $\Delta(\psi) \geq \beta$ , where  $\Delta$  is a guaranteed (anti-)possibility measure characterized by  $\Delta(\phi \vee \psi) = \min(\Delta(\phi), \Delta(\psi))$  and  $\Delta(\emptyset) = 1$ . In such a context, a base  $B^\Delta = \{[\psi_i, \beta_i] \mid i = 1, \dots, n\}$  is associated to the distribution

$$\pi_B^\Delta(\omega) = \max_{i=1, \dots, n} \pi_{[\psi_i, \beta_i]}(\omega)$$

with  $\pi_{[\psi_i, \beta_i]}(\omega) = \beta_i$  if  $\omega \in M(\psi_i)$  and  $\pi_{[\psi_i, \beta_i]}(\omega) = 0$  otherwise. If  $\omega \in M(\psi_i)$ ,  $\pi_B^\Delta(\omega) \geq \beta_i$ , and if  $\omega \in \bigcup_{i \in I} M(\psi_i)$ ,  $\pi_B^\Delta(\omega) \geq \max_{i \in I} \beta_i$ . So this base is a description “from below” of  $\pi_B^\Delta$ . A dual possibilistic base  $B^\Delta$  can always be transformed in a base in which the formulas  $\psi_j$  are conjunctions of literals (cubes) without altering  $\pi_B^\Delta$ . So  $B^N$  can be seen as a weighted combination of cubes, i.e., as an extension of the disjunctive normal form.

A possibilistic logic base  $B^\Delta$  expressed in terms of a guaranteed possibility measure can always be rewritten equivalently in terms of a standard possibilistic logic base  $B^N$  based on necessity measures [2,1] and conversely with the equality  $\pi_B^N = \pi_B^\Delta$ . This transformation is similar to a description from below of  $\pi_B^N$ . Let us note that

- if  $\omega \in M(\varphi_j)$ ,  $\pi_B^N(\omega) \geq \min_{k \neq j}(1 - \alpha_k)$ ,
- and more generally si  $\omega \in \bigcap_{j \in J} M(\varphi_j)$ ,  $\pi_B^N(\omega) \geq \min_{k \notin J}(1 - \alpha_k)$ ,

so  $\pi_B^N$  can be rewritten in a max-min form (equivalent to the previous one):

$$\pi_B^N(\omega) = \max_{J \subseteq \{1, \dots, m\}} \min_{k \notin J} (\min(1 - \alpha_k), I_{M(\bigwedge_{j \in J} \varphi_j)}(\omega))$$

where  $I_{M(\bigwedge_{j \in J} \varphi_j)}(\omega) = \min_{j \in J} I_{M(\varphi_j)}(\omega)$ , and  $\min_{\emptyset} X = 1$ . This transformation corresponds to writing the min-max expression of  $\pi_B^N$  as a max-min expression by applying the distributivity of min to max. The base obtained is

$$B^\Delta = \{[\bigwedge_{j \in J} \varphi_j, \min_{k \notin J}(1 - \alpha_k)], J \subseteq \{1, \dots, m\}\}.$$

Note that this procedure generalizes the transformation of a conjunctive normal form into a disjunctive normal form to the gradual case.

In the following, for convenience, we use a possibilistic logic encoding of the type  $\Delta(\psi) \geq \beta$ .

## 2.2 Algebraic Framework

We consider a set of criteria  $\mathcal{C} = \{1, \dots, n\}$ . Objects are evaluated using these criteria. The evaluation scale,  $\mathcal{L}$ , associated to each criterion is totally ordered. It may be finite or be the interval  $[0, 1]$ . Then an object is represented by its evaluation on the different criteria, i.e., by  $f = (f_1, \dots, f_n) \in \mathcal{L}^n$ . Moreover we consider  $\mathcal{L}$  as a Heyting algebra i.e as a complete residuated lattice with a greatest element denoted by 1 and a least element denoted by 0. More precisely,  $\langle \mathcal{L}, \wedge, \vee, \rightarrow, 0, 1 \rangle$  is a complete lattice:  $\langle \mathcal{L}, \wedge, 1 \rangle$  is a commutative monoid (i.e  $\wedge$  is associative, commutative and for all  $a \in \mathcal{L}$ ,  $a \wedge 1 = a$ ). The operator denoted by  $\rightarrow$  will be the Gödel implication defined by  $a \rightarrow b = 1$  if  $a \leq b$  and  $b$  otherwise.

In the following, we consider positive criteria and negative criteria. In the latter case, 0 will be a good evaluation, 1 will be a bad evaluation and the scale will be said decreasing (the scale is increasing in the case of positive criteria). To handle the directionality of the scale, we also need an operation that reverses the scale. This operation (a decreasing involution) defined on  $\mathcal{L}$  is denoted by  $1-$ . We can then define integrals and desintegrals. A particular integral is Sugeno integral for which the possibilistic logic counterpart has been studied recently [8].

