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Supplementary material

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Supplementary material

Sara Elbouch

We show the intermediate steps that have led to the computation of the variance. We detail each moment that intervenes in the computation of the variance in the scalar case ($p = 1$). For the sake of readability in the bivariate case ($p = 2$), we detail those moments for the particular case of a bivariate-embedding of a scalar process (because having $\gamma_{11}(\tau) = \gamma_{22}(\tau)$ reduces the length of the expressions involved). In addition, we expose the expression of the mean of \hat{B}_3 in the particular case of a tri-variate embedding ($p = 3$) of a scalar process. If requested, the variance of \hat{B}_3 can be given in a handwritten format.

1. Computation of the variance of \hat{B}_p

1.1. Scalar case ($p = 1$)

$$\mathbb{E}\{A_{nn}^2 A_{kk}^2\} = 9 + 72 \frac{S(n-k)^2}{S^2} + 24 \frac{S(n-k)^4}{S^4} \quad (1)$$

$$\begin{aligned} \mathbb{E}\{A_{jj}^2 A_{nn} A_{ni}^2\} &= 9 + 72 \frac{S(n-j)^2}{S^2} + 24 \frac{S(n-j)^4}{S^4} + 36 \frac{S(n-i)}{S^2} + 36 \frac{S(j-i)}{S} \\ &+ 144 \frac{S(n-j)^2 S(n-i)^2}{S^4} + 144 \frac{S(n-j)^2 S(j-i)^2}{S^4} \\ &+ 288 \frac{S(n-j) S(n-i) S(j-i)}{S^3} \\ &+ 192 \frac{S(n-j)^3 S(n-i) S(j-i)}{C_0^2} \end{aligned} \quad (2)$$

$$\begin{aligned} \mathbb{E}\{A_{nn} A_{kk} A_{ni}^2 A_{kj}^2\} &= 9 + 18 \frac{S(i-j)^2}{S^2} + 36 \frac{S(k-i)^2}{S^2} + 36 \frac{S(k-j)^2}{S^2} + 36 \frac{S(n-i)^2}{S^2} \\ &+ 36 \frac{S(n-j)^2}{S^2} + 72 \frac{S(n-k)^2}{S^2} + 24 \frac{S(n-k)^4}{S^4} \\ &+ 144 \frac{S(k-i) \gamma_0(i-j) S(k-j)}{C_0^3} + 144 \frac{S(n-j) S(n-i) S(i-j)}{S^3} \\ &+ 48 \frac{S(n-k)^4 S(i-j)^2}{S^6} + 192 \frac{S(n-k)^3 S(n-i) S(k-i)}{S^5} \\ &+ 192 \frac{S(n-k)^3 S(k-i) S(n-i)}{C_0^5} \end{aligned} \quad (3)$$

$$\begin{aligned}\mathbb{E}\{A_{kk}^2 A_{ni}^2 A_{nj}^2\} &= 9 + 18 \frac{S(i-j)^2}{S^2} + 36 \frac{S(k-i)^2}{S^2} + 36 \frac{S(k-j)^2}{S^2} + 36 \frac{S(n-i)^2}{S^2} \\ &\quad + 36 \frac{S(n-j)^2}{S^2} + 72 \frac{S(n-k)^2}{S^2} + 24 \frac{S(n-k)^4}{S^4}\end{aligned}\quad (4)$$

$$\begin{aligned}\mathbb{E}\{A_{\ell\ell}^2 A_{nm} A_{nj} A_{jk} A_{kn}\} &= 9 + 18 \frac{S(j-k)^2}{S^2} + 36 \frac{S(\ell-j)^2}{S^2} + 36 \frac{S(\ell-k)^2}{S^2} + 36 \frac{S(n-j)^2}{S^2} \\ &\quad + 36 \frac{S(n-k)^2}{S^2} + 72 \frac{S(n-\ell)^2}{S^2} + 24 \frac{S(n-\ell)^4}{S^4}\end{aligned}\quad (5)$$

$$\begin{aligned}\mathbb{E}\{A_{kk} A_{kl}^2 A_{ni}^2 A_{nj}^2\} &= 9 + 24 \frac{S(n-k)^4}{S^4} + 36 \frac{S(k-i)^2}{S^2} + 36 \frac{S(n-i)^2}{S^2} + 36 \frac{S(k-j)^2}{S^2} \\ &\quad + 36 \frac{S(n-j)^2}{S^2} + 36 \frac{S(k-\ell)^2}{S^2} + 36 \frac{S(n-\ell)^2}{S^2} + 18 \frac{S(i-j)^2}{S^2} \\ &\quad + 18 \frac{S(i-\ell)^2}{S^2} + 18 \frac{S(j-\ell)^2}{S^2} + 72 \frac{S(n-k)^2}{S^2}\end{aligned}\quad (6)$$

$$\begin{aligned}\mathbb{E}\{A_{ni}^2 A_{m\ell}^2 A_{nj}^2 A_{mq}^2\} &= 9 + 24 \frac{S(n-m)^4}{S^4} + 36 \frac{S(n-i)^2}{S^2} + 36 \frac{S(m-i)^2}{S^2} + 36 \frac{S(n-j)^2}{S^2} \\ &\quad + 36 \frac{S(m-j)^2}{S^2} + 36 \frac{S(m-\ell)^2}{S^2} + 36 \frac{S(n-\ell)^2}{S^2} + 36 \frac{S(k-n)^2}{S^2} \\ &\quad + 36 \frac{S(n-q)^2}{S^2} + 72 \frac{S(n-m)^2}{S^2} + 18 \frac{S(i-j)^2}{S^2} + 18 \frac{S(i-\ell)^2}{S^2} \\ &\quad + 18 \frac{S(i-q)^2}{S^2} + 18 \frac{S(j-\ell)^2}{S^2} + 18 \frac{S(j-q)^2}{S^2} + 18 \frac{S(\ell-q)^2}{S^2}\end{aligned}\quad (7)$$

