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Non-Asymptotic Analysis of Stochastic Approximation Algorithms for Streaming Data

Antoine Godichon-Baggioni, Nicklas Werge*, Olivier Wintenberger

LPSM, Sorbonne Université, 4 place Jussieu, 75005 Paris, France

Abstract

We consider the stochastic approximation problem in a streaming framework where an objective is minimized through unbiased estimates of its gradients. In this streaming framework, we consider time-varying data streams that must be processed sequentially. Our methods are Stochastic Gradient (SG) based due to their applicability and computational advantages. We provide a non-asymptotic analysis of the convergence of various SG-based methods; this includes the famous SG descent (a.k.a. Robbins-Monro algorithm), constant and time-varying mini-batch SG methods, and their averaged estimates (a.k.a. Polyak-Ruppert averaging). Our analysis suggests choosing the learning rate according to the expected data streams, which can speed up the convergence. In addition, we show how the averaged estimate can achieve optimal convergence in terms of attaining Cramer-Rao's lower bound while being robust to any data stream rate. In particular, our analysis shows how Polyak-Ruppert averaging of time-varying mini-batches can provide variance reduction and accelerate convergence simultaneously, which is advantageous for large-scale learning problems. These theoretical results are illustrated for various data streams, showing the effectiveness of the proposed algorithms.

Keywords: stochastic optimization, machine learning, stochastic algorithms, online learning, streaming, mini-batch

1. Introduction

Machine learning and artificial intelligence have become an integral part of modern society. This massive utilization of intelligent systems generates an endless sequence of data, many of which come as *streaming data* such as weather, traffic, stock trade, or other real-time sensor data. These continuously generated data should be processed sequentially with the property that the data stream may change over time. Such a streaming framework requires computationally efficient and robust algorithms that can quickly update the model as more data arrives.

Stochastic approximation algorithms have proven effective in handling large amounts of data; Bottou et al. [3] reviews such stochastic algorithms for large-scale machine learning, including noise reduction and second-order methods, among others. Among these, the most well-known is presumably the Stochastic Gradient (SG) descent proposed by [27], which is used for many models within machine learning. Since its introduction, much work has been spent on analyzing, developing, and improving various SG-based methods, e.g., see Kushner and Yin [16], Lan [17], Shalev-Shwartz et al. [30]. An essential extension is the Polyak-Ruppert procedure (ASG) proposed by Polyak and Juditsky [26], Ruppert [28], which guarantees optimal statistical efficiency without jeopardizing the computational complexity; this average aggregates the estimates sequentially, which reduces the estimate variance while accelerating convergence [26].

Contributions. A fundamental aspect of this paper is to explore how changing data streams affect these stochastic algorithms. Our analysis extends the work of Moulines and Bach [21] to a streaming framework. We investigate several kinds of data streams, from vanilla SG descent and ASG to more exotic learning designs such as time-varying mini-batch SG and ASG. Our main theoretical contribution is the non-asymptotic analysis of SG-based methods in

*Corresponding author

Email addresses: antoine.godichon_baggioni@upmc.fr (Antoine Godichon-Baggioni), nicklas.werge@upmc.fr (Nicklas Werge), olivier.wintenberger@upmc.fr (Olivier Wintenberger)

this streaming framework. Our results show a noticeable improvement in convergence rates by having learning rates that adapt to the expected data streams. In particular, we show how to obtain *optimal* convergence rates *robust* to any data streaming rate.

Organization. Section 2 presents the streaming framework on which the non-asymptotic analysis relies. Our convergence results are presented in Sections 3 and 4, with and without averaging. Both sections includes analysis of unbounded and uniform bounded gradients. These theoretical results are illustrated in Section 5 for a variety of data streams. At last, some final remarks are done in Section 6.

2. Problem Formulation

The analysis of statistical and machine learning models often involves some form of optimization [3, 11, 18]. Many of these optimization problems aim to minimize functions of the form,

$$L(\theta) = \mathbb{E}[l_t(\theta)], \quad (1)$$

with respect to $\theta \in \Theta$, where $\Theta \subseteq \mathbb{R}^d$ is a convex body; a convex body in \mathbb{R}^d is a compact convex set with non-empty interior. For streaming (or large-scale) problems, it would be too expensive to compute the full gradient $\nabla_{\theta} L(\theta)$. Instead, the minimization of L in (1) is achieved without evaluating it directly but by unbiased estimates of its gradients, namely, through $\nabla_{\theta} l_t : \Theta \rightarrow \mathbb{R}$. Observe that the principles for biased functions (l_t) are rather different [6, 29]. Thus, we let (l_t) constitute a sequence of independent differentiable random (possibly non-convex) functions and their gradients unbiased estimates of $\nabla_{\theta} L$, e.g., see Nesterov et al. [24] for definitions and properties of such functions.

Many problems, from classification, and regression to ranking, can be written on this form (1), e.g., see Teo et al. [33] for examples of scalar and vectorial loss functions and their derivatives. For example, consider the simple case where we have some samples $\{(X_t, Y_t)\}$, $t = 1, \dots, n$. Our interest is to find predictor h_{θ} over some parameterization $\{h_{\theta}\}_{\theta \in \Theta}$, by minimizing (1) with $l_t(\theta) = l(h_{\theta}(X_t), Y_t) + \lambda \Omega(\theta)$, where l is some loss function, $\lambda > 0$ a regularizer parameter, and $\Omega : \Theta \rightarrow \mathbb{R}$ some regularizer, e.g., the l_1 or l_2 regularization; here the loss l could be the quadratic, logistic, (squared) hinge, or Huber's (robust) loss, but it depends on the experiments that one wants to perform [3, 17, 24].

Streaming framework. Let us now describe our streaming framework in which we will solve our problem in (1): at each time $t \in \mathbb{N}$, a *block* consisting of $n_t \in \mathbb{N}$ random functions $l_t = (l_{t,1}, \dots, l_{t,n_t})$ arrive. To solve this, we introduce the Stochastic Streaming Gradient (SSG), defined as

$$\theta_t = \theta_{t-1} - \frac{\gamma_t}{n_t} \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}), \quad \theta_0 \in \Theta, \quad (2)$$

where (γ_t) is a decreasing sequence of positive numbers also referred to as the *learning rate*, satisfying $\sum_{i=1}^t \gamma_i = \infty$ and $\sum_{i=1}^t \gamma_i^2 < \infty$ for $t \rightarrow \infty$ [27]. In the same way, we introduce the Projected SSG (PSSG), defined by

$$\theta_t = \mathcal{P}_{\Theta} \left(\theta_{t-1} - \frac{\gamma_t}{n_t} \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right), \quad \theta_0 \in \Theta, \quad (3)$$

where \mathcal{P}_{Θ} denotes the Euclidean projection onto the convex body $\Theta \subseteq \mathbb{R}^d$, i.e., $\mathcal{P}_{\Theta}(\theta) = \arg \min_{\theta' \in \Theta} \|\theta - \theta'\|_2$. The PSSG estimate in (3) is very convenient for models with conditions on the parameters space, and thereby, requires a projection of the parameters. Next, to guarantee optimal convergence properties [26, 28], we introduce the Polyak-Ruppert average of (2), called Averaged SSG (ASSG), given as

$$\bar{\theta}_t = \frac{1}{N_t} \sum_{i=0}^{t-1} n_{i+1} \theta_i, \quad \bar{\theta}_0 = 0, \quad (4)$$

where $N_t = \sum_{i=1}^t n_i$ denotes the accumulated sum of observations. Similarly, we define the (Polyak-Ruppert) Average PSSG (APSSG) estimate as when $(\bar{\theta}_t)$ (in (4)) is derived using (3).

2.1. Quasi-strong Convex and Lipschitz Smooth Objectives

Following Moulines and Bach [21], Sridharan et al. [31], we make the following assumptions about the objective function L : assume that $\theta^* \in \Theta$ is the unique global minimizer of L with $\nabla_{\theta}L(\theta^*) = 0$. Also, let L be μ -quasi-strong convex [13, 23] with convexity constant $\mu > 0$, that is, $\forall \theta \in \Theta$ the following inequality holds,

$$L(\theta^*) \geq L(\theta) + \langle \nabla_{\theta}L(\theta), \theta - \theta^* \rangle + \frac{\mu}{2} \|\theta - \theta^*\|^2. \quad (5)$$

Teo et al. [33] provides a comprehensive record of various convex functions L used in machine learning applications. Milder degrees of convexity have been studied by, e.g., Karimi et al. [13], which studied stochastic gradient methods under the Polyak-Łojasiewicz condition [19, 25], or Gadat and Panloup [7], which studied the Ruppert-Polyak averaging estimate under some Kurdyka-Łojasiewicz-type condition [15, 19]. Next, let the function $\nabla_{\theta}L$ be C_{∇} -Lipschitz continuous, i.e., there exists $C_{\nabla} > 0$ such that $\forall \theta, \theta' \in \Theta$,

$$\|\nabla_{\theta}L(\theta) - \nabla_{\theta}L(\theta')\| \leq C_{\nabla} \|\theta - \theta'\|. \quad (6)$$

Furthermore, for the averaging estimate in (4), we need the function L to be twice differentiable with C'_{∇} -Lipschitz continuous Hessian operator ∇_{θ}^2L , meaning, there exists $C'_{\nabla} \geq 0$ such that $\forall \theta, \theta' \in \Theta$,

$$\|\nabla_{\theta}^2L(\theta) - \nabla_{\theta}^2L(\theta')\| \leq C'_{\nabla} \|\theta - \theta'\|. \quad (7)$$

Note that (6) and (7) only needs to hold for $\theta' = \theta^*$.

3. Stochastic Streaming Gradients

This section considers the SSG and PSSG methods with streaming batches arriving in constant and time-varying streams. Our aim is to provide bounds on the quadratic mean $\mathbb{E}[\|\theta_t - \theta^*\|^2]$, which depends explicitly upon the problem's parameters. In order to do this, we assume the following about the function $l_{t,i}$ for each $t \in \mathbb{N}$ with $i = 1, \dots, n_t$:

Assumption 1. For each $\theta \in \Theta$, the random variable $\nabla_{\theta}l_{t,i}(\theta)$ is square-integrable and $\forall \theta \in \Theta$, $\mathbb{E}[\nabla_{\theta}l_{t,i}(\theta)] = \nabla_{\theta}L(\theta)$.

Assumption 2-p (C_l -expected smoothness). There exists a positive integer p such that $\forall \theta, \theta' \in \Theta$, $\mathbb{E}[\|\nabla_{\theta}l_{t,i}(\theta) - \nabla_{\theta}l_{t,i}(\theta')\|^p] \leq C_l^p \mathbb{E}[\|\theta - \theta'\|^p]$ with $C_l \in \mathbb{R}_+$.

Assumption 3-p (σ -gradient noise). There exists a positive integer p such that $\mathbb{E}[\|\nabla_{\theta}l_{t,i}(\theta^*)\|^p] \leq \sigma^p$ with $\sigma \in \mathbb{R}_+$.

These assumptions are modified versions of the standard assumptions for stochastic approximations [2, 16] as they hold for any $i = 1, \dots, n_t$. Note that Assumption 2-p only needs to hold for $\theta' = \theta^*$. By the smoothness assumption (Assumption 2-p), we avoid the unfavorable uniformly bounded gradients assumption, which is too restrictive and only holds for a few losses. Assumption 3-p is a weak assumption that should be seen as an assumption on Θ rather than on $(l_{t,i})$. For SSG and PSSG, we only need Assumptions 2-p and 3-p to hold for $p = 2$, whereas, for ASSG and APSSG, we need $p = 4$ in order to bound the fourth-order moment.

Our streaming framework include classic examples: stochastic approximation (Robbins-Monro setting [27]) and learning from i.i.d. data, such as linear regression, logistic regression, general ridge regressions and quantile regression, p -means, and softmax regression, under regularity conditions [5, 32, 33].

In the following theorem, we derive an explicit upper bound on the t -th estimate of (2) and (3) for any learning rate (γ_t) using classical techniques from stochastic approximations [2, 16].

Theorem 1 (SSG/PSSG). Denote $\delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^2]$ for some $\delta_0 \geq 0$, where (θ_t) follows (2) or (3). Under Assumption 1, Assumptions 2-p and 3-p with $p = 2$, we have for any learning rate (γ_t) that

$$\delta_t \leq \exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) \pi_t^{\delta} + \frac{2\sigma^2}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i}{n_i}, \quad (8)$$

with $\pi_t^{\delta} = \exp(4C_l^2 \sum_{i=1}^t \gamma_i^2/n_i) \exp(2C_{\nabla}^2 \sum_{i=1}^t \mathbb{1}_{\{n_i > 1\}} \gamma_i^2) (\delta_0 + 2\sigma^2/C_l^2)$.

Sketch of proof. Under Assumption 1, Assumptions 2-p and 3-p with $p = 2$, we derive from (2) that (δ_t) satisfies the recursive relation

$$\delta_t \leq [1 - 2\mu\gamma_t + (2C_l^2 + (n_t - 1)C_\nabla^2)n_t^{-1}\gamma_t^2]\delta_{t-1} + 2\sigma^2n_t^{-1}\gamma_t^2, \quad (9)$$

for any (n_t) and (γ_t) fulfilling the conditions imposed on the learning rate [27]. This recursive relation is then bounded in a non-asymptotic manner using Proposition 5. Bounding the projected estimate in (3) follows directly from the fact that $\mathbb{E}[\|\mathcal{P}_\Theta(\theta) - \theta^*\|^2] \leq \mathbb{E}[\|\theta - \theta^*\|^2]$, $\forall \theta \in \Theta$ [35]. Alternatively, the projected estimate can also be shown without Assumptions 2-p and 3-p but instead with a bounded gradient assumption (Assumption 5), e.g., see Moulines and Bach [21].

Related work. When $n_t = 1$ in (8), we obtain the usual SG descent studied in Moulines and Bach [21]. Similarly, Theorem 1 provides an upper bound on the function values, $\mathbb{E}[L(\theta_t) - L(\theta^*)] \leq C_l\delta_t/2$; this follows by Cauchy-Schwarz inequality and Assumption 2-p.

Natural decay imposed by Robbins and Monro [27]. The learning rate (γ_t) should satisfy the following requirements: $\sum_{i=1}^t \gamma_i = \infty$ and $\sum_{i=1}^t \gamma_i^2/n_i \leq \sum_{i=1}^t \gamma_i^2 < \infty$ for $t \rightarrow \infty$. These conditions directly imply that $\pi_t^\delta < \infty$ as $t \rightarrow \infty$. Thus, our attention is on reducing the noise term $\max_{t/2 \leq i \leq t} \gamma_i/n_i$ without damaging the natural decay of the sub-exponential term $\exp(-\mu \sum_{i=t/2}^t \gamma_i)$. In particular, this non-asymptotic bound shows convergence in quadratic mean for any learning rate, fulfilling these conditions. In addition, the scaling with (n_t) in the noise term shows an apparent variance reduction when we increase the streaming batches (n_t) .

Throughout this paper, we will consider learning rates on the form $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with hyper-parameters $C_\gamma > 0$, $\beta \in [0, 1]$, and α chosen accordingly to the expected streaming batches denoted by n_t . We start by considering constant streaming batches (i.e., mini-batch SG), where n_t follows the constant streaming batch size $C_\rho \in \mathbb{N}$:

Corollary 1 (SSG/PSSG, constant streaming batches). *Denote $\delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^2]$, where (θ_t) follows (2) or (3). Suppose $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with $n_t = C_\rho$ for $C_\rho \in \mathbb{N}$, such that $\alpha \in (1/2, 1)$. Under Assumption 1, Assumptions 2-p and 3-p with $p = 2$, we have*

$$\delta_t \leq \exp\left(-\frac{\mu C_\gamma N_t^{1-\alpha}}{2^{1-\alpha} C_\rho^{1-\alpha-\beta}}\right) \pi_\infty^c + \frac{2^{1+\alpha} \sigma^2 C_\gamma}{\mu C_\rho^{1-\alpha-\beta} N_t^\alpha}, \quad (10)$$

where $\pi_\infty^c = \exp(4\alpha C_\gamma^2 (2C_l^2 + C_\rho \mathbb{1}_{\{C_\rho > 1\}} C_\nabla^2) / (2\alpha - 1) C_\rho^{1-2\beta}) (\delta_0 + 2\sigma^2 / C_l^2)$ is a finite constant.

Decay of the initial conditions. The bound in Corollary 1 depends on the initial condition $\delta_0 = \|\theta_0 - \theta^*\|^2$ and the variance σ^2 in the noise term. The initial condition δ_0 vanish sub-exponentially fast for $\alpha \in (1/2, 1)$. Thus, the asymptotic term is $2^{1+\alpha} \sigma^2 C_\gamma / \mu C_\rho^{1-\alpha-\beta} N_t^\alpha$, i.e., $\delta_t = \mathcal{O}(N_t^{-\alpha})$. Moreover, the bound in (10) is optimal (up to some constants) for quadratic functions $(I_{t,i})$, since the deterministic recursion in (9) would be with equality. It is worth noting that if $C_\gamma C_l$ or $C_\gamma C_\nabla$ is chosen too large, they may produce a large π_∞^c constant. In addition, π_∞^c is positively affected by C_ρ when $\beta < 1/2$. Obviously, the hyper-parameter β only comes into play if the streaming batch size is larger than one, i.e., $C_\rho > 1$. Nonetheless, the effect of π_∞^c will decrease exponentially fast due to the sub-exponentially decaying factor in front.

Variance reduction. The asymptotic term is divided by $C_\rho^{1-\alpha-\beta}$, implying we could achieve variance reduction by taking $\alpha + \beta \leq 1$ when C_ρ is large. Taking a large streaming batch size, e.g., $C_\rho = t$, one accelerates the vanilla SG descent convergence rate to $\mathcal{O}(N_t^{1-\beta})$. However, this large streaming batch size would be unsuitable in practice, and it would mean that we would take few steps until convergence is achieved.

The *safe* choice of having $\beta = 0$ functions well for the SSG method for any streaming batch size C_ρ , but fixed-sized streaming batches are not the most realistic streaming setting. These streaming batches are far more likely to vary in size depending on the data streams. For the sake of simplicity, we consider time-varying streaming batches where n_t are on the form $C_\rho t^\rho$ with $C_\rho \in \mathbb{N}$ and $\rho \in (-1, 1)$ such that $n_t \geq 1$ for all t . We will refer to ρ as the *streaming rate*. For the convenience of notation, let $\tilde{\rho} = \rho \mathbb{1}_{\{\rho \geq 0\}}$.

Corollary 2 (SSG/PSSG, time-varying streaming batches). *Denote $\delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^2]$, where (θ_t) follows (2) or (3). Suppose $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with $n_t = C_\rho t^\rho$ for $C_\rho \in \mathbb{N}$ and $\rho \in (-1, 1)$, such that $\alpha - \beta\tilde{\rho} \in (1/2, 1)$. Under Assumption 1,*

Assumptions 2-p and 3-p with $p = 2$, we have

$$\delta_t \leq \exp\left(-\frac{\mu C_\gamma N_t^{1-\phi}}{2^{(2+\rho)(1-\phi)} C_\rho^{1-\beta-\phi}}\right) \pi_\infty^v + \frac{2^{1+(2+\rho)\phi} \sigma^2 C_\gamma}{\mu C_\rho^{(1-\beta)\mathbb{1}_{\{\rho \geq 0\}} - \phi} N_t^\phi}, \quad (11)$$

where $\phi = ((1 - \beta)\bar{\rho} + \alpha)/(1 + \bar{\rho})$ and $\pi_\infty^v = \exp(4(\alpha - \beta\bar{\rho})C_\gamma^2 C_\rho^{2\beta} (2C_l^2 + C_\nabla^2)/(2(\alpha - \beta\bar{\rho}) - 1))(\delta_0 + 2\sigma^2/C_l^2)$ is a finite constant.

