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► **To cite this version:**

Patrick Cattiaux, Laetitia Colombani, Manon Costa. LARGE DEVIATION PRINCIPLES FOR CUMULATIVE PROCESSES AND APPLICATIONS. 2021. hal-03344425v1

HAL Id: hal-03344425

<https://hal.science/hal-03344425v1>

Preprint submitted on 15 Sep 2021 (v1), last revised 13 Apr 2023 (v2)

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LARGE DEVIATION PRINCIPLES FOR CUMULATIVE PROCESSES AND APPLICATIONS

PATRICK CATTIAUX , LAETITIA COLOMBANI , AND MANON COSTA

UNIVERSITÉ DE TOULOUSE

ABSTRACT. The aim of this paper is to prove a Large Deviation Principle (LDP) for cumulative processes also known as compound renewal processes. These processes cumulate independent random variables occurring in time interval given by a renewal process. Our result extends the one obtained in [13] in the sense that we impose no specific dependency between the cumulated random variables and the renewal process. The proof is inspired from [13] but deals with additional difficulties due to the general framework that is considered here. In the companion paper [6] we apply this principle to Hawkes processes with inhibition. Under some assumptions Hawkes processes are indeed cumulative processes, but they do not enter the framework of [13].

Key words : Cumulative processes, large deviation, deviation inequalities, Hawkes processes

MSC 2010 : 60F10, 60K15 .

1. INTRODUCTION.

1.1. Cumulative processes. Cumulative processes have been introduced by Smith [14] and are applied in many purposes, such as finance where they are called *compound-renewal* processes or *compound Markov renewal* processes. Indeed these continuous time processes cumulate independent random variables occurring in time interval given by a renewal process. To be more specific a real valued process $(Z_t)_{t \geq 0}$ is called *a cumulative process* if the following properties are satisfied:

- (1) $Z_0 = 0$,
- (2) there exists a renewal process $(S_n)_n$ such that for any n , $(Z_{S_n+t} - Z_{S_n})_{t \geq 0}$ is independent of S_0, \dots, S_n and $(Z_s)_{s < S_n}$,
- (3) the distribution of $(Z_{S_n+t} - Z_{S_n})_{t \geq 0}$ is independent of n .

To study such processes, we write for all $t \geq 0$

$$Z_t = W_0(t) + W_1 + \dots + W_{M_t} + r_t$$

where $W_0(t) = Z_{t \wedge S_0}$, $W_n = Z_{S_n} - Z_{S_{n-1}}$, $r_t = Z_t - Z_{M_t}$, where M_t is defined by

$$M_t = \sup \{n \geq 0, S_n \leq t\}.$$

The $(W_k)_{k \geq 1}$'s are thus i.i.d.

We denote by $(\tau_i)_i$ the times associated to the renewal process $\tau_n = S_n - S_{n-1}$ and $\tau_0 = 0$.

Date: September 15, 2021.

It is worth noticing that τ_i and W_i can be dependent.

In the sequel we suppress the subscript i when dealing with the distribution (and all associated quantities like expectation, variance ...) of (τ_i, W_i) and simply use (τ, W) .

A simple example of cumulative process is $Z_t = \int_0^t f(X_s)ds$ where $(X_t)_t$ is a regenerative process with i.i.d. cycles [11]. Markov additive processes are other classical examples of cumulative process.

In [7] the authors exhibited a renewal structure for some Hawkes processes. This description is extensively used in our companion paper [6] in order to describe such processes as cumulative processes, and to study their asymptotic behaviour.

For cumulative processes, the law of large numbers (assuming that $\mathbb{E}[|W|]$ and $\mathbb{E}[\tau]$ are not infinite)

$$\frac{Z_t}{t} \xrightarrow[t \rightarrow \infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \text{ if and only if } \mathbb{E} \left(\max_{S_0 \leq t < S_1} |r_t| \right) < \infty,$$

and the central limit theorem (assuming $\text{Var}(W) < \infty$ and $\text{Var}(\tau) < \infty$)

$$\frac{\left(Z_t - t \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \right)}{\sqrt{t}} \xrightarrow[t \rightarrow \infty]{} \mathcal{N} \left(0, \sigma^2 \right) \text{ where } \sigma^2 = \frac{1}{\mathbb{E}(\tau)} \text{Var} \left(W - \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \tau \right)$$

can be found in Asmussen [1], theorem 3.1 and theorem 3.2.

Brown and Ross [5] have proved equivalent of the Blackwell theorem and of the key renewal theorem for a subclass of cumulative processes, since cumulative processes are a generalization of renewal processes. Glynn and Whitt have focused in [11] on cumulative processes associated to a regenerative process and have proved law of large numbers (strong and weak), law of the iterated logarithm, central limit theorem and functional generalizations of these properties.

The aim of this work is to establish a large deviation principle (LDP) for cumulative processes. Some works have already been done. Duffy and Metcalfe [10] have considered the estimation of a rate function for a cumulative process (if it admits a LDP). In a series of papers, Borovkov and Mogulskii ([2], [3], [4]) have studied the (LDP) (they use the term compound-renewal theorem), under some assumptions of comparison between the values (θ_1, θ_2) for which $\mathbb{E} \left(e^{\theta_1 \tau + \theta_2 W} \right)$ is finite and the value θ for which $\mathbb{E} \left(e^{\theta \tau} \right)$ is finite (a comparison between the joint distribution of (τ, W) and the law of τ). Actually, some points in their approach are not clear for us.

Lefevre, Mariani and Zambotti [13] worked on specific cumulative process where $W_i = F(\tau_i)$ for some deterministic function F which is assumed to be non-negative, bounded and continuous. We choose to follow their approach. Our proof of the LDP has the same skeleton, but in a general framework for the pair (τ, W) .

In this paper, we show a LDP for Z_t/t in the case $r_t = 0$. This assumption can be relaxed if r_t/t tends to 0 quickly enough, as it will be the case for the application to Hawkes process (see [6]). For example, if for all $\delta > 0$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\frac{|r_t|}{t} > \delta \right) = -\infty,$$

then Z_t/t and $(Z_t - r_t)/t$ are exponentially equivalent and satisfy the same LDP (if $(Z_t - r_t)/t$ have one) and have the same rate function.

1.2. Hawkes processes. A Hawkes process is a point process on the real line \mathbb{R} characterized by its intensity process $t \mapsto \Lambda(t)$. We consider an appropriate filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual assumptions.

Definition 1.1. Let $\lambda > 0$ and $h : (0, +\infty) \rightarrow \mathbb{R}$ a signed measurable function. Let N^0 a locally finite point process on $(-\infty, 0]$ with law \mathbf{m} .

The point process N^h on \mathbb{R} is a Hawkes process on $(0, +\infty)$, with initial condition N^0 and reproduction measure $\mu(dt) = h(t)dt$ if:

- $N^h|_{(-\infty, 0]} = N^0$,
- the conditional intensity measure of $N^h|_{(0, +\infty)}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$ is absolutely continuous w.r.t the Lebesgue measure and has density:

$$\Lambda^h : t \in (0, +\infty) \mapsto f \left(\lambda + \int_{(-\infty, t)} h(t-u) N^h(du) \right). \quad (1.1)$$

for some non-negative function f .

Hawkes processes have been introduced by Hawkes [12]. Most of the literature concerned with the large time behaviour of $N_t^h = N^h([0, t])$ is dedicated to the case $h \geq 0$ (self excitation). This behaviour is studied in details in [6] when h is a signed (the negative part modelling self inhibition) compactly supported function, and the function f (called the jump rate function) is given by

$$f(u) = \max(0, u).$$

In this situation one gets a description of N_t^h as a cumulative process (see [6] subsection 2.3) with few information on the joint law of (τ, W) . This was the initial motivation for the present work. We refer to [6] for a more complete overview and explicit results in this situation. In the remaining part of the paper we will come back to more general cumulative processes.

2. NOTATIONS AND MAIN RESULT

2.1. First notations.

We consider $(\tau_1, W_1), (\tau_2, W_2), \dots$ an i.i.d. sequence of pairs of random variables built on some probability space (Ω, \mathbb{P}) . The law of (τ_i, W_i) is an arbitrary probability ψ on $(0, +\infty) \times \mathbb{R}$. We denote this by: $(\tau_i, W_i) \sim \psi$. In the sequel we generically use the notation (τ, W) for a pair with the same distribution as (τ_i, W_i) . Notice that we assume in particular that $\mathbb{E}(\tau) > 0$.

We denote by $\mathcal{M}^1(\mathcal{X})$ the space of probability measure on some measurable space \mathcal{X} .

We consider the renewal process associated with $(\tau_i)_{i \geq 1}$:

$$S_0 = 0, \quad S_n = \sum_{k=1}^n \tau_k,$$

$$M_t = \sup \{n \geq 0, S_n \leq t\}.$$

We will study the quantity:

$$Z_t = \sum_{i=1}^{M_t} W_i. \quad (2.1)$$

The main goal of this paper is to prove a Large Deviation Principle for the process Z_t/t . Let us recall some basic definitions in large deviation theory (we refer to [9]).

A family of probability measures $(\eta_t)_{t \in \mathbb{R}^+}$ on a topological space $(\mathcal{X}, T_{\mathcal{X}})$ satisfies the Large Deviations Principle (LDP) with rate function $J(\cdot)$ and speed $\gamma(t) = t$ if J is lower semi-continuous from \mathcal{X} to $[0, +\infty]$, and the following holds

$$-\inf_{x \in \mathcal{O}} J(x) \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \eta_t(\mathcal{O}) \quad \text{for all open subset } \mathcal{O}, \quad (2.2)$$

and

$$-\inf_{x \in \mathcal{C}} J(x) \geq \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \eta_t(\mathcal{C}) \quad \text{for all closed subset } \mathcal{C}. \quad (2.3)$$

We shall sometimes say that $(\eta_t)_{t \in \mathbb{R}^+}$ satisfies the *full* LDP when (2.2) and (2.3) are satisfied, while we will use *weak* LDP when \mathcal{C} closed is replaced by \mathcal{C} compact in (2.3). When η_t is the distribution of some random variable Y_t (for instance Z_t/t) we shall say that the family $(Y_t)_t$ satisfies a LDP.

Since J is lower semi-continuous the level sets $\{x, J(x) \leq a\}$ are closed. If in addition they are compact, then J is said to be a good rate function.

In this paper we only consider the speed function $\gamma(t) = t$ so that we will no more refer to it.

A particularly important notion for our purpose is the notion of *exponentially good approximation*.

Definition 2.1. Assume that (\mathcal{X}, d) is a metric space. A family of random variables $Y_{n,t}$ ($n \in \mathbb{N}$) is an exponentially good approximation of Y_t (all these variables being defined on the same probability space (Ω, \mathbb{P})), if for all $\delta > 0$ it holds

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}(d(Y_{n,t}, Y_t) > \delta) = -\infty.$$

The key result is then

Theorem 2.2. *In the framework of definition 2.1, assume that $Y_{n,t}$ is an exponentially good approximation of Y_t . Then the following statements hold true.*

- (1) *If $Y_{n,t}$ satisfies a full LDP with rate function J^n then Y_t satisfies a weak LDP with rate function*

$$J(x) = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{d(y,x) < \delta} J^n(y).$$

- (2) *If $\mathcal{X} = \mathbb{R}^k$ equipped with any norm, then the same conclusion is true when $Y_{n,t}$ satisfies only a weak LDP.*

- (3) *If J (defined above) is a good rate function such that for any closed set F ,*

$$\inf_{y \in F} J(y) \leq \limsup_{n \rightarrow \infty} \inf_{y \in F} J^n(y)$$

then Y_t satisfies a full LDP with rate function J .

The first and last points in the previous Theorem are contained in [9] Theorem 4.2.16. The second one is a consequence of the fact that closed balls are compact sets in \mathbb{R}^k . Usually, the Theorem is sufficient to prove a full LDP. Nevertheless, in some cases, the study of the rate function J is difficult. The lemma below gives an alternative, using exponential tightness which is easy to obtain with our assumptions.

Lemma 2.3. *If Y_t satisfies a weak LDP with a rate function I and is exponentially tight, i.e. for all $\alpha > 0$, there exists a compact set K_α such that*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(Y_t \notin K_\alpha^c) < -\alpha,$$

then Y_t satisfies a full LDP and I is a good rate function.

This Lemma is a consequence of the Lemma 1.2.18 in [9].

Theorem 2.2 as well as Lemma 2.3 will be used at the end of the article, in the section 9 which proves the main Theorem 2.5.

2.2. Main results.

Assumption 2.4. We will make the following set of assumptions:

- i) $\exists \theta_0 \in (0, +\infty]$ such that $\mathbb{E}[e^{\theta\tau}] < \infty$ for $\theta < \theta_0$,
- ii) $\exists \beta_0 \in (0, +\infty]$ such that $\mathbb{E}[e^{\beta|W|}] < \infty$, for $\beta < \beta_0$,
- iii) for all interval \mathcal{I} such that $\mathbb{P}(W \in \mathcal{I}) > 0$, it holds

$$\mathbb{P}(\tau > t, W \in \mathcal{I}) > 0 \quad \text{for all } t \geq 0.$$

We will introduce the classical Cramer transforms, for $(a, b) \in \mathbb{R}^2$,

$$\Lambda^*(a, b) = \sup_{x, y} \{ax + by - \ln \mathbb{E}(e^{x\tau + yW})\} \quad (2.4)$$

$$\Lambda_n^*(a, b) = \sup_{x, y} \left\{ ax + by - \ln \mathbb{E} \left(e^{x\tau + yW^n} \right) \right\}, \quad (2.5)$$

where W^n is a well-chosen reduction of W . We finally introduce the rate functions

$$J^n(m) = \inf_{\beta > 0} \beta \Lambda_n^* \left(\frac{1}{\beta}, \frac{m}{\beta} \right) \quad , \quad J(m) = \inf_{\beta > 0} \beta \Lambda^* \left(\frac{1}{\beta}, \frac{m}{\beta} \right), \quad (2.6)$$

and

$$\tilde{J} = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{|m-z| < \delta} J^n(z). \quad (2.7)$$

We then may state

Theorem 2.5. *Assume that Assumption 2.4 is fulfilled. Let J given by (2.6).*

- *If $\beta_0 = +\infty$ (in particular if W is bounded) then Z_t/t satisfies a full LDP with good rate function \tilde{J} . We also have the following inequalities*

$$\mathbb{P} \left(\frac{Z_t}{t} \geq m + a \right) \leq - \inf_{z \geq m+a} J(z), \quad (2.8)$$

$$\mathbb{P} \left(\frac{Z_t}{t} \leq m - a \right) \leq - \inf_{z \leq m-a} J(z). \quad (2.9)$$

- If $\beta_0 < +\infty$, denoting $m = \mathbb{E}(W)/\mathbb{E}(\tau)$ we have for all $a > 0$

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P} \left(\frac{Z_t}{t} \geq m + a \right) \leq - \min \left[\inf_{z \geq m + (a/2)} J(z), \beta_0 a/4 \right], \quad (2.10)$$

and

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P} \left(\frac{Z_t}{t} \leq m - a \right) \leq - \min \left[\inf_{z \geq m - (a/2)} J(z), \beta_0 a/4 \right]. \quad (2.11)$$

We may replace $a/2$ and $\beta_0/2$ by κa and $(1 - \kappa)\beta_0$ for any $\kappa \in (0, 1)$.

Remark 2.6. The previous inequalities (2.8) and (2.9) actually hold true with function \tilde{J} as well as function J since $J \leq \tilde{J}$ (see Lemma 9.5. However, since J is clearly more easy to calculate than \tilde{J} , we prefer to write the inequalities with J .

2.3. The scheme of proof.

The proof will be divided in several steps. In the next section we will reduce the problem first to bounded W 's replacing the original W by W^n , and then to finite valued W 's. This will be done by using exponentially good approximations and Theorem 2.2.

Following [13] we introduce in section 4 (for finite valued W 's) an associated empirical measure and state the LDP for this measure (theorem 4.3) and the LDP for the cumulative process (theorem 4.4), the latter being obtained via the contraction principle. The following sections are devoted to the proofs of these theorems.

Section 5 is devoted to the study of the rate function in theorem 4.3. This is the most technical part of this work. The idea is that one can adapt the arguments in [13] by conditioning w.r.t. each value of W .

At this point the fact that W is finite valued will provide us both with simple conditioning and with the necessary compactness arguments, since $\mathcal{M}_1(\{1, \dots, n\})$ is compact.

If many arguments are close to those in [13], some have to be written in detail. We have decided to refer to the corresponding statements in [13] only when they can be reproduced line by line in our case. Some topological points also have to be clarified. In section 6 we show useful auxiliary lemmata. The next section 7 is devoted to the proof of 4.3. In section 8 we deduce theorem 4.4. The final section 9 is devoted to the study of the rate function J and to the proof of theorem 2.5.

3. REDUCTION OF THE PROBLEM TO FINITE VALUED W 'S.

3.1. First reduction to bounded W 's.

To Z_t defined by (2.1) we associate

$$Z_t^n = \sum_{i=1}^{M_t} (W_i \wedge n \vee (-n)),$$

so that

$$|Z_t - Z_t^n| = \sum_{i=1}^{M_t} (W_i - n)_+ + \sum_{i=1}^{M_t} (W_i + n)_- \quad (3.1)$$

where $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. We then have

Lemma 3.1. *If Assumption 2.4 is fulfilled, for all $\delta > 0$,*

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) \leq -\frac{\beta_0 \delta}{2}.$$

In particular if $\beta_0 = +\infty$, Z_t^n/t is an exponentially good approximation of Z_t/t .