$$\begin{aligned}
\mathbb{E}\{A_{nn}A_{nj}A_{jk}A_{kn}A_{mm}A_{ml}A_{\ell q}A_{qm}\} &= 9 + 24\frac{S(n-m)^4}{S^4} + 36\frac{S(n-i)^2}{S^2} + 36\frac{S(m-i)^2}{S^2} \\
&+ 36\frac{S(n-j)^2}{S^2} + 36\frac{S(m-j)^2}{S^2} + 36\frac{S(m-\ell)^2}{S^2} \\
&+ 36\frac{S(n-\ell)^2}{S^2} + 36\frac{S(k-n)^2}{S^2} + 36\frac{S(n-q)^2}{S^2} \\
&+ 72\frac{S(n-m)^2}{S^2} + 18\frac{S(i-j)^2}{S^2} + 18\frac{S(i-\ell)^2}{S^2} \\
&+ 18\frac{S(i-q)^2}{S^2} + 18\frac{S(j-\ell)^2}{S^2} + 18\frac{S(j-q)^2}{S^2} \\
&+ 18\frac{S(\ell-q)^2}{S^2} \tag{8}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{A_{ml}^2A_{mq}^2A_{nn}A_{nj}A_{jk}A_{kn}\} &= 9 + 24\frac{S(n-m)^4}{S^4} + 36\frac{S(n-i)^2}{S^2} + 36\frac{S(m-i)^2}{S^2} \\
&+ 36\frac{S(n-j)^2}{S^2} + 36\frac{S(m-j)^2}{S^2} + 36\frac{S(m-\ell)^2}{S^2} \\
&+ 36\frac{S(n-\ell)^2}{S^2} + 36\frac{S(k-n)^2}{S^2} + 36\frac{S(n-q)^2}{S^2} \\
&+ 36\frac{S(n-\ell)^2}{S^2} + 72\frac{S(n-m)^2}{S^2} + 18\frac{S(i-j)^2}{S^2} \\
&+ 18\frac{S(i-\ell)^2}{S^2} + 18\frac{S(i-q)^2}{S^2} + 18\frac{S(j-\ell)^2}{S^2} \\
&+ 18\frac{S(j-q)^2}{S^2} + 18\frac{S(\ell-q)^2}{S^2} \tag{9}
\end{aligned}$$

The second-order moment of \hat{B}_1 reads:

$$\boxed{\mathbb{E}\{\hat{B}_1^2\} = 9 - \frac{36}{N^2} \sum_{n,k} \frac{S(n-k)^2}{S^2} + \frac{24}{N^2} \sum_{n,k} \frac{S(n-k)^4}{S^4} + o\left(\frac{1}{N}\right)} \tag{10}$$

and by using the Proposition 5.1. $Var\{\hat{B}_1\} = \mathbb{E}\{\hat{B}_1^2\} - 9 + \frac{36}{N^2} \sum_{n,k} \frac{S(n-k)^2}{S^2} + o\left(\frac{1}{N}\right)$

$$\boxed{Var\{\hat{B}_1\} = \frac{24}{N^2} \sum_{n,k=1}^N \frac{S(n-k)^4}{S^4} = \frac{24}{N} \left[1 + \frac{2}{N} \sum_{\tau=1}^{T-1} (N-\tau) \frac{S(\tau)^4}{S^4}\right] + o\left(\frac{1}{N}\right)} \tag{11}$$

1.2. Bivariate case ($p = 2$)

For the sake of readability, we restrict details about computation to the case of a bivariate embedding of scalar process (because this particular case where $\gamma_{11}(\tau) = \gamma_{22}(\tau)$ simplifies the length of the expressions).

$$\begin{aligned}
\mathbb{E}\{A_{nn}^2 A_{kk}^2\} &= 64 + \frac{1}{(C_0^2 - C_1^2)^4} \left[64(\gamma_0^2(n-k) - \gamma_1^2(n-k)\gamma_{-1}^2(n-k))^2 \right. \\
&\quad + 24(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))^2 + 96\gamma_0^2(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
&\quad - 96\gamma_1^2(n-k)\gamma_{-1}^2(n-k)]C_0^4 + [256\gamma_0^4(n-k) \\
&\quad + 96 \times (5\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
&\quad - 128\gamma_1(n-k)\gamma_{-1}(n-k)(\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))]C_0^2C_1^2 \\
&\quad + [64\gamma_0^4(n-k) + 256\gamma_0^2(n-k)\gamma_1(n-k)\gamma_{-1}(n-k) + 64\gamma_1^2(n-k)\gamma_{-1}^2(n-k)] \\
&\quad \times C_1^4 - 192C_0C_1 \times \gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))[(2\gamma_0^2(n-k) \\
&\quad + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k))C_0^2 + 2(\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))C_1^2] \\
&\quad + \frac{128}{(C_0^2 - C_1^2)^2} \left[C_0^2[2\gamma_0^2(n-k) + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k)] \right. \\
&\quad + C_1^2 \times [2\gamma_0^2(n-k) + 2(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))] \\
&\quad \left. - 4C_0C_1\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k)) \right] \tag{12}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{A_{jj}^2 A_{nm} A_{ni}^2\} = & 64 + \frac{1}{(C_0^2 - C_1^2)^4} \left[64(\gamma_0^2(n-j) - \gamma_1^2(n-j)\gamma_{-1}^2(n-j))^2 \right. \\
& + 24(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j))^2 + 96\gamma_0^2(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))^2 \\
& - 96\gamma_1^2(n-j)\gamma_{-1}^2(n-j) \Big] C_0^4 + [256\gamma_0^4(n-j) \\
& + 96 \times (5\gamma_0^2(n-j) + \gamma_1(n-j)\gamma_{-1}(n-j))(\gamma_1(n-j) + \gamma_{-1}(n-j))^2 \\
& - 128\gamma_1(n-j)\gamma_{-1}(n-j)(\gamma_0^2(n-j) + \gamma_1(n-j)\gamma_{-1}(n-j))] C_0^2 C_1^2 \\
& + [64\gamma_0^4(n-k) + 256\gamma_0^2(n-k)\gamma_1(n-k)\gamma_{-1}(n-j) + 64\gamma_1^2(n-j)\gamma_{-1}^2(n-j)] \\
& \times C_1^4 - 192C_0C_1 \times \gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))[(2\gamma_0^2(n-k) \\
& + \gamma_1^2(n-j) + \gamma_{-1}^2(n-j))C_0^2 + 2(\gamma_0^2(n-j) + \gamma_1(n-j)\gamma_{-1}(n-j))C_1^2] \\
& + \frac{128}{(C_0^2 - C_1^2)^2} \left[C_0^2[2\gamma_0^2(n-j) + \gamma_1^2(n-j) + \gamma_{-1}^2(n-j)] \right. \\
& + C_1^2 \times [2\gamma_0^2(n-j) + 2(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j))] \\
& \left. - 4C_0C_1\gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j)) \right] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-i) + 16(\gamma_1^2(n-i) + \gamma_{-1}^2(n-i)) \\
& + 96\gamma_1(n-k)\gamma_{-1}(n-i)) + C_1^2(96(\gamma_1^2(n-i) + \gamma_{-1}^2(n-i)) \\
& + 128\gamma_1(n-i)\gamma_{-1}(n-i) + 128\gamma_0^2(n-i)) \\
& - 448C_0C_1\gamma_0(n-i)(\gamma_1(n-i) + \gamma_{-1}(n-i))] + \\
& \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(j-i) + 64(\gamma_1^2(j-i) + \gamma_{-1}^2(j-i))) \\
& + C_1^2(128\gamma_0^2(j-i) + 128\gamma_1(j-i)\gamma_{-1}(j-i)) \\
& \left. - 256C_0C_1\gamma_0(j-i)(\gamma_1(j-i) + \gamma_{-1}(j-i))] \right]
\end{aligned} \tag{13}$$