### 3 Qualitative Integrals and Desintegrals

Now we introduce two qualitative integrals and two qualitative desintegrals. See [9] for a more comprehensive framework. This should not be confused with a proposal made in [3] where the integrals considered are generalizations of Sugeno integral on a De Morgan-like algebra, and where the idea of desintegrals does not appear, neither the use of a residuated structure, nor the concern for a weighted logic counterpart.

#### 3.1 Qualitative Integrals and Increasing Scale

In this part the criteria are evaluated on an increasing scale and the global evaluation is also on an increasing scale. An importance factor  $\pi_i$  is assigned to each criterion  $i$ . It is all the higher as the criterion  $i$  is important. We assume that  $\bigvee_{i=1,\dots,n} \pi_i = 1$ . We view an object as described by a mapping  $f : \mathcal{C} \rightarrow \mathcal{L}$ .

In a loose aggregation of type max-priority,  $\bigvee_{i=1,\dots,n} f_i \wedge \pi_i$ ,  $\pi_i$  is the maximum possible global score due to the only criterion  $i$ . Indeed, we obtain  $\pi_i$  if  $f_i = 1$  and  $f_j = 0$  if  $j \neq i$ . A criterion is all the more important as it can contribute to a higher global evaluation. A demanding aggregation is of the min-priority form  $\bigwedge_{i=1,\dots,n} f_i \vee (1 - \pi_i)$ , where we consider  $1 - \pi_i$  as the minimum possible global evaluation solely due to criterion  $i$  (we obtain  $1 - \pi_i$  if  $f_i = 0$  and  $f_j = 1$  if  $j \neq i$ ). A criterion is all the more important as it can lead to a lower global evaluation. In this setting, the importance factors act as saturation levels.

We can generalize importance factors from individual criteria to sets thereof by means of a capacity  $\mu : 2^{\mathcal{C}} \rightarrow \mathcal{L}$ :  $\mu(A)$  is the importance of set  $A$ .  $\mu$  is an increasing set function such that  $\mu(\emptyset) = 0$  and  $\mu(\mathcal{C}) = 1$ .

An important class of aggregation functions, used in a qualitative framework is the so-called Sugeno integral:

$$\oint_{\mu}(f) = \bigvee_{A \subseteq \mathcal{C}} \mu(A) \wedge (\bigwedge_{i \in A} f_i). \quad (1)$$

This integral generalizes the prioritized max and min respectively obtained if  $\mu$  is a possibility measure or a necessity measure in (1). We can check that  $\oint_{\mu}(I_A) = \mu(A)$  where  $I_A$  is the characteristic function of  $A$ .

Another viewpoint is to consider that the importance factor  $\pi_i$  acts as follows on the evaluations of an object  $f$ : If  $f_i \geq \pi_i$  then the evaluation becomes 1 and it becomes  $1 - \pi_i$  otherwise. Therefore, in this case the evaluation scale of the criterion  $i$  is reduced to  $\{1 - \pi_i, 1\}$ . If  $f_i$  is greater than  $\pi_i$ , we consider that the criterion  $i$  is satisfied. If  $\pi_i$  is high and  $f_i$  is less than  $\pi_i$ , then the value of  $f_i$  is drastically reduced to  $1 - \pi_i$ . Conversely, if  $\pi_i$  is small, and  $f_i$  is less than  $\pi_i$ , then  $f_i$  is upgraded to  $1 - \pi_i$ . In this case, importance factors correspond to tolerance levels.

In such a context, partial evaluations can be aggregated using the minimum:  $\bigwedge_{i=1,\dots,n} (1 - f_i) \rightarrow (1 - \pi_i)$ . More generally, if the groups of criteria are weighted, we obtain the following integral with respect to a capacity  $\mu$ :

$$\oint_{\mu}^{\uparrow}(f) = \wedge_{A \subseteq \mathcal{C}} (\wedge_{i \in A} (1 - f_i)) \rightarrow \mu(\overline{A}) \quad (2)$$

We can check that  $\oint_{\mu}^{\uparrow}(I_B) = \wedge_{A \subseteq \overline{B}} \mu(\overline{A}) = \mu(B)$ .

### 3.2 Qualitative Desintegrals and Decreasing Scale

In this part, the evaluation scale for each criterion is decreasing, i.e., 0 is better than 1, but the scale for the global evaluation is increasing. In this case the aggregation functions must be decreasing and the capacities are replaced by decreasing set functions  $\nu$  such that  $\nu(\emptyset) = 1$  and  $\nu(\mathcal{C}) = 0$ , called anti-measures.  $\nu(A)$  is the level of tolerance of  $A$ : the greater  $\nu(A)$ , the less important is  $A$ .

