



**HAL**  
open science

# LARGE DEVIATION PRINCIPLES FOR CUMULATIVE PROCESSES AND APPLICATIONS

Patrick Cattiaux, Laetitia Colombani, Manon Costa

► **To cite this version:**

Patrick Cattiaux, Laetitia Colombani, Manon Costa. LARGE DEVIATION PRINCIPLES FOR CUMULATIVE PROCESSES AND APPLICATIONS. 2021. hal-03344425v1

**HAL Id: hal-03344425**

**<https://hal.science/hal-03344425v1>**

Preprint submitted on 15 Sep 2021 (v1), last revised 13 Apr 2023 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LARGE DEVIATION PRINCIPLES FOR CUMULATIVE PROCESSES AND APPLICATIONS

PATRICK CATTIAUX , LAETITIA COLOMBANI , AND MANON COSTA

UNIVERSITÉ DE TOULOUSE

ABSTRACT. The aim of this paper is to prove a Large Deviation Principle (LDP) for cumulative processes also known as compound renewal processes. These processes cumulate independent random variables occurring in time interval given by a renewal process. Our result extends the one obtained in [13] in the sense that we impose no specific dependency between the cumulated random variables and the renewal process. The proof is inspired from [13] but deals with additional difficulties due to the general framework that is considered here. In the companion paper [6] we apply this principle to Hawkes processes with inhibition. Under some assumptions Hawkes processes are indeed cumulative processes, but they do not enter the framework of [13].

*Key words* : Cumulative processes, large deviation, deviation inequalities, Hawkes processes

*MSC 2010* : 60F10, 60K15 .

## 1. INTRODUCTION.

**1.1. Cumulative processes.** Cumulative processes have been introduced by Smith [14] and are applied in many purposes, such as finance where they are called *compound-renewal* processes or *compound Markov renewal* processes. Indeed these continuous time processes cumulate independent random variables occurring in time interval given by a renewal process. To be more specific a real valued process  $(Z_t)_{t \geq 0}$  is called *a cumulative process* if the following properties are satisfied:

- (1)  $Z_0 = 0$ ,
- (2) there exists a renewal process  $(S_n)_n$  such that for any  $n$ ,  $(Z_{S_n+t} - Z_{S_n})_{t \geq 0}$  is independent of  $S_0, \dots, S_n$  and  $(Z_s)_{s < S_n}$ ,
- (3) the distribution of  $(Z_{S_n+t} - Z_{S_n})_{t \geq 0}$  is independent of  $n$ .

To study such processes, we write for all  $t \geq 0$

$$Z_t = W_0(t) + W_1 + \dots + W_{M_t} + r_t$$

where  $W_0(t) = Z_{t \wedge S_0}$ ,  $W_n = Z_{S_n} - Z_{S_{n-1}}$ ,  $r_t = Z_t - Z_{M_t}$ , where  $M_t$  is defined by

$$M_t = \sup \{n \geq 0, S_n \leq t\}.$$

The  $(W_k)_{k \geq 1}$ 's are thus i.i.d.

We denote by  $(\tau_i)_i$  the times associated to the renewal process  $\tau_n = S_n - S_{n-1}$  and  $\tau_0 = 0$ .

---

*Date*: September 15, 2021.

It is worth noticing that  $\tau_i$  and  $W_i$  can be dependent.

In the sequel we suppress the subscript  $i$  when dealing with the distribution (and all associated quantities like expectation, variance ...) of  $(\tau_i, W_i)$  and simply use  $(\tau, W)$ .

A simple example of cumulative process is  $Z_t = \int_0^t f(X_s)ds$  where  $(X_t)_t$  is a regenerative process with i.i.d. cycles [11]. Markov additive processes are other classical examples of cumulative process.

In [7] the authors exhibited a renewal structure for some Hawkes processes. This description is extensively used in our companion paper [6] in order to describe such processes as cumulative processes, and to study their asymptotic behaviour.

For cumulative processes, the law of large numbers (assuming that  $\mathbb{E}[|W|]$  and  $\mathbb{E}[\tau]$  are not infinite)

$$\frac{Z_t}{t} \xrightarrow[t \rightarrow \infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \text{ if and only if } \mathbb{E} \left( \max_{S_0 \leq t < S_1} |r_t| \right) < \infty,$$

and the central limit theorem (assuming  $\text{Var}(W) < \infty$  and  $\text{Var}(\tau) < \infty$ )

$$\frac{\left( Z_t - t \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \right)}{\sqrt{t}} \xrightarrow[t \rightarrow \infty]{} \mathcal{N} \left( 0, \sigma^2 \right) \text{ where } \sigma^2 = \frac{1}{\mathbb{E}(\tau)} \text{Var} \left( W - \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \tau \right)$$

can be found in Asmussen [1], theorem 3.1 and theorem 3.2.

Brown and Ross [5] have proved equivalent of the Blackwell theorem and of the key renewal theorem for a subclass of cumulative processes, since cumulative processes are a generalization of renewal processes. Glynn and Whitt have focused in [11] on cumulative processes associated to a regenerative process and have proved law of large numbers (strong and weak), law of the iterated logarithm, central limit theorem and functional generalizations of these properties.

The aim of this work is to establish a large deviation principle (LDP) for cumulative processes. Some works have already been done. Duffy and Metcalfe [10] have considered the estimation of a rate function for a cumulative process (if it admits a LDP). In a series of papers, Borovkov and Mogulskii ([2], [3], [4]) have studied the (LDP) (they use the term compound-renewal theorem), under some assumptions of comparison between the values  $(\theta_1, \theta_2)$  for which  $\mathbb{E} \left( e^{\theta_1 \tau + \theta_2 W} \right)$  is finite and the value  $\theta$  for which  $\mathbb{E} \left( e^{\theta \tau} \right)$  is finite (a comparison between the joint distribution of  $(\tau, W)$  and the law of  $\tau$ ). Actually, some points in their approach are not clear for us.

Lefevre, Mariani and Zambotti [13] worked on specific cumulative process where  $W_i = F(\tau_i)$  for some deterministic function  $F$  which is assumed to be non-negative, bounded and continuous. We choose to follow their approach. Our proof of the LDP has the same skeleton, but in a general framework for the pair  $(\tau, W)$ .

In this paper, we show a LDP for  $Z_t/t$  in the case  $r_t = 0$ . This assumption can be relaxed if  $r_t/t$  tends to 0 quickly enough, as it will be the case for the application to Hawkes process (see [6]). For example, if for all  $\delta > 0$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left( \frac{|r_t|}{t} > \delta \right) = -\infty,$$

then  $Z_t/t$  and  $(Z_t - r_t)/t$  are exponentially equivalent and satisfy the same LDP (if  $(Z_t - r_t)/t$  have one) and have the same rate function.

**1.2. Hawkes processes.** A Hawkes process is a point process on the real line  $\mathbb{R}$  characterized by its intensity process  $t \mapsto \Lambda(t)$ . We consider an appropriate filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the usual assumptions.

**Definition 1.1.** Let  $\lambda > 0$  and  $h : (0, +\infty) \rightarrow \mathbb{R}$  a signed measurable function. Let  $N^0$  a locally finite point process on  $(-\infty, 0]$  with law  $\mathbf{m}$ .

The point process  $N^h$  on  $\mathbb{R}$  is a Hawkes process on  $(0, +\infty)$ , with initial condition  $N^0$  and reproduction measure  $\mu(dt) = h(t)dt$  if:

- $N^h|_{(-\infty, 0]} = N^0$ ,
- the conditional intensity measure of  $N^h|_{(0, +\infty)}$  with respect to  $(\mathcal{F}_t)_{t \geq 0}$  is absolutely continuous w.r.t the Lebesgue measure and has density:

$$\Lambda^h : t \in (0, +\infty) \mapsto f \left( \lambda + \int_{(-\infty, t)} h(t-u) N^h(du) \right). \quad (1.1)$$

for some non-negative function  $f$ .

Hawkes processes have been introduced by Hawkes [12]. Most of the literature concerned with the large time behaviour of  $N_t^h = N^h([0, t])$  is dedicated to the case  $h \geq 0$  (self excitation). This behaviour is studied in details in [6] when  $h$  is a signed (the negative part modelling self inhibition) compactly supported function, and the function  $f$  (called the jump rate function) is given by

$$f(u) = \max(0, u).$$

In this situation one gets a description of  $N_t^h$  as a cumulative process (see [6] subsection 2.3) with few information on the joint law of  $(\tau, W)$ . This was the initial motivation for the present work. We refer to [6] for a more complete overview and explicit results in this situation. In the remaining part of the paper we will come back to more general cumulative processes.

## 2. NOTATIONS AND MAIN RESULT

### 2.1. First notations.

We consider  $(\tau_1, W_1), (\tau_2, W_2), \dots$  an i.i.d. sequence of pairs of random variables built on some probability space  $(\Omega, \mathbb{P})$ . The law of  $(\tau_i, W_i)$  is an arbitrary probability  $\psi$  on  $(0, +\infty) \times \mathbb{R}$ . We denote this by:  $(\tau_i, W_i) \sim \psi$ . In the sequel we generically use the notation  $(\tau, W)$  for a pair with the same distribution as  $(\tau_i, W_i)$ . Notice that we assume in particular that  $\mathbb{E}(\tau) > 0$ .

We denote by  $\mathcal{M}^1(\mathcal{X})$  the space of probability measure on some measurable space  $\mathcal{X}$ .

We consider the renewal process associated with  $(\tau_i)_{i \geq 1}$  :

$$S_0 = 0, \quad S_n = \sum_{k=1}^n \tau_k,$$

$$M_t = \sup \{n \geq 0, S_n \leq t\}.$$

We will study the quantity:

$$Z_t = \sum_{i=1}^{M_t} W_i. \quad (2.1)$$

The main goal of this paper is to prove a Large Deviation Principle for the process  $Z_t/t$ . Let us recall some basic definitions in large deviation theory (we refer to [9]).

A family of probability measures  $(\eta_t)_{t \in \mathbb{R}^+}$  on a topological space  $(\mathcal{X}, T_{\mathcal{X}})$  satisfies the Large Deviations Principle (LDP) with rate function  $J(\cdot)$  and speed  $\gamma(t) = t$  if  $J$  is lower semi-continuous from  $\mathcal{X}$  to  $[0, +\infty]$ , and the following holds

$$-\inf_{x \in \mathcal{O}} J(x) \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \eta_t(\mathcal{O}) \quad \text{for all open subset } \mathcal{O}, \quad (2.2)$$

and

$$-\inf_{x \in \mathcal{C}} J(x) \geq \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \eta_t(\mathcal{C}) \quad \text{for all closed subset } \mathcal{C}. \quad (2.3)$$

We shall sometimes say that  $(\eta_t)_{t \in \mathbb{R}^+}$  satisfies the *full* LDP when (2.2) and (2.3) are satisfied, while we will use *weak* LDP when  $\mathcal{C}$  closed is replaced by  $\mathcal{C}$  compact in (2.3). When  $\eta_t$  is the distribution of some random variable  $Y_t$  (for instance  $Z_t/t$ ) we shall say that the family  $(Y_t)_t$  satisfies a LDP.

Since  $J$  is lower semi-continuous the level sets  $\{x, J(x) \leq a\}$  are closed. If in addition they are compact, then  $J$  is said to be a good rate function.

In this paper we only consider the speed function  $\gamma(t) = t$  so that we will no more refer to it.

A particularly important notion for our purpose is the notion of *exponentially good approximation*.

**Definition 2.1.** Assume that  $(\mathcal{X}, d)$  is a metric space. A family of random variables  $Y_{n,t}$  ( $n \in \mathbb{N}$ ) is an exponentially good approximation of  $Y_t$  (all these variables being defined on the same probability space  $(\Omega, \mathbb{P})$ ), if for all  $\delta > 0$  it holds

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}(d(Y_{n,t}, Y_t) > \delta) = -\infty.$$

The key result is then

**Theorem 2.2.** *In the framework of definition 2.1, assume that  $Y_{n,t}$  is an exponentially good approximation of  $Y_t$ . Then the following statements hold true.*

- (1) *If  $Y_{n,t}$  satisfies a full LDP with rate function  $J^n$  then  $Y_t$  satisfies a weak LDP with rate function*

$$J(x) = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{d(y,x) < \delta} J^n(y).$$

- (2) *If  $\mathcal{X} = \mathbb{R}^k$  equipped with any norm, then the same conclusion is true when  $Y_{n,t}$  satisfies only a weak LDP.*

- (3) *If  $J$  (defined above) is a good rate function such that for any closed set  $F$ ,*

$$\inf_{y \in F} J(y) \leq \limsup_{n \rightarrow \infty} \inf_{y \in F} J^n(y)$$

*then  $Y_t$  satisfies a full LDP with rate function  $J$ .*

The first and last points in the previous Theorem are contained in [9] Theorem 4.2.16. The second one is a consequence of the fact that closed balls are compact sets in  $\mathbb{R}^k$ . Usually, the Theorem is sufficient to prove a full LDP. Nevertheless, in some cases, the study of the rate function  $J$  is difficult. The lemma below gives an alternative, using exponential tightness which is easy to obtain with our assumptions.

**Lemma 2.3.** *If  $Y_t$  satisfies a weak LDP with a rate function  $I$  and is exponentially tight, i.e. for all  $\alpha > 0$ , there exists a compact set  $K_\alpha$  such that*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(Y_t \notin K_\alpha^c) < -\alpha,$$

*then  $Y_t$  satisfies a full LDP and  $I$  is a good rate function.*

This Lemma is a consequence of the Lemma 1.2.18 in [9].

Theorem 2.2 as well as Lemma 2.3 will be used at the end of the article, in the section 9 which proves the main Theorem 2.5.

## 2.2. Main results.

**Assumption 2.4.** We will make the following set of assumptions:

- i)  $\exists \theta_0 \in (0, +\infty]$  such that  $\mathbb{E}[e^{\theta\tau}] < \infty$  for  $\theta < \theta_0$ ,
- ii)  $\exists \beta_0 \in (0, +\infty]$  such that  $\mathbb{E}[e^{\beta|W|}] < \infty$ , for  $\beta < \beta_0$ ,
- iii) for all interval  $\mathcal{I}$  such that  $\mathbb{P}(W \in \mathcal{I}) > 0$ , it holds

$$\mathbb{P}(\tau > t, W \in \mathcal{I}) > 0 \quad \text{for all } t \geq 0.$$

We will introduce the classical Cramer transforms, for  $(a, b) \in \mathbb{R}^2$ ,

$$\Lambda^*(a, b) = \sup_{x, y} \{ax + by - \ln \mathbb{E}(e^{x\tau + yW})\} \quad (2.4)$$

$$\Lambda_n^*(a, b) = \sup_{x, y} \left\{ ax + by - \ln \mathbb{E} \left( e^{x\tau + yW^n} \right) \right\}, \quad (2.5)$$

where  $W^n$  is a well-chosen reduction of  $W$ . We finally introduce the rate functions

$$J^n(m) = \inf_{\beta > 0} \beta \Lambda_n^* \left( \frac{1}{\beta}, \frac{m}{\beta} \right) \quad , \quad J(m) = \inf_{\beta > 0} \beta \Lambda^* \left( \frac{1}{\beta}, \frac{m}{\beta} \right), \quad (2.6)$$

and

$$\tilde{J} = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{|m-z| < \delta} J^n(z). \quad (2.7)$$

We then may state

**Theorem 2.5.** *Assume that Assumption 2.4 is fulfilled. Let  $J$  given by (2.6).*

- *If  $\beta_0 = +\infty$  (in particular if  $W$  is bounded) then  $Z_t/t$  satisfies a full LDP with good rate function  $\tilde{J}$ . We also have the following inequalities*

$$\mathbb{P} \left( \frac{Z_t}{t} \geq m + a \right) \leq - \inf_{z \geq m+a} J(z), \quad (2.8)$$

$$\mathbb{P} \left( \frac{Z_t}{t} \leq m - a \right) \leq - \inf_{z \leq m-a} J(z). \quad (2.9)$$

- If  $\beta_0 < +\infty$ , denoting  $m = \mathbb{E}(W)/\mathbb{E}(\tau)$  we have for all  $a > 0$

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P} \left( \frac{Z_t}{t} \geq m + a \right) \leq - \min \left[ \inf_{z \geq m + (a/2)} J(z), \beta_0 a/4 \right], \quad (2.10)$$

and

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P} \left( \frac{Z_t}{t} \leq m - a \right) \leq - \min \left[ \inf_{z \geq m - (a/2)} J(z), \beta_0 a/4 \right]. \quad (2.11)$$

We may replace  $a/2$  and  $\beta_0/2$  by  $\kappa a$  and  $(1 - \kappa)\beta_0$  for any  $\kappa \in (0, 1)$ .

*Remark 2.6.* The previous inequalities (2.8) and (2.9) actually hold true with function  $\tilde{J}$  as well as function  $J$  since  $J \leq \tilde{J}$  (see Lemma 9.5. However, since  $J$  is clearly more easy to calculate than  $\tilde{J}$ , we prefer to write the inequalities with  $J$ .

### 2.3. The scheme of proof.

The proof will be divided in several steps. In the next section we will reduce the problem first to bounded  $W$ 's replacing the original  $W$  by  $W^n$ , and then to finite valued  $W$ 's. This will be done by using exponentially good approximations and Theorem 2.2.

Following [13] we introduce in section 4 (for finite valued  $W$ 's) an associated empirical measure and state the LDP for this measure (theorem 4.3) and the LDP for the cumulative process (theorem 4.4), the latter being obtained via the contraction principle. The following sections are devoted to the proofs of these theorems.

Section 5 is devoted to the study of the rate function in theorem 4.3. This is the most technical part of this work. The idea is that one can adapt the arguments in [13] by conditioning w.r.t. each value of  $W$ .

*At this point the fact that  $W$  is finite valued will provide us both with simple conditioning and with the necessary compactness arguments, since  $\mathcal{M}_1(\{1, \dots, n\})$  is compact.*

If many arguments are close to those in [13], some have to be written in detail. We have decided to refer to the corresponding statements in [13] only when they can be reproduced line by line in our case. Some topological points also have to be clarified. In section 6 we show useful auxiliary lemmata. The next section 7 is devoted to the proof of 4.3. In section 8 we deduce theorem 4.4. The final section 9 is devoted to the study of the rate function  $J$  and to the proof of theorem 2.5.

## 3. REDUCTION OF THE PROBLEM TO FINITE VALUED $W$ 'S.

### 3.1. First reduction to bounded $W$ 's.

To  $Z_t$  defined by (2.1) we associate

$$Z_t^n = \sum_{i=1}^{M_t} (W_i \wedge n \vee (-n)),$$

so that

$$|Z_t - Z_t^n| = \sum_{i=1}^{M_t} (W_i - n)_+ + \sum_{i=1}^{M_t} (W_i + n)_- \quad (3.1)$$

where  $u_+ = \max(u, 0)$  and  $u_- = \max(-u, 0)$ . We then have

**Lemma 3.1.** *If Assumption 2.4 is fulfilled, for all  $\delta > 0$ ,*

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) \leq -\frac{\beta_0 \delta}{2}.$$

*In particular if  $\beta_0 = +\infty$ ,  $Z_t^n/t$  is an exponentially good approximation of  $Z_t/t$ .*

*Proof.* It is enough to look at

$$\mathbb{P} \left( \sum_{i=1}^{M_t} (W_i - n)_+ > \delta t \right),$$

since the other term in (3.1) can be treated similarly. To conclude it is enough to use the elementary

$$\ln(a + b) \leq \max(\ln(2a), \ln(2b)). \quad (3.2)$$

Using that the  $W_i$ 's are i.i.d. we may write for  $\delta > 0$  and  $c > 0$ , (as usual an empty sum is equal to 0 by convention)

$$\begin{aligned} & \mathbb{P} \left( \sum_{i=1}^{M_t} (W_i - n)_+ > \delta t \right) \\ & \leq \mathbb{P} \left( \sum_{i=1}^{ct} (W_i - n)_+ > \frac{\delta t}{2} \right) + \mathbb{P} \left( \sum_{i=ct+1}^{M_t} (W_i - n)_+ > \frac{\delta t}{2} \right) \\ & \leq \mathbb{P} \left( \sum_{i=1}^{ct} (W_i - n)_+ > \frac{\delta t}{2} \right) + \mathbb{P} \left( \left\{ \sum_{i=ct+1}^{M_t} (W_i - n)_+ > \frac{\delta t}{2} \right\} \cap \{1 + ct \leq M_t < 2ct\} \right) \\ & \quad + \mathbb{P} \left( \left\{ \sum_{i=ct+1}^{M_t} (W_i - n)_+ > \frac{\delta t}{2} \right\} \cap \{M_t \geq 2ct\} \right) \\ & \leq 2\mathbb{P} \left( \sum_{j=1}^{ct} (W_j - n)_+ > \frac{\delta t}{2} \right) + \mathbb{P}(M_t \geq 2ct) \end{aligned}$$

**Study of  $\mathbb{P}(M_t \geq 2ct)$**  . Start with the second term in the sum above. According to theorem 2.3 in [15], we know that  $M_t/t$  satisfies a LDP with rate function  $J_\tau$  given by

$$J_\tau(x) = \begin{cases} \sup_{\lambda} \{\lambda - x \ln \mathbb{E}(e^{\lambda\tau})\} & \text{if } x \geq 0 \\ \infty & \text{if } x < 0 \end{cases}$$

Notice that  $J_\tau(x) = x \Lambda^*(1/x, 0)$  for  $x > 0$ . In addition (see Lemma 2.6 in [15]) the supremum is achieved for  $\lambda \leq 0$  if  $x \in (1/\mathbb{E}(\tau), +\infty)$  and  $J_\tau$  is non-decreasing on this interval.