$$\begin{aligned}
\mathbb{E}\{A_{nn}A_{kk}A_{ni}^2A_{kj}^2\} &= 64 + \frac{1}{(C_0^2 - C_1^2)^4} \left[[64(\gamma_0^2(n-k) - \gamma_1^2(n-k)\gamma_{-1}^2(n-k))^2 \right. \\
&\quad + 24(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))^2 + 96\gamma_0^2(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
&\quad - 96\gamma_1^2(n-k)\gamma_{-1}^2(n-k)]C_0^4 + [256\gamma_0^4(n-k) \\
&\quad + 96 \times (5\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
&\quad - 128\gamma_1(n-k)\gamma_{-1}(n-k)(\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))]C_0^2C_1^2 \\
&\quad + [64\gamma_0^4(n-k) + 256\gamma_0^2(n-k)\gamma_1(n-k)\gamma_{-1}(n-k) + 64\gamma_1^2(n-k) \\
&\quad \times \gamma_{-1}^2(n-k)]C_1^4 - 192C_0C_1 \times \gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))] \\
&\quad (2\gamma_0^2(n-k) + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k))C_0^2 + 2(\gamma_0^2(n-k) \\
&\quad + \gamma_1(n-k)\gamma_{-1}(n-k))C_1^2] \\
&\quad + \frac{128}{(C_0^2 - C_1^2)^2} \left[C_0^2[2\gamma_0^2(n-k) + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k)] \right. \\
&\quad + C_1^2 \times [2\gamma_0^2(n-k) + 2(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))] \\
&\quad \left. - 4C_0C_1\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k)) \right] \\
&\quad + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(i-j) + 32(\gamma_1^2(i-j) + \gamma_{-1}^2(i-j))) \\
&\quad + C_1^2(64\gamma_0^2(i-j) + 64\gamma_1(i-j)\gamma_{-1}(i-j)) \\
&\quad - 128C_0C_1\gamma_0(i-j)(\gamma_1(i-j) + \gamma_{-1}(i-j))] \\
&\quad + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n-i) + 64(\gamma_1^2(n-i) + \gamma_{-1}^2(n-i))) \\
&\quad + C_1^2(128\gamma_0^2(n-i) + 128\gamma_1(n-i)\gamma_{-1}(n-i)) \\
&\quad - 256C_0C_1\gamma_0(n-i)(\gamma_1(n-i) + \gamma_{-1}(n-i))] \\
&\quad + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n-j) + 64(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j))) \\
&\quad + C_1^2(128\gamma_0^2(n-j) + 128\gamma_1(n-j)\gamma_{-1}(n-j)) \\
&\quad - 256C_0C_1\gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))] \\
&\quad + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(k-i) + 64(\gamma_1^2(k-i) + \gamma_{-1}^2(k-i))) \\
&\quad + C_1^2(128\gamma_0^2(k-i) + 128\gamma_1(k-i)\gamma_{-1}(k-i)) \\
&\quad - 256C_0C_1\gamma_0(k-i)(\gamma_1(k-i) + \gamma_{-1}(k-i))] \\
&\quad + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(k-j) + 64(\gamma_1^2(k-j) + \gamma_{-1}^2(k-j))) \\
&\quad + C_1^2(128\gamma_0^2(k-j) + 128\gamma_1(k-j)\gamma_{-1}(k-j)) \\
&\quad - 256C_0C_1\gamma_0(k-j)(\gamma_1(k-j) + \gamma_{-1}(k-j))]
\end{aligned}$$

