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► **To cite this version:**

Benoît Claudon, Patrick Graf, Henri Guenancia. Numerical characterization of complex torus quotients. 2021. hal-03344158v1

HAL Id: hal-03344158

<https://hal.science/hal-03344158v1>

Preprint submitted on 14 Sep 2021 (v1), last revised 20 Jul 2022 (v2)

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NUMERICAL CHARACTERIZATION OF COMPLEX TORUS QUOTIENTS

BENOÎT CLAUDON, PATRICK GRAF, AND HENRI GUENANCIA

ABSTRACT. This article gives a characterization of quotients of complex tori by finite groups acting freely in codimension two in terms of a numerical vanishing condition on the first and second Chern class. This generalizes results previously obtained by Greb–Kebekus–Peternell in the projective setting, and by Kirschner and the second author in dimension three. As a key ingredient to the proof, we obtain a version of the Bogomolov–Gieseker inequality for stable sheaves on singular spaces, including a discussion of the case of equality.

1. INTRODUCTION

Let X be a compact Kähler manifold of dimension n such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. An important application of Yau’s resolution of the Calabi conjecture [Yau78] is the following uniformization result: X is isomorphic to the quotient T/G of a complex torus T by a finite group G acting freely on T if and only if there exists a Kähler class $\alpha \in H^2(X, \mathbb{R})$ such that $c_2(X) \cdot \alpha^{n-2} = 0$.

In recent years, a lot of effort has been devoted to generalizing the above uniformization criterion to a class of singular varieties arising naturally in the Minimal Model Program, called varieties with Kawamata log terminal singularities (klt for short). A first roadblock consists in defining Chern classes, e.g. $c_2(X)$. This is quite delicate for singular varieties, as several possible definitions exist that do not coincide in general. We will gloss over this problem in the introduction and refer to [Section 2](#) and the references therein for a more in-depth discussion.

In the projective case, i.e. when X is a projective variety with klt singularities with $c_1(X) = 0$ and $\alpha = c_1(\mathcal{L})$ is the class of an ample line bundle \mathcal{L} , the uniformization problem has been solved by [GKP16a] and [LT18]. More precisely, Greb–Kebekus–Peternell [GKP16a] proved that if X is smooth in codimension two, then X is the quotient of an abelian variety provided that $c_2(X) \cdot \alpha^{n-2} = 0$. Later on, Lu–Taji [LT18] were able to lift the assumption about the codimension of the singular locus using the theory of orbifold Chern classes initiated by Mumford [Mum83].

In the transcendental case however, the slicing arguments used in *loc. cit.* to reduce to a complete intersection surface are certainly not available anymore, and new ideas are required. The threefold case has recently been settled by Kirschner and the second

Date: September 14, 2021.

2010 Mathematics Subject Classification. 32J27, 14J60.

Key words and phrases. Complex tori, vanishing Chern classes, klt singularities, stable sheaves, Bogomolov–Gieseker inequality.

author [GK20], but the key techniques therein do not seem to generalize to the higher dimensional case. In this paper, we rely on the recent Beauville–Bogomolov decomposition theorem [BGL21] as well as our Bogomolov–Gieseker inequality, [Theorem B](#) below, to settle the uniformization problem in the transcendental case, assuming that X is smooth in codimension two.

Theorem A. *Let X be a compact complex space of dimension n with klt singularities, smooth in codimension two. The following are equivalent:*

(A.1) *We have $c_1(X) = 0 \in H^2(X, \mathbb{R})$, and there exists a Kähler class $\alpha \in H^2(X, \mathbb{R})$ as well as a resolution of singularities $f: Y \rightarrow X$ which is an isomorphism over X_{reg} such that*

$$\int_Y c_2(Y) \wedge (f^* \alpha)^{n-2} = 0.$$

(A.2) *There exists a complex n -torus T and a holomorphic action of a finite group $G \curvearrowright T$, free in codimension two, such that $X \cong T/G$.*

Strategy of proof of [Theorem A](#). In the smooth case, the cohomological assumption $c_1(X) = 0$ can be turned into the existence of a Ricci-flat Kähler metric ω in the class α [Yau78]. The numerical condition $c_2(X) \cdot \alpha^{n-2} = 0$ can then be easily translated into the vanishing of the full curvature tensor of ω [CO75]. When X is merely a compact klt Kähler space with vanishing first Chern class, it is known that X admits a singular Ricci-flat metric [EGZ09], but the curvature tensor of (X_{reg}, ω) is not known to be L^2 -integrable in general, and there does not seem to be a way to compute the quantity $c_2(X) \cdot \alpha^{n-2}$ using ω .

To circumvent this intrinsic difficulty, in addition to the Beauville–Bogomolov Decomposition Theorem cited above, we use the following statement that generalizes the classical Bogomolov–Gieseker inequality to the singular case. We refer to [Section 2](#) for a more detailed discussion of the notion of Chern classes used below.

Theorem B (Bogomolov–Gieseker inequality). *Let X be a normal compact Kähler space of dimension n together with a Kähler class $\alpha \in H^2(X, \mathbb{R})$. Assume that X is smooth in codimension two. Furthermore, let \mathcal{E} be a rank r reflexive coherent sheaf on X which is slope stable with respect to α .*

(B.1) *The discriminant $\Delta(\mathcal{E}) := 2r c_2(\mathcal{E}) - (r-1) c_1^2(\mathcal{E})$ satisfies the inequality*

$$\Delta(\mathcal{E}) \cdot \alpha^{n-2} \geq 0.$$

(B.2) *If equality holds in (B.1), then we have*

$$\Delta(\mathcal{E}) \cdot \beta^{n-2} = 0$$

for any Kähler class $\beta \in H^2(X, \mathbb{R})$.

(B.3) *If $c_2(\mathcal{E}) \cdot \alpha^{n-2} = c_1^2(\mathcal{E}) \cdot \alpha^{n-2} = 0$, then we have*

$$c_2(\mathcal{E}) \cdot \beta^{n-2} = c_1^2(\mathcal{E}) \cdot \beta^{n-2} = 0$$

for any Kähler class $\beta \in H^2(X, \mathbb{R})$.

Remark. The definition of the Chern classes used in [Theorem B](#) does not rely on the assumption that X is smooth in codimension two, and the statements continue to hold

(with the same proof) if that assumption is dropped. However, if X has singularities in codimension two, these notions can behave quite counterintuitively. For example, it may happen that $c_1(\mathcal{E})$ is zero, but $c_1^2(\mathcal{E})$ is not (cf. [Example 2.6](#)). For this reason, we have chosen to include the smoothness in codimension two assumption in the above statement.

With [Theorem B](#) at hand, the roadmap to proving [Theorem A](#) is then the following: take X as in the statement, i.e. klt, smooth in codimension two and with $c_1(X) = 0$.

- Deduce from [\(B.1\)](#) the semipositivity of the second Chern class, i.e. $c_2(X) \cdot \alpha^{n-2} \geq 0$ for any Kähler class α . This is not entirely immediate because the tangent sheaf \mathcal{T}_X will in general not be stable, but see [Corollary 3.6](#).
- Decompose a cover of X as a product of a complex torus, irreducible Calabi–Yau (ICY) and irreducible holomorphic symplectic (IHS) varieties. Use the first step to show that if $c_2(X)$ vanishes against a Kähler class, then the same is true of each factor in the above decomposition. We are thus reduced to showing that for ICY and IHS varieties, c_2 is *strictly* positive against any Kähler class.
- If X is an ICY variety (hence projective), argue by contradiction. Assuming that $c_2(X)$ is zero against some Kähler class, use [\(B.3\)](#) to obtain vanishing against an *ample* class. This contradicts the projective case of [Theorem A](#).
- If X is an IHS variety, the required positivity is achieved via a complete description of $c_2(X)$ using the (generalized) Fujiki relations, cf. [Proposition 4.4](#).

Relation to earlier and recent work. As mentioned above, the projective case of [Theorem A](#) was settled by Shepherd-Barron and Wilson [[SBW94](#)] in dimension three, and by Greb–Kebekus–Peternell [[GKP16a](#)] in general, always assuming that α is a rational class. Lu–Taji [[LT18](#)] later removed the smoothness in codimension two assumption.

Inequality [\(B.1\)](#) has recently been obtained by Wu [[Wu21](#)], but he does not discuss the case of equality. Note that his result is formulated for polystable sheaves, but the proof actually assumes stability. His line of argument is very similar to ours: pull everything back to a resolution and prove an openness of stability type statement there. This basic idea goes back at least to [[DP03](#), Proposition 6.9].