A first desintegral is obtained by a saturation effect on a reversed scale:

$$\oint_{\nu}(f) = \vee_{A \subseteq \mathcal{C}} \nu(\overline{A}) \wedge (\wedge_{i \in A} (1 - f_i)) \quad (3)$$

where  $\nu$  is an anti-measure. We can check that  $\oint_{\nu}^{\downarrow}(I_A) = \nu(A)$ . We recognize the definition of Sugeno integral  $\oint_{\nu(\tau)}(1-f)$ . Note that  $\oint_{\nu}^{\downarrow}(f) = 1$  if there exists a non important subset of criteria (because completely tolerant) and the evaluations of  $f$  with respect to the other criteria are equal to 0.

Moreover, we can verify that  $\oint_{\nu}^{\downarrow}(f) = \wedge_{i=1}^n (1 - f_i) \vee t_i$  if the anti-measure  $\nu$  is a guaranteed possibility, i.e.,  $\nu(A) = \wedge_{i \in A} t_i =_{def} \Delta_T(A)$  where  $t_i$  is the tolerance of criterion  $i$  (the greater is  $t_i$  the more tolerant is the criterion  $i$ ). Moreover,  $\oint_{\nu}^{\downarrow}(f) = \vee_{i=1}^n (1 - t_i) \wedge (1 - f_i)$  if  $\nu(A) = \vee_{i \in \overline{A}} (1 - t_i) =_{def} \nabla_t(A) = 1 - \Delta_T(\overline{A})$ . This is the counterpart of the fact that the Sugeno integral gives the max or min with priority when the measure is a possibility measure or a necessity measure.

The other viewpoint is to consider that if  $f_i > t_i$  then the local evaluation is bad and  $f_i$  becomes  $t_i$ . Otherwise the local evaluation is good and  $f_i$  becomes 1. This corresponds to the use of the Gödel implication and the global evaluation  $\wedge_{i=1, \dots, n} f_i \rightarrow t_i$  which is generalized by the following desintegral:

$$\oint_{\nu}^{\downarrow}(f) = \wedge_{A \subseteq \mathcal{C}} (\wedge_{i \in A} f_i) \rightarrow \nu(A) \quad (4)$$

where  $\nu$  is an anti-measure (with the convention  $\wedge_{i \in \emptyset} f_i = 0$ ). We can check that  $\oint_{\nu}^{\downarrow}(I_A) = \nu(A)$ . Note that  $\oint_{\nu}^{\downarrow}(f) = 1$  if for each subset of criteria, at least one criterion has an evaluation lower than the tolerance of this subset.

In the following the values of the function  $f$  are in  $\{0, 1\}$  and the values of the set functions are in  $[0, 1]$ . In this context, the criteria are binary and  $f_i = 1$  (resp.  $f_i = 0$ ) can be encoded as a proposition  $f_i$  (resp.  $\neg f_i$ ) expressing that the criterion  $i$  is satisfied (resp. not satisfied).

### 3.3 Relation between Integrals and Desintegrals

If the criteria are represented on a binary scale then there exist some links between integrals and desintegrals.

**Proposition 1.** *If the values of the functions  $f$  are in  $\{0, 1\}$  then  $f$  is a characteristic function  $I_A$  and we can check that*

- for any capacity  $\mu$ ,  $\oint_{\mu}(I_A) = \oint_{\mu}^{\uparrow}(I_A) = \mu(A)$ ;
- for any anti-measure  $\nu$ ,  $\oint_{\nu}(I_A) = \oint_{\nu}^{\downarrow}(I_A) = \nu(A)$ .

These relations are not true in the general case because if  $f$  is such that  $f_i < 1, \forall i$ , then  $\oint_{\mu}^{\uparrow}(f) = 0$  since  $\bigwedge_{i \in \mathcal{C}} (1 - f_i) \rightarrow \mu(\overline{\mathcal{C}}) = \alpha \rightarrow 0 = 0$  with  $\alpha > 0$ . But generally we have  $\oint_{\mu}(f) \neq 0$ . Similarly, if  $f_i > 0, \forall i$ ,  $\bigwedge_{i \in \mathcal{C}} f_i \rightarrow \nu(\mathcal{C}) = \beta \rightarrow 0 = 0$  with  $\beta > 0$ . Hence  $\oint_{\nu}^{\downarrow}(f) = 0$ , but generally  $\oint_{\nu}(f) \neq 0$ .