It follows that for  $2c > 1/\mathbb{E}(\tau)$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}(M_t \geq 2ct) = -J_\tau(2c). \quad (3.3)$$



In order to get  $\lim_n \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}(M_t \geq 2c_n t) = -\infty$  for some sequence  $c_n$  (to be chosen later) it remains to show that

$$J_\tau(x) \xrightarrow{x \rightarrow \infty} +\infty.$$

Since  $\tau$  is a non-negative random variable one can find  $\lambda_\tau$  such that

$$\mathbb{E}(e^{\lambda_\tau \tau}) = e^{-1}.$$

Let  $x \in \mathbb{R}^+$ , we have:

$$\begin{aligned} J_\tau(x) &= \sup_\lambda \{\lambda - x \ln \mathbb{E}(e^{\lambda \tau})\} \\ &\geq \lambda_\tau - x \ln \mathbb{E}(e^{\lambda_\tau \tau}) \\ &\geq x + \lambda_\tau \end{aligned}$$

yielding the desired result.

**Study of**  $\mathbb{P}\left(\sum_{j=1}^{ct} (W_j - n)_+ > \frac{\delta t}{2}\right)$ . Denote as usual by  $\lfloor ct \rfloor$  the integer part of  $ct$ . We have

$$\mathbb{P}\left(\sum_{j=1}^{ct} (W_j - n)_+ > \frac{\delta t}{2}\right) = \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \frac{\delta t}{2}\right)$$

so that we may use this time the usual Cramer's theorem. Defining

$$\begin{aligned} \Psi_n(\lambda) &= \ln \mathbb{E}\left[e^{\lambda(W-n)_+}\right] \\ \Psi_n^*(x) &= \sup_\lambda \{\lambda x - \Psi_n(\lambda)\}, \end{aligned}$$

we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t/2\right) &= \limsup_{t \rightarrow \infty} \frac{c}{\lfloor ct \rfloor} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t/2\right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{c}{\lfloor ct \rfloor} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta \lfloor ct \rfloor / 2c\right) \\ &\leq -c \inf_{x \in [\delta/2c, +\infty)} \Psi_n^*(x). \end{aligned}$$

As the function  $x \mapsto \Psi_n^*(x)$  is non-decreasing on  $[\mathbb{E}((W-n)_+), +\infty)$ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t/2\right) = -c \Psi_n^*(\delta/2c),$$

provided  $\delta/2c \geq \mathbb{E}((W-n)_+)$ . Notice that for  $\lambda < \beta_0$ ,

$$c \Psi_n^*(\delta/2c) \geq \frac{\lambda \delta}{2} - c \ln \left(1 + \mathbb{E}\left[(e^{\lambda(W-n)} - 1) \mathbf{1}_{W > n}\right]\right)$$

so that choosing  $c_n$  growing to infinity and such that

$$c_n \mathbb{E}((W-n)_+) \rightarrow 0 \text{ and } c_n \ln \left(1 + \mathbb{E}\left[(e^{\lambda(W-n)} - 1) \mathbf{1}_{W > n}\right]\right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

which is always possible since both  $\mathbb{E}((W - n)_+)$  and  $\ln \left(1 + \mathbb{E} \left[ (e^{\lambda(W-n)} - 1) \mathbf{1}_{W > n} \right] \right)$  are going to 0, we get

$$\lim_n \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left( \sum_{j=1}^{\lfloor ct \rfloor} (W_j - n)_+ > \delta t / 2 \right) \geq \frac{\lambda \delta}{2}.$$

We may optimize in  $\lambda$  and plugging the same sequence  $c_n$  in (3.3) ends the proof.  $\square$

### 3.2. Second reduction to finite valued $W$ 's.

Starting with a bounded  $W$  such that  $-K < W < K$  almost surely we define a discretized version of  $W$  by

$$W^n = \sum_{j=-n}^{n-1} \frac{jK}{n} \mathbf{1}_{W \in [jK/n, (j+1)K/n[}.$$

It clearly holds

$$|W - W^n| \leq \frac{K}{n}.$$

We thus introduce

$$\bar{Z}_t^n = \sum_{i=1}^{M_t} W_i^n$$

so that

$$|\bar{Z}_t^n - Z_t| \leq \frac{K M_t}{n}.$$

According to the study in the previous section

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left( \left| \frac{\bar{Z}_t^n}{t} - \frac{Z_t}{t} \right| > \delta \right) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} (M_t \geq \delta t n / K) = -J_\tau(\delta n / K)$$

so that, since  $J_\tau$  grows to infinity

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left( \left| \frac{\bar{Z}_t^n}{t} - \frac{Z_t}{t} \right| > \delta \right) = -\infty.$$

We have thus shown

**Lemma 3.2.** *If Assumption 2.4 is fulfilled, and  $W$  is almost surely bounded,  $\bar{Z}_t^n/t$  defined above is an exponentially good approximation of  $Z_t/t$ .*

*Remark 3.3.* If  $W$  isn't bounded nor discrete, a double reduction can be done in one step. We obtain the same results as doing successively both the reductions, but it allows formulating the rate function more easily, when there is one.

Let's consider

$$\tilde{W}^n = -n \mathbf{1}_{W < -n} + n \mathbf{1}_{W \geq n} + \sum_{j=-n^2}^{n^2-1} \frac{j}{n} \mathbf{1}_{W \in [\frac{j}{n}, \frac{j+1}{n})}.$$

By denoting  $\tilde{Z}_t^n = \sum_{i=1}^{M_t} \tilde{W}_i^n$ , we have, for  $\delta > 0$

$$\begin{aligned} \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\tilde{Z}_t^n}{t}\right| > \delta\right) &\leq \mathbb{P}\left(\sum_{i=1}^{M_t} (W_i - n)_+ > \delta t\right) + \mathbb{P}\left(\sum_{i=1}^{M_t} (W_i + n)_+ > \delta t\right) \\ &\quad + \mathbb{P}\left(\sum_{i=1}^{M_t} \sum_{j=-n^2}^{n^2-1} \left(W_i - \frac{j}{n}\right) \mathbb{1}_{W_i \in [\frac{j}{n}, \frac{j+1}{n}]} > \delta t\right). \end{aligned}$$

Since

$$\left|\sum_{i=1}^{M_t} \sum_{j=-n^2}^{n^2-1} \left(W_i - \frac{j}{n}\right) \mathbb{1}_{W_i \in [\frac{j}{n}, \frac{j+1}{n}]} \right| \leq \frac{M_t}{n},$$

we obtain

$$\mathbb{P}\left(\sum_{i=1}^{M_t} \sum_{j=-n^2}^{n^2-1} \left(W_i - \frac{j}{n}\right) \mathbb{1}_{W_i \in [\frac{j}{n}, \frac{j+1}{n}]} > \delta t\right) \leq \mathbb{P}(M_t > \delta t n).$$

By the proofs of Lemmas 3.1 and 3.2, we finally obtain

**Lemma 3.4.** *If Assumption 2.4 is fulfilled, for all  $\delta > 0$ ,*

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\tilde{Z}_t^n}{t}\right| > \delta\right) \leq -\frac{\beta_0 \delta}{2}.$$

*In particular if  $\beta_0 = +\infty$ ,  $\tilde{Z}_t^n/t$  is an exponentially good approximation of  $Z_t/t$ .*

*Remark 3.5.* The proofs in the previous section suggest a direct naive approach in order to get deviation bounds. Indeed we may write

$$\left|\frac{Z_t}{t} - \frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right| \leq \frac{1}{t} \left|\sum_{i=1}^{\lfloor M_t \rfloor} (W_i - \mathbb{E}(W))\right| + \mathbb{E}(W) \left|\frac{\lfloor M_t \rfloor}{t} - \frac{1}{\mathbb{E}(\tau)}\right|$$

so that following the lines of the proof of lemma 3.1, we may write

$$\begin{aligned} \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right| \geq \delta\right) &\leq \mathbb{P}\left(\frac{1}{t} \left|\sum_{i=1}^{\lfloor M_t \rfloor} (W_i - \mathbb{E}(W))\right| \geq \delta/2\right) + \mathbb{P}\left(\mathbb{E}(W) \left|\frac{\lfloor M_t \rfloor}{t} - \frac{1}{\mathbb{E}(\tau)}\right| \geq \delta/2\right) \\ &\leq 2\mathbb{P}\left(\left|\sum_{i=1}^{\lfloor M_t \rfloor} (W_i - \mathbb{E}(W))\right| > \delta t/4\right) + \mathbb{P}(\lfloor M_t \rfloor > 2ct) + \\ &\quad + \mathbb{P}\left(\mathbb{E}(W) \left|\frac{\lfloor M_t \rfloor}{t} - \frac{1}{\mathbb{E}(\tau)}\right| \geq \delta/2\right). \end{aligned}$$

Introducing the Cramer transform of  $W$ ,

$$\Psi^*(x) = \sup_{\lambda} \{\lambda x - \ln \mathbb{E}(e^{\lambda(W - \mathbb{E}(W))})\}$$

we thus deduce, if Assumption 2.4 is fulfilled

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{\mathbb{E}(W)}{\mathbb{E}(\tau)}\right| \geq \delta\right) \leq -A(\delta), \quad (3.4)$$

where

$$\begin{aligned} A(\delta) &= \min(A_1(\delta), A_2(\delta)) \\ A_1(\delta) &= \min\left(J_\tau\left(\frac{1}{\mathbb{E}(\tau)} + \frac{\delta}{2\mathbb{E}(W)}\right), J_\tau\left(\frac{1}{\mathbb{E}(\tau)} - \frac{\delta}{2\mathbb{E}(W)}\right)\right) \end{aligned} \quad (3.5)$$

$$A_2(\delta) = \sup_{2c > 1/\mathbb{E}(\tau)} \min(J_\tau(2c), c \min(\psi^*(\delta/4c), \psi^*(-\delta/4c))). \quad (3.6)$$

This time however we cannot let  $c$  go to infinity so that  $A_2(\delta)$  will furnish a worse bound than  $\beta_0\delta/2$  (the correcting term in Lemma 3.1). So even when  $\beta_0 < +\infty$  the previous bound will be worse than the one obtained by combining Lemma 3.1 and the LDP for finite valued  $W$ .

This kind of approach should be useful in order to get non asymptotic bounds.  $\diamond$

#### 4. LARGE DEVIATIONS FOR A FINITE VALUED $W$ .

From now on we will assume that  $W$  takes its values in a set of cardinal  $n$  (fixed). However we will keep a superscript  $n$  to remember that this is a reduction of the general case, so we shall write  $W^n$ . For simplicity we rename the values of  $W^n$  as  $1, \dots, n$ . According to Assumption 2.4 iii) and the approximation made in subsection 3.2 we may assume that

$$\forall j = 1, \dots, n, \forall t > 0, \psi^n(\{\tau > t\} \cap \{w = j\}) = \mathbb{P}(\tau > t, W^n = j) > 0.$$

This condition will replace iii) in Assumption 2.4 in all the next statements.

Some results are still true when  $W$  takes its value in some  $\mathbb{W} \subset \mathbb{R}$  which is compact, in particular if  $W$  is bounded. We will mention these results in related remarks each time it is possible.

##### 4.1. More notations.

As we said we will now closely follow the approach in [13]. Consider the backward recurrence time process  $(A_t)_{t \geq 0}$ , the forward recurrence time process  $(B_t)_{t \geq 0}$  and the process  $(C_t^n)_{t \geq 0}$ , defined by:

$$A_t = t - S_{M_t}, \quad B_t = S_{M_{t+1}} - t, \quad C_t^n = W_{M_{t+1}}^n,$$

and the associated empirical measure:

$$\mu_t^n := \frac{1}{t} \int_{[0,t)} \delta_{(A_s, B_s, C_s^n)} ds \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}).$$

We denote by  $\mathbb{P}_t^n$  the law of  $\mu_t^n$ .

This empirical measure is a measure on three coordinates, whereas Lefevre, Mariani and Zambotti [13] consider a measure on two coordinates since  $W = F(\tau)$  in their work. Looking at  $(0, +\infty]$  allows us to avoid integrability considerations at infinity since  $A_s$  and  $B_s$  can be as large as we want simultaneously.

Choosing as metric  $d(t, t') = |\frac{1}{t} - \frac{1}{t'}|$  on  $(0, +\infty]$  and the usual one on  $\{1, \dots, n\}$  we immediately see that  $\mathcal{Y} = (0, +\infty]^2 \times \{1, \dots, n\}$  is a Polish space, so that  $\mathcal{X} = \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  is also Polish. In addition  $(a, b, c) \rightarrow \frac{1}{a+b}$  is continuous (of course it is equal to 0 if either  $a$  or  $b$  equals  $+\infty$ ).

For  $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ , we denote:

$$\mu(1/\tau) = \mu(1/(a+b)) = \int_{(0, +\infty)^2 \times \{1, \dots, n\}} \frac{1}{a+b} \mu(da, db, dc)$$

and for  $\pi \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$ , we denote:

$$\pi(1/\tau) = \int_{(0, +\infty] \times \{1, \dots, n\}} \frac{1}{\tau} \pi(d\tau, dW).$$

Let us define  $\Delta_0^n \subset \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  as :

$$\Delta_0^n = \{\mu_0 \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), \quad (4.1)$$

$$\mu_0(da, db, dc) = \int_{[0, 1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW),$$

$$\pi \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\}), \pi(1/\tau) < +\infty\}$$

We also define  $\Delta^n$ :

$$\begin{aligned} \Delta^n = \{ & \mu(da, db, dc) = \alpha \mu_0(da, db, dc) + (1 - \alpha) \delta_{(+\infty, +\infty)}(da, db) \otimes \eta(dc), \\ & \mu_0 \in \Delta_0^n, \alpha \in [0, 1], \eta \in \mathcal{M}^1(\{1, \dots, n\}) \}. \end{aligned} \quad (4.2)$$

The following will be used several times in what follows

**Lemma 4.1.** *For  $M > 0$ , the set  $\mathcal{X}_M = \{\nu \in \mathcal{X}, \nu(1/(a+b)) \leq M\}$  is compact in  $\mathcal{X}$ .*

*Proof.* First the set  $\{a+b \geq \delta\}$  is compact in  $\mathcal{Y}$  for  $\delta \geq 0$ . It follows that  $\mathcal{X}_M$  is a uniformly integrable set, hence is relatively compact (tight). Indeed for  $\nu \in \mathcal{X}_M$

$$\nu(\{a+b < \delta\}) \leq \delta \nu(1/(a+b)) \leq \delta M.$$

Notice that this set is also closed, so that it is compact. Indeed if  $\nu$  belongs to  $\mathcal{X}_M$ ,  $\nu(\frac{1}{a+b+\varepsilon}) \leq M$  for all  $\varepsilon > 0$ . Any weak limit  $\nu$  of such  $\nu_k$ 's satisfies  $\nu(\frac{1}{a+b+\varepsilon}) \leq M$  since  $(a, b, c) \mapsto 1/(a+b+\varepsilon)$  is bounded and continuous. It remains to apply the monotone convergence theorem.  $\square$

*Remark 4.2.* This lemma and its proof are still true if we replace  $\{1, \dots, n\}$  by a compact set  $\mathbb{W}$ .  $\diamond$

Recall that  $\psi^n$  denotes the joint distribution of  $(\tau, W^n)$ , so that its first marginal does not depend on  $n$ . We may thus skip the superscript  $n$  when dealing with quantities that only depend on this first marginal. To simplify the notation we will denote by  $\xi$  the  $\theta_0$  in Assumption 2.4, i.e.

$$\xi = \sup\{c \in \mathbb{R}, \psi(e^{c\tau}) < \infty\} \in [0, +\infty]. \quad (4.3)$$

For  $\pi \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$ , satisfying  $\pi(1/\tau) \in (0, +\infty)$ , we define  $\tilde{\pi} \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$  as:

$$\tilde{\pi}(d\tau, dW) = \frac{1}{\pi(1/\tau)} \frac{1}{\tau} \pi(d\tau, dW).$$

$\tilde{\pi}$  has no weight on  $\{+\infty\} \times \{1, \dots, n\}$ .

We also define the functional  $I_0^n, I^n : \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}) \rightarrow [0, +\infty]$  by:

$$I_0^n(\mu) = \begin{cases} \pi(1/\tau)H(\tilde{\pi}|\psi^n) & \text{if } \mu \in \Delta_0^n \\ \infty & \text{otherwise} \end{cases}, \quad (4.4)$$

$$I^n(\mu) = \begin{cases} \alpha\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \alpha)\xi & \text{if } \mu \in \Delta^n \\ \infty & \text{otherwise} \end{cases}, \quad (4.5)$$

where  $H$  is the relative entropy defined by

$$H(\nu|\mu) = \begin{cases} \int \ln\left(\frac{d\nu}{d\mu}\right) d\mu & \text{if } \nu \text{ is absolutely continuous w.r.t. } \mu \\ \infty & \text{otherwise.} \end{cases}$$

We denote by  $\mathcal{C}_b((0, +\infty]^2 \times \{1, \dots, n\})$  the set of bounded and continuous functions on  $(0, +\infty]^2 \times \{1, \dots, n\}$ .

For a bounded measurable  $f : (0, +\infty) \times (0, +\infty) \times \{1, \dots, n\} \rightarrow \mathbb{R}$ , we set :

$$\bar{f}(x, \tau, W) = \int_0^x f(u\tau, (1-u)\tau, W) du \quad (4.6)$$

We define:

$$C_{n,f} = \int_{(0,+\infty) \times \{1,\dots,n\}} e^{\tau \bar{f}(1,\tau,W)} \psi^n(d\tau, dW), \quad (4.7)$$

and

$$D_{n,f} = \sup_{s>0} \int_{(s,+\infty)} e^{\tau \bar{f}(s/\tau,\tau,W)} \psi^n(d\tau, dW) .. \quad (4.8)$$

We consider

$$\Gamma := \{f : (0, +\infty]^2 \times \{1, \dots, n\} \rightarrow \mathbb{R}, \text{ bounded, lower semicontinuous, } C_{n,f} < 1, D_{n,f} < +\infty\}. \quad (4.9)$$

The reasons for introducing such quantities are detailed in [13] section 1, see in particular subsection 1.5 (and more particularly Remark 1.5) and subsection 1.6.