Even more recently, Chen and Wentworth [[CW21](#)] have likewise obtained a Bogomolov–Gieseker inequality in a setting similar to ours. Their results, once combined with our previous paper [[CGGN20](#)], readily imply that if X is a compact Kähler space with klt singularities, smooth in codimension two and $c_1(X) = 0$, then there exists an admissible Hermite–Einstein metric h on $\mathcal{T}_{X_{\text{reg}}}$. Assuming additionally that $c_2(X) \cdot \alpha^{n-2} = 0$ for some Kähler class α , this *should* imply that $(\mathcal{T}_{X_{\text{reg}}}, h)$ is hermitian flat and therefore X is a torus quotient by [[CGGN20](#), Theorem D]. There are at least two reasons why [Theorem A](#) is still not an immediate consequence of [[CW21](#)] and [[CGGN20](#)]:

- In [[CW21](#)] the authors make the quite strong assumption that there is a global embedding $X \hookrightarrow M$ into a compact Kähler manifold M , and that the Kähler metric on X extends to M . This is used in order to define the second Chern class and also to relate Chern numbers to integrals of Chern forms against powers of Kähler metrics. We do not know any natural condition that would guarantee the existence of such an embedding.

- A delicate point is to prove that $c_2(X) \cdot \alpha^{n-2} = \int_{X_{\text{reg}}} c_2(\mathcal{T}_{X_{\text{reg}}}, h) \wedge \omega^{n-2}$ if ω is a Kähler metric representing α . This is currently not completely clear to us.

Acknowledgements. The second author would like to thank Philipp Naumann, Mihai Păun and Thomas Peternell for inspiring discussions.

2. CHERN CLASSES ON SINGULAR SPACES

We will use the following notions of Chern classes on singular spaces. In what follows, X denotes an n -dimensional connected normal compact complex space.

Definition 2.1 (Chern classes of sheaves). Let \mathcal{E} be a torsion-free coherent sheaf on X , and let $f: Y \rightarrow X$ be a resolution such that $f^\sharp \mathcal{E} := f^* \mathcal{E} / \text{tor}(f^* \mathcal{E})$ is locally free. For any number $1 \leq i \leq n$ and any class $a \in H^{2n-2i}(X, \mathbb{R})$, we set

$$c_i(\mathcal{E}) \cdot a := c_i(f^\sharp \mathcal{E}) \cdot f^*(a) \in H^{2n}(Y, \mathbb{R}) = \mathbb{R}.$$

The Chern classes $c_i(\mathcal{E})$ thus defined are elements of $H^{2n-2i}(X, \mathbb{R})^\vee = H_{2n-2i}(X, \mathbb{R})$. For the existence of a resolution with the required property, see [Ros68, Theorem 3.5]. Well-definedness follows from the simple observation that if $h: Z \rightarrow X$ is a resolution which factors as $Z \xrightarrow{g} Y \xrightarrow{f} X$, then there is an exact sequence

$$g^*(\text{tor}(f^* \mathcal{E})) \rightarrow h^* \mathcal{E} \rightarrow g^*(f^\sharp \mathcal{E}) \rightarrow 0,$$

hence $h^\sharp \mathcal{E} = g^*(f^\sharp \mathcal{E})$ and consequently

$$c_i(h^\sharp \mathcal{E}) \cdot h^*(a) = g^* c_i(f^\sharp \mathcal{E}) \cdot g^*(f^* a) = c_i(f^\sharp \mathcal{E}) \cdot f^*(a).$$

Remark 2.2 (Polynomial combinations of Chern classes). The above definition applies more generally to weighted homogeneous polynomials in the Chern classes, where c_i has degree i . For example, we may set $c_1^2(\mathcal{E}) \cdot a := c_1^2(f^\sharp \mathcal{E}) \cdot f^*(a)$ and then $c_1^2(\mathcal{E})$ will be an element of $H_{2n-4}(X, \mathbb{R})$. Note that we cannot directly define “ $c_1^2(\mathcal{E}) := c_1(\mathcal{E}) \cdot c_1(\mathcal{E})$ ” because in general there is no ring structure on the homology $H_*(X, \mathbb{R})$.

In a similar vein, we can also define combinations of Chern classes of different sheaves. E.g. if \mathcal{E}, \mathcal{F} are torsion-free, we pick a resolution $f: Y \rightarrow X$ such that both $f^\sharp \mathcal{E}$ and $f^\sharp \mathcal{F}$ are locally free. Then $c_1(\mathcal{E}) \cdot c_1(\mathcal{F}) \in H_{2n-4}(X, \mathbb{R})$ is defined by setting $c_1(\mathcal{E}) \cdot c_1(\mathcal{F}) \cdot a := c_1(f^\sharp \mathcal{E}) \cdot c_1(f^\sharp \mathcal{F}) \cdot f^*(a)$.

Remark. While the above definitions are sufficient for our purposes, they turn out to be rather moot for general coherent sheaves. In particular, they do not behave nicely in short exact sequences. For example, the Chern classes of a torsion sheaf would obviously all vanish.

In the case where X is klt and $\mathcal{E} = \mathcal{T}_X$ is the tangent sheaf, there is another reasonable way to define the second Chern class.

Definition 2.3 (Second Chern class of X). Assume that X has klt singularities, and let $f: Y \rightarrow X$ be a resolution which is minimal in codimension two. For any class $a \in H^{2n-4}(X, \mathbb{R})$, we set

$$c_2(X) \cdot a := c_2(Y) \cdot f^*(a) \in H^{2n}(Y, \mathbb{R}) = \mathbb{R}.$$

For existence of such a resolution and well-definedness of $c_2(X)$, see [GK20, Proposition 5.3]. Again, we have $c_2(X) \in H^{2n-4}(X, \mathbb{R})^\vee = H_{2n-4}(X, \mathbb{R})$.

Remark. The klt assumption in Definition 2.3 can be weakened. In fact, the proof of independence of Y does not use it, and hence the only question is whether a resolution minimal in codimension two actually exists. This is true e.g. whenever X has klt singularities in codimension two (equivalently, quotient singularities in codimension two) and thus in particular if X is smooth in codimension two.

The next result, together with Example 2.6, clarifies the relationship between Definitions 2.1 and 2.3.

Proposition 2.4 (Compatibility, I). *If X has klt singularities and is smooth in codimension two, i.e. $\text{codim}_X(X_{\text{sg}}) \geq 3$, then we have $c_2(X) = c_2(\mathcal{T}_X)$ as elements of $H_{2n-4}(X, \mathbb{R})$.*

Proof. Let $f: Y \rightarrow X$ be a resolution such that $f^\# \mathcal{T}_X$ is locally free. It is clear from the construction in [Ros68] that f can be chosen to be an isomorphism over X_{reg} , since there \mathcal{T}_X is already locally free. As X is assumed to be smooth in codimension two, such a resolution f will automatically be minimal in codimension two. So it is sufficient to show that

$$c_2(f^\# \mathcal{T}_X) \cdot f^* a = c_2(Y) \cdot f^* a$$

for all $a \in H^{2n-4}(X, \mathbb{R})$. This follows from Lemma 2.7 below applied with $Z = \text{Exc}(f)$, once we know that $(f^* a)|_Z = 0$. But $(f^* a)|_Z = f^*(a|_{f(Z)}) = 0$ because $f(Z) = X_{\text{sg}}$ and $H^{2n-4}(X_{\text{sg}}, \mathbb{R}) = 0$ by dimension reasons (the real dimension is at most $2n - 6$). \square

Going back to the original setup (where X is only assumed to be normal), recall that the determinant of the rank r torsion-free sheaf \mathcal{E} is defined as $\det \mathcal{E} := (\bigwedge^r \mathcal{E})^{\vee\vee}$, where $(-)^{\vee\vee}$ denotes the reflexive hull (= double dual). By definition, this is a rank one reflexive sheaf. We are interested in situations where it is actually \mathbb{Q} -Cartier. If this is the case, we may as usual consider $c_1(\det \mathcal{E}) \in H^2(X, \mathbb{R})$, and more generally $c_1^k(\det \mathcal{E}) \in H^{2k}(X, \mathbb{R})$. By abuse of notation, we will also consider

$$c_1^k(\det \mathcal{E}) \in H_{2n-2k}(X, \mathbb{R})$$

via the natural map $H^{2k}(X, \mathbb{R}) \rightarrow H^{2n-2k}(X, \mathbb{R})^\vee = H_{2n-2k}(X, \mathbb{R})$ given by the cup product pairing (or equivalently, by cap product with the fundamental class of X). We compare this notion to Definition 2.1:

Proposition 2.5 (Compatibility, II). *Assume that X is normal and smooth in codimension $k \geq 1$, that \mathcal{E} is locally free in codimension k , and that $\det \mathcal{E}$ is \mathbb{Q} -Cartier. Then we have*

$$c_1^\ell(\mathcal{E}) = c_1^\ell(\det \mathcal{E}) \quad \text{for any } \ell \leq k$$

as elements of $H_{2n-2\ell}(X, \mathbb{R})$. In particular, if X is smooth in codimension two, \mathcal{E} is reflexive, and $\det \mathcal{E} \cong \mathcal{O}_X$, then $c_1^2(\mathcal{E}) = 0$.