**Example 1.** *We consider two criteria denoted by  $a$  and  $b$ , the function  $f$  is such that  $f(a) = 0.5$  and  $f(b) = 0.6$ .*

*If  $\mu$  is the capacity  $\mu(a) = 0.4$ ,  $\mu(b) = 0.5$  then  $\oint_{\mu}(f) = \vee(0.5, 0.6 \wedge 0.5) = 0.5$  and  $\oint_{\mu}^{\uparrow}(f) = \wedge(0.5 \rightarrow 0.5, 0.4 \rightarrow 0.4, 0.4 \rightarrow 0) = 0$ .*

*If  $\nu$  is an anti-measure  $\nu(a) = \nu(b) = 1$  then  $\oint_{\nu}(f) = \vee(0.5, 0.4, 0.4) = 0.5$  and  $\oint_{\nu}^{\downarrow}(f) = \wedge(0.5 \rightarrow 1, 0.6 \rightarrow 1, 0.5 \rightarrow 0) = 0$ .*

**Proposition 2.** *If  $\nu$  is an anti-measure, then there exists a capacity  $\mu$  such that for all  $f$  (Boolean or not)  $1 - \oint_{\mu}(f) = \oint_{\nu}^{\downarrow}(f)$  and conversely.*

**Proof.**  $1 - \oint_{\nu}(f) = 1 - \vee_{A \subseteq \mathcal{C}} \nu(\overline{A}) \wedge \bigwedge_{i \in A} (1 - f_i) = \bigwedge_{A \subseteq \mathcal{C}} (1 - \nu(\overline{A})) \vee \bigvee_{i \in A} f_i = \vee_{A \subseteq \mathcal{C}} 1 - \nu(A) \wedge \bigwedge_{i \in A} f_i = \oint_{1-\nu}^{\uparrow}(f)$  because of a Sugeno integral property [12].

So, Sugeno integral is the complement to 1 of its dual desintegral by replacing  $\mu$  by  $\nu = 1 - \mu$ . This result is general for any function  $f$ . In the Boolean case, the two propositions are summarized in

$$\oint_{\mu}(I_A) = \oint_{\mu}^{\uparrow}(I_A) = \mu(A) = 1 - \oint_{1-\mu}(I_A) = 1 - \oint_{1-\mu}^{\downarrow}(I_A).$$

The relation  $\oint_{\mu}^{\uparrow}(f) = 1 - \oint_{1-\mu}^{\downarrow}(f)$  is not true in the general case:

**Example 2.** *Let  $\mathcal{C} = \{a, b\}$ ,  $f(a) = 1$ ,  $f(b) = 0.8$ ,  $\mu(a) = 0$   $\mu(b) = 0.5$ .*

*$1 - \oint_{1-\mu}^{\downarrow}(f) = \vee_{A \subseteq \mathcal{C}} (1 - ((\bigwedge_{i \in A} f_i) \rightarrow (1 - \mu(A)))) = \max(0, 0.5, 1) = 1$ , and  $\oint_{\mu}^{\uparrow}(f) = \bigwedge_{A \subseteq \mathcal{C}} (\bigwedge_{i \in A} (1 - f_i)) \rightarrow \mu(\overline{A}) = \min(0 \rightarrow 0.5, 0.2 \rightarrow 0, 0 \rightarrow 0) = \min(1, 0, 1) = 0$ .*

## 4 Qualitative Integrals as Possibilistic Bases

This section recalls the method presented in [8] for interpreting a Sugeno integral as a possibilistic base when  $f$  is a Boolean function. Next we will study the case of a desintegral. More precisely, this section presents logical representations for capacities and anti-measures.

## 4.1 Logical Framework for Sugeno Integral in a Boolean Context

The Sugeno integral  $\oint_{\mu}$  is used to classify objects  $f$  according to their evaluation  $\oint_{\mu}(f)$ . A possibilistic logic framework can be constructed as follows.

Looking at each criterion  $i$  as a predicate  $P_i$ ,  $P_i(f_i)$  indicates that the evaluation with respect to the criterion  $i$  is  $f_i \in L$ . Hence the object  $f = (f_1, \dots, f_n)$  is represented by the logic Boolean formula  $P_1(f_1) \wedge \dots \wedge P_n(f_n)$ . Boolean criteria are assumed so we can simplify  $P_i(1)$  into  $f_i$  and  $P_i(0)$  into  $\neg f_i$  according to whether criterion  $i$  is satisfied or not. If  $f = I_A$ , then  $P_1(f_1) \wedge \dots \wedge P_n(f_n) = \bigwedge_{i \in A} f_i \wedge \bigwedge_{i \notin A} \neg f_i$ . So any object is encoded as an interpretation of the language induced by the variables associated to the criteria.

In this context, a logical formula corresponds to a set of objects that must satisfy (or not) some criteria. The global evaluation  $\oint_{\mu}(I_A)$  can be seen as a degree of guaranteed possibility of  $\bigwedge_{i \in A} f_i \wedge \bigwedge_{i \notin A} \neg f_i$  (view from below) or as a degree of standard possibility (view from above).