## 4.2. Large deviations principle for cumulative process when $W$ is finite valued.

We will first prove a LDP for the empirical measure  $\mu_t^n$ :

**Theorem 4.3.** *Let  $\mu_t^n$  the empirical measure of a cumulative process satisfying Assumptions 2.4, and such that  $W^n$  takes its values in  $\{1, \dots, n\}$ . The family  $(\mathbb{P}_t^n)_{t \geq 0}$  of probability distributions of  $\mu_t^n$  satisfies a large deviations principle with good rate function  $I^n$  as  $t \rightarrow \infty$  with speed  $t$ , where  $I^n$  is defined in (4.5).*

Applying the contraction principle we will then deduce

**Theorem 4.4.** *Let  $(\tau_i, W_i^n)_i$  an i.i.d. sequence of couples of random variables in  $(0, +\infty) \times \{1, \dots, n\}$  following the law  $\psi^n$ . Let us define  $Z_t^n = \sum_{i=1}^{M_t} W_i^n$  the associated cumulative process. If  $\psi^n$  satisfies Assumptions 2.4, then the law of  $Z_t^n/t$  satisfies a large deviation principle with good rate function  $J^n$  (given by (2.6)).*

5. THE FUNCTIONAL  $I^n$ 

This section aims at proving that  $I^n$ , defined in (4.5), is a good rate function i.e proposition 5.2 (analogue to Proposition 1.3 in [13]). Three propositions are necessary to prove it: the propositions 5.3, 5.4 and 5.6 which are the analogues of Proposition 2.1 in [13]. In order to prove these three propositions some additional lemmata are necessary.

*Remark 5.1.* If the scheme of proof is close to the one in [13], the proofs of this section need some new ideas. In particular, we will use conditioning by sets like  $\{W = j\}$ . This induces to use disintegration and uniform controls in  $j$ . That is why it is useful to work with a finite valued  $W$ . In addition the compactness of  $\mathcal{M}_1(\{1, \dots, n\})$  will help to control the third coordinate (which does not exist in [13]). Except two auxiliary lemmata, we decided to give self contained proofs.  $\diamond$

**Proposition 5.2.** *The function  $I^n$  is a good rate function. Moreover,  $I^n$  is the lower semicontinuous envelope of  $I_0^n$ .*

For the proof we need the next three propositions. We need another notation: if  $\psi_k$  is a sequence in  $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ ,  $\xi_k$  and  $I_k^n$  are defined as in (4.3) and (4.5) respectively, with  $\psi^n$  replaced by  $\psi_k$ .

**Proposition 5.3.** *Let  $\psi_k$  be a sequence in  $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ . Assume that  $\psi_k \rightharpoonup \psi^n$ , and  $\xi_k \rightarrow \xi$  as  $k \rightarrow +\infty$ . Then any sequence  $(\mu_k)_k$  in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  such that  $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < \infty$  is tight and thus, relatively compact in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ . In particular, for  $k$  large enough, such a sequence is in  $\Delta^n$ . If we consider  $\pi_k$  and  $\alpha_k$  the associated quantities, we have:*

$$\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < +\infty. \quad (5.1)$$

**Proposition 5.4.** *Let  $\psi_k$  be a sequence in  $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ . Assume that  $\psi_k \rightharpoonup \psi^n$ , and  $\xi_k \rightarrow \xi$  as  $k \rightarrow +\infty$ . Then for any  $\mu$  and any sequence  $(\mu_k)_k$  in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ , such that  $\mu_k \rightharpoonup \mu$ , we have  $\liminf_{k \rightarrow \infty} I_k^n(\mu_k) \geq I^n(\mu)$ .*

*Remark 5.5.* Both previous propositions are still true when replacing  $\{1, \dots, n\}$  by  $\mathbb{W}$  a compact subset of  $\mathbb{R}$  as it will be clear looking at their proof.  $\diamond$

**Proposition 5.6.** *Let  $\psi_k$  be a sequence in  $\mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ . Assume that  $\psi_k \rightharpoonup \psi^n$ , and  $\xi_k \rightarrow \xi$  as  $k \rightarrow +\infty$ . Then for any  $\mu$  in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  with  $I^n(\mu) < \infty$ , there exists a sequence  $(\mu_k)_k$  such that  $\mu_k \rightharpoonup \mu$ ,  $\mu_k \in \Delta_0^n$  for all  $k$  and  $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) \leq I^n(\mu)$ . Moreover, we have  $\mu_k \left( \frac{1}{a+b} \right) \xrightarrow[k \rightarrow \infty]{} \mu \left( \frac{1}{a+b} \right)$ .*

For this last proposition we need to work with  $W$  taking values in a finite set.

Now we can prove the main Proposition of this section.

*Proof of Proposition 5.2.* We want to prove that  $I^n$  is a good rate function and is the lower semicontinuous envelope of  $I_0^n$ .

We apply Proposition 5.3 with  $\psi_k = \psi^n$  : for all  $(\mu_k)_k$  sequence of  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ , if  $\limsup_{k \rightarrow \infty} I^n(\mu_k) < +\infty$ , then the family  $(\mu_k)_k$  is tight in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ . We can deduce that  $I^n$  has relatively compact sublevel sets (and is coercive).

Thanks to Proposition 5.4 we have, for any  $\mu$  and any sequence  $\mu_k$  in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  such that  $\mu_k \rightharpoonup \mu$ ,  $\liminf_{k \rightarrow \infty} I^n(\mu_k) \geq I^n(\mu)$ . Thus  $I^n$  has closed sublevel sets. It also means that  $I^n$  is lower semicontinuous.

$I^n$  has relatively compact and closed sublevel sets, so  $I^n$  is a good rate function.

We already know that  $I^n \leq I_0^n$  since  $\Delta_0^n \subset \Delta^n$ , and  $I^n$  is lower semicontinuous, so  $I^n$  is smaller than or equal to the lower semicontinuous envelope of  $I_0^n$ .

Thanks to proposition 5.6, for any  $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  with  $I^n(\mu) < \infty$ , there exists  $(\mu_k)_k$  a sequence in  $\Delta_0^n$  such that  $\mu_k \rightharpoonup \mu$  and  $\limsup_{k \rightarrow \infty} I^n(\mu_k) \leq I^n(\mu)$ . In fact,  $\limsup_{k \rightarrow \infty} I_0^n(\mu_k) \leq I^n(\mu)$ . So  $I^n$  is greater or equal to the lower semicontinuous envelope of  $I_0^n$  and finally  $I^n$  is the lower semicontinuous envelope of  $I_0^n$ .  $\square$

We turn to the proof of the three auxiliary propositions.

*Proof of proposition 5.3.* Choose a sequence  $(\mu_k)_k$  in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  such that  $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < \infty$ . We want to prove that this sequence is tight and thus, relatively compact in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ .

Since  $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < \infty$ , for  $k$  large enough,  $\mu_k \in \Delta^n$ . Then, there exist  $\alpha_k \in [0, 1]$ ,  $\pi_k \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$  with  $\pi_k(1/\tau) < +\infty$  and  $\eta_k \in \mathcal{M}^1(\{1, \dots, n\})$ .

First, we prove equation (5.1):

$$\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < +\infty.$$

We have  $I_k^n(\mu_k) = \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k | \psi_k) + (1 - \alpha) \xi_k \geq \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k | \psi_k)$ . Therefore

$$\alpha_k \pi_k(1/\tau) \leq \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k | \psi_k)} \quad (\text{we consider } H(\tilde{\pi}_k | \psi_k) \neq 0).$$

Moreover

$$\alpha_k \pi_k(1/\tau) = \alpha_k \frac{1}{\tilde{\pi}_k(\tau)}.$$

Then

$$\alpha_k \pi_k(1/\tau) \leq \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k | \psi_k)} \wedge \frac{\alpha_k}{\tilde{\pi}_k(\tau)}$$

and  $\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) \leq \limsup_{k \rightarrow \infty} \frac{I_k^n(\mu_k)}{H(\tilde{\pi}_k | \psi_k)} \wedge \frac{\alpha_k}{\tilde{\pi}_k(\tau)}$ .



$\alpha_k \in [0, 1]$  and  $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) < +\infty$ , so there exists a constant  $C < \infty$  such that  $\forall k, \alpha_k \leq C$  and  $I_k^n(\mu_k) \leq C$ . Then

$$\begin{aligned} \limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) &\leq \limsup_{k \rightarrow \infty} \frac{C}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{C}{\tilde{\pi}_k(\tau)} \\ &\leq C \limsup_{k \rightarrow \infty} \frac{1}{H(\tilde{\pi}_k|\psi_k)} \wedge \frac{1}{\tilde{\pi}_k(\tau)} \\ &\leq C \limsup_{k \rightarrow \infty} \frac{1}{H(\tilde{\pi}_k|\psi_k) \vee \tilde{\pi}_k(\tau)} \\ &\leq \frac{C}{\liminf_{k \rightarrow \infty} H(\tilde{\pi}_k|\psi_k) \vee \tilde{\pi}_k(\tau)}. \end{aligned}$$

If there exists a subsequence  $k_j$  such that  $H(\tilde{\pi}_{k_j}|\psi_{k_j}) \rightarrow 0$ , then  $\lim \tilde{\pi}_{k_j} = \lim \psi_{k_j} = \psi^n$ . Thus, since  $[0, 1]$  and  $\mathcal{M}^1(\{1, \dots, n\})$  are compact, there exist subsequences of  $\alpha_k$  and  $\eta_k$  which converge. Eventually, there exists a subsequence of  $\mu_k$  which converges, so this sequence is relatively compact. In particular, (5.1) is true because  $\liminf_{j \rightarrow \infty} \tilde{\pi}_{k_j}(\tau) = \psi^n(\tau) > 0$ .

Else,  $\liminf_{k \rightarrow \infty} H(\tilde{\pi}_k|\psi_k) > 0$  and  $\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < \infty$ . Then (5.1) is true. Moreover, for  $M$  large enough and  $k$  large enough,  $\mu_k(1/(a+b)) = \alpha_k \pi_k(1/\tau) \leq M$ . So  $\mu_k \in \mathcal{X}_M$  which is a compact set by Lemma 4.1. Then, this sequence is relatively compact. In both cases, the sequence is relatively compact and (5.1) is true.  $\square$

*Remark 5.7.* As already said we only used the compactness of  $\mathcal{M}^1(\mathbb{W})$  and lemma 4.1 so that the above proof immediately extends to  $\mathbb{W}$  compact.  $\diamond$

*Proof of proposition 5.4.* Let  $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  and let  $(\mu_k)_k$  in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ , such that  $\mu_k \rightarrow \mu$ . We want to prove that  $\liminf_{k \rightarrow \infty} I_k^n(\mu_k) \geq I^n(\mu)$ .

Since  $\mu_k \rightarrow \mu$ , we may replace  $\mu_k$  by subsequences again denoted  $\mu_k$  in what follows. We assume  $\sup_k I_k^n(\mu_k) < \infty$  (otherwise, if  $\liminf_{k \rightarrow \infty} I_k^n(\mu_k) = +\infty$ ,  $+\infty \geq I^n(\mu)$ ). Then  $\mu_k \in \Delta^n$  and we denote by  $\alpha_k, \pi_k, \eta_k$  and  $\mu_{0,k}$  the corresponding quantities, with  $\alpha_k \in [0, 1]$  and  $\pi_k(1/\tau) < \infty$ .

First, if  $\limsup_{k \rightarrow \infty} \alpha_k = 0$ , we may assume taking a subsequence that  $\lim \alpha_k = 0$ . So  $\mu = \delta_{(+\infty, +\infty)} \otimes (\lim_k \eta_k)$ . Since  $\mathcal{M}^1(\{1, \dots, n\})$  is compact,  $(\lim_k \eta_k) \in \mathcal{M}^1(\{1, \dots, n\})$ . Therefore

$$\liminf_{k \rightarrow \infty} I_k^n(\mu_k) = \liminf_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) H(\tilde{\pi}_k|\psi_k) + (1 - \alpha_k) \xi_k \geq \liminf_{k \rightarrow \infty} (1 - \alpha_k) \xi_k = \xi = I^n(\mu).$$

Secondly, we have  $\limsup_{k \rightarrow \infty} \alpha_k = \bar{\alpha} > 0$ . Again we may assume that  $\lim \alpha_k = \bar{\alpha}$ .

We begin by studying the sequence  $\mu_k$  and  $\pi_k$  to have some information about  $\liminf_{k \rightarrow \infty} I_k^n(\mu_k)$ . Then, we prove that  $\mu$  is in  $\Delta^n$ , and finally we work on  $I^n(\mu)$ .

Since  $\sup_k I_k^n(\mu_k) < \infty$ , we can apply the proposition 5.3 and in particular the equation (5.1):  $\limsup_{k \rightarrow \infty} \alpha_k \pi_k(1/\tau) < \infty$ . Since  $\bar{\alpha} > 0$ , we have:  $\limsup_{k \rightarrow \infty} \pi_k(1/\tau) < \infty$ . For  $k$  large enough and  $M$  large enough, we have  $\pi_k(1/\tau) \leq M$ .

As

$$\left\{ \nu \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\}), \nu \left( \frac{1}{\tau} \right) \leq r \right\}$$

is tight for all  $r > 0$ , there exists a subsequence of  $\pi_k$  which converges in  $\mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$ . We can write the limit of  $\pi_k$  (or its subsequence) as  $\beta\pi + (1 - \beta)\delta_{(+\infty, +\infty)} \otimes \eta_0$ , for some  $\beta \in [0, 1]$  and  $\eta_0 \in \mathcal{M}^1(\{1, \dots, n\})$ .

If  $\beta > 0$ ,  $\pi(1/\tau) \leq \frac{1}{\beta} \limsup_{k \rightarrow \infty} \pi_k(1/\tau) < \infty$ . If  $\beta = 0$  we choose an arbitrary  $\pi$  such that  $\pi(1/\tau) < \infty$ .

Now, we prove that  $\mu$  is in  $\Delta^n$ , where  $\mu$  is defined by:

$$\begin{aligned} \mu &= \lim_k \mu_k \\ &= \lim_k (\alpha_k \mu_{0,k} + (1 - \alpha_k) \delta_{(+\infty, +\infty)} \otimes \eta_k) \\ &= \bar{\alpha} \lim_k \mu_{0,k} + (1 - \bar{\alpha}) \delta_{(+\infty, +\infty)} \otimes (\lim_k \eta_k). \end{aligned}$$

It holds

$$\begin{aligned} \mu_{0,k} &= \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)} du \otimes \pi_k(d\tau, dW) \\ &\xrightarrow{k \rightarrow \infty} \beta \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)} du \otimes \pi(d\tau, dW) + (1 - \beta) \delta_{(+\infty, +\infty)} \otimes \eta_0. \end{aligned}$$

Let  $\mu_0$  and  $\eta$  be defined as:

$$\begin{aligned} \mu_0 &:= \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)} du \otimes \pi(d\tau, dW) \\ \eta &:= \frac{\alpha(1 - \beta)}{1 - \bar{\alpha}\beta} \eta_0 + \frac{1 - \bar{\alpha}}{1 - \bar{\alpha}\beta} (\lim_k \eta_k). \end{aligned}$$

We get

$$\begin{aligned} \mu &= \bar{\alpha}\beta\mu_0 + \bar{\alpha}(1 - \beta)\delta_{(+\infty, +\infty)} \otimes \eta_0 + (1 - \bar{\alpha})\delta_{(+\infty, +\infty)} \otimes (\lim_k \eta_k) \\ &= \bar{\alpha}\beta\mu_0 + (1 - \bar{\alpha}\beta)\delta_{(+\infty, +\infty)} \otimes \eta, \end{aligned}$$

and in particular  $\mu \in \Delta^n$  with  $\alpha = \beta\bar{\alpha}$ .

Eventually

$$\begin{aligned} I^n(\mu) &= \beta\bar{\alpha}\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta\bar{\alpha})\xi \\ &= \bar{\alpha}[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi] + (1 - \bar{\alpha})\xi \\ &= \bar{\alpha}\pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) + (1 - \bar{\alpha})\xi \\ &\quad + \bar{\alpha}[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)] \\ &\leq \liminf_{k \rightarrow \infty} I_k^n(\mu_k) + \bar{\alpha}[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)]. \end{aligned}$$

In particular, we can apply Lemma 2.4 from [13] (the proof can be directly adapted in our case), since the hypotheses therein are satisfied:

**Lemma 5.8.** *Let  $\pi_k \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$  be such that  $\pi_k(1/\tau) < \infty$  such that*

$$\lim_k \pi_k(d\tau, dW) = \beta\pi(d\tau, dW) + (1 - \beta)\delta_{(+\infty)}(d\tau) \otimes \eta_0(dW) \quad (5.2)$$

for some  $\beta \in [0, 1]$ ,  $\pi \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$  such that  $\pi(1/\tau) < \infty$  and  $\eta_0 \in \mathcal{M}^1(\{1, \dots, n\})$ . Then

$$\liminf_{k \rightarrow \infty} \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k) \geq \beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi.$$

Eventually

$$\begin{aligned} I^n(\mu) &\leq \liminf_{k \rightarrow \infty} I_k^n(\mu_k) + \bar{\alpha} \underbrace{[\beta\pi(1/\tau)H(\tilde{\pi}|\psi^n) + (1 - \beta)\xi - \pi_k(1/\tau)H(\tilde{\pi}_k|\psi_k)]}_{<0 \text{ by Lemma 5.8}} \\ &\leq \liminf_{k \rightarrow \infty} I_k^n(\mu_k). \end{aligned}$$

□

*Remark 5.9.* Once again we only used the fact that  $\{1, \dots, n\}$  is compact so that we may replace it by any compact  $\mathbb{W}$ . ◇

*Proof of proposition 5.6.* Let  $\mu$  in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  with  $I^n(\mu) < \infty$ . We want to prove that there exists a sequence  $(\mu_k)$  such that  $\mu_k \rightharpoonup \mu$ ,  $\mu_k \in \Delta_0^n$  for all  $k$  and  $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) \leq I^n(\mu)$ .

Since  $I^n(\mu) < \infty$ ,  $\mu \in \Delta^n$ . Let  $\alpha$ ,  $\pi$  and  $\eta$  corresponding to  $\mu$ :

$$\begin{aligned} \mu(da, db, dc) &= \alpha \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW) \\ &\quad + (1 - \alpha)\delta_{(+\infty, +\infty)}(da, db) \otimes \eta(dc). \end{aligned}$$

The aim of this (technical) proof is to construct a sequence of laws  $\pi_k$ , depending on  $\psi_k$ , which satisfies a condition on its limit (the condition (5.6) described below) and a condition on its entropy with respect to  $\psi_k$  (the condition (5.7) described below). Then, we will construct a sequence of measure  $\mu_k$  which will satisfy the wished conditions. The next paragraphs give the details.

We denote  $\psi^n(d\tau)$  the marginal law of  $\tau$  in  $\psi^n$ , and  $\psi_k(d\tau)$  the corresponding law associated to  $\psi_k$  to simplify notations. In particular,  $\psi^n(d\tau|W = j)$  is the marginal law of  $\tau$  given  $W = j$ . We write  $q_j$  the weight of  $j$  for  $\eta$ :  $q_j = \eta(\{W = j\})$ .

Fix  $\rho > 0$ ,  $L > M > 1$  such that  $\mathbb{P}_{\psi^n}(\tau = 1/M) = \mathbb{P}_{\psi^n}(\tau = M) = \mathbb{P}_{\psi^n}(\tau = L) = 0$ . Then there exist  $N \in \mathbb{N}$  and  $1/M = T_1 < T_2 < \dots < T_N = M$  such that  $T_{i+1} - T_i \leq \rho$  and  $\mathbb{P}_{\psi^n}(\tau = T_i) = 0$ .

Here of course  $N$  and  $T_i$  depend on  $M$  and  $\rho$ . We also use the shorthand notation  $A_i = [T_i, T_{i+1})$  and  $A = \cup_{i=1}^N A_i$  in this proof. Then for  $L > M$  define  $\pi_{0,k}^{\rho, L, M}(d\tau, dW)$ ,  $\eta_k^{L, M}(d\tau, dW)$

and  $\pi_k^{\rho, M, L}(d\tau, dW) \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$  as:

$$\pi_{0,k}^{\rho, M}(d\tau, dW) = \frac{1}{\beta_k^{\rho, M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \tau \psi_k(d\tau, dW | \tau \in A_i, W = j), \quad (5.3)$$

$$\eta_k^{L, M}(d\tau, dW) = \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbb{1}_{\tau \in [M, L]} | W = j)} \tau \mathbb{1}_{\tau \in [M, L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW), \quad (5.4)$$

$$\pi_k^{\rho, L, M}(d\tau, dW) = \alpha \pi_{0,k}^{\rho, M}(d\tau, dW) + (1 - \alpha) \eta_k^{L, M}(d\tau, dW), \quad (5.5)$$

where  $\beta_k^{\rho, M}$  is the normalizing constant such that  $\pi_{0,k}^{\rho, M}$  is a probability measure.