Proof. Let $f: Y \rightarrow X$ be a resolution such that $f^\# \mathcal{E}$ is locally free. As before, we may assume that f is an isomorphism over the locus where both X is smooth and \mathcal{E} is locally

free. Pick $m > 0$ such that $\mathcal{L} := ((\det \mathcal{E})^{\otimes m})^{\vee\vee}$ is a line bundle. Then we need to show that

$$c_1^\ell(\det f^\# \mathcal{E}) \cdot f^* a = \frac{1}{m^\ell} c_1^\ell(f^* \mathcal{L}) \cdot f^* a$$

for all $a \in \mathbb{H}^{2n-2\ell}(X, \mathbb{R})$. This follows from [Lemma 2.7](#) applied with $Z = \text{Exc}(f)$, since $(\det f^\# \mathcal{E})^{\otimes m}$ and $f^* \mathcal{L}$ are isomorphic outside of Z and $(f^* a)|_Z = 0$ as in the proof of [Proposition 2.4](#).

For the second statement, it suffices to show that \mathcal{E} is locally free in codimension two. After discarding X_{sg} , this follows from the fact that a reflexive sheaf on a *smooth* space enjoys this property [[OSS80](#), Lemma 1.1.10]. \square

Example 2.6. Without assuming that X is smooth in codimension two, the Chern classes $c_2(X)$ and $c_2(\mathcal{T}_X)$ are in general different. For instance let $f: Y \rightarrow X$ be the minimal resolution of a Kummer surface $X = A/\pm 1$. The surface Y being K3, we have $c_2(X) = 24$. On the other hand, from $f^\# \mathcal{T}_X = \mathcal{T}_Y(-\log E)$ we get that

$$c_2(\mathcal{T}_X) = c_2(\mathcal{T}_Y(-\log E)) = -8 \neq 24.$$

The same example also shows that [Proposition 2.5](#) fails if $\ell > k$. Indeed, the sheaf $\mathcal{E} = \mathcal{T}_X$ has trivial determinant, hence $c_1^2(\det \mathcal{T}_X) = 0$, but

$$c_1^2(\mathcal{T}_X) = c_1^2(\mathcal{T}_Y(-\log E)) = (-(K_Y + E))^2 = E^2 = -32 \neq 0,$$

as E is the disjoint union of sixteen (-2) -curves.

Lemma 2.7. *Let X be a compact complex manifold of dimension n , $Z \subset X$ an analytic subset and $U := X \setminus Z$. Suppose two locally free coherent sheaves \mathcal{F}, \mathcal{G} on X are given such that there is an isomorphism $\mathcal{F}|_U \cong \mathcal{G}|_U$. Then*

$$c_i(\mathcal{F}) \cdot a = c_i(\mathcal{G}) \cdot a \quad \text{for any } a \in \mathbb{H}^{2n-2i}(X, \mathbb{R}) \text{ with } a|_Z = 0.$$

A similar statement holds for degree i polynomials in the Chern classes.

Proof. Pick a sufficiently small neighborhood $Z \subset W \subset X$ which deformation retracts onto Z via a map $r: W \rightarrow Z$. This exists because we can find a triangulation of X such that Z is a subcomplex [[Loj64](#), Theorems 2 and 3]. Set $W^\times := W \setminus Z$, and note that the inclusion $j: (W, W^\times) \hookrightarrow (X, U)$ satisfies excision. We define a map $\varphi: \mathbb{H}^{2i}(X, U) \rightarrow \mathbb{H}_{2n-2i}(Z)$ as follows (everything is with real coefficients):

$$\mathbb{H}^{2i}(X, U) \xrightarrow{j^*} \mathbb{H}^{2i}(W, W^\times) \xrightarrow{\cap [W]_Z} \mathbb{H}_{2n-2i}(W) \xrightarrow{r_*} \mathbb{H}_{2n-2i}(Z).$$

The first map is an isomorphism by excision. The second map is cap product with the fundamental class¹ $[W]_Z \in \mathbb{H}_{2n}(W, W^\times)$. The third map r_* is an isomorphism because r is a deformation retraction. (The map φ itself is also an isomorphism, but we do not need this. The reason is that the dual map $\varphi^\vee: \mathbb{H}^{2n-2i}(Z) \rightarrow \mathbb{H}_{2i}(X, U)$ is an isomorphism by [[Hat02](#), Proposition 3.46].)

¹Recall that for an oriented n -manifold M and a compact subset $K \subset M$, the fundamental class $[M]_K$ is the unique element of $\mathbb{H}_n(M, M \setminus K)$ which at each point $x \in K$ induces the local orientation in $\mathbb{H}_n(M, M \setminus \{x\})$ given by the orientation of M (here we are using integer coefficients). In case $K = M$ is compact, this reduces to the usual notion. Cf. [[Hat02](#), Lemma 3.27] for proofs.

We now have a commutative diagram

$$\begin{array}{ccccc}
 \mathrm{H}^{2i}(X, U) & \xrightarrow{p^*} & \mathrm{H}^{2i}(X) & \xrightarrow{\iota_U^*} & \mathrm{H}^{2i}(U) \\
 \varphi \downarrow & & \downarrow \cap [X] & & \\
 \mathrm{H}_{2n-2i}(Z) & \xrightarrow{(\iota_Z)_*} & \mathrm{H}_{2n-2i}(X) & &
 \end{array}$$

where the upper row is the long exact sequence of relative cohomology. To see commutativity, first note that under the isomorphism $\mathrm{H}_{2n}(W, W^\times) \cong \mathrm{H}_{2n}(X, U)$, the fundamental classes $[W]_Z$ and $[X]_Z$ correspond to each other. Now pick an arbitrary class $\alpha \in \mathrm{H}^{2i}(X, U)$ and calculate

$$\begin{aligned}
 (\iota_Z)_* \varphi(\alpha) &= (\iota_Z)_* r_* (j^* \alpha \cap [W]_Z) && \text{by definition} \\
 &= j_* (j^* \alpha \cap [W]_Z) && r \text{ is a deformation retraction} \\
 &= \alpha \cap j_* [W]_Z && \text{projection formula} \\
 &= \alpha \cap [X]_Z && \text{remark above} \\
 &= \alpha \cap p_* [X] && \text{where } p_* : \mathrm{H}_{2n}(X) \rightarrow \mathrm{H}_{2n}(X, U) \\
 &= p^* \alpha \cap [X] && \text{by definition.}
 \end{aligned}$$

Let us now return to the original setting. The assumption on \mathcal{F} and \mathcal{G} clearly implies that $\iota_U^*(c_i(\mathcal{F})) = \iota_U^*(c_i(\mathcal{G}))$, hence the difference $c := c_i(\mathcal{F}) - c_i(\mathcal{G})$ lies in the image of $\mathrm{H}^{2i}(X, U)$. By the above diagram, $c \cap [X] = (\iota_Z)_* \sigma$ for some $\sigma \in \mathrm{H}_{2n-2i}(Z)$. The claim now follows easily:

$$c \cdot a = \langle c \cap [X], a \rangle = \langle (\iota_Z)_* \sigma, a \rangle = \langle \sigma, \iota_Z^* a \rangle = 0,$$

since $\iota_Z^* a = a|_Z = 0$ by assumption. Here $\langle -, - \rangle$ denotes the natural pairing between homology and cohomology. The proof for arbitrary polynomials in the Chern classes is the same, and hence is omitted. \square

Remark. In [Lemma 2.7](#), we do *not* require the existence of a global map $\mathcal{F} \rightarrow \mathcal{G}$ whose restriction to U is an isomorphism. This greatly simplifies the proof of [Proposition 2.4](#). Indeed, as the sheaf of Kähler differentials Ω_X^1 is not locally free, from the natural map $f^* \Omega_X^1 \rightarrow \Omega_Y^1$ we cannot by dualizing deduce a map $\mathcal{T}_Y \rightarrow f^* \mathcal{T}_X$. If we choose f to be the functorial resolution, there is indeed a map $f^* \mathcal{T}_X \rightarrow \mathcal{T}_Y$, but then we do not know whether $f^\# \mathcal{T}_X$ is always locally free.

Slopes and stability. If $\alpha \in \mathrm{H}^2(X, \mathbb{R})$ is a Kähler (or merely big and nef) class on X , one can define the *slope* (with respect to α) of a nonzero torsion-free sheaf \mathcal{E} as

$$\mu_\alpha(\mathcal{E}) := \frac{1}{\mathrm{rk} \mathcal{E}} c_1(\mathcal{E}) \cdot \alpha^{n-1}.$$

In case $\det \mathcal{E}$ is a \mathbb{Q} -line bundle, one recovers the usual definition of the slope (cf. e.g. [\[GKP16b\]](#)) thanks to [Proposition 2.5](#). We say that \mathcal{E} is (*slope*) *stable* with respect to α if for any nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ of rank strictly less than $\mathrm{rk} \mathcal{E}$, we have $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$.

As before, let $f: Y \rightarrow X$ be a resolution such that $f^\# \mathcal{E}$ is locally free. Given a subsheaf $\mathcal{F} \subset \mathcal{E}$ of rank $s > 0$, $f^\# \mathcal{F}$ will in general not be locally free, but its determinant is a

line bundle since Y is smooth. Thanks to the observations above, we can still compute the slope of \mathcal{F} as $\mu_\alpha(\mathcal{F}) = \frac{1}{s} c_1(f^\sharp \mathcal{F}) \cdot f^* \alpha^{n-1}$.