**Example 3.** *We consider three criteria or properties  $a, b, c$ . Some objects are evaluated with respect to these criteria. The evaluation scale for each criterion is  $\{0, 1\}$ , an object  $f$  is represented by the characteristic function of the subset of criteria (or properties) that it satisfies.*

*Capacities (and anti-measures) are valued in  $\mathcal{L} = \{1, 1 - \beta, \beta, 0\}$  with  $1 > 1 - \beta > \beta > 0$ .*

*We consider the capacity  $\mu$  defined by:  $\mu(a) = \mu(b) = \beta$ ,  $\mu(c) = 1 - \beta$ ,  $\mu(\{a, b\}) = \beta$ ,  $\mu(\{a, c\}) = 1$ ,  $\mu(\{b, c\}) = 1 - \beta$ ,  $\mu(\{a, b, c\}) = 1$ .*

*In this context if we consider  $f = I_{\{a, b\}}$  then  $\oint_{\mu}(f) = \mu(\{a, b\})$ . As the capacity  $\mu$  is increasing, if the properties  $a$  and  $b$  are satisfied by an object  $g$  then  $\oint_{\mu}(g) \geq \mu(\{a, b\})$ . This inequality corresponds to a  $\Delta$  possibilistic base on language generated by  $\{a, b, c\}$ , as we shall see.*

## 4.2 Construction of a Possibilistic Base Associated to a Capacity

This section presents the possibilistic base associated to a Sugeno integral in the particular case of binary values i.e, a capacity. The general case was presented in [8]. The following property characterizes the set of objects  $f$  solutions of the inequality  $\oint_{\mu}(f) \geq \gamma$ ,  $\gamma \in \mathcal{L}$  when the criteria are not Boolean.

**Proposition 3.**  $\{f \mid \oint_{\mu}(f) \geq \gamma\} = \{f \mid \exists A \text{ s.t. } \mu(A) \geq \gamma \text{ and } \forall i \in A, f_i \geq \gamma\}$ .

**Proof.**  $\oint_{\mu}(f) = \bigvee_{A \subseteq C} \mu(A) \wedge \bigwedge_{i \in A} f_i \geq \gamma$  iff  $\exists A, \mu(A) \wedge \bigwedge_{i \in A} f_i \geq \gamma$  i.e.,  $\exists A$  such that  $\mu(A) \geq \gamma$  and  $\forall i \in A, f_i \geq \gamma$ .

Particularly, the non-trivial case is  $\gamma > 0$ , so if the evaluation scale is Boolean we must suppose  $f_i = 1$  in the previous proposition and therefore  $\{f \mid \oint_{\mu}(f) \geq \gamma\} = \{I_B \mid \exists A \text{ s.t. } \mu(A) \geq \gamma \text{ and } A \subseteq B\}$ . The monotony of  $\mu$  and the fact that the set  $\mathcal{F}_{\mu}^{\gamma} = \{A, \mu(A) \geq \gamma > 0\}$  is closed under inclusion entails that  $\mathcal{F}_{\mu}^{\gamma}$  has least elements  $A_k^{\gamma}, k = 1, \dots, p^{\gamma}$  such that  $\mu(A) \geq \gamma \iff \exists k, A_k^{\gamma} \subseteq A$ . In logical terms, the constraint  $\mu(A) \geq \gamma$  can be represented with the base

$$B_{\gamma}^{\Delta} = \{[\bigwedge_{i \in A_k^{\gamma}} f_i, \gamma], k = 1, \dots, p^{\gamma}\}.$$

Hence it is obvious that  $\mathcal{F}_\mu(f) \geq \gamma$ , with  $f = I_A$  if and only if  $\bigwedge_{i \in A} f_i \wedge \bigwedge_{i \notin A} \neg f_i \models \bigvee_{k=1, \dots, p} \bigwedge_{i \in A} f_i$  which can be written with a  $\Delta$  possibilistic base  $B_\gamma^\Delta \vdash [f, \gamma]$ . So the capacity  $\mu$  can be represented by a  $\Delta$  possibilistic base:  $B^\Delta = \bigcup_{\gamma \in \mu(2^{\mathcal{C}})} B_\gamma^\Delta$ .