Decay of the initial conditions. As mentioned for Corollary 1, the condition of having $\alpha - \beta\bar{\rho} \in (1/2, 1)$ is a natural restriction from Robbins and Monro [27], which relaxes the usual condition of having $\alpha \in (1/2, 1)$ for ρ non-negative. For $\rho \in (-1, 1/2)$, setting $\alpha = 2/3$ and $\beta = 1/3$ would give same decay rate, $\delta_t = \mathcal{O}(N_t^{-2/3})$ as we saw for Corollary 1 when $\alpha = 2/3$. However, accelerated convergence could be achieved by, e.g., setting $\alpha = 1$ and $\beta = 1/2$ for streaming rate $\rho \in (0, 1)$, giving us $\delta_t = \mathcal{O}(N_t^{-(1+\rho/2)/(1+\rho)})$.

Variance reduction. Similarly to Corollary 1, the sub-exponential and asymptotic term is scaled by $C_\rho^{1-\beta-\phi}$ for $\rho \geq 0$, implying we should take $\alpha + \beta \leq 1$ to obtain variance reduction. These conclusions will change when we consider the averaging estimate in Section 4.

The reasoning in Corollary 2 could be expanded to include *random* streaming batches where n_t is given such that $C_L t^{\rho_L} \leq n_t \leq C_H t^{\rho_H}$ with $\rho_L, \rho_H \in (-1, 1)$ and $C_L, C_H \geq 1$. This yields the modified rate $\phi' = ((1 - \beta)\rho_L + \alpha)/(1 + \rho_H)$; nevertheless, we will leave the proof to the reader.

4. Averaged Stochastic Streaming Gradients

In what follows, we consider the averaging estimate $(\bar{\theta}_n)$ given in (4) derived with use of (θ_t) from (2) (Section 4.1) or (3) (Section 4.2). Besides having Assumptions 2-p and 3-p to hold for $p = 4$, an additional assumption is needed for bounding the *rest* term of the averaging estimate.

Assumption 4. *There exists a non-negative self-adjoint operator Σ such that $\mathbb{E}[\nabla_{\theta} l_{t,i}(\theta^*) \nabla_{\theta} l_{t,i}(\theta^*)^\top] \leq \Sigma$.*

Note that the operator Σ always exists when σ is finite for order $p = 4$ in Assumption 3-p.

4.1. Unbounded Gradients

As in Section 3, we conduct a general study for any learning rate (γ_t) when applying the Polyak-Ruppert averaging estimate from (4):

Theorem 2 (ASSG). *Denote $\bar{\delta}_t = \mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$ with $(\bar{\theta}_t)$ given by (4), where (θ_t) follows (2). Under Assumption 1, Assumptions 2-p and 3-p with $p = 4$, and Assumption 4, we have for any learning rate (γ_t) that*

$$\bar{\delta}_t^{1/2} \leq \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{1}{\mu N_t} \sum_{i=1}^{t-1} \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| \delta_i^{1/2} + \frac{n_t}{\mu \gamma_t N_t} \delta_t^{1/2} + \frac{n_1}{\mu N_t} \left(\frac{1}{\gamma_1} + C_l \right) \delta_0^{1/2} + \frac{C_l}{\mu N_t} \left(\sum_{i=1}^{t-1} n_{i+1} \delta_i \right)^{1/2} + \frac{C_\nabla}{\mu N_t} \sum_{i=0}^{t-1} n_{i+1} \Delta_i^{1/2}, \quad (12)$$

where $\Lambda = \text{Tr}(\nabla_{\theta}^2 L(\theta^*)^{-1} \Sigma \nabla_{\theta}^2 L(\theta^*)^{-1})$ and $\Delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^4]$ for some $\Delta_0 \geq 0$.

As noticed in Polyak and Juditsky [26], the leading term Λ/N_t achieves the Cramer-Rao lower bound [7, 22]. Note that the leading term Λ/N_t is invariant of the learning rate (γ_t) . Moreover, this bound of $\mathcal{O}(N_t^{-1})$ is achieved without inverting the Hessian. Next, the processes (δ_t) and (Δ_t) can be bounded by the recursive relations in (8) and (22). There are no sub-exponential decaying terms for the initial conditions in Theorem 2, which is a common problem for averaging. However, as mentioned previously, we are more interested in advancing the decay of the asymptotic terms. To ease notation, we make use of the functions $\psi_x^y(t) : \mathbb{R} \rightarrow \mathbb{R}$, given as

$$\psi_x^y(t) = \begin{cases} t^{(1-x)/(1+y)}/(1-x) & \text{if } x < 1, \\ (1+y) \log(t) & \text{if } x = 1, \\ x/(x-1) & \text{if } x > 1, \end{cases}$$

with $y \in \mathbb{R}_+$, such that $\sum_{i=1}^t i^{-x} \leq \psi_x^0(t)$ for any $x \in \mathbb{R}_+$. Note that $\psi_x^y(t)/t = O(t^{-(x+y)/(1+y)})$ if $x < 1$, $\psi_x^y(t)/t = O(\log(t)t^{-1})$ if $x = 1$, and $\psi_x^y(t)/t = O(t^{-1})$ if $x > 1$. Hence, for any $x, y \in \mathbb{R}_+$, $\psi_x^y(t)/t = \tilde{O}(t^{-(x+y)/(1+y)})$, where the $\tilde{O}(\cdot)$ notation hides logarithmic factors.

Corollary 3 (ASSG, constant streaming batches). *Denote $\bar{\delta}_t = \mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$ with $(\bar{\theta}_t)$ given by (4), where (θ_t) follows (2). Suppose $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with $n_t = C_\rho$ for $C_\rho \in \mathbb{N}$, such that $\alpha \in (1/2, 1)$. Under Assumption 1, Assumptions 2-p and 3-p with $p = 4$, and Assumption 4, we have*

$$\begin{aligned} \bar{\delta}_t^{1/2} \leq & \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{6\sigma C_\rho^{(1-\alpha-\beta)/2}}{\mu^{3/2} C_\gamma^{1/2} N_t^{1-\alpha/2}} + \frac{2^\alpha 6 C_\nabla' \sigma^2 C_\gamma}{\mu^2 C_\rho^{1-\alpha-\beta} N_t^\alpha} + \frac{2C_l \sigma C_\gamma^{1/2}}{\mu^{3/2} C_\rho^{(1-\alpha-\beta)/2} N_t^{(1+\alpha)/2}} + \frac{C_\rho \Gamma_c}{\mu N_t} \\ & + \frac{C_\rho^{2-\alpha-\beta} \sqrt{\pi_\infty^c} A_\infty^c}{\mu C_\gamma N_t^{2-\alpha}} + \frac{(6 + 7 \mathbb{1}_{\{C_\rho > 1\}}) 2^{3\alpha/2} C_\nabla' \sigma^2 C_\gamma^{3/2} C_\rho^{3\beta/2} \psi_{3\alpha/2}^0(N_t/C_\rho)}{\mu^{3/2} N_t}, \end{aligned}$$

with Γ_c given by $(1/C_\gamma C_\rho^\beta + C_l) \delta_0^{1/2} + C_l \sqrt{\pi_\infty^c A_\infty^c / C_\rho} + \sqrt{\pi_\infty^c A_\infty^c} / C_\gamma C_\rho^\beta + C_\nabla' \sqrt{\Pi_\infty^c A_\infty^c}$, consisting of the finite constants π_∞^c , Π_∞^c and A_∞^c , that only depends on μ , δ_0 , Δ_0 , C_l , σ , C_∇ , C_∇' , C_γ , C_ρ , β and α .

Accelerated decay the initial conditions. By averaging, we have increased the rate of convergence from $O(N_t^{-\alpha})$ to the optimal rate $O(N_t^{-1})$. The two subsequent terms are the main remaining terms decaying at the rate $O(N_t^{\alpha-2})$ and $O(N_t^{-2\alpha})$, which suggests setting $\alpha = 2/3$ would be *optimal*. The remaining terms are negligible. Next, it is worth noting that having $\alpha + \beta = 1$ in Corollary 3, we would give no impact in the main remaining terms from the streaming batch size C_ρ . Moreover, taking $\alpha = 2/3$ and $\beta \leq 1/3$ would be an *optimal* choice of hyper-parameters such that the streaming batch size C_ρ have a positive or no impact. At last, as we do not rely on sub-exponentially decaying terms, we need to be more careful when picking our hyper-parameters, e.g., taking $C_\gamma C_l$ too large may cause Γ_c to be significant. Nevertheless, the term consisting of Γ_c decay at a rate of at least $O(N_t^{-2})$.

Corollary 4 (ASSG, time-varying streaming batches). *Denote $\bar{\delta}_t = \mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$ with $(\bar{\theta}_t)$ given by (4), where (θ_t) follows (2). Suppose $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with $n_t = C_\rho t^\rho$ for $C_\rho \in \mathbb{N}$ and $\rho \in (-1, 1)$, such that $\alpha - \beta\bar{\rho} \in (1/2, 1)$. Under Assumption 1, Assumptions 2-p and 3-p with $p = 4$, and Assumption 4, we have*

$$\begin{aligned} \bar{\delta}_t^{1/2} \leq & \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{2^{3+\phi(1+\bar{\rho})} \sigma C_\rho^{(1-\phi-\bar{\rho})/2} \mathbb{1}_{\{\rho \geq 0\}}}{\mu^{3/2} C_\gamma^{1/2} N_t^{1-\phi/2}} + \frac{2^{(1+\phi)(1+\bar{\rho})-2} C_\nabla' \sigma^2 C_\gamma}{\mu^2 C_\rho^{1-\phi-\beta} N_t^\phi} + \frac{2^{\phi(1+\bar{\rho})/2} C_l \sigma C_\gamma^{1/2}}{\mu^{3/2} C_\rho^{(1-\phi-\bar{\rho})/2} \mathbb{1}_{\{\rho \geq 0\}} N_t^{(1+\phi)/2}} + \frac{C_\rho \Gamma_v}{\mu N_t} \\ & + \frac{C_\rho^{2-\phi-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{\mu C_\gamma N_t^{2-\phi}} + \frac{2^{3(1+\phi)(1+\bar{\rho})/2} C_\nabla' \sigma^2 C_\gamma^{3/2} C_\rho^{1+3\beta/2} \psi_{3(\alpha-\beta\bar{\rho})/2}^{\bar{\rho}}(N_t/C_\rho)}{\mu^{3/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t}, \end{aligned}$$

with Γ_v given by $(1/C_\gamma C_\rho^\beta + C_l) \delta_0^{1/2} + 2^{\bar{\rho}} C_l \sqrt{\pi_\infty^v A_\infty^v / C_\rho} + 2 \sqrt{\pi_\infty^v A_\infty^v} / C_\gamma C_\rho^\beta + 2^{\bar{\rho}} C_\nabla' \sqrt{\Pi_\infty^v A_\infty^v}$, consisting of the finite constants π_∞^v , Π_∞^v and A_∞^v , that only depends on μ , δ_0 , Δ_0 , C_l , σ , C_∇ , C_∇' , C_γ , C_ρ , β and α .

Robustness towards streaming rates ρ : Following the arguments above, the two main remainder terms reveal that $\phi = 2/3 \Leftrightarrow \alpha - \beta\bar{\rho} = (2 - \bar{\rho})/3$, e.g., by setting $\beta = 0$, we should pick $\alpha = (2 - \bar{\rho})/3$. Likewise, if $\rho = 0$, we yield the same conclusion as in Corollary 3, namely $\alpha = 2/3$. However, these hyper-parameter choices are not resilient against any arrival schedule ρ . Nonetheless, we can *robustly* achieve $\phi = 2/3$ for any $\rho \in (-1, 1)$ by setting $\alpha = 2/3$ and $\beta = 1/3$. In other words, we can achieve *optimal* convergence for any data stream by having $\alpha = 2/3$ and $\beta = 1/3$.

4.2. Bounded Gradients

In what follows, we consider the averaging estimate $\bar{\theta}_n$ given in (4) but with the use of the projected estimate PSSG from (3). To avoid calculating the six-order moment, we make the unnecessary assumption that $\|\nabla_{\theta} l_{t,i}(\theta)\|$ is uniformly bounded for any $\theta \in \Theta$; the derivation of the six-order moment can be found in Godichon-Baggioli [9].

Assumption 5. *Let $D_\theta = \inf_{\theta \in \partial\Theta} \|\theta - \theta^*\| > 0$ with $\partial\Theta$ denoting the frontier of Θ . Moreover, there exists $G_\Theta > 0$ such that $\forall t \geq 1$, $\sup_{\theta \in \Theta} \|\nabla_{\theta} l_{t,i}(\theta)\|^2 \leq G_\Theta^2$ a.s., with $i = 1, \dots, n_t$.*

Corollary 5 (APSSG, constant streaming batches). Denote $\bar{\delta}_t = \mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$ with $(\bar{\theta}_t)$ given by (4), where (θ_t) follows (3). Suppose $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with $n_t = C_\rho$ for $C_\rho \in \mathbb{N}$, such that $\alpha \in (1/2, 1)$. Under Assumption 1, Assumptions 2-p and 3-p with $p = 4$, Assumptions 4 and 5, we have

$$\begin{aligned} \bar{\delta}_t^{1/2} \leq & \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{6\sigma C_\rho^{(1-\alpha-\beta)/2}}{\mu^{3/2} C_\gamma^{1/2} N_t^{1-\alpha/2}} + \frac{2^\alpha 6 C_\nabla' \sigma^2 C_\gamma}{\mu^2 C_\rho^{1-\alpha-\beta} N_t^\alpha} + \frac{2 C_l \sigma C_\gamma^{1/2}}{\mu^{3/2} C_\rho^{(1-\alpha-\beta)/2} N_t^{(1+\alpha)/2}} + \frac{C_\rho \Gamma_c}{\mu N_t} \\ & + \frac{C_\rho^{2-\alpha-\beta} \sqrt{\pi_\infty^c} A_\infty^c}{\mu C_\gamma N_t^{2-\alpha}} + \frac{(6 + 7 \mathbb{1}_{\{C_\rho > 1\}}) 2^{3\alpha/2} C_\nabla'' \sigma^2 C_\gamma^{3/2} C_\rho^{3\beta/2} \psi_{3\alpha/2}^0(N_t/C_\rho)}{\mu^{3/2} N_t}, \end{aligned}$$

with $C_\nabla'' = C_\nabla' + 2^2 G_\Theta / D_\theta^2$ and Γ_c given by $(1/C_\gamma C_\rho^\beta + C_l) \delta_0^{1/2} + C_l \sqrt{\pi_\infty^c A_\infty^c / C_\rho} + \sqrt{\pi_\infty^c A_\infty^c} / C_\gamma C_\rho^\beta + C_\nabla' \sqrt{\pi_\infty^c A_\infty^c}$, consisting of the finite constants π_∞^c , Π_∞^c and A_∞^c , that only depends on μ , δ_0 , Δ_0 , C_l , σ , C_∇ , C_∇' , C_γ , C_ρ , β and α .

Corollary 6 (APSSG, time-varying streaming batches). Denote $\bar{\delta}_t = \mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$ with $(\bar{\theta}_t)$ given by (4), where (θ_t) follows (3). Suppose $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with $n_t = C_\rho t^\rho$ for $C_\rho \in \mathbb{N}$ and $\rho \in (-1, 1)$, such that $\alpha - \beta \tilde{\rho} \in (1/2, 1)$. Under Assumption 1, Assumptions 2-p and 3-p with $p = 4$, Assumptions 4 and 5, we have

$$\begin{aligned} \bar{\delta}_t^{1/2} \leq & \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{2^{3+\phi(1+\tilde{\rho})} \sigma C_\rho^{(1-\phi-\beta)/2} \mathbb{1}_{\{\rho \geq 0\}}}{\mu^{3/2} C_\gamma^{1/2} N_t^{1-\phi/2}} + \frac{2^{(1+\phi)(1+\tilde{\rho})-2} C_\nabla' \sigma^2 C_\gamma}{\mu^2 C_\rho^{1-\phi-\beta} N_t^\phi} + \frac{2^{\phi(1+\tilde{\rho})/2} C_l \sigma C_\gamma^{1/2}}{\mu^{3/2} C_\rho^{(1-\phi-\beta)/2} \mathbb{1}_{\{\rho \geq 0\}} N_t^{(1+\phi)/2}} + \frac{C_\rho \Gamma_v}{\mu N_t} \\ & + \frac{C_\rho^{2-\phi-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{\mu C_\gamma N_t^{2-\phi}} + \frac{2^{3(1+\phi)(1+\tilde{\rho})/2} C_\nabla'' \sigma^2 C_\gamma^{3/2} C_\rho^{-1+3\beta/2} \psi_{3(\alpha-\beta\tilde{\rho})/2}^{\tilde{\rho}}(N_t/C_\rho)}{\mu^{3/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t}, \end{aligned}$$

with $C_\nabla'' = C_\nabla' + 2^2 G_\Theta / D_\theta^2$ and Γ_v given by $(1/C_\gamma C_\rho^\beta + C_l) \delta_0^{1/2} + 2^{\tilde{\rho}} C_l \sqrt{\pi_\infty^v A_\infty^v / C_\rho} + 2 \sqrt{\pi_\infty^v A_\infty^v} / C_\gamma C_\rho^\beta + 2^{\tilde{\rho}} C_\nabla' \sqrt{\pi_\infty^v A_\infty^v}$, consisting of the finite constants π_∞^v , Π_∞^v and A_∞^v , that only depends on μ , δ_0 , Δ_0 , C_l , σ , C_∇ , C_∇' , C_γ , C_ρ , β and α .

5. Experiments

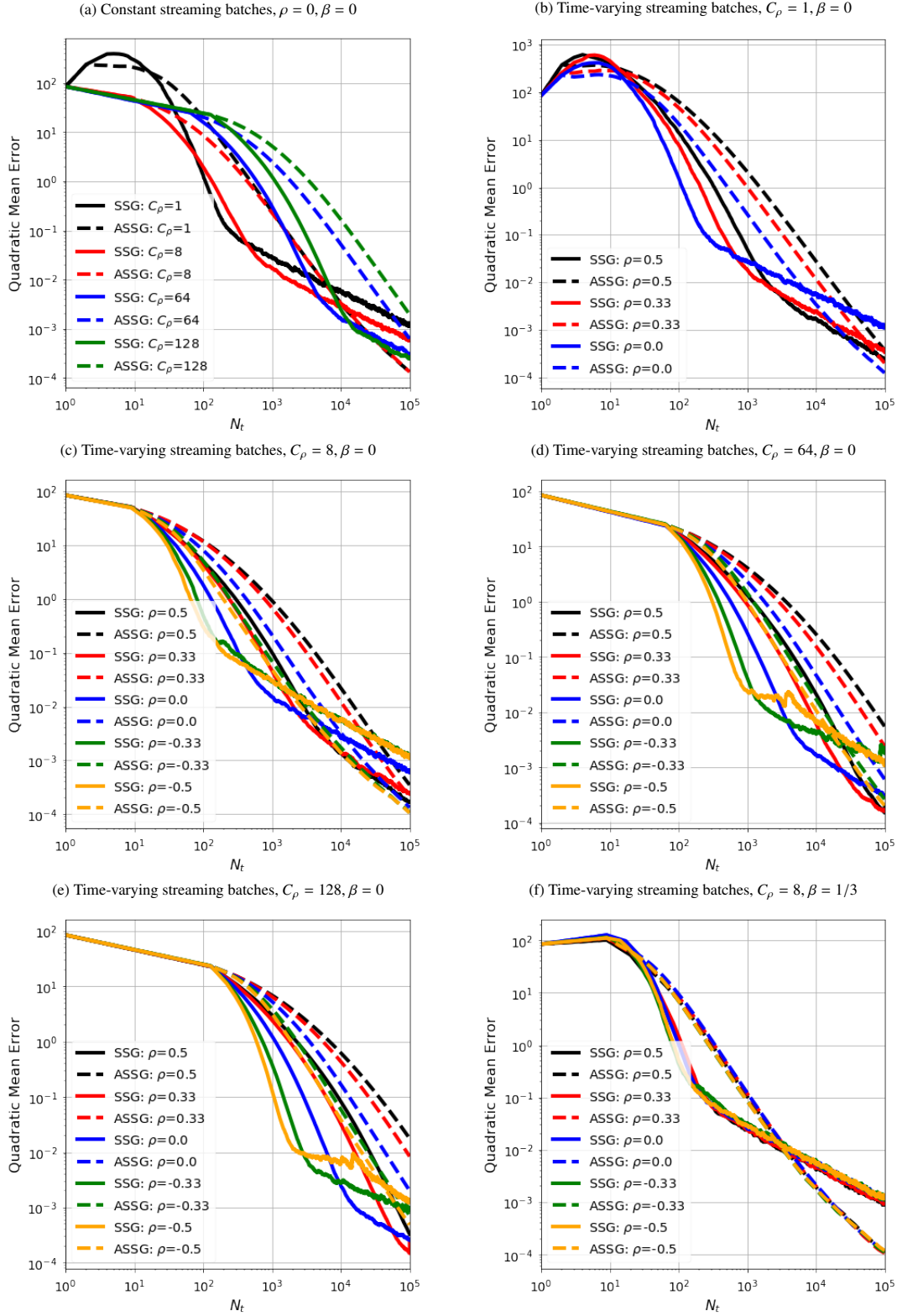
In this section, we demonstrate the theoretical results presented in Sections 3 and 4 for various data streams. In Section 5.1, we illustrate the unbounded gradient case (Sections 3 and 4.1) using linear regression. Where in Section 5.2, we present the bounded gradient case (Sections 3 and 4.2) by considering the geometric median. To measure the performance, we use the quadratic mean error of the parameter estimates over one-hundred replications, given by $(\mathbb{E}[\|\theta_{N_t} - \theta^*\|^2])_{t \geq 1}$. Note that averaging over several iterations gives a reduction in variability, which mainly benefits the SSG and PSSG.