$$\begin{aligned}
& + C_1^2(128\gamma_0^2(k-j) + 128\gamma_1(k-j)\gamma_{-1}(k-j)) \\
& \quad - 256C_0C_1\gamma_0(k-j)(\gamma_1(k-j) + \gamma_{-1}(k-j))] \tag{14} \\
\mathbb{E}\{A_{kk}^2 A_{ni}^2 A_{nj}^2\} = & 64 + \frac{1}{(C_0^2 - C_1^2)^4} \left[64(\gamma_0^2(n-k) - \gamma_1^2(n-k)\gamma_{-1}^2(n-k))^2 \right. \\
& + 24(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))^2 + 96\gamma_0^2(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
& - 96\gamma_1^2(n-k)\gamma_{-1}^2(n-k)]C_0^4 + [256\gamma_0^4(n-k) \\
& + 96 \times (5\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
& - 128\gamma_1(n-k)\gamma_{-1}(n-k)(\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))]C_0^2C_1^2 \\
& + [64\gamma_0^4(n-k) + 256\gamma_0^2(n-k)\gamma_1(n-k)\gamma_{-1}(n-k) + 64\gamma_1^2(n-k)\gamma_{-1}^2(n-k)] \\
& \times C_1^4 - 192C_0C_1 \times \gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))[(2\gamma_0^2(n-k) \\
& + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k))C_0^2 + 2(\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))C_1^2] \\
& + \frac{128}{(C_0^2 - C_1^2)^2} \left[C_0^2[2\gamma_0^2(n-k) + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k)] \right. \\
& + C_1^2 \times [2\gamma_0^2(n-k) + 2(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))] \\
& \left. - 4C_0C_1\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k)) \right] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(16(\gamma_1^2(i-j) + \gamma_{-1}^2(i-j)) + 32\gamma_1(i-j)\gamma_{-1}(i-j) \\
& + 128\gamma_0^2(i-j)) + C_1^2(32(\gamma_1^2(i-j) + \gamma_{-1}^2(i-j)) + 64\gamma_1(i-j)\gamma_{-1}(i-j) \\
& + 64\gamma_1(i-j)\gamma_{-1}(i-j)) - 192C_0C_1\gamma_0(i-j)(\gamma_1(i-j) + \gamma_{-1}(i-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-i) + 16(\gamma_1^2(n-i) + \gamma_{-1}^2(n-i)) \\
& + 96\gamma_1(n-i)\gamma_{-1}(n-i)) + C_1^2(96(\gamma_1^2(n-i) + \gamma_{-1}^2(n-i)) \\
& + 128\gamma_1(n-i)\gamma_{-1}(n-i) + 128\gamma_0^2(n-i)) \\
& - 448C_0C_1\gamma_0(n-i)(\gamma_1(n-i) + \gamma_{-1}(n-i))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-j) + 16(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j)) \\
& + 96\gamma_1(n-j)\gamma_{-1}(n-j)) + C_1^2(96(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j)) \\
& + 128\gamma_1(n-j)\gamma_{-1}(n-j) + 128\gamma_0^2(n-j)) \\
& - 448C_0C_1\gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(k-i) + 64(\gamma_1^2(k-i) + \gamma_{-1}^2(k-i))) \\
& + C_1^2(128\gamma_0^2(k-i) + 128\gamma_1(k-i)\gamma_{-1}(k-i)) \\
& - 256C_0C_1\gamma_0(k-i)(\gamma_1(k-i) + \gamma_{-1}(k-i))]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(k-j) + 64(\gamma_1^2(k-j) + \gamma_{-1}^2(k-j))) \\
& + C_1^2(128\gamma_0^2(k-j) + 128\gamma_1(k-j)\gamma_{-1}(k-j)) \\
& - 256C_0C_1\gamma_0(k-j)(\gamma_1(k-j) + \gamma_{-1}(k-j))] \tag{16}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{A_{\ell\ell}^2 A_{nn} A_{nj} A_{jk} A_{kn}\} & = 64 + \frac{1}{(C_0^2 - C_1^2)^4} \left[[64(\gamma_0^2(n-\ell) - \gamma_1^2(n-\ell)\gamma_{-1}^2(n-\ell))^2 \right. \\
& + 24(\gamma_1^2(n-\ell) + \gamma_{-1}^2(n-\ell))^2 + 96\gamma_0^2(n-\ell)(\gamma_1(n-\ell) + \gamma_{-1}(n-\ell))^2 \\
& - 96\gamma_1^2(n-\ell)\gamma_{-1}^2(n-\ell)]C_0^4 + [256\gamma_0^4(n-\ell) \\
& + 96 \times (5\gamma_0^2(n-\ell) + \gamma_1(n-\ell)\gamma_{-1}(n-\ell))(\gamma_1(n-\ell) + \gamma_{-1}(n-\ell))^2 \\
& - 128\gamma_1(n-\ell)\gamma_{-1}(n-\ell)(\gamma_0^2(n-\ell) + \gamma_1(n-\ell)\gamma_{-1}(n-\ell))]C_0^2C_1^2 \\
& + [64\gamma_0^4(n-\ell) + 256\gamma_0^2(n-\ell)\gamma_1(n-\ell)\gamma_{-1}(n-\ell) + 64\gamma_1^2(n-\ell)\gamma_{-1}^2(n-\ell) \\
& \times C_1^4 - 192C_0C_1 \times \gamma_0(n-\ell)(\gamma_1(n-\ell) + \gamma_{-1}(n-\ell))[(2\gamma_0^2(n-\ell) \\
& + \gamma_1^2(n-\ell) + \gamma_{-1}^2(n-\ell))C_0^2 + 2(\gamma_0^2(n-\ell) + \gamma_1(n-\ell)\gamma_{-1}(n-\ell))C_1^2] \\
& + \frac{128}{(C_0^2 - C_1^2)^2} \left[C_0^2[2\gamma_0^2(n-\ell) + \gamma_1^2(n-\ell) + \gamma_{-1}^2(n-\ell)] \right. \\
& + C_1^2 \times [2\gamma_0^2(n-\ell) + 2(\gamma_1^2(n-\ell) + \gamma_{-1}^2(n-\ell))] \\
& \left. - 4C_0C_1\gamma_0(n-\ell)(\gamma_1(n-\ell) + \gamma_{-1}(n-\ell)) \right] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(192\gamma_0^2(j-k) + 64\gamma_1(j-k)\gamma_{-1}(j-k)) \\
& + C_1^2(64\gamma_0^2(j-k) + 64(\gamma_1^2(j-k) + \gamma_{-1}^2(j-k)) \\
& + 64\gamma_1(j-k)\gamma_{-1}(j-k)) \\
& - 256C_0C_1\gamma_0(j-k)(\gamma_1(j-k) + \gamma_{-1}(j-k))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-k) + 16(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k)) \\
& + 96\gamma_1(n-k)\gamma_{-1}(n-k)) + C_1^2(96(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k)) \\
& + 128\gamma_1(n-k)\gamma_{-1}(n-k) + 128\gamma_0^2(n-k)) \\
& - 448C_0C_1\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-j) + 16(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j))) \\
