Submersions, Fibrations and Bundles

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Mathematics Subject Classification : 55R05, 55R10

to appear in Transactions of the A.M.S.

Abstract — When does a submersion have the homotopy lifting property ? When is it a locally trivial fibre bundle ? We establish characterizations in terms of consistency in the topology of the neighbouring fibres.

In differential topology, one meets nonproper submersive maps, and hope that they will be fibrations (resp. fibre bundles) under hypotheses of consistency between the homotopy type (resp. topology) of the neighbouring fibres. The aim of this paper is to give suitable characterizations.

I — Submersions & Fibrations.

This first part belongs to the most elementary homotopy theory. Our purpose is to establish the following homotopy lifting characterization, and a few corollaries.

THEOREM A — *A surjective map is a fibration if and only if it verifies those three conditions: it is a homotopic submersion, all vanishing cycles of all dimensions are trivial, and all emerging cycles of all dimensions are trivial.*

Let us first specify definitions, conventions and notations.

I.1. — *Definitions.* Throughout this paper, "space" means Hausdorff topological space, "map" means continuous map, "polytope" means finite simplicial complex. For every $p \geq 0$, denote $\mathbf{B}^p$ the compact p-ball and $\mathbf{S}^p = \partial \mathbf{B}^{p+1}$ the p-sphere. Fix a basepoint $* \in \mathbf{S}^p$.

Let E, B be two spaces and $\pi : E \to B$ a map.

As usually, by a *homotopy* for the map $f : X \to Y$, we mean a map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ for every $x \in X$; and we call (E, π), or π, a *fibration,* or equivalently we say that it has the *homotopy lifting property,* if for every map $f : X \to E$ whose source X is a polytope, every homotopy for $\pi \circ f$ lifts to a homotopy for f. More generally, call π a *r-fibration* if this is true for every polytope X of dimension at most r . A 0-fibration is also said to have the *path lifting property.*

Here is another generalization of fibrations. Two homotopies

$$F, F' : X \times [0,1] \to Y$$

are said to *have the same germ* if they coincide in a neighborhood of the subspace $X \times \{0\}$.

DEFINITION 1 — *Call π a* homotopic submersion, *or equivalently say that it has the* germ-of-homotopy lifting property, *if for every map $f : X \to E$ whose source X is a polytope, every germ-of-homotopy for $\pi \circ f$ lifts to a germ-of-homotopy for f.*

Every fibration is a homotopic submersion (obvious) as well as every smooth submersion in the sense of Differential geometry, and more generally every topological submersion (after lemma 6 below.)

A "fibre" is the inverse image in E of a point $b \in B$, denoted $E_b = \pi^{-1}(b)$. A map $X \to E$ is *vertical* if it sends X into a single fibre. A map $f : X \times Y \to E$ is *fibred* if for each point y in Y, the map

$$f_y : X \to E : x \mapsto f(x, y)$$

is vertical.

DEFINITION 2 — *A* vanishing p-cycle *is a fibred map*

$$f : \mathbf{S}^p \times [0, 1] \to E$$

such that, for each $t > 0$, the map f_t is null-homotopic in its fibre. Call f trivial *if f_0 is also null-homotopic in its fibre.*

Observe that f has to be continuous, but that we don't demand the homotopies the vertical maps $g_t : \mathbf{B}^{p+1} \to E$ such that $\partial g_t = f_t$, to depend continuously on t . Dually,

DEFINITION 3 — *An* emerging p-cycle *is a fibred map*

$$f : \mathbf{S}^p \times]0, 1] \to E$$

such that $f(*, t)$ has a limit for $t \to 0$ (recall that $*$ denotes the basepoint in $\mathbf{S}^p$.) Call it trivial *if there exists $\epsilon > 0$ and a fibred map*

$$f' : \mathbf{S}^p \times [0, \epsilon[\to E$$

such that for each $0 < t < \epsilon$, one has $f'(*, t) = f(*, t)$, and such that the maps f_t and f'_t are homotopic to each other in their common fibre, relatively to the basepoint $f(*, t)$.

Here also, we don't ask any continuity, relatively to t , of the homotopy linking f_t to f'_t .

Thus a nontrivial vanishing (resp. emerging) cycle is a lack of injectivity (resp. surjectivity) in the homotopy groups of fibres, when moving from a given fibre to immediately neighbouring ones over some path in the base.