5.1. Linear Regression

Consider the linear regression defined by $y_t = X_t^T \theta + \epsilon_t$, where $X_t \in \mathbb{R}^d$ is a random features vector, $\theta \in \mathbb{R}^d$ is the parameters vector, and ϵ_t is a random variable with zero mean, independent from X_t . Moreover, $(X_t, \epsilon_t)_{t \geq 1}$ are independent and identically distributed. Thus, θ^* is the minimizer of $L(\theta) = \mathbb{E}[(y_t - X_t^T \theta)^2]$. In this example, we fix $d = 10$, set $\theta = (-4, -3, 2, 1, 0, 1, 2, 3, 4, 5)^T \in \mathbb{R}^{10}$, and let (X_t) and (ϵ_t) be standard Gaussian. It is well-known that C_γ can substantially impact convergence; when C_γ is too large, instability can occur, leading to an explosion during the first iterations. If C_γ is too small, the convergence can become very slow and destroy the desired rate α . To focus on the various data streams, we set $C_\gamma = 1/2$ and $\alpha = 2/3$.

In Figure 1a, we consider constant data streams to illustrate the results in Corollaries 1 and 3. The figures show a solid decay rate proportional to $\alpha = 2/3$ for any streaming batch size $C_\rho \in \{1, 8, 64, 128\}$ with $\beta = 0$, as shown in Corollary 1. In addition, we see an acceleration in decay by averaging, as explained in Corollary 3. Both methods show a noticeable reduction in variance when C_ρ increases which are particularly beneficial in the beginning. Moreover, as mentioned in Remark 1, the *stationary* phase may also commence earlier when we raise the streaming batch size C_ρ . Next, in Figures 1b to 1e, we vary the streaming rate ρ for streaming batch sizes $C_\rho = 1, 8, 64$, and 128, respectively, with $\beta = 0$. These figures shows an increase in decay of the SSG when the streaming rate ρ increase. Despite this, we still achieve better convergence for the ASSG method, which seems more immune to the different choices of

Figure 1: Linear regression for various data streams $n_t = C_\rho t^\rho$. See Section 5.1 for details.



streaming rate ρ , e.g., see the discussion after Corollary 4. We know this from Corollary 2, as $\phi = (\tilde{\rho} + \alpha)/(1 + \tilde{\rho}) \geq \alpha$ for $\beta = 0$. In addition, we see that C_ρ has a positive effect on the noise (i.e., variance reduction), but if C_ρ becomes too large, it may slow down convergence (as seen in Figure 1e). Alternatively, we could think around the problem in another way: how can we choose α and β such that we have *optimal* decay of $\phi = 2/3$ for any ρ . In other words, for any arrival schedule that may occur, how should we choose our hyper-parameters such that we achieve optimal decay of $\phi = 2/3$. As discussed after Corollary 4, one example of this could be achieved by setting $\alpha = 2/3$ and $\beta = 1/3$ such that $\phi = 2/3$ for any ρ . Figure 1f shows an example of this where we (indeed) achieve the same decay rate for any streaming rate ρ .

5.2. Geometric Median

The geometric median is a generalization of the real median introduced by Haldane [12]. Robust estimators such as the geometric median may be preferred over the mean when the data is noisy. Moreover, in our streaming framework, stochastic algorithms are preferred as they efficiently handle large samples of high-dimensional data [5, 9]. The geometric median of $X \in \mathbb{R}^d$ is defined by $\theta^* \in \mathbb{R}^d$ which minimizes the convex function $L(\theta) = \mathbb{E}[\|X - \theta\| - \|X\|]$, e.g., see Gervini [8], Kemperman [14] for properties such as existence, uniqueness, and robustness (breakdown point). Thus, the gradient $\nabla_\theta L(\theta) = \mathbb{E}[\nabla_\theta l_t(\theta)]$ with $\nabla_\theta l_t(\theta) = -(X_t - \theta)/\|X_t - \theta\|$ is bounded as $\|\nabla_\theta l_t(\theta)\| \leq 1$. As discussed in Gadat and Panloup [7], this boundedness implies that it is not necessary to project our estimates explicitly. Similarly to above, we fix $d = 10$ and let (X_t) be standard Gaussian centered at $\theta = (-4, -3, 2, 1, 0, 1, 2, 3, 4, 5)^T \in \mathbb{R}^{10}$. Moreover, following the reasoning of Cardot et al. [5], we set $C_\gamma = \sqrt{d} = \sqrt{10}$, and let $\alpha = 2/3$.

Figure 2a shows the variance reduction effect for different constant streaming batches C_ρ with $\beta = 0$. However, the robustness of the geometric median leaves only a small positive impact for further variance reduction. Thus, too large (constant) streaming batch sizes C_ρ hinders the convergence as we make too few iterations. These findings can be extended to Figures 2b to 2e, where we vary the streaming rate ρ for streaming batch sizes $C_\rho = 1, 8, 64$, and 128, respectively, with $\beta = 0$. The lack of convergence improvements comes from $\beta = 0$, which means we do not exploit the potential of using more observations to accelerate convergence. As shown in Figure 2f, we can achieve this acceleration by simply taking $\beta = 1/3$. In addition, $\beta = 1/3$ provides optimal convergence robust to any streaming rate ρ . Choosing a proper $\beta > 0$ is particularly important when C_ρ is large, as robustness is an integral part of the geometric median method.

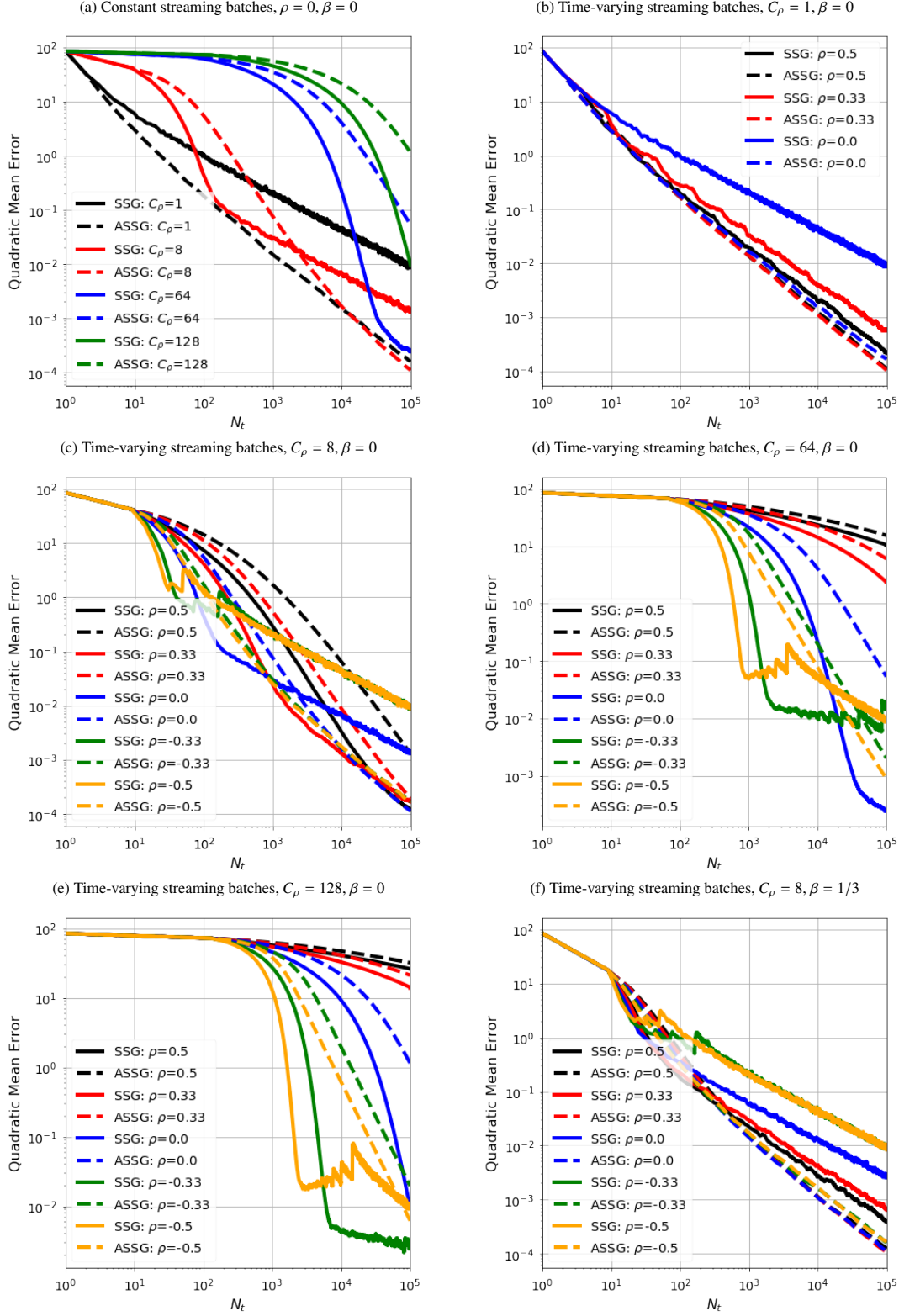
6. Conclusions

We considered the stochastic approximation problem in a streaming framework where we had to minimize convex objectives using only unbiased estimates of its gradients. We introduced and studied the convergence rates of the stochastic streaming algorithms in a non-asymptotic manner. This investigation was derived using learning rates of the form $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ under time-varying data streams (n_t) . The theoretical results and our experiments showed a noticeable improvement in the convergence rate by choosing the learning rate (hyper-parameters) according to the expected data streams. For ASSG and APSSG, we showed that this choice of learning rate led to optimal convergence rates and was robust to any data stream rate we may encounter. Moreover, in large-scale learning problems, we know how to accelerate convergence and reduce variance through the learning rate and the treatment pattern of the data.

There are several ways to expand our work but let us give some examples: first, we can extend our analysis to include streaming batches of any size in the spirit of the discussion after Corollary 2. Second, many machine learning problems encounter correlated variables and high-dimensional data, making an extension to non-strongly convex objectives advantageous. Bach and Moulines [1], e.g., in Werge and Wintenberger [34], they use SG-based optimization methods for volatility prediction through GARCH modeling. Third, Assumption 1 requires unbiased (and independent) gradient estimates, thus, an obvious extension could incorporate a more realistic dependency assumption, thereby increasing the applicability for more models. Moreover, studying dependence may give insight into how to process dependent information *optimally*. Next, a natural extension would be to modify our averaging estimate from (4) to a weighted averaged version (WASSG) proposed by Mokkadem and Pelletier [20] and Boyer and Godichon-Baggioni [4], given as

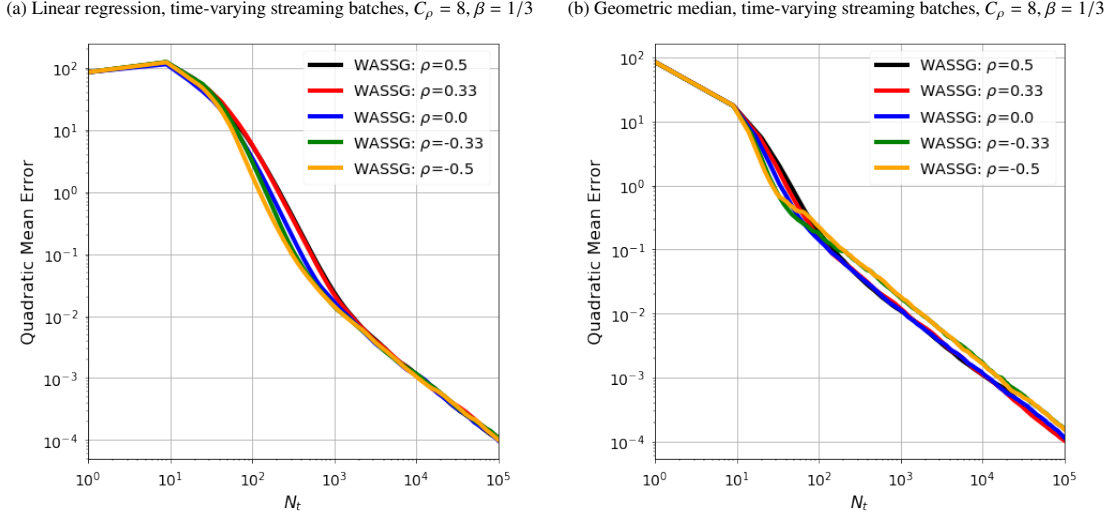
$$\bar{\theta}_{t,\lambda} = \frac{1}{\sum_{i=1}^t n_i \log(1+i)^\lambda} \sum_{i=1}^t n_i \log(1+i)^\lambda \theta_{i-1}, \quad \bar{\theta}_{0,\lambda} = 0, \quad (13)$$

Figure 2: Geometric median for various data streams $n_t = C_\rho t^\rho$. See Section 5.2 for details.



for $\lambda > 0$ with (θ_t) following (2) or (3). We can limit the effect of bad initializations by placing more weight on the newest estimates. Following the demonstrations in Section 5, an example of this WASSG estimate $(\bar{\theta}_{t,\lambda})$ can be found in Figure 3 with use of $\lambda = 2$. Here we see that although the WASSG estimate in (13) may not achieve a better final error (compared to the ASSG and APSSG estimates in Figures 1f and 2f), but it still achieves a better decay along the way, often referred to as *parameter tracking*.

Figure 3: WASSG for various data streams $n_t = C_\rho t^\rho$. See Section 6 for details.



7. Proofs

In this section, we provide detailed proofs of the results presented in the manuscript. Purely technical results used in the proofs can be found in Appendix A. Let $(\mathcal{F}_t)_{t \geq 1}$ be an increasing family of σ -fields, namely $\mathcal{F}_t = \sigma(l_1, \dots, l_t)$ with $l_t = (l_{t,1}, \dots, l_{t,n_t})$. Furthermore, we expand the notation with $\mathcal{F}_{t-1,i} = \sigma(l_{1,1}, \dots, l_{t-1,n_{t-1}}, l_{t,1}, \dots, l_{t,i})$ such that $\mathcal{F}_{t-1,0} = \mathcal{F}_{t-1}$. Meaning, $\forall 0 \leq i < j$, we have $\mathcal{F}_{t-1,0} \subseteq \mathcal{F}_{t-1,i} \subseteq \mathcal{F}_{t-1,j}$. Thus, by the independence of the random (differentiable) functions $(l_{t,i})$, Assumption 1 yields that $\forall t \geq 1$, $\mathbb{E}[\nabla_{\theta} l_{t,i}(\theta_{t-1}) | \mathcal{F}_{t-1,i-1}] = \nabla_{\theta} L(\theta_{t-1})$ with $i = 1, \dots, n_t$.

7.1. Proofs for Section 3

The section is structured such that we start by analyzing the recursive relations and bounding them for every choice of learning rate. Next, we look at specific choices of learning rates.

Proof of Theorem 1. Taking the quadratic norm on both sides of (2), expanding it, and take the conditional expectation, yields

$$\mathbb{E}[\|\theta_t - \theta^*\|^2 | \mathcal{F}_{t-1}] = \|\theta_{t-1} - \theta^*\|^2 + \frac{\gamma_t^2}{n_t^2} \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^2 \middle| \mathcal{F}_{t-1} \right] - \frac{2\gamma_t}{n_t} \sum_{i=1}^{n_t} \mathbb{E}[\langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \theta_{t-1} - \theta^* \rangle | \mathcal{F}_{t-1}]. \quad (14)$$

To bound the second term (on the right-hand side) of (14), we first expand it as follows,

$$\sum_{i=1}^{n_t} \mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^2 | \mathcal{F}_{t-1}] + \sum_{i \neq j} \mathbb{E}[\langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \nabla_{\theta} l_{t,j}(\theta_{t-1}) \rangle | \mathcal{F}_{t-1}]. \quad (15)$$

For first term of (15), we utilize the Lipschitz continuity of $\nabla_{\theta} l_{t,i}$, together with Assumptions 1 to 3-p, to obtain

$$\mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^2 | \mathcal{F}_{t-1}] \leq 2\mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta_{t-1}) - \nabla_{\theta} l_{t,i}(\theta^*)\|^2 | \mathcal{F}_{t-1}] + 2\mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta^*)\|^2 | \mathcal{F}_{t-1}] \leq 2C_l^2 \|\theta_{t-1} - \theta^*\|^2 + 2\sigma^2, \quad (16)$$

using $\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$. Next, for the second term in (15): as $\mathcal{F}_{t-1} \subseteq \mathcal{F}_{t-1,i} \subset \mathcal{F}_{t-1,j}$ for all $0 \leq i < j$, we have

$$\mathbb{E}[\langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \nabla_{\theta} l_{t,j}(\theta_{t-1}) \rangle | \mathcal{F}_{t-1}] = \mathbb{E}[\mathbb{E}[\langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \nabla_{\theta} L(\theta_{t-1}) \rangle | \mathcal{F}_{t-1,i-1}] | \mathcal{F}_{t-1}],$$

since θ_{t-1} and $l_{t,i}$ are $\mathcal{F}_{t-1,j-1}$ -measurable for all $0 \leq i < j$, and similarly, as θ_{t-1} is \mathcal{F}_{t-1} -measurable and $\mathcal{F}_{t-1,i-1}$ -measurable for all $i \geq 0$, we also have

$$\mathbb{E}[\mathbb{E}[\langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \nabla_{\theta} L(\theta_{t-1}) \rangle | \mathcal{F}_{t-1,i-1}] | \mathcal{F}_{t-1}] = \mathbb{E}[\langle \mathbb{E}[\nabla_{\theta} l_{t,i}(\theta_{t-1}) | \mathcal{F}_{t-1,i-1}], \nabla_{\theta} L(\theta_{t-1}) \rangle | \mathcal{F}_{t-1}] = \|\nabla_{\theta} L(\theta_{t-1})\|^2,$$

where $\|\nabla_{\theta} L(\theta_{t-1})\|^2 \leq C_{\nabla}^2 \|\theta_{t-1} - \theta^*\|^2$ as $\nabla_{\theta} L$ is C_{∇} -Lipschitz continuous and $\nabla_{\theta} L(\theta^*) = 0$. Thus, we obtained a bound for the second term (on the right-hand side) of (14) using the bounds of the two terms in (15):

$$\sum_{i=1}^{n_t} (2C_l^2 \|\theta_{t-1} - \theta^*\|^2 + 2\sigma^2) + \sum_{i \neq j}^{n_t} C_{\nabla}^2 \|\theta_{t-1} - \theta^*\|^2 = (2C_l^2 n_t + C_{\nabla}^2 (n_t - 1)n_t) \|\theta_{t-1} - \theta^*\|^2 + 2\sigma^2 n_t. \quad (17)$$

For the third term (on the right-hand side) of (14) we use that L is μ -quasi-strong convex and θ_{t-1} is \mathcal{F}_{t-1} -measurable,

$$\mathbb{E}[\langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \theta_{t-1} - \theta^* \rangle | \mathcal{F}_{t-1}] = \langle \mathbb{E}[\nabla_{\theta} l_{t,i}(\theta_{t-1}) | \mathcal{F}_{t-1}], \theta_{t-1} - \theta^* \rangle = \langle \nabla_{\theta} L(\theta_{t-1}), \theta_{t-1} - \theta^* \rangle \geq \mu \|\theta_{t-1} - \theta^*\|^2, \quad (18)$$

by Assumption 1. Combining inequalities from (17) and (18) into (14) and taking the expectation on both sides of the inequality, yields the recursive relation (9):

$$\delta_t \leq [1 - 2\mu\gamma_t + (2C_l^2 + (n_t - 1)C_{\nabla}^2)n_t^{-1}\gamma_t^2]\delta_{t-1} + 2\sigma^2 n_t^{-1}\gamma_t^2,$$

with $\delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^2]$ with some $\delta_0 \geq 0$. At last, by Proposition 5, we obtain the desired inequality in (8), namely

$$\delta_t \leq \exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) \exp\left(4C_l^2 \sum_{i=1}^t \frac{\gamma_i^2}{n_i}\right) \exp\left(2C_{\nabla}^2 \sum_{i=1}^t \mathbb{1}_{\{n_i > 1\}} \gamma_i^2\right) \left(\delta_0 + \frac{2\sigma^2}{C_l^2}\right) + \frac{2\sigma^2}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i}{n_i}.$$

using that $(n_t - 1)n_t^{-1} \leq \mathbb{1}_{\{n_t > 1\}}$, $n_t \geq 1$, and that $\max_{1 \leq i \leq t} 2\sigma^2 / (2C_l^2 + (n_i - 1)C_{\nabla}^2) \leq \max_{1 \leq i \leq t} 2\sigma^2 / 2C_l^2 = \sigma^2 / C_l^2$. \square

Remark 1. The decrease of $(2C_l^2 + (n_t - 1)C_{\nabla}^2)n_t^{-1}\gamma_t$ determines when the *stationary* phase occurs. This is more clearly seen in Proposition 4, where the inner terms directly depend on the inception of the stationary phase. Thus, by increasing n_t , we decrease $(2C_l^2 + (n_t - 1)C_{\nabla}^2)n_t^{-1}\gamma_t$, and especially it dominates the constant C_l .