& + 96\gamma_1(n-j)\gamma_{-1}(n-j) + C_1^2(96(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j))) \\
& + 128\gamma_1(n-j)\gamma_{-1}(n-j) + 128\gamma_0^2(n-j)) \\
& - 448C_0C_1\gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(\ell-k) + 64(\gamma_1^2(\ell-k) + \gamma_{-1}^2(\ell-k))) \\
& + C_1^2(128\gamma_0^2(\ell-k) + 128\gamma_1(\ell-k)\gamma_{-1}(\ell-k)) \\
& - 256C_0C_1\gamma_0(\ell-k)(\gamma_1(\ell-k) + \gamma_{-1}(\ell-k))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(\ell-j) + 64(\gamma_1^2(\ell-j) + \gamma_{-1}^2(\ell-j))) \\
& + C_1^2(128\gamma_0^2(\ell-k) + 128\gamma_1(\ell-j)\gamma_{-1}(\ell-j)) \\
& - 256C_0C_1\gamma_0(\ell-j)(\gamma_1(\ell-j) + \gamma_{-1}(\ell-j))] \tag{17}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{A_{kk}A_{kl}^2A_{ni}^2A_{nj}^2\} & = 64 + \frac{1}{(C_0^2 - C_1^2)^4} \left[[64(\gamma_0^2(n-k) - \gamma_1^2(n-k)\gamma_{-1}^2(n-k))^2 \right. \\
& + 24(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))^2 + 96\gamma_0^2(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
& - 96\gamma_1^2(n-k)\gamma_{-1}^2(n-k)]C_0^4 + [256\gamma_0^4(n-k) \\
& + 96 \times (5\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 \\
& - 128\gamma_1(n-k)\gamma_{-1}(n-k)(\gamma_0^2(n-k) + \gamma_1(n-k)\gamma_{-1}(n-k))]C_0^2C_1^2 \\
& + [64\gamma_0^4(n-k) + 256\gamma_0^2(n-k)\gamma_1(n-k)\gamma_{-1}(n-k) + 64\gamma_1^2(n-k) \\
& \gamma_{-1}^2(n-k)]C_1^4 - 192C_0C_1 \times \gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k)) [\\
& (2\gamma_0^2(n-k) + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k))C_0^2 + 2(\gamma_0^2(n-k) + \gamma_1(n-k) \\
& \gamma_{-1}(n-k))C_1^2] \left. \right] \\
& + \frac{128}{(C_0^2 - C_1^2)^2} \left[C_0^2[2\gamma_0^2(n-k) + \gamma_1^2(n-k) + \gamma_{-1}^2(n-k)] \right. \\
& + C_1^2 \times [2\gamma_0^2(n-k) + 2(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k))] \\
& \left. - 4C_0C_1\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k)) \right] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(i-\ell) + 32(\gamma_1^2(i-\ell) + \gamma_{-1}^2(i-\ell)^2)) \\
& + C_1^2(64\gamma_0^2(i-\ell) + 64\gamma_1(i-\ell)\gamma_{-1}(i-\ell)) \\
& - 128C_0C_1\gamma_0(i-\ell)(\gamma_1(i-\ell) + \gamma_{-1}(i-\ell))]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(\ell - j) + 32(\gamma_1^2(\ell - j) + \gamma_{-1}(\ell - j)^2)) \\
& + C_1^2(64\gamma_0^2(\ell - j) + 64\gamma_1(\ell - j)\gamma_{-1}(\ell - j)) \\
& - 128C_0C_1\gamma_0(\ell - j)(\gamma_1(\ell - j) + \gamma_{-1}(\ell - j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(16(\gamma_1^2(i - j) + \gamma_{-1}^2(i - j)) + 32\gamma_1(i - j)\gamma_{-1}(i - j) \\
& + 128\gamma_0^2(i - j)) + C_1^2(32(\gamma_1^2(i - j) + \gamma_{-1}^2(i - j)) \\
& + 64\gamma_1(i - j)\gamma_{-1}(i - j) + 64\gamma_1(i - j)\gamma_{-1}(i - j)) \\
& - 192C_0C_1\gamma_0(i - j)(\gamma_1(i - j) + \gamma_{-1}(i - j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n - j) + 16(\gamma_1^2(n - j) + \gamma_{-1}^2(n - j)) \\
& + 96\gamma_1(n - j)\gamma_{-1}(n - j)) + C_1^2(96(\gamma_1^2(n - j) + \gamma_{-1}^2(n - j)) + 128\gamma_1(n - j)\gamma_{-1}(n - j) \\
& + 128\gamma_0^2(n - j)) - 448C_0C_1\gamma_0(n - j)(\gamma_1(n - j) + \gamma_{-1}(n - j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n - i) + 16(\gamma_1^2(n - i) + \gamma_{-1}^2(n - i)) \\
& + 96\gamma_1(n - i)\gamma_{-1}(n - i)) + C_1^2(96(\gamma_1^2(n - i) + \gamma_{-1}^2(n - i)) + 128\gamma_1(n - i)\gamma_{-1}(n - i) \\
& + 128\gamma_0^2(n - i)) - 448C_0C_1\gamma_0(n - i)(\gamma_1(n - i) + \gamma_{-1}(n - i))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(k - \ell) + 16(\gamma_1^2(k - \ell) + \gamma_{-1}^2(k - \ell)) \\
& + 96\gamma_1(k - \ell)\gamma_{-1}(k - \ell)) + C_1^2(96(\gamma_1^2(k - \ell) + \gamma_{-1}^2(k - \ell)) + 128\gamma_1(k - \ell)\gamma_{-1}(k - \ell) \\
& + 128\gamma_0^2(k - \ell)) - 448C_0C_1\gamma_0(k - \ell)(\gamma_1(k - \ell) + \gamma_{-1}(k - \ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n - \ell) + 64(\gamma_1^2(n - \ell) + \gamma_{-1}^2(n - \ell))) \\
& + C_1^2(128\gamma_0^2(n - \ell) + 128\gamma_1(n - \ell)\gamma_{-1}(n - \ell)) \\
& - 256C_0C_1\gamma_0(n - \ell)(\gamma_1(n - \ell) + \gamma_{-1}(n - \ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n - \ell) + 64(\gamma_1^2(n - \ell) + \gamma_{-1}^2(n - \ell))) \\
& + C_1^2(128\gamma_0^2(n - \ell) + 128\gamma_1(n - \ell)\gamma_{-1}(n - \ell)) \\
& - 256C_0C_1\gamma_0(n - \ell)(\gamma_1(n - \ell) + \gamma_{-1}(n - \ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(k - i) + 64(\gamma_1^2(k - i) + \gamma_{-1}^2(k - i))) \\
& + C_1^2(128\gamma_0^2(k - i) + 128\gamma_1(k - i)\gamma_{-1}(k - i)) \\
& - 256C_0C_1\gamma_0(k - i)(\gamma_1(k - i) + \gamma_{-1}(k - i))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(k - j) + 64(\gamma_1^2(k - j) + \gamma_{-1}^2(k - j)))
\end{aligned}$$