**Example 4.** We consider the capacity  $\mu$  of the previous example. Let us lay bare:

- objects  $f$  such that  $\mathcal{F}_\mu(f) \geq \beta$ : we find  $\{A \subseteq \mathcal{C} | \mu(A) \geq \beta\} = 2^{\mathcal{C}} \setminus \{\emptyset\}$ . So  $f = I_A$  with  $a \in A$ , or  $b \in A$  or  $c \in A$ . These three vectors correspond to the formulas  $[a, \beta]$ ,  $[b, \beta]$  and  $[c, \beta]$  respectively.
- objects  $f$  such that  $\mathcal{F}_\mu(f) \geq 1 - \beta$ . We find  $\{A \subseteq \mathcal{C} | \mu(A) \geq 1 - \beta\} = \{\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} = \{A, c \in A\}$ . So  $f = I_A$  with  $c \in A$ . which corresponds to the formula  $[c, 1 - \beta]$ .
- objects  $f$  such that  $\mathcal{F}_\mu(f) \geq 1$ . We find  $\{A \subseteq \mathcal{P} | \mu(A) \geq 1\} = \{\{a, c\}, \{a, b, c\}\}$ . So  $f = I_A$  with  $\{a, c\} \subseteq A$  which corresponds to the formula  $[a \wedge c, 1]$ .
- According to the definition all objects  $f$  are such that  $\mathcal{F}_\mu(f) \geq 0$ . We have  $\{A \subseteq \mathcal{C} | \mu(A) \geq 0\} = 2^{\mathcal{C}} = \{A : \emptyset \subseteq A\}$ , which entails no  $\Delta$ -formula because  $[\perp, 0]$  is a tautology.

The associated possibilistic  $\Delta$  base is  $K_\mu^\Delta = \{[a, \beta], [b, \beta], [c, 1 - \beta], [a \wedge c, 1]\}$ .

### 4.3 Qualitative Desintegrals as Possibilistic Bases

In this part we consider  $\nu$  an anti-measure. Similarly as the positive case, each object  $f = (f_1, \dots, f_n)$  represents a logical interpretation  $f_1^{\epsilon_1} \wedge \dots \wedge f_n^{\epsilon_n}$  where  $f_i^{\epsilon_i} \in \{f_i, \neg f_i\}$ ; so  $\mathcal{F}_\nu^\downarrow(f) = \mathcal{F}_\nu(f) = \nu(A)$  with  $f = I_A$ . But in this context it is  $1 - \mathcal{F}_\nu$  which is viewed as the possibilistic degree corresponding to the formula. So the inequality  $1 - \mathcal{F}_\nu \leq \gamma$ , i.e.  $\mathcal{F}_\nu^\downarrow \geq 1 - \gamma$ , is linked with a  $N$  possibilistic base; and the inequality  $\mathcal{F}_\nu \leq 1 - \gamma$ , is associated to a  $\Delta$  possibilistic base. We are going to use the relationship between the desintegral  $\mathcal{F}_\nu^\downarrow$  and Sugeno integral to come back to the  $\Delta$  possibilistic base of a Sugeno integral. In practice in the Boolean case,  $\mathcal{F}_\nu^\downarrow(f) \leq 1 - \gamma$  means  $\nu(A) \leq 1 - \gamma$  i.e.  $\mu(A) = 1 - \nu(A) \geq \gamma$ . Formally we come back to the previous case, but the meaning is totally different. Indeed  $f_i = 1$  means that the object is bad for the criterion  $i$  (it is a defect) and  $\nu(A)$  shows how  $f = I_A$  is good. So  $\mu(A) = 1 - \nu(A)$  measures how much the object  $f$  is bad with respect to the evaluations of its defects. We evaluate the unattractiveness of an object with respect to its defects, in the same way as we evaluate its attractiveness with a Sugeno integral.

**Example 5.** Let us come back to the example 2 but with an anti-measure. For example we consider  $\nu = 1 - \mu$  where  $\mu$  is the capacity of the example 2:

$$\nu(a) = \nu(b) = 1 - \beta, \nu(c) = \beta, \nu(\{a, b\}) = 1 - \beta, \nu(\{a, c\}) = 0, \nu(\{b, c\}) = \beta, \nu(\{a, b, c\}) = 0.$$

In such a context, if we consider  $f = I_{\{a, b\}}$ ,  $\mathcal{F}_\nu^\downarrow(f) = \nu(\{a, b\})$ . As the desintegral is decreasing, if the properties  $a$  and  $b$  are satisfied for an object  $g$  we have  $1 - \mathcal{F}_\nu^\downarrow(g) \geq 1 - \nu(\{a, b\})$ . This inequality corresponds to a  $\Delta$  possibilistic base.

We use the proposition 2 :  $1 - \mathfrak{f}_\nu^{\downarrow} = \mathfrak{f}_{1-\nu}$ . As  $\nu$  is defined by  $1 - \mu$  we recover  $\mathfrak{f}_\mu$  and the possibilistic base associated to the Sugeno integral of example 3.

**Example 6.** We consider the anti-measure  $\nu$  defined by  $\nu(a) = 1 - \beta$ ,  $\nu(b) = 1$ ,  $\nu(c) = 1 - \beta$ ,  $\nu(\{a, b\}) = 1 - \beta$ ,  $\nu(\{a, c\}) = 1 - \beta$ ,  $\nu(\{b, c\}) = \beta$ ,  $\nu(\{a, b, c\}) = 0$ .