3. THE BOGOMOLOV–GIESEKER INEQUALITY

In this section, we prove [Theorem B](#) and give an application to varieties with vanishing first Chern class ([Corollary 3.6](#)).

Setup 3.1. Let X be a normal compact Kähler space of dimension n , let α be a Kähler class represented by a Kähler metric ω and let \mathcal{E} be a reflexive coherent sheaf of rank r on X . In particular, the locus

$$Z := X_{\text{sing}} \cup \{x \in X; \mathcal{E} \text{ is not locally free near } x\}$$

satisfies $\text{codim}_X(Z) \geq 2$; one sets $X^\circ := X \setminus Z$. Once and for all, we fix a log resolution $f: \widehat{X} \rightarrow X$ of (X, Z) such that $\widehat{\mathcal{E}} := f^\sharp \mathcal{E} = f^* \mathcal{E} / \text{tor } f^* \mathcal{E}$ is locally free. The exceptional divisor of f is denoted by $F = \sum F_i$. Finally, we set

$$\Delta(\mathcal{E}) \cdot \alpha^{n-2} := (2r c_2(\mathcal{E}) - (r-1) c_1^2(\mathcal{E})) \cdot \alpha^{n-2}$$

to be the Bogomolov–Gieseker discriminant of \mathcal{E} against the class $\alpha^{n-2} \in H^{2n-4}(X, \mathbb{R})$. Let us emphasize that in this section, we do not assume that X is smooth in codimension two, unless explicitly mentioned otherwise.

3.A. Openness of stability. The goal of this subsection is to prove an openness-type property for stable sheaves on a resolution of singularities, cf. [Proposition 3.4](#). This was previously observed in the smooth case in [[Cao13](#), Proposition 2.1]. As an immediate consequence, we obtain [\(B.1\)](#) from the introduction.

Lemma 3.2. *In [Setup 3.1](#) above, \mathcal{E} is stable with respect to α if and only if $\widehat{\mathcal{E}}$ is stable with respect to $f^* \alpha$.*

Proof. Assume that \mathcal{E} is stable and let $\widehat{\mathcal{G}} \subset \widehat{\mathcal{E}}$ be a proper subsheaf. Note that by reflexivity of \mathcal{E} , the natural morphism $\mathcal{E} \rightarrow f_* f^* \mathcal{E}$ induces an isomorphism $(f_* f^* \mathcal{E})^{**} \cong \mathcal{E}$. Similarly, taking the direct image of the canonical surjection $f^* \mathcal{E} \rightarrow \widehat{\mathcal{E}}$ and dualizing induces an isomorphism $(f_* \widehat{\mathcal{E}})^{**} \cong \mathcal{E}$. Now the subsheaf $\mathcal{G} := (f_* \widehat{\mathcal{G}})^{**} \subset (f_* \widehat{\mathcal{E}})^{**} \cong \mathcal{E}$ is such that $c_1(f^\sharp \mathcal{G}) - c_1(\widehat{\mathcal{G}})$ is a linear combination of the F_i , and therefore

$$\mu_{f^* \alpha}(\widehat{\mathcal{G}}) = \mu_\alpha(\mathcal{G}) < \mu_\alpha(\mathcal{E}) = \mu_{f^* \alpha}(\widehat{\mathcal{E}}).$$

In the other direction, assume that $\widehat{\mathcal{E}}$ is stable with respect to $f^* \alpha$ and let $\mathcal{F} \subset \mathcal{E}$ be a proper subsheaf. The image of $f^* \mathcal{F} \rightarrow f^\sharp \mathcal{E}$ yields a subsheaf $\widehat{\mathcal{F}}$ of $\widehat{\mathcal{E}}$ which coincides with $f^* \mathcal{F}$ away from F . In particular, $c_1(f^\sharp \mathcal{F}) - c_1(\widehat{\mathcal{F}})$ is supported on F , hence

$$\mu_\alpha(\mathcal{F}) = \mu_{f^* \alpha}(\widehat{\mathcal{F}}) < \mu_{f^* \alpha}(\widehat{\mathcal{E}}) = \mu_\alpha(\mathcal{E})$$

and the lemma follows. \square

Lemma 3.3. *In [Setup 3.1](#) above, assume that \mathcal{E} is stable with respect to α . Let β be a Kähler class on \widehat{X} . Then,*

(3.3.1) *The sheaf $\widehat{\mathcal{E}}$ is stable with respect to $f^* \alpha$.*

(3.3.2) *There exists a constant $C > 0$ such that for any subsheaf $\widehat{\mathcal{F}} \subset \widehat{\mathcal{E}}$ and any integer $k \in [0, n-1]$, one has*

$$c_1(\widehat{\mathcal{F}}) \cdot \alpha^k \cdot \beta^{n-1-k} \leq C.$$

(3.3.3) *There exists $\delta > 0$ such that for every subsheaf $\widehat{\mathcal{F}} \subset \widehat{\mathcal{E}}$ with $\text{rk}(\widehat{\mathcal{F}}) < \text{rk}(\widehat{\mathcal{E}})$, we have*

$$\mu_{f^*\alpha}(\widehat{\mathcal{F}}) \leq \mu_{f^*\alpha}(\widehat{\mathcal{E}}) - \delta.$$

Proof. The first item is an immediate consequence of [Lemma 3.2](#).

The second item can be proved in a similar way as the case of a single Kähler polarization, see e.g. [\[Kob87, Lemma 7.16\]](#). For that reason, we restrict to the case where $\widehat{\mathcal{F}}$ is the sheaf of sections of a sub-vector bundle \widehat{F} of the vector bundle \widehat{E} associated to $\widehat{\mathcal{E}}$. Let us pick a Kähler metric $\widehat{\omega} \in \beta$ and a smooth hermitian metric $h_{\widehat{E}}$ on \widehat{E} ; it induces a hermitian metric $h_{\widehat{F}}$ on \widehat{F} and their respective Chern curvature forms satisfy

$$i\Theta(\widehat{F}, h_{\widehat{F}}) \leq \text{pr}_{\widehat{F}} i\Theta(\widehat{E}, h_{\widehat{E}})|_{\widehat{F}}$$

Taking the (endomorphism) trace and wedging with $f^*\omega^k \wedge \widehat{\omega}^{n-1-k}$, we get

$$c_1(\widehat{F}, h_{\widehat{F}}) \wedge f^*\omega^k \wedge \widehat{\omega}^{n-1-k} \leq C \|i\Theta(\widehat{E}, h_{\widehat{E}})\|_{h_{\widehat{E}}, \widehat{\omega}} \cdot \widehat{\omega}^n$$

where C depends only on $\text{tr}_{\widehat{\omega}}(f^*\omega)$. The right-hand side does not depend on \widehat{F} anymore; integrating over \widehat{X} yields the expected result.

The last item is a consequence of the proof of [Lemma 3.2](#) and the fact that the statement is true for subsheaves \mathcal{F} of \mathcal{E} with rank strictly less than $\text{rank}(\mathcal{E})$. As for the latter fact, it derives e.g. from the finiteness of components of the Douady space of quotients of \mathcal{E} with slope at most $\mu_{\alpha}(\mathcal{E}) + 1$, cf e.g. [\[Tom19, Corollary 6.3\]](#) applied to $S = \text{pt}$. \square

Proposition 3.4 (Bogomolov–Gieseker inequality). *In [Setup 3.1](#) above, assume that \mathcal{E} is stable with respect to α . Then, there exists $\varepsilon_0 > 0$ such that the sheaf $\widehat{\mathcal{E}}$ is stable with respect to $f^*\alpha + \varepsilon\beta$ for any $0 \leq \varepsilon \leq \varepsilon_0$. In particular, one has the Bogomolov–Gieseker inequality*

$$\Delta(\mathcal{E}) \cdot \alpha^{n-2} = (2rc_2(\mathcal{E}) - (r-1)c_1^2(\mathcal{E})) \cdot \alpha^{n-2} \geq 0.$$

Proof. With the notation of [Lemma 3.3](#) above, let us set $\varepsilon_0 := \frac{\delta}{2(n-1)C}$. Then, one has for $\varepsilon \leq \varepsilon_0$ and any subsheaf $\widehat{\mathcal{F}} \subset \widehat{\mathcal{E}}$ of rank strictly less than $\text{rk}(\widehat{\mathcal{E}})$:

$$\begin{aligned} \mu_{f^*\alpha + \varepsilon\beta}(\widehat{\mathcal{F}}) &= \mu_{f^*\alpha}(\widehat{\mathcal{F}}) + \frac{1}{r} \sum_{k=0}^{n-2} \varepsilon^{n-1-k} c_1(\widehat{\mathcal{F}}) \cdot f^*\alpha^k \cdot \beta^{n-1-k} \\ &\leq \mu_{f^*\alpha}(\widehat{\mathcal{E}}) - \delta + (n-1)C\varepsilon \\ &\leq \mu_{f^*\alpha}(\widehat{\mathcal{E}}) - \delta/2, \end{aligned}$$

where the first inequality follows from [\(3.3.2\)](#) and [\(3.3.3\)](#) in [Lemma 3.3](#).