Proof. It is enough to look at

$$\mathbb{P} \left(\sum_{i=1}^{M_t} (W_i - n)_+ > \delta t \right),$$

since the other term in (3.1) can be treated similarly. To conclude it is enough to use the elementary

$$\ln(a + b) \leq \max(\ln(2a), \ln(2b)). \quad (3.2)$$

Using that the W_i 's are i.i.d. we may write for $\delta > 0$ and $c > 0$, (as usual an empty sum is equal to 0 by convention)

$$\begin{aligned} & \mathbb{P} \left(\sum_{i=1}^{M_t} (W_i - n)_+ > \delta t \right) \\ & \leq \mathbb{P} \left(\sum_{i=1}^{ct} (W_i - n)_+ > \frac{\delta t}{2} \right) + \mathbb{P} \left(\sum_{i=ct+1}^{M_t} (W_i - n)_+ > \frac{\delta t}{2} \right) \\ & \leq \mathbb{P} \left(\sum_{i=1}^{ct} (W_i - n)_+ > \frac{\delta t}{2} \right) + \mathbb{P} \left(\left\{ \sum_{i=ct+1}^{M_t} (W_i - n)_+ > \frac{\delta t}{2} \right\} \cap \{1 + ct \leq M_t < 2ct\} \right) \\ & \quad + \mathbb{P} \left(\left\{ \sum_{i=ct+1}^{M_t} (W_i - n)_+ > \frac{\delta t}{2} \right\} \cap \{M_t \geq 2ct\} \right) \\ & \leq 2\mathbb{P} \left(\sum_{j=1}^{ct} (W_j - n)_+ > \frac{\delta t}{2} \right) + \mathbb{P}(M_t \geq 2ct) \end{aligned}$$

Study of $\mathbb{P}(M_t \geq 2ct)$. Start with the second term in the sum above. According to theorem 2.3 in [15], we know that M_t/t satisfies a LDP with rate function J_τ given by

$$J_\tau(x) = \begin{cases} \sup_\lambda \{\lambda - x \ln \mathbb{E}(e^{\lambda\tau})\} & \text{if } x \geq 0 \\ \infty & \text{if } x < 0 \end{cases}$$

Notice that $J_\tau(x) = x \Lambda^*(1/x, 0)$ for $x > 0$. In addition (see Lemma 2.6 in [15]) the supremum is achieved for $\lambda \leq 0$ if $x \in (1/\mathbb{E}(\tau), +\infty)$ and J_τ is non-decreasing on this interval.

It follows that for $2c > 1/\mathbb{E}(\tau)$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}(M_t \geq 2ct) = -J_\tau(2c). \quad (3.3)$$

In order to get $\lim_n \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}(M_t \geq 2c_n t) = -\infty$ for some sequence c_n (to be chosen later) it remains to show that

$$J_\tau(x) \xrightarrow{x \rightarrow \infty} +\infty.$$

Since τ is a non-negative random variable one can find λ_τ such that

$$\mathbb{E}(e^{\lambda_\tau \tau}) = e^{-1}.$$

Let $x \in \mathbb{R}^+$, we have:

$$\begin{aligned} J_\tau(x) &= \sup_\lambda \{\lambda - x \ln \mathbb{E}(e^{\lambda \tau})\} \\ &\geq \lambda_\tau - x \ln \mathbb{E}(e^{\lambda_\tau \tau}) \\ &\geq x + \lambda_\tau \end{aligned}$$

yielding the desired result.

Study of $\mathbb{P}\left(\sum_{j=1}^{ct} (W_j - n)_+ > \frac{\delta t}{2}\right)$. Denote as usual by $\lfloor ct \rfloor$ the integer part of ct . We have

$$\mathbb{P}\left(\sum_{j=1}^{ct} (W_j - n)_+ > \frac{\delta t}{2}\right) = \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \frac{\delta t}{2}\right)$$

so that we may use this time the usual Cramer's theorem. Defining

$$\begin{aligned} \Psi_n(\lambda) &= \ln \mathbb{E}\left[e^{\lambda(W-n)_+}\right] \\ \Psi_n^*(x) &= \sup_\lambda \{\lambda x - \Psi_n(\lambda)\}, \end{aligned}$$

we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t/2\right) &= \limsup_{t \rightarrow \infty} \frac{c}{\lfloor ct \rfloor} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t/2\right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{c}{\lfloor ct \rfloor} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta \lfloor ct \rfloor / 2c\right) \\ &\leq -c \inf_{x \in [\delta/2c, +\infty)} \Psi_n^*(x). \end{aligned}$$

As the function $x \mapsto \Psi_n^*(x)$ is non-decreasing on $[\mathbb{E}((W-n)_+), +\infty)$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t/2\right) = -c \Psi_n^*(\delta/2c),$$

provided $\delta/2c \geq \mathbb{E}((W-n)_+)$. Notice that for $\lambda < \beta_0$,

$$c \Psi_n^*(\delta/2c) \geq \frac{\lambda \delta}{2} - c \ln \left(1 + \mathbb{E}\left[(e^{\lambda(W-n)} - 1) \mathbf{1}_{W > n}\right]\right)$$

so that choosing c_n growing to infinity and such that

$$c_n \mathbb{E}((W-n)_+) \rightarrow 0 \text{ and } c_n \ln \left(1 + \mathbb{E}\left[(e^{\lambda(W-n)} - 1) \mathbf{1}_{W > n}\right]\right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

which is always possible since both $\mathbb{E}((W - n)_+)$ and $\ln \left(1 + \mathbb{E} \left[(e^{\lambda(W-n)} - 1) \mathbf{1}_{W > n} \right] \right)$ are going to 0, we get

$$\lim_n \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t / 2 \right) \geq \frac{\lambda \delta}{2}.$$

We may optimize in λ and plugging the same sequence c_n in (3.3) ends the proof. \square

3.2. Second reduction to finite valued W 's.

Starting with a bounded W such that $-K < W < K$ almost surely we define a discretized version of W by

$$W^n = \sum_{j=-n}^{n-1} \frac{jK}{n} \mathbf{1}_{W \in [jK/n, (j+1)K/n[}.$$

It clearly holds

$$|W - W^n| \leq \frac{K}{n}.$$

We thus introduce

$$\bar{Z}_t^n = \sum_{i=1}^{M_t} W_i^n$$

so that

$$|\bar{Z}_t^n - Z_t| \leq \frac{K M_t}{n}.$$

According to the study in the previous section

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\left| \frac{\bar{Z}_t^n}{t} - \frac{Z_t}{t} \right| > \delta \right) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} (M_t \geq \delta t n / K) = -J_\tau(\delta n / K)$$

so that, since J_τ grows to infinity

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\left| \frac{\bar{Z}_t^n}{t} - \frac{Z_t}{t} \right| > \delta \right) = -\infty.$$

We have thus shown

Lemma 3.2. *If Assumption 2.4 is fulfilled, and W is almost surely bounded, \bar{Z}_t^n/t defined above is an exponentially good approximation of Z_t/t .*

Remark 3.3. If W isn't bounded nor discrete, a double reduction can be done in one step. We obtain the same results as doing successively both the reductions, but it allows formulating the rate function more easily, when there is one.

Let's consider

$$\tilde{W}^n = -n \mathbf{1}_{W < -n} + n \mathbf{1}_{W \geq n} + \sum_{j=-n^2}^{n^2-1} \frac{j}{n} \mathbf{1}_{W \in [\frac{j}{n}, \frac{j+1}{n})}.$$

By denoting $\tilde{Z}_t^n = \sum_{i=1}^{M_t} \tilde{W}_i^n$, we have, for $\delta > 0$

$$\begin{aligned} \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\tilde{Z}_t^n}{t}\right| > \delta\right) &\leq \mathbb{P}\left(\sum_{i=1}^{M_t} (W_i - n)_+ > \delta t\right) + \mathbb{P}\left(\sum_{i=1}^{M_t} (W_i + n)_+ > \delta t\right) \\ &\quad + \mathbb{P}\left(\sum_{i=1}^{M_t} \sum_{j=-n^2}^{n^2-1} \left(W_i - \frac{j}{n}\right) \mathbb{1}_{W_i \in [\frac{j}{n}, \frac{j+1}{n}]} > \delta t\right). \end{aligned}$$

Since

$$\left|\sum_{i=1}^{M_t} \sum_{j=-n^2}^{n^2-1} \left(W_i - \frac{j}{n}\right) \mathbb{1}_{W_i \in [\frac{j}{n}, \frac{j+1}{n}]}\right| \leq \frac{M_t}{n},$$

we obtain

$$\mathbb{P}\left(\sum_{i=1}^{M_t} \sum_{j=-n^2}^{n^2-1} \left(W_i - \frac{j}{n}\right) \mathbb{1}_{W_i \in [\frac{j}{n}, \frac{j+1}{n}]} > \delta t\right) \leq \mathbb{P}(M_t > \delta t n).$$

By the proofs of Lemmas 3.1 and 3.2, we finally obtain

Lemma 3.4. *If Assumption 2.4 is fulfilled, for all $\delta > 0$,*

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\tilde{Z}_t^n}{t}\right| > \delta\right) \leq -\frac{\beta_0 \delta}{2}.$$

In particular if $\beta_0 = +\infty$, \tilde{Z}_t^n/t is an exponentially good approximation of Z_t/t .

Remark 3.5. The proofs in the previous section suggest a direct naive approach in order to get deviation bounds. Indeed we may write

$$\left|\frac{Z_t}{t} - \frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right| \leq \frac{1}{t} \left|\sum_{i=1}^{\lfloor M_t \rfloor} (W_i - \mathbb{E}(W))\right| + \mathbb{E}(W) \left|\frac{\lfloor M_t \rfloor}{t} - \frac{1}{\mathbb{E}(\tau)}\right|$$

so that following the lines of the proof of lemma 3.1, we may write

$$\begin{aligned} \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right| \geq \delta\right) &\leq \mathbb{P}\left(\frac{1}{t} \left|\sum_{i=1}^{\lfloor M_t \rfloor} (W_i - \mathbb{E}(W))\right| \geq \delta/2\right) + \mathbb{P}\left(\mathbb{E}(W) \left|\frac{\lfloor M_t \rfloor}{t} - \frac{1}{\mathbb{E}(\tau)}\right| \geq \delta/2\right) \\ &\leq 2\mathbb{P}\left(\left|\sum_{i=1}^{\lfloor M_t \rfloor} (W_i - \mathbb{E}(W))\right| > \delta t/4\right) + \mathbb{P}(\lfloor M_t \rfloor > 2ct) + \\ &\quad + \mathbb{P}\left(\mathbb{E}(W) \left|\frac{\lfloor M_t \rfloor}{t} - \frac{1}{\mathbb{E}(\tau)}\right| \geq \delta/2\right). \end{aligned}$$

Introducing the Cramer transform of W ,

$$\Psi^*(x) = \sup_{\lambda} \{\lambda x - \ln \mathbb{E}(e^{\lambda(W - \mathbb{E}(W))})\}$$

we thus deduce, if Assumption 2.4 is fulfilled

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right| \geq \delta\right) \leq -A(\delta), \quad (3.4)$$

where

$$\begin{aligned} A(\delta) &= \min(A_1(\delta), A_2(\delta)) \\ A_1(\delta) &= \min\left(J_\tau\left(\frac{1}{\mathbb{E}(\tau)} + \frac{\delta}{2\mathbb{E}(W)}\right), J_\tau\left(\frac{1}{\mathbb{E}(\tau)} - \frac{\delta}{2\mathbb{E}(W)}\right)\right) \end{aligned} \quad (3.5)$$

$$A_2(\delta) = \sup_{2c > 1/\mathbb{E}(\tau)} \min(J_\tau(2c), c \min(\psi^*(\delta/4c), \psi^*(-\delta/4c))). \quad (3.6)$$

This time however we cannot let c go to infinity so that $A_2(\delta)$ will furnish a worse bound than $\beta_0\delta/2$ (the correcting term in Lemma 3.1). So even when $\beta_0 < +\infty$ the previous bound will be worse than the one obtained by combining Lemma 3.1 and the LDP for finite valued W .

This kind of approach should be useful in order to get non asymptotic bounds. \diamond

4. LARGE DEVIATIONS FOR A FINITE VALUED W .

From now on we will assume that W takes its values in a set of cardinal n (fixed). However we will keep a superscript n to remember that this is a reduction of the general case, so we shall write W^n . For simplicity we rename the values of W^n as $1, \dots, n$. According to Assumption 2.4 iii) and the approximation made in subsection 3.2 we may assume that

$$\forall j = 1, \dots, n, \forall t > 0, \psi^n(\{\tau > t\} \cap \{w = j\}) = \mathbb{P}(\tau > t, W^n = j) > 0.$$

This condition will replace iii) in Assumption 2.4 in all the next statements.

Some results are still true when W takes its value in some $\mathbb{W} \subset \mathbb{R}$ which is compact, in particular if W is bounded. We will mention these results in related remarks each time it is possible.

4.1. More notations.

As we said we will now closely follow the approach in [13]. Consider the backward recurrence time process $(A_t)_{t \geq 0}$, the forward recurrence time process $(B_t)_{t \geq 0}$ and the process $(C_t^n)_{t \geq 0}$, defined by:

$$A_t = t - S_{M_t}, \quad B_t = S_{M_{t+1}} - t, \quad C_t^n = W_{M_{t+1}}^n,$$

and the associated empirical measure:

$$\mu_t^n := \frac{1}{t} \int_{[0,t)} \delta_{(A_s, B_s, C_s^n)} ds \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}).$$

We denote by \mathbb{P}_t^n the law of μ_t^n .

This empirical measure is a measure on three coordinates, whereas Lefevre, Mariani and Zambotti [13] consider a measure on two coordinates since $W = F(\tau)$ in their work. Looking at $(0, +\infty]$ allows us to avoid integrability considerations at infinity since A_s and B_s can be as large as we want simultaneously.

Choosing as metric $d(t, t') = |\frac{1}{t} - \frac{1}{t'}|$ on $(0, +\infty]$ and the usual one on $\{1, \dots, n\}$ we immediately see that $\mathcal{Y} = (0, +\infty]^2 \times \{1, \dots, n\}$ is a Polish space, so that $\mathcal{X} = \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ is also Polish. In addition $(a, b, c) \rightarrow \frac{1}{a+b}$ is continuous (of course it is equal to 0 if either a or b equals $+\infty$).

For $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$, we denote:

$$\mu(1/\tau) = \mu(1/(a+b)) = \int_{(0, +\infty)^2 \times \{1, \dots, n\}} \frac{1}{a+b} \mu(da, db, dc)$$

and for $\pi \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$, we denote:

$$\pi(1/\tau) = \int_{(0, +\infty] \times \{1, \dots, n\}} \frac{1}{\tau} \pi(d\tau, dW).$$

Let us define $\Delta_0^n \subset \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ as :

$$\Delta_0^n = \{\mu_0 \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), \quad (4.1)$$

$$\mu_0(da, db, dc) = \int_{[0, 1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW),$$

$$\pi \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\}), \pi(1/\tau) < +\infty\}$$

We also define Δ^n :

$$\begin{aligned} \Delta^n = \{ & \mu(da, db, dc) = \alpha \mu_0(da, db, dc) + (1 - \alpha) \delta_{(+\infty, +\infty)}(da, db) \otimes \eta(dc), \\ & \mu_0 \in \Delta_0^n, \alpha \in [0, 1], \eta \in \mathcal{M}^1(\{1, \dots, n\}) \}. \end{aligned} \quad (4.2)$$

The following will be used several times in what follows

Lemma 4.1. *For $M > 0$, the set $\mathcal{X}_M = \{\nu \in \mathcal{X}, \nu(1/(a+b)) \leq M\}$ is compact in \mathcal{X} .*

Proof. First the set $\{a+b \geq \delta\}$ is compact in \mathcal{Y} for $\delta \geq 0$. It follows that \mathcal{X}_M is a uniformly integrable set, hence is relatively compact (tight). Indeed for $\nu \in \mathcal{X}_M$

$$\nu(\{a+b < \delta\}) \leq \delta \nu(1/(a+b)) \leq \delta M.$$

Notice that this set is also closed, so that it is compact. Indeed if ν belongs to \mathcal{X}_M , $\nu(\frac{1}{a+b+\varepsilon}) \leq M$ for all $\varepsilon > 0$. Any weak limit ν of such ν_k 's satisfies $\nu(\frac{1}{a+b+\varepsilon}) \leq M$ since $(a, b, c) \mapsto 1/(a+b+\varepsilon)$ is bounded and continuous. It remains to apply the monotone convergence theorem. \square

Remark 4.2. This lemma and its proof are still true if we replace $\{1, \dots, n\}$ by a compact set \mathbb{W} . \diamond

Recall that ψ^n denotes the joint distribution of (τ, W^n) , so that its first marginal does not depend on n . We may thus skip the superscript n when dealing with quantities that only depend on this first marginal. To simplify the notation we will denote by ξ the θ_0 in Assumption 2.4, i.e.

$$\xi = \sup\{c \in \mathbb{R}, \psi(e^{c\tau}) < \infty\} \in [0, +\infty]. \quad (4.3)$$

For $\pi \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$, satisfying $\pi(1/\tau) \in (0, +\infty)$, we define $\tilde{\pi} \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$ as:

$$\tilde{\pi}(d\tau, dW) = \frac{1}{\pi(1/\tau)} \frac{1}{\tau} \pi(d\tau, dW).$$

$\tilde{\pi}$ has no weight on $\{+\infty\} \times \{1, \dots, n\}$.

We also define the functional $I_0^n, I^n : \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}) \rightarrow [0, +\infty]$ by:

$$I_0^n(\mu) = \begin{cases} \pi(1/\tau)H(\tilde{\pi}|\psi^n) & \text{if } \mu \in \Delta_0^n \\ \infty & \text{otherwise} \end{cases}, \quad (4.4)$$

$$I^n(\mu) = \begin{cases} \alpha\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \alpha)\xi & \text{if } \mu \in \Delta^n \\ \infty & \text{otherwise} \end{cases}, \quad (4.5)$$

where H is the relative entropy defined by

$$H(\nu|\mu) = \begin{cases} \int \ln \left(\frac{d\nu}{d\mu} \right) d\mu & \text{if } \nu \text{ is absolutely continuous w.r.t. } \mu \\ \infty & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{C}_b((0, +\infty]^2 \times \{1, \dots, n\})$ the set of bounded and continuous functions on $(0, +\infty]^2 \times \{1, \dots, n\}$.

For a bounded measurable $f : (0, +\infty) \times (0, +\infty) \times \{1, \dots, n\} \rightarrow \mathbb{R}$, we set :

$$\bar{f}(x, \tau, W) = \int_0^x f(u\tau, (1-u)\tau, W) du \quad (4.6)$$

We define:

$$C_{n,f} = \int_{(0,+\infty) \times \{1, \dots, n\}} e^{\tau \bar{f}(1, \tau, W)} \psi^n(d\tau, dW), \quad (4.7)$$

and

$$D_{n,f} = \sup_{s>0} \int_{(s,+\infty)} e^{\tau \bar{f}(s/\tau, \tau, W)} \psi^n(d\tau, dW) .. \quad (4.8)$$

We consider

$$\Gamma := \{f : (0, +\infty]^2 \times \{1, \dots, n\} \rightarrow \mathbb{R}, \text{ bounded, lower semicontinuous, } C_{n,f} < 1, D_{n,f} < +\infty\}. \quad (4.9)$$

The reasons for introducing such quantities are detailed in [13] section 1, see in particular subsection 1.5 (and more particularly Remark 1.5) and subsection 1.6.