We have  $1 - \mathfrak{f}_\nu^{\downarrow} = \mathfrak{f}_{\mu'}$ , with  $\mu'(a) = \beta$ ,  $\mu'(b) = 0$ ,  $\mu'(c) = \beta$ ,  $\mu'(\{a, b\}) = \beta$ ,  $\mu'(\{a, c\}) = \beta$ ,  $\mu'(\{b, c\}) = 1 - \beta$ ,  $\mu'(\{a, b, c\}) = 1$  and define the following sets:

$\{A | \mu'(A) \geq \beta\} = \{\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$  which corresponds to the  $\Delta$ -formulas  $[a, \beta]$  and  $[c, \beta]$ .

$\{A | \mu'(A) \geq 1 - \beta\} = \{\{b, c\}, \{a, b, c\}\}$  which corresponds to the  $\Delta$ -formula  $[b \wedge c, 1 - \beta]$ .

$\{A | \mu'(A) \geq 1\} = \{\{a, b, c\}\}$  which corresponds to the  $\Delta$ -formulas  $[a \wedge b \wedge c, 1]$ .

So the  $\Delta$  possibilistic base associated to  $\mathfrak{f}_\nu^{\downarrow}$  is  $K_\nu^\Delta = \{[a, \beta], [c, \beta], [b \wedge c, 1 - \beta], [a \wedge b \wedge c, 1]\}$ .

	a	b	c	d	$\mathfrak{f}_\nu^{\downarrow}(I_S)$	$\mathfrak{f}_\mu(I_S)$
1	1	1	1	1	$\theta$	$\eta$
2	1	1	1	0	$\theta$	0
3	1	1	0	1	$\theta$	$\eta$
4	1	1	0	0	$\theta$	0
5	1	0	1	1	$\lambda$	$\eta$
6	1	0	1	0	$\lambda$	0
7	1	0	0	1	1	$\eta$
8	1	0	0	0	1	0
9	0	1	1	1	$\lambda$	$\rho$
10	0	1	1	0	$\lambda$	0
11	0	1	0	1	1	$\rho$
12	0	1	0	0	1	0
13	0	0	1	1	$\lambda$	$\rho$
14	0	0	1	0	$\lambda$	0
15	0	0	0	1	1	$\rho$
16	0	0	0	0	1	0

**Fig. 2.** Values of the desintegral and the integral for the different subsets of possible satisfied properties for an object

## 5 Towards the Description of Acceptable Objects

A previous work [13] has already proposed to use a pair of evaluations made of a Sugeno integral and a Sugeno integral reversed by complementation to 1 (which corresponds to a desintegral), in order to describe acceptable objects in terms of properties they must have and of properties that they must avoid. In that work, all properties were assumed to be partitioned between these two categories for a given object. In the general case, as suggested by the example presented in

the introduction, the fact that a property is desirable or undesirable depends on the context of other satisfied properties. In this example the property  $a$  is undesirable if  $b$  is satisfied but the property  $d$ , which is good by itself, is desirable if the property  $a$  is satisfied. Let us now encode the example of the introduction in terms of an integral and a desintegral.

First let us consider the positive aspects. An object is satisfactory to degree  $\rho$  (at least) if the property  $d$  is satisfied, and to degree  $\eta > \rho$  (at least) if the properties  $a$  and  $d$  are satisfied. So we consider the fuzzy measure<sup>1</sup>  $\mu$  such that  $\mu(\{d\}) = \rho$  and  $\mu(\{a, d\}) = \eta$ . Moreover,  $\forall A \neq \mathcal{C}, A \supseteq \{a, d\}, \mu(A) = \eta$  and  $\forall A \supseteq \{d\}, A \not\supseteq \{a\}, \mu(A) = \rho$  and  $\mu(B) = 0$  otherwise. The value of the integral  $\oint_{\mu}(I_S)$  is given in Figure 2 for the different possible subsets of properties satisfied by an object. The larger  $\oint_{\mu}(I_S)$ , the more satisfactory the object characterized by  $S$ . Now let us consider the negative aspects. The object is only satisfactory to a degree at most  $\lambda$  if the property  $c$  is satisfied, and satisfactory to a degree at most  $\theta < \lambda$  if the properties  $a$  and  $b$  are satisfied. This leads us to consider an anti-measure<sup>2</sup>  $\nu$  such that  $\nu(\{c\}) = \lambda$  and  $\nu(\{a, b\}) = \theta$ . Moreover,  $\forall A \neq \emptyset, A \subseteq \{a, b\}, \nu(A) = \theta$ , and  $\nu(B) = 0$  otherwise. The value of the desintegral  $\oint_{\nu}^{\downarrow}(I_S)$  is given in Figure 2 for the different possible subsets of properties satisfied by an object. The less  $\oint_{\nu}^{\downarrow}(I_S)$ , the less satisfactory the object characterized by  $S$ . An acceptable object needs to be fully satisfactory w.r.t. its negative aspects, i.e.  $\oint_{\nu}^{\downarrow}(I_S) = 1$ , which means that the object should have no potential defect, and to be as satisfactory as possible w.r.t. its positive aspects, i.e. with  $\oint_{\mu}(I_S)$  maximal. In our example, we can verify in Figure 2 that the objects that satisfy  $a$  and  $d$ , but not  $b$  or  $c$  are the most acceptable, and that the objects which satisfy  $b$  and  $d$  and not  $a$  nor  $c$  or  $d$  and not  $a$  nor  $b$  nor  $c$  are slightly less acceptable. There is no other acceptable objects. The values of  $1 - \oint_{\nu}^{\downarrow}(I_S)$  indicate how much an object should be rejected. Note that some objects (in this example, there are 5 cases (lines 1, 3, 5, 9 and 13 of the table in Figure 2) are both in some respects satisfactory and in other respects unsatisfactory. It comes close to the problem of choosing objects described by binary properties from the pros and cons of arguments [5].