For the second assertion, one can use the Kobayashi–Hitchin correspondence to obtain for any $\varepsilon > 0$ a Hermite–Einstein metric h_{ε} on $\widehat{\mathcal{E}}$ with respect to a Kähler metric ω_{ε} of

the form $\omega_\varepsilon := f^*\omega + \varepsilon\widehat{\omega}$ where $\omega \in \alpha$ (resp. $\widehat{\omega} \in \beta$) is a Kähler metric on X (resp. on \widehat{X}). It is classic to get the inequality

$$\left(2rc_2(\widehat{\mathcal{E}}, h_\varepsilon) - (r-1)c_1(\widehat{\mathcal{E}}, h_\varepsilon)^2\right) \wedge \omega_\varepsilon^{n-2} \geq 0$$

pointwise, for any $\varepsilon > 0$. Integrating over \widehat{X} and letting ε go to zero, one gets the expected inequality. \square

3.B. Application to varieties with trivial first Chern class. Let X be a compact Kähler space with klt singularities such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. In that setting, the Abundance conjecture is known, i.e. K_X is a torsion \mathbb{Q} -line bundle, cf. e.g. [CGP19, Corollary 1.18]. If we assume additionally that X is smooth in codimension two, we immediately infer that $c_1^2(\mathcal{T}_X) \cdot \alpha^{n-2} = 0$ for any Kähler class α thanks to Proposition 2.5. Moreover, the recent Decomposition Theorem [BGL21, Theorem A] asserts that up to a quasi-étale cover, X splits as a product of a complex torus, irreducible Calabi–Yau varieties and irreducible holomorphic symplectic varieties, where the latter two are defined as follows.

Definition 3.5 (ICY and IHS varieties). Let X be a compact Kähler space of dimension $n \geq 2$ with canonical singularities and $\omega_X \cong \mathcal{O}_X$.

(3.5.1) We call X *irreducible Calabi–Yau* (ICY) if $H^0(Y, \Omega_Y^{[p]}) = 0$ for all integers $0 < p < n$ and all quasi-étale covers $Y \rightarrow X$, in particular for X itself.

(3.5.2) We call X *irreducible holomorphic symplectic* (IHS) if there exists a holomorphic symplectic two-form $\sigma \in H^0(X, \Omega_X^{[2]})$ such that for all quasi-étale covers $\gamma: Y \rightarrow X$, the exterior algebra of global reflexive differential forms is generated by $\gamma^{[*]}\sigma$.

Given the Bochner principle [CGGN20, Theorem A], it is relatively easy to show that the tangent sheaf \mathcal{T}_X of an IHS variety X (resp. ICY variety X) is stable with respect to any polarization. The following result is then an immediate consequence of the Decomposition Theorem and Proposition 3.4. However, we will give a more pedestrian proof relying mostly on the polystability of \mathcal{T}_X rather than a global structure result.

Corollary 3.6 (Semipositivity of c_2). *Let X be a compact Kähler space with klt singularities such $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Assume furthermore that X is smooth in codimension two. Then, one has*

$$c_2(X) \cdot \alpha^{n-2} \geq 0$$

for any Kähler class $\alpha \in H^2(X, \mathbb{R})$.

Proof. It follows from [Gue16, Theorem A] and [CGGN20, Theorem C] that up to replacing X by a further quasi-étale cover $\widetilde{X} \rightarrow X$ (which does not affect semipositivity of the second Chern class by [GK20, Proposition 5.6]), one can decompose the tangent sheaf of X as

$$(3.6.1) \quad \mathcal{T}_X = \bigoplus_{i \in I} \mathcal{E}_i$$

where the \mathcal{E}_i are subsheaves of rank r_i that are α -stable with slope zero. Moreover, $\mathcal{E}_i|_{X_{\text{reg}}}$ is parallel with respect to the singular Ricci flat metric $\omega \in \alpha$ and its holonomy group

lies in $SU(r_i)$. This implies readily that $\det \mathcal{E}_i \cong \mathcal{O}_X$, and therefore $c_1^2(\mathcal{E}_i) \cdot \alpha^{n-2} = 0$ by [Proposition 2.5](#). Similarly, we get $c_1(\mathcal{E}_i) \cdot c_1(\mathcal{E}_j) \cdot \alpha^{n-2} = 0$ for any indices $i, j \in I$. Combining this identity with [Proposition 2.4](#), we find

$$\begin{aligned} c_2(X) \cdot \alpha^{n-2} &= \sum_{i \in I} c_2(\mathcal{E}_i) \cdot \alpha^{n-2} \\ &= \sum_{i \in I} \frac{1}{2r_i} \Delta(\mathcal{E}_i) \cdot \alpha^{n-2} \\ &\geq 0 \end{aligned}$$

where the last inequality follows from [Proposition 3.4](#). The corollary is now proved. \square

Remark. It is instructive to return to the Kummer surface X of [Example 2.6](#) to see how the above proof fails if $\text{codim}_X(X_{\text{sg}}) = 2$. In that example, $\Delta(\mathcal{T}_X) = 4 \cdot (-8) - (-32) = 0$, but $c_1^2(\mathcal{T}_X)$ is nonzero and $c_2(\mathcal{T}_X)$ ends up being negative. Of course, the statement of [Proposition 3.4](#) itself still holds in this example.

Remark. In the spirit of [[Miy87a](#), Theorem 6.6], it would be interesting to prove the inequality $c_2(X) \cdot \alpha^{n-2} \geq 0$ under the weaker assumption that K_X is nef, although this is probably quite challenging.

3.C. The case of equality. In this subsection, we prove [\(B.2\)](#) and [\(B.3\)](#). That is, we discuss what happens if equality holds in the Bogomolov–Gieseker inequality. [Proposition 3.7](#) below asserts that if $c_1^2(\mathcal{E})$ and $c_2(\mathcal{E})$, seen as symmetric multilinear forms on $H^2(X, \mathbb{R})$, vanish against *one* Kähler class, then they vanish against *any* Kähler class. In case X has rational singularities, the Hodge structure on $H^2(X, \mathbb{C})$ is pure and the statement has a nice reformulation in Hodge-theoretic terms: the Chern classes in question vanish against $H^{1,1}(X)$.

Proposition 3.7 (Vanishing discriminant, I). *In [Setup 3.1](#), assume that \mathcal{E} is stable with respect to α . Let β be an arbitrary Kähler class.*

(3.7.1) *If $\Delta(\mathcal{E}) \cdot \alpha^{n-2} = 0$, then $\Delta(\mathcal{E}) \cdot \beta^{n-2} = 0$.*

(3.7.2) *If $c_1^2(\mathcal{E}) \cdot \alpha^{n-2} = c_2(\mathcal{E}) \cdot \alpha^{n-2} = 0$, then $c_1^2(\mathcal{E}) \cdot \beta^{n-2} = c_2(\mathcal{E}) \cdot \beta^{n-2} = 0$.*

Corollary 3.8 (Vanishing discriminant, II). *In [Setup 3.1](#), assume that X has rational singularities and that \mathcal{E} is stable with respect to α .*

(3.8.1) *If $\Delta(\mathcal{E}) \cdot \alpha^{n-2} = 0$, then we have*

$$\Delta(\mathcal{E}) \cdot \alpha_1 \cdots \alpha_{n-2} = 0$$

for any $\alpha_1, \dots, \alpha_{n-2} \in H^{1,1}(X, \mathbb{R}) := F^1 H^2(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$.

(3.8.2) *If $c_1^2(\mathcal{E}) \cdot \alpha^{n-2} = c_2(\mathcal{E}) \cdot \alpha^{n-2} = 0$, then*

$$c_1^2(\mathcal{E}) \cdot \alpha_1 \cdots \alpha_{n-2} = c_2(\mathcal{E}) \cdot \alpha_1 \cdots \alpha_{n-2} = 0$$

for any $\alpha_1, \dots, \alpha_{n-2} \in H^{1,1}(X, \mathbb{R})$.

Remark 3.9 (Topological vanishing). In [Corollary 3.8](#), it would be very desirable to show the vanishing on all of $H^2(X, \mathbb{R})$, if only because it would drastically simplify the proof of [Theorem A](#). The reason is that the vanishing would then be a topological statement and hence propagate to any locally trivial algebraic deformation. In particular, we would

not need the full force of the Decomposition Theorem, but only the affirmative answer to the Kodaira problem [BGL21, Theorem B].