Some  $q_j$ 's can be equal to 0. In these cases, we consider in fact the sum on  $j$  such that the  $q_j$  aren't zero: all the following calculation are then true. Thanks to Assumption 2.4, we know that for each  $j \in \{1, \dots, n\}$ ,  $\forall t \in \mathbb{R}^+$ ,  $\psi^n(\{\tau > t\} \cap \{W = j\}) > 0$ , so the measure  $\eta_k^{L, M}$  exists for adequate  $L$  and  $M$ . To simplify the notation, we consider in the following calculus that  $\eta_k^{L, M}$  is the sum for  $j \in \{1, \dots, n\}$ .

The above definition makes sense if  $L > M$  is large enough, and  $k$  is large enough depending on  $L$  and  $M$  ( $k$  will be sent to  $+\infty$  before  $L$ , and  $L$  before  $M$ ), and there is no problem with the conditioning. As  $\psi^n(\tau \in \partial A_i) = 0$ , if  $\psi_k(A_i \times \{j\}) = 0$  for each  $k$  large enough, then  $\psi(A_i \times \{j\}) = 0$ . Since  $I^n(\mu) < \infty$ , then  $H(\tilde{\pi} | \psi^n) < \infty$  and  $\tilde{\pi}(A_i \times \{j\}) = 0$ . The term in  $(i, j)$  in (5.3) would be considered equal to 0. Similarly if  $\psi_k([M, L] \times \{n\}) = 0$  then the term in  $j$  in (5.4) vanishes. If each terms of one sum is zero, then  $\alpha$  is equal to 0 or 1, by the same arguments on the relative entropy.

We want to prove:

$$\lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \lim_{k \rightarrow \infty} \pi_k^{\rho, L, M} = \alpha \pi + (1 - \alpha) \delta_{(+\infty)} \otimes \eta, \quad (5.6)$$

$$\liminf_{M \rightarrow \infty} \liminf_{L \rightarrow \infty} \liminf_{\rho \rightarrow 0} \limsup_{k \rightarrow \infty} \pi_k^{\rho, L, M} (1/\tau) H(\tilde{\pi}_k^{\rho, L, M} | \psi_k) \leq \alpha \pi (1/\tau) H(\tilde{\pi} | \psi^n) + (1 - \alpha) \xi = I^n(\mu), \quad (5.7)$$

where the limits in  $M$  and  $L$  are understood to run over  $M$  and  $L$  satisfying the above conditions.

Once (5.6) and (5.7) are proved, we can consider sequences  $\rho_k \rightarrow 0$ ,  $L_k \rightarrow \infty$ ,  $M_k \rightarrow \infty$  (such that  $\rho_k$ ,  $L_k$  and  $M_k$  satisfy the above conditions), we can define  $\pi_k = \pi_k^{\rho_k, L_k, M_k}$  which satisfy:  $\pi_k \rightarrow \alpha \pi + (1 - \alpha) \delta_{(+\infty)} \otimes \eta$  and

$$\limsup_{k \rightarrow \infty} \pi_k (1/\tau) H(\tilde{\pi}_k | \psi_k) \leq I^n(\mu).$$

Then  $\mu_k$  defined by:

$$\mu_k(da, db, dc) := \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi_k(d\tau, dW)$$

satisfies the proposition.

First, we prove the convergence (5.6) step by step. First, we consider  $\tilde{\pi}_{0,k}^{\rho,M}$ .

The normalisation constant  $\beta_k^{\rho,M}$  satisfies:

$$\begin{aligned}\beta_k^{\rho,M} &= \int_{(0,+\infty) \times \{1,\dots,n\}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \tau \psi_k(d\tau, dW | \tau \in A_i, W = j) \\ &= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \int_{A_i \times \{j\}} \tau \psi_k(d\tau, dW | \tau \in A_i, W = j) \\ &= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi_k(\tau | \tau \in A_i, W = j)\end{aligned}$$

Since  $\psi_k \xrightarrow[k \rightarrow \infty]{} \psi$ , then  $\forall i \leq N, \forall j \leq n, \psi_k(\tau \in A_i, W = j) \xrightarrow[k \rightarrow \infty]{} \psi^n(\tau \in A_i, W = j)$ . By bounded convergence (because  $\tau \in A_i \Rightarrow \tau \leq T_{i+1}$ ,  $\forall j \leq n, \forall f \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\})$ ),

$$\int_{A_i \times \{j\}} \tau f(\tau, W) \psi_k(d\tau, dW) \xrightarrow[k \rightarrow \infty]{} \int_{A_i \times \{j\}} \tau f(\tau, W) \psi^n(d\tau, dW).$$

So:

$$\beta_k^{\rho,M} \rightarrow \beta^{\rho,M} := \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(\tau | \tau \in A_i, W = j),$$

and

$$\pi_{0,k}^{\rho,M}(d\tau, dW) \rightarrow \pi_0^{\rho,M}(d\tau, dW) := \frac{1}{\beta^{\rho,M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \tau \psi^n(d\tau, dW | \tau \in A_i, W = j).$$

Let  $f \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\})$ . On  $A = [1/M, M)$ , for each  $j \leq n$ ,  $\tau \mapsto \tau f(\tau, j)$  is uniformly continuous, so there exists a modulus of continuity  $\omega_f$ , such that  $\forall (x, y) \in A, \forall j \in \{1, \dots, n\}, |xf(x, j) - yf(y, j)| \leq \omega_f(|x - y|)$ . In fact,  $\omega_f$  is the maximum of the  $n$  modulus of continuity for functions  $\tau \mapsto \tau f(\tau, j)$ .

For all  $\rho > 0$ ,  $(T_i)_i$  defined as before,

$$\begin{aligned}\int_{A \times \{1,\dots,n\}} \frac{f(\tau, W)\tau}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) &= \sum_{j=1}^n \int_{A \times \{j\}} \frac{f(\tau, j)\tau}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\ &= \sum_{j=1}^n \sum_{i=1}^N \int_{A_i \times \{j\}} \frac{f(\tau, j)\tau}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW).\end{aligned}$$

Then:

$$\begin{aligned}
\sum_{j=1}^n \sum_{i=1}^N \int_{A_i \times \{j\}} \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) &\leq \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\
&\leq \sum_{j=1}^n \sum_{i=1}^N \int_{A_i \times \{j\}} \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\
\sum_{j=1}^n \sum_{i=1}^N \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) &\leq \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(d\tau, dW) \\
&\leq \sum_{j=1}^n \sum_{i=1}^N \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j)
\end{aligned}$$

Let us remark that

$$\begin{aligned}
&\sum_{j=1}^n \sum_{i=1}^N \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) - \sum_{j=1}^n \sum_{i=1}^N \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) \\
&= \frac{1}{\tilde{\pi}(\tau \in A)} \sum_{j=1}^n \sum_{i=1}^N \tilde{\pi}(\tau \in A_i, W = j) \left( \max_{\tau \in A_i}(\tau f(\tau, j)) - \min_{\tau \in A_i}(\tau f(\tau, j)) \right) \\
&\leq \frac{1}{\tilde{\pi}(A)} \sum_{j=1}^n \sum_{i=1}^N \tilde{\pi}(\tau \in A_i, W = j) \omega_f(\rho) = \omega_f(\rho) \xrightarrow{\rho \rightarrow 0} 0.
\end{aligned}$$

Then

$$\sum_{j=1}^n \sum_{i=1}^N \frac{\min_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) \xrightarrow{\rho \rightarrow 0} \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW)$$

and

$$\sum_{j=1}^n \sum_{i=1}^N \frac{\max_{\tau \in A_i}(\tau f(\tau, j))}{\tilde{\pi}(\tau \in A)} \tilde{\pi}(\tau \in A_i, W = j) \xrightarrow{\rho \rightarrow 0} \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW).$$

Then, by studying  $\beta^{\rho, M} \pi_0^{\rho, M}(f)$ :

$$\begin{aligned}
&\sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \min_{\tau \in A_i}(\tau f(\tau, j)) \\
&\leq \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \int \tau f(\tau, W) \psi^n(d\tau, dW | \tau \in A_i, W = j) \\
&\leq \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \max_{\tau \in A_i}(\tau f(\tau, j)).
\end{aligned}$$

So:

$$\forall f \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\}), \beta^{\rho, M} \pi_0^{\rho, M}(f) \xrightarrow{\rho \rightarrow 0} \int_{A \times \{1, \dots, n\}} \frac{\tau f(\tau, W)}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW). \quad (5.8)$$

In particular, for  $f = \mathbb{1}$ ,  $\pi_0^{\rho,M}(\mathbb{1}) = 1$  so:  $\beta^{\rho,M} \xrightarrow{\rho \rightarrow 0} \beta^M := \int_{A \times \{1, \dots, n\}} \frac{\tau}{\tilde{\pi}(A)} \tilde{\pi}(d\tau, dW) = \tilde{\pi}(\tau | \tau \in A)$ . So  $\pi_0^{\rho,M}(d\tau, dW) \rightarrow \pi_0^M(d\tau, dW) := \frac{1}{\beta^M} \tau \tilde{\pi}(d\tau, dW | \tau \in A)$ .

When  $M \rightarrow \infty$ ,  $\pi(\tau \in A) = \pi(\tau \in [\frac{1}{M}, M]) \rightarrow 1$ . So:  $\beta^M \xrightarrow{M \rightarrow \infty} \tilde{\pi}(\tau)$  and:  $\pi_0^M(d\tau, dW) \rightarrow \tau \tilde{\pi}(d\tau, dW) / \tilde{\pi}(\tau) = \pi(d\tau, dW)$ .

Now, we study  $\eta_k^{L,M}$ :

$$\eta_k^{L,M}(d\tau, dW) = \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \tau \mathbb{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW).$$

We have  $\psi_k(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \xrightarrow{k \rightarrow \infty} \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)$ , so:

$$\eta_k^{L,M}(d\tau, dW) \rightarrow \eta^{L,M}(d\tau, dW) := \sum_{j=1}^n \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \tau \mathbb{1}_{\tau \in [M,L]} q_j \psi^n(d\tau | W = j) \otimes \delta_j(dW).$$

When  $L \rightarrow \infty$ ,

$$\eta^{L,M}(d\tau, dW) \rightarrow \eta^M(d\tau, dW) := \sum_{j=1}^n \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = j)} \tau \mathbb{1}_{\tau \in [M,+\infty]} q_j \psi^n(d\tau | W = j) \otimes \delta_j(dW).$$

For  $g \in \mathcal{C}_b((0, +\infty) \times \{1, \dots, n\})$  with a compact support,

$$\eta^M(g) = \sum_{j=1}^n \int_{[M,+\infty)} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = j)} \tau g(\tau, j) q_j \psi^n(d\tau | W = j).$$

For  $M$  large enough,  $\eta^M(g) = 0$ .

Moreover, for  $i \in \{1, \dots, n\}$ ,

$$\eta^M(\mathbb{1}_{W=i}) = \int_{[M,+\infty)} \frac{1}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = i)} \tau q_i \psi^n(d\tau | W = i) = q_i \frac{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = i)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,+\infty]} | W = i)} = q_i.$$

Then:  $\eta^M(d\tau, dW) \rightarrow \delta_{(+\infty)}(d\tau) \otimes \eta(dW)$ .

Now we deduce the convergence of  $\pi_k^{\rho,L,M}$ :

$$\lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \lim_{k \rightarrow \infty} \pi_k^{\rho,L,M} = \alpha \pi + (1 - \alpha) \delta_{(+\infty)} \otimes \eta.$$

(5.6) is then proved.

Now we prove the equation (5.7). We define  $\tilde{\pi}_k^{\rho,L,M}$ ,  $\tilde{\pi}_{0,k}^{\rho,M}$  and  $\tilde{\eta}_k^{L,M}$  by:

$$\tilde{\pi}_k^{\rho,L,M}(d\tau, dW) := \frac{1}{\pi_k^{\rho,L,M}(1/\tau)} \frac{1}{\tau} \pi_k^{\rho,L,M}(d\tau, dW), \quad (5.9)$$

$$\tilde{\pi}_{0,k}^{\rho,M}(d\tau, dW) := \frac{1}{\pi_{0,k}^{\rho,M}(1/\tau)} \frac{1}{\tau} \pi_{0,k}^{\rho,M}(d\tau, dW), \quad (5.10)$$

$$\tilde{\eta}_k^{L,M}(d\tau, dW) := \frac{1}{\eta_k^{L,M}(1/\tau)} \frac{1}{\tau} \eta_k^{L,M}(d\tau, dW). \quad (5.11)$$

In particular:

$$\begin{aligned}
\eta_k^{L,M}(1/\tau) &= \sum_{j=1}^n \int \frac{1}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \frac{\tau}{\tau} \mathbf{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW) \\
&= \sum_{j=1}^n q_j \frac{\psi_k(\mathbf{1}_{\tau \in [M,L]} | W = j)}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \left( = \sum_{j=1}^n q_j \frac{1}{\psi_k(\tau | \tau \in [M,L], W = j)} \right) \\
\tilde{\eta}_k^{L,M}(d\tau, dW) &= \frac{1}{\eta_k^{L,M}(1/\tau)} \frac{1}{\tau} \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \tau \mathbf{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW) \\
&= \frac{1}{\eta_k^{L,M}(1/\tau)} \sum_{j=1}^n \frac{1}{\psi_k(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \mathbf{1}_{\tau \in [M,L]} q_j \psi_k(d\tau | W = j) \otimes \delta_j(dW).
\end{aligned}$$

Using Lemma 2.3 from [13] (the proof of which can be easily adapted in this situation) ,  $\pi \mapsto \pi(1/\tau)H(\tilde{\pi}|\psi_k) = \frac{1}{\tilde{\pi}(\tau)}H(\tilde{\pi}|\psi_k)$  is convex, so:

$$\pi_k^{\rho,L,M}(1/\tau)H(\tilde{\pi}_k^{\rho,L,M}|\psi_k) \leq \alpha \frac{1}{\tilde{\pi}_{0,k}^{\rho,M}(\tau)}H(\tilde{\pi}_{0,k}^{\rho,M}|\psi_k) + (1-\alpha) \frac{1}{\tilde{\eta}_k^{L,M}(\tau)}H(\tilde{\eta}_k^{L,M}|\psi_k).$$

First we consider the second term: since  $\psi^n(\{\tau = M\}) = \psi^n(\{\tau = L\}) = 0$ , we have:

$$\lim_k \frac{1}{\tilde{\eta}_k^{L,M}(\tau)}H(\tilde{\eta}_k^{L,M}|\psi_k) = \frac{1}{\tilde{\eta}^{L,M}(\tau)}H(\tilde{\eta}^{L,M}|\psi^n)$$

$$\begin{aligned}
\tilde{\eta}^{L,M}(\tau) &= \eta^{L,M}(1/\tau)^{-1} \\
\tilde{\eta}^{L,M}(d\tau, dW) &= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n \frac{1}{\psi^n(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \mathbf{1}_{\tau \in [M,L]} q_j \psi^n(d\tau | W = j) \otimes \delta_j(dW).
\end{aligned}$$

We have

$$\begin{aligned}
&H(\tilde{\eta}^{L,M}|\psi^n) \\
&= \int_{[M,L] \times \{1, \dots, n\}} \ln \left( \frac{\tilde{\eta}^{L,M}(d\tau, dW)}{\psi^n(d\tau, dW)} \right) \tilde{\eta}^{L,M}(d\tau, dW) \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n \int_{[M,L] \times \{j\}} \ln \left( \frac{\tilde{\eta}^{L,M}(d\tau, dW)}{\psi^n(d\tau, dW)} \right) \frac{\mathbf{1}_{\tau \in [M,L]} q_j}{\psi^n(\tau \mathbf{1}_{\tau \in [M,L]} | W = j)} \psi^n(d\tau | W = j) \otimes \delta_j(dW).
\end{aligned}$$

We can decompose  $\psi^n$  in the following form:

$$\psi^n(d\tau, dW) = \sum_{j=1}^n p_j \psi^n(d\tau | W = j) \otimes \delta_j(d\tau),$$



where  $p_j = \psi^n(W = j)$ . By definition of  $\Delta^n$  and  $\{1, \dots, n\}$ , we know that  $p_i = 0$  implies  $q_i = 0$ . Then:

$$\begin{aligned}
& H(\tilde{\eta}^{L,M} | \psi^n) \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n \int_{[M,L] \times \{j\}} \ln \left( \frac{1}{\eta^{L,M}(1/\tau)} \frac{\mathbb{1}_{\tau \in [M,L]} q_j}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \frac{\psi^n(d\tau | W = j) \otimes \delta_j(dW)}{p_j \psi^n(d\tau | W = j) \otimes \delta_j(d\tau)} \right) \\
&\quad \times \frac{\mathbb{1}_{\tau \in [M,L]} q_j}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \psi^n(d\tau | W = j) \otimes \delta_j(dW) \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \int_{[M,L]} \ln \left( \frac{q_j}{p_j \eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \right) \frac{\psi^n(d\tau | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \ln \left( \frac{q_j}{p_j \eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \ln \left( \frac{q_j}{p_j} \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&\quad - \frac{1}{\eta^{L,M}(1/\tau)} \sum_{j=1}^n q_j \ln \left( \eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)}.
\end{aligned}$$

Then we study

$$\begin{aligned}
\frac{1}{\tilde{\eta}_k^{L,M}(\tau)} H(\tilde{\eta}_k^{L,M} | \psi_k) &= \sum_{j=1}^n q_j \ln \left( \frac{q_j}{p_j} \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&\quad - \sum_{j=1}^n q_j \ln \left( \eta^{L,M}(1/\tau) \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)}.
\end{aligned}$$

First, we have  $\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \geq M$ , and  $\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j) \leq 1$ , so

$$\begin{aligned}
\frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \left| \sum_{j=1}^n q_j \ln \left( \frac{q_j}{p_j} \right) \right| &\leq \frac{1}{M} \left| \sum_{j=1}^n q_j \ln \left( \frac{q_j}{p_j} \right) \right| \\
&\xrightarrow{M \rightarrow \infty} 0.
\end{aligned}$$

Secondly, we have:

$$\begin{aligned}
& \sum_{j=1}^n q_j \ln \left( \psi^n(\eta^{L,M}(1/\tau) \tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \sum_{j=1}^n q_j \ln \left( \eta^{L,M}(1/\tau) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&\quad + \sum_{j=1}^n q_j \ln \left( \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right) \frac{\psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \\
&= \ln \left( \eta^{L,M}(1/\tau) \right) \times \eta^{L,M}(1/\tau) + \sum_{j=1}^n q_j \frac{\ln \left( \psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j) \right)}{\psi^n(\tau \mathbb{1}_{\tau \in [M,L]} | W = j)} \psi^n(\mathbb{1}_{\tau \in [M,L]} | W = j).
\end{aligned}$$

Moreover,  $\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j) \geq M$ , so  $\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j) \xrightarrow{M, L \rightarrow +\infty} +\infty$ . Since  $\frac{\ln x}{x} \xrightarrow{x \rightarrow +\infty} 0$  and  $\psi^n(\mathbf{1}_{\tau \in [M, +\infty)} | W = j) \xrightarrow{M \rightarrow +\infty} 0$ , we have:

$$\sum_{j=1}^n q_j \frac{\ln(\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j))}{\psi^n(\tau \mathbf{1}_{\tau \in [M, L]} | W = j)} \psi^n(\mathbf{1}_{\tau \in [M, L]} | W = j) \xrightarrow{M, L \rightarrow \infty} 0.$$

Eventually,  $\eta^{L, M}(1/\tau) = \int_{[M, L]} \frac{1}{\tau} \eta^{L, M}(d\tau) \leq \frac{1}{M}$ . Then in particular

$$\lim_{M \rightarrow +\infty} \lim_{L \rightarrow \infty} \eta^{L, M} \left( \frac{1}{\tau} \right) = 0. \quad (5.12)$$

Since  $x \ln x \xrightarrow{x \rightarrow 0} 0$ ,  $\eta^{L, M}(1/\tau) \times \ln(\eta^{L, M}(1/\tau)) \xrightarrow{M, L \rightarrow +\infty} 0$ .