Proof of Corollary 1. By Theorem 1, we have the upper bound giving as

$$\delta_t \leq \exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) \pi_t^c + \frac{2\sigma^2}{\mu C_{\rho}} \max_{t/2 \leq i \leq t} \gamma_i. \quad (19)$$

as $n_t = C_{\rho}$, with $\pi_t^c = \exp((4C_l^2 / C_{\rho}) \sum_{i=1}^t \gamma_i^2) \exp(2C_{\nabla}^2 \mathbb{1}_{\{C_{\rho} > 1\}} \sum_{i=1}^t \gamma_i^2) (\delta_0 + \sigma^2 / C_l^2)$. The sum term $\sum_{i=1}^t \gamma_i^2 = C_{\gamma}^2 C_{\rho}^{2\beta} \sum_{i=1}^t i^{-2\alpha}$ in π_t^c can be bounded with the help of integral tests for convergence, $\sum_{i=1}^t i^{-2\alpha} = 1 + \sum_{i=2}^t i^{-2\alpha} \leq 1 + \int_1^t x^{-2\alpha} dx \leq 1 + 1/(2\alpha - 1) = 2\alpha / (2\alpha - 1)$, as $\alpha \in (1/2, 1)$. Likewise, plugging $\gamma_t = C_{\gamma} C_{\rho}^{\beta} t^{-\alpha}$ into the first term of (19), gives

$$\exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) = \exp\left(-\mu C_{\gamma} C_{\rho}^{\beta} \sum_{i=t/2}^t i^{-\alpha}\right) \leq \exp\left(-\mu C_{\gamma} C_{\rho}^{\beta} \int_{t/2}^t x^{-\alpha} dx\right) \leq \exp\left(-\frac{\mu C_{\gamma} C_{\rho}^{\beta} t^{1-\alpha}}{2^{1-\alpha}}\right),$$

using the integral test for convergence. Next, as $(\gamma_t)_{t \geq 1}$ is decreasing, then $\max_{t/2 \leq i \leq t} \gamma_i = \gamma_{t/2}$. Combining all these findings into (19), gives us

$$\delta_t \leq \exp\left(-\frac{\mu C_{\gamma} C_{\rho}^{\beta} t^{1-\alpha}}{2^{1-\alpha}}\right) \pi_{\infty}^c + \frac{2^{1+\alpha} \sigma^2 C_{\gamma}}{\mu C_{\rho}^{1-\beta} t^{\alpha}}, \quad (20)$$

with $\pi_{\infty}^c = \exp(4\alpha C_{\gamma}^2 (2C_l^2 + C_{\rho} \mathbb{1}_{\{C_{\rho} > 1\}} C_{\nabla}^2) / (2\alpha - 1) C_{\rho}^{1-2\beta}) (\delta_0 + 2\sigma^2 / C_l^2)$. At last, converting (20) into terms of N_t using $N_t = C_{\rho} t$, yields the desired. \square

Proof of Corollary 2. For convenience, we divided the proof into two cases to comprehend that $n_t \geq 1$ for all t : first, we bound each term of (8) (from Theorem 1) after inserting, $\gamma_t = C_\gamma n_t^\beta t^{-\alpha} = C_\gamma C_\rho^\beta t^{\beta\rho-\alpha}$ if $\rho \geq 0$, or $\gamma_t \geq C_\gamma t^{-\alpha}$ if $\rho < 0$ (using that $\beta \geq 0$) into the inequality. If $\rho \geq 0$, the first term of (8) can be bounded, as follows:

$$\exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) = \exp\left(-\mu C_\gamma C_\rho^\beta \sum_{i=t/2}^t i^{\beta\rho-\alpha}\right) \leq \exp\left(-\frac{\mu C_\gamma C_\rho^\beta t^{1+\beta\rho-\alpha}}{2^{1+\beta\rho-\alpha}}\right),$$

using that $\alpha - \beta\rho \in (1/2, 1)$ and the integral test for convergence. In a same way, if $\rho < 0$, one has

$$\exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) \leq \exp\left(-\mu C_\gamma \sum_{i=t/2}^t i^{-\alpha}\right) \leq \exp\left(-\frac{\mu C_\gamma t^{1-\alpha}}{2^{1-\alpha}}\right).$$

Likewise, with the help of integral tests for convergence, we have for $\rho \geq 0$, that $\sum_{i=1}^t \gamma_i^2/n_i \leq \sum_{i=1}^t \gamma_i^2 \leq 2(\alpha - \beta\rho)C_\gamma^2 C_\rho^{2\beta}/(2(\alpha - \beta\rho) - 1)$, as $n_t \geq 1$ and $\alpha - \rho\beta > 1/2$. If $\rho < 0$, one has $\sum_{i=1}^t \gamma_i^2/n_i \leq \sum_{i=1}^t \gamma_i^2 \leq 2\alpha C_\gamma^2 C_\rho^{2\beta}/(2\alpha - 1)$ since $C_\rho \geq n_t \geq 1$. Next, as $(1 - \beta)\rho + \alpha > 0$ for $\rho \geq 0$, then we can bound the last term of (8) by

$$\frac{2\sigma^2}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i}{n_i} = \frac{2\sigma^2 C_\gamma}{\mu C_\rho^{1-\beta}} \max_{t/2 \leq i \leq t} \frac{1}{i^{(1-\beta)\rho+\alpha}} \leq \frac{2^{1+(1-\beta)\rho+\alpha} \sigma^2 C_\gamma}{\mu C_\rho^{1-\beta} t^{(1-\beta)\rho+\alpha}}.$$

Likewise, if $\rho < 0$, we have

$$\frac{2\sigma^2}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i}{n_i} = \frac{2\sigma^2 C_\gamma}{\mu} \max_{t/2 \leq i \leq t} \frac{1}{n_i^{1-\beta} i^\alpha} \leq \frac{2^{1+\alpha} \sigma^2 C_\gamma}{\mu t^\alpha},$$

since $n_t \geq 1$ and $\beta \leq 1$. Combining all these findings gives

$$\delta_t \leq \exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}} t^{(1-\phi)(1+\tilde{\rho})}}}{2^{(1-\phi)(1+\tilde{\rho})}}\right) \pi_\infty^\nu + \frac{2^{1+\phi(1+\tilde{\rho})} \sigma^2 C_\gamma}{\mu C_\rho^{(1-\beta) \mathbb{1}_{\{\rho \geq 0\}} t^{\phi(1+\tilde{\rho})}}}, \quad (21)$$

where $\pi_\infty^\nu = \exp(4(\alpha - \beta\tilde{\rho})C_\gamma^2 C_\rho^{2\beta} (2C_t^2 + C_\rho^2)/2(\alpha - \beta\tilde{\rho}) - 1)$ with $\tilde{\rho} = \rho \mathbb{1}_{\{\rho \geq 0\}}$ and $\phi = ((1 - \beta)\tilde{\rho} + \alpha)/(1 + \tilde{\rho})$. To write this in terms of N_t , we use that $N_t = \sum_{i=1}^t n_i = C_\rho \sum_{i=1}^t i^\rho = C_\rho(t^\rho + \sum_{i=1}^{t-1} i^\rho) \leq C_\rho(t^\rho + \int_1^t x^\rho dx) \leq C_\rho(t^\rho + t^\rho \int_1^t dx) = C_\rho(t^\rho + t^{1+\rho}) \leq 2C_\rho t^{1+\rho}$, for $\rho \geq 0$, thus, $t \geq (N_t/2C_\rho)^{1/(1+\rho)}$. For $\rho < 0$, we have $N_t \leq C_\rho t$, i.e., $t \geq N_t/C_\rho$. \square

7.2. Proofs for Section 4

Lemma 1 (ASSG/APSSG). Denote $\Delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^4]$ for some $\Delta_0 \geq 0$, where (θ_t) follows (2) or (3). Under Assumption 1, Assumptions 2-p and 3-p with $p = 4$ and Assumption 4, we have for any learning rate (γ_t) that

$$\Delta_t \leq \exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) \Pi_t^\Delta + \frac{32\sigma^4}{\mu^2} \max_{t/2 \leq i \leq t} \frac{\gamma_i^2}{n_i^2} + \frac{48\sigma^4}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i^3}{n_i^3} + \frac{114\sigma^4}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i^3 \mathbb{1}_{\{n_i > 1\}}}{n_i^2}, \quad (22)$$

with Π_t^Δ given in (30).

Proof of Lemma 1. We will now derive the recursive step sequence for the fourth-order moment using the same arguments as in proof for Theorem 1. Thus, one can show that

$$\begin{aligned} \mathbb{E}[\|\theta_t - \theta^*\|^4 | \mathcal{F}_{t-1}] &\leq \|\theta_{t-1} - \theta^*\|^4 + \frac{\gamma_t^4}{n_t^4} \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^4 \middle| \mathcal{F}_{t-1} \right] + \frac{4\gamma_t^2}{n_t^2} \mathbb{E} \left[\left\langle \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}), \theta_{t-1} - \theta^* \right\rangle^2 \middle| \mathcal{F}_{t-1} \right] \\ &+ \frac{2\gamma_t^2}{n_t^2} \|\theta_{t-1} - \theta^*\|^2 \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^2 \middle| \mathcal{F}_{t-1} \right] - \frac{4\gamma_t}{n_t} \|\theta_{t-1} - \theta^*\|^2 \sum_{i=1}^{n_t} \langle \mathbb{E}[\nabla_{\theta} l_{t,i}(\theta_{t-1}) | \mathcal{F}_{t-1}], \theta_{t-1} - \theta^* \rangle \\ &+ \frac{4\gamma_t^3}{n_t^3} \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^2 \left\langle \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}), \theta_{t-1} - \theta^* \right\rangle \middle| \mathcal{F}_{t-1} \right], \end{aligned}$$

using θ_{t-1} is \mathcal{F}_{t-1} -measurable. Note, by Assumption 1, we have

$$\langle \mathbb{E}[\nabla_{\theta} l_{t,i}(\theta_{t-1}) | \mathcal{F}_{t-1}], \theta_{t-1} - \theta^* \rangle = \langle \nabla_{\theta} L(\theta_{t-1}), \theta_{t-1} - \theta^* \rangle \geq \mu \|\theta_{t-1} - \theta^*\|^2,$$

as L is μ -quasi-strong convex. Combining this with the Cauchy-Schwarz inequality $\langle x, y \rangle \leq \|x\| \|y\|$, we obtain the simplified expression:

$$\begin{aligned} \mathbb{E}[\|\theta_t - \theta^*\|^4 | \mathcal{F}_{t-1}] &\leq \|\theta_{t-1} - \theta^*\|^4 + \frac{\gamma_t^4}{n_t^4} \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^4 \middle| \mathcal{F}_{t-1} \right] + \frac{6\gamma_t^2}{n_t^2} \|\theta_{t-1} - \theta^*\|^2 \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^2 \middle| \mathcal{F}_{t-1} \right] \\ &\quad - 4\mu\gamma_t \|\theta_{t-1} - \theta^*\|^4 + \frac{4\gamma_t^3}{n_t^3} \|\theta_{t-1} - \theta^*\| \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^3 \middle| \mathcal{F}_{t-1} \right]. \end{aligned}$$

Next, recall Young's Inequality, i.e., for any $a_t, b_t, c_t > 0$ we have $a_t b_t \leq a_t^2 c_t^2 / 2 + b_t^2 / 2c_t^2$,

$$\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^3 \leq \frac{\gamma_t}{2n_t \|\theta_{t-1} - \theta^*\|} \left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^4 + \frac{2n_t \|\theta_{t-1} - \theta^*\|}{\gamma_t} \left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^2,$$

giving us

$$\mathbb{E}[\|\theta_t - \theta^*\|^4 | \mathcal{F}_{t-1}] \leq (1 - 4\mu\gamma_t) \|\theta_{t-1} - \theta^*\|^4 + \frac{3\gamma_t^4}{n_t^4} \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^4 \middle| \mathcal{F}_{t-1} \right] + \frac{8\gamma_t^2}{n_t^2} \|\theta_{t-1} - \theta^*\|^2 \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^2 \middle| \mathcal{F}_{t-1} \right]. \quad (23)$$

To bound the second and fourth-order terms in (23), we would need to study the recursive sequences: firstly, utilizing the Lipschitz continuity of $\nabla_{\theta} l_{t,i}$, together with Assumptions 2-p and 3-p, and that θ_{t-1} is \mathcal{F}_{t-1} -measurable (Assumption 1), we obtain

$$\mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^p | \mathcal{F}_{t-1}] \leq 2^{p-1} [\mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta_{t-1}) - \nabla_{\theta} l_{t,i}(\theta^*)\|^p | \mathcal{F}_{t-1}] + \mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta^*)\|^p | \mathcal{F}_{t-1}]] \leq 2^{p-1} [C_t^p \|\theta_{t-1} - \theta^*\|^p + \sigma^p], \quad (24)$$

for any $p \in [1, 4]$ using the bound $\|x + y\|^p \leq 2^{p-1} (\|x\|^p + \|y\|^p)$. Thus, we can bound the second-order term in (23) by

$$\mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^2 \middle| \mathcal{F}_{t-1} \right] \leq [2C_t^2 n_t + C_{\nabla}^2 (n_t - 1)n_t] \|\theta_{t-1} - \theta^*\|^2 + 2\sigma^2 n_t \leq [2C_t^2 n_t + C_{\nabla}^2 n_t^2 \mathbb{1}_{\{n_t > 1\}}] \|\theta_{t-1} - \theta^*\|^2 + 2\sigma^2 n_t, \quad (25)$$

following the same steps in the proof of Theorem 1, but with use of (24). Bounding the fourth-order term is a bit heavier computationally, but let us recall that $\|\sum_i x_i\|^2 = \sum_i \|x_i\|^2 + \sum_{i \neq j} \langle x_i, x_j \rangle = \sum_i \|x_i\|^2 + 2 \sum_{i < j} \langle x_i, x_j \rangle$. Then, we have that

$$\begin{aligned} \left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^4 &= \left(\sum_{i=1}^{n_t} \|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^2 + \sum_{i \neq j} \langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \nabla_{\theta} l_{t,j}(\theta_{t-1}) \rangle \right)^2 \\ &\leq 2 \left(\sum_{i=1}^{n_t} \|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^2 \right)^2 + 4 \left(\sum_{i < j} \langle \nabla_{\theta} l_{t,i}(\theta_{t-1}), \nabla_{\theta} l_{t,j}(\theta_{t-1}) \rangle \right)^2, \end{aligned} \quad (26)$$

as $(x + y)^2 \leq 2x^2 + 2y^2$. For the first term of (26), we have

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i=1}^{n_t} \|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^2 \right)^2 \middle| \mathcal{F}_{t-1} \right] &= \sum_{i=1}^{n_t} \mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^4 | \mathcal{F}_{t-1}] + \sum_{i \neq j} \mathbb{E}[\|\nabla_{\theta} l_{t,i}(\theta_{t-1})\|^2 \|\nabla_{\theta} l_{t,j}(\theta_{t-1})\|^2 | \mathcal{F}_{t-1}] \\ &\leq 8n_t [C_t^4 \|\theta_{t-1} - \theta^*\|^4 + \sigma^4] + 4n_t^2 \mathbb{1}_{\{n_t > 1\}} [C_t^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2]^2, \end{aligned}$$

using the bound from (24), $n_t(n_t - 1) \leq n_t^2 \mathbb{1}_{\{n_t > 1\}}$, and that $\mathcal{F}_{t-1} \subseteq \mathcal{F}_{t-1,i} \subset \mathcal{F}_{t-1,j}$ for all $0 \leq i < j$. To bound the second term of (26), we ease notation by denoting $\nabla_{\theta} l_{t,i}(\theta_{t-1})$ by v_i , giving us

$$\left(\sum_{i < j} \langle v_i, v_j \rangle \right)^2 = \sum_{i < j} \langle v_i, v_j \rangle^2 + \underbrace{\sum_{\substack{i < j, k < l \\ (i,j) \neq (k,l)}} \langle v_i, v_j \rangle \langle v_k, v_l \rangle}_{A} = \sum_{i < j} \langle v_i, v_j \rangle^2 + \underbrace{\sum_{\substack{i < j, k < l \\ (i,j) \neq (k,l), j=l}} \langle v_i, v_j \rangle \langle v_k, v_l \rangle}_{B} + \underbrace{\sum_{\substack{i < j, k < l \\ (i,j) \neq (k,l), j \neq l}} \langle v_i, v_j \rangle \langle v_k, v_l \rangle}_{C}.$$