If X is smooth, the classical argument shows that both classes are actually zero as elements of $H^4(X, \mathbb{R})$ and, in particular, they are also zero as symmetric multilinear forms on $H^2(X, \mathbb{R})$. However, we are not able to derive the latter conclusion in our setting. For instance, we do not know about the vanishing of $c_2(\mathcal{E}) \cdot (\sigma + \bar{\sigma})^{n-2}$ if $\sigma \in H^{2,0}(X)$ is nonzero. The underlying difficulty here is that reflexive forms, which naturally represent classes in $H^{p,0}(X)$, may not be smooth, i.e. they may not be the restriction of smooth forms under local embeddings $X \xrightarrow[\text{loc}]{} \mathbb{C}^N$. As a result, we do not have in general

$$(\sigma + \bar{\sigma})^{n-2} \lesssim \omega^{n-2}$$

if σ is a reflexive 2-form and ω a Kähler metric, thus preventing the argument below from going through.

Proof of Proposition 3.7. Denote by \widehat{E} the vector bundle on \widehat{X} associated to $\widehat{\mathcal{E}}$, and let $\omega \in \alpha$ (resp. $\omega' \in \beta$) be a Kähler metric on X . As in the proof of Proposition 3.4, we pick an Hermite–Einstein metric h_ε on E , i.e.

$$i\Theta(\widehat{E}, h_\varepsilon) \wedge \omega_\varepsilon^{n-1} = \lambda_\varepsilon \text{Id}_{\widehat{E}} \omega_\varepsilon^n.$$

where $\omega_\varepsilon = f^*\omega + \varepsilon\omega_{\widehat{X}}$, for $\omega_{\widehat{X}}$ an arbitrary Kähler metric on \widehat{X} , and $\lambda_\varepsilon = \frac{c_1(\widehat{E}) \cdot [\omega_\varepsilon]^{n-1}}{r[\omega_\varepsilon]^n}$. A standard computation (see e.g. [Kob87, Theorem 4.7]) using the Hermitian–Einstein condition shows that

$$\int_{\widehat{X}} \left(2rc_2(\widehat{E}, h_\varepsilon) - (r-1)c_1(\widehat{E}, h_\varepsilon)^2 \right) \wedge \omega_\varepsilon^{n-2} = c_n \int_{\widehat{E}} \|\Theta_\varepsilon^\circ\|_{h_\varepsilon, \omega_\varepsilon}^2 \cdot \omega_\varepsilon^n$$

and

$$\int_{\widehat{X}} c_1(\widehat{E}, h_\varepsilon)^2 \wedge \omega_\varepsilon^{n-2} = -c_n \int_{\widehat{X}} \left\| \frac{1}{r} \text{tr}_{\text{End}}(\Theta_\varepsilon) \cdot \text{Id}_{\widehat{E}} \right\|_{h_\varepsilon, \omega_\varepsilon}^2 \cdot \omega_\varepsilon^n$$

where $\Theta_\varepsilon := \Theta(\widehat{E}, h_\varepsilon)$ and $\Theta_\varepsilon^\circ = \Theta_\varepsilon - \frac{1}{r} \text{tr}_{\text{End}}(\Theta_\varepsilon) \cdot \text{Id}_{\widehat{E}}$ and $c_n = \frac{1}{4\pi^2 n(n-1)}$.

Proof of (3.7.2). Assume $c_1(\mathcal{E})^2 \cdot \alpha^{n-2} = c_2(\mathcal{E}) \cdot \alpha^{n-2} = 0$.

Given the assumptions above, one finds respectively

$$\begin{aligned} c_1(\widehat{E})^2 \cdot [\omega_\varepsilon]^{n-2} &= c_1(\widehat{E})^2 \cdot (f^*\alpha)^{n-2} + O(\varepsilon) = O(\varepsilon) \\ c_2(\widehat{E}) \cdot [\omega_\varepsilon]^{n-2} &= c_2(\widehat{E}) \cdot (f^*\alpha)^{n-2} + O(\varepsilon) = O(\varepsilon). \end{aligned}$$

Combined with the identities above, one finds that there is a constant $C_1 > 0$ such that

$$(3.9.1) \quad \int_{\widehat{X}} \|\Theta_\varepsilon\|_{h_\varepsilon, \omega_\varepsilon}^2 \cdot \omega_\varepsilon^n \leq C_1 \varepsilon.$$

Let us set $\omega'_\varepsilon := f^*\omega' + \varepsilon\omega_{\widehat{X}}$. Clearly, there exists $C_2 > 0$ such that $C_2^{-1}\omega_\varepsilon \leq \omega'_\varepsilon \leq C_2\omega_\varepsilon$, which yields another constant C_3 satisfying

$$C_3^{-1} \|\Theta_\varepsilon\|_{h_\varepsilon, \omega_\varepsilon}^2 \leq \|\Theta_\varepsilon\|_{h_\varepsilon, \omega'_\varepsilon}^2 \leq C_3 \|\Theta_\varepsilon\|_{h_\varepsilon, \omega_\varepsilon}^2.$$

Given (3.9.1), we find

$$(3.9.2) \quad \lim_{\varepsilon \rightarrow 0} \int_{\widehat{X}} \|\Theta_\varepsilon\|_{h_\varepsilon, \omega'_\varepsilon}^2 \cdot \omega'_\varepsilon^n = 0.$$

Now, write $c_2(\widehat{E}) \cdot (f^*\beta)^{n-2} = \lim_{\varepsilon \rightarrow 0} \int_{\widehat{X}} c_2(\widehat{E}, h_\varepsilon) \wedge \omega_\varepsilon^{m-2}$ and remember that up to some dimensional constants, one has $c_2(\widehat{E}, h_\varepsilon) = \mathrm{tr}_{\mathrm{End}}(\Theta_\varepsilon \wedge \Theta_\varepsilon) - \mathrm{tr}_{\mathrm{End}}(\Theta_\varepsilon)^2$ so that

$$\left| \int_{\widehat{X}} c_2(\widehat{E}, h_\varepsilon) \wedge \omega_\varepsilon^{m-2} \right| \leq C_4 \int_{\widehat{X}} \|\Theta_\varepsilon\|_{h_\varepsilon, \omega'_\varepsilon}^2 \cdot \omega_\varepsilon^m$$

and (3.7.2) follows from (3.9.2).

Proof of (3.7.1). Assume $\Delta(\mathcal{E}) \cdot \alpha^{n-2} = 0$.

Observe that by a standard computation, one has $\Delta(\mathrm{End}(\widehat{E})) = 2r^2 \Delta(\widehat{E})$. Moreover, $c_1(\mathrm{End}(\widehat{E})) = 0$ in $H^2(\widehat{X}, \mathbb{R})$ so that the assumptions yield

$$(3.9.3) \quad c_1(\mathrm{End}(\widehat{E}))^2 \cdot f^*\alpha^{n-2} = 0, \quad \text{and} \quad c_2(\mathrm{End}(\widehat{E})) \cdot f^*\alpha^{n-2} = 0.$$

The Hermite–Einstein metric h_ε on \widehat{E} with respect to ω_ε above yields a Hermite–Einstein metric $\widetilde{h}_\varepsilon$ on $\mathrm{End}(\widehat{E})$. From (3.9.3) and the proof of (3.7.2), it follows that the curvature tensor of $\widetilde{h}_\varepsilon$ converges to zero in L^2 norm with respect to ω_ε , or equivalently with respect to ω'_ε . (3.7.1) now follows. \square

Proof of Corollary 3.8. Let $f: Y \rightarrow X$ be a resolution of singularities, where Y is Kähler. Since X has rational singularities, we have the following diagram, where the horizontal maps are induced by multiplication with $i = \sqrt{-1}$:

$$\begin{array}{ccc} H^2(Y, \mathbb{R}) & \longrightarrow & H^2(Y, \mathcal{O}_Y) \\ f^* \uparrow & & \uparrow \wr \\ H^2(X, \mathbb{R}) & \xrightarrow{\beta} & H^2(X, \mathcal{O}_X) \end{array}$$

This implies immediately that $H^{1,1}(X, \mathbb{R}) \subset \ker \beta$. The other inclusion follows because the morphism f^* is strict with respect to the Hodge filtration. Now let PH_X be the sheaf of real-valued pluriharmonic functions on X . By [BGL21, Proposition 6.3], the natural map $H^1(X, \mathrm{PH}_X) \rightarrow H^2(X, \mathbb{R})$ is injective with image equal to $\ker \beta$ and we obtain

$$H^{1,1}(X, \mathbb{R}) = H^1(X, \mathrm{PH}_X).$$

Recall also that the Kähler cone is open in $H^1(X, \mathrm{PH}_X)$ by [GK20, Proposition 3.8].

For the actual proof of Corollary 3.8, we focus on (3.8.1), since the proof of (3.8.2) is entirely similar. By (3.7.1) and the above remarks, the assumptions of (3.8.1) first imply that $\Delta(\mathcal{E}) \cdot \beta^{n-2} = 0$ for any $\beta \in H^{1,1}(X, \mathbb{R})$. The vanishing for possibly different classes $\alpha_1, \dots, \alpha_{n-2} \in H^{1,1}(X, \mathbb{R})$ then follows by a standard polarization argument [Tho14]. \square

4. THE SECOND CHERN CLASS OF IHS VARIETIES

In this section, we discuss the second Chern class of singular holomorphic symplectic varieties X . The main result is as follows. Note that here we do not assume X to be smooth in codimension two.