Then

$$\lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{H(\tilde{\eta}^{L, M} | \psi^n)}{\tilde{\eta}^{L, M}(\tau)} = 0 \leq \xi.$$

We focus on the first term. We know that  $\tilde{\pi}_{0, k}^{\rho, M}(\tau) = 1/\pi_{0, k}^{\rho, M}(1/\tau)$ . We have:

$$\begin{aligned} \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \lim_{k \rightarrow \infty} \pi_{0, k}^{\rho, M}(1/\tau) &= \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \frac{1}{\beta^{\rho, M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} = \frac{1}{\beta^{\rho, M}} \\ &= \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{1}{\beta^M} \\ &= \frac{1}{\tilde{\pi}(\tau)} \end{aligned} \quad (5.13)$$

Then:  $\frac{1}{\tilde{\pi}_{0, k}^{\rho, M}(\tau)} \xrightarrow{M, \rho, k} \frac{1}{\tilde{\pi}(\tau)}$ . Now, we study  $H(\tilde{\pi}_{0, k}^{\rho, M} | \psi_k)$ :

As  $\pi_0^{\rho, M}(1/\tau) = 1/\beta^{\rho, M}$ , we have

$$\begin{aligned} \tilde{\pi}_0^{\rho, M}(d\tau, dW) &= \frac{1}{\pi_0^{\rho, M}(1/\tau)} \frac{1}{\tau} \pi_0^{\rho, M}(d\tau, dW) \\ &= \beta^{\rho, M} \times \frac{1}{\beta^{\rho, M}} \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(d\tau, dW | \tau \in A_i, W = j) \\ &= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \psi^n(d\tau, dW | \tau \in A_i, W = j). \end{aligned}$$

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} H(\tilde{\pi}_{0,k}^{\rho, M} | \psi_k) \\
&= H(\tilde{\pi}_0^{\rho, M} | \psi^n) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \times \\
&\quad \int \ln \left( \frac{\sum_{l=1}^N \sum_{m=1}^n \frac{\tilde{\pi}(\tau \in A_l, W = m)}{\tilde{\pi}(A)} \frac{\psi^n(d\tau, dW | \tau \in A_l, W = m)}{\psi^n(d\tau, dW)}}{\psi^n(d\tau, dW | \tau \in A_i, W = j)} \right) \psi^n(d\tau, dW | \tau \in A_i, W = j) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \times \\
&\quad \int_{A_i \times \{j\}} \ln \left( \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \frac{\psi^n(d\tau, dW)}{\psi^n(\tau \in A_i, W = j) \psi^n(d\tau, dW)} \right) \frac{\psi^n(d\tau, dW)}{\psi^n(\tau \in A_i, W = j)} \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \times \int_{A_i \times \{j\}} \ln \left( \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \right) \psi^n(d\tau, dW) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \ln \left( \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A) \psi^n(\tau \in A_i, W = j)} \right) \\
&= \sum_{i=1}^N \sum_{j=1}^n \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\tilde{\pi}(A)} \ln \left( \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) - \ln(\tilde{\pi}(A)) \\
&= \frac{1}{\tilde{\pi}(A)} \sum_{i=1}^N \sum_{j=1}^n \tilde{\pi}(\tau \in A_i, W = j) \ln \left( \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) - \ln(\tilde{\pi}(A)) \\
&= \frac{1}{\tilde{\pi}(A)} \left[ \tilde{\pi}(\tau \in A^c) \ln \left( \frac{\tilde{\pi}(\tau \in A^c)}{\psi^n(\tau \in A^c)} \right) + \sum_{i=1}^N \sum_{j=1}^n \tilde{\pi}(\tau \in A_i, W = j) \ln \left( \frac{\tilde{\pi}(\tau \in A_i, W = j)}{\psi^n(\tau \in A_i, W = j)} \right) \right] \\
&\quad - \left( \ln(\tilde{\pi}(A)) + \tilde{\pi}(A^c) \ln \left( \frac{\tilde{\pi}(A^c)}{\psi^n(A^c)} \right) \right)
\end{aligned}$$

The well known decomposition of entropy property tells us that if  $(B_m)_{m \leq M}$  is a partition of  $(0, +\infty) \times \{1, \dots, n\}$ :

$$H(\tilde{\pi} | \psi^n) \geq \sum_{m=1}^M \tilde{\pi}(B_m) \ln \left( \frac{\tilde{\pi}(B_m)}{\psi^n(B_m)} \right).$$

Here,  $(A_i \times \{j\})_{i,j}$  and  $A^c$  are a partition of  $(0, +\infty) \times \{1, \dots, n\}$ . We have:

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} H(\tilde{\pi}_{0,k}^{\rho, M} | \psi_k) = H(\tilde{\pi}_0^{\rho, M} | \psi^n) \\
& \leq \frac{1}{\tilde{\pi}(A)} H(\tilde{\pi} | \psi^n) - \left( \ln(\tilde{\pi}(A)) + \tilde{\pi}(A^c) \ln \left( \frac{\tilde{\pi}(A^c)}{\psi^n(A^c)} \right) \right).
\end{aligned}$$

Since  $\tilde{\pi}(A) \rightarrow 1$  and  $\tilde{\pi}(A^c) \rightarrow 0$  as  $M \rightarrow \infty$ :

$$\limsup_{M \rightarrow \infty} \sup_{\rho < 1} \lim_k H(\tilde{\pi}_{0,k}^{\rho, M} | \psi_k) \leq H(\tilde{\pi} | \psi^n).$$

Now, we can construct a sequence  $\pi_k (= \pi_k^{\rho_k, L_k, M_k})$  such that:  $\pi_k \rightarrow \alpha\pi + (1 - \alpha)\delta_\infty \otimes \eta$  and  $\limsup_{k \rightarrow \infty} \pi_k(1/\tau) H(\tilde{\pi}_k | \psi_k) \leq I^n(\mu)$ .

Eventually, by (5.12) and (5.13),

$$\begin{aligned} \mu_k \left( \frac{1}{a+b} \right) &= \pi_k \left( \frac{1}{\tau} \right) \\ &= \alpha \pi_{0,k}^{\rho_k, M_k} \left( \frac{1}{\tau} \right) + (1 - \alpha) \eta_k^{L_k, M_k} \left( \frac{1}{\tau} \right) \\ &\xrightarrow{k \rightarrow \infty} \alpha \tilde{\pi} \left( \frac{1}{\tau} \right) = \mu \left( \frac{1}{a+b} \right). \end{aligned}$$

□

## 6. SOME ADDITIONAL TECHNICAL TOPOLOGICAL LEMMATA.

Before proving the upper bound and the lower bound of the LDP for empirical measures, we add two useful lemmata. The first one will be used several times for the upper bound. Once again its proof uses conditioning.

**Lemma 6.1.** *Let  $\mathcal{D}_M^n = \left\{ \mu \in \Delta^n, \mu \left( \frac{1}{a+b} \right) \leq M \right\}$ . Then  $\mathcal{D}_M^n$  is a compact set.*

*Proof.* Since  $\mathcal{D}_M^n \subset \mathcal{X}_M$  we already know that it is relatively compact. But since  $\Delta^n$  is not clearly closed, we have to show that  $\mathcal{D}_M^n$  is closed.

Let  $(\mu_k)_k$  be a sequence of measures in  $\mathcal{D}_M^n$ . We will prove that some subsequence again denoted  $(\mu_k)_k$  converges in  $\mathcal{D}_M^n$ . We denote by  $\alpha_k, \mu_{0,k}, \pi_k$  and  $\eta_k$  the corresponding quantities in the definition.

For any  $k, \alpha_k \in [0, 1]$  and  $\eta_k \in \mathcal{M}^1(\{1, \dots, n\})$  which is compact, so there exists a subsequence  $\alpha_{\varphi(k)}, \eta_{\varphi(k)}$  and  $\alpha \in [0, 1]$  and  $\eta \in \mathcal{M}^1(\{1, \dots, n\})$  such that  $\alpha_{\varphi(k)} \xrightarrow{k \rightarrow \infty} \alpha$  and  $\eta_{\varphi(k)} \rightarrow \eta$ .

We remind that, for any  $r > 0$ ,  $\left\{ \nu \in \mathcal{M}^1((0, +\infty]), \nu \left( \frac{1}{x} \right) \leq r \right\}$  is a compact set.

If  $\alpha = 0$  : then for  $f \in \mathcal{C}_b((0, +\infty]^2 \times \{1, \dots, n\})$ :

$$\begin{aligned} \mu_{\varphi(k)}(f) &= \underbrace{\alpha_{\varphi(k)}}_{\rightarrow 0} \underbrace{\mu_{0, \varphi(k)}(f)}_{\in [-\|f\|_\infty, \|f\|_\infty]} + \underbrace{(1 - \alpha_{\varphi(k)})}_{\rightarrow 1} (\delta_{(+\infty, +\infty)} \otimes \underbrace{\eta_{\varphi(k)}}_{\rightarrow \eta})(f) \\ &\xrightarrow{k \rightarrow \infty} (\delta_{(+\infty, +\infty)} \otimes \eta)(f). \end{aligned}$$

Since  $(\delta_{(+\infty, +\infty)} \otimes \eta) \in \mathcal{D}_M^n$ ,  $\mu_{\varphi(k)}$  converges in  $\mathcal{D}_M^n$ .

If  $\alpha > 0$  : then for  $k$  large enough, we have :  $|\alpha_{\varphi(k)} - \alpha| < \alpha/2$ , so  $\alpha_{\varphi(k)} > \alpha/2$ . Then, we have:

$$\pi_{\varphi(k)} \left( \frac{1}{\tau} \right) = \mu_{0, \varphi(k)} \left( \frac{1}{a+b} \right) = \frac{1}{\alpha_{\varphi(k)}} \mu_{\varphi(k)} \left( \frac{1}{a+b} \right) \leq \frac{M}{\alpha_{\varphi(k)}} \leq \frac{2M}{\alpha}.$$

We will prove that for all  $i$ ,  $\pi_{\varphi(k)}|_{W=i}$ , which is the conditional law  $\pi$  given  $W = i$ , belongs to  $\left\{ \nu \in \mathcal{M}^1((0, +\infty]), \nu\left(\frac{1}{x}\right) \leq r \right\}$  for a well-chosen  $r$ .

For any  $k$ ,  $\pi_{\varphi(k)} \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$ , let define  $(q_{j, \varphi(k)})_{j \in \{1, \dots, n\}}$  the weights such that  $q_{j, \varphi(k)} = \mathbb{P}_{\pi_{\varphi(k)}}(W = j)$ . Then  $\pi_{\varphi(k)}(1/\tau)$  can be written with its marginal law:

$$\pi_{\varphi(k)}\left(\frac{1}{\tau}\right) = \sum_{j \in \{1, \dots, n\}} q_{j, \varphi(k)} \mathbb{E}_{\pi_{\varphi(k)}} \left[ \frac{1}{\tau} \middle| W = j \right].$$

The  $((q_{j, \varphi(k)})_{j \in \{1, \dots, n\}})_k$  are at most  $n$  sequences of weights in  $[0, 1]$ . There exists a subsequence  $(q_{j, \varphi_2(k)})_{j \in \{1, \dots, n\}}$  and a  $(q_j)_{j \in \{1, \dots, n\}}$  with  $q_j \in [0, 1]$  such that  $\sum q_j = 1$  and  $(q_{j, \varphi_2(k)})_{j \in \{1, \dots, n\}}$  tends to  $(q_j)_{j \in \{1, \dots, n\}}$ . Let  $J$  the set of indices  $j$  such that  $q_j \neq 0$ . Let  $p_{\min}$  the minimum of the  $q_j$  for  $j \in J$ :

$$p_{\min} = \min\{q_j, j \in J\} \neq 0.$$

Then, for  $k$  large enough, we have:

$$\forall j \in J, q_{j, \varphi_2(k)} > \frac{p_{\min}}{2}.$$

Then, we have, for each  $j \in J$

$$\pi_{\varphi_2(k)}\left(\frac{1}{\tau}\right) = \sum_{i \in \{1, \dots, n\}} q_{i, \varphi_2(k)} \mathbb{E}_{\pi_{\varphi_2(k)}} \left[ \frac{1}{\tau} \middle| W = i \right] \geq \frac{p_{\min}}{2} \mathbb{E}_{\pi_{\varphi_2(k)}} \left[ \frac{1}{\tau} \middle| W = j \right]$$

and  $\pi_{\varphi_2(k)}\left(\frac{1}{\tau}\right) \leq \frac{2M}{\alpha}$ , then:

$$\forall j \in J, \mathbb{E}_{\pi_{\varphi_2(k)}} \left[ \frac{1}{\tau} \middle| W = j \right] \leq \frac{4M}{\alpha p_{\min}}.$$

So  $\forall j \in J, \pi_{\varphi_2(k)}|_{W=j} \in \left\{ \nu \in \mathcal{M}^1((0, +\infty]), \nu\left(\frac{1}{x}\right) \leq 4M/(\alpha p_{\min}) \right\}$  which is a compact set.

There exists a subsequence of  $(\pi_{\varphi_2(k)}|_{W=j})_k$  which converges to a  $\pi^j$  for each  $j \in J$ . By a diagonal argument, we consider an extraction  $\varphi_3$  such that:

$$\forall j \in J, \pi_{\varphi_3(k)}|_{W=j} \rightharpoonup \pi^j.$$

We define then  $\pi$  corresponding to  $(q_j)_j$  and  $(\pi^j)_j$ :

$$\pi(d\tau, dW) = \sum_{j \in J} q_j \delta_{(W=j)} \pi^j(d\tau)$$

$$\forall f \in \mathcal{C}_b((0, +\infty] \times \{1, \dots, n\}), \pi(f) = \sum_{j \in J} q_j \pi^j(f(\cdot, j)).$$

Then,  $\pi_{\varphi_3(k)} \rightharpoonup \pi$ .

It remains to show that  $\mu$  to belong to  $\mathcal{D}_M^n$  i.e to  $\Delta^n$ .

$\pi$  can be written as:

$$\pi(d\tau, dW) = \pi(\tau < +\infty) \times \pi(d\tau, dW | \tau < \infty) + (1 - \pi(\tau < \infty)) \pi(d\tau, dW | \tau = \infty).$$

Let  $\pi^*(d\tau, dW) = \pi(d\tau, dW | \tau < \infty)$ ,  $\beta^* = \pi(\tau < +\infty)$  and  $\eta^*$  such that  $\pi(d\tau, dW | \tau = \infty) = \delta_{(+\infty)} \otimes \eta^*$ .

Moreover,  $\mu_{0,\varphi(k)}$  is the product measure of  $du$  and  $\pi_{\varphi(k)}$ .  $\pi_{\varphi_3(k)}$  weakly converges to  $\pi$ , so  $\mu_{0,\varphi_3(k)}$  weakly converges to the product measure of  $du$  and  $\pi$ . We denote by  $\mu_0$  the product measure of  $du$  and  $\pi^*$ . Then  $\mu_{\varphi(k)}$  tends to

$$\mu = \alpha\beta^*\mu_0 + \alpha(1 - \beta^*)\delta_{(+\infty,+\infty)} \otimes \eta^* + (1 - \alpha)\delta_{(+\infty,+\infty)} \otimes \eta.$$

With

$$\eta_0 = \frac{\alpha(1 - \beta^*)}{(1 - \alpha\beta^*)}\eta^* + \frac{(1 - \alpha)}{(1 - \alpha\beta^*)}\eta,$$

we can write  $\mu = \alpha\beta^*\mu_0 + (1 - \alpha\beta^*)\delta_{(+\infty,+\infty)} \otimes \eta_0$ , so  $\mu \in \Delta^n$ .

□

The next lemma introduces  $\bar{\Delta}^n$ , a closed set which contains  $\Delta^n$  (but is not its closure despite the notation). This will be used directly in the proof of the upper bound of the large deviation principle. A similar set is discussed in [13] lemma 2.5. However we have here to carefully manage the third coordinate, so that the proof in [13] has to be rewritten.

**Lemma 6.2.** *The set*

$$\begin{aligned} \bar{\Delta}^n &:= \{ \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}) : \mu = \alpha\mu_0 + (1 - \alpha)\delta_{(+\infty,+\infty)} \otimes \eta, \alpha \in [0, 1], \\ &\mu_0(da, db, dc) = \int_{[0,1] \times (0, \infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW), \\ &\pi \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\}), \eta \in \mathcal{M}^1(\{1, \dots, n\}) \} \end{aligned} \quad (6.1)$$

is closed in  $\mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ .

*Proof.* Let  $\mu_k \in \bar{\Delta}^n$  such that  $\mu_k$  converges to a  $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$  when  $k$  tends to  $\infty$ . We will prove that  $\mu$  is in  $\bar{\Delta}^n$ . Let  $\alpha_k$ ,  $\pi_k$  and  $\eta_k$  be the corresponding quantities to  $\mu_k$  as in the definition of  $\bar{\Delta}^n$ .

$(\alpha_k)_k \in [0, 1]$  admits a subsequence which converges to a  $\alpha \in [0, 1]$ .  $\eta_k$  is a sequence of measure on the finite set  $\{1, \dots, n\}$ , so a subsequence of  $\eta_k$  converges to  $\eta \in \mathcal{M}^1$ . As usual we identify the subsequence and the sequence.

$(\pi_k)_k \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\}) \subset \mathcal{M}^1([0, +\infty] \times \{1, \dots, n\})$  also admits a subsequence which tends to a limit  $\pi \in \mathcal{M}^1([0, +\infty] \times \{1, \dots, n\})$ .

Then

$$\begin{aligned} \mu(da, db, dc) &= \alpha \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW) \\ &\quad + (1 - \alpha)\delta_{(+\infty,+\infty)}(da, db) \otimes \eta(dc). \end{aligned}$$

We need to prove that  $\mu$  is in  $\bar{\Delta}^n$ . First we verify that  $\pi$  has no weight in 0, and secondly we prove that by rewriting  $\mu$ , we can consider  $\bar{\pi} \in \mathcal{M}^1((0, +\infty) \times \{1, \dots, n\})$ ,  $\bar{\alpha}$  and  $\bar{\eta}$  such that  $\mu$  have the good form.

First, we consider the weight of 0 for  $\pi$ . By Skorohod's representation theorem, there exists a sequence  $(P_k, V_k)_k$  and  $X_k$  of random variables such that  $(P_k, V_k) \in (0, +\infty) \times \{1, \dots, n\}$  has the law  $\pi_k$ ,  $X_k$  has the law  $\eta_k$  and  $(P_k, V_k)$  converges a.s. to  $(P, V)$  of law  $\pi$ ,  $X_k$  converges

a.s. to  $X$  of law  $\eta$ . Let  $U$  be an uniform random variable independent of  $(P_k)_k$  and  $P$ . For any  $f \in \mathcal{C}_b([0, +\infty]^2 \times \{1, \dots, n\})$  we obtain

$$\begin{aligned} \mu_k(f) &= \alpha_k \mathbb{E}[f(UP_k, (1-U)P_k, V_k)] + (1 - \alpha_k) \mathbb{E}[f(+\infty, +\infty, X_k)] \\ &\xrightarrow[k \rightarrow \infty]{} \alpha \mathbb{E}[f(UP, (1-U)P, V)] + (1 - \alpha) \mathbb{E}[f(+\infty, +\infty, X)] \end{aligned}$$

and this limit is equal to  $\mu(f)$ . For  $f_\varepsilon(a, b, c) = \mathbb{1}_{a+b < \varepsilon}$ , since  $\mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\})$ :  $\alpha\pi(\{P < \varepsilon\}) = \mu(f_\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{} 0$ . Then  $\pi \in \mathcal{M}^1((0, +\infty] \times \{1, \dots, n\})$ .

Now we prove that we can write  $\mu$  in the same form that in  $\bar{\Delta}^n$ . Let  $\beta = \pi(P = +\infty)$ . If  $\beta = 0$ , then

$$\alpha \int_{[0,1] \times (0, +\infty) \times \{1, \dots, n\}} \delta_{(u\tau, (1-u)\tau, W)}(da, db, dc) du \otimes \pi(d\tau, dW) + (1-\alpha) \delta_{(+\infty, +\infty)}(da, db) \otimes \eta(dc)$$

is already in the good form.