4.2. Large deviations principle for cumulative process when W is finite valued.

We will first prove a LDP for the empirical measure μ_t^n :

Theorem 4.3. *Let μ_t^n the empirical measure of a cumulative process satisfying Assumptions 2.4, and such that W^n takes its values in $\{1, \dots, n\}$. The family $(\mathbb{P}_t^n)_{t \geq 0}$ of probability distributions of μ_t^n satisfies a large deviations principle with good rate function I^n as $t \rightarrow \infty$ with speed t , where I^n is defined in (4.5).*

Applying the contraction principle we will then deduce

Theorem 4.4. *Let $(\tau_i, W_i^n)_i$ an i.i.d. sequence of couples of random variables in $(0, +\infty) \times \{1, \dots, n\}$ following the law ψ^n . Let us define $Z_t^n = \sum_{i=1}^{M_t} W_i^n$ the associated cumulative process. If ψ^n satisfies Assumptions 2.4, then the law of Z_t^n/t satisfies a large deviation principle with good rate function J^n (given by (2.6)).*

5. THE FUNCTIONAL I^n

This section aims at proving that I^n , defined in (4.5), is a good rate function i.e proposition 5.2 (analogue to Proposition 1.3 in [13]). Three propositions are necessary to prove it: the propositions 5.3, 5.4 and 5.6 which are the analogues of Proposition 2.1 in [13]. In order to prove these three propositions some additional lemmata are necessary.

Remark 5.1. If the scheme of proof is close to the one in [13], the proofs of this section need some new ideas. In particular, we will use conditioning by sets like $\{W = j\}$. This induces to use disintegration and uniform controls in j . That is why it is useful to work with a finite valued W . In addition the compactness of $\mathcal{M}_1(\{1, \dots, n\})$ will help to control the third coordinate (which does not exist in [13]). Except two auxiliary lemmata, we decided to give self contained proofs. \diamond

Proposition 5.2. *The function I^n is a good rate function. Moreover, I^n is the lower semicontinuous envelope of I_0^n .*

For the proof we need the next three propositions. We need another notation: if ψ_k is a sequence in $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$, ξ_k and I_k^n are defined as in (4.3) and (4.5) respectively, with ψ^n replaced by ψ_k .

Proposition 5.3. *Let ψ_k be a sequence in $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$. Assume that $\psi_k \rightharpoonup \psi^n$, and $\xi_k \rightarrow \xi$ as $k \rightarrow +\infty$. Then any sequence $(\mu_k)_k$ in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ such that $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < \infty$ is tight and thus, relatively compact in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$. In particular, for k large enough, such a sequence is in Δ^n . If we consider π_k and α_k the associated quantities, we have:*

$$\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < +\infty. \quad (5.1)$$

Proposition 5.4. *Let ψ_k be a sequence in $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$. Assume that $\psi_k \rightharpoonup \psi^n$, and $\xi_k \rightarrow \xi$ as $k \rightarrow +\infty$. Then for any μ and any sequence $(\mu_k)_k$ in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$, such that $\mu_k \rightharpoonup \mu$, we have $\liminf_{k \rightarrow \infty} I_k^n(\mu_k) \geq I^n(\mu)$.*

Remark 5.5. Both previous propositions are still true when replacing $\{1, \dots, n\}$ by \mathbb{W} a compact subset of \mathbb{R} as it will be clear looking at their proof. \diamond

Proposition 5.6. *Let ψ_k be a sequence in $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$. Assume that $\psi_k \rightharpoonup \psi^n$, and $\xi_k \rightarrow \xi$ as $k \rightarrow +\infty$. Then for any μ in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ with $I^n(\mu) < \infty$, there exists a sequence $(\mu_k)_k$ such that $\mu_k \rightharpoonup \mu$, $\mu_k \in \Delta_0^n$ for all k and $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) \leq I^n(\mu)$. Moreover, we have $\mu_k \left(\frac{1}{a+b} \right) \xrightarrow[k \rightarrow \infty]{} \mu \left(\frac{1}{a+b} \right)$.*

For this last proposition we need to work with W taking values in a finite set.

Now we can prove the main Proposition of this section.

Proof of Proposition 5.2. We want to prove that I^n is a good rate function and is the lower semicontinuous envelope of I_0^n .

We apply Proposition 5.3 with $\psi_k = \psi^n$: for all $(\mu_k)_k$ sequence of $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$, if $\limsup_{k \rightarrow \infty} I^n(\mu_k) < +\infty$, then the family $(\mu_k)_k$ is tight in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$. We can deduce that I^n has relatively compact sublevel sets (and is coercive).

Thanks to Proposition 5.4 we have, for any μ and any sequence μ_k in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ such that $\mu_k \rightharpoonup \mu$, $\liminf_{k \rightarrow \infty} I^n(\mu_k) \geq I^n(\mu)$. Thus I^n has closed sublevel sets. It also means that I^n is lower semicontinuous.

I^n has relatively compact and closed sublevel sets, so I^n is a good rate function.

We already know that $I^n \leq I_0^n$ since $\Delta_0^n \subset \Delta^n$, and I^n is lower semicontinuous, so I^n is smaller than or equal to the lower semicontinuous envelope of I_0^n .

Thanks to proposition 5.6, for any $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ with $I^n(\mu) < \infty$, there exists $(\mu_k)_k$ a sequence in Δ_0^n such that $\mu_k \rightharpoonup \mu$ and $\limsup_{k \rightarrow \infty} I^n(\mu_k) \leq I^n(\mu)$. In fact, $\limsup_{k \rightarrow \infty} I_0^n(\mu_k) \leq I^n(\mu)$. So I^n is greater or equal to the lower semicontinuous envelope of I_0^n and finally I^n is the lower semicontinuous envelope of I_0^n . \square

We turn to the proof of the three auxiliary propositions.

Proof of proposition 5.3. Choose a sequence $(\mu_k)_k$ in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ such that $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < \infty$. We want to prove that this sequence is tight and thus, relatively compact in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$.

Since $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < \infty$, for k large enough, $\mu_k \in \Delta^n$. Then, there exist $\alpha_k \in [0, 1]$, $\pi_k \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ with $\pi_k(1/\tau) < +\infty$ and $\eta_k \in \mathcal{M}^1(\{1, \dots, n\})$.

First, we prove equation (5.1):

$$\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < +\infty.$$

We have $I_k^n(\mu_k) = \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k | \psi_k) + (1 - \alpha) \xi_k \geq \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k | \psi_k)$. Therefore

$$\alpha_k \pi_k(1/\tau) \leq \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k | \psi_k)} \quad (\text{we consider } H(\tilde{\pi}_k | \psi_k) \neq 0).$$

Moreover

$$\alpha_k \pi_k(1/\tau) = \alpha_k \frac{1}{\tilde{\pi}_k(\tau)}.$$

Then

$$\alpha_k \pi_k(1/\tau) \leq \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k | \psi_k)} \wedge \frac{\alpha_k}{\tilde{\pi}_k(\tau)}$$

and $\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) \leq \limsup_{k \rightarrow \infty} \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k | \psi_k)} \wedge \frac{\alpha_k}{\tilde{\pi}_k(\tau)}$.

$\alpha_k \in [0, 1]$ and $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < +\infty$, so there exists a constant $C < \infty$ such that $\forall k, \alpha_k \leq C$ and $I_k^n(\mu_k) \leq C$. Then

$$\begin{aligned} \limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) &\leq \limsup_{k \rightarrow \infty} \frac{C}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{C}{\tilde{\pi}_k(\tau)} \\ &\leq C \limsup_{k \rightarrow \infty} \frac{1}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{1}{\tilde{\pi}_k(\tau)} \\ &\leq C \limsup_{k \rightarrow \infty} \frac{1}{H(\tilde{\pi}_k|\psi_k) \vee \tilde{\pi}_k(\tau)} \\ &\leq \frac{C}{\liminf_{k \rightarrow \infty} H(\tilde{\pi}_k|\psi_k) \vee \tilde{\pi}_k(\tau)}. \end{aligned}$$

If there exists a subsequence k_j such that $H(\tilde{\pi}_{k_j}|\psi_{k_j}) \rightarrow 0$, then $\lim \tilde{\pi}_{k_j} = \lim \psi_{k_j} = \psi^n$. Thus, since $[0, 1]$ and $\mathcal{M}^1(\{1, \dots, n\})$ are compact, there exist subsequences of α_k and η_k which converge. Eventually, there exists a subsequence of μ_k which converges, so this sequence is relatively compact. In particular, (5.1) is true because $\liminf_{j \rightarrow \infty} \tilde{\pi}_{k_j}(\tau) = \psi^n(\tau) > 0$.

Else, $\liminf_{k \rightarrow \infty} H(\tilde{\pi}_k|\psi_k) > 0$ and $\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < \infty$. Then (5.1) is true. Moreover, for M large enough and k large enough, $\mu_k(1/(a+b)) = \alpha_k \pi_k(1/\tau) \leq M$. So $\mu_k \in \mathcal{X}_M$ which is a compact set by Lemma 4.1. Then, this sequence is relatively compact. In both cases, the sequence is relatively compact and (5.1) is true. \square

Remark 5.7. As already said we only used the compactness of $\mathcal{M}^1(\mathbb{W})$ and lemma 4.1 so that the above proof immediately extends to \mathbb{W} compact. \diamond

Proof of proposition 5.4. Let $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ and let $(\mu_k)_k$ in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$, such that $\mu_k \rightarrow \mu$. We want to prove that $\liminf_{k \rightarrow \infty} I_k^n(\mu_k) \geq I^n(\mu)$.

Since $\mu_k \rightarrow \mu$, we may replace μ_k by subsequences again denoted μ_k in what follows. We assume $\sup_k I_k^n(\mu_k) < \infty$ (otherwise, if $\liminf_{k \rightarrow \infty} I_k^n(\mu_k) = +\infty$, $+\infty \geq I^n(\mu)$). Then $\mu_k \in \Delta^n$ and we denote by α_k, π_k, η_k and $\mu_{0,k}$ the corresponding quantities, with $\alpha_k \in [0, 1]$ and $\pi_k(1/\tau) < \infty$.

First, if $\limsup_{k \rightarrow \infty} \alpha_k = 0$, we may assume taking a subsequence that $\lim \alpha_k = 0$. So $\mu = \delta_{(+\infty, +\infty)} \otimes (\lim_k \eta_k)$. Since $\mathcal{M}^1(\{1, \dots, n\})$ is compact, $(\lim_k \eta_k) \in \mathcal{M}^1(\{1, \dots, n\})$. Therefore

$$\liminf_{k \rightarrow \infty} I_k^n(\mu_k) = \liminf_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k|\psi_k) + (1 - \alpha_k) \xi_k \geq \liminf_{k \rightarrow \infty} (1 - \alpha_k) \xi_k = \xi = I^n(\mu).$$

Secondly, we have $\limsup_{k \rightarrow \infty} \alpha_k = \bar{\alpha} > 0$. Again we may assume that $\lim \alpha_k = \bar{\alpha}$.

We begin by studying the sequence μ_k and π_k to have some information about $\liminf_{k \rightarrow \infty} I_k^n(\mu_k)$. Then, we prove that μ is in Δ^n , and finally we work on $I^n(\mu)$.

Since $\sup_k I_k^n(\mu_k) < \infty$, we can apply the proposition 5.3 and in particular the equation (5.1): $\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < \infty$. Since $\bar{\alpha} > 0$, we have: $\limsup_{k \rightarrow \infty} \pi_k(1/\tau) < \infty$. For k large enough and M large enough, we have $\pi_k(1/\tau) \leq M$.

As

$$\left\{ \nu \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\}), \nu \left(\frac{1}{\tau} \right) \leq r \right\}$$

is tight for all $r > 0$, there exists a subsequence of π_k which converges in $\mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$. We can write the limit of π_k (or its subsequence) as $\beta\pi + (1 - \beta)\delta_{(+\infty, +\infty)} \otimes \eta_0$, for some $\beta \in [0, 1]$ and $\eta_0 \in \mathcal{M}^1(\{1, \dots, n\})$.

If $\beta > 0$, $\pi(1/\tau) \leq \frac{1}{\beta} \limsup_{k \rightarrow \infty} \pi_k(1/\tau) < \infty$. If $\beta = 0$ we choose an arbitrary π such that $\pi(1/\tau) < \infty$.

Now, we prove that μ is in Δ^n , where μ is defined by:

$$\begin{aligned} \mu &= \lim_k \mu_k \\ &= \lim_k (\alpha_k \mu_{0,k} + (1 - \alpha_k) \delta_{(+\infty, +\infty)} \otimes \eta_k) \\ &= \bar{\alpha} \lim_k \mu_{0,k} + (1 - \bar{\alpha}) \delta_{(+\infty, +\infty)} \otimes (\lim_k \eta_k). \end{aligned}$$

It holds

$$\begin{aligned} \mu_{0,k} &= \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)} du \otimes \pi_k(d\tau, dW) \\ &\xrightarrow{k \rightarrow \infty} \beta \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)} du \otimes \pi(d\tau, dW) + (1 - \beta) \delta_{(+\infty, +\infty)} \otimes \eta_0. \end{aligned}$$

Let μ_0 and η be defined as:

$$\begin{aligned} \mu_0 &:= \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)} du \otimes \pi(d\tau, dW) \\ \eta &:= \frac{\alpha(1 - \beta)}{1 - \bar{\alpha}\beta} \eta_0 + \frac{1 - \bar{\alpha}}{1 - \bar{\alpha}\beta} (\lim_k \eta_k). \end{aligned}$$

We get

$$\begin{aligned} \mu &= \bar{\alpha}\beta\mu_0 + \bar{\alpha}(1 - \beta)\delta_{(+\infty, +\infty)} \otimes \eta_0 + (1 - \bar{\alpha})\delta_{(+\infty, +\infty)} \otimes (\lim_k \eta_k) \\ &= \bar{\alpha}\beta\mu_0 + (1 - \bar{\alpha}\beta)\delta_{(+\infty, +\infty)} \otimes \eta, \end{aligned}$$

and in particular $\mu \in \Delta^n$ with $\alpha = \beta\bar{\alpha}$.

Eventually

$$\begin{aligned} I^n(\mu) &= \beta\bar{\alpha}\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta\bar{\alpha})\xi \\ &= \bar{\alpha}[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi] + (1 - \bar{\alpha})\xi \\ &= \bar{\alpha}\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) + (1 - \bar{\alpha})\xi \\ &\quad + \bar{\alpha}[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)] \\ &\leq \liminf_{k \rightarrow \infty} I_k^n(\mu_k) + \bar{\alpha}[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)]. \end{aligned}$$

In particular, we can apply Lemma 2.4 from [13] (the proof can be directly adapted in our case), since the hypotheses therein are satisfied:

Lemma 5.8. *Let $\pi_k \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ be such that $\pi_k(1/\tau) < \infty$ such that*

$$\lim_k \pi_k(d\tau, dW) = \beta\pi(d\tau, dW) + (1 - \beta)\delta_{(+\infty)}(d\tau) \otimes \eta_0(dW) \quad (5.2)$$

for some $\beta \in [0, 1]$, $\pi \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ such that $\pi(1/\tau) < \infty$ and $\eta_0 \in \mathcal{M}^1(\{1, \dots, n\})$. Then

$$\liminf_{k \rightarrow \infty} \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) \geq \beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi.$$

Eventually

$$\begin{aligned} I^n(\mu) &\leq \liminf_{k \rightarrow \infty} I_k^n(\mu_k) + \bar{\alpha} \underbrace{[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)]}_{<0 \text{ by Lemma 5.8}} \\ &\leq \liminf_{k \rightarrow \infty} I_k^n(\mu_k). \end{aligned}$$

□

Remark 5.9. Once again we only used the fact that $\{1, \dots, n\}$ is compact so that we may replace it by any compact \mathbb{W} . ◇

Proof of proposition 5.6. Let μ in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ with $I^n(\mu) < \infty$. We want to prove that there exists a sequence (μ_k) such that $\mu_k \rightharpoonup \mu$, $\mu_k \in \Delta_0^n$ for all k and $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) \leq I^n(\mu)$.

Since $I^n(\mu) < \infty$, $\mu \in \Delta^n$. Let α , π and η corresponding to μ :

$$\begin{aligned} \mu(da, db, dc) &= \alpha \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW) \\ &\quad + (1 - \alpha)\delta_{(+\infty, +\infty)}(da, db) \otimes \eta(dc). \end{aligned}$$

The aim of this (technical) proof is to construct a sequence of laws π_k , depending on ψ_k , which satisfies a condition on its limit (the condition (5.6) described below) and a condition on its entropy with respect to ψ_k (the condition (5.7) described below). Then, we will construct a sequence of measure μ_k which will satisfy the wished conditions. The next paragraphs give the details.

We denote $\psi^n(d\tau)$ the marginal law of τ in ψ^n , and $\psi_k(d\tau)$ the corresponding law associated to ψ_k to simplify notations. In particular, $\psi^n(d\tau|W = j)$ is the marginal law of τ given $W = j$. We write q_j the weight of j for η : $q_j = \eta(\{W = j\})$.

Fix $\rho > 0$, $L > M > 1$ such that $\mathbb{P}_{\psi^n}(\tau = 1/M) = \mathbb{P}_{\psi^n}(\tau = M) = \mathbb{P}_{\psi^n}(\tau = L) = 0$. Then there exist $N \in \mathbb{N}$ and $1/M = T_1 < T_2 < \dots < T_N = M$ such that $T_{i+1} - T_i \leq \rho$ and $\mathbb{P}_{\psi^n}(\tau = T_i) = 0$.

Here of course N and T_i depend on M and ρ . We also use the shorthand notation $A_i = [T_i, T_{i+1})$ and $A = \cup_{i=1}^N A_i$ in this proof. Then for $L > M$ define $\pi_{0,k}^{\rho, L, M}(d\tau, dW)$, $\eta_k^{L, M}(d\tau, dW)$

and $\pi_k^{\rho, M, L}(d\tau, dW) \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ as:

$$\pi_{0,k}^{\rho, M}(d\tau, dW) = \frac{1}{\beta_k^{\rho, M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \tau \psi_k(d\tau, dW | \tau \in A_i, W = j), \quad (5.3)$$

$$\eta_k^{L, M}(d\tau, dW) = \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbb{1}_{\tau \in [M, L]} | W = j)} \tau \mathbb{1}_{\tau \in [M, L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW), \quad (5.4)$$

$$\pi_k^{\rho, L, M}(d\tau, dW) = \alpha \pi_{0,k}^{\rho, M}(d\tau, dW) + (1 - \alpha) \eta_k^{L, M}(d\tau, dW), \quad (5.5)$$

where $\beta_k^{\rho, M}$ is the normalizing constant such that $\pi_{0,k}^{\rho, M}$ is a probability measure.