$$\begin{aligned}
& + C_1^2(128\gamma_0^2(k-j) + 128\gamma_1(k-j)\gamma_{-1}(k-j)) \\
& \quad - 256C_0C_1\gamma_0(k-j)(\gamma_1(k-j) + \gamma_{-1}(k-j))] \quad (18) \\
\mathbb{E}\{A_{mm}A_{ml}^2A_{nn}A_{nj}A_{jk}A_{kn}\} & = \mathbb{E}\{A_{nn}^2A_{mm}^2\} + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(\ell-j) \\
& + 32(\gamma_1^2(\ell-j) + \gamma_{-1}(\ell-j)^2)) + C_1^2(64\gamma_0^2(\ell-j) \\
& + 64\gamma_1(\ell-j)\gamma_{-1}(\ell-j)) \\
& - 128C_0C_1\gamma_0(\ell-j)(\gamma_1(\ell-j) + \gamma_{-1}(\ell-j)) \\
& + 32(\gamma_1^2(\ell-k) + \gamma_{-1}(\ell-k)^2)) + C_1^2(64\gamma_0^2(\ell-k) \\
& + 64\gamma_1(\ell-k)\gamma_{-1}(\ell-k)) \\
& - 128C_0C_1\gamma_0(\ell-k)(\gamma_1(\ell-k) + \gamma_{-1}(\ell-k)) \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(192\gamma_0^2(j-k) + 64\gamma_1(j-k)\gamma_{-1}(j-k)) \\
& + C_1^2(64\gamma_0^2(j-k) + 64(\gamma_1^2(j-k) + \gamma_{-1}^2(j-k)) \\
& + 64\gamma_1(j-k)\gamma_{-1}(j-k)) \\
& - 256C_0C_1\gamma_0(j-k)(\gamma_1(j-k) + \gamma_{-1}(j-k))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(m-\ell) + 16(\gamma_1^2(m-\ell) \\
& + \gamma_{-1}^2(m-\ell)) + 96\gamma_1(m-\ell)\gamma_{-1}(m-\ell)) \\
& + C_1^2(96(\gamma_1^2(m-\ell) + \gamma_{-1}^2(m-\ell)) \\
& + 128\gamma_1(m-\ell)\gamma_{-1}(m-\ell) + 128\gamma_0^2(m-\ell)) \\
& - 448C_0C_1\gamma_0(m-\ell)(\gamma_1(m-\ell) + \gamma_{-1}(m-\ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-j) + 16(\gamma_1^2(n-j) \\
& + \gamma_{-1}^2(n-j)) + 96\gamma_1(n-j)\gamma_{-1}(n-j)) \\
& + C_1^2(96(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j)) \\
& + 128\gamma_1(n-j)\gamma_{-1}(n-j) + 128\gamma_0^2(n-j)) \\
& - 448C_0C_1\gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-k) + 16(\gamma_1^2(n-k) \\
& + \gamma_{-1}^2(n-k)) + 96\gamma_1(n-k)\gamma_{-1}(n-k)) \\
& + C_1^2(96(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k)) \\
& + 128\gamma_1(n-k)\gamma_{-1}(n-k) + 128\gamma_0^2(n-k)) \\
& - 448C_0C_1\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))] \quad (19)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n - \ell) + 64(\gamma_1^2(n - \ell) + \gamma_{-1}^2(n - \ell))) \\
& + C_1^2(128\gamma_0^2(n - \ell) + 128\gamma_1(n - \ell)\gamma_{-1}(n - \ell)) \\
& - 256C_0C_1\gamma_0(n - \ell)(\gamma_1(n - \ell) + \gamma_{-1}(n - \ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(m - j) + 64(\gamma_1^2(m - j) + \gamma_{-1}^2(m - j))) \\
& + C_1^2(128\gamma_0^2(m - j) + 128\gamma_1(m - j)\gamma_{-1}(m - j)) \\
& - 256C_0C_1\gamma_0(m - j)(\gamma_1(m - j) + \gamma_{-1}(m - j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(m - k) + 64(\gamma_1^2(m - k) + \gamma_{-1}^2(m - k))) \\
& + C_1^2(128\gamma_0^2(m - k) + 128\gamma_1(m - k)\gamma_{-1}(m - k)) \\
& - 256C_0C_1\gamma_0(m - k)(\gamma_1(m - k) + \gamma_{-1}(m - k))] \tag{20}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{A_{ni}^2 A_{ml}^2 A_{nj}^2 A_{mq}^2\} & = \mathbb{E}\{A_{nm}^2 A_{mm}^2\} + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(i - \ell) \\
& + 32(\gamma_1^2(i - \ell) + \gamma_{-1}(i - \ell)^2)) + C_1^2(64\gamma_0^2(i - \ell) \\
& + 64\gamma_1(i - \ell)\gamma_{-1}(i - \ell)) \\
& - 128C_0C_1\gamma_0(i - \ell)(\gamma_1(i - \ell) + \gamma_{-1}(i - \ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(\ell - j) \\
& + 32(\gamma_1^2(\ell - j) + \gamma_{-1}(\ell - j)^2)) + C_1^2(64\gamma_0^2(\ell - j) \\
& + 64\gamma_1(\ell - j)\gamma_{-1}(\ell - j)) \\
& - 128C_0C_1\gamma_0(\ell - j)(\gamma_1(\ell - j) + \gamma_{-1}(\ell - j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(i - q) \\
& + 32(\gamma_1^2(i - q) + \gamma_{-1}(i - q)^2)) + C_1^2(64\gamma_0^2(i - q) \\
& + 64\gamma_1(i - q)\gamma_{-1}(i - q)) \\
& - 128C_0C_1\gamma_0(i - q)(\gamma_1(i - q) + \gamma_{-1}(i - q))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(16(\gamma_1^2(\ell - q) + \gamma_{-1}^2(\ell - q)) + 32\gamma_1(\ell - q)\gamma_{-1}(\ell - q) \\
& + 128\gamma_0^2(\ell - q)) + C_1^2(32(\gamma_1^2(\ell - q) + \gamma_{-1}^2(\ell - q)) \\
& + 64\gamma_1(\ell - q)\gamma_{-1}(\ell - q) + 64\gamma_1(\ell - q)\gamma_{-1}(\ell - q)) \\
& - 192C_0C_1\gamma_0(\ell - q)(\gamma_1(\ell - q) + \gamma_{-1}(\ell - q))]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(16(\gamma_1^2(i-j) + \gamma_{-1}^2(i-j)) + 32\gamma_1(i-j)\gamma_{-1}(i-j)) \\
& \quad + 128\gamma_0^2(i-j) + C_1^2(32(\gamma_1^2(i-j) + \gamma_{-1}^2(i-j)) \\
& \quad + 64\gamma_1(i-j)\gamma_{-1}(i-j) + 64\gamma_1(i-j)\gamma_{-1}(i-j)) \\
& \quad - 192C_0C_1\gamma_0(i-j)(\gamma_1(i-j) + \gamma_{-1}(i-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-i) + 16(\gamma_1^2(n-i) + \gamma_{-1}^2(n-i)) \\
& \quad + 96\gamma_1(n-i)\gamma_{-1}(n-i) + C_1^2(96(\gamma_1^2(n-i) + \gamma_{-1}^2(n-i)) \\
& \quad + 128\gamma_1(n-i)\gamma_{-1}(n-i) + 128\gamma_0^2(n-i)) \\
& \quad - 448C_0C_1\gamma_0(n-i)(\gamma_1(n-i) + \gamma_{-1}(n-i))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-j) + 16(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j)) \\
& \quad + 96\gamma_1(n-j)\gamma_{-1}(n-j) + C_1^2(96(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j)) \\
& \quad + 128\gamma_1(n-j)\gamma_{-1}(n-j) + 128\gamma_0^2(n-j)) \\
& \quad - 448C_0C_1\gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(m-\ell) + 16(\gamma_1^2(m-\ell) + \gamma_{-1}^2(m-\ell)) \\
& \quad + 96\gamma_1(m-\ell)\gamma_{-1}(m-\ell) + C_1^2(96(\gamma_1^2(m-\ell) + \gamma_{-1}^2(m-\ell)) \\
& \quad + 128\gamma_1(m-\ell)\gamma_{-1}(m-\ell) + 128\gamma_0^2(m-\ell)) \\
& \quad - 448C_0C_1\gamma_0(m-\ell)(\gamma_1(m-\ell) + \gamma_{-1}(m-\ell))] + \\
& \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(m-q) + 16(\gamma_1^2(m-q) + \gamma_{-1}^2(m-q)) \\
& \quad + 96\gamma_1(m-q)\gamma_{-1}(m-q) + C_1^2(96(\gamma_1^2(m-q) + \gamma_{-1}^2(m-q)) \\
& \quad + 128\gamma_1(m-q)\gamma_{-1}(m-q) + 128\gamma_0^2(m-q)) \\
& \quad - 448C_0C_1\gamma_0(m-q)(\gamma_1(m-q) + \gamma_{-1}(m-q))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n-\ell) + 64(\gamma_1^2(n-\ell) + \gamma_{-1}^2(n-\ell))) \\
& \quad + C_1^2(128\gamma_0^2(n-\ell) + 128\gamma_1(n-\ell)\gamma_{-1}(n-\ell)) \\
& \quad - 256C_0C_1\gamma_0(n-\ell)(\gamma_1(n-\ell) + \gamma_{-1}(n-\ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n-q) + 64(\gamma_1^2(n-q) + \gamma_{-1}^2(n-q))) \\
& \quad + C_1^2(128\gamma_0^2(n-q) + 128\gamma_1(n-q)\gamma_{-1}(n-q))]
\end{aligned}$$