By Cauchy-Schwarz inequality, we can bound the first term A, by

$$\mathbb{E}[A|\mathcal{F}_{t-1}] \leq \sum_{i < j} \mathbb{E}[\|v_i\|^2 \|v_j\|^2 | \mathcal{F}_{t-1}] \leq 2n_t(n_t - 1)[C_l^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2]^2 \leq 2n_t^2 \mathbb{1}_{\{n_t > 1\}} [C_l^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2]^2,$$

using that $\mathcal{F}_{t-1} \subseteq \mathcal{F}_{t-1,i} \subset \mathcal{F}_{t-1,j}$ for all $0 \leq i < j$. Next, since $l = j$ implies $i \neq k$, we have

$$\begin{aligned} \mathbb{E}[B|\mathcal{F}_{t-1}] &= \sum_{i < j, k < l, i \neq k, j=l} \mathbb{E}[\langle v_i, v_j \rangle \langle v_k, v_l \rangle | \mathcal{F}_{t-1}] = \sum_{i < j, k < l, i \neq k, j=l} \mathbb{E}[\mathbb{E}[\langle \mathbb{E}[v_i | \mathcal{F}_{t-1,i-1}], v_j \rangle \langle \mathbb{E}[v_k | \mathcal{F}_{t-1,k-1}], v_l \rangle | \mathcal{F}_{t-1,l-1}] | \mathcal{F}_{t-1}] \\ &= \sum_{i < j, k < l, i \neq k, j=l} \mathbb{E}[\mathbb{E}[\langle \nabla_{\theta} L(\theta_{t-1}), v_l \rangle^2 | \mathcal{F}_{t-1,l-1}] | \mathcal{F}_{t-1}] \leq \sum_{i < j, k < l, i \neq k, j=l} \mathbb{E}[\|\nabla_{\theta} L(\theta_{t-1})\|^2 \mathbb{E}[\|v_l\|^2 | \mathcal{F}_{t-1,l-1}] | \mathcal{F}_{t-1}] \\ &\leq \sum_{i < j, k < l, i \neq k, j=l} 2C_{\nabla}^2 \|\theta_{t-1} - \theta^*\|^2 [C_l^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2] = n_t(n_t - 1)(n_t - 2)C_{\nabla}^2 \|\theta_{t-1} - \theta^*\|^2 [C_l^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2] \\ &\leq n_t^3 \mathbb{1}_{\{n_t > 1\}} C_{\nabla}^2 \|\theta_{t-1} - \theta^*\|^2 [C_l^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2], \end{aligned}$$

using the Cauchy-Schwarz inequality and the bound in (24). In the same way, as $j \neq l$ includes $(i, j) \neq (k, l)$, we can rewrite C as

$$C = \sum_{i < j, k < l, j \neq l} \langle v_i, v_j \rangle \langle v_k, v_l \rangle = \underbrace{\sum_{i < j, k < l, i=k, j \neq l} \langle v_i, v_j \rangle \langle v_k, v_l \rangle}_{C_1} + \underbrace{\sum_{i < j, k < l, i \neq k, j \neq l} \langle v_i, v_j \rangle \langle v_k, v_l \rangle}_{C_2},$$

where $\mathbb{E}[C_1 | \mathcal{F}_{t-1}] = \mathbb{E}[B | \mathcal{F}_{t-1}]$. Finally, we can rewrite C_2 as

$$C_2 = \underbrace{\sum_{i < j, k < l, i \neq k, j \neq l, i=l, j \neq k} \langle v_i v_j \rangle \langle v_k v_l \rangle}_{C_{2,1}} + \underbrace{\sum_{i < j, k < l, i \neq k, j \neq l, i \neq l, j=k} \langle v_i v_j \rangle \langle v_k v_l \rangle}_{C_{2,2}} + \underbrace{\sum_{i < j, k < l, i \neq j \neq k \neq l} \langle v_i v_j \rangle \langle v_k v_l \rangle}_{C_{2,3}},$$

where $\mathbb{E}[C_{2,1} | \mathcal{F}_{t-1}] = \mathbb{E}[C_{2,2} | \mathcal{F}_{t-1}] = \mathbb{E}[B | \mathcal{F}_{t-1}]$, and

$$\mathbb{E}[C_{2,3} | \mathcal{F}_{t-1}] = \sum_{i < j, k < l, i \neq j \neq k \neq l} \mathbb{E}[\|\nabla_{\theta} L(\theta_{t-1})\|^4 | \mathcal{F}_{t-1}] \leq n_t(n_t - 1)(n_t - 2)(n_t - 3)C_{\nabla}^4 \|\theta_{t-1} - \theta^*\|^4 \leq n_t^4 \mathbb{1}_{\{n_t > 1\}} C_{\nabla}^4 \|\theta_{t-1} - \theta^*\|^4.$$

Thus, the fourth-order term of (23), is bounded by

$$\begin{aligned} \mathbb{E} \left[\left\| \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \right\|^4 \middle| \mathcal{F}_{t-1} \right] &\leq 16n_t [C_l^4 \|\theta_{t-1} - \theta^*\|^4 + \sigma^4] + 16n_t^2 \mathbb{1}_{\{n_t > 1\}} [C_l^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2]^2 \\ &\quad + 12n_t^3 \mathbb{1}_{\{n_t > 1\}} C_{\nabla}^2 \|\theta_{t-1} - \theta^*\|^2 [C_l^2 \|\theta_{t-1} - \theta^*\|^2 + \sigma^2] + 4n_t^4 \mathbb{1}_{\{n_t > 1\}} C_{\nabla}^4 \|\theta_{t-1} - \theta^*\|^4 \\ &\leq [16C_l^4 n_t + 16C_l^4 n_t^2 \mathbb{1}_{\{n_t > 1\}} + 12C_{\nabla}^2 C_l^2 n_t^3 \mathbb{1}_{\{n_t > 1\}} + 4C_{\nabla}^4 n_t^4 \mathbb{1}_{\{n_t > 1\}}] \|\theta_{t-1} - \theta^*\|^4 \\ &\quad + [32C_l^2 \sigma^2 n_t^2 \mathbb{1}_{\{n_t > 1\}} + 12C_{\nabla}^2 \sigma^2 n_t^3 \mathbb{1}_{\{n_t > 1\}}] \|\theta_{t-1} - \theta^*\|^2 + 16\sigma^4 n_t + 16\sigma^4 n_t^2 \mathbb{1}_{\{n_t > 1\}}. \end{aligned} \quad (27)$$

Combining the bound from (25) and (27) into (23), we obtain the recursive relation for the fourth-order moment:

$$\begin{aligned} \mathbb{E}[\|\theta_t - \theta^*\|^4 | \mathcal{F}_{t-1}] &\leq [1 - 4\mu\gamma_t + 8C_{\nabla}^2 \mathbb{1}_{\{n_t > 1\}} \gamma_t^2 + 16C_l^2 n_t^{-1} \gamma_t^2 + 48C_l^4 n_t^{-3} \gamma_t^4 + 48C_l^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 + 36C_{\nabla}^2 C_l^2 n_t^{-1} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 \\ &\quad + 12C_{\nabla}^4 \mathbb{1}_{\{n_t > 1\}} \gamma_t^4] \|\theta_{t-1} - \theta^*\|^4 + [16\sigma^2 n_t^{-1} \gamma_t^2 + 96C_l^2 \sigma^2 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 + 36C_{\nabla}^2 \sigma^2 n_t^{-1} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4] \|\theta_{t-1} - \theta^*\|^2 \\ &\quad + 48\sigma^4 n_t^{-3} \gamma_t^4 + 48\sigma^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4. \end{aligned}$$

Using the Young's inequalities, $2C_{\nabla}^2 C_l^2 \leq n_t C_{\nabla}^4 + n_t^{-1} C_l^4$, $16\sigma^2 n_t^{-1} \gamma_t^2 \|\theta_{t-1} - \theta^*\|^2 \leq 2\mu\gamma_t \|\theta_{t-1} - \theta^*\|^4 + 32\sigma^4 \mu^{-1} n_t^{-2} \gamma_t^3$, $2C_l^2 \sigma^2 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 \|\theta_{t-1} - \theta^*\|^2 \leq C_l^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 \|\theta_{t-1} - \theta^*\|^4 + \sigma^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4$, and $2C_{\nabla}^2 \sigma^2 n_t^{-1} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 \|\theta_{t-1} - \theta^*\|^2 \leq C_{\nabla}^4 \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 \|\theta_{t-1} - \theta^*\|^4 + \sigma^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4$, yields,

$$\begin{aligned} \mathbb{E}[\|\theta_t - \theta^*\|^4 | \mathcal{F}_{t-1}] &\leq [1 - 2\mu\gamma_t + 8C_{\nabla}^2 \mathbb{1}_{\{n_t > 1\}} \gamma_t^2 + 16C_l^2 n_t^{-1} \gamma_t^2 + 48C_l^4 n_t^{-3} \gamma_t^4 + 114C_l^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 + 48C_{\nabla}^4 \mathbb{1}_{\{n_t > 1\}} \gamma_t^4] \|\theta_{t-1} - \theta^*\|^4 \\ &\quad + 32\mu^{-1} \sigma^4 n_t^{-2} \gamma_t^3 + 48\sigma^4 n_t^{-3} \gamma_t^4 + 114\sigma^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4. \end{aligned} \quad (28)$$

Taking, the expectation on both sides of the inequality in (28) yields the recursive relation for the fourth-order moment:

$$\begin{aligned} \Delta_t &\leq [1 - 2\mu\gamma_t + 8C_{\nabla}^2 \mathbb{1}_{\{n_t > 1\}} \gamma_t^2 + 16C_l^2 n_t^{-1} \gamma_t^2 + 48C_l^4 n_t^{-3} \gamma_t^4 + 114C_l^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4 + 48C_{\nabla}^4 \mathbb{1}_{\{n_t > 1\}} \gamma_t^4] \Delta_{t-1} \\ &\quad + 32\mu^{-1} \sigma^4 n_t^{-2} \gamma_t^3 + 48\sigma^4 n_t^{-3} \gamma_t^4 + 114\sigma^4 n_t^{-2} \mathbb{1}_{\{n_t > 1\}} \gamma_t^4. \end{aligned} \quad (29)$$

with $\Delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^4]$ for some $\Delta_0 \geq 0$. By Proposition 5, we achieve the (upper) bound of Δ_t in (29), given as

$$\Delta_t \leq \exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) \Pi_t^\Delta + \frac{32\sigma^4}{\mu^2} \max_{t/2 \leq i \leq t} \frac{\gamma_i^2}{n_i^2} + \frac{48\sigma^4}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i^3}{n_i^3} + \frac{114\sigma^4}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i^3 \mathbb{1}_{\{n_i > 1\}}}{n_i^2}.$$

where Π_t^Δ is given by

$$\begin{aligned} &\exp\left(32C_l^2 \sum_{i=1}^t \frac{\gamma_i^2}{n_i}\right) \exp\left(96C_l^4 \sum_{i=1}^t \frac{\gamma_i^4}{n_i^3}\right) \exp\left(228C_l^4 \sum_{i=1}^t \frac{\mathbb{1}_{\{n_i > 1\}} \gamma_i^4}{n_i^2}\right) \\ &\exp\left(16C_{\nabla}^2 \sum_{i=1}^t \mathbb{1}_{\{n_i > 1\}} \gamma_i^2\right) \exp\left(96C_{\nabla}^4 \sum_{i=1}^t \mathbb{1}_{\{n_i > 1\}} \gamma_i^4\right) \left(\Delta_0 + \frac{2\sigma^4}{C_l^4} + \frac{4\sigma^4 \gamma_1}{\mu C_l^2 n_1}\right), \end{aligned} \quad (30)$$

with use of

$$\max_{1 \leq i \leq t} \frac{32\mu^{-1} \sigma^4 n_i^{-2} \gamma_i + 48\sigma^4 n_i^{-3} \gamma_i^2 + 114\sigma^4 n_i^{-2} \mathbb{1}_{\{n_i > 1\}} \gamma_i^2}{8C_{\nabla}^2 \mathbb{1}_{\{n_i > 1\}} + 16C_l^2 n_i^{-1} + 48C_l^4 n_i^{-3} \gamma_i^2 + 114C_l^4 n_i^{-2} \mathbb{1}_{\{n_i > 1\}} \gamma_i^2 + 48C_{\nabla}^4 \mathbb{1}_{\{n_i > 1\}} \gamma_i^2} \leq \frac{\sigma^4}{C_l^4} + \frac{2\sigma^4 \gamma_1}{\mu C_l^2 n_1}.$$

At last, bounding the projected estimate (3) follows from that $\mathbb{E}[\|\mathcal{P}_\Theta(\theta) - \theta^*\|^2] \leq \mathbb{E}[\|\theta - \theta^*\|^2]$, $\forall \theta \in \Theta$. \square

7.2.1. Proofs for Section 4.1

Proof of Theorem 2. Following Polyak and Juditsky [26], we rewrite (2) to

$$\theta_t = \theta_{t-1} - \frac{\gamma_t}{n_t} \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1}) \iff \frac{1}{\gamma_t} (\theta_{t-1} - \theta_t) = \nabla_{\theta} l_t(\theta_{t-1}), \quad (31)$$

where $\nabla_{\theta} l_t(\theta_{t-1})$ denotes $n_t^{-1} \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1})$. Note $\nabla_{\theta} l_t(\theta_{t-1}) \approx \nabla_{\theta} l_t(\theta^*) + \nabla_{\theta}^2 l_t(\theta^*)(\theta_{t-1} - \theta^*)$, and that $\nabla_{\theta} l_t(\theta^*)$ and $\nabla_{\theta} l_t(\theta) - \nabla_{\theta} L(\theta)$ behaves almost like an i.i.d. sequences with zero mean. Thus, $\theta_t - \theta^*$ behaves like $-\nabla_{\theta} L(\theta^*)^{-1} N_t^{-1} \sum_{i=1}^t n_i \nabla_{\theta} l_i(\theta^*)$ leading to a bound in $\mathcal{O}(\sqrt{N_t})$. Observe that

$$\nabla_{\theta}^2 L(\theta^*)(\theta_{t-1} - \theta^*) = \nabla_{\theta} l_t(\theta_{t-1}) - \nabla_{\theta} l_t(\theta^*) - \underbrace{[\nabla_{\theta} l_t(\theta_{t-1}) - \nabla_{\theta} l_t(\theta^*) - \nabla_{\theta} L(\theta_{t-1})]}_{\text{martingale term}} - \underbrace{[\nabla_{\theta} L(\theta_{t-1}) - \nabla_{\theta}^2 L(\theta^*)(\theta_{t-1} - \theta^*)]}_{\text{rest term}},$$

where $\nabla_{\theta}^2 L(\theta^*)$ is invertible with lowest eigenvalue greater than μ , i.e., $\nabla_{\theta}^2 L(\theta^*) \geq \mu$. Thus, summing the parts and using the Minkowski's inequality, we obtain the inequality:

$$\begin{aligned} \left(\mathbb{E} \left[\|\bar{\theta}_t - \theta^*\|^2 \right] \right)^{\frac{1}{2}} &\leq \left(\mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i \nabla_{\theta} l_i(\theta^*) \right\|^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i \nabla_{\theta} l_i(\theta_{i-1}) \right\|^2 \right] \right)^{\frac{1}{2}} \\ &\quad + \left(\mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i [\nabla_{\theta} l_i(\theta_{i-1}) - \nabla_{\theta} l_i(\theta^*) - \nabla_{\theta} L(\theta_{i-1})] \right\|^2 \right] \right)^{\frac{1}{2}} \\ &\quad + \left(\mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i [\nabla_{\theta} L(\theta_{i-1}) - \nabla_{\theta}^2 L(\theta^*)(\theta_{i-1} - \theta^*)] \right\|^2 \right] \right)^{\frac{1}{2}}. \end{aligned}$$

As $(\nabla_{\theta} l_{t,i}(\theta^*))$ is a square-integrable martingale increment sequences on \mathbb{R}^d (Assumption 1), we have

$$\mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i \nabla_{\theta} l_i(\theta^*) \right\|^2 \right] \leq \frac{1}{N_t^2} \sum_{i=1}^t \sum_{j=1}^{n_i} \mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \nabla_{\theta} l_{i,j}(\theta^*) \right\|^2 \right] \leq \frac{\text{Tr} \left[\nabla_{\theta}^2 L(\theta^*)^{-1} \Sigma \nabla_{\theta}^2 L(\theta^*)^{-1} \right]}{N_t}, \quad (32)$$

using Assumption 4. To ease notation, we denote $\text{Tr}[\nabla_{\theta}^2 L(\theta^*)^{-1} \Sigma \nabla_{\theta}^2 L(\theta^*)^{-1}]$ by Λ . Next, note that for all $t \geq 1$, we have the relation in (31), giving us

$$\frac{1}{N_t} \sum_{i=1}^t n_i \nabla_{\theta} l_i(\theta_{i-1}) = \frac{1}{N_t} \sum_{i=1}^t \frac{n_i}{\gamma_i} (\theta_{i-1} - \theta_i) = \frac{1}{N_t} \sum_{i=1}^{t-1} (\theta_i - \theta^*) \left(\frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right) - \frac{1}{N_t} (\theta_t - \theta^*) \frac{n_t}{\gamma_t} + \frac{1}{N_t} (\theta_0 - \theta^*) \frac{n_1}{\gamma_1},$$

leading to

$$\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i \nabla_{\theta} l_i(\theta_{i-1}) \right\| \leq \frac{1}{N_t \mu} \sum_{i=1}^{t-1} \|\theta_i - \theta^*\| \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| + \frac{1}{N_t \mu} \|\theta_t - \theta^*\| \frac{n_t}{\gamma_t} + \frac{1}{N_t \mu} \|\theta_0 - \theta^*\| \frac{n_1}{\gamma_1}.$$

Hence, with the notion of $\delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^2]$ this expression can be simplified to

$$\left(\mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i \nabla_{\theta} l_i(\theta_{i-1}) \right\|^2 \right] \right)^{\frac{1}{2}} \leq \frac{1}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{\frac{1}{2}} \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| + \frac{n_t}{N_t \gamma_t \mu} \delta_t^{\frac{1}{2}} + \frac{n_1}{N_t \gamma_1 \mu} \delta_0^{\frac{1}{2}}. \quad (33)$$

For the martingale term, we have

$$\begin{aligned} \mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i [\nabla_{\theta} l_i(\theta_{i-1}) - \nabla_{\theta} l_i(\theta^*) - \nabla_{\theta} L(\theta_{i-1})] \right\|^2 \right] &\leq \frac{1}{N_t^2 \mu^2} \sum_{i=1}^t n_i^2 \mathbb{E} \left[\|\nabla_{\theta} l_i(\theta_{i-1}) - \nabla_{\theta} l_i(\theta^*)\|^2 \right] \\ &= \frac{1}{N_t^2 \mu^2} \sum_{i=1}^t \mathbb{E} \left[\left\| \sum_{j=1}^{n_i} \nabla_{\theta} l_{i,j}(\theta_{i-1}) - \nabla_{\theta} l_{i,j}(\theta^*) \right\|^2 \right] \leq \frac{1}{N_t^2 \mu^2} \sum_{i=1}^t \sum_{j=1}^{n_i} \left(\mathbb{E} \left[\|\nabla_{\theta} l_{i,j}(\theta_{i-1}) - \nabla_{\theta} l_{i,j}(\theta^*)\|^2 \right] \right)^{\frac{1}{2}} \leq \frac{C_l^2}{N_t^2 \mu^2} \sum_{i=1}^t n_i \delta_{i-1}, \end{aligned} \quad (34)$$

by the Cauchy-Schwarz inequality and Assumption 2-p. For all $t \geq 1$, the rest term is directly bounded by (7):

$$\left(\mathbb{E} \left[\left\| \nabla_{\theta}^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t n_i [\nabla_{\theta} L(\theta_{i-1}) - \nabla_{\theta}^2 L(\theta^*)(\theta_{i-1} - \theta^*)] \right\|^2 \right] \right)^{\frac{1}{2}} \leq \frac{C'_V}{N_t \mu} \sum_{i=1}^t n_i \Delta_{i-1}^{\frac{1}{2}}, \quad (35)$$

with the notion $\Delta_t = \mathbb{E}[\|\theta_t - \theta^*\|^4]$. Finally, combining the terms from (32) to (35), gives us

$$\bar{\delta}_t^{1/2} \leq \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{1}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| + \frac{n_t}{N_t \gamma_t \mu} \delta_t^{1/2} + \frac{n_1}{N_t \gamma_1 \mu} \delta_0^{1/2} + \frac{C_l}{N_t \mu} \left(\sum_{i=1}^t n_i \delta_{i-1} \right)^{1/2} + \frac{C'_V}{N_t \mu} \sum_{i=1}^t n_i \Delta_{i-1}^{1/2}, \quad (36)$$

where $\bar{\delta}_t = \mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$, which can be simplified into (12) by shifting the indices and collecting the δ_0 terms. \square