Some q_j 's can be equal to 0. In these cases, we consider in fact the sum on j such that the q_j aren't zero: all the following calculation are then true. Thanks to Assumption 2.4, we know that for each $j \in \{1, \dots, n\}$, $\forall t \in \mathbb{R}^+$, $\psi^n(\{\tau > t\} \cap \{W = j\}) > 0$, so the measure $\eta_k^{L, M}$ exists for adequate L and M . To simplify the notation, we consider in the following calculus that $\eta_k^{L, M}$ is the sum for $j \in \{1, \dots, n\}$.

The above definition makes sense if $L > M$ is large enough, and k is large enough depending on L and M (k will be sent to $+\infty$ before L , and L before M), and there is no problem with the conditioning. As $\psi^n(\tau \in \partial A_i) = 0$, if $\psi_k(A_i \times \{j\}) = 0$ for each k large enough, then $\psi(A_i \times \{j\}) = 0$. Since $I^n(\mu) < \infty$, then $H(\tilde{\pi} | \psi^n) < \infty$ and $\tilde{\pi}(A_i \times \{j\}) = 0$. The term in (i, j) in (5.3) would be considered equal to 0. Similarly if $\psi_k([M, L] \times \{n\}) = 0$ then the term in j in (5.4) vanishes. If each terms of one sum is zero, then α is equal to 0 or 1, by the same arguments on the relative entropy.

We want to prove:

$$\lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \lim_{k \rightarrow \infty} \pi_k^{\rho, L, M} = \alpha \pi + (1 - \alpha) \delta_{(+\infty)} \otimes \eta, \quad (5.6)$$

$$\liminf_{M \rightarrow \infty} \liminf_{L \rightarrow \infty} \liminf_{\rho \rightarrow 0} \limsup_{k \rightarrow \infty} \pi_k^{\rho, L, M} (1/\tau) H(\tilde{\pi}_k^{\rho, L, M} | \psi_k) \leq \alpha \pi (1/\tau) H(\tilde{\pi} | \psi^n) + (1 - \alpha) \xi = I^n(\mu), \quad (5.7)$$

where the limits in M and L are understood to run over M and L satisfying the above conditions.

Once (5.6) and (5.7) are proved, we can consider sequences $\rho_k \rightarrow 0$, $L_k \rightarrow \infty$, $M_k \rightarrow \infty$ (such that ρ_k , L_k and M_k satisfy the above conditions), we can define $\pi_k = \pi_k^{\rho_k, L_k, M_k}$ which satisfy: $\pi_k \rightarrow \alpha \pi + (1 - \alpha) \delta_{(+\infty)} \otimes \eta$ and

$$\limsup_{k \rightarrow \infty} \pi_k (1/\tau) H(\tilde{\pi}_k | \psi_k) \leq I^n(\mu).$$

Then μ_k defined by:

$$\mu_k(da, db, dc) := \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi_k(d\tau, dW)$$

satisfies the proposition.

First, we prove the convergence (5.6) step by step. First, we consider $\tilde{\pi}_{0,k}^{\rho,M}$.

The normalisation constant $\beta_k^{\rho,M}$ satisfies:

$$\begin{aligned}\beta_k^{\rho,M} &= \int_{(0,+\infty) \times \{1,\dots,n\}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \tau \psi_k(d\tau, dW | \tau \in A_i, W = j) \\ &= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \int_{A_i \times \{j\}} \tau \psi_k(d\tau, dW | \tau \in A_i, W = j) \\ &= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi_k(\tau | \tau \in A_i, W = j)\end{aligned}$$

Since $\psi_k \xrightarrow[k \rightarrow \infty]{} \psi$, then $\forall i \leq N, \forall j \leq n, \psi_k(\tau \in A_i, W = j) \xrightarrow[k \rightarrow \infty]{} \psi^n(\tau \in A_i, W = j)$. By bounded convergence (because $\tau \in A_i \Rightarrow \tau \leq T_{i+1}$, $\forall j \leq n, \forall f \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\})$),

$$\int_{A_i \times \{j\}} \tau f(\tau, W) \psi_k(d\tau, dW) \xrightarrow[k \rightarrow \infty]{} \int_{A_i \times \{j\}} \tau f(\tau, W) \psi^n(d\tau, dW).$$

So:

$$\beta_k^{\rho,M} \rightarrow \beta^{\rho,M} := \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(\tau | \tau \in A_i, W = j),$$

and

$$\pi_{0,k}^{\rho,M}(d\tau, dW) \rightarrow \pi_0^{\rho,M}(d\tau, dW) := \frac{1}{\beta^{\rho,M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \tau \psi^n(d\tau, dW | \tau \in A_i, W = j).$$

Let $f \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\})$. On $A = [1/M, M)$, for each $j \leq n$, $\tau \mapsto \tau f(\tau, j)$ is uniformly continuous, so there exists a modulus of continuity ω_f , such that $\forall (x, y) \in A, \forall j \in \{1, \dots, n\}, |xf(x, j) - yf(y, j)| \leq \omega_f(|x - y|)$. In fact, ω_f is the maximum of the n modulus of continuity for functions $\tau \mapsto \tau f(\tau, j)$.

For all $\rho > 0$, $(T_i)_i$ defined as before,

$$\begin{aligned}\int_{A \times \{1,\dots,n\}} \frac{f(\tau, W)\tau}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) &= \sum_{j=1}^n \int_{A \times \{j\}} \frac{f(\tau, j)\tau}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\ &= \sum_{j=1}^n \sum_{i=1}^N \int_{A_i \times \{j\}} \frac{f(\tau, j)\tau}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW).\end{aligned}$$

Then:

$$\begin{aligned}
\sum_{j=1}^n \sum_{i=1}^N \int_{A_i \times \{j\}} \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) &\leq \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\
&\leq \sum_{j=1}^n \sum_{i=1}^N \int_{A_i \times \{j\}} \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\
\sum_{j=1}^n \sum_{i=1}^N \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) &\leq \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\
&\leq \sum_{j=1}^n \sum_{i=1}^N \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j)
\end{aligned}$$

Let us remark that

$$\begin{aligned}
&\sum_{j=1}^n \sum_{i=1}^N \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) - \sum_{j=1}^n \sum_{i=1}^N \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) \\
&= \frac{1}{\tilde{\pi}(\tau \in A)} \sum_{j=1}^n \sum_{i=1}^N \tilde{\pi}(\tau \in A_i, W = j) \left(\max_{\tau \in A_i}(\tau f(\tau, j)) - \min_{\tau \in A_i}(\tau f(\tau, j)) \right) \\
&\leq \frac{1}{\tilde{\pi}(A)} \sum_{j=1}^n \sum_{i=1}^N \tilde{\pi}(\tau \in A_i, W = j) \omega_f(\rho) = \omega_f(\rho) \xrightarrow{\rho \rightarrow 0} 0.
\end{aligned}$$

Then

$$\sum_{j=1}^n \sum_{i=1}^N \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) \xrightarrow{\rho \rightarrow 0} \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW)$$

and

$$\sum_{j=1}^n \sum_{i=1}^N \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) \xrightarrow{\rho \rightarrow 0} \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW).$$

Then, by studying $\beta^{\rho, M} \pi_0^{\rho, M}(f)$:

$$\begin{aligned}
&\sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \min_{\tau \in A_i}(\tau f(\tau, j)) \\
&\leq \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \int \tau f(\tau, W) \psi^n(d\tau, dW | \tau \in A_i, W = j) \\
&\leq \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \max_{\tau \in A_i}(\tau f(\tau, j)).
\end{aligned}$$

So:

$$\forall f \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\}), \beta^{\rho, M} \pi_0^{\rho, M}(f) \xrightarrow{\rho \rightarrow 0} \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW). \quad (5.8)$$

In particular, for $f = \mathbb{1}$, $\pi_0^{\rho,M}(\mathbb{1}) = 1$ so: $\beta^{\rho,M} \xrightarrow{\rho \rightarrow 0} \beta^M := \int_{A \times \{1, \dots, n\}} \frac{\tau}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW) = \tilde{\pi}(\tau | \tau \in A)$. So $\pi_0^{\rho,M}(d\tau, dW) \rightarrow \pi_0^M(d\tau, dW) := \frac{1}{\beta^M} \tau \tilde{\pi}(d\tau, dW | \tau \in A)$.

When $M \rightarrow \infty$, $\pi(\tau \in A) = \pi(\tau \in [\frac{1}{M}, M]) \rightarrow 1$. So: $\beta^M \xrightarrow{M \rightarrow \infty} \tilde{\pi}(\tau)$ and: $\pi_0^M(d\tau, dW) \rightarrow \tau \tilde{\pi}(d\tau, dW) / \tilde{\pi}(\tau) = \pi(d\tau, dW)$.

Now, we study $\eta_k^{L,M}$:

$$\eta_k^{L,M}(d\tau, dW) = \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \tau \mathbb{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW).$$

We have $\psi_k(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \xrightarrow{k \rightarrow \infty} \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)$, so:

$$\eta_k^{L,M}(d\tau, dW) \rightarrow \eta^{L,M}(d\tau, dW) := \sum_{j=1}^n \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \tau \mathbb{1}_{\tau \in [M,L]} q_j \psi^n(d\tau | W = j) \otimes \delta_j(dW).$$

When $L \rightarrow \infty$,

$$\eta^{L,M}(d\tau, dW) \rightarrow \eta^M(d\tau, dW) := \sum_{j=1}^n \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = j)} \tau \mathbb{1}_{\tau \in [M,+\infty]} q_j \psi^n(d\tau | W = j) \otimes \delta_j(dW).$$

For $g \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\})$ with a compact support,

$$\eta^M(g) = \sum_{j=1}^n \int_{[M, \infty)} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = j)} \tau g(\tau, j) q_j \psi^n(d\tau | W = j).$$

For M large enough, $\eta^M(g) = 0$.

Moreover, for $i \in \{1, \dots, n\}$,

$$\eta^M(\mathbb{1}_{W=i}) = \int_{[M, +\infty)} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = i)} \tau q_i \psi^n(d\tau | W = i) = q_i \frac{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = i)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = i)} = q_i.$$

Then: $\eta^M(d\tau, dW) \rightarrow \delta_{(+\infty)}(d\tau) \otimes \eta(dW)$.

Now we deduce the convergence of $\pi_k^{\rho,L,M}$:

$$\lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \lim_{k \rightarrow \infty} \pi_k^{\rho,L,M} = \alpha \pi + (1 - \alpha) \delta_{(+\infty)} \otimes \eta.$$

(5.6) is then proved.

Now we prove the equation (5.7). We define $\tilde{\pi}_k^{\rho,L,M}$, $\tilde{\pi}_{0,k}^{\rho,M}$ and $\tilde{\eta}_k^{L,M}$ by:

$$\tilde{\pi}_k^{\rho,L,M}(d\tau, dW) := \frac{1}{\pi_k^{\rho,L,M}(1/\tau)} \frac{1}{\tau} \pi_k^{\rho,L,M}(d\tau, dW), \quad (5.9)$$

$$\tilde{\pi}_{0,k}^{\rho,M}(d\tau, dW) := \frac{1}{\pi_{0,k}^{\rho,M}(1/\tau)} \frac{1}{\tau} \pi_{0,k}^{\rho,M}(d\tau, dW), \quad (5.10)$$

$$\tilde{\eta}_k^{L,M}(d\tau, dW) := \frac{1}{\eta_k^{L,M}(1/\tau)} \frac{1}{\tau} \eta_k^{L,M}(d\tau, dW). \quad (5.11)$$

In particular:

$$\begin{aligned}
\eta_k^{L,M}(1/\tau) &= \sum_{j=1}^n \int \frac{1}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \frac{\tau}{\tau} \mathbf{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW) \\
&= \sum_{j=1}^n q_j \frac{\psi_k(\mathbf{1}_{\tau \in [M,L]} | W = j)}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \left(= \sum_{j=1}^n q_j \frac{1}{\psi_k(\tau | \tau \in [M,L], W = j)} \right) \\
\tilde{\eta}_k^{L,M}(d\tau, dW) &= \frac{1}{\eta_k^{L,M}(1/\tau)} \frac{1}{\tau} \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \tau \mathbf{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW) \\
&= \frac{1}{\eta_k^{L,M}(1/\tau)} \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \mathbf{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW).
\end{aligned}$$

Using Lemma 2.3 from [13] (the proof of which can be easily adapted in this situation) , $\pi \mapsto \pi(1/\tau)H(\tilde{\pi}|\psi_k) = \frac{1}{\tilde{\pi}(\tau)}H(\tilde{\pi}|\psi_k)$ is convex, so:

$$\pi_k^{\rho,L,M}(1/\tau)H(\tilde{\pi}_k^{\rho,L,M}|\psi_k) \leq \alpha \frac{1}{\tilde{\pi}_{0,k}^{\rho,M}(\tau)}H(\tilde{\pi}_{0,k}^{\rho,M}|\psi_k) + (1-\alpha) \frac{1}{\tilde{\eta}_k^{L,M}(\tau)}H(\tilde{\eta}_k^{L,M}|\psi_k).$$

First we consider the second term: since $\psi^n(\{\tau = M\}) = \psi^n(\{\tau = L\}) = 0$, we have:

$$\lim_k \frac{1}{\tilde{\eta}_k^{L,M}(\tau)}H(\tilde{\eta}_k^{L,M}|\psi_k) = \frac{1}{\tilde{\eta}^{L,M}(\tau)}H(\tilde{\eta}^{L,M}|\psi^n)$$

$$\begin{aligned}
\tilde{\eta}^{L,M}(\tau) &= \eta^{L,M}(1/\tau)^{-1} \\
\tilde{\eta}^{L,M}(d\tau, dW) &= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n \frac{1}{\psi^n(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \mathbf{1}_{\tau \in [M,L]} q_j \psi^n(d\tau | W = j) \otimes \delta_j(dW).
\end{aligned}$$

We have

$$\begin{aligned}
&H(\tilde{\eta}^{L,M}|\psi^n) \\
&= \int_{[M,L] \times \{1, \dots, n\}} \ln \left(\frac{\tilde{\eta}^{L,M}(d\tau, dW)}{\psi^n(d\tau, dW)} \right) \tilde{\eta}^{L,M}(d\tau, dW) \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n \int_{[M,L] \times \{j\}} \ln \left(\frac{\tilde{\eta}^{L,M}(d\tau, dW)}{\psi^n(d\tau, dW)} \right) \frac{\mathbf{1}_{\tau \in [M,L]} q_j}{\psi^n(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \psi^n(d\tau | W = j) \otimes \delta_j(dW).
\end{aligned}$$

We can decompose ψ^n in the following form:

$$\psi^n(d\tau, dW) = \sum_{j=1}^n p_j \psi^n(d\tau | W = j) \otimes \delta_j(d\tau),$$

where $p_j = \psi^n(W = j)$. By definition of Δ^n and $\{1, \dots, n\}$, we know that $p_i = 0$ implies $q_i = 0$. Then:

$$\begin{aligned}
& H(\tilde{\eta}^{L,M} | \psi^n) \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n \int_{[M,L] \times \{j\}} \ln \left(\frac{1}{\eta^{L,M}(1/\tau)} \frac{\mathbb{1}_{\tau \in [M,L]} q_j}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \frac{\psi^n(d\tau | W = j) \otimes \delta_j(dW)}{p_j \psi^n(d\tau | W = j) \otimes \delta_j(d\tau)} \right) \\
&\quad \times \frac{\mathbb{1}_{\tau \in [M,L]} q_j}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \psi^n(d\tau | W = j) \otimes \delta_j(dW) \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \int_{[M,L]} \ln \left(\frac{q_j}{p_j \eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \right) \frac{\psi^n(d\tau | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \ln \left(\frac{q_j}{p_j \eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \ln \left(\frac{q_j}{p_j} \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&\quad - \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \ln \left(\eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)}.
\end{aligned}$$

Then we study

$$\begin{aligned}
\frac{1}{\tilde{\eta}_k^{L,M}(\tau)} H(\tilde{\eta}_k^{L,M} | \psi_k) &= \sum_{j=1}^n q_j \ln \left(\frac{q_j}{p_j} \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&\quad - \sum_{j=1}^n q_j \ln \left(\eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)}.
\end{aligned}$$

First, we have $\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \geq M$, and $\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j) \leq 1$, so

$$\begin{aligned}
\frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \left| \sum_{j=1}^n q_j \ln \left(\frac{q_j}{p_j} \right) \right| &\leq \frac{1}{M} \left| \sum_{j=1}^n q_j \ln \left(\frac{q_j}{p_j} \right) \right| \\
&\xrightarrow{M \rightarrow \infty} 0.
\end{aligned}$$

Secondly, we have:

$$\begin{aligned}
& \sum_{j=1}^n q_j \ln \left(\psi^n(\eta^{L,M}(1/\tau) \tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \sum_{j=1}^n q_j \ln \left(\eta^{L,M}(1/\tau) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&\quad + \sum_{j=1}^n q_j \ln \left(\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \ln \left(\eta^{L,M}(1/\tau) \right) \times \eta^{L,M}(1/\tau) + \sum_{j=1}^n q_j \frac{\ln \left(\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j).
\end{aligned}$$

Moreover, $\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j) \geq M$, so $\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j) \xrightarrow{M, L \rightarrow +\infty} +\infty$. Since $\frac{\ln x}{x} \xrightarrow{x \rightarrow +\infty} 0$ and $\psi^n(\mathbf{1}_{\tau \in [M, +\infty)} | W = j) \xrightarrow{M \rightarrow +\infty} 0$, we have:

$$\sum_{j=1}^n q_j \frac{\ln(\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j))}{\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j)} \psi^n(\mathbf{1}_{\tau \in [M, L]} | W = j) \xrightarrow{M, L \rightarrow \infty} 0.$$

Eventually, $\eta^{L, M}(1/\tau) = \int_{[M, L]} \frac{1}{\tau} \eta^{L, M}(d\tau) \leq \frac{1}{M}$. Then in particular

$$\lim_{M \rightarrow +\infty} \lim_{L \rightarrow \infty} \eta^{L, M} \left(\frac{1}{\tau} \right) = 0. \quad (5.12)$$

Since $x \ln x \xrightarrow{x \rightarrow 0} 0$, $\eta^{L, M}(1/\tau) \times \ln(\eta^{L, M}(1/\tau)) \xrightarrow{M, L \rightarrow +\infty} 0$.