Proposition 4.1 (Positivity of c_2). *In Setup 4.2 below, we have $c_2(X) \cdot b^{2n-2} > 0$ for any class $b \in H^2(X, \mathbb{R})$ with $q_X(b) > 0$. In particular, this holds whenever b is a Kähler class.*

Remark. If X satisfies the condition $\text{codim}_X(X_{\text{sg}}) \geq 3$ (which, by results of Namikawa and Kaledin, is equivalent to $\text{codim}_X(X_{\text{sg}}) \geq 4$), the proof of [Proposition 4.1](#) can be somewhat simplified. To be more precise, from [Proposition 4.4](#) we only need the existence of the Fujiki constant C , but not its positivity and deformation invariance. Instead, we can obtain $C \geq 0$ from [Corollary 3.6](#) and $C \neq 0$ from [Lemma 4.3](#). In particular, in this case we do not need the fact that X can be deformed to a projective variety.

For the rest of this section, we work in the following setup.

Setup 4.2. Let X be an IHS variety of complex dimension $2n \geq 2$ in the sense of [Definition 3.5](#). We denote by $\sigma \in H^0(X, \Omega_X^{[2]})$ a holomorphic symplectic 2-form, which is unique up to a scalar. Furthermore, we denote by $q_X: H^2(X, \mathbb{C}) \rightarrow \mathbb{C}$ the BBF (= Beauville–Bogomolov–Fujiki) form of X . We will always normalize q_X in such a way that it comes from an *indivisible* integral quadratic form $H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$, cf. [\[BL18, Lemma 5.7\]](#). With this convention, the BBF form is a topological invariant of X . In particular, it is invariant under locally trivial deformations.

Lemma 4.3 (Non-vanishing of c_2). *We have $c_2(X) \neq 0$ on $H^2(X, \mathbb{R})$. That is, there exists a class $a \in H^2(X, \mathbb{R})$ such that $c_2(X) \cdot a^{2n-2} \neq 0$.*

Proof. By [\[BL18, Corollary 1.4\]](#), X admits a locally trivial algebraic approximation $\mathfrak{X} \rightarrow \Delta$, where Δ is smooth. Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be the simultaneous resolution obtained in [\[GS20, Lemma 4.2\]](#), and let X_t, Y_t be the fibres of the respective maps. Note that the fibrewise resolutions $Y_t \rightarrow X_t$ are then minimal in codimension two. For any $t \neq 0$, we have a commutative diagram

$$\begin{array}{ccccc} H^*(Y_0, \mathbb{R}) & \xleftarrow{\sim} & H^*(\mathfrak{Y}, \mathbb{R}) & \xrightarrow{\sim} & H^*(Y_t, \mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \\ H^*(X_0, \mathbb{R}) & \xleftarrow{\sim} & H^*(\mathfrak{X}, \mathbb{R}) & \xrightarrow{\sim} & H^*(X_t, \mathbb{R}) \end{array}$$

where the horizontal maps are isomorphisms due to the topological triviality of the locally trivial maps $\mathfrak{Y} \rightarrow \Delta$ and $\mathfrak{X} \rightarrow \Delta$. Note that the relative tangent sheaf $\mathcal{T}_{\mathfrak{Y}/\Delta}$ is locally free, so we can consider its second Chern class $c_2(\mathcal{T}_{\mathfrak{Y}/\Delta}) \in H^4(\mathfrak{Y}, \mathbb{R})$. By construction, this class gets mapped to $c_2(Y_0)$ and $c_2(Y_t)$, respectively, under the upper horizontal maps in the above diagram. Since $X_0 = X$, this shows that the following conditions are equivalent:

$$(4.3.1) \text{ For any } a \in H^2(X, \mathbb{R}), \text{ we have } c_2(X) \cdot a^{2n-2} = 0.$$

$$(4.3.2) \text{ For any } a \in H^2(\mathfrak{X}, \mathbb{R}), \text{ we have } c_2(\mathcal{T}_{\mathfrak{Y}/\Delta}) \cdot a^{2n-2} = 0 \in H^{4n}(\mathfrak{Y}, \mathbb{R}) = \mathbb{R}.$$

$$(4.3.3) \text{ For any } a \in H^2(X_t, \mathbb{R}), \text{ we have } c_2(X_t) \cdot a^{2n-2} = 0.$$

We now argue by contradiction and assume that condition [\(4.3.1\)](#) is satisfied. Let $t \in \Delta$ be such that X_t is projective. Pick an arbitrary ample divisor H on X_t . Then $c_2(X_t) \cdot H^{2n-2} = 0$ by [\(4.3.3\)](#). Applying [\[GKP16b, Theorem 7.1\]](#), we obtain a finite quasi-étale cover $A_t \rightarrow X_t$, where A_t is an abelian variety.² By the argument in the proof

²The cited reference makes the extra assumption that X_t be \mathbb{Q} -factorial. However, this is not used in the proof. In fact, a general complete intersection surface $S \subset X_t$ will again have canonical singularities and in particular be rational and \mathbb{Q} -factorial. Therefore [\[GKP16b, Lemma 7.2\]](#) can be applied to S .

of [CGGN20, Lemma 8.8], this quasi-étale cover can be extended to a quasi-étale cover $\mathfrak{A} \rightarrow \mathfrak{X}$ such that the induced map $\mathfrak{A} \rightarrow \Delta$ is again locally trivial. In the situation at hand, $\mathfrak{A} \rightarrow \Delta$ will even be smooth since A_t is smooth. By the local constancy of Hodge numbers in smooth families, we see that

$$h^0(A_0, \Omega_{A_0}^1) = h^0(A_t, \Omega_{A_t}^1) = 2n > 0.$$

On the other hand, $h^0(A_0, \Omega_{A_0}^1) = 0$ because $A_0 \rightarrow X_0 = X$ is a quasi-étale cover and X is IHS. This is the desired contradiction. \square

The following result is an adaptation of a well-known property of Chern classes on smooth IHS manifolds, cf. e.g. [O'G12, Proposition 2.2]. It has to be noticed that the proof given in [BL18, Proposition 5.20] has a different flavor. Also, the first result in this direction (under stronger assumptions) appears to be [Mat01, Lemma 2.4].

Proposition 4.4 (Fujiki relations for c_2). *There exists a positive rational constant $C = C(X) \in \mathbb{Q}^+$, called the Fujiki constant with respect to $c_2(X)$, such that for any $a \in H^2(X, \mathbb{R})$, we have*

$$c_2(X) \cdot a^{2n-2} = C \cdot q_X(a)^{n-1}.$$

Furthermore, $C(X)$ is constant in locally trivial families. More precisely, if $\mathfrak{X} \rightarrow B$ is a locally trivial deformation over a (reduced and connected) base B , then $C(X_t) = C(X_s)$ for all $t, s \in B$.

Proof. We mimic the proof of [O'G12], claiming no originality. Let $\mathfrak{X} \rightarrow \Delta$ be a representative of $\text{Def}^{\text{lt}}(X)$ (Δ is thus smooth according to [BL18, Theorem 4.7]). We consider the period map

$$\pi : \begin{cases} \Delta & \longrightarrow \Omega(X) := \{x \in \mathbb{P}(H^2(X, \mathbb{C})) \mid q_X(x) = 0\} \\ t & \longmapsto \mathbb{P}(H^{2,0}(X_t)) \end{cases}$$

that is a local isomorphism as stated in [BL18, Proposition 5.5]. Let us now consider the degree $2(n-1)$ homogeneous polynomial defined by

$$G(\alpha) := c_2(X) \cdot \alpha^{2n-2}$$

for any $\alpha \in H^2(X, \mathbb{C})$. Type considerations (and Gauss–Manin invariance of the second Chern class) yield that

$$\forall \alpha_1, \dots, \alpha_{n-2} \in H^2(X, \mathbb{C}), \quad \int_{X_t} c_2(X_t) \wedge \sigma_t^n \wedge \alpha_1 \wedge \dots \wedge \alpha_{n-2} = 0$$

where σ_t is the symplectic form on X_t (for $t \in \Delta$). The latter can be interpreted as saying that all the derivatives of G up to order $n-2$ vanish along the image of π . The Zariski closure of the image of π being $\Omega(X)$, the zero locus of the quadratic polynomial q_X (see above), we infer that G has to be of the form

$$(4.4.1) \quad G = C \cdot q_X^{n-1}$$

with $C \in \mathbb{C}$ a constant. **Lemma 4.3** immediately implies $C \neq 0$, and by evaluating (4.4.1) at some $a \in H^2(X, \mathbb{Q})$ with $q_X(a) \neq 0$, we see that $C \in \mathbb{Q}$. This argument also shows that C remains constant in a locally trivial family, using the fact that q_X is unchanged under such a deformation.