Proof of Corollary 3. As $n_t = C_\rho$ for all $t \geq 1$, we simplify the bound for $\bar{\delta}_t$ in (12) to

$$\bar{\delta}_t^{1/2} \leq \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{C_\rho}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} \left| \frac{1}{\gamma_{i+1}} - \frac{1}{\gamma_i} \right| + \frac{C_\rho}{N_t \gamma_t \mu} \delta_t^{1/2} + \frac{C_\rho}{N_t \mu} \left(\frac{1}{\gamma_1} + C_l \right) \delta_0^{1/2} + \frac{C_l C_\rho^{1/2}}{N_t \mu} \left(\sum_{i=1}^{t-1} \delta_i \right)^{1/2} + \frac{C_\rho' C_\rho}{N_t \mu} \sum_{i=0}^{t-1} \Delta_i^{1/2}. \quad (37)$$

The second-order moment δ_t is bounded by Corollary 1 but with use of (20) as we work in terms of t . The fourth-order moment Δ_t from Lemma 1 can be simplified to:

$$\begin{aligned} \Delta_t &\leq \exp \left(-\mu \sum_{i=t/2}^t \gamma_i \right) \Pi_\infty^c + \frac{1}{\mu} \left(\frac{32\sigma^4}{\mu C_\rho^2} \max_{t/2 \leq i \leq t} \gamma_i^2 + \frac{48\sigma^4}{C_\rho^3} \max_{t/2 \leq i \leq t} \gamma_i^3 + \frac{114\sigma^4 \mathbb{1}_{\{C_\rho > 1\}}}{C_\rho^2} \max_{t/2 \leq i \leq t} \gamma_i^3 \right) \\ &\leq \exp \left(-\frac{\mu C_\gamma C_\rho^\beta t^{1-\alpha}}{2^{1-\alpha}} \right) \Pi_\infty^c + \frac{1}{\mu} \left(\frac{2^{2\alpha} 32\sigma^4 C_\gamma^2 C_\rho^{2\beta}}{\mu C_\rho^2 t^{2\alpha}} + \frac{2^{3\alpha} 48\sigma^4 C_\gamma^3 C_\rho^{3\beta}}{C_\rho^3 t^{3\alpha}} + \frac{2^{3\alpha} 114\sigma^4 C_\gamma^3 C_\rho^{3\beta} \mathbb{1}_{\{C_\rho > 1\}}}{C_\rho^2 t^{3\alpha}} \right), \end{aligned}$$

using that $\gamma_t = C_\gamma C_\rho^\beta t^{-\alpha}$ is decreasing as $\alpha \in (1/2, 1)$. Regarding Π_t^A defined in (30), we can bound it by

$$\begin{aligned} \Pi_\infty^c &= \exp \left(\frac{64\alpha C_l^2 C_\gamma^2 C_\rho^{2\beta}}{(2\alpha - 1) C_\rho} \right) \exp \left(\frac{(192 + 456C_\rho \mathbb{1}_{\{C_\rho > 1\}}) C_l^4 C_\gamma^4 C_\rho^{4\beta}}{C_\rho^3} \right) \exp \left(\frac{32\alpha C_\rho^2 C_\gamma^2 C_\rho^{2\beta} \mathbb{1}_{\{C_\rho > 1\}}}{2\alpha - 1} \right) \\ &\quad \exp \left(192 C_\rho^4 C_\gamma^4 C_\rho^{4\beta} \mathbb{1}_{\{C_\rho > 1\}} \right) \left(\Delta_0 + \frac{2\sigma^4}{C_l^4} + \frac{4\sigma^4 C_\gamma}{\mu C_l^2 C_\rho^{1-\beta}} \right), \end{aligned}$$

using $\sum_{i=1}^t i^{-2\alpha} \leq 2\alpha/(2\alpha - 1)$ and $\sum_{i=1}^t i^{-4\alpha} \leq 2$. Note that Π_∞^c is a finite constant, independent of t . To bound the first term of (37), namely $\frac{C_\rho}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} |\gamma_{i+1}^{-1} - \gamma_i^{-1}|$, we remark that $|\gamma_{t+1}^{-1} - \gamma_t^{-1}| \leq C_\gamma^{-1} C_\rho^{-\beta} \alpha t^{\alpha-1}$, one has (since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$),

$$\frac{C_\rho}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} \left| \frac{1}{\gamma_{i+1}} - \frac{1}{\gamma_i} \right| \leq \frac{C_\rho^{1-\beta} \alpha}{C_\gamma \mu N_t} \sum_{i=1}^t i^{\alpha-1} \left(\exp \left(-\frac{\mu C_\gamma C_\rho^\beta i^{1-\alpha}}{2^{2-\alpha}} \right) \sqrt{\pi_\infty^c} + \frac{2^{\frac{1+\alpha}{2}} \sigma \sqrt{C_\gamma}}{\sqrt{\mu} C_\rho^{\frac{1-\beta}{2}} i^{\alpha/2}} \right). \quad (38)$$

For simplicity, let us denote

$$A_\infty^c = \sum_{i=0}^{\infty} \exp \left(-\frac{\mu C_\gamma C_\rho^\beta i^{1-\alpha}}{2^{2-\alpha}} \right) \geq \sum_{i=0}^{\infty} i^{\alpha-1} \exp \left(-\frac{\mu C_\gamma C_\rho^\beta i^{1-\alpha}}{2^{2-\alpha}} \right),$$

as $\alpha < 1$. Thus, the first part of (38) is bounded as follows:

$$\frac{C_\rho^{1-\beta} \alpha \sqrt{\pi_\infty^c}}{C_\gamma \mu N_t} \sum_{i=1}^t i^{\alpha-1} \exp \left(-\frac{\mu C_\gamma C_\rho^\beta i^{1-\alpha}}{2^{2-\alpha}} \right) \leq \frac{C_\rho^{1-\beta} \alpha \sqrt{\pi_\infty^c} A_\infty^c}{C_\gamma \mu N_t}.$$

Furthermore, with the help of an integral test for convergence, one has $\sum_{i=1}^t i^{\alpha/2-1} \leq 1 + \int_1^t s^{\alpha/2-1} ds = 1 + (2/\alpha)t^{\alpha/2} - (2/\alpha) \leq (2/\alpha)t^{\alpha/2}$, such that the second part of (38) can be bounded by

$$\frac{2^{\frac{1+\alpha}{2}} \sigma C_\rho^{\frac{1-\beta}{2}} \alpha}{C_\gamma^{1/2} \mu^{3/2} N_t} \sum_{i=1}^t i^{\alpha/2-1} \leq \frac{2^{\frac{3+\alpha}{2}} \sigma C_\rho^{\frac{1-\beta}{2}} t^{\alpha/2}}{C_\gamma^{1/2} \mu^{3/2} N_t} = \frac{2^{\frac{3+\alpha}{2}} \sigma C_\rho^{\frac{1-\alpha-\beta}{2}}}{C_\gamma^{1/2} \mu^{3/2} N_t^{1-\alpha/2}}.$$

By combining this, we get

$$\frac{C_\rho}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} \left| \frac{1}{\gamma_{i+1}} - \frac{1}{\gamma_i} \right| \leq \frac{C_\rho^{1-\beta} \alpha \sqrt{\pi_\infty^c} A_\infty^c}{C_\gamma \mu N_t} + \frac{2^{\frac{3+\alpha}{2}} \sigma C_\rho^{\frac{1-\alpha-\beta}{2}}}{\sqrt{C_\gamma} \mu^{3/2} N_t^{1-\alpha/2}}. \quad (39)$$

Similarly, second term of (37), can be bounded by

$$\frac{C_\rho}{N_t \gamma_t \mu} \delta_t^{\frac{1}{2}} \leq \frac{C_\rho^{1-\alpha-\beta}}{C_\gamma \mu N_t^{1-\alpha}} \left(\exp\left(-\frac{\mu C_\gamma C_\rho^\beta t^{1-\alpha}}{2^{2-\alpha}}\right) \sqrt{\pi_\infty^c} + \frac{2^{\frac{1+\alpha}{2}} \sigma \sqrt{C_\gamma}}{\sqrt{\mu} C_\rho^{\frac{1-\beta}{2}} t^{\alpha/2}} \right) \leq \frac{C_\rho^{2-\alpha-\beta} \sqrt{\pi_\infty^c} A_\infty^c}{C_\gamma \mu N_t^{2-\alpha}} + \frac{2^{\frac{1+\alpha}{2}} C_\rho^{\frac{1-\alpha-\beta}{2}} \sigma}{\sqrt{C_\gamma} \mu^{3/2} N_t^{1-\alpha/2}},$$

using $\exp(-\mu C_\gamma C_\rho^\beta t^{1-\alpha}/2^{2-\alpha}) = A_t^c \leq t^{-1} \sum_{i=1}^t A_i^c \leq t^{-1} A_\infty^c$ as A_t^c is decreasing. In a same way, one has

$$\frac{C_l C_\rho^{\frac{1}{2}}}{N_t \mu} \left(\sum_{i=1}^{t-1} \delta_i \right)^{\frac{1}{2}} \leq \frac{C_l C_\rho^{\frac{1}{2}}}{N_t \mu} \left(A_\infty^c \pi_\infty^c + \frac{2^{1+\alpha} \sigma^2 C_\gamma t^{1-\alpha}}{(1-\alpha) \mu C_\rho^{1-\beta}} \right)^{1/2} \leq \frac{C_l C_\rho^{\frac{1}{2}} \sqrt{\pi_\infty^c} \sqrt{A_\infty^c}}{N_t \mu} + \frac{2^{\frac{1+\alpha}{2}} C_l \sigma \sqrt{C_\gamma}}{C_\rho^{\frac{1-\alpha-\beta}{2}} \mu^{3/2} N_t^{\frac{1+\alpha}{2}}}.$$

Bound the last term of (37), is done as follows,

$$\begin{aligned} \frac{C'_\nabla C_\rho}{N_t \mu} \sum_{i=0}^{t-1} \Delta_i^{\frac{1}{2}} &\leq \frac{C'_\nabla C_\rho}{N_t \mu} \sum_{i=0}^{t-1} \exp\left(-\frac{\mu C_\gamma C_\rho^\beta i^{1-\alpha}}{2^{2-\alpha}}\right) \sqrt{\pi_\infty^c} + \frac{2^\alpha 6 C'_\nabla \sigma^2 C_\gamma C_\rho^\beta}{N_t \mu^2} \sum_{i=1}^{t-1} i^{-\alpha} + \frac{(6 + 7 \mathbb{1}_{\{C_\rho > 1\}}) 2^{3\alpha/2} C'_\nabla \sigma^2 C_\gamma^{3/2} C_\rho^{3\beta/2}}{N_t \mu^{3/2}} \sum_{i=1}^{t-1} i^{-3\alpha/2} \\ &\leq \frac{C'_\nabla C_\rho \sqrt{\pi_\infty^c} A_\infty^c}{N_t \mu} + \frac{2^\alpha 6 C'_\nabla \sigma^2 C_\gamma}{C_\rho^{1-\alpha-\beta} \mu^2 N_t^\alpha} + \frac{(6 + 7 \mathbb{1}_{\{C_\rho > 1\}}) 2^{3\alpha/2} C'_\nabla \sigma^2 C_\gamma^{3/2} C_\rho^{3\beta/2} \psi_{3\alpha/2}^0(N_t/C_\rho)}{\mu^{3/2} N_t}. \end{aligned}$$

Thus, by collecting the terms above, we obtain:

$$\bar{\delta}_t^{1/2} \leq \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{6\sigma C_\rho^{\frac{1-\alpha-\beta}{2}}}{\sqrt{C_\gamma} \mu^{3/2} N_t^{1-\alpha/2}} + \frac{2^\alpha 6 C'_\nabla \sigma^2 C_\gamma}{C_\rho^{1-\alpha-\beta} \mu^2 N_t^\alpha} + \frac{C_\rho^{2-\alpha-\beta} \sqrt{\pi_\infty^c} A_\infty^c}{C_\gamma \mu N_t^{2-\alpha}} + \frac{2^{\frac{1+\alpha}{2}} C_l \sigma \sqrt{C_\gamma}}{C_\rho^{\frac{1-\alpha-\beta}{2}} \mu^{3/2} N_t^{\frac{1+\alpha}{2}}} + \frac{C_\rho \Gamma_c}{\mu N_t} + \frac{(6 + 7 \mathbb{1}_{\{C_\rho > 1\}}) 2^{3\alpha/2} C'_\nabla \sigma^2 C_\gamma^{3/2} C_\rho^{3\beta/2}}{\mu^{3/2} \psi_{3\alpha/2}^0(N_t/C_\rho)^{-1} N_t},$$

where $\Gamma_c = (1/C_\gamma C_\rho^\beta + C_l) \delta_0^{1/2} + C_l \sqrt{\pi_\infty^c} A_\infty^c / C_\rho^{1/2} + \sqrt{\pi_\infty^c} A_\infty^c / C_\gamma C_\rho^\beta + C'_\nabla \sqrt{\pi_\infty^c} A_\infty^c$. \square

Proof of Corollary 4. The steps of the proof follows the ones of Corollary 3 with the smart notation of ϕ and $\tilde{\rho}$: The bound for $\bar{\delta}_t$ in (12) is given by

$$\bar{\delta}_t^{1/2} \leq \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{1}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| + \frac{n_t}{N_t \gamma_t \mu} \delta_t^{1/2} + \frac{n_1}{N_t \mu} \left(\frac{1}{\gamma_1} + C_l \right) \delta_0^{1/2} + \frac{C_l}{N_t \mu} \left(\sum_{i=1}^{t-1} n_{i+1} \delta_i \right)^{1/2} + \frac{C'_\nabla}{N_t \mu} \sum_{i=0}^{t-1} n_{i+1} \Delta_i^{1/2}, \quad (40)$$

where the learning rate is on the form $\gamma_t = C_\gamma n_t^\beta t^{-\alpha}$ with $n_t = C_\rho t^\rho$. The second-order moment δ_t is upper bounded by (21) from Corollary 2. The fourth-order moment Δ_t from Lemma 1 can be simplified as follows,

$$\Delta_t \leq \exp\left(-\mu \sum_{i=t/2}^t \gamma_i\right) \Pi_\infty^v + \frac{32\sigma^4}{\mu^2} \max_{t/2 \leq i \leq t} \frac{\gamma_i^2}{n_i^2} + \frac{162\sigma^4}{\mu} \max_{t/2 \leq i \leq t} \frac{\gamma_i^3}{n_i^2},$$

as $n_t \geq 1$ for any $t \geq 1$ and $\beta \leq 1$, and

$$\Pi_\infty^v = \exp\left(\frac{32(\alpha - \beta \tilde{\rho}) C_\gamma^2 C_\rho^{2\beta} (2C_l^2 + C_\nabla^2)}{2(\alpha - \beta \tilde{\rho}) - 1}\right) \exp\left(192 C_\gamma^4 C_\rho^{4\beta} (4C_l^4 + C_\nabla^4)\right) \left(\Delta_0 + \frac{2\sigma^4}{C_l^4} + \frac{4\sigma^4 C_\gamma}{\mu C_l^2 C_\rho^{1-\beta}} \right)$$

using that $\sum_{i=1}^t i^{-a} \leq 2$ for $a \geq 2$. Next, for $\rho \geq 0$, we have

$$\Delta_t \leq \exp\left(-\frac{\mu C_\gamma C_\rho^\beta t^{1+\beta\rho-\alpha}}{2^{1+\beta\rho-\alpha}}\right) \Pi_\infty^v + \frac{2^{2\alpha-2\beta\rho+2\rho} 32\sigma^4 C_\gamma^2 C_\rho^{2\beta}}{\mu^2 C_\rho^{2\rho} t^{2\alpha-2\beta\rho+2\rho}} + \frac{2^{3\alpha-3\beta\rho+2\rho} 162\sigma^4 C_\gamma^3 C_\rho^{3\beta}}{\mu C_\rho^{2\rho} t^{3\alpha-3\beta\rho+2\rho}},$$

using that $\alpha - \beta\rho \in (1/2, 1)$. If $\rho < 0$, one directly have

$$\Delta_t \leq \exp\left(-\frac{\mu C_\gamma C_\rho^\beta t^{1-\alpha}}{2^{1-\alpha}}\right) \Pi_\infty^v + \frac{2^{2\alpha} 32\sigma^4 C_\gamma^2 C_\rho^{2\beta}}{\mu^2 t^{2\alpha}} + \frac{2^{3\alpha} 162\sigma^4 C_\gamma^3 C_\rho^{3\beta}}{\mu t^{3\alpha}}.$$

With the notion of ϕ and $\tilde{\rho}$, we can combine the two ρ -cases as follows:

$$\Delta_t \leq \exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} t^{(1-\phi)(1+\tilde{\rho})}}{2^{(1-\phi)(1+\tilde{\rho})}}\right) \Pi_\infty^v + \frac{2^{2\phi(1+\tilde{\rho})} 32\sigma^4 C_\gamma^2 C_\rho^{2\beta}}{\mu^2 C_\rho^{2 \mathbb{1}_{\{\rho \geq 0\}}} t^{2\phi(1+\tilde{\rho})}} + \frac{2^{3\phi(1+\tilde{\rho})-\tilde{\rho}} 162\sigma^4 C_\gamma^3 C_\rho^{3\beta}}{\mu C_\rho^{2 \mathbb{1}_{\{\rho \geq 0\}}} t^{3\phi(1+\tilde{\rho})-\tilde{\rho}}}.$$

We will in the following bound the terms for t but afterwards we will translate it to terms in N_t . If $\rho \geq 0$, the first relation is $t \geq (N_t/2C_\rho)^{1/(1+\rho)}$ since $N_t = C_\rho(t^\rho + \sum_{i=1}^{t-1} i^\rho) \leq C_\rho(t^\rho + \int_1^t x^\rho dx) \leq C_\rho(t^\rho + t^\rho \int_1^t dx) = C_\rho(t^\rho + t^{1+\rho}) \leq 2C_\rho t^{1+\rho}$ by use the integral test for convergence. Similarly, $N_t = C_\rho \sum_{i=1}^t i^\rho \geq C_\rho \int_0^t x^\rho dx = C_\rho t^{\rho+1}$, thus, $t \leq (N_t/C_\rho)^{1/(1+\rho)}$. If $\rho < 0$, one has $t \leq N_t$ and $N_t \leq C_\rho t$, i.e., $t \geq N_t/C_\rho$.