Then

$$\lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{H(\tilde{\eta}^{L, M} | \psi^n)}{\tilde{\eta}^{L, M}(\tau)} = 0 \leq \xi.$$

We focus on the first term. We know that $\tilde{\pi}_{0, k}^{\rho, M}(\tau) = 1/\pi_{0, k}^{\rho, M}(1/\tau)$. We have:

$$\begin{aligned} \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \lim_{k \rightarrow \infty} \pi_{0, k}^{\rho, M}(1/\tau) &= \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \frac{1}{\beta^{\rho, M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} = \frac{1}{\beta^{\rho, M}} \\ &= \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{1}{\beta^M} \\ &= \frac{1}{\tilde{\pi}(\tau)} \end{aligned} \quad (5.13)$$

Then: $\frac{1}{\tilde{\pi}_{0, k}^{\rho, M}(\tau)} \xrightarrow{M, \rho, k} \frac{1}{\tilde{\pi}(\tau)}$. Now, we study $H(\tilde{\pi}_{0, k}^{\rho, M} | \psi_k)$:

As $\pi_0^{\rho, M}(1/\tau) = 1/\beta^{\rho, M}$, we have

$$\begin{aligned} \tilde{\pi}_0^{\rho, M}(d\tau, dW) &= \frac{1}{\pi_0^{\rho, M}(1/\tau)} \frac{1}{\tau} \pi_0^{\rho, M}(d\tau, dW) \\ &= \beta^{\rho, M} \times \frac{1}{\beta^{\rho, M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(d\tau, dW | \tau \in A_i, W = j) \\ &= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(d\tau, dW | \tau \in A_i, W = j). \end{aligned}$$

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} H(\tilde{\pi}_{0,k}^{\rho,M} | \psi_k) \\
&= H(\tilde{\pi}_0^{\rho,M} | \psi^n) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \times \\
&\quad \int \ln \left(\frac{\sum_{l=1}^N \sum_{m=1}^n \frac{\tilde{\pi}(\tau \in A_l, W = m)}{\tilde{\pi}(A)} \frac{\psi^n(d\tau, dW | \tau \in A_l, W = m)}{\psi^n(d\tau, dW)}}{\psi^n(d\tau, dW | \tau \in A_i, W = j)} \right) \psi^n(d\tau, dW | \tau \in A_i, W = j) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \times \\
&\quad \int_{A_i \times \{j\}} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \frac{\psi^n(d\tau, dW)}{\psi^n(\tau \in A_i, W = j) \psi^n(d\tau, dW)} \right) \frac{\psi^n(d\tau, dW)}{\psi^n(\tau \in A_i, W = j)} \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \times \int_{A_i \times \{j\}} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \right) \psi^n(d\tau, dW) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \right) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) - \ln(\tilde{\pi}(A)) \\
&= \frac{1}{\tilde{\pi}(A)} \sum_{i=1}^N \sum_{j=1}^n \tilde{\pi}(\tau \in A_i, W = j) \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) - \ln(\tilde{\pi}(A)) \\
&= \frac{1}{\tilde{\pi}(A)} \left[\tilde{\pi}(\tau \in A^c) \ln \left(\frac{\tilde{\pi}(\tau \in A^c)}{\psi^n(\tau \in A^c)} \right) + \sum_{i=1}^N \sum_{j=1}^n \tilde{\pi}(\tau \in A_i, W = j) \ln \left(\frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) \right] \\
&\quad - \left(\ln(\tilde{\pi}(A)) + \tilde{\pi}(A^c) \ln \left(\frac{\tilde{\pi}(A^c)}{\psi^n(A^c)} \right) \right)
\end{aligned}$$

The well known decomposition of entropy property tells us that if $(B_m)_{m \leq M}$ is a partition of $(0, +\infty) \times \{1, \dots, n\}$:

$$H(\tilde{\pi} | \psi^n) \geq \sum_{m=1}^M \tilde{\pi}(B_m) \ln \left(\frac{\tilde{\pi}(B_m)}{\psi^n(B_m)} \right).$$

Here, $(A_i \times \{j\})_{i,j}$ and A^c are a partition of $(0, +\infty) \times \{1, \dots, n\}$. We have:

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} H(\tilde{\pi}_{0,k}^{\rho,M} | \psi_k) = H(\tilde{\pi}_0^{\rho,M} | \psi^n) \\
& \leq \frac{1}{\tilde{\pi}(A)} H(\tilde{\pi} | \psi^n) - \left(\ln(\tilde{\pi}(A)) + \tilde{\pi}(A^c) \ln \left(\frac{\tilde{\pi}(A^c)}{\psi^n(A^c)} \right) \right).
\end{aligned}$$

Since $\tilde{\pi}(A) \rightarrow 1$ and $\tilde{\pi}(A^c) \rightarrow 0$ as $M \rightarrow \infty$:

$$\limsup_{M \rightarrow \infty} \sup_{\rho < 1} \lim_k H(\tilde{\pi}_{0,k}^{\rho, M} | \psi_k) \leq H(\tilde{\pi} | \psi^n).$$

Now, we can construct a sequence $\pi_k (= \pi_k^{\rho_k, L_k, M_k})$ such that: $\pi_k \rightarrow \alpha\pi + (1 - \alpha)\delta_\infty \otimes \eta$ and $\limsup_{k \rightarrow \infty} \pi_k(1/\tau) H(\tilde{\pi}_k | \psi_k) \leq I^n(\mu)$.

Eventually, by (5.12) and (5.13),

$$\begin{aligned} \mu_k \left(\frac{1}{a+b} \right) &= \pi_k \left(\frac{1}{\tau} \right) \\ &= \alpha \pi_{0,k}^{\rho_k, M_k} \left(\frac{1}{\tau} \right) + (1 - \alpha) \eta_k^{L_k, M_k} \left(\frac{1}{\tau} \right) \\ &\xrightarrow{k \rightarrow \infty} \alpha \tilde{\pi} \left(\frac{1}{\tau} \right) = \mu \left(\frac{1}{a+b} \right). \end{aligned}$$

□

6. SOME ADDITIONAL TECHNICAL TOPOLOGICAL LEMMATA.

Before proving the upper bound and the lower bound of the LDP for empirical measures, we add two useful lemmata. The first one will be used several times for the upper bound. Once again its proof uses conditioning.

Lemma 6.1. *Let $\mathcal{D}_M^n = \left\{ \mu \in \Delta^n, \mu \left(\frac{1}{a+b} \right) \leq M \right\}$. Then \mathcal{D}_M^n is a compact set.*

Proof. Since $\mathcal{D}_M^n \subset \mathcal{X}_M$ we already know that it is relatively compact. But since Δ^n is not clearly closed, we have to show that \mathcal{D}_M^n is closed.

Let $(\mu_k)_k$ be a sequence of measures in \mathcal{D}_M^n . We will prove that some subsequence again denoted $(\mu_k)_k$ converges in \mathcal{D}_M^n . We denote by $\alpha_k, \mu_{0,k}, \pi_k$ and η_k the corresponding quantities in the definition.

For any $k, \alpha_k \in [0, 1]$ and $\eta_k \in \mathcal{M}^1(\{1, \dots, n\})$ which is compact, so there exists a subsequence $\alpha_{\varphi(k)}, \eta_{\varphi(k)}$ and $\alpha \in [0, 1]$ and $\eta \in \mathcal{M}^1(\{1, \dots, n\})$ such that $\alpha_{\varphi(k)} \xrightarrow{k \rightarrow \infty} \alpha$ and $\eta_{\varphi(k)} \rightarrow \eta$.

We remind that, for any $r > 0$, $\left\{ \nu \in \mathcal{M}^1((0, +\infty]), \nu \left(\frac{1}{x} \right) \leq r \right\}$ is a compact set.

If $\alpha = 0$: then for $f \in \mathcal{C}_b((0, +\infty]^2 \times \{1, \dots, n\})$:

$$\begin{aligned} \mu_{\varphi(k)}(f) &= \underbrace{\alpha_{\varphi(k)}}_{\rightarrow 0} \underbrace{\mu_{0, \varphi(k)}(f)}_{\in [-\|f\|_\infty, \|f\|_\infty]} + \underbrace{(1 - \alpha_{\varphi(k)})}_{\rightarrow 1} (\delta_{(+\infty, +\infty)} \otimes \underbrace{\eta_{\varphi(k)}}_{\rightarrow \eta})(f) \\ &\xrightarrow{k \rightarrow \infty} (\delta_{(+\infty, +\infty)} \otimes \eta)(f). \end{aligned}$$

Since $(\delta_{(+\infty, +\infty)} \otimes \eta) \in \mathcal{D}_M^n$, $\mu_{\varphi(k)}$ converges in \mathcal{D}_M^n .

If $\alpha > 0$: then for k large enough, we have : $|\alpha_{\varphi(k)} - \alpha| < \alpha/2$, so $\alpha_{\varphi(k)} > \alpha/2$. Then, we have:

$$\pi_{\varphi(k)} \left(\frac{1}{\tau} \right) = \mu_{0, \varphi(k)} \left(\frac{1}{a+b} \right) = \frac{1}{\alpha_{\varphi(k)}} \mu_{\varphi(k)} \left(\frac{1}{a+b} \right) \leq \frac{M}{\alpha_{\varphi(k)}} \leq \frac{2M}{\alpha}.$$

We will prove that for all i , $\pi_{\varphi(k)}|_{W=i}$, which is the conditional law π given $W = i$, belongs to $\left\{ \nu \in \mathcal{M}^1((0, +\infty]), \nu\left(\frac{1}{x}\right) \leq r \right\}$ for a well-chosen r .

For any k , $\pi_{\varphi(k)} \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$, let define $(q_{j, \varphi(k)})_{j \in \{1, \dots, n\}}$ the weights such that $q_{j, \varphi(k)} = \mathbb{P}_{\pi_{\varphi(k)}}(W = j)$. Then $\pi_{\varphi(k)}(1/\tau)$ can be written with its marginal law:

$$\pi_{\varphi(k)}\left(\frac{1}{\tau}\right) = \sum_{j \in \{1, \dots, n\}} q_{j, \varphi(k)} \mathbb{E}_{\pi_{\varphi(k)}} \left[\frac{1}{\tau} \middle| W = j \right].$$

The $((q_{j, \varphi(k)})_{j \in \{1, \dots, n\}})_k$ are at most n sequences of weights in $[0, 1]$. There exists a subsequence $(q_{j, \varphi_2(k)})_{j \in \{1, \dots, n\}}$ and a $(q_j)_{j \in \{1, \dots, n\}}$ with $q_j \in [0, 1]$ such that $\sum q_j = 1$ and $(q_{j, \varphi_2(k)})_{j \in \{1, \dots, n\}}$ tends to $(q_j)_{j \in \{1, \dots, n\}}$. Let J the set of indices j such that $q_j \neq 0$. Let p_{\min} the minimum of the q_j for $j \in J$:

$$p_{\min} = \min\{q_j, j \in J\} \neq 0.$$

Then, for k large enough, we have:

$$\forall j \in J, q_{j, \varphi_2(k)} > \frac{p_{\min}}{2}.$$

Then, we have, for each $j \in J$

$$\pi_{\varphi_2(k)}\left(\frac{1}{\tau}\right) = \sum_{i \in \{1, \dots, n\}} q_{i, \varphi_2(k)} \mathbb{E}_{\pi_{\varphi_2(k)}} \left[\frac{1}{\tau} \middle| W = i \right] \geq \frac{p_{\min}}{2} \mathbb{E}_{\pi_{\varphi_2(k)}} \left[\frac{1}{\tau} \middle| W = j \right]$$

and $\pi_{\varphi_2(k)}\left(\frac{1}{\tau}\right) \leq \frac{2M}{\alpha}$, then:

$$\forall j \in J, \mathbb{E}_{\pi_{\varphi_2(k)}} \left[\frac{1}{\tau} \middle| W = j \right] \leq \frac{4M}{\alpha p_{\min}}.$$

So $\forall j \in J, \pi_{\varphi_2(k)}|_{W=j} \in \left\{ \nu \in \mathcal{M}^1((0, +\infty]), \nu\left(\frac{1}{x}\right) \leq 4M/(\alpha p_{\min}) \right\}$ which is a compact set.

There exists a subsequence of $(\pi_{\varphi_2(k)}|_{W=j})_k$ which converges to a π^j for each $j \in J$. By a diagonal argument, we consider an extraction φ_3 such that:

$$\forall j \in J, \pi_{\varphi_3(k)}|_{W=j} \rightharpoonup \pi^j.$$

We define then π corresponding to $(q_j)_j$ and $(\pi^j)_j$:

$$\begin{aligned} \pi(d\tau, dW) &= \sum_{j \in J} q_j \delta_{(W=j)} \pi^j(d\tau) \\ \forall f \in \mathcal{C}_b((0, +\infty] \times \{1, \dots, n\}), \pi(f) &= \sum_{j \in J} q_j \pi^j(f(\cdot, j)). \end{aligned}$$

Then, $\pi_{\varphi_3(k)} \rightharpoonup \pi$.

It remains to show that μ to belong to \mathcal{D}_M^n i.e to Δ^n .

π can be written as:

$$\pi(d\tau, dW) = \pi(\tau < +\infty) \times \pi(d\tau, dW | \tau < \infty) + (1 - \pi(\tau < \infty)) \pi(d\tau, dW | \tau = \infty).$$

Let $\pi^*(d\tau, dW) = \pi(d\tau, dW | \tau < \infty)$, $\beta^* = \pi(\tau < +\infty)$ and η^* such that $\pi(d\tau, dW | \tau = \infty) = \delta_{(+\infty)} \otimes \eta^*$.

Moreover, $\mu_{0,\varphi(k)}$ is the product measure of du and $\pi_{\varphi(k)}$. $\pi_{\varphi_3(k)}$ weakly converges to π , so $\mu_{0,\varphi_3(k)}$ weakly converges to the product measure of du and π . We denote by μ_0 the product measure of du and π^* . Then $\mu_{\varphi(k)}$ tends to

$$\mu = \alpha\beta^*\mu_0 + \alpha(1 - \beta^*)\delta_{(+\infty,+\infty)} \otimes \eta^* + (1 - \alpha)\delta_{(+\infty,+\infty)} \otimes \eta.$$

With

$$\eta_0 = \frac{\alpha(1 - \beta^*)}{(1 - \alpha\beta^*)}\eta^* + \frac{(1 - \alpha)}{(1 - \alpha\beta^*)}\eta,$$

we can write $\mu = \alpha\beta^*\mu_0 + (1 - \alpha\beta^*)\delta_{(+\infty,+\infty)} \otimes \eta_0$, so $\mu \in \Delta^n$.

□

The next lemma introduces $\bar{\Delta}^n$, a closed set which contains Δ^n (but is not its closure despite the notation). This will be used directly in the proof of the upper bound of the large deviation principle. A similar set is discussed in [13] lemma 2.5. However we have here to carefully manage the third coordinate, so that the proof in [13] has to be rewritten.

Lemma 6.2. *The set*

$$\begin{aligned} \bar{\Delta}^n &:= \{ \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}) : \mu = \alpha\mu_0 + (1 - \alpha)\delta_{(+\infty,+\infty)} \otimes \eta, \alpha \in [0, 1], \\ &\mu_0(da, db, dc) = \int_{[0,1] \times (0, \infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW), \\ &\pi \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\}), \eta \in \mathcal{M}^1(\{1, \dots, n\}) \} \end{aligned} \quad (6.1)$$

is closed in $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$.

Proof. Let $\mu_k \in \bar{\Delta}^n$ such that μ_k converges to a $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ when k tends to ∞ . We will prove that μ is in $\bar{\Delta}^n$. Let α_k , π_k and η_k be the corresponding quantities to μ_k as in the definition of $\bar{\Delta}^n$.

$(\alpha_k)_k \in [0, 1]$ admits a subsequence which converges to a $\alpha \in [0, 1]$. η_k is a sequence of measure on the finite set $\{1, \dots, n\}$, so a subsequence of η_k converges to $\eta \in \mathcal{M}^1$. As usual we identify the subsequence and the sequence.

$(\pi_k)_k \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\}) \subset \mathcal{M}^1([0, +\infty] \times \{1, \dots, n\})$ also admits a subsequence which tends to a limit $\pi \in \mathcal{M}^1([0, +\infty] \times \{1, \dots, n\})$.

Then

$$\begin{aligned} \mu(da, db, dc) &= \alpha \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW) \\ &\quad + (1 - \alpha)\delta_{(+\infty,+\infty)}(da, db) \otimes \eta(dc). \end{aligned}$$

We need to prove that μ is in $\bar{\Delta}^n$. First we verify that π has no weight in 0, and secondly we prove that by rewriting μ , we can consider $\bar{\pi} \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$, $\bar{\alpha}$ and $\bar{\eta}$ such that μ have the good form.

First, we consider the weight of 0 for π . By Skorohod's representation theorem, there exists a sequence $(P_k, V_k)_k$ and X_k of random variables such that $(P_k, V_k) \in (0, +\infty) \times \{1, \dots, n\}$ has the law π_k , X_k has the law η_k and (P_k, V_k) converges a.s. to (P, V) of law π , X_k converges

a.s. to X of law η . Let U be an uniform random variable independent of $(P_k)_k$ and P . For any $f \in \mathcal{C}_b([0, +\infty]^2 \times \{1, \dots, n\})$ we obtain

$$\begin{aligned} \mu_k(f) &= \alpha_k \mathbb{E}[f(UP_k, (1-U)P_k, V_k)] + (1 - \alpha_k) \mathbb{E}[f(+\infty, +\infty, X_k)] \\ &\xrightarrow[k \rightarrow \infty]{} \alpha \mathbb{E}[f(UP, (1-U)P, V)] + (1 - \alpha) \mathbb{E}[f(+\infty, +\infty, X)] \end{aligned}$$

and this limit is equal to $\mu(f)$. For $f_\varepsilon(a, b, c) = \mathbb{1}_{a+b < \varepsilon}$, since $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$: $\alpha\pi(\{P < \varepsilon\}) = \mu(f_\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{} 0$. Then $\pi \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$.

Now we prove that we can write μ in the same form that in $\bar{\Delta}^n$. Let $\beta = \pi(P = +\infty)$. If $\beta = 0$, then

$$\alpha \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW) + (1-\alpha) \delta_{(+\infty, +\infty)}(da, db) \otimes \eta(dc)$$

is already in the good form.