$$\begin{aligned}
& -256C_0C_1\gamma_0(n-q)(\gamma_1(n-q) + \gamma_{-1}(n-q))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n-q) + 64(\gamma_1^2(n-q) + \gamma_{-1}^2(n-q))) \\
& + C_1^2(128\gamma_0^2(n-q) + 128\gamma_1(n-q)\gamma_{-1}(n-q)) \\
& - 256C_0C_1\gamma_0(n-q)(\gamma_1(n-q) + \gamma_{-1}(n-q))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(m-j) + 64(\gamma_1^2(m-j) + \gamma_{-1}^2(m-j))) \\
& + C_1^2(128\gamma_0^2(m-j) + 128\gamma_1(m-j)\gamma_{-1}(n-q)) \\
& - 256C_0C_1\gamma_0(m-j)(\gamma_1(m-j) + \gamma_{-1}(m-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(m-i) + 64(\gamma_1^2(m-i) + \gamma_{-1}^2(m-i))) \\
& + C_1^2(128\gamma_0^2(m-i) + 128\gamma_1(m-i)\gamma_{-1}(m-i)) \\
& - 256C_0C_1\gamma_0(m-i)(\gamma_1(m-i) + \gamma_{-1}(m-i))] \tag{21}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\{A_{nn}A_{nj}A_{jk}A_{kn}A_{mm}A_{m\ell}A_{lq}A_{qm}\} &= \mathbb{E}\{A_{nn}^2A_{mm}^2\} + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(192 \\
&\gamma_0^2(j-k) + 64\gamma_1(j-k)\gamma_{-1}(j-k)) + C_1^2 \\
&(64\gamma_0^2(j-k) + 64(\gamma_1^2(j-k) + \gamma_{-1}^2(j-k)) \\
&+ 64\gamma_1(j-k)\gamma_{-1}(j-k)) \\
&- 256C_0C_1\gamma_0(j-k)(\gamma_1(j-k) \\
&+ \gamma_{-1}(j-k))] \\
&+ \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(192 \\
&\gamma_0^2(\ell-q) + 64\gamma_1(\ell-q)\gamma_{-1}(\ell-q)) + C_1^2 \\
&(64\gamma_0^2(\ell-q) + 64(\gamma_1^2(\ell-q) + \gamma_{-1}^2(\ell-q)) \\
&+ 64\gamma_1(\ell-q)\gamma_{-1}(\ell-q)) \\
&- 256C_0C_1\gamma_0(\ell-q)(\gamma_1(\ell-q) \\
&+ \gamma_{-1}(\ell-q))] \\
&+ \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(\ell-k) + 32(\gamma_1^2(\ell-k) \\
&+ \gamma_{-1}(\ell-k)^2)) + C_1^2(64\gamma_0^2(\ell-k) + 64\gamma_1(\ell-k) \\
&\gamma_{-1}(\ell-k)) \\
&- 128C_0C_1\gamma_0(\ell-k)(\gamma_1(\ell-k) + \gamma_{-1}(\ell-k))] \\
&+ \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(j-\ell) + 32(\gamma_1^2(j-\ell) \\
&+ \gamma_{-1}(j-\ell)^2)) + C_1^2(64\gamma_0^2(j-\ell) + 64\gamma_1(j-\ell) \\
&\gamma_{-1}(j-\ell)) \\
&- 128C_0C_1\gamma_0(j-\ell)(\gamma_1(j-\ell) + \gamma_{-1}(j-\ell))] \\
&+ \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(j-q) + 32(\gamma_1^2(j-q) \\
&+ \gamma_{-1}(j-q)^2)) + C_1^2(64\gamma_0^2(j-q) + 64\gamma_1(j-q) \\
&\gamma_{-1}(j-q)) \\
&- 128C_0C_1\gamma_0(j-q)(\gamma_1(j-q) + \gamma_{-1}(j-q))] \\
&+ \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(64\gamma_0^2(q-k) + 32(\gamma_1^2(q-k) \\
&+ \gamma_{-1}(q-k)^2)) + C_1^2(64\gamma_0^2(q-k) + 64\gamma_1(q-k) \\
&\gamma_{-1}(q-k)) \\
&- 128C_0C_1\gamma_0(q-k)(\gamma_1(q-k) + \gamma_{-1}(q-k))]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-j) + 16(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j)) + 96\gamma_1(n-j)\gamma_{-1}(n-j)) \\
& + C_1^2(96(\gamma_1^2(n-j) + \gamma_{-1}^2(n-j)) + 128\gamma_1(n-j)\gamma_{-1}(n-j) + 128\gamma_0^2(n-j)) \\
& - 448C_0C_1\gamma_0(n-j)(\gamma_1(n-j) + \gamma_{-1}(n-j))] \\
& \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(n-k) + 16(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k)) + 96\gamma_1(n-k)\gamma_{-1}(n-k)) \\
& + C_1^2(96(\gamma_1^2(n-k) + \gamma_{-1}^2(n-k)) + 128\gamma_1(n-k)\gamma_{-1}(n-k) + 128\gamma_0^2(n-k)) \\
& - 448C_0C_1\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))] \\
& \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(m-\ell) + 16(\gamma_1^2(m-\ell) + \gamma_{-1}^2(m-\ell)) + 96\gamma_1(m-\ell)\gamma_{-1}(m-\ell)) \\
& + C_1^2(96(\gamma_1^2(m-\ell) + \gamma_{-1}^2(m-\ell)) + 128\gamma_1(m-\ell)\gamma_{-1}(m-\ell) + 128\gamma_0^2(m-\ell)) \\
& - 448C_0C_1\gamma_0(m-\ell)(\gamma_1(m-\ell) + \gamma_{-1}(m-\ell))] \\
& \frac{1}{(C_0^2 - C_1^2)^2} \times [C_0^2(320\gamma_0^2(m-q) + 16(\gamma_1^2(m-q) + \gamma_{-1}^2(m-q)) + 96\gamma_1(m-q)\gamma_{-1}(m-q)) \\
& + C_1^2(96(\gamma_1^2(m-q) + \gamma_{-1}^2(m-q)) + 128\gamma_1(m-q)\gamma_{-1}(m-q) + 128\gamma_0^2(m-q)) \\
& - 448C_0C_1\gamma_0(m-q)(\gamma_1(m-q) + \gamma_{-1}(m-q))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(m-j) + 64(\gamma_1^2(m-j) + \gamma_{-1}^2(m-j))) + C_1^2(128\gamma_0^2(m-j) \\
& + 128\gamma_1(m-j)\gamma_{-1}(m-j)) - 256C_0C_1\gamma_0(m-j)(\gamma_1(m-j) + \gamma_{-1}(m-j))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n-\ell) + 64(\gamma_1^2(n-\ell) + \gamma_{-1}^2(n-\ell))) + C_1^2(128\gamma_0^2(n-\ell) \\
& + 128\gamma_1(n-\ell)\gamma_{-1}(n-\ell)) - 256C_0C_1\gamma_0(n-\ell)(\gamma_1(n-\ell) + \gamma_{-1}(n-\ell))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(n-q) + 64(\gamma_1^2(n-q) + \gamma_{-1}^2(n-q))) + C_1^2(128\gamma_0^2(n-q) \\
& + 128\gamma_1(n-q)\gamma_{-1}(n-q)) - 256C_0C_1\gamma_0(n-q)(\gamma_1(n-q) + \gamma_{-1}(n-q))] \\
& + \frac{1}{(C_0^2 - C_1^2)^2} [C_0^2(128\gamma_0^2(m-k) + 64(\gamma_1^2(m-k) + \gamma_{-1}^2(m-k))) + C_1^2(128\gamma_0^2(m-k) \\
& + 128\gamma_1(m-k)\gamma_{-1}(m-k)) - 256C_0C_1\gamma_0(m-k)(\gamma_1(m-k) + \gamma_{-1}(m-k))] \tag{22}
\end{aligned}$$