## 6 Conclusion

We have given the definitions and some properties of qualitative integrals and desintegrals which extend the classical Sugeno integral. We have also studied the particular case of binary properties and we have proposed a logical possibilistic view for these aggregation functions. The general case of gradual properties remains to be studied. Generally speaking, we may think of a level-cut based approach, and we may also take lessons of the logical approach to qualitative decision under uncertainty [6] where two logical bases are used for preferences and knowledge, making a classical parallel between multiple criteria decision and

<sup>1</sup> We give the smallest one.

<sup>2</sup> We give the greatest one.

decision under uncertainty. A first attempt at providing a logical representation in the general case has been presented in [8] for Sugeno integrals only, and another type of representation of positive and negative synergy between properties, which is not bipolar (for example you want an object which satisfies  $a$  or  $b$ , but not  $a$  and  $b$ ) has been also indicated in [8]. The relation between the two types of representation is a topic for further research.

## References

1. Benferhat, S., Dubois, D., Kaci, S., Prade, H.: Modeling positive and negative information in possibility theory. *Int. J. of Intellig. Syst.* 23(10), 1094–1118 (2008)
2. Benferhat, S., Kaci, S.: Logical representation and fusion of prioritized information based on guaranteed possibility measures: Application to the distance-based merging of classical bases. *Artificial Intelligence* 148, 291–333 (2003)
3. de Cooman, G., de Baets, B.: Implicator and coimplicator integrals. In: *Proc. 6th Inter. Conf. on Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 1996)*, Granada, July 1-5, vol. III, pp. 1433–1438 (1996)
4. Domshlak, C., Hüllermeier, E., Kaci, S., Prade, H.: Preferences in AI: An overview. *Artif. Intell.* 175(7-8), 1037–1052 (2011)
5. Dubois, D., Fargier, H., Bonnefon, J.F.: On the comparison of decisions having positive and negative features. *J. of Artif. Intellig. Res.* 32, 385–417 (2008)
6. Dubois, D., Le Berre, D., Prade, H., Sabbadin, R.: Using possibilistic logic for modeling qualitative decision: ATMS-based algorithms. *Fund. Inform.* 37, 1–30 (1999)
7. Dubois, D., Prade, H.: Possibilistic logic: a retrospective and prospective view. *Fuzzy Sets and Systems* 144, 3–23 (2004)
8. Dubois, D., Prade, H., Rico, A.: A Possibilistic Logic View of Sugeno Integrals. In: Melo-Pinto, P., Couto, P., Serôdio, C., Fodor, J., De Baets, B. (eds.) *Eurofuse 2011. AISC*, vol. 107, pp. 19–30. Springer, Heidelberg (2011)
9. Dubois, D., Prade, H., Rico, A.: Qualitative Integrals and Desintegrals: How to Handle Positive and Negative Scales in Evaluation. In: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (eds.) *IPMU 2012, Part III. CCIS*, vol. 299, pp. 306–316. Springer, Heidelberg (2012)
10. Grabisch, M., Labreuche, C.: A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *Annals of Oper. Res.* 175, 247–286 (2010)
11. Hadjali, A., Kaci, S., Prade, H.: Database preference queries - A possibilistic logic approach with symbolic priorities. *Ann. Math. in AI* (2012)
12. Marichal, J.-L.: *Aggregation Operations for Multicriteria Decision Aid*. Ph.D.Thesis, University of Liège, Belgium (1998)
13. Prade, H., Rico, A.: Describing acceptable objects by means of Sugeno integrals. In: *Proc. 2nd Inter. IEEE Conf. on Soft Computing and Pattern Recognition (SoCPaR 2010)*, Cergy-Pontoise, pp. 6–11 (December 2010)