Else,

$$\begin{aligned} \mu(f) &= \alpha\beta \mathbb{E}[f(UP, (1-U)P, V) | P < \infty] + \alpha(1-\beta) \mathbb{E}[f(+\infty, +\infty, V) | P = \infty] \\ &\quad + (1-\alpha) \mathbb{E}[f(+\infty, +\infty, X)] \\ &= \alpha\beta \mathbb{E}[f(UP, (1-U)P, V) | P < \infty] + \\ &\quad (1-\alpha\beta) \left(\frac{\alpha(1-\beta)}{1-\alpha\beta} \mathbb{E}[f(+\infty, +\infty, V) | P = \infty] + \frac{1-\alpha}{1-\alpha\beta} \mathbb{E}[f(+\infty, +\infty, X)] \right). \end{aligned}$$

Let $\bar{\eta}(dW) = \frac{\alpha(1-\beta)}{1-\alpha\beta} \pi(dW | \tau = \infty) + \frac{1-\alpha}{1-\alpha\beta} \eta$, $\bar{\alpha} = \alpha\beta$ and $\bar{\pi}(d\tau, dW) = \pi(d\tau, dW | \tau < \infty)$. Therefore $\mu \in \bar{\Delta}^n$. \square

Remark 6.3. Here again one can replace $\{1, \dots, n\}$ by a compact subset \mathbb{W} . \diamond

We finally set two important results. The first one is identical to Lemma 2.6 and Lemma 2.7 in [13]:

Lemma 6.4. *If $\mu \in \Delta^n$ then $I^n(\mu) \leq \sup_{f \in \Gamma} \mu(f)$ and if $\mu \in \bar{\Delta}^n \setminus \Delta^n$ then $\sup_{f \in \Gamma} \mu(f) = +\infty$. (recall that Γ is defined in (4.9))*

The proof of the Lemma is almost identical to [13]. Indeed it is enough to consider a third variable $c \in \{1, \dots, n\}$. Similarly to the proof of Lemma 2.6 we have to introduce a function

$$f_{d,\varphi,M}(a, b, c) = \frac{\varphi(a+b, c)}{a+b} + d\mathbb{1}_{(M, +\infty]}(a+b)$$

for a continuous and compactly supported φ . $f_{d,\varphi,M}$ is lower semi-continuous. One can then copy the proof of Lemma 2.6 in [13]. For the second part we may consider the same function f_ε as in the proof of Lemma 2.7 in [13].

Notice that again we may replace $\{1, \dots, n\}$ by a compact \mathbb{W} .

The second one (and its proof) is identical to Proposition 3.3 in [13]

Lemma 6.5. *For all $f \in \Gamma$ and all $t > 0$, $\mathbb{E}(e^{t\mu_t^n(f)}) \leq \frac{D_{n,f}}{1-C_{n,f}} < +\infty$.*

7. PROOF OF THEOREM 4.3.

In this section, we prove the LDP for \mathbb{P}_t^n that denotes the \mathbb{P} distribution of μ_t^n . This time the introduction of the third coordinate W replacing $F(\tau)$ does not create any new difficulty. However some points in the proofs of [13] are not clear for us and we will give the details for some points.

The proof of the upper bound is made in several steps: the proof of a weak principle, for compact sets \mathcal{C} (itself divided in several steps), and the proof that μ_t^n is an exponentially tight family, i.e. satisfies: for all $\alpha \in \mathbb{R}^+$ there exists some compact set \mathcal{K}_α with

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n(\mathcal{K}_\alpha^c) < -\alpha.$$

The (full) upper bound for closed sets \mathcal{C} then follows from [9]. These steps are described in [13] section 3.

Exponential tightness is the aim of Lemma 3.1 in [13]. The first step is the following lemma

Lemma 7.1. *We have:*

$$\lim_{M \rightarrow +\infty} \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P} \left(\mu_t^n \left(\frac{1}{a+b} \right) > M \right) = -\infty.$$

whose proof is unchanged in our case. Since $\{\nu \in \mathcal{M}_1((0, +\infty]^2 \times \{1, \dots, n\}), \nu(1/(a+b)) \leq M\}$ is compact (see lemma 4.1), exponential tightness follows.

The proof of the upper bound for compact subsets is done in [13] p. 2261 and 2262 beginning with the definition of the set

$$\Delta_{M,g,\delta} = \{\mu \in \mathcal{M}_1((0, +\infty]^2), \exists \nu \in \Delta, |\nu(g) - \mu(g)| \leq \delta, \mu(1/(a+b)) \leq M\}$$

for some continuous and bounded g , δ and M positive. We do not see an immediate argument showing that this set is closed (hence compact since it is relatively compact). We will thus slightly modify the proof in [13].

We introduce a modified set for M , δ and g as before,

$$\begin{aligned} \Delta_{M,g,\delta}^n &= \{\mu \in \mathcal{M}_1((0, +\infty]^2), \exists \nu \in \Delta^n, |\nu(g) - \mu(g)| \leq \delta, \\ &\quad \mu(1/(a+b)) \leq M, \nu(1/(a+b)) \leq M + \delta\}. \end{aligned} \quad (7.1)$$

We also define

$$R_{M,g,\delta}^n := - \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n((\Delta_{M,g,\delta}^n)^c). \quad (7.2)$$

One can of course replace g by a finite number of continuous and bounded g_i 's.

Lemma 7.2. $\Delta_{M,g,\delta}^n$ is a compact set.

Proof. Notice that if $\mu_k \in \Delta_{M,g,\delta}^n$ weakly converges to some μ , $\mu \in \mathcal{D}_M^n$ according to lemma 6.1. The corresponding sequence ν_k in Δ^n actually belongs to $\mathcal{D}_{M+\delta}^n$ so that one can find a subsequence still denoted ν_k that converges to some $\nu \in \mathcal{D}_{M+\delta}^n$ (this is the key difference with $\Delta_{M,g,\delta}$). Since g is bounded and continuous, by taking limits we have $|\nu(g) - \mu(g)| \leq \delta$. Of course for $\varepsilon > 0$,

$$\nu(1/(a+b+\varepsilon)) = \lim_k \nu_k(1/(a+b+\varepsilon)) \leq \lim_k \nu_k(1/(a+b)) \leq M + \delta,$$

and $\nu(1/(a+b)) \leq M + \delta$ follows by letting ε go to 0 thanks to the monotone convergence theorem. Compactness follows since $\Delta_{M,g,\delta}^n \subset \mathcal{D}_M^n$ which is compact. \square

The next step consists in showing that the empirical measure μ_t^n is close to Δ^n more precisely

Lemma 7.3. *For $f \in C_b((0, +\infty]^2 \times \{1, \dots, n\})$ we define*

$$\nu_t^n(f) := \frac{1}{t} \sum_{i=1}^{M_t} \tau_i \int_0^1 f(u\tau_i, (1-u)\tau_i, W_i^n) du + \frac{t - S_{M_t}}{t} f(+\infty, +\infty, W_{M_t+1}^n).$$

Then $\nu_t^n \in \Delta^n$ almost surely.

For all g, δ there exists some $t(g, \delta)$ such that for $t \geq t(g, \delta)$, the events $\{|\mu_t^n(g) - \nu_t^n(g)| > \delta\}$ and $\left\{\left|\mu_t^n\left(\frac{1}{a+b}\right) - \nu_t^n\left(\frac{1}{a+b}\right)\right| > \delta\right\}$ are almost surely empty.

Proof. This lemma is the analogue of Lemma 3.2 in [13] but due to the modification of our $\Delta_{M,g,\delta}^n$ we have to complete the proof therein.

Arguing as in the proof of Lemma 3.2 in [13] one shows that for t large enough the set $\{|\mu_t^n(g) - \nu_t^n(g)| > \delta\}$ is almost surely empty. For the other term,

$$\begin{aligned} & \left| \mu_t^n\left(\frac{1}{a+b}\right) - \nu_t^n\left(\frac{1}{a+b}\right) \right| \\ &= \left| \left(\frac{1}{t} \sum_{i=1}^{M_t} \tau_i \times \frac{1}{\tau_i} + \frac{\tau_{M_t+1}}{t} \times \frac{t - S_{M_t}}{\tau_{M_t+1}} \times \frac{1}{\tau_{M_t+1}} \right) - \frac{1}{t} \sum_{i=1}^{M_t} \tau_i \times \frac{1}{\tau_i} \right| \\ &= \frac{1}{t} \frac{t - S_{M_t}}{\tau_{M_t+1}} \leq \frac{1}{t}. \end{aligned}$$

Hence for $t > 1/\delta$ we have $\left| \mu_t^n\left(\frac{1}{a+b}\right) - \nu_t^n\left(\frac{1}{a+b}\right) \right| \leq \delta$. \square

Corollary 7.4. $\lim_{M \rightarrow +\infty} R_{M,\delta,g}^n = +\infty$.

Proof. According to lemma 7.3, for t large enough (that does not depend on M but only on δ and g) the set $\{\mu_t^n(1/(a+b)) \leq M\} \cap (\Delta_{M,g,\delta}^n)^c$ is almost surely empty. Hence for t large enough

$$\mathbb{P}^n((\Delta_{M,g,\delta}^n)^c) \leq \mathbb{P}^n(\mu_t^n(1/(a+b)) > M)$$

and the result follows from lemma 7.1. \square

We may now follow the proof in [13]. First exactly as in [13], thanks to Lemma 6.5, for all open set \mathcal{O} , all g, M, δ and all $f \in \Gamma$,

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n(\mathcal{O}) \leq - \inf_{\nu \in \mathcal{O}} I_{f,M,g,\delta}^n(\mu)$$

where

$$I_{f,M,g,\delta}^n(\mu) = \begin{cases} \mu(f) \wedge R_{M,g,\delta}^n & \text{if } \mu \in \Delta_{M,g,\delta}^n \\ +\infty & \text{otherwise.} \end{cases}, \quad (7.3)$$

Since f is lower semicontinuous, $I_{f,M,g,\delta}^n(\mu)$ is also lower semicontinuous thanks to the compactness of $\Delta_{M,g,\delta}^n$. One can thus deduce as in [13] that for all compact subset \mathcal{K} ,

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n(K) \leq - \inf_{\mu \in \mathcal{K}} \sup_{f,M,g,\delta} I_{f,M,g,\delta}^n(\mu) := - \inf_{\mu \in \mathcal{K}} \tilde{I}(\mu).$$

If $\mu \notin \bar{\Delta}^n$ which is closed according to lemma 6.2, one can find an open neighborhood \mathcal{U} of μ such that $\mathcal{U} \cap \bar{\Delta}^n = \emptyset$. We may choose \mathcal{U} of the form

$$\mathcal{U} = \bigcap_{i=1}^k \{\nu, |\nu(g_i) - \mu(g_i)| \leq \delta\}$$

for some family g_1, \dots, g_k of bounded and continuous functions, so that for at least one of the g_i 's denoted by g , $\mu \notin \Delta_{M,g,\delta}^n$. Hence $\bigcap_{g,\delta} \Delta_{M,g,\delta}^n \subset \bar{\Delta}^n$ and $\tilde{I}(\mu) = +\infty$ if $\mu \notin \bar{\Delta}^n$. Together with corollary 7.4 we deduce that

$$\tilde{I}(\mu) \geq \sup_{f \in \Gamma} I_f(\mu) \quad \text{where} \quad I_f(\mu) = \mu(f) \text{ if } \mu \in \bar{\Delta}^n, \quad I_f(\mu) = +\infty \text{ otherwise.}$$

Thus, according to lemma 6.4, $\tilde{I}(\mu) \geq I^n(\mu)$ and the upper bound is proved.

The proof of the lower bound is similar to [13], just replacing the sample τ_i by a sample (τ_i, W_i) , and thus is omitted.

8. PROOF OF THEOREM 4.4.

We turn to the proof of theorem 4.4. We first will deduce a LDP from theorem 4.3 by using the contraction principle, and then identify the rate function.

8.1. The contraction principle.

Define $\varphi(a, b, c) = \frac{c}{a+b}$ which is continuous on $(0, +\infty]^2 \times \{1, \dots, n\}$. Then

$$\mu_t^n(\varphi) = \frac{1}{t} \sum_{i=1}^{M_t} W_i^n + \frac{(t - S_{M_t})W_{M_t+1}^n}{t \tau_{M_t+1}}.$$

Remark that

$$\left| \frac{(t - S_{M_t})W_{M_t+1}^n}{t \tau_{M_t+1}} \right| \leq \frac{n}{t}$$

so that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\left| \mu_t^n(\varphi) - \frac{1}{t} \sum_{i=1}^{M_t} W_i^n \right| > \delta \right) = -\infty$$

showing that $\mu_t^n(\varphi)$ and $\frac{1}{t} \sum_{i=1}^{M_t} W_i^n = Z_t^n/t$ will satisfy the same LDP. The contraction principle (Theorem 4.2.1 in [9]) should thus furnish some LDP for Z_t^n/t . Unfortunately φ is not bounded so that $\mu \mapsto \mu(\varphi)$ is not continuous from $\mathcal{M}_1(0, +\infty]^2 \times \{1, \dots, n\}$ to \mathbb{R} and the contraction principle does not apply directly. We are thus obliged one more time to use an approximation procedure replacing φ by

$$\varphi^\varepsilon(a, b, c) = \frac{c}{(a+b) \vee \varepsilon}$$

for $\varepsilon > 0$. Replacing τ_i by $\tau_i^\varepsilon = \tau_i \vee \varepsilon$ we may introduce M_t^ε , A_t^ε , B_t^ε , $C_t^{n,\varepsilon}$ and $\mu_t^{n,\varepsilon}$ as in subsection 4.1. The results of the previous section apply to this new process (introducing a rate function $I^{n,\varepsilon}(\mu)$) This time we may apply the contraction principle so that $\mu_t^{\varepsilon,n}(\varphi^\varepsilon)$ satisfies a LDP with rate function

$$J^{\varepsilon,n}(m) = \inf\{I^{n,\varepsilon}(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi^\varepsilon)\}. \quad (8.1)$$

We now have

$$\mu_t^{\varepsilon,n}(\varphi^\varepsilon) = \frac{1}{t} \sum_{i=1}^{M_t^\varepsilon} W_i^n + \frac{(t - S_{M_t^\varepsilon}^\varepsilon)W_{M_t^\varepsilon+1}^n}{t \tau_{M_t^\varepsilon+1}^\varepsilon},$$

so that, since $M_t^\varepsilon \leq M_t$,

$$\begin{aligned} \left| \mu_t^{\varepsilon,n}(\varphi^\varepsilon) - \frac{1}{t} \sum_{i=1}^{M_t} W_i^n \right| &\leq \frac{1}{t} \left| \sum_{i=M_t^\varepsilon+1}^{M_t} W_i^n \right| + \left| \frac{(t - S_{M_t^\varepsilon}^\varepsilon)W_{M_t^\varepsilon+1}^n}{t \tau_{M_t^\varepsilon+1}^\varepsilon} \right| \\ &\leq \frac{n}{t} ((M_t - M_t^\varepsilon) + 1). \end{aligned}$$

But according to [13] Lemma 5.4, M_t^ε is an exponentially good approximation of M_t . It follows that $\mu_t^{\varepsilon,n}(\varphi^\varepsilon)$ is an exponentially good approximation of Z_t^n/t so that, finally, thanks to Theorem 2.2, Z_t^n/t satisfies a weak LDP with rate function

$$\tilde{J}^n(m) = \sup_{\delta > 0} \liminf_{\varepsilon \rightarrow 0} \inf_{|z-m| < \delta} J^{\varepsilon,n}(m). \quad (8.2)$$

In particular we know that \tilde{J}^n is l.s.c. so that its level sets are closed.

8.2. Study of the rate function.

The goal of this subsection is to show the following lemma (partly close to Lemma 5.1 in [13]) explaining the various forms of the rate function (recall that $\varphi(a, b, c) = c/(a + b)$).

Lemma 8.1. *We define, for all $m > 0$, $\bar{J}^n(m) := \inf\{I^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\}$. Then*

i) $\bar{J}^n = J^n$ (J^n is defined in (2.6)). In addition

$$\bar{J}^n(m) = \inf\{I_0^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\}.$$

ii) We also have:

$$\bar{J}^n(m) = \min\{I^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\} \quad (8.3)$$

$$= \min\{I_0^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\}. \quad (8.4)$$

iii) Finally,

$$J^n = \bar{J}^n = \tilde{J}^n.$$

Proof. Proof of i) We have

$$\begin{aligned}
\bar{J}^n(m) &:= \inf \left\{ I^n(\mu), \mu \in \mathcal{M}_1((0, +\infty]^2 \times \{1, \dots, n\}), \int \frac{c}{a+b} \mu(da, db, dc) = m \right\} \\
&= \inf \left\{ I^n(\mu), \mu \in \Delta^n, \int \frac{c}{a+b} \mu(da, db, dc) = m \right\} \\
&= \inf \left\{ \alpha \pi \left(\frac{1}{\tau} \right) H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \mu_0 \in \Delta_0^n, \alpha \int \frac{c}{a+b} \mu_0(da, db, dc) = m \right\} \\
&= \inf \left\{ \alpha \pi \left(\frac{1}{\tau} \right) H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \mu_0 \in \Delta_0^n, \alpha \pi \left(\frac{W}{\tau} \right) = m \right\} \\
&= \inf \left\{ \alpha \frac{1}{\tilde{\pi}(\tau)} H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \tilde{\pi} \in \mathcal{M}_1((0, +\infty) \times \{1, \dots, n\}), \alpha \frac{\tilde{\pi}(W)}{\tilde{\pi}(\tau)} = m \right\} \\
&= \inf \left\{ \frac{\alpha}{\beta} H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \beta > 0, \tilde{\pi}(\tau) = \beta, \alpha \frac{\tilde{\pi}(W)}{\beta} = m \right\}
\end{aligned}$$

Let $p(a, b) = \inf \{ H(\nu|\psi^n), \nu(\tau) = a, \nu(W) = b \}$. We have $p = \Lambda_n^*$ according to Csiszar I-projection theorem (Theorem 3 in [8]). As in [13] (proof of Lemma 5.1) another way is to directly prove the dual equality $p^* = \Lambda^n$.