Bounding $\frac{1}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} |n_{i+1}/\gamma_{i+1} - n_i/\gamma_i|$, we first note $n_t/\gamma_t = C_\gamma^{-1} C_\rho^{1-\beta} t^{(1-\beta)\rho+\alpha}$ for $\rho \geq 0$. Thus, by the mean value theorem, we obtain:

$$\left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| \leq ((1-\beta)\rho + \alpha) \frac{C_\rho^{1-\beta}}{C_\gamma} \sup_{v \in (i, i+1)} |v^{(1-\beta)\rho+\alpha-1}| \leq \frac{((1-\beta)\rho + \alpha) C_\rho^{1-\beta}}{C_\gamma i^{1-(1-\beta)\rho-\alpha}}, \quad (41)$$

as $\alpha + (1-\beta)\rho \leq 1 - \rho$ since $\alpha - \beta\rho \in (1/2, 1)$. For $\rho < 0$, the mean value theorem gives us

$$\left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| = \frac{1}{C_\gamma} \left| n_{i+1}^{1-\beta} (i+1)^\alpha - n_i^{1-\beta} i^\alpha \right| \leq \frac{C_\rho^{1-\beta}}{C_\gamma} |(i+1)^\alpha - i^\alpha| \leq \frac{\alpha C_\rho^{1-\beta}}{C_\gamma} \sup_{v \in (i, i+1)} |v^{\alpha-1}| \leq \frac{\alpha C_\rho^{1-\beta}}{C_\gamma i^{1-\alpha}},$$

as $(n_t)_{t \geq 1}$ is a decreasing sequence and $\beta \leq 1$. Thus, for any $\rho \in (-1, 1)$, we have

$$\left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| \leq \frac{\phi(1+\tilde{\rho}) C_\rho^{1-\beta}}{C_\gamma i^{1-\phi(1+\tilde{\rho})}}.$$

By using this, we obtain:

$$\frac{1}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{1/2} \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| \leq \frac{\phi(1+\tilde{\rho}) C_\rho^{1-\beta}}{N_t \mu C_\gamma} \sum_{i=1}^t i^{\phi(1+\tilde{\rho})-1} \left(\exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\tilde{\rho})}}{2^{1+(1-\phi)(1+\tilde{\rho})}}\right) \sqrt{\pi_\infty^v} + \frac{2^{\frac{1+\phi(1+\tilde{\rho})}{2}} \sigma \sqrt{C_\gamma}}{\sqrt{\mu C_\rho^{\frac{(1-\beta)}{2} \mathbb{1}_{\{\rho \geq 0\}}} i^{\frac{\phi(1+\tilde{\rho})}{2}}}} \right).$$

Next, let us denote

$$A_\infty^v = \sum_{i=0}^{\infty} i^{\tilde{\rho}} \exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\tilde{\rho})}}{2^{1+(1-\phi)(1+\tilde{\rho})}}\right) \geq \sum_{i=0}^{\infty} i^{\phi(1+\tilde{\rho})-1} \exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\tilde{\rho})}}{2^{1+(1-\phi)(1+\tilde{\rho})}}\right),$$

since $\phi(1+\tilde{\rho}) - 1 = \alpha + (1-\beta)\tilde{\rho} - 1 \leq \tilde{\rho}$. Thus,

$$\frac{\phi(1+\tilde{\rho}) C_\rho^{1-\beta} \sqrt{\pi_\infty^v}}{N_t \mu C_\gamma} \sum_{i=1}^t i^{\phi(1+\tilde{\rho})-1} \exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\tilde{\rho})}}{2^{1+(1-\phi)(1+\tilde{\rho})}}\right) \leq \frac{\phi(1+\tilde{\rho}) C_\rho^{1-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{N_t \mu C_\gamma}.$$

Furthermore, with the help of an integral test for convergence, we have

$$\frac{\phi(1+\tilde{\rho}) 2^{\frac{1+\phi(1+\tilde{\rho})}{2}} \sigma C_\rho^{\frac{1-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}}}{\mu^{3/2} \sqrt{C_\gamma} N_t} \sum_{i=1}^t i^{\frac{\phi(1+\tilde{\rho})}{2}-1} \leq \frac{2^{\frac{3+\phi(1+\tilde{\rho})}{2}} \sigma C_\rho^{\frac{1-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}} t^{\frac{\phi(1+\tilde{\rho})}{2}}}{\mu^{3/2} \sqrt{C_\gamma} N_t} \leq \frac{2^{\frac{3+\phi(1+\tilde{\rho})}{2}} \sigma C_\rho^{\frac{1-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}}}{\mu^{3/2} \sqrt{C_\gamma} N_t^{1-\phi/2}}.$$

Summarising, with use of $\phi(1 + \bar{\rho}) < 2$, we obtain

$$\frac{1}{N_t \mu} \sum_{i=1}^{t-1} \delta_i^{\frac{1}{2}} \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| \leq \frac{\phi(1 + \bar{\rho}) C_\rho^{1-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{N_t \mu C_\gamma} + \frac{2^{\frac{3+\phi(1+\bar{\rho})}{2}} \sigma C_\rho^{\frac{1-\phi-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}}}{\mu^{3/2} \sqrt{C_\gamma} N_t^{1-\phi/2}} \leq \frac{2 C_\rho^{1-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{\mu C_\gamma N_t} + \frac{2^{\frac{3+\phi(1+\bar{\rho})}{2}} \sigma C_\rho^{\frac{1-\phi-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}}}{\mu^{3/2} \sqrt{C_\gamma} N_t^{1-\phi/2}}.$$

Similarly, for $\frac{n_t}{N_t \gamma_t \mu} \delta_t^{1/2}$, one have

$$\frac{n_t}{N_t \gamma_t \mu} \delta_t^{\frac{1}{2}} \leq \frac{C_\rho^{1-\beta} \sqrt{\pi_\infty^v} t^{\phi(1+\bar{\rho})}}{N_t C_\gamma \mu} \exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} t^{(1-\phi)(1+\bar{\rho})}}{2^{1+(1-\phi)(1+\bar{\rho})}}\right) + \frac{2^{\frac{1+\phi(1+\bar{\rho})}{2}} \sigma C_\rho^{\frac{1-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}} t^{\frac{\phi(1+\bar{\rho})}{2}}}{\mu^{3/2} \sqrt{C_\gamma} N_t} \leq \frac{C_\rho^{2-\phi-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{\mu C_\gamma N_t^{2-\phi}} + \frac{2^{\frac{1+\phi(1+\bar{\rho})}{2}} \sigma C_\rho^{\frac{1-\phi-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}}}{\mu^{3/2} \sqrt{C_\gamma} N_t^{1-\phi/2}}.$$

For $\frac{n_1}{N_1 \mu} (\gamma_1^{-1} + C_l) \delta_0^{1/2}$, we insert the definition of our learning functions, giving us

$$\frac{n_1}{N_1 \mu} \left(\frac{1}{\gamma_1} + C_l \right) \delta_0^{1/2} = \frac{C_\rho}{N_1 \mu} \left(\frac{1}{C_\gamma C_\rho^\beta} + C_l \right) \delta_0^{1/2}.$$

Bounding $\frac{C_l}{N_t \mu} (\sum_{i=1}^{t-1} n_{i+1} \delta_i)^{1/2}$, follows the ideas from above, using that $n_{t+1} \leq 2^{\bar{\rho}} n_t$, to obtain

$$\begin{aligned} \frac{C_l}{N_t \mu} \left(\sum_{i=1}^{t-1} n_{i+1} \delta_i \right)^{\frac{1}{2}} &\leq \frac{2^{\bar{\rho}/2} C_l}{N_t \mu} \left(C_\rho \sum_{i=1}^t i^{\bar{\rho}} \left(\exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\bar{\rho})}}{2^{(1-\phi)(1+\bar{\rho})}}\right) \pi_\infty^v + \frac{2^{1+\phi(1+\bar{\rho})} \sigma^2 C_\gamma}{\mu C_\rho^{(1-\beta) \mathbb{1}_{\{\rho \geq 0\}}} i^{\phi(1+\bar{\rho})}} \right) \right)^{\frac{1}{2}} \\ &= \frac{2^{\bar{\rho}/2} C_l}{N_t \mu} \left(C_\rho \pi_\infty^v \sum_{i=1}^t i^{\bar{\rho}} \exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\bar{\rho})}}{2^{(1-\phi)(1+\bar{\rho})}}\right) + \frac{2^{1+\phi(1+\bar{\rho})} \sigma^2 C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}}}{\mu} \sum_{i=1}^t i^{\beta \bar{\rho} - \alpha} \right)^{\frac{1}{2}} \\ &\leq \frac{2^{\bar{\rho}/2} C_l}{N_t \mu} \left(C_\rho \pi_\infty^v A_\infty^v + \frac{2^{\phi(1+\bar{\rho})} \sigma^2 C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} t^{(1-\phi)(1+\bar{\rho})}}{\mu} \right)^{\frac{1}{2}} \\ &\leq \frac{2^{\bar{\rho}/2} C_l \sqrt{C_\rho} \sqrt{\pi_\infty^v} \sqrt{A_\infty^v}}{\mu N_t} + \frac{2^{\frac{\phi(1+\bar{\rho})}{2}} C_l \sigma \sqrt{C_\gamma} C_\rho^{\beta/2 \mathbb{1}_{\{\rho \geq 0\}}} t^{\frac{(1-\phi)(1+\bar{\rho})}{2}}}{\mu^{3/2} N_t} \\ &\leq \frac{2^{\bar{\rho}/2} C_l \sqrt{C_\rho} \sqrt{\pi_\infty^v} \sqrt{A_\infty^v}}{\mu N_t} + \frac{2^{\frac{\phi(1+\bar{\rho})}{2}} C_l \sigma \sqrt{C_\gamma}}{\mu^{3/2} C_\rho^{\frac{1-\phi-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}} N_t^{\frac{1+\phi}{2}}}. \end{aligned}$$

Likewise, for $\frac{C'_v}{N_t \mu} \sum_{i=0}^{t-1} n_{i+1} \Delta_i^{1/2}$, we get

$$\begin{aligned} \frac{C'_v}{N_t \mu} \sum_{i=0}^{t-1} n_{i+1} \Delta_i^{\frac{1}{2}} &\leq \frac{2^{\bar{\rho}} C'_v C_\rho}{N_t \mu} \sum_{i=1}^{t-1} i^{\bar{\rho}} \left(\exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\bar{\rho})}}{2^{(1-\phi)(1+\bar{\rho})}}\right) \Pi_\infty^v + \frac{2^{2\phi(1+\bar{\rho})} 32 \sigma^4 C_\gamma^2 C_\rho^{2\beta}}{\mu^2 C_\rho^{2 \mathbb{1}_{\{\rho \geq 0\}}} i^{2\phi(1+\bar{\rho})}} + \frac{2^{3\phi(1+\bar{\rho})-\bar{\rho}} 162 \sigma^4 C_\gamma^3 C_\rho^{3\beta}}{\mu C_\rho^{2 \mathbb{1}_{\{\rho \geq 0\}}} i^{3\phi(1+\bar{\rho})-\bar{\rho}}} \right)^{\frac{1}{2}} \\ &\leq \frac{2^{\bar{\rho}} C'_v C_\rho}{N_t \mu} \sum_{i=1}^{t-1} i^{\bar{\rho}} \left(\exp\left(-\frac{\mu C_\gamma C_\rho^{\beta \mathbb{1}_{\{\rho \geq 0\}}} i^{(1-\phi)(1+\bar{\rho})}}{2^{1+(1-\phi)(1+\bar{\rho})}}\right) \sqrt{\Pi_\infty^v} + \frac{2^{\phi(1+\bar{\rho})} 6 \sigma^2 C_\gamma C_\rho^\beta}{\mu C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} i^{\phi(1+\bar{\rho})}} + \frac{2^{3\phi(1+\bar{\rho})/2-\bar{\rho}/2} 13 \sigma^2 C_\gamma^3/2 C_\rho^{3\beta/2}}{\mu^{1/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} i^{3\phi(1+\bar{\rho})/2}} \right) \\ &\leq \frac{2^{\bar{\rho}} C'_v C_\rho \sqrt{\Pi_\infty^v} A_\infty^v}{\mu N_t} + \frac{2^{\phi(1+\bar{\rho})+\bar{\rho}} C'_v \sigma^2 C_\gamma C_\rho^{1+\beta}}{\mu^2 C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t} \sum_{i=1}^{t-1} i^{\beta \bar{\rho} - \alpha} + \frac{2^{3\phi(1+\bar{\rho})/2+\bar{\rho}/2} C'_v \sigma^2 C_\gamma^3/2 C_\rho^{1+3\beta/2}}{\mu^{3/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t} \sum_{i=1}^{t-1} i^{3(\beta \bar{\rho} - \alpha)/2}, \end{aligned}$$

where the second term can be bounded as

$$\frac{2^{(1+\phi)(1+\bar{\rho})-1} C'_v \sigma^2 C_\gamma C_\rho^{1+\beta}}{\mu^2 C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t} \sum_{i=1}^{t-1} i^{\beta \bar{\rho} - \alpha} \leq \frac{2^{(1+\phi)(1+\bar{\rho})-1} C'_v \sigma^2 C_\gamma C_\rho^{1+\beta} t^{1+\beta \bar{\rho} - \alpha}}{(1 + \beta \bar{\rho} - \alpha) \mu^2 C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t} \leq \frac{2^{(1+\phi)(1+\bar{\rho})-2} C'_v \sigma^2 C_\gamma}{\mu^2 C_\rho^{1-\phi-\beta} N_t^\phi},$$

and the third term by

$$\frac{2^{3(1+\phi)(1+\bar{p})/2} C'_\nabla \sigma^2 C_\gamma^{3/2} C_\rho^{1+3\beta/2}}{\mu^{3/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t} \sum_{i=1}^{t-1} i^{3(\beta\bar{p}-\alpha)/2} \leq \frac{2^{3(1+\phi)(1+\bar{p})/2} C'_\nabla \sigma^2 C_\gamma^{3/2} C_\rho^{1+3\beta/2} \psi_{3(\alpha-\beta\bar{p})/2}^{\bar{p}} (N_t/C_\rho)}{\mu^{3/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t}.$$

By collecting these bounds, we get

$$\frac{C'_\nabla}{N_t \mu} \sum_{i=0}^{t-1} n_{i+1} \Delta_i^{\frac{1}{2}} \leq \frac{2^{\bar{p}} C'_\nabla C_\rho \sqrt{\Pi_\infty^v} A_\infty^v}{\mu N_t} + \frac{2^{(1+\phi)(1+\bar{p})-2} C'_\nabla \sigma^2 C_\gamma}{\mu^2 C_\rho^{1-\phi-\beta} N_t^\phi} + \frac{2^{3(1+\phi)(1+\bar{p})/2} C'_\nabla \sigma^2 C_\gamma^{3/2} C_\rho^{1+3\beta/2} \psi_{3(\alpha-\beta\bar{p})/2}^{\bar{p}} (N_t/C_\rho)}{\mu^{3/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t}.$$

Combining our findings from above, we have

$$\begin{aligned} \bar{\delta}_t^{1/2} &\leq \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{2C_\rho^{1-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{\mu C_\gamma N_t} + \frac{2^{\frac{3+\phi(1+\bar{p})}{2}} \sigma C_\rho^{\frac{1-\phi-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}}}{\mu^{3/2} \sqrt{C_\gamma} N_t^{1-\phi/2}} + \frac{C_\rho^{2-\phi-\beta} \sqrt{\pi_\infty^v} A_\infty^v}{\mu C_\gamma N_t^{2-\phi}} + \frac{2^{\frac{1+\phi(1+\bar{p})}{2}} \sigma C_\rho^{\frac{1-\phi-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}}}{\mu^{3/2} \sqrt{C_\gamma} N_t^{1-\phi/2}} \\ &\quad + \frac{C_\rho}{N_t \mu} \left(\frac{1}{C_\gamma C_\rho^\beta} + C_l \right) \delta_0^{\frac{1}{2}} + \frac{2^{\bar{p}/2} C_l \sqrt{C_\rho} \sqrt{\pi_\infty^v} \sqrt{A_\infty^v}}{\mu N_t} + \frac{2^{\frac{\phi(1+\bar{p})}{2}} C_l \sigma \sqrt{C_\gamma}}{\mu^{3/2} C_\rho^{\frac{1-\phi-\beta}{2} \mathbb{1}_{\{\rho \geq 0\}}} N_t^{\frac{1+\phi}{2}}} + \frac{2^{\bar{p}} C'_\nabla C_\rho \sqrt{\Pi_\infty^v} A_\infty^v}{\mu N_t} \\ &\quad + \frac{2^{(1+\phi)(1+\bar{p})-2} C'_\nabla \sigma^2 C_\gamma}{\mu^2 C_\rho^{1-\phi-\beta} N_t^\phi} + \frac{2^{3(1+\phi)(1+\bar{p})/2} C'_\nabla \sigma^2 C_\gamma^{3/2} C_\rho^{1+3\beta/2} \psi_{3(\alpha-\beta\bar{p})/2}^{\bar{p}} (N_t/C_\rho)}{\mu^{3/2} C_\rho^{\mathbb{1}_{\{\rho \geq 0\}}} N_t}. \end{aligned}$$

This can be simplified to the desired using Γ_v given by $(1/C_\gamma C_\rho^\beta + C_l) \delta_0^{1/2} + 2^{\bar{p}} C_l \sqrt{\pi_\infty^v} A_\infty^v / C_\rho^{1/2} + 2 \sqrt{\pi_\infty^v} A_\infty^v / C_\gamma C_\rho^\beta + 2^{\bar{p}} C'_\nabla \sqrt{\Pi_\infty^v} A_\infty^v$, consisting of the finite constants π_∞^v , Π_∞^v and A_∞^v . \square

7.2.2. Proofs for Section 4.2

Theorem 3 (APSSG). Denote $\bar{\delta}_t = \mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$ with $(\bar{\theta}_t)$ given by (4) using (θ_t) from (3). Under Assumption 1, Assumptions 2-p and 3-p with $p = 4$, Assumptions 4 and 5, we have for any learning rate (γ_t) that

$$\bar{\delta}_t^{1/2} \leq \frac{\Lambda^{1/2}}{N_t^{1/2}} + \frac{1}{N_t \mu} \sum_{i=1}^{t-1} \left| \frac{n_{i+1}}{\gamma_{i+1}} - \frac{n_i}{\gamma_i} \right| \delta_i^{1/2} + \frac{n_t}{N_t \gamma_t \mu} \delta_t^{1/2} + \frac{n_1}{N_t \mu} \left(\frac{1}{\gamma_1} + C_l \right) \delta_0^{1/2} + \frac{C_l}{N_t \mu} \left(\sum_{i=1}^{t-1} n_{i+1} \delta_i \right)^{1/2} + \frac{C''_\nabla}{N_t \mu} \sum_{i=0}^t n_{i+1} \Delta_i^{1/2}$$

where $\Lambda = \text{Tr}(\nabla_\theta^2 L(\theta^*)^{-1} \Sigma \nabla_\theta^2 L(\theta^*)^{-1})$ and $C''_\nabla = C'_\nabla + 2^2 G_\Theta / D_\theta^2$.

Proof of Theorem 3. Denote $\mathbb{E}[\|\bar{\theta}_t - \theta^*\|^2]$ by $\bar{\delta}_t$ with $(\bar{\theta}_t)$ given by (4) using (θ_t) from (3). As in the proof Theorem 2, we follow the steps of Polyak and Juditsky [26], in which, we can rewrite (3) to

$$\frac{1}{\gamma_t} (\theta_{t-1} - \theta_t) = \nabla_{\theta} l_t(\theta_{t-1}) - \frac{1}{\gamma_t} \Omega_t,$$

where $\nabla_{\theta} l_t(\theta_{t-1}) = n_t^{-1} \sum_{i=1}^{n_t} \nabla_{\theta} l_{t,i}(\theta_{t-1})$ and $\Omega_t = \mathcal{P}_\Theta(\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1})) - (\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1}))$. Thus, summing the parts, using the Minkowski's inequality, and bounding each term gives us the same bound as in Theorem 2, but with an additional term regarding Ω_t , namely

$$\left(\mathbb{E} \left[\left\| \nabla_\theta^2 L(\theta^*)^{-1} \frac{1}{N_t} \sum_{i=1}^t \frac{n_i}{\gamma_i} \Omega_i \right\|^2 \right] \right)^{\frac{1}{2}} \leq \frac{1}{\mu N_t} \sum_{i=1}^t \frac{n_i}{\gamma_i} \sqrt{\mathbb{E}[\|\Omega_i\|^2]} = \frac{1}{\mu N_t} \sum_{i=1}^t \frac{n_i}{\gamma_i} \sqrt{\mathbb{E}[\|\Omega_i\|^2 \mathbb{1}_{\{\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1}) \notin \Theta\}}]}, \quad (42)$$

using Godichon-Baggioni [9, Lemma 4.3]. Next, we note that $\mathbb{E}[\|\Omega_t\|^2 \mathbb{1}_{\{\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1}) \notin \Theta\}}] = 4\gamma_t^2 G_\Theta^2 \mathbb{P}[\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1}) \notin \Theta]$, since

$$\begin{aligned} \|\Omega_t\|^2 &\leq 2 \|\mathcal{P}_\Theta(\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1})) - \theta_{t-1}\|^2 + 2\gamma_t^2 \|\nabla_{\theta} l_t(\theta_{t-1})\|^2 = 2 \|\mathcal{P}_\Theta(\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1})) - \mathcal{P}_\Theta(\theta_{t-1})\|^2 + 2\gamma_t^2 \|\nabla_{\theta} l_t(\theta_{t-1})\|^2 \\ &\leq 2 \|\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1}) - \theta_{t-1}\|^2 + 2\gamma_t^2 \|\nabla_{\theta} l_t(\theta_{t-1})\|^2 = 4\gamma_t^2 \|\nabla_{\theta} l_t(\theta_{t-1})\|^2 \leq 4\gamma_t^2 G_\Theta^2, \end{aligned}$$

as \mathcal{P}_Θ is Lipschitz and $\|\nabla_{\theta} l_{t,i}(\theta)\|^2 \leq G_\Theta^2$ for any $\theta \in \Theta$. Moreover, as in Godichon-Baggioni and Portier [10, Theorem 4.2], we know that $\mathbb{P}[\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1}) \notin \Theta] \leq \Delta_t / D_\Theta^4$, where $D_\Theta = \inf_{\theta \in \partial\Theta} \|\theta - \theta^*\|$ with $\partial\Theta$ denoting the frontier of Θ . Thus, (42) can then be bounded by

$$\frac{1}{\mu N_t} \sum_{i=1}^t \frac{n_i}{\gamma_i} \sqrt{\mathbb{E}[\|\Omega_i\|^2 \mathbb{1}_{\{\theta_{t-1} - \gamma_t \nabla_{\theta} l_t(\theta_{t-1}) \notin \Theta\}}]} \leq \frac{2G_\Theta}{\mu D_\Theta^2 N_t} \sum_{i=1}^t n_i \Delta_i^{1/2} \leq \frac{2^2 G_\Theta}{\mu D_\Theta^2 N_t} \sum_{i=1}^t n_{i+1} \Delta_i^{1/2},$$

using that the sequence (n_t) is either constant or time-varying, meaning $n_{t+1}/n_t \leq 2$. \square

Proof of Corollary 5. The proof follows directly from Corollary 3 but with use of Theorem 3. \square

Proof of Corollary 6. The proof follows directly from Corollary 4 but with use of Theorem 3. \square

A. Technical Proofs

Appendix A contains purely technical results used in the proofs presented in Section 7. In what follows, we use the convention $\inf \emptyset = 0$, $\sum_{i=1}^0 = 0$, and $\prod_{i=1}^0 = 1$.