We sum now over all the time indices according to proposition 5.1.

$$\boxed{\mathbb{E}\{\hat{B}_2^2\} = 64 - \frac{1}{N^2} \sum_{n,k} \frac{Q_1(n-k)}{(C_0^2 - C_1^2)^2} + \frac{1}{N^2} \sum_{n,k} \frac{Q_2(n-k)}{(C_0^2 - C_1^2)^4}} \tag{23}$$

$$\begin{aligned}
Q_1(n-k) &= [32(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 + 256\gamma_0(n-k)^2]C_0^2 + \\
&\quad [64(\gamma_1(n-k) + \gamma_{-1}(n-k))^2 + 128\gamma_0(n-k)^2]C_1^2 \\
&\quad - 384\gamma_0(n-k)(\gamma_1(n-k) + \gamma_{-1}(n-k))
\end{aligned} \tag{24}$$

$$\mathbb{E}[\hat{B}_2]^2 = 64 - \frac{1}{N^2} \sum_{n,k} \frac{Q_1(n-k)}{(C_0^2 - C_1^2)^2} + o(1/N) \tag{25}$$

$$\begin{aligned}
\text{Var}[\hat{B}_2] &= \mathbb{E}[\hat{B}_2^2] - \mathbb{E}[\hat{B}_2]^2 \\
&= \frac{1}{N^2} \sum_{n,k} \frac{Q_2(n-k)}{(C_0^2 - C_1^2)^4} + o\left(\frac{1}{N}\right)
\end{aligned}$$

2. Mean of tri-variate embedding of a scalar process ($p = 3$)

$$\boxed{\mathbb{E}[\hat{B}_3] = 15 - \frac{90}{N} + \frac{2}{N^2} \sum_{s=1}^{N-1} \frac{(N-s)Q(s)}{(C_0 - C_2)^2(C_0^2 + C_2C_0 - 2C_1^2)^2} + o\left(\frac{1}{N}\right)} \tag{26}$$

Where :

$$\begin{aligned}
Q(s) = & C_0^4[-90\gamma_0(s)^2 + 4(\gamma_1(s) + \gamma_{-1}(s))^2 + 2(\gamma_2(s) + \gamma_{-2}(s))^2 \\
& - 20(\gamma_2(s)\gamma_{-2}(s) + 2\gamma_1(s)\gamma_{-1}(s))] \\
& - 4C_0^2C_1^2[16(\gamma_1(s) + \gamma_{-1}(s))^2 + (\gamma_2(s) + \gamma_{-2}(s))^2 \\
& - 20\gamma_1(s)\gamma_{-1}(s) - 10\gamma_2(s)\gamma_{-2}(s) + 20\gamma_0(s)(\gamma_2(s) + \gamma_{-2}(s)) \\
& - 24\gamma_0(s)^2] - 4C_0^2C_2^2[-12\gamma_0(s)^2 + (\gamma_1(s) + \gamma_{-1}(s))^2 + 4(\gamma_2(s) + \gamma_{-2}(s))^2 \\
& - 5(\gamma_2(s)\gamma_{-2}(s) + 2\gamma_1(s)\gamma_{-1}(s))] - 4C_1^2C_2^2[14\gamma_0(s)^2 + 6(\gamma_1(s) + \gamma_{-1}(s))^2 \\
& + 20\gamma_1(s)\gamma_{-1}(s) - \gamma_0(s)(\gamma_2(s) + \gamma_{-2}(s))] + 4C_1^2C_0C_2[18\gamma_0(s)^2 \\
& + 22(\gamma_1 + \gamma_{-1})^2 + 4(\gamma_2(s) + \gamma_{-2}(s))^2 - 10\gamma_2(s)\gamma_{-2}(s) \\
& - 8\gamma_0(s)(\gamma_2(s) + \gamma_{-2}(s))] + 8C_0^3C_1[4\gamma_0(s)(\gamma_1(s) + \gamma_{-1}(s)) \\
& + 4(\gamma_2(s)\gamma_{-1}(s) + \gamma_{-2}(s)\gamma_1(s)) - (\gamma_2(s)\gamma_1(s) + \gamma_{-2}(s)\gamma_{-1}(s))] \\
& - 56C_0^2C_1C_2[3\gamma_0(s)(\gamma_1(s) + \gamma_{-1}(s)) + (\gamma_2(s)\gamma_1(s) \\
& + \gamma_2(s)\gamma_{-1}(s) + \gamma_{-2}(s)\gamma_1(s) + \gamma_{-2}(s)\gamma_{-1}(s))] + 8C_0^3C_2[9\gamma_0(s)(\gamma_2 + \gamma_{-2}) \\
& + 2(\gamma_1(s) + \gamma_{-1}(s))^2 - 5\gamma_1(s)\gamma_{-1}(s)] - 8C_0C_2^3[2\gamma_0(s)(\gamma_2 + \gamma_{-2}) \\
& + 2(\gamma_1(s) + \gamma_{-1}(s))^2 - 5\gamma_1(s)\gamma_{-1}(s)] + 56C_1C_2^3\gamma_0(s)^2 \\
& + 8C_1^3(C_2 - C_0)[8\gamma_0(s)(\gamma_1(s) + \gamma_{-1}(s)) - 9(\gamma_2(s)\gamma_1(s) \\
& + \gamma_{-2}\gamma_{-1}) + (\gamma_{-1}(s)\gamma_2(s) + \gamma_{-2}(s)\gamma_1(s))] + 8C_0C_1C_2^2[-4\gamma_0(s)(\gamma_1(s) \\
& + \gamma_{-1}(s)) + 8(\gamma_2(s)\gamma_1(s) + \gamma_{-2}(s)\gamma_{-1}(s)) + 3(\gamma_{-2}(s)\gamma_1(s) + \gamma_2(s)\gamma_{-1}(s))] \\
& + C_1^4[-14(\gamma_2(s) + \gamma_{-2}(s))^2 - 56\gamma_0(s)^2 + 56\gamma_0(s)(\gamma_2(s) + \gamma_{-2}(s))] \\
& - 14C_2^4\gamma_0(s)^2 \tag{27}
\end{aligned}$$