We thus have

$$\bar{J}^n(m) = \inf \left\{ \frac{\alpha}{\beta} \Lambda_n^* \left(\beta, \frac{m\beta}{\alpha} \right) + (1-\alpha)\xi, \alpha \in [0, 1], \beta > 0 \right\}.$$

But

$$\begin{aligned}
\frac{\alpha}{\beta} \Lambda_n^* \left(\beta, \frac{m\beta}{\alpha} \right) &= \sup_{x,y} \left\{ \alpha x + m y - \frac{\alpha}{\beta} \Lambda_n(x, y) \right\} \\
&= \beta' \Lambda_n^* \left(\frac{\alpha}{\beta'}, \frac{m}{\beta'} \right) \text{ where } \beta' = \frac{\alpha}{\beta}.
\end{aligned}$$

Thus

$$\bar{J}^n(m) = \inf \left\{ \beta \Lambda_n^* \left(\frac{\alpha}{\beta}, \frac{m}{\beta} \right) + (1-\alpha)\xi, \alpha \in [0, 1], \beta > 0 \right\}.$$

We will show that :

$$\inf_{\alpha \in [0, 1]} \left\{ \beta \Lambda_n^* \left(\frac{\alpha}{\beta}, \frac{m}{\beta} \right) + (1-\alpha)\xi \right\} = \beta \Lambda_n^* \left(\frac{1}{\beta}, \frac{m}{\beta} \right).$$

Taking $\alpha = 1$, we see that the left hand side is less than or equal to the right hand side. To show the converse inequality, pick $\alpha \in [0, 1]$:

$$\begin{aligned}
\beta \Lambda_n^* \left(\frac{\alpha}{\beta}, \frac{m}{\beta} \right) + (1-\alpha)\xi &= \sup_{x,y} \{ \alpha x + (1-\alpha)\xi + m y - \beta \Lambda_n(x, y) \} \\
&\geq \sup_{x,y} \{ x \wedge \xi + m y - \beta \Lambda_n(x, y) \}.
\end{aligned}$$

Since W is bounded, $e^{yW} \geq C(y) > 0$ for all y , so that we have for all $x > \xi$ and all y ,

$$\psi^n(e^{x\tau + yW}) \geq C(y) \psi^n(e^{x\tau}) = +\infty.$$

This shows that $\Lambda_n(x, y) = +\infty$, for all $x > \xi$ and for all y . Hence, the supremum on x can be restricted to the supremum on $\{x \leq \xi\}$:

$$\begin{aligned} \beta\Lambda_n^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1 - \alpha)\xi &\geq \sup_{x,y} \{x \wedge \xi + my - \beta\Lambda_n(x, y)\} \\ &= \sup_{x \leq \xi, y} \{x + my - \beta\Lambda_n(x, y)\} \\ &= \beta\Lambda_n^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right) \end{aligned}$$

and the desired inequality is proved.

Notice that during the proof we have seen than the minimization is obtained looking only at $\mu \in \Delta_0^n$ so that we may replace I^n by I_0^n in the definition of \bar{J}^n .

Proof of ii). This part is completely similar to the corresponding one in the proof of Lemma 5.1 in [13]. The only thing to see is that we may replace $\|F\|_\infty$ in [13] by K if $|W| \leq K$.

Proof of iii). To prove this equality, we first prove the inequality $\bar{J}^n(m) \leq \tilde{J}^n(m)$.

Let $m \in \mathbb{R}^+$ and $\varepsilon > 0$. First, remark that $\bar{J}^n(m) \leq J^{n,\varepsilon}(m)$. Indeed, if $\mu \in \Delta^n$, with associated π , α and η , is such that $H(\tilde{\pi}|\psi^{n,\varepsilon}) < +\infty$ and $\mu(\varphi^\varepsilon) = m$, then $\tilde{\pi}$ has its support included in $(\varepsilon, +\infty) \times \{1, \dots, n\}$. Hence $I^n(\mu) = I^{n,\varepsilon}(\mu)$. Moreover, $\mu(\varphi) = \mu\left(\frac{1}{a+b}\right) = m$. This yields

$$\begin{aligned} \bar{J}^n(m) &= \inf\{I^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\} \\ &\leq \inf\{I^n(\mu), \mu \in \mathcal{M}^1(\{(a, b, c) \in (0, +\infty)^2 \times \{1, \dots, n\}\}, \mu(\{a + b \leq \varepsilon\}) = 0, m = \mu(\varphi)\} \\ &= \inf\{I^{n,\varepsilon}(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi^\varepsilon)\} = J^{n,\varepsilon}. \end{aligned}$$

We will take the limit as $\varepsilon \rightarrow 0$. To this end we may write for $\delta > 0$,

$$\inf_{z, |z-m| \leq \delta} \bar{J}^n(z) \leq \inf_{z, |z-m| \leq \delta} J^{n,\varepsilon}(z)$$

so that

$$\inf_{z, |z-m| \leq \delta} \bar{J}^n(z) \leq \liminf_{\varepsilon \rightarrow 0} \inf_{z, |z-m| \leq \delta} J^{n,\varepsilon}(z) \leq \tilde{J}^n(m)$$

Since $\bar{J}^n = J^n$, it is clearly lower semicontinuous so that

$$\bar{J}^n(m) \leq \liminf_{\delta \rightarrow 0} \inf_{z, |z-m| \leq \delta} \bar{J}^n(z) \leq \tilde{J}^n(m).$$

Finally $\bar{J}^n \leq \tilde{J}^n$

Now, we prove that $\bar{J}^n \geq \tilde{J}^n$. Since $\bar{J}^n(m)$ is a minimum, let $\bar{\mu}$ a measure such that $\bar{J}^n(m) = I^n(\bar{\mu})$ and $\bar{\mu}(\varphi) = m$. For a sequence ε_k going to 0, we consider ψ_k the distribution of $(\tau \vee \varepsilon_k, W)$. Using proposition 5.6, there exists $\mu_k \in \Delta_0^n$ with associated to ψ_k such that $\mu_k \rightarrow \bar{\mu}$ and $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) \leq I^n(\bar{\mu})$. Here $I_k^n = I^n$, so $\limsup_{k \rightarrow \infty} I^n(\mu_k) \leq I^n(\bar{\mu})$.

By construction $\mu_k(\{a + b < \varepsilon_k\}) = 0$ and for $\tilde{\pi}_k$ associated to μ_k , we have $H(\tilde{\pi}_k|\psi^n) < +\infty$. Since $\tilde{\pi}_k$ is a measure on $(\varepsilon_k, +\infty) \times \{1, \dots, n\}$, $H(\tilde{\pi}_k|\psi_k) = H(\tilde{\pi}_k|\psi^n)$, and it follows $I^n(\mu_k) = I^{n,\varepsilon_k}(\mu_k)$.

As a consequence of proposition 5.6 it holds $\mu_k\left(\frac{1}{a+b}\right) \rightarrow \bar{\mu}\left(\frac{1}{a+b}\right)$. Hence, $\delta_k = \left|\mu_k\left(\frac{1}{a+b}\right) - \bar{\mu}\left(\frac{1}{a+b}\right)\right| \rightarrow 0$.

We can now write

$$\inf_{z, |z-m| \leq \delta_k} J^{n, \varepsilon_k}(z) \leq I^{n, \varepsilon_k}(\mu_k) = I^n(\mu_k)$$

so that

$$\limsup_{k \rightarrow \infty} \inf_{z, |z-m| \leq \delta_k} J^{n, \varepsilon_k}(z) \leq \limsup_{k \rightarrow \infty} I^n(\mu_k) \leq I^n(\bar{\mu}) = \bar{J}^n(m)$$

and finally

$$\tilde{J}^n(m) = \sup_{\delta > 0} \liminf_{\varepsilon \rightarrow 0} \inf_{z, |z-m| < \delta} J^{n, \varepsilon}(z) \leq \bar{J}^n(m).$$

□

Remark 8.2. The first two items i) and ii) in the previous lemma are still hold true if we replace $\{1, \dots, n\}$ by a compact set \mathbb{W} . ◇

8.3. End of the proof.

At this point we have obtained that Z_t^n/t satisfies a weak LDP with rate function J^n . In order to get the full LDP we have to show that the level sets are bounded (since we know that they are closed). Recall that it is not a direct consequence of the contraction principle since $\mu \mapsto \mu(\varphi)$ is not continuous.

First since $J^n \leq J^{n, \varepsilon}$ for all $\varepsilon > 0$, for all closed set F it holds $\limsup_{\varepsilon \rightarrow 0} \inf_{y \in F} J^{n, \varepsilon}(y) \geq \inf_{y \in F} J^n(y)$.

It remains to show that J^n is a good rate function i.e. that it has compact level sets. According to the previous subsection for a sequence m_k such that $J^n(m_k) \leq \beta$, we may find a sequence $\mu_k \in \Delta_0^n$ such that $\mu_k(\varphi) = m_k$ and $I_0^n(\mu_k) \leq \beta$. The corresponding π_k satisfies $\limsup \pi_k(1/(a+b)) < +\infty$ according to proposition 5.3 (here $\alpha_k = 1$). Since $|\varphi(a, b, c)| \leq C/(a+b)$, m_k is bounded and one can find a convergent subsequence.

9. PROOF OF THEOREM 2.5.

When W is bounded and discrete, the full LDP is already given by Theorem 4.4. If W isn't bounded, or isn't discrete, we need to differentiate whether $\beta_0 = \infty$ or not.

In the first case, when $\beta_0 = \infty$, we are able to prove a Large Deviation principle. Since the rate function \tilde{J} is difficult to calculate, we compare \tilde{J} with an other function J to simplify some inequalities. This work is done in the following subsection 9.1.

In the second case, since the approximation Z_t^n/t isn't an exponentially good approximation of Z_t/t , we only prove some useful inequalities of deviations, but we're not able to prove the Large Deviation Principle.

9.1. Case A: $\beta_0 = \infty$. In this case, Z_t^n/t is an exponentially good approximation of Z_t/t (see Lemma 3.1, 3.2 or 3.4 depending on the approximation strategy). Combining the LDP principle obtained for Z_t^n/t (Theorem 4.4) with Theorem 2.2 (1) we obtain that Z_t/t satisfies a weak LDP with rate function

$$\tilde{J}(m) = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{|m-z| < \delta} J^n(z).$$

In order to obtain a full LDP, we use lemma 2.3. Therefore it remains to show that Z_t/t is exponentially tight.

Lemma 9.1. *Assume that Assumption 2.4 is fulfilled, then $(Z_t/t)_{t \geq 0}$ is exponentially tight, i.e. for all $\alpha > 0$, there exists a compact set K_α such that*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t}{t} \notin K_\alpha^c \right) < -\alpha.$$

Proof. Since Z_t^n/t is an approximation of Z_t/t and satisfies a full LDP, we can decompose the probability as following: for each n , and for all δ :

$$\begin{aligned} \mathbb{P} \left(\frac{Z_t}{t} \notin [-A, A] \right) &\leq \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) + \mathbb{P} \left(\frac{Z_t^n}{t} \notin [-A + \delta, A - \delta] \right) \\ &\leq \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) + \mathbb{P} \left(\frac{Z_t^n}{t} < -A + \delta \right) + \mathbb{P} \left(\frac{Z_t^n}{t} > A - \delta \right). \\ &\leq 3 \max \left(\mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right), \mathbb{P} \left(\frac{Z_t^n}{t} < -A + \delta \right), \mathbb{P} \left(\frac{Z_t^n}{t} > A - \delta \right) \right). \end{aligned} \quad (9.1)$$

By Lemma 3.1, Z_t^n/t and Z_t/t satisfies

$$\forall \delta > 0, \lim_n \overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) = -\frac{\beta_0 \delta}{2},$$

i.e.

$$\forall \alpha > 0, \forall \delta > \frac{2\alpha}{\beta_0}, \exists n(\alpha, \delta), \forall n > n(\alpha, \delta), \overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) \leq -\alpha. \quad (9.2)$$

(If $\beta_0 = +\infty$, we can consider all $\delta > 0$.)

We just have to study $\mathbb{P} \left(\frac{Z_t^n}{t} > A - \delta \right)$ and the symmetric case. We know from Theorem 4.4 that:

$$\overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t^n}{t} > B \right) \leq - \inf_{m > B} J^n(m).$$

It remains to show that $\forall \alpha > 0$ one can choose a level B_α such that $\forall m > B_\alpha, J^n(m) > \alpha$.

Remind that

$$\begin{aligned} J^n(m) &= \inf_{\beta > 0} \left\{ \beta \lambda_n^* \left(\frac{1}{\beta}, \frac{m}{\beta} \right) \right\} \\ &= \inf_{\beta > 0} \sup_{x, y} \left\{ x + my - \beta \log \mathbb{E}[e^{x\tau + yW_n}] \right\}, \end{aligned}$$

(where $W_n = W \wedge n \vee (-n)$).

Since $-|W| \leq W_n \leq |W|$, we obtain by differentiating whether y is positive or negative that

$$x + my - \beta \log \mathbb{E}[e^{x\tau + yW_n}] \geq x + my - \beta \log \mathbb{E}[e^{x\tau + |y||W|}].$$

Therefore, we deduce the lower bound

$$J^n(m) \geq J^{|\cdot|}(m) := \inf_{\beta > 0} \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + |m|y - \beta \log \mathbb{E}[e^{x\tau + y|W|}] \right\}.$$

Remark that $J^{|\cdot|}$ is an even function, thus by symmetry, we can assume $m \geq 0$.
Now, using Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} \sup_{x \in \mathbb{R}, y \geq 0} \{x + my - \beta \log \mathbb{E}[e^{x\tau + y|W|}]\} &= \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \log \mathbb{E}[e^{x\tau + y|W|}]^2 \right\} \\ &\geq \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \log \mathbb{E}[e^{2x\tau}] - \frac{\beta}{2} \log \mathbb{E}[e^{2y|W|}] \right\} \\ &\geq \sup_{x \in \mathbb{R}} \left\{ x - \frac{\beta}{2} \log \mathbb{E}[e^{2x\tau}] \right\} + \sup_{y \geq 0} \left\{ my - \frac{\beta}{2} \log \mathbb{E}[e^{2y|W|}] \right\} \end{aligned}$$

There exists $x_0 < 0$ such that $\mathbb{E}[e^{2x_0\tau}] \leq e^{-1}$, and since $\beta_0 > 0$, $y \mapsto \mathbb{E}[e^{y|W|}]$ is strictly increasing and continuous on $[0, \beta_0)$. Then there exists $y_0 > 0$ such that $1 < \mathbb{E}[e^{2y_0|W|}] \leq e$. Therefore

$$\begin{aligned} \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \log \mathbb{E}[e^{x\tau + y|W|}]^2 \right\} &\geq x_0 - \frac{\beta}{2} \log \mathbb{E}[e^{2x_0\tau}] + my_0 - \frac{\beta}{2} \log \mathbb{E}[e^{2y_0|W|}] \\ &\geq x_0 + my_0 + \frac{\beta}{2} (1 - \log \mathbb{E}[e^{2y_0|W|}]) \end{aligned}$$

Finally, since $1 - \log \mathbb{E}[e^{2y_0|W|}] > 0$, we conclude that

$$J^{|\cdot|}(m) \geq x_0 + my_0 \xrightarrow{m \rightarrow \infty} +\infty.$$

In conclusion, let $\alpha > 0$ and $\delta > \frac{2\alpha}{\beta_0}$ (or $\delta > 0$ if $\beta_0 = +\infty$), let $n(\alpha, \delta)$ defined in (9.2) and $n \geq n(\alpha, \delta)$, let $A = \frac{\alpha - x_0}{y_0} + \delta$ (x_0 and y_0 are determined by the law of τ and W). We have

$$\begin{aligned} \overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t^n}{t} > A - \delta \right) &\leq - \inf_{m > A - \delta} J^n(m) \\ &\leq - \inf_{m > -A + \delta} J^{|\cdot|}(m) \\ &\leq - \inf_{m > A - \delta} x_0 + my_0 \\ &\leq -x_0 - y_0(A - \delta) \\ &\leq -\alpha \end{aligned}$$

and using the evenness of $J^{|\cdot|}$,

$$\overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t^n}{t} < -A + \delta \right) \leq -\alpha.$$

Eventually combining these bounds with (9.2) in (9.1), and deduce that for $n \geq n(\alpha, \delta)$,

$$\overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t}{t} \notin [-A, A] \right) \leq -\alpha.$$

□

Now, we have a full LDP for Z_t/t , but the expression of \tilde{J} isn't very convenient. In particular, this expression can depend on the reduction used. In order to simplify some inequality and obtain Equations (2.8) and (2.9), we prove $\tilde{J} \geq J$ defined in (2.6).

We consider two cases: in the first one, Lemma 9.2, we assume W is bounded by K and consider the discretization W^n defined in subsection 3.2. Then in the second one, Lemma

9.5, we will examine the general case.

Contrary to [13] we do not know a priori that \tilde{J} is a good rate function. However, we use results of Lemma 8.1 replacing $\{1, \dots, n\}$ by $[-K, K]$. In particular we deduce that

$$J(m) = \bar{J}(m) := \min\{I(\mu); \mu \in \mathcal{M}^1((0, +\infty]^2 \times [-K, K]), \mu(\varphi) = m\}.$$

We want to prove

Lemma 9.2. *If W is bounded, it holds $J \leq \tilde{J}$.*

Proof. We may of course assume that $\tilde{J}(m) < +\infty$. One can thus find sequences m_n and $\varepsilon_n \leq 1$ such that $\varepsilon_n \rightarrow 0$, $m_n \rightarrow m$ and $J^n(m_n) \leq \tilde{J}(m) + \varepsilon_n$. Since $J^n = \bar{J}^n$, one can thus find a sequence $\mu_n \in \mathcal{M}^1((0, +\infty]^2 \times [-K, K])$ such that $I_n(\mu_n) < \tilde{J}(m) + 1$. According to proposition 5.3 (recall that it is true here, see remark 5.5) one can thus find some subsequence still denoted by μ_n converging to some μ in $\mathcal{M}^1((0, +\infty]^2 \times [-K, K])$. Now (again recall remark 5.5) according to proposition 5.4 we have $\liminf_n I_n(\mu_n) \geq I(\mu)$ so that $I(\mu) \leq \tilde{J}(m)$.

In order to see that $I(\mu) \geq J(m)$ it remains to show that $\mu(c/(a+b)) = m$ and to apply $J = \bar{J}$.

Notice that thanks to ii) in lemma 8.1 we may have chosen $\mu_n \in \Delta_0^n$ so that proposition 5.3 tells us in addition that $\limsup_n \mu_n(1/(a+b)) = M < +\infty$ (since the sequence $\alpha_n = 1$). It follows

$$\mu(1/(a+b+\varepsilon)) = \lim_n \mu_n(1/(a+b+\varepsilon)) \leq \limsup_n \mu_n(1/(a+b)) = M$$

so that again using monotone convergence we deduce $\mu(1/(a+b)) \leq M$. Hence, since c is μ a.s. bounded, $\mu(c/(a+b))$ is well defined.

To calculate $\mu(c/(a+b))$ we need more. Using the definition of \bar{J}^n and what precedes the sequence μ_n satisfies

$$H(\tilde{\pi}^n | \psi^n) = \frac{m_n}{\mu_n(1/(a+b))} \leq C$$

for some $C < +\infty$ since m_n is bounded and $\lim_n \mu_n(1/(a+b)) = \mu(1/(a+b)) > 0$. We will deduce that $\tilde{\pi}^n$ is tight. Indeed a standard application of the Orlicz-Hölder inequality (with the conjugate pair $u \mapsto e^u - 1 - u$ and $u \mapsto u \mapsto u \ln u - u$) shows that

$$\tilde{\pi}^n(A) \leq \kappa C \frac{1}{\ln(1/\psi^n(A))}$$

for some universal constant κ . Choosing $A = \{a+b < \varepsilon\}$ so that $\psi^n(A) = \psi(A)$ we get the desired result choosing ε small enough.