It remains to be seen that $C \geq 0$. To this end, let $\mathfrak{X} \rightarrow \Delta$ be as above, and pick $t \in \Delta$ such that $Y := X_t$ is projective. Let H be an ample divisor on Y . By the above observations, it suffices to show that $C(Y) \geq 0$. Since $q_Y(H) > 0$, this is equivalent to $c_2(Y) \cdot H^{2n-2} \geq 0$, which is what we will show. By [Miy87b, Corollary 8.6], the cotangent sheaf $\Omega_Y^{[1]}$ is generically nef, hence H -semistable, as $c_1(Y) = 0$. By [Fle84, Theorem 1.2], the restriction $\Omega_Y^{[1]}|_S$ to a general complete intersection surface $S \subset Y$ remains semistable. Since S has only quotient singularities, $\Omega_Y^{[1]}|_S$ is automatically a \mathbb{Q} -vector bundle and we may apply [Kaw92, Lemma 2.5]. This yields $c_2(Y) \cdot H^{2n-2} \geq 0$, as desired. \square

Proof of Proposition 4.1. The first part of the statement is clear from Proposition 4.4. It only remains to be seen that $q_X(b) > 0$ for any Kähler class $b \in H^2(X, \mathbb{R})$. To this end, note that the “usual” degree zero Fujiki relations on X yield $\int_X b^{2n} = \mu q_X(b)^n$ for some $\mu > 0$, see [Sch20, Theorem 2]. We conclude by noting that the left-hand side is strictly positive. Alternatively, one may also resort to the original definition of q_X as a certain integral on a resolution of X . There, one uses the Hodge–Riemann bilinear relations, cf. the proof of [GS21, (4.5.1)]. \square

5. CHARACTERIZATION OF TORUS QUOTIENTS

We are now in a position to prove Theorem A.

Setup 5.1. Let X be a normal compact Kähler space of dimension n with klt singularities and trivial first Chern class $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Assume that X is smooth in codimension two.

Theorem 5.2. *In Setup 5.1, assume that there exists a Kähler class $a \in H^2(X, \mathbb{R})$ such that $c_2(X) \cdot a^{n-2} = 0$. Then there exists a complex torus T and a holomorphic action of a finite group $G \subset T$, free in codimension two, such that $X \cong T/G$.*

Proof. We proceed in three steps.

Step 1: Reduction to the split case. By [BGL21, Theorem A], there exists a finite quasi-étale cover $p: \tilde{X} \rightarrow X$ which decomposes as

$$\tilde{X} = T \times \prod_{i \in I} Y_i \times \prod_{j \in J} Z_j,$$

where T is a complex torus, the Y_i are ICY (= irreducible Calabi–Yau) varieties and the Z_j are IHS (= irreducible holomorphic symplectic) varieties, cf. Definition 3.5. In particular, \tilde{X} has canonical singularities because all its factors do.

Since X is smooth in codimension two, the quasi-étale map p is necessarily étale in codimension two, and hence \tilde{X} is still smooth in codimension two. Therefore

$$c_2(\tilde{X}) \cdot (p^*a)^{n-2} = \deg(p) \cdot c_2(X) \cdot a^{n-2} = 0$$

by [GK20, Proposition 5.6]. Also, p^*a is a Kähler class by [GK20, Proposition 3.6]. Finally, if the conclusion of Theorem 5.2 holds for \tilde{X} , then it also holds for X , by taking Galois closure [GK20, Lemma 7.4]. We may and will therefore replace X by \tilde{X} (and a by p^*a) for the remaining argument. In order to finish the proof, it is sufficient to show

that $I = J = \emptyset$ in the above notation.

Step 2: Chern class computations. The following calculation gets slightly messy due to the fact that we need to work on a resolution, but the basic idea is very simple. — Since $H^1(Y_i, \mathbb{R}) = H^1(Z_j, \mathbb{R}) = 0$ for all $i \in I, j \in J$, the Künneth formula implies that the class a decomposes as

$$(5.2.1) \quad a = p_T^* a_T + \sum_{i \in I} p_i^* b_i + \sum_{j \in J} p_j^* c_j,$$

where a_T (resp. b_i, c_j) is a Kähler class on the torus T (resp. on Y_i, Z_j) and the maps p_\bullet are the projections. We pick strong log resolutions $\widehat{Y}_i \rightarrow Y_i, \widehat{Z}_j \rightarrow Z_j$ and set $f: \widehat{X} \rightarrow X$ where $\widehat{X} := T \times \prod_{i \in I} \widehat{Y}_i \times \prod_{j \in J} \widehat{Z}_j$, with projections \widehat{p}_\bullet . The first Chern class of each factor is (either zero or) supported on the exceptional divisor of f and as X is smooth in codimension two and $c_2(T) = 0$, we have

$$(5.2.2) \quad c_2(\widehat{X}) \cdot f^*(a^{n-2}) = \sum_{i \in I} \widehat{p}_i^* c_2(\widehat{Y}_i) \cdot f^*(a^{n-2}) + \sum_{j \in J} \widehat{p}_j^* c_2(\widehat{Z}_j) \cdot f^*(a^{n-2}).$$

Here we have used [Lemma 2.7](#) to see that the “mixed terms” such as $\widehat{p}_i^* c_1(\widehat{Y}_i) \cdot \widehat{p}_j^* c_1(\widehat{Z}_j)$ vanish against $f^*(a^{n-2})$. Putting together [\(5.2.2\)](#) and the pullback of [\(5.2.1\)](#) to \widehat{X} , and remembering our very definition of c_2 , as a result one gets

$$(5.2.3) \quad c_2(X) \cdot a^{n-2} = \sum_{i \in I} \lambda_i c_2(Y_i) \cdot b_i^{\dim Y_i - 2} + \sum_{j \in J} \mu_j c_2(Z_j) \cdot c_j^{\dim Z_j - 2}.$$

for some positive constants $\lambda_i, \mu_j > 0$. More precisely, for $i_0 \in I, j_0 \in J$, we have up to some binomial coefficient $\lambda_{i_0} = a_T^{\dim T} \cdot \prod_{i \neq i_0} b_i^{\dim Y_i} \cdot \prod_{j \in J} c_j^{\dim Z_j}$ and similarly for μ_{j_0} .

We have seen in [Corollary 3.6](#) that the numbers $c_2(Y_i) \cdot b_i^{\dim Y_i - 2}$ and $c_2(Z_j) \cdot c_j^{\dim Z_j - 2}$ are non-negative. As the left-hand side of [\(5.2.3\)](#) is zero, they must therefore all vanish.

Step 3: Eliminating the non-torus factors. By [Proposition 4.1](#), one must have $J = \emptyset$. Assuming $i \in I \neq \emptyset$, note that the ICY variety Y_i is projective by Kodaira’s Embedding Theorem because $H^2(Y_i, \mathcal{O}_{Y_i}) = 0$. The vanishing $c_2(Y_i) \cdot b_i^{\dim Y_i - 2} = 0$ implies the vanishing of $c_2(Y_i)$ against any ample class by [Proposition 3.7](#). Applying [[GKP16b](#), Theorem 7.1] yields that Y_i is a torus quotient, contradicting the definition of ICY varieties. Hence $I = \emptyset$. In particular, $X = T$ and the theorem is proved. \square

Proof of Theorem A. The implication “[\(A.1\)](#) \Rightarrow [\(A.2\)](#)” follows immediately from [Theorem 5.2](#). More precisely, as X is smooth in codimension two, the resolution f is clearly minimal in codimension two, being an isomorphism over X_{reg} . Therefore the assumptions of [\(A.1\)](#) imply that $c_2(X) \cdot \alpha^{n-2} = 0$.

The other direction “[\(A.2\)](#) \Rightarrow [\(A.1\)](#)” can be proven exactly as in [[GK20](#), Theorem 1.1]: let $\pi: T \rightarrow T/G \cong X$ be the quotient map. Since $\omega_X^{[|G|]} \cong \mathcal{O}_X$, we have $c_1(X) = 0 \in H^2(X, \mathbb{R})$. By [[Var89](#), Chapter IV, Corollary 1.2], X is Kähler. As π is étale in codimension two, we have

$$0 = c_2(T) \cdot \pi^* \alpha^{n-2} = \deg(\pi) \cdot c_2(X) \cdot \alpha^{n-2}$$

for any Kähler class α on X , according to [GK20, Proposition 5.6]. **Theorem A** is thus proved. \square

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UNIV RENNES, CNRS, IRMAR — UMR 6625, F-35000 RENNES, FRANCE ET INSTITUT UNIVERSITAIRE DE FRANCE

Email address: benoit.claudon@univ-rennes1.fr

URL: perso.univ-rennes1.fr/benoit.claudon/

LEHRSTUHL FÜR MATHEMATIK I, UNIVERSITÄT BAYREUTH, 95440 BAYREUTH, GERMANY

Email address: patrick.graf@uni-bayreuth.de

URL: www.graficland.uni-bayreuth.de

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ PAUL SABATIER, 31062 TOULOUSE CEDEX 9, FRANCE

Email address: henri.guenancia@math.cnrs.fr

URL: hguenancia.perso.math.cnrs.fr/