Else,

$$\begin{aligned} \mu(f) &= \alpha\beta \mathbb{E}[f(UP, (1-U)P, V) | P < \infty] + \alpha(1-\beta) \mathbb{E}[f(+\infty, +\infty, V) | P = \infty] \\ &\quad + (1-\alpha) \mathbb{E}[f(+\infty, +\infty, X)] \\ &= \alpha\beta \mathbb{E}[f(UP, (1-U)P, V) | P < \infty] + \\ &\quad (1-\alpha\beta) \left( \frac{\alpha(1-\beta)}{1-\alpha\beta} \mathbb{E}[f(+\infty, +\infty, V) | P = \infty] + \frac{1-\alpha}{1-\alpha\beta} \mathbb{E}[f(+\infty, +\infty, X)] \right). \end{aligned}$$

Let  $\bar{\eta}(dW) = \frac{\alpha(1-\beta)}{1-\alpha\beta} \pi(dW | \tau = \infty) + \frac{1-\alpha}{1-\alpha\beta} \eta$ ,  $\bar{\alpha} = \alpha\beta$  and  $\bar{\pi}(d\tau, dW) = \pi(d\tau, dW | \tau < \infty)$ . Therefore  $\mu \in \bar{\Delta}^n$ .  $\square$

*Remark 6.3.* Here again one can replace  $\{1, \dots, n\}$  by a compact subset  $\mathbb{W}$ .  $\diamond$

We finally set two important results. The first one is identical to Lemma 2.6 and Lemma 2.7 in [13]:

**Lemma 6.4.** *If  $\mu \in \Delta^n$  then  $I^n(\mu) \leq \sup_{f \in \Gamma} \mu(f)$  and if  $\mu \in \bar{\Delta}^n \setminus \Delta^n$  then  $\sup_{f \in \Gamma} \mu(f) = +\infty$ . (recall that  $\Gamma$  is defined in (4.9))*

The proof of the Lemma is almost identical to [13]. Indeed it is enough to consider a third variable  $c \in \{1, \dots, n\}$ . Similarly to the proof of Lemma 2.6 we have to introduce a function

$$f_{d,\varphi,M}(a, b, c) = \frac{\varphi(a+b, c)}{a+b} + d\mathbb{1}_{(M, +\infty]}(a+b)$$

for a continuous and compactly supported  $\varphi$ .  $f_{d,\varphi,M}$  is lower semi-continuous. One can then copy the proof of Lemma 2.6 in [13]. For the second part we may consider the same function  $f_\varepsilon$  as in the proof of Lemma 2.7 in [13].

Notice that again we may replace  $\{1, \dots, n\}$  by a compact  $\mathbb{W}$ .

The second one (and its proof) is identical to Proposition 3.3 in [13]

**Lemma 6.5.** *For all  $f \in \Gamma$  and all  $t > 0$ ,  $\mathbb{E}(e^{t\mu_t^n(f)}) \leq \frac{D_{n,f}}{1-C_{n,f}} < +\infty$ .*

## 7. PROOF OF THEOREM 4.3.

In this section, we prove the LDP for  $\mathbb{P}_t^n$  that denotes the  $\mathbb{P}$  distribution of  $\mu_t^n$ . This time the introduction of the third coordinate  $W$  replacing  $F(\tau)$  does not create any new difficulty. However some points in the proofs of [13] are not clear for us and we will give the details for some points.

The proof of the upper bound is made in several steps: the proof of a weak principle, for compact sets  $\mathcal{C}$  (itself divided in several steps), and the proof that  $\mu_t^n$  is an exponentially tight family, i.e. satisfies: for all  $\alpha \in \mathbb{R}^+$  there exists some compact set  $\mathcal{K}_\alpha$  with

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n(\mathcal{K}_\alpha^c) < -\alpha.$$

The (full) upper bound for closed sets  $\mathcal{C}$  then follows from [9]. These steps are described in [13] section 3.

Exponential tightness is the aim of Lemma 3.1 in [13]. The first step is the following lemma

**Lemma 7.1.** *We have:*

$$\lim_{M \rightarrow +\infty} \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P} \left( \mu_t^n \left( \frac{1}{a+b} \right) > M \right) = -\infty.$$

whose proof is unchanged in our case. Since  $\{\nu \in \mathcal{M}_1((0, +\infty]^2 \times \{1, \dots, n\}), \nu(1/(a+b)) \leq M\}$  is compact (see lemma 4.1), exponential tightness follows.

The proof of the upper bound for compact subsets is done in [13] p. 2261 and 2262 beginning with the definition of the set

$$\Delta_{M,g,\delta} = \{\mu \in \mathcal{M}_1((0, +\infty]^2), \exists \nu \in \Delta, |\nu(g) - \mu(g)| \leq \delta, \mu(1/(a+b)) \leq M\}$$

for some continuous and bounded  $g$ ,  $\delta$  and  $M$  positive. We do not see an immediate argument showing that this set is closed (hence compact since it is relatively compact). We will thus slightly modify the proof in [13].

We introduce a modified set for  $M$ ,  $\delta$  and  $g$  as before,

$$\begin{aligned} \Delta_{M,g,\delta}^n &= \{\mu \in \mathcal{M}_1((0, +\infty]^2), \exists \nu \in \Delta^n, |\nu(g) - \mu(g)| \leq \delta, \\ &\quad \mu(1/(a+b)) \leq M, \nu(1/(a+b)) \leq M + \delta\}. \end{aligned} \quad (7.1)$$

We also define

$$R_{M,g,\delta}^n := - \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n((\Delta_{M,g,\delta}^n)^c). \quad (7.2)$$

One can of course replace  $g$  by a finite number of continuous and bounded  $g_i$ 's.

**Lemma 7.2.**  $\Delta_{M,g,\delta}^n$  is a compact set.

*Proof.* Notice that if  $\mu_k \in \Delta_{M,g,\delta}^n$  weakly converges to some  $\mu$ ,  $\mu \in \mathcal{D}_M^n$  according to lemma 6.1. The corresponding sequence  $\nu_k$  in  $\Delta^n$  actually belongs to  $\mathcal{D}_{M+\delta}^n$  so that one can find a subsequence still denoted  $\nu_k$  that converges to some  $\nu \in \mathcal{D}_{M+\delta}^n$  (this is the key difference with  $\Delta_{M,g,\delta}$ ). Since  $g$  is bounded and continuous, by taking limits we have  $|\nu(g) - \mu(g)| \leq \delta$ . Of course for  $\varepsilon > 0$ ,

$$\nu(1/(a+b+\varepsilon)) = \lim_k \nu_k(1/(a+b+\varepsilon)) \leq \lim_k \nu_k(1/(a+b)) \leq M + \delta,$$



and  $\nu(1/(a+b)) \leq M + \delta$  follows by letting  $\varepsilon$  go to 0 thanks to the monotone convergence theorem. Compactness follows since  $\Delta_{M,g,\delta}^n \subset \mathcal{D}_M^n$  which is compact.  $\square$

The next step consists in showing that the empirical measure  $\mu_t^n$  is close to  $\Delta^n$  more precisely

**Lemma 7.3.** *For  $f \in C_b((0, +\infty]^2 \times \{1, \dots, n\})$  we define*

$$\nu_t^n(f) := \frac{1}{t} \sum_{i=1}^{M_t} \tau_i \int_0^1 f(u\tau_i, (1-u)\tau_i, W_i^n) du + \frac{t - S_{M_t}}{t} f(+\infty, +\infty, W_{M_t+1}^n).$$

Then  $\nu_t^n \in \Delta^n$  almost surely.

For all  $g, \delta$  there exists some  $t(g, \delta)$  such that for  $t \geq t(g, \delta)$ , the events  $\{|\mu_t^n(g) - \nu_t^n(g)| > \delta\}$  and  $\left\{ \left| \mu_t^n\left(\frac{1}{a+b}\right) - \nu_t^n\left(\frac{1}{a+b}\right) \right| > \delta \right\}$  are almost surely empty.

*Proof.* This lemma is the analogue of Lemma 3.2 in [13] but due to the modification of our  $\Delta_{M,g,\delta}^n$  we have to complete the proof therein.

Arguing as in the proof of Lemma 3.2 in [13] one shows that for  $t$  large enough the set  $\{|\mu_t^n(g) - \nu_t^n(g)| > \delta\}$  is almost surely empty. For the other term,

$$\begin{aligned} & \left| \mu_t^n\left(\frac{1}{a+b}\right) - \nu_t^n\left(\frac{1}{a+b}\right) \right| \\ &= \left| \left( \frac{1}{t} \sum_{i=1}^{M_t} \tau_i \times \frac{1}{\tau_i} + \frac{\tau_{M_t+1}}{t} \times \frac{t - S_{M_t}}{\tau_{M_t+1}} \times \frac{1}{\tau_{M_t+1}} \right) - \frac{1}{t} \sum_{i=1}^{M_t} \tau_i \times \frac{1}{\tau_i} \right| \\ &= \frac{1}{t} \frac{t - S_{M_t}}{\tau_{M_t+1}} \leq \frac{1}{t}. \end{aligned}$$

Hence for  $t > 1/\delta$  we have  $\left| \mu_t^n\left(\frac{1}{a+b}\right) - \nu_t^n\left(\frac{1}{a+b}\right) \right| \leq \delta$ .  $\square$

**Corollary 7.4.**  $\lim_{M \rightarrow +\infty} R_{M,\delta,g}^n = +\infty$ .

*Proof.* According to lemma 7.3, for  $t$  large enough (that does not depend on  $M$  but only on  $\delta$  and  $g$ ) the set  $\{\mu_t^n(1/(a+b)) \leq M\} \cap (\Delta_{M,g,\delta}^n)^c$  is almost surely empty. Hence for  $t$  large enough

$$\mathbb{P}^n((\Delta_{M,g,\delta}^n)^c) \leq \mathbb{P}^n(\mu_t^n(1/(a+b)) > M)$$

and the result follows from lemma 7.1.  $\square$

We may now follow the proof in [13]. First exactly as in [13], thanks to Lemma 6.5, for all open set  $\mathcal{O}$ , all  $g, M, \delta$  and all  $f \in \Gamma$ ,

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n(\mathcal{O}) \leq - \inf_{\nu \in \mathcal{O}} I_{f,M,g,\delta}^n(\mu)$$

where

$$I_{f,M,g,\delta}^n(\mu) = \begin{cases} \mu(f) \wedge R_{M,g,\delta}^n & \text{if } \mu \in \Delta_{M,g,\delta}^n \\ +\infty & \text{otherwise.} \end{cases}, \quad (7.3)$$

Since  $f$  is lower semicontinuous,  $I_{f,M,g,\delta}^n(\mu)$  is also lower semicontinuous thanks to the compactness of  $\Delta_{M,g,\delta}^n$ . One can thus deduce as in [13] that for all compact subset  $\mathcal{K}$ ,

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_t^n(K) \leq - \inf_{\mu \in \mathcal{K}} \sup_{f,M,g,\delta} I_{f,M,g,\delta}^n(\mu) := - \inf_{\mu \in \mathcal{K}} \tilde{I}(\mu).$$

If  $\mu \notin \bar{\Delta}^n$  which is closed according to lemma 6.2, one can find an open neighborhood  $\mathcal{U}$  of  $\mu$  such that  $\mathcal{U} \cap \bar{\Delta}^n = \emptyset$ . We may choose  $\mathcal{U}$  of the form

$$\mathcal{U} = \bigcap_{i=1}^k \{\nu, |\nu(g_i) - \mu(g_i)| \leq \delta\}$$

for some family  $g_1, \dots, g_k$  of bounded and continuous functions, so that for at least one of the  $g_i$ 's denoted by  $g$ ,  $\mu \notin \Delta_{M,g,\delta}^n$ . Hence  $\bigcap_{g,\delta} \Delta_{M,g,\delta}^n \subset \bar{\Delta}^n$  and  $\tilde{I}(\mu) = +\infty$  if  $\mu \notin \bar{\Delta}^n$ . Together with corollary 7.4 we deduce that

$$\tilde{I}(\mu) \geq \sup_{f \in \Gamma} I_f(\mu) \quad \text{where} \quad I_f(\mu) = \mu(f) \text{ if } \mu \in \bar{\Delta}^n, \quad I_f(\mu) = +\infty \text{ otherwise.}$$

Thus, according to lemma 6.4,  $\tilde{I}(\mu) \geq I^n(\mu)$  and the upper bound is proved.

The proof of the lower bound is similar to [13], just replacing the sample  $\tau_i$  by a sample  $(\tau_i, W_i)$ , and thus is omitted.

## 8. PROOF OF THEOREM 4.4.

We turn to the proof of theorem 4.4. We first will deduce a LDP from theorem 4.3 by using the contraction principle, and then identify the rate function.

### 8.1. The contraction principle.

Define  $\varphi(a, b, c) = \frac{c}{a+b}$  which is continuous on  $(0, +\infty]^2 \times \{1, \dots, n\}$ . Then

$$\mu_t^n(\varphi) = \frac{1}{t} \sum_{i=1}^{M_t} W_i^n + \frac{(t - S_{M_t})W_{M_t+1}^n}{t \tau_{M_t+1}}.$$

Remark that

$$\left| \frac{(t - S_{M_t})W_{M_t+1}^n}{t \tau_{M_t+1}} \right| \leq \frac{n}{t}$$

so that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left( \left| \mu_t^n(\varphi) - \frac{1}{t} \sum_{i=1}^{M_t} W_i^n \right| > \delta \right) = -\infty$$

showing that  $\mu_t^n(\varphi)$  and  $\frac{1}{t} \sum_{i=1}^{M_t} W_i^n = Z_t^n/t$  will satisfy the same LDP. The contraction principle (Theorem 4.2.1 in [9]) should thus furnish some LDP for  $Z_t^n/t$ . Unfortunately  $\varphi$  is not bounded so that  $\mu \mapsto \mu(\varphi)$  is not continuous from  $\mathcal{M}_1(0, +\infty]^2 \times \{1, \dots, n\}$  to  $\mathbb{R}$  and the contraction principle does not apply directly. We are thus obliged one more time to use an approximation procedure replacing  $\varphi$  by

$$\varphi^\varepsilon(a, b, c) = \frac{c}{(a+b) \vee \varepsilon}$$

for  $\varepsilon > 0$ . Replacing  $\tau_i$  by  $\tau_i^\varepsilon = \tau_i \vee \varepsilon$  we may introduce  $M_t^\varepsilon$ ,  $A_t^\varepsilon$ ,  $B_t^\varepsilon$ ,  $C_t^{n,\varepsilon}$  and  $\mu_t^{n,\varepsilon}$  as in subsection 4.1. The results of the previous section apply to this new process (introducing a rate function  $I^{n,\varepsilon}(\mu)$ ) This time we may apply the contraction principle so that  $\mu_t^{\varepsilon,n}(\varphi^\varepsilon)$  satisfies a LDP with rate function

$$J^{\varepsilon,n}(m) = \inf\{I^{n,\varepsilon}(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi^\varepsilon)\}. \quad (8.1)$$

We now have

$$\mu_t^{\varepsilon,n}(\varphi^\varepsilon) = \frac{1}{t} \sum_{i=1}^{M_t^\varepsilon} W_i^n + \frac{(t - S_{M_t^\varepsilon}^\varepsilon) W_{M_t^\varepsilon+1}^n}{t \tau_{M_t^\varepsilon+1}^\varepsilon},$$

so that, since  $M_t^\varepsilon \leq M_t$ ,

$$\begin{aligned} \left| \mu_t^{\varepsilon,n}(\varphi^\varepsilon) - \frac{1}{t} \sum_{i=1}^{M_t} W_i^n \right| &\leq \frac{1}{t} \left| \sum_{i=M_t^\varepsilon+1}^{M_t} W_i^n \right| + \left| \frac{(t - S_{M_t^\varepsilon}^\varepsilon) W_{M_t^\varepsilon+1}^n}{t \tau_{M_t^\varepsilon+1}^\varepsilon} \right| \\ &\leq \frac{n}{t} ((M_t - M_t^\varepsilon) + 1). \end{aligned}$$

But according to [13] Lemma 5.4,  $M_t^\varepsilon$  is an exponentially good approximation of  $M_t$ . It follows that  $\mu_t^{\varepsilon,n}(\varphi^\varepsilon)$  is an exponentially good approximation of  $Z_t^n/t$  so that, finally, thanks to Theorem 2.2,  $Z_t^n/t$  satisfies a weak LDP with rate function

$$\tilde{J}^n(m) = \sup_{\delta > 0} \liminf_{\varepsilon \rightarrow 0} \inf_{|z-m| < \delta} J^{\varepsilon,n}(m). \quad (8.2)$$

In particular we know that  $\tilde{J}^n$  is l.s.c. so that its level sets are closed.

## 8.2. Study of the rate function.

The goal of this subsection is to show the following lemma (partly close to Lemma 5.1 in [13]) explaining the various forms of the rate function (recall that  $\varphi(a, b, c) = c/(a + b)$ ).

**Lemma 8.1.** *We define, for all  $m > 0$ ,  $\bar{J}^n(m) := \inf\{I^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\}$ . Then*

i)  $\bar{J}^n = J^n$  ( $J^n$  is defined in (2.6)). In addition

$$\bar{J}^n(m) = \inf\{I_0^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\}.$$

ii) We also have:

$$\bar{J}^n(m) = \min\{I^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\} \quad (8.3)$$

$$= \min\{I_0^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\}. \quad (8.4)$$

iii) Finally,

$$J^n = \bar{J}^n = \tilde{J}^n.$$

*Proof. Proof of i)* We have

$$\begin{aligned}
\bar{J}^n(m) &:= \inf \left\{ I^n(\mu), \mu \in \mathcal{M}_1((0, +\infty]^2 \times \{1, \dots, n\}), \int \frac{c}{a+b} \mu(da, db, dc) = m \right\} \\
&= \inf \left\{ I^n(\mu), \mu \in \Delta^n, \int \frac{c}{a+b} \mu(da, db, dc) = m \right\} \\
&= \inf \left\{ \alpha \pi \left( \frac{1}{\tau} \right) H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \mu_0 \in \Delta_0^n, \alpha \int \frac{c}{a+b} \mu_0(da, db, dc) = m \right\} \\
&= \inf \left\{ \alpha \pi \left( \frac{1}{\tau} \right) H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \mu_0 \in \Delta_0^n, \alpha \pi \left( \frac{W}{\tau} \right) = m \right\} \\
&= \inf \left\{ \alpha \frac{1}{\tilde{\pi}(\tau)} H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \tilde{\pi} \in \mathcal{M}_1((0, +\infty) \times \{1, \dots, n\}), \alpha \frac{\tilde{\pi}(W)}{\tilde{\pi}(\tau)} = m \right\} \\
&= \inf \left\{ \frac{\alpha}{\beta} H(\tilde{\pi}|\psi^n) + (1-\alpha)\xi, \alpha \in [0, 1], \beta > 0, \tilde{\pi}(\tau) = \beta, \alpha \frac{\tilde{\pi}(W)}{\beta} = m \right\}
\end{aligned}$$

Let  $p(a, b) = \inf \{ H(\nu|\psi^n), \nu(\tau) = a, \nu(W) = b \}$ . We have  $p = \Lambda_n^*$  according to Csiszar I-projection theorem (Theorem 3 in [8]). As in [13] (proof of Lemma 5.1) another way is to directly prove the dual equality  $p^* = \Lambda^n$ .