It follows

$$\begin{aligned} \left| \mu_n \left(\frac{c}{a+b} \left(\varepsilon^{-1}(a+b) \mathbf{1}_{a+b \leq \varepsilon} + \mathbf{1}_{a+b > \varepsilon} \right) \right) - m_n \right| &\leq K \mu_n(1/(a+b)) C \kappa \mu_n(\{a+b \leq \varepsilon\}) \\ &\leq C' \mu(\{a+b \leq \varepsilon\}). \end{aligned}$$

Since the integrated function is bounded and continuous we can pass to the limit in n first, and then in ε using Lebesgue's bounded convergence theorem since

$$\left| \frac{c}{a+b} \left(\varepsilon^{-1}(a+b) \mathbf{1}_{a+b \leq \varepsilon} + \mathbf{1}_{a+b > \varepsilon} \right) \right| \leq \frac{K}{a+b}$$

which is μ integrable according to what we did before. This finally shows that $\mu(c/(a+b)) = m$ and concludes the proof. \square

We do not know about the converse inequality.

Remark 9.3. We will see below another way to prove this result (in an even more general context). Nevertheless we have given this proof in order to complete the picture in the [13] context. \diamond

We will now directly study the function J . Recall that

$$J(m) = \inf_{\beta > 0} \sup_{x, y} \left(x + my - \beta \ln \mathbb{E}(e^{x\tau + yW}) \right) := \inf_{\beta > 0} \sup_{x, y} \Lambda(m, \beta, x, y).$$

We do not assume here that W is bounded.

Lemma 9.4. *J is a good rate function.*

Proof. First we remark that

$$\sup_{x, y} \Lambda(m, \beta, x, y) \geq \sup_x \Lambda(m, \beta, x, 0) = \sup_x (x - \beta \ln \mathbb{E}(e^{x\tau})).$$

Since $\tau \geq 0$ (and supposed not to be identically 0) one can find $x_\tau < 0$ such that $\mathbb{E}(e^{x_\tau \tau}) = e^{-1}$ (we already use this in the proof of lemma 3.1), so that

$$\sup_{x, y} \Lambda(m, \beta, x, y) \geq x_\tau + \beta.$$

Let $\{J \leq M\}$ be some level set of J . For $\beta > M - x_\tau := \beta_\tau$ one has $\sup_{x, y} \Lambda(m, \beta, x, y) > M$ so that for all $m \in \{J \leq M\}$ it holds

$$J(m) = \inf_{0 < \beta \leq \beta_\tau} \sup_{x, y} \Lambda(m, \beta, x, y).$$

Now remark that

$$\sup_{x, y} \Lambda(m, \beta, x, y) \geq \sup_y \Lambda(m, \beta, 0, y) \geq \Lambda(m, \beta, 0, \kappa)$$

where $0 \leq \kappa < 1 \wedge \beta_0$, β_0 being defined in Assumption 2.4 *ii*). For $\beta \leq \beta_\tau$, $\beta \ln \mathbb{E}(e^{\kappa W})$ is thus bounded by C , so that

$$J(m) \geq m\kappa - C$$

showing that $\{J \leq M\}$ is bounded.

It remains to show that the level sets are closed. Let $m_n \rightarrow m$ be a sequence in $\{J \leq M\}$. According to what precedes we know that the infimum in β has to be taken in a bounded interval, so that one can find a (sub)-sequence β_n converging to β such that

$$J(m_n) \leq \sup_{x, y} \Lambda(m_n, \beta_n, x, y) + \varepsilon_n$$

with ε_n going to 0. This implies that for all (x, y)

$$x + m_n y - \beta_n \ln \mathbb{E}(e^{x\tau + yW}) \leq M + \varepsilon_n$$

which implies $J(m) \leq M$ by taking the limit in n and then the supremum w.r.t. (x, y) . \square

For a general W , consider

$$\tilde{J}(m) = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{|m-z| < \delta} J^n(z).$$

We now know that Z_t^n/t satisfies a full LDP with good rate function J^n . We may state

Lemma 9.5. *It holds $J \leq \tilde{J}$.*

Proof. As for the proof of lemma 9.2, we may of assume that $\tilde{J}(m) < +\infty$ and thus find sequences m_n and $\varepsilon_n \leq 1$ such that $\varepsilon_n \rightarrow 0$, $m_n \rightarrow m$ and $J^n(m_n) \leq \tilde{J}(m) + \varepsilon_n$. We denote $\delta_n = |m - m_n|$. Define $M = \tilde{J}(m) + 1$. For n large enough, m_n belongs to $\{J^n \leq M\}$. We have seen in the proof of lemma 9.4 that $J_n(m_n)$ is thus given by the infimum for $0 < \beta \leq M - |x_\tau| = \beta_\tau$ where $\mathbb{E}(e^{x_\tau \tau}) = e^{-1}$, i.e x_τ does not depend on n . The same holds with $J(m)$. If $\beta \in (0, \beta_\tau]$ and $\varepsilon > 0$ we may found $(x_\varepsilon, y_\varepsilon)$ such that

$$\begin{aligned} \beta \Lambda^*(1/\beta, m/\beta) &\leq x_\varepsilon + m y_\varepsilon - \beta \ln \left(\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon \\ &\leq x_\varepsilon + m_n y_\varepsilon - \beta \ln \left(\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon + \delta_n |y_\varepsilon|. \end{aligned}$$

Since

$$\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \rightarrow \mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W} \right]$$

as n grows to infinity, for n large enough the difference is less than ε so that

$$\begin{aligned} \beta \Lambda^*(1/\beta, m/\beta) &\leq x_\varepsilon + m_n y_\varepsilon - \beta \ln \left(\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \right) + (1 + \beta_\tau) \varepsilon + \delta_n |y_\varepsilon| \\ &\leq \beta \Lambda_n^*(1/\beta, m_n/\beta) + (1 + \beta_\tau) \varepsilon + \delta_n |y_\varepsilon|. \end{aligned}$$

Taking the infimum in β and then the \liminf_n we get

$$J(m) \leq \liminf_n J_n(m_n) + (1 + \beta_\tau) \varepsilon \leq \tilde{J}(m) + (1 + \beta_\tau) \varepsilon.$$

It remains to let ε go to 0 to conclude. \square

Remark 9.6. When $W = 1$ a.s. it is easily seen that $\sup_{x,y} \Lambda(m, \beta, x, y) = +\infty$ except for $m = \beta$ yielding $J(m) = \sup_x (x - m \ln \mathbb{E}(e^{x\tau}))$ as expected.

Since J is defined on \mathbb{R} one can expect some monotonicity on intervals delimited by the asymptotic mean $\mathbb{E}(W)/\mathbb{E}(\tau)$. We were not able to prove this monotonicity. \diamond

9.2. Case B: $\beta_0 < \infty$. We remind that W^n is a reduction of W such that Z_t^n/t is an approximation of Z_t/t and satisfies a full LDP, as proved in Theorem 4.4. However, this approximation is not an exponentially good approximation, therefore the LDP cannot be transferred to Z_t/t . In this case, we prove the deviation Inequalities (2.10) and (2.11). We will focus on the first one, $\mathbb{P} \left(\frac{Z_t}{t} \geq m + a \right)$, since the proof is exactly the same for the second one, $\mathbb{P} \left(\frac{Z_t}{t} \leq m - a \right)$.

For each $n \in \mathbb{N}^*$, for $\kappa \in (0, 1)$, we have

$$\begin{aligned} \mathbb{P} \left(\frac{Z_t}{t} \geq m + a \right) &\leq \mathbb{P} \left(\frac{Z_t^n}{t} \geq m + \kappa a \right) + \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| \geq (1 - \kappa) a \right) \\ &\leq 2 \max \left[\mathbb{P} \left(\frac{Z_t^n}{t} \geq m + \kappa a \right), \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| \geq (1 - \kappa) a \right) \right] \end{aligned}$$

Then, for each $n \in \mathbb{N}^*$ and $\kappa \in (0, 1)$, we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t}{t} \geq m + a \right) \\ & \leq \max \left[\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t^n}{t} \geq m + \kappa a \right), \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left(\left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| \geq (1 - \kappa)a \right) \right]. \end{aligned}$$

We handle the first term with the full LDP for Z_t^n/t with the rate function J^n (Theorem 4.4) and the second term with Lemma 3.1 (or 3.2, or 3.4 depending on the approximation strategy). We then have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t}{t} \geq m + a \right) \\ & \leq \max \left[\liminf_{n \rightarrow \infty} \left(- \inf_{z \geq m + \kappa a} J^n(z) \right), -\beta_0 \frac{(1 - \kappa)a}{2} \right] \\ & \leq - \min \left[\limsup_{n \rightarrow \infty} \inf_{z \geq m + \kappa a} J^n(z), \beta_0 \frac{(1 - \kappa)a}{2} \right]. \end{aligned}$$

Since this inequality is satisfied for each n , we can then apply the Lemma 9.8, proved below, which gives us

$$\limsup_{n \rightarrow \infty} \inf_{z \geq m + \kappa a} J^n(z) \geq \inf_{z \geq m + \kappa a} J(z),$$

and we obtain the Inequation (2.10): for all $\kappa \in (0, 1)$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left(\frac{Z_t}{t} \geq m + a \right) \leq - \min \left[\inf_{z \geq m + \kappa a} J(z), \beta_0 \frac{(1 - \kappa)a}{2} \right].$$

Remark 9.7. Notice J^n isn't the same in each case. J^n can be written

$$J^n(z) = \inf_{\beta > 0} \sup_{x, y} \{x + my - \beta \ln \mathbb{E} (e^{x\tau + yW^n})\},$$

but the definition of W^n depends on the reduction:

- If W is only bounded by K , a reduction with finite valued W^n was necessary:

$$W^n = \sum_{j=-n}^{n-1} \frac{jK}{n} \mathbf{1}_{W \in [jK/n, (j+1)K/n]}.$$

- If W is only discrete, a reduction with W^n bounded by $-n$ and n have been done: $W^n = (W \vee n) \wedge (-n)$.
- If W isn't bounded nor discrete, another reduction is necessary:

$$\tilde{W}^n = -n \mathbf{1}_{W < -n} + n \mathbf{1}_{W \geq n} + \sum_{j=-n^2}^{n^2-1} \frac{j}{n} \mathbf{1}_{W \in [\frac{j}{n}, \frac{j+1}{n}]}.$$

Since in each case, W^n is discrete and bounded a.s. and $E (e^{x\tau + yW^n}) \rightarrow_{n \rightarrow \infty} E (e^{x\tau + yW})$ for (x, y) such that $E (e^{x\tau + yW}) < \infty$, the proof is the same.

Lemma 9.8. *Assume $(W^n)_n$ is a sequence of random variables which converges almost surely to W and such that for each n , W^n is discrete and bounded a.s. Then, we have, for all $z_0 \in \mathbb{R}$,*

$$\limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) \geq \inf_{z \geq z_0} J(z),$$

where J^n is defined by

$$J^n(z) = \inf_{\beta > 0} \sup_{x, y} \{x + my - \beta \ln \mathbb{E} (e^{x\tau + yW^n})\}.$$

Proof. Let $(W^n)_n$ be a sequence of random variables which converges almost surely to W , such that for each n , W^n is discrete and bounded a.s. Let $z_0 \in \mathbb{R}$. Thanks to Lemma 8.1, J^n has different forms. Here we will use the form

$$J^n(z) = \inf_{\beta > 0} \beta \Lambda_n^* \left(\frac{1}{\beta}, \frac{m}{\beta} \right).$$

If $\limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) = \infty$, then the inequality is satisfied.

Let assume $\limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) < \infty$. We denote $C(z_0)$ this quantity.

There exists a sequence $n_k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} \inf_{z \geq z_0} J^{n_k}(z) = C(z_0)$. For each $k \in \mathbb{N}$, there exists a sequence $(z_i^k)_i$ in $[z_0, +\infty)$ such that $\inf_{z \geq z_0} J^{n_k}(z) = \lim_i J^{n_k}(z_i^k)$.

Let $\delta > 0$. There exists K such that $\forall k \geq K$, $\inf_{z \geq z_0} J^{n_k}(z) \in [C(z_0) - \delta/2, C(z_0) + \delta/2]$ and I_k such that $\forall i \geq I_k$, $J^{n_k}(z_i^k) \in [C(z_0) - \delta, C(z_0) + \delta]$.

Then, for $k \geq K$, and $i \geq I_k$, z_i^k is in $\{J^{n_k} \leq C(z_0) + \delta\}$.

Moreover, for each $n \in N$, $J^n \geq J^{|\cdot|}$ by the proof of the Lemma 9.1, where

$$J^{|\cdot|}(m) := \inf_{\beta > 0} \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + |m|y - \beta \log \mathbb{E}[e^{x\tau + y|W|}] \right\}.$$

So for $k \geq K$, and $i \geq I_k$, z_i^k is in $\{J^{|\cdot|} \leq C(z_0) + \delta\}$.

By the exact same argument than for J in the proof of Lemma 9.4, $\{J^{|\cdot|} \leq C(z_0) + \delta\}$ is a compact level set of $J^{|\cdot|}$.

Then, for $k \geq K$ and $i \geq I_k$, z_i^k is in a compact. There exists at least an adherent point z_{lim} in this compact, and a subsequence $z_{i_j}^{k_j} \rightarrow_{k_j, i_j \rightarrow \infty} z_{\text{lim}}$. In particular, $z_{\text{lim}} \geq z_0$.

Like in the proof of Lemma 9.5, $J^{n_{k_j}}(z_{i_j}^{k_j})$ is given by the infimum for $\beta \in (0, \beta_\tau)$. If $\beta \in (0, \beta_\tau)$ and $\varepsilon > 0$, we may found $(x_\varepsilon, y_\varepsilon)$ such that

$$\begin{aligned} \beta \Lambda^*(1/\beta, z_{\text{lim}}/\beta) &\leq x_\varepsilon + my_\varepsilon - \beta \ln \left(\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon \\ &\leq x_\varepsilon + z_{i_j}^{k_j} y_\varepsilon - \beta \ln \left(\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon + |z_{\text{lim}} - z_{i_j}^{k_j}| |y_\varepsilon| \end{aligned}$$

Since

$$\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \rightarrow \mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W} \right]$$

as n grows to infinity, for n large enough the difference is less than ε so that

$$\begin{aligned} \beta \Lambda^*(1/\beta, z_{\text{lim}}/\beta) &\leq x_\varepsilon + z_{i_j}^{k_j} y_\varepsilon - \beta \ln \left(\mathbb{E} \left[e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \right) + (1 + \beta_\tau) \varepsilon + |z_{\text{lim}} - z_{i_j}^{k_j}| |y_\varepsilon| \\ &\leq \beta \Lambda_n^*(1/\beta, z_{i_j}^{k_j}/\beta) + (1 + \beta_\tau) \varepsilon + |z_{\text{lim}} - z_{i_j}^{k_j}| |y_\varepsilon| \end{aligned}$$

Taking the infimum in β and then the \lim_j we get

$$J(z_{\text{lim}}) \leq \liminf_j J^{n_{k_j}}(z_{i_j}^{k_j}) + (1 + \beta_\tau) \varepsilon \leq \limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) + (1 + \beta_\tau) \varepsilon.$$

By letting ε go to 0 we obtain:

$$J(z_{\text{lim}}) \leq \limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z).$$

Then,

$$\inf_{z \geq z_0} J(z) \leq J(z_{\text{lim}}) \leq \limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z).$$

□

REFERENCES

- [1] Søren Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.
- [2] Alexander A. Borovkov and Anatolii A. Mogulskii. Large deviation principles for the finite-dimensional distributions of compound renewal processes. *Sib. Math. J.*, 56(1):28–53, 2015.
- [3] Alexander A. Borovkov and Anatolii A. Mogulskii. Large Deviation Principles for Trajectories of Compound Renewal Processes. I. *Theory Probab. Appl.*, 60(2):207–224, 2016.
- [4] Alexander A. Borovkov and Anatolii A. Mogulskii. Large Deviation Principles for Trajectories of Compound Renewal Processes. II. *Theory Probab. Appl.*, 60(3):349–366, 2016.
- [5] Mark Brown and Sheldon M. Ross. Asymptotic Properties of Cumulative Processes. *SIAM J. Appl. Math.*, 22(1):93–105, 1972.
- [6] Patrick Cattiaux, Laetitia Colombani, and Manon Costa. Limit theorems for Hawkes processes including inhibition. Preprint, 2021.
- [7] Manon Costa, Carl Graham, Laurence Marsalle, and Viet Chi Tran. Renewal in Hawkes processes with self-excitation and inhibition. *Adv. in Appl. Probab.*, 52(3):879–915, 2020.
- [8] Imre Csiszár. Sanov property, generalized I -projection and a conditional limit theorem. *Ann. Probab.*, 12(3):768–793, 1984.
- [9] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications*. Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin Heidelberg, 2 edition, 2010.
- [10] Ken Duffy and Anthony P. Metcalfe. How to estimate the rate function of a cumulative process. *J. Appl. Probab.*, 42(4):1044–1052, 2005.
- [11] Peter W. Glynn and Ward Whitt. Limit theorems for cumulative processes. *Stochastic Processes Appl.*, 47(2):299–314, 1993.
- [12] Alan G. Hawkes. Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58:83–90, 1971.
- [13] Raphaël Lefevre, Mauro Mariani, and Lorenzo Zambotti. Large deviations for renewal processes. *Stochastic Processes Appl.*, 121(10):2243–2271, 2011.
- [14] Walter L. Smith. Regenerative stochastic processes. *Proc. Roy. Soc. London Ser. A.*, 232(1188):6–31, 1955.
- [15] Jiang Tiefeng. Large deviations for renewal processes. *Stochastic Processes and their Applications*, 50(1):57–71, March 1994.

Patrick CATTIAUX, INSTITUT DE MATHÉMATIQUES DE TOULOUSE. CNRS UMR 5219., UNIVERSITÉ PAUL SABATIER., 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09.

E-mail address: patrick.cattiaux@math.univ-toulouse.fr

Laetitia COLOMBANI, INSTITUT DE MATHÉMATIQUES DE TOULOUSE. CNRS UMR 5219., UNIVERSITÉ PAUL SABATIER., 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09.

E-mail address: laetitia.colombani@math.univ-toulouse.fr

Manon COSTA, INSTITUT DE MATHÉMATIQUES DE TOULOUSE. CNRS UMR 5219., UNIVERSITÉ PAUL SABATIER., 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09.

E-mail address: manon.costa@math.univ-toulouse.fr