Proposition 1. *Let $(\gamma_t)_{t \geq 1}$ be a positive sequence. For any $k \leq t$, and $\omega > 0$, we have*

$$\sum_{i=k}^t \prod_{j=i+1}^t [1 + \omega \gamma_j] \gamma_i \leq \frac{1}{\omega} \prod_{j=k}^t [1 + \omega \gamma_j] \leq \frac{1}{\omega} \exp\left(\omega \sum_{j=k}^t \gamma_j\right). \quad (\text{A.1})$$

Proof of Proposition 1. We begin with considering the first inequality in (A.1), which follows by expanding the sum of product:

$$\begin{aligned} \sum_{i=k}^t \prod_{j=i+1}^t [1 + \omega \gamma_j] \gamma_i &= \frac{1}{\omega} \sum_{i=k}^t \prod_{j=i+1}^t [1 + \omega \gamma_j] \omega \gamma_i = \frac{1}{\omega} \sum_{i=k}^t \prod_{j=i+1}^t [1 + \omega \gamma_j] [1 + \omega \gamma_i - 1] \\ &= \frac{1}{\omega} \sum_{i=k}^t \left[\prod_{j=i+1}^t [1 + \omega \gamma_j] [1 + \omega \gamma_i] - \prod_{j=i+1}^t [1 + \omega \gamma_j] \right] = \frac{1}{\omega} \sum_{i=k}^t \left[\prod_{j=i}^t [1 + \omega \gamma_j] - \prod_{j=i+1}^t [1 + \omega \gamma_j] \right]. \end{aligned}$$

As the (positive) terms cancel out, we end up with the first inequality in (A.1):

$$\begin{aligned} \frac{1}{\omega} \sum_{i=k}^t \left[\prod_{j=i}^t [1 + \omega \gamma_j] - \prod_{j=i+1}^t [1 + \omega \gamma_j] \right] &= \frac{1}{\omega} \left[\prod_{j=k}^t [1 + \omega \gamma_j] - \prod_{j=k+1}^t [1 + \omega \gamma_j] + \dots - \prod_{j=t+1}^t [1 + \omega \gamma_j] \right] \\ &= \frac{1}{\omega} \left[\prod_{j=k}^t [1 + \omega \gamma_j] - \prod_{j=t+1}^t [1 + \omega \gamma_j] \right] = \frac{1}{\omega} \left[\prod_{j=k}^t [1 + \omega \gamma_j] - 1 \right] \leq \frac{1}{\omega} \prod_{j=k}^t [1 + \omega \gamma_j], \end{aligned}$$

as $\prod_{j=t+1}^t = 1$ for all $t \in \mathbb{N}$. Using the (simple) bound of $1 + t \leq \exp(t)$ for all $t \in \mathbb{R}$, we obtain the second inequality of (A.1):

$$\frac{1}{\omega} \prod_{j=k}^t [1 + \omega \gamma_j] \leq \frac{1}{\omega} \prod_{j=k}^t \exp(\omega \gamma_j) = \frac{1}{\omega} \exp\left(\omega \sum_{j=k}^t \gamma_j\right).$$

\square

Proposition 2. *Let $(\gamma_t)_{t \geq 1}$ be a positive sequence. Let $\omega > 0$ and $k \leq t$ such that for all $i \geq k$, $\omega \gamma_i \leq 1$, then*

$$\sum_{i=k}^t \prod_{j=i+1}^t [1 - \omega \gamma_j] \gamma_i \leq \frac{1}{\omega}. \quad (\text{A.2})$$

Proof of Proposition 2. We start with expanding the sums of products term in (A.2), given us

$$\begin{aligned} \sum_{i=k}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] \gamma_i &= -\frac{1}{\omega} \sum_{i=k}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] [1 - \omega\gamma_i - 1] = -\frac{1}{\omega} \sum_{i=k}^t \left[\prod_{j=i+1}^t [1 - \omega\gamma_j] [1 - \omega\gamma_i] - \prod_{j=i+1}^t [1 - \omega\gamma_j] \right] \\ &= -\frac{1}{\omega} \sum_{i=k}^t \left[\prod_{j=i}^t [1 - \omega\gamma_j] - \prod_{j=i+1}^t [1 - \omega\gamma_j] \right] = \frac{1}{\omega} \sum_{i=k}^t \left[\prod_{j=i+1}^t [1 - \omega\gamma_j] - \prod_{j=i}^t [1 - \omega\gamma_j] \right]. \end{aligned}$$

As we only have positive terms, we can upper bound the term:

$$\frac{1}{\omega} \sum_{i=k}^t \left[\prod_{j=i+1}^t [1 - \omega\gamma_j] - \prod_{j=i}^t [1 - \omega\gamma_j] \right] \leq \frac{1}{\omega} \left[1 - \prod_{j=k}^t [1 - \omega\gamma_j] \right] \leq \frac{1}{\omega},$$

using $\prod_{j=k}^t [1 - \omega\gamma_j] \geq 0$, showing the inequality in (A.2). \square

Proposition 3. Let $(\gamma_t)_{t \geq 1}$ and $(\eta_t)_{t \geq 1}$ be positive sequences. For any $k \leq t$, we can obtain the (upper) bounds:

$$\sum_{i=k}^t \prod_{j=i+1}^t [1 + \omega\gamma_j] \eta_i \gamma_i \leq \frac{1}{\omega} \max_{k \leq i \leq t} \eta_i \exp \left(\omega \sum_{j=k}^t \gamma_j \right), \quad (\text{A.3})$$

with $\omega > 0$. Furthermore, suppose that for all $i \geq k$, $\omega\gamma_i \leq 1$, then

$$\sum_{i=k}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] \eta_i \leq \frac{1}{\omega} \max_{k \leq i \leq t} \eta_i. \quad (\text{A.4})$$

Proof of Proposition 3. We obtain the inequality in (A.3) directly by Proposition 1:

$$\sum_{i=k}^t \prod_{j=i+1}^t [1 + \omega\gamma_j] \eta_i \gamma_i \leq \max_{k \leq i \leq t} \eta_i \sum_{i=k}^t \prod_{j=i+1}^t [1 + \omega\gamma_j] \gamma_i \leq \frac{1}{\omega} \max_{k \leq i \leq t} \eta_i \prod_{j=k}^t [1 + \omega\gamma_j] \leq \frac{1}{\omega} \max_{k \leq i \leq t} \eta_i \exp \left(\omega \sum_{j=k}^t \gamma_j \right).$$

Similarly, for the inequality in (A.4), we have

$$\sum_{i=k}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] \eta_i \gamma_i \leq \max_{k \leq i \leq t} \eta_i \sum_{i=k}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] \gamma_i \leq \frac{1}{\omega} \max_{k \leq i \leq t} \eta_i,$$

by Proposition 2. \square

Proposition 4. Let $(\delta_t)_{t \geq 0}$, $(\gamma_t)_{t \geq 1}$, $(\eta_t)_{t \geq 1}$, and $(\nu_t)_{t \geq 1}$ be some positive sequences satisfying the recursive relation:

$$\delta_t \leq (1 - 2\omega\gamma_t + \eta_t\gamma_t) \delta_{t-1} + \nu_t\gamma_t, \quad (\text{A.5})$$

with $\delta_0 \geq 0$ and $\omega > 0$. Denote $t_0 = \inf \{t \geq 1 : \eta_t \leq \omega\}$, and suppose that for all $t \geq t_0 + 1$, one has $\omega\gamma_t \leq 1$. Then, for γ_t and η_t decreasing, we have the upper bound on (δ_t) :

$$\delta_t \leq \exp \left(-\omega \sum_{i=t/2}^t \gamma_i \right) \left[\exp \left(\sum_{i=1}^{t_0} \eta_i \gamma_i \right) \left(\delta_0 + \max_{1 \leq i \leq t_0} \frac{\nu_i}{\eta_i} \right) + \sum_{i=t_0+1}^{t/2-1} \nu_i \gamma_i \right] + \frac{1}{\omega} \max_{t/2 \leq i \leq t} \nu_i, \quad (\text{A.6})$$

for all $t \in \mathbb{N}$ with the convention that $\sum_{i=t_0}^{t/2} = 0$ if $t/2 < t_0$.

Proof of Proposition 4. Applying the recursive relation from (A.5) t times, we derive:

$$\delta_t \leq \underbrace{\prod_{i=1}^t [1 - 2\omega\gamma_i + \eta_i\gamma_i]}_{B_t} \delta_0 + \underbrace{\sum_{i=1}^t \prod_{j=i+1}^t [1 - 2\omega\gamma_j + \eta_j\gamma_j]}_{A_t} \nu_i \gamma_i,$$

where B_t can be seen as a transient term only depending on the initialisation δ_0 , and a stationary term A_t . The transient term B_t can be divided into two products, before and after t_0 ,

$$B_t = \prod_{i=1}^t [1 - 2\omega\gamma_i + \eta_i\gamma_i] = \left(\prod_{i=1}^{t_0} [1 - 2\omega\gamma_i + \eta_i\gamma_i] \right) \left(\prod_{i=t_0+1}^t [1 - 2\omega\gamma_i + \eta_i\gamma_i] \right).$$

Using that $t_0 = \inf \{t \geq 1 : \eta_t \leq \omega\}$, and since for all $t \geq t_0 + 1$, we have $2\omega\gamma_t - \eta_t\gamma_t \geq \omega\gamma_t$, it comes

$$\begin{aligned} B_t &\leq \left(\prod_{i=1}^{t_0} [1 - 2\omega\gamma_i + \eta_i\gamma_i] \right) \left(\prod_{i=t_0+1}^t [1 - \omega\gamma_i] \right) \leq \left(\prod_{i=1}^{t_0} \exp(-2\omega\gamma_i + \eta_i\gamma_i) \right) \left(\prod_{i=t_0+1}^t \exp(-\omega\gamma_i) \right) \\ &= \exp\left(-2\omega \sum_{i=1}^{t_0} \gamma_i\right) \exp\left(\sum_{i=1}^{t_0} \eta_i\gamma_i\right) \exp\left(-\omega \sum_{i=t_0+1}^t \gamma_i\right) \leq \exp\left(-\omega \sum_{i=1}^t \gamma_i\right) \exp\left(\sum_{i=1}^{t_0} \eta_i\gamma_i\right) \end{aligned}$$

by applying the (simple) bound $1 + t \leq \exp(t)$ for all $t \in \mathbb{R}$. We derive that

$$B_t \leq \exp\left(-\omega \sum_{i=t/2}^t \gamma_i\right) \exp\left(\sum_{i=1}^{t_0} \eta_i\gamma_i\right). \quad (\text{A.7})$$

Next, the stationary term A_t can (similarly) be divided into two sums (after and before t_0):

$$A_t = \underbrace{\sum_{i=t_0+1}^t \prod_{j=i+1}^t [1 - 2\omega\gamma_j + \eta_j\gamma_j] v_i \gamma_i}_{A_{t,1}} + \underbrace{\sum_{i=1}^{t_0} \prod_{j=i+1}^t [1 - 2\omega\gamma_j + \eta_j\gamma_j] v_i \gamma_i}_{A_{t,2}}.$$

The first stationary term $A_{t,1}$ (with $t > t_0$) can be bounded as follows: if $t/2 \leq t_0 + 1$, we have

$$A_{t,1} \leq \max_{t_0+1 \leq i \leq t} v_i \sum_{i=t_0+1}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] \gamma_i = \frac{1}{\omega} \max_{t_0+1 \leq i \leq t} v_i \leq \frac{1}{\omega} \max_{t/2 \leq i \leq t} v_i,$$

by Proposition 3. Furthermore, if $t/2 > t_0 + 1$, we get

$$\begin{aligned} A_{t,1} &\leq \sum_{i=t_0+1}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] v_i \gamma_i = \sum_{i=t_0+1}^{t/2-1} \prod_{j=i+1}^t [1 - \omega\gamma_j] v_i \gamma_i + \sum_{i=t/2}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] v_i \gamma_i \\ &\leq \sum_{i=t_0+1}^{t/2-1} \prod_{j=i+1}^t [1 - \omega\gamma_j] v_i \gamma_i + \max_{t/2 \leq i \leq t} v_i \sum_{i=t/2}^t \prod_{j=i+1}^t [1 - \omega\gamma_j] \gamma_i = \prod_{j=t/2}^t [1 - \omega\gamma_j] \sum_{i=t_0+1}^{t/2-1} v_i \gamma_i + \frac{1}{\omega} \max_{t/2 \leq i \leq t} v_i, \end{aligned}$$

where $\prod_{j=t/2}^t [1 - \omega\gamma_j] \leq \exp(-\omega \sum_{j=t/2}^t \gamma_j)$ as $1 + t \leq \exp(t)$ for all $t \in \mathbb{R}$. Thus, for all $t \in \mathbb{R}$,

$$A_{t,1} \leq \exp\left(-\omega \sum_{j=t/2}^t \gamma_j\right) \sum_{i=t_0+1}^{t/2-1} v_i \gamma_i + \frac{1}{\omega} \max_{t/2 \leq i \leq t} v_i, \quad (\text{A.8})$$

where $\sum_{i=t_0+1}^{t/2-1} v_i \gamma_i = 0$ if $t/2 < t_0$. The second stationary term $A_{t,2}$ can be bounded, thanks to Proposition 1, as follows:

$$\begin{aligned} A_{t,2} &= \sum_{i=1}^{t_0} \prod_{j=i+1}^t [1 - 2\omega\gamma_j + \eta_j\gamma_j] v_i \gamma_i = \left(\prod_{j=t_0+1}^t [1 - 2\omega\gamma_j + \eta_j\gamma_j] \right) \sum_{i=1}^{t_0} \prod_{j=i+1}^{t_0} [1 - 2\omega\gamma_j + \eta_j\gamma_j] v_i \gamma_i \\ &\leq \left(\prod_{j=t_0+1}^t [1 - \omega\gamma_j] \right) \sum_{i=1}^{t_0} \prod_{j=i+1}^{t_0} [1 + \eta_j\gamma_j] v_i \gamma_i \leq \exp\left(-\omega \sum_{j=t_0+1}^t \gamma_j\right) \max_{1 \leq i \leq t_0} \frac{v_i}{\eta_i} \sum_{i=1}^{t_0} \prod_{j=i+1}^{t_0} [1 + \eta_j\gamma_j] \eta_i \gamma_i \\ &\leq \exp\left(-\omega \sum_{j=t_0+1}^t \gamma_j\right) \max_{1 \leq i \leq t_0} \frac{v_i}{\eta_i} \exp\left(\sum_{i=1}^{t_0} \eta_i \gamma_i\right) \leq \exp\left(-\omega \sum_{j=1}^t \gamma_j\right) \max_{1 \leq i \leq t_0} \frac{v_i}{\eta_i} \exp\left(2 \sum_{i=1}^{t_0} \eta_i \gamma_i\right), \end{aligned}$$

by the definition of t_0 , thus

$$A_{t,2} \leq \exp\left(-\omega \sum_{j=1}^t \gamma_j\right) \max_{1 \leq i \leq t_0} \frac{v_i}{\eta_i} \exp\left(2 \sum_{i=1}^{t_0} \eta_i \gamma_i\right) \leq \exp\left(-\omega \sum_{j=t/2}^t \gamma_j\right) \max_{1 \leq i \leq t_0} \frac{v_i}{\eta_i} \exp\left(2 \sum_{i=1}^{t_0} \eta_i \gamma_j\right). \quad (\text{A.9})$$

Then, using the bound for $A_{t,1}$ in (A.8) and $A_{t,2}$ in (A.9), we can bound A_t by

$$A_t \leq \exp\left(-\omega \sum_{j=t/2}^t \gamma_j\right) \left[\exp\left(2 \sum_{i=1}^{t_0} \eta_i \gamma_j\right) \max_{1 \leq i \leq t_0} \frac{v_i}{\eta_i} + \sum_{i=t_0+1}^{t/2-1} v_i \gamma_i \right] + \frac{1}{\omega} \max_{t/2 \leq i \leq t} v_i. \quad (\text{A.10})$$

Finally, combining the bound for B_t in (A.7) and A_t in (A.10), we achieve the bound for $\delta_t \leq B_t \delta_0 + A_t$, namely the upper bound in (A.6). \square

The following proposition is a more simplistic but rougher version of the bound in Proposition 4.

Proposition 5. *Let $(\delta_t)_{t \geq 0}$, $(\gamma_t)_{t \geq 1}$, $(\eta_t)_{t \geq 1}$, and $(v_t)_{t \geq 1}$ be some positive sequences satisfying the recursive relation in (A.5). Denote $t_0 = \inf\{t \geq 1 : \eta_t \leq \omega\}$, and suppose that for all $t \geq t_0 + 1$, one has $\omega \gamma_t \leq 1$. Then, for γ_t and η_t decreasing, we have for all $t \in \mathbb{N}$,*

$$\delta_t \leq \exp\left(-\omega \sum_{i=t/2}^t \gamma_i\right) \exp\left(2 \sum_{i=1}^{t_0} \eta_i \gamma_i\right) \left(\delta_0 + 2 \max_{1 \leq i \leq t} \frac{v_i}{\eta_i} \right) + \frac{1}{\omega} \max_{t/2 \leq i \leq t} v_i. \quad (\text{A.11})$$

Proof of Proposition 5. The resulting (upper) bound in (A.11) follows directly from (A.6) by noting that $t_0 \leq t$, giving us $\sum_{i=t_0+1}^{t/2-1} v_i \gamma_i \leq \sum_{i=1}^t v_i \gamma_i \leq \max_{1 \leq i \leq t} (v_i / \eta_i) \sum_{i=1}^t \eta_i \gamma_i \leq \max_{1 \leq i \leq t} (v_i / \eta_i) \exp(2 \sum_{i=1}^t \eta_i \gamma_i)$, as (v_t) and (γ_t) are positive sequences. \square

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