We thus have

$$\bar{J}^n(m) = \inf \left\{ \frac{\alpha}{\beta} \Lambda_n^* \left( \beta, \frac{m\beta}{\alpha} \right) + (1-\alpha)\xi, \alpha \in [0, 1], \beta > 0 \right\}.$$

But

$$\begin{aligned}
\frac{\alpha}{\beta} \Lambda_n^* \left( \beta, \frac{m\beta}{\alpha} \right) &= \sup_{x,y} \left\{ \alpha x + m y - \frac{\alpha}{\beta} \Lambda_n(x, y) \right\} \\
&= \beta' \Lambda_n^* \left( \frac{\alpha}{\beta'}, \frac{m}{\beta'} \right) \text{ where } \beta' = \frac{\alpha}{\beta}.
\end{aligned}$$

Thus

$$\bar{J}^n(m) = \inf \left\{ \beta \Lambda_n^* \left( \frac{\alpha}{\beta}, \frac{m}{\beta} \right) + (1-\alpha)\xi, \alpha \in [0, 1], \beta > 0 \right\}.$$

We will show that :

$$\inf_{\alpha \in [0, 1]} \left\{ \beta \Lambda_n^* \left( \frac{\alpha}{\beta}, \frac{m}{\beta} \right) + (1-\alpha)\xi \right\} = \beta \Lambda_n^* \left( \frac{1}{\beta}, \frac{m}{\beta} \right).$$

Taking  $\alpha = 1$ , we see that the left hand side is less than or equal to the right hand side. To show the converse inequality, pick  $\alpha \in [0, 1]$ :

$$\begin{aligned}
\beta \Lambda_n^* \left( \frac{\alpha}{\beta}, \frac{m}{\beta} \right) + (1-\alpha)\xi &= \sup_{x,y} \{ \alpha x + (1-\alpha)\xi + m y - \beta \Lambda_n(x, y) \} \\
&\geq \sup_{x,y} \{ x \wedge \xi + m y - \beta \Lambda_n(x, y) \}.
\end{aligned}$$

Since  $W$  is bounded,  $e^{yW} \geq C(y) > 0$  for all  $y$ , so that we have for all  $x > \xi$  and all  $y$ ,

$$\psi^n(e^{x\tau + yW}) \geq C(y) \psi^n(e^{x\tau}) = +\infty.$$

This shows that  $\Lambda_n(x, y) = +\infty$ , for all  $x > \xi$  and for all  $y$ . Hence, the supremum on  $x$  can be restricted to the supremum on  $\{x \leq \xi\}$ :

$$\begin{aligned} \beta\Lambda_n^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1 - \alpha)\xi &\geq \sup_{x,y} \{x \wedge \xi + my - \beta\Lambda_n(x, y)\} \\ &= \sup_{x \leq \xi, y} \{x + my - \beta\Lambda_n(x, y)\} \\ &= \beta\Lambda_n^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right) \end{aligned}$$

and the desired inequality is proved.

Notice that during the proof we have seen than the minimization is obtained looking only at  $\mu \in \Delta_0^n$  so that we may replace  $I^n$  by  $I_0^n$  in the definition of  $\bar{J}^n$ .

*Proof of ii).* This part is completely similar to the corresponding one in the proof of Lemma 5.1 in [13]. The only thing to see is that we may replace  $\|F\|_\infty$  in [13] by  $K$  if  $|W| \leq K$ .

*Proof of iii).* To prove this equality, we first prove the inequality  $\bar{J}^n(m) \leq \tilde{J}^n(m)$ .

Let  $m \in \mathbb{R}^+$  and  $\varepsilon > 0$ . First, remark that  $\bar{J}^n(m) \leq J^{n,\varepsilon}(m)$ . Indeed, if  $\mu \in \Delta^n$ , with associated  $\pi$ ,  $\alpha$  and  $\eta$ , is such that  $H(\tilde{\pi}|\psi^{n,\varepsilon}) < +\infty$  and  $\mu(\varphi^\varepsilon) = m$ , then  $\tilde{\pi}$  has its support included in  $(\varepsilon, +\infty) \times \{1, \dots, n\}$ . Hence  $I^n(\mu) = I^{n,\varepsilon}(\mu)$ . Moreover,  $\mu(\varphi) = \mu\left(\frac{1}{a+b}\right) = m$ . This yields

$$\begin{aligned} \bar{J}^n(m) &= \inf\{I^n(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi)\} \\ &\leq \inf\{I^n(\mu), \mu \in \mathcal{M}^1(\{(a, b, c) \in (0, +\infty)^2 \times \{1, \dots, n\}\}, \mu(\{a + b \leq \varepsilon\}) = 0, m = \mu(\varphi)\} \\ &= \inf\{I^{n,\varepsilon}(\mu), \mu \in \mathcal{M}^1((0, +\infty]^2 \times \{1, \dots, n\}), m = \mu(\varphi^\varepsilon)\} = J^{n,\varepsilon}. \end{aligned}$$

We will take the limit as  $\varepsilon \rightarrow 0$ . To this end we may write for  $\delta > 0$ ,

$$\inf_{z, |z-m| \leq \delta} \bar{J}^n(z) \leq \inf_{z, |z-m| \leq \delta} J^{n,\varepsilon}(z)$$

so that

$$\inf_{z, |z-m| \leq \delta} \bar{J}^n(z) \leq \liminf_{\varepsilon \rightarrow 0} \inf_{z, |z-m| \leq \delta} J^{n,\varepsilon}(z) \leq \tilde{J}^n(m)$$

Since  $\bar{J}^n = J^n$ , it is clearly lower semicontinuous so that

$$\bar{J}^n(m) \leq \liminf_{\delta \rightarrow 0} \inf_{z, |z-m| \leq \delta} \bar{J}^n(z) \leq \tilde{J}^n(m).$$

Finally  $\bar{J}^n \leq \tilde{J}^n$

Now, we prove that  $\bar{J}^n \geq \tilde{J}^n$ . Since  $\bar{J}^n(m)$  is a minimum, let  $\bar{\mu}$  a measure such that  $\bar{J}^n(m) = I^n(\bar{\mu})$  and  $\bar{\mu}(\varphi) = m$ . For a sequence  $\varepsilon_k$  going to 0, we consider  $\psi_k$  the distribution of  $(\tau \vee \varepsilon_k, W)$ . Using proposition 5.6, there exists  $\mu_k \in \Delta_0^n$  with associated to  $\psi_k$  such that  $\mu_k \rightarrow \bar{\mu}$  and  $\limsup_{k \rightarrow \infty} I_k^n(\mu_k) \leq I^n(\bar{\mu})$ . Here  $I_k^n = I^n$ , so  $\limsup_{k \rightarrow \infty} I^n(\mu_k) \leq I^n(\bar{\mu})$ .

By construction  $\mu_k(\{a + b < \varepsilon_k\}) = 0$  and for  $\tilde{\pi}_k$  associated to  $\mu_k$ , we have  $H(\tilde{\pi}_k|\psi^n) < +\infty$ . Since  $\tilde{\pi}_k$  is a measure on  $(\varepsilon_k, +\infty) \times \{1, \dots, n\}$ ,  $H(\tilde{\pi}_k|\psi_k) = H(\tilde{\pi}_k|\psi^n)$ , and it follows  $I^n(\mu_k) = I^{n,\varepsilon_k}(\mu_k)$ .

As a consequence of proposition 5.6 it holds  $\mu_k\left(\frac{1}{a+b}\right) \rightarrow \bar{\mu}\left(\frac{1}{a+b}\right)$ . Hence,  $\delta_k = \left|\mu_k\left(\frac{1}{a+b}\right) - \bar{\mu}\left(\frac{1}{a+b}\right)\right| \rightarrow 0$ .

We can now write

$$\inf_{z, |z-m| \leq \delta_k} J^{n, \varepsilon_k}(z) \leq I^{n, \varepsilon_k}(\mu_k) = I^n(\mu_k)$$

so that

$$\limsup_{k \rightarrow \infty} \inf_{z, |z-m| \leq \delta_k} J^{n, \varepsilon_k}(z) \leq \limsup_{k \rightarrow \infty} I^n(\mu_k) \leq I^n(\bar{\mu}) = \bar{J}^n(m)$$

and finally

$$\tilde{J}^n(m) = \sup_{\delta > 0} \liminf_{\varepsilon \rightarrow 0} \inf_{z, |z-m| < \delta} J^{n, \varepsilon}(z) \leq \bar{J}^n(m).$$

□

*Remark 8.2.* The first two items i) and ii) in the previous lemma are still hold true if we replace  $\{1, \dots, n\}$  by a compact set  $\mathbb{W}$ . ◇

### 8.3. End of the proof.

At this point we have obtained that  $Z_t^n/t$  satisfies a weak LDP with rate function  $J^n$ . In order to get the full LDP we have to show that the level sets are bounded (since we know that they are closed). Recall that it is not a direct consequence of the contraction principle since  $\mu \mapsto \mu(\varphi)$  is not continuous.

First since  $J^n \leq J^{n, \varepsilon}$  for all  $\varepsilon > 0$ , for all closed set  $F$  it holds  $\limsup_{\varepsilon \rightarrow 0} \inf_{y \in F} J^{n, \varepsilon}(y) \geq \inf_{y \in F} J^n(y)$ .

It remains to show that  $J^n$  is a good rate function i.e. that it has compact level sets. According to the previous subsection for a sequence  $m_k$  such that  $J^n(m_k) \leq \beta$ , we may find a sequence  $\mu_k \in \Delta_0^n$  such that  $\mu_k(\varphi) = m_k$  and  $I_0^n(\mu_k) \leq \beta$ . The corresponding  $\pi_k$  satisfies  $\limsup \pi_k(1/(a+b)) < +\infty$  according to proposition 5.3 (here  $\alpha_k = 1$ ). Since  $|\varphi(a, b, c)| \leq C/(a+b)$ ,  $m_k$  is bounded and one can find a convergent subsequence.

## 9. PROOF OF THEOREM 2.5.

When  $W$  is bounded and discrete, the full LDP is already given by Theorem 4.4. If  $W$  isn't bounded, or isn't discrete, we need to differentiate whether  $\beta_0 = \infty$  or not.

In the first case, when  $\beta_0 = \infty$ , we are able to prove a Large Deviation principle. Since the rate function  $\tilde{J}$  is difficult to calculate, we compare  $\tilde{J}$  with an other function  $J$  to simplify some inequalities. This work is done in the following subsection 9.1.

In the second case, since the approximation  $Z_t^n/t$  isn't an exponentially good approximation of  $Z_t/t$ , we only prove some useful inequalities of deviations, but we're not able to prove the Large Deviation Principle.

**9.1. Case A:**  $\beta_0 = \infty$ . In this case,  $Z_t^n/t$  is an exponentially good approximation of  $Z_t/t$  (see Lemma 3.1, 3.2 or 3.4 depending on the approximation strategy). Combining the LDP principle obtained for  $Z_t^n/t$  (Theorem 4.4) with Theorem 2.2 (1) we obtain that  $Z_t/t$  satisfies a weak LDP with rate function

$$\tilde{J}(m) = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{|m-z| < \delta} J^n(z).$$

In order to obtain a full LDP, we use lemma 2.3. Therefore it remains to show that  $Z_t/t$  is exponentially tight.

**Lemma 9.1.** *Assume that Assumption 2.4 is fulfilled, then  $(Z_t/t)_{t \geq 0}$  is exponentially tight, i.e. for all  $\alpha > 0$ , there exists a compact set  $K_\alpha$  such that*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t}{t} \notin K_\alpha^c \right) < -\alpha.$$

*Proof.* Since  $Z_t^n/t$  is an approximation of  $Z_t/t$  and satisfies a full LDP, we can decompose the probability as following: for each  $n$ , and for all  $\delta$ :

$$\begin{aligned} \mathbb{P} \left( \frac{Z_t}{t} \notin [-A, A] \right) &\leq \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) + \mathbb{P} \left( \frac{Z_t^n}{t} \notin [-A + \delta, A - \delta] \right) \\ &\leq \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) + \mathbb{P} \left( \frac{Z_t^n}{t} < -A + \delta \right) + \mathbb{P} \left( \frac{Z_t^n}{t} > A - \delta \right). \\ &\leq 3 \max \left( \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right), \mathbb{P} \left( \frac{Z_t^n}{t} < -A + \delta \right), \mathbb{P} \left( \frac{Z_t^n}{t} > A - \delta \right) \right). \end{aligned} \quad (9.1)$$

By Lemma 3.1,  $Z_t^n/t$  and  $Z_t/t$  satisfies

$$\forall \delta > 0, \lim_n \overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) = -\frac{\beta_0 \delta}{2},$$

i.e.

$$\forall \alpha > 0, \forall \delta > \frac{2\alpha}{\beta_0}, \exists n(\alpha, \delta), \forall n > n(\alpha, \delta), \overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| > \delta \right) \leq -\alpha. \quad (9.2)$$

(If  $\beta_0 = +\infty$ , we can consider all  $\delta > 0$ .)

We just have to study  $\mathbb{P} \left( \frac{Z_t^n}{t} > A - \delta \right)$  and the symmetric case. We know from Theorem 4.4 that:

$$\overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t^n}{t} > B \right) \leq - \inf_{m > B} J^n(m).$$

It remains to show that  $\forall \alpha > 0$  one can choose a level  $B_\alpha$  such that  $\forall m > B_\alpha, J^n(m) > \alpha$ .

Remind that

$$\begin{aligned} J^n(m) &= \inf_{\beta > 0} \left\{ \beta \lambda_n^* \left( \frac{1}{\beta}, \frac{m}{\beta} \right) \right\} \\ &= \inf_{\beta > 0} \sup_{x, y} \left\{ x + my - \beta \log \mathbb{E}[e^{x\tau + yW_n}] \right\}, \end{aligned}$$

(where  $W_n = W \wedge n \vee (-n)$ ).

Since  $-|W| \leq W_n \leq |W|$ , we obtain by differentiating whether  $y$  is positive or negative that

$$x + my - \beta \log \mathbb{E}[e^{x\tau + yW_n}] \geq x + my - \beta \log \mathbb{E}[e^{x\tau + |y||W|}].$$

Therefore, we deduce the lower bound

$$J^n(m) \geq J^{|\cdot|}(m) := \inf_{\beta > 0} \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + |m|y - \beta \log \mathbb{E}[e^{x\tau + y|W|}] \right\}.$$

Remark that  $J^{|\cdot|}$  is an even function, thus by symmetry, we can assume  $m \geq 0$ . Now, using Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} \sup_{x \in \mathbb{R}, y \geq 0} \{x + my - \beta \log \mathbb{E}[e^{x\tau + y|W|}]\} &= \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \log \mathbb{E}[e^{x\tau + y|W|}]^2 \right\} \\ &\geq \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \log \mathbb{E}[e^{2x\tau}] - \frac{\beta}{2} \log \mathbb{E}[e^{2y|W|}] \right\} \\ &\geq \sup_{x \in \mathbb{R}} \left\{ x - \frac{\beta}{2} \log \mathbb{E}[e^{2x\tau}] \right\} + \sup_{y \geq 0} \left\{ my - \frac{\beta}{2} \log \mathbb{E}[e^{2y|W|}] \right\} \end{aligned}$$

There exists  $x_0 < 0$  such that  $\mathbb{E}[e^{2x_0\tau}] \leq e^{-1}$ , and since  $\beta_0 > 0$ ,  $y \mapsto \mathbb{E}[e^{y|W|}]$  is strictly increasing and continuous on  $[0, \beta_0)$ . Then there exists  $y_0 > 0$  such that  $1 < \mathbb{E}[e^{2y_0|W|}] \leq e$ . Therefore

$$\begin{aligned} \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + my - \frac{\beta}{2} \log \mathbb{E}[e^{x\tau + y|W|}]^2 \right\} &\geq x_0 - \frac{\beta}{2} \log \mathbb{E}[e^{2x_0\tau}] + my_0 - \frac{\beta}{2} \log \mathbb{E}[e^{2y_0|W|}] \\ &\geq x_0 + my_0 + \frac{\beta}{2} (1 - \log \mathbb{E}[e^{2y_0|W|}]) \end{aligned}$$

Finally, since  $1 - \log \mathbb{E}[e^{2y_0|W|}] > 0$ , we conclude that

$$J^{|\cdot|}(m) \geq x_0 + my_0 \xrightarrow{m \rightarrow \infty} +\infty.$$

In conclusion, let  $\alpha > 0$  and  $\delta > \frac{2\alpha}{\beta_0}$  (or  $\delta > 0$  if  $\beta_0 = +\infty$ ), let  $n(\alpha, \delta)$  defined in (9.2) and  $n \geq n(\alpha, \delta)$ , let  $A = \frac{\alpha - x_0}{y_0} + \delta$  ( $x_0$  and  $y_0$  are determined by the law of  $\tau$  and  $W$ ). We have

$$\begin{aligned} \overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t^n}{t} > A - \delta \right) &\leq - \inf_{m > A - \delta} J^n(m) \\ &\leq - \inf_{m > -A + \delta} J^{|\cdot|}(m) \\ &\leq - \inf_{m > A - \delta} x_0 + my_0 \\ &\leq -x_0 - y_0(A - \delta) \\ &\leq -\alpha \end{aligned}$$

and using the evenness of  $J^{|\cdot|}$ ,

$$\overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t^n}{t} < -A + \delta \right) \leq -\alpha.$$

Eventually combining these bounds with (9.2) in (9.1), and deduce that for  $n \geq n(\alpha, \delta)$ ,

$$\overline{\lim}_t \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t}{t} \notin [-A, A] \right) \leq -\alpha.$$

□

Now, we have a full LDP for  $Z_t/t$ , but the expression of  $\tilde{J}$  isn't very convenient. In particular, this expression can depend on the reduction used. In order to simplify some inequality and obtain Equations (2.8) and (2.9), we prove  $\tilde{J} \geq J$  defined in (2.6).

We consider two cases: in the first one, Lemma 9.2, we assume  $W$  is bounded by  $K$  and consider the discretization  $W^n$  defined in subsection 3.2. Then in the second one, Lemma



9.5, we will examine the general case.

Contrary to [13] we do not know a priori that  $\tilde{J}$  is a good rate function. However, we use results of Lemma 8.1 replacing  $\{1, \dots, n\}$  by  $[-K, K]$ . In particular we deduce that

$$J(m) = \bar{J}(m) := \min\{I(\mu); \mu \in \mathcal{M}^1((0, +\infty]^2 \times [-K, K]), \mu(\varphi) = m\}.$$

We want to prove

**Lemma 9.2.** *If  $W$  is bounded, it holds  $J \leq \tilde{J}$ .*

*Proof.* We may of course assume that  $\tilde{J}(m) < +\infty$ . One can thus find sequences  $m_n$  and  $\varepsilon_n \leq 1$  such that  $\varepsilon_n \rightarrow 0$ ,  $m_n \rightarrow m$  and  $J^n(m_n) \leq \tilde{J}(m) + \varepsilon_n$ . Since  $J^n = \bar{J}^n$ , one can thus find a sequence  $\mu_n \in \mathcal{M}^1((0, +\infty]^2 \times [-K, K])$  such that  $I_n(\mu_n) < \tilde{J}(m) + 1$ . According to proposition 5.3 (recall that it is true here, see remark 5.5) one can thus find some subsequence still denoted by  $\mu_n$  converging to some  $\mu$  in  $\mathcal{M}^1((0, +\infty]^2 \times [-K, K])$ . Now (again recall remark 5.5) according to proposition 5.4 we have  $\liminf_n I_n(\mu_n) \geq I(\mu)$  so that  $I(\mu) \leq \tilde{J}(m)$ .

In order to see that  $I(\mu) \geq J(m)$  it remains to show that  $\mu(c/(a+b)) = m$  and to apply  $J = \bar{J}$ .

Notice that thanks to ii) in lemma 8.1 we may have chosen  $\mu_n \in \Delta_0^n$  so that proposition 5.3 tells us in addition that  $\limsup_n \mu_n(1/(a+b)) = M < +\infty$  (since the sequence  $\alpha_n = 1$ ). It follows

$$\mu(1/(a+b+\varepsilon)) = \lim_n \mu_n(1/a+b+\varepsilon) \leq \limsup_n \mu_n(1/(a+b)) = M$$

so that again using monotone convergence we deduce  $\mu(1/(a+b)) \leq M$ . Hence, since  $c$  is  $\mu$  a.s. bounded,  $\mu(c/(a+b))$  is well defined.

To calculate  $\mu(c/(a+b))$  we need more. Using the definition of  $\bar{J}^n$  and what precedes the sequence  $\mu_n$  satisfies

$$H(\tilde{\pi}^n | \psi^n) = \frac{m_n}{\mu_n(1/(a+b))} \leq C$$

for some  $C < +\infty$  since  $m_n$  is bounded and  $\lim_n \mu_n(1/(a+b)) = \mu(1/(a+b)) > 0$ . We will deduce that  $\tilde{\pi}^n$  is tight. Indeed a standard application of the Orlicz-Hölder inequality (with the conjugate pair  $u \mapsto e^u - 1 - u$  and  $u \mapsto u \mapsto u \ln u - u$ ) shows that

$$\tilde{\pi}^n(A) \leq \kappa C \frac{1}{\ln(1/\psi^n(A))}$$

for some universal constant  $\kappa$ . Choosing  $A = \{a+b < \varepsilon\}$  so that  $\psi^n(A) = \psi(A)$  we get the desired result choosing  $\varepsilon$  small enough.

It follows

$$\begin{aligned} \left| \mu_n \left( \frac{c}{a+b} \left( \varepsilon^{-1}(a+b) \mathbf{1}_{a+b \leq \varepsilon} + \mathbf{1}_{a+b > \varepsilon} \right) \right) - m_n \right| &\leq K \mu_n(1/(a+b)) C \kappa \mu_n(\{a+b \leq \varepsilon\}) \\ &\leq C' \mu(\{a+b \leq \varepsilon\}). \end{aligned}$$

Since the integrated function is bounded and continuous we can pass to the limit in  $n$  first, and then in  $\varepsilon$  using Lebesgue's bounded convergence theorem since

$$\left| \frac{c}{a+b} \left( \varepsilon^{-1}(a+b) \mathbf{1}_{a+b \leq \varepsilon} + \mathbf{1}_{a+b > \varepsilon} \right) \right| \leq \frac{K}{a+b}$$

which is  $\mu$  integrable according to what we did before. This finally shows that  $\mu(c/(a+b)) = m$  and concludes the proof.  $\square$

We do not know about the converse inequality.

*Remark 9.3.* We will see below another way to prove this result (in an even more general context). Nevertheless we have given this proof in order to complete the picture in the [13] context.  $\diamond$

We will now directly study the function  $J$ . Recall that

$$J(m) = \inf_{\beta > 0} \sup_{x, y} \left( x + my - \beta \ln \mathbb{E}(e^{x\tau + yW}) \right) := \inf_{\beta > 0} \sup_{x, y} \Lambda(m, \beta, x, y).$$

We do not assume here that  $W$  is bounded.

**Lemma 9.4.**  *$J$  is a good rate function.*

*Proof.* First we remark that

$$\sup_{x, y} \Lambda(m, \beta, x, y) \geq \sup_x \Lambda(m, \beta, x, 0) = \sup_x (x - \beta \ln \mathbb{E}(e^{x\tau})).$$

Since  $\tau \geq 0$  (and supposed not to be identically 0) one can find  $x_\tau < 0$  such that  $\mathbb{E}(e^{x_\tau \tau}) = e^{-1}$  (we already use this in the proof of lemma 3.1), so that

$$\sup_{x, y} \Lambda(m, \beta, x, y) \geq x_\tau + \beta.$$

Let  $\{J \leq M\}$  be some level set of  $J$ . For  $\beta > M - x_\tau := \beta_\tau$  one has  $\sup_{x, y} \Lambda(m, \beta, x, y) > M$  so that for all  $m \in \{J \leq M\}$  it holds

$$J(m) = \inf_{0 < \beta \leq \beta_\tau} \sup_{x, y} \Lambda(m, \beta, x, y).$$

Now remark that

$$\sup_{x, y} \Lambda(m, \beta, x, y) \geq \sup_y \Lambda(m, \beta, 0, y) \geq \Lambda(m, \beta, 0, \kappa)$$

where  $0 \leq \kappa < 1 \wedge \beta_0$ ,  $\beta_0$  being defined in Assumption 2.4 *ii*). For  $\beta \leq \beta_\tau$ ,  $\beta \ln \mathbb{E}(e^{\kappa W})$  is thus bounded by  $C$ , so that

$$J(m) \geq m\kappa - C$$

showing that  $\{J \leq M\}$  is bounded.

It remains to show that the level sets are closed. Let  $m_n \rightarrow m$  be a sequence in  $\{J \leq M\}$ . According to what precedes we know that the infimum in  $\beta$  has to be taken in a bounded interval, so that one can find a (sub)-sequence  $\beta_n$  converging to  $\beta$  such that

$$J(m_n) \leq \sup_{x, y} \Lambda(m_n, \beta_n, x, y) + \varepsilon_n$$

with  $\varepsilon_n$  going to 0. This implies that for all  $(x, y)$

$$x + m_n y - \beta_n \ln \mathbb{E}(e^{x\tau + yW}) \leq M + \varepsilon_n$$

which implies  $J(m) \leq M$  by taking the limit in  $n$  and then the supremum w.r.t.  $(x, y)$ .  $\square$

For a general  $W$ , consider

$$\tilde{J}(m) = \sup_{\delta > 0} \liminf_{n \rightarrow \infty} \inf_{|m-z| < \delta} J^n(z).$$

We now know that  $Z_t^n/t$  satisfies a full LDP with good rate function  $J^n$ . We may state

**Lemma 9.5.** *It holds  $J \leq \tilde{J}$ .*

*Proof.* As for the proof of lemma 9.2, we may of assume that  $\tilde{J}(m) < +\infty$  and thus find sequences  $m_n$  and  $\varepsilon_n \leq 1$  such that  $\varepsilon_n \rightarrow 0$ ,  $m_n \rightarrow m$  and  $J^n(m_n) \leq \tilde{J}(m) + \varepsilon_n$ . We denote  $\delta_n = |m - m_n|$ . Define  $M = \tilde{J}(m) + 1$ . For  $n$  large enough,  $m_n$  belongs to  $\{J^n \leq M\}$ . We have seen in the proof of lemma 9.4 that  $J_n(m_n)$  is thus given by the infimum for  $0 < \beta \leq M - |x_\tau| = \beta_\tau$  where  $\mathbb{E}(e^{x_\tau \tau}) = e^{-1}$ , i.e  $x_\tau$  does not depend on  $n$ . The same holds with  $J(m)$ . If  $\beta \in (0, \beta_\tau]$  and  $\varepsilon > 0$  we may found  $(x_\varepsilon, y_\varepsilon)$  such that

$$\begin{aligned} \beta \Lambda^*(1/\beta, m/\beta) &\leq x_\varepsilon + m y_\varepsilon - \beta \ln \left( \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon \\ &\leq x_\varepsilon + m_n y_\varepsilon - \beta \ln \left( \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon + \delta_n |y_\varepsilon|. \end{aligned}$$

Since

$$\mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \rightarrow \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W} \right]$$

as  $n$  grows to infinity, for  $n$  large enough the difference is less than  $\varepsilon$  so that

$$\begin{aligned} \beta \Lambda^*(1/\beta, m/\beta) &\leq x_\varepsilon + m_n y_\varepsilon - \beta \ln \left( \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \right) + (1 + \beta_\tau) \varepsilon + \delta_n |y_\varepsilon| \\ &\leq \beta \Lambda_n^*(1/\beta, m_n/\beta) + (1 + \beta_\tau) \varepsilon + \delta_n |y_\varepsilon|. \end{aligned}$$

Taking the infimum in  $\beta$  and then the  $\liminf_n$  we get

$$J(m) \leq \liminf_n J_n(m_n) + (1 + \beta_\tau) \varepsilon \leq \tilde{J}(m) + (1 + \beta_\tau) \varepsilon.$$

It remains to let  $\varepsilon$  go to 0 to conclude.  $\square$

*Remark 9.6.* When  $W = 1$  a.s. it is easily seen that  $\sup_{x,y} \Lambda(m, \beta, x, y) = +\infty$  except for  $m = \beta$  yielding  $J(m) = \sup_x (x - m \ln \mathbb{E}(e^{x\tau}))$  as expected.

Since  $J$  is defined on  $\mathbb{R}$  one can expect some monotonicity on intervals delimited by the asymptotic mean  $\mathbb{E}(W)/\mathbb{E}(\tau)$ . We were not able to prove this monotonicity.  $\diamond$

**9.2. Case B:**  $\beta_0 < \infty$ . We remind that  $W^n$  is a reduction of  $W$  such that  $Z_t^n/t$  is an approximation of  $Z_t/t$  and satisfies a full LDP, as proved in Theorem 4.4. However, this approximation is not an exponentially good approximation, therefore the LDP cannot be transferred to  $Z_t/t$ . In this case, we prove the deviation Inequalities (2.10) and (2.11). We will focus on the first one,  $\mathbb{P} \left( \frac{Z_t}{t} \geq m + a \right)$ , since the proof is exactly the same for the second one,  $\mathbb{P} \left( \frac{Z_t}{t} \leq m - a \right)$ .

For each  $n \in \mathbb{N}^*$ , for  $\kappa \in (0, 1)$ , we have

$$\begin{aligned} \mathbb{P} \left( \frac{Z_t}{t} \geq m + a \right) &\leq \mathbb{P} \left( \frac{Z_t^n}{t} \geq m + \kappa a \right) + \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| \geq (1 - \kappa) a \right) \\ &\leq 2 \max \left[ \mathbb{P} \left( \frac{Z_t^n}{t} \geq m + \kappa a \right), \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| \geq (1 - \kappa) a \right) \right] \end{aligned}$$

Then, for each  $n \in \mathbb{N}^*$  and  $\kappa \in (0, 1)$ , we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t}{t} \geq m + a \right) \\ & \leq \max \left[ \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t^n}{t} \geq m + \kappa a \right), \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left( \left| \frac{Z_t}{t} - \frac{Z_t^n}{t} \right| \geq (1 - \kappa)a \right) \right]. \end{aligned}$$

We handle the first term with the full LDP for  $Z_t^n/t$  with the rate function  $J^n$  (Theorem 4.4) and the second term with Lemma 3.1 (or 3.2, or 3.4 depending on the approximation strategy). We then have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t}{t} \geq m + a \right) \\ & \leq \max \left[ \liminf_{n \rightarrow \infty} \left( - \inf_{z \geq m + \kappa a} J^n(z) \right), -\beta_0 \frac{(1 - \kappa)a}{2} \right] \\ & \leq - \min \left[ \limsup_{n \rightarrow \infty} \inf_{z \geq m + \kappa a} J^n(z), \beta_0 \frac{(1 - \kappa)a}{2} \right]. \end{aligned}$$

Since this inequality is satisfied for each  $n$ , we can then apply the Lemma 9.8, proved below, which gives us

$$\limsup_{n \rightarrow \infty} \inf_{z \geq m + \kappa a} J^n(z) \geq \inf_{z \geq m + \kappa a} J(z),$$

and we obtain the Inequation (2.10): for all  $\kappa \in (0, 1)$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{Z_t}{t} \geq m + a \right) \leq - \min \left[ \inf_{z \geq m + \kappa a} J(z), \beta_0 \frac{(1 - \kappa)a}{2} \right].$$

*Remark 9.7.* Notice  $J^n$  isn't the same in each case.  $J^n$  can be written

$$J^n(z) = \inf_{\beta > 0} \sup_{x, y} \{x + my - \beta \ln \mathbb{E} (e^{x\tau + yW^n})\},$$

but the definition of  $W^n$  depends on the reduction:

- If  $W$  is only bounded by  $K$ , a reduction with finite valued  $W^n$  was necessary:

$$W^n = \sum_{j=-n}^{n-1} \frac{jK}{n} \mathbf{1}_{W \in [jK/n, (j+1)K/n]}.$$

- If  $W$  is only discrete, a reduction with  $W^n$  bounded by  $-n$  and  $n$  have been done:  $W^n = (W \vee n) \wedge (-n)$ .
- If  $W$  isn't bounded nor discrete, another reduction is necessary:

$$\tilde{W}^n = -n \mathbf{1}_{W < -n} + n \mathbf{1}_{W \geq n} + \sum_{j=-n^2}^{n^2-1} \frac{j}{n} \mathbf{1}_{W \in [\frac{j}{n}, \frac{j+1}{n}]}.$$

Since in each case,  $W^n$  is discrete and bounded a.s. and  $E (e^{x\tau + yW^n}) \rightarrow_{n \rightarrow \infty} E (e^{x\tau + yW})$  for  $(x, y)$  such that  $E (e^{x\tau + yW}) < \infty$ , the proof is the same.

**Lemma 9.8.** *Assume  $(W^n)_n$  is a sequence of random variables which converges almost surely to  $W$  and such that for each  $n$ ,  $W^n$  is discrete and bounded a.s. Then, we have, for all  $z_0 \in \mathbb{R}$ ,*

$$\limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) \geq \inf_{z \geq z_0} J(z),$$

where  $J^n$  is defined by

$$J^n(z) = \inf_{\beta > 0} \sup_{x, y} \{x + my - \beta \ln \mathbb{E} (e^{x\tau + yW^n})\}.$$

*Proof.* Let  $(W^n)_n$  be a sequence of random variables which converges almost surely to  $W$ , such that for each  $n$ ,  $W^n$  is discrete and bounded a.s. Let  $z_0 \in \mathbb{R}$ . Thanks to Lemma 8.1,  $J^n$  has different forms. Here we will use the form

$$J^n(z) = \inf_{\beta > 0} \beta \Lambda_n^* \left( \frac{1}{\beta}, \frac{m}{\beta} \right).$$

If  $\limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) = \infty$ , then the inequality is satisfied.

Let assume  $\limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) < \infty$ . We denote  $C(z_0)$  this quantity.

There exists a sequence  $n_k \rightarrow \infty$  such that  $\lim_{k \rightarrow \infty} \inf_{z \geq z_0} J^{n_k}(z) = C(z_0)$ . For each  $k \in \mathbb{N}$ , there exists a sequence  $(z_i^k)_i$  in  $[z_0, +\infty)$  such that  $\inf_{z \geq z_0} J^{n_k}(z) = \lim_i J^{n_k}(z_i^k)$ .

Let  $\delta > 0$ . There exists  $K$  such that  $\forall k \geq K$ ,  $\inf_{z \geq z_0} J^{n_k}(z) \in [C(z_0) - \delta/2, C(z_0) + \delta/2]$  and  $I_k$  such that  $\forall i \geq I_k$ ,  $J^{n_k}(z_i^k) \in [C(z_0) - \delta, C(z_0) + \delta]$ .

Then, for  $k \geq K$ , and  $i \geq I_k$ ,  $z_i^k$  is in  $\{J^{n_k} \leq C(z_0) + \delta\}$ .

Moreover, for each  $n \in N$ ,  $J^n \geq J^{|\cdot|}$  by the proof of the Lemma 9.1, where

$$J^{|\cdot|}(m) := \inf_{\beta > 0} \sup_{x \in \mathbb{R}, y \geq 0} \left\{ x + |m|y - \beta \log \mathbb{E}[e^{x\tau + y|W|}] \right\}.$$

So for  $k \geq K$ , and  $i \geq I_k$ ,  $z_i^k$  is in  $\{J^{|\cdot|} \leq C(z_0) + \delta\}$ .

By the exact same argument than for  $J$  in the proof of Lemma 9.4,  $\{J^{|\cdot|} \leq C(z_0) + \delta\}$  is a compact level set of  $J^{|\cdot|}$ .

Then, for  $k \geq K$  and  $i \geq I_k$ ,  $z_i^k$  is in a compact. There exists at least an adherent point  $z_{\text{lim}}$  in this compact, and a subsequence  $z_{i_j}^{k_j} \rightarrow_{k_j, i_j \rightarrow \infty} z_{\text{lim}}$ . In particular,  $z_{\text{lim}} \geq z_0$ .

Like in the proof of Lemma 9.5,  $J^{n_{k_j}}(z_{i_j}^{k_j})$  is given by the infimum for  $\beta \in (0, \beta_\tau)$ . If  $\beta \in (0, \beta_\tau)$  and  $\varepsilon > 0$ , we may found  $(x_\varepsilon, y_\varepsilon)$  such that

$$\begin{aligned} \beta \Lambda^*(1/\beta, z_{\text{lim}}/\beta) &\leq x_\varepsilon + my_\varepsilon - \beta \ln \left( \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon \\ &\leq x_\varepsilon + z_{i_j}^{k_j} y_\varepsilon - \beta \ln \left( \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W} \right] \right) + \varepsilon + |z_{\text{lim}} - z_{i_j}^{k_j}| |y_\varepsilon| \end{aligned}$$

Since

$$\mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \rightarrow \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W} \right]$$

as  $n$  grows to infinity, for  $n$  large enough the difference is less than  $\varepsilon$  so that

$$\begin{aligned} \beta \Lambda^*(1/\beta, z_{\text{lim}}/\beta) &\leq x_\varepsilon + z_{i_j}^{k_j} y_\varepsilon - \beta \ln \left( \mathbb{E} \left[ e^{x_\varepsilon \tau + y_\varepsilon W^n} \right] \right) + (1 + \beta_\tau)\varepsilon + |z_{\text{lim}} - z_{i_j}^{k_j}| |y_\varepsilon| \\ &\leq \beta \Lambda_n^*(1/\beta, z_{i_j}^{k_j}/\beta) + (1 + \beta_\tau)\varepsilon + |z_{\text{lim}} - z_{i_j}^{k_j}| |y_\varepsilon| \end{aligned}$$

Taking the infimum in  $\beta$  and then the  $\lim_j$  we get

$$J(z_{\text{lim}}) \leq \liminf_j J^{n_{k_j}}(z_{i_j}^{k_j}) + (1 + \beta_\tau)\varepsilon \leq \limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z) + (1 + \beta_\tau)\varepsilon.$$

By letting  $\varepsilon$  go to 0 we obtain:

$$J(z_{\text{lim}}) \leq \limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z).$$

Then,

$$\inf_{z \geq z_0} J(z) \leq J(z_{\text{lim}}) \leq \limsup_{n \rightarrow \infty} \inf_{z \geq z_0} J^n(z).$$

□

## REFERENCES

- [1] Søren Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.
- [2] Alexander A. Borovkov and Anatolii A. Mogulskii. Large deviation principles for the finite-dimensional distributions of compound renewal processes. *Sib. Math. J.*, 56(1):28–53, 2015.
- [3] Alexander A. Borovkov and Anatolii A. Mogulskii. Large Deviation Principles for Trajectories of Compound Renewal Processes. I. *Theory Probab. Appl.*, 60(2):207–224, 2016.
- [4] Alexander A. Borovkov and Anatolii A. Mogulskii. Large Deviation Principles for Trajectories of Compound Renewal Processes. II. *Theory Probab. Appl.*, 60(3):349–366, 2016.
- [5] Mark Brown and Sheldon M. Ross. Asymptotic Properties of Cumulative Processes. *SIAM J. Appl. Math.*, 22(1):93–105, 1972.
- [6] Patrick Cattiaux, Laetitia Colombani, and Manon Costa. Limit theorems for Hawkes processes including inhibition. Preprint, 2021.
- [7] Manon Costa, Carl Graham, Laurence Marsalle, and Viet Chi Tran. Renewal in Hawkes processes with self-excitation and inhibition. *Adv. in Appl. Probab.*, 52(3):879–915, 2020.
- [8] Imre Csiszár. Sanov property, generalized  $I$ -projection and a conditional limit theorem. *Ann. Probab.*, 12(3):768–793, 1984.
- [9] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications*. Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin Heidelberg, 2 edition, 2010.
- [10] Ken Duffy and Anthony P. Metcalfe. How to estimate the rate function of a cumulative process. *J. Appl. Probab.*, 42(4):1044–1052, 2005.
- [11] Peter W. Glynn and Ward Whitt. Limit theorems for cumulative processes. *Stochastic Processes Appl.*, 47(2):299–314, 1993.
- [12] Alan G. Hawkes. Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58:83–90, 1971.
- [13] Raphaël Lefevre, Mauro Mariani, and Lorenzo Zambotti. Large deviations for renewal processes. *Stochastic Processes Appl.*, 121(10):2243–2271, 2011.
- [14] Walter L. Smith. Regenerative stochastic processes. *Proc. Roy. Soc. London Ser. A.*, 232(1188):6–31, 1955.
- [15] Jiang Tiefeng. Large deviations for renewal processes. *Stochastic Processes and their Applications*, 50(1):57–71, March 1994.

**Patrick CATTIAUX**, INSTITUT DE MATHÉMATIQUES DE TOULOUSE. CNRS UMR 5219., UNIVERSITÉ PAUL SABATIER., 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09.

*E-mail address:* patrick.cattiaux@math.univ-toulouse.fr

**Laetitia COLOMBANI**, INSTITUT DE MATHÉMATIQUES DE TOULOUSE. CNRS UMR 5219., UNIVERSITÉ PAUL SABATIER., 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09.

*E-mail address:* laetitia.colombani@math.univ-toulouse.fr

**Manon COSTA**, INSTITUT DE MATHÉMATIQUES DE TOULOUSE. CNRS UMR 5219., UNIVERSITÉ PAUL SABATIER., 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09.

*E-mail address:* manon.costa@math.univ-toulouse.fr