Write $V(X)$ for the space of vertical maps from X into E , with the compact-open topology. Thus there is a canonical map :

$$\pi_X : V(X) \to B$$

Obviously, every p-dimensionnal vanishing (resp. emerging) cycle for the map π can as well be considered as a 0-dimensionnal vanishing (resp. emerging) cycle for the map $\pi_{\mathbf{S}^p}$. They are simultaneously trivial.

To end with precisions, the pullback of π by some map $f : B' \to B$ is, as usually, defined as $f^*\pi : f^*E \to B'$, where f^*E is the set of couples $(e, b') \in E \times B'$ such that $\pi(e) = f(b')$, and where $(f^*\pi)(e, b') = b'$.

It is immediate that if π is a fibration, or a r-fibration, or a homotopic submersion, or has only trivial p-vanishing, or emerging, cycles, then every pullback enjoys the same property.

EXAMPLE 4 — The three conditions in theorem A are independent. Indeed, let $E \subset [0, 1]^2$ be defined by the condition $xy = 0$ (resp. $x \neq 1/2$ or $y \neq 1/2$) (resp. $y = 0$ or ($y = 1$ and $x \neq 0$)). Let $B = [0, 1]$ and π be simply $(x, y) \mapsto x$. Then the first (resp. second) (resp. third) condition is not verified, but the two other ones are.

I.2 — *Proof of theorem A.*

I.2.1 : three general remarks about homotopic submersions.

An immediate and well-known fact about fibrations is that they automatically also have the *relative* homotopy lifting property. The same argument proves that :

LEMMA 5 — *Every homotopic submersion verifies the* relative *germ-of-homotopy lifting property : for every polytope X , every subpolytope $X' \subset X$, every map $f : X \to E$, every germ-of-homotopy F for $\pi \circ f$ and every germ-of-homotopy $\bar{F}'$ for $f|_{X'}$, if $\pi \circ \bar{F}' = F|_{X'}$ then $\bar{F}'$ extends to a germ-of-homotopy $\bar{F}$ for f such that $\pi \circ \bar{F} = F$.*

LEMMA 6 — *Assume that π is open and is a local homotopic submersion, i.e. that each $x \in E$ has a neighborhood $U(x) \subset E$ such that the restriction $\pi|_{U(x)}$ is a homotopic submersion of $U(x)$ onto $\pi(U(x))$. Then π is a (global) homotopic submersion.*

Proof. Consider a polytope X , a map $f : X \to E$ and a germ-of-homotopy $F : X \times [0, \epsilon[\to B$ for $\pi \circ f$. Proceed by induction on $p = \dim X$. The case $p = 0$ is trivial. Subdividing X we can assume that for each p-simplex σ , its image $f(\sigma)$ is contained in an open subset $U(\sigma) \subset E$ such that $\pi|_{U(\sigma)}$ verifies the germ-of-homotopy lifting property. By induction, we have a lifting $\bar{F}'$ for F restricted to the $(p-1)$-skeletton of X . Since after the previous lemma $\pi|_{U(\sigma)}$ has the relative germ-of-homotopy lifting property, $\bar{F}'$ extends to the interior of σ . •

A second immediate corollary of lemma 5, is the following tool for handling sections.

LEMMA 7 — *Let $\pi : E \to B$ be a surjective homotopic submersion whose base B is a polytope. Let $Y \subset B$ be a subpolytope. Denote $\Gamma(Y)$ the space of (continuous) sections of π over Y . Then :*

(i: local section extension) Each $s \in \Gamma(Y)$ extends over some neighborhood of Y ; in particular each point of B has a neighborhood over which there exists a section;

(ii: sections pasting) Let $s \in \Gamma(B)$ and let $s' \in \Gamma(Y)$ be homotopic to $s|_Y$. Then there exists $s'' \in \Gamma(B)$ which coincides with s over the complement of an arbitrary small neighborhood of Y , and with s' on Y .

I.2.2 : Haefliger's lemma.

The following fact, due to Haefliger [**4**] , is central in the demonstration of theorem A. We give a proof since it seems simpler than the original one.

LEMMA 8 — *If a surjective homotopic submersion $\pi : E \to B$ has a polytope base B and weakly contractible fibres E_b , then it has a section.*

Proof. By induction on $q = \dim B$. Obvious if $q = 0$. Assume that the lemma is true for $q-1$. Since a surjective homotopic submersion has local sections, subdividing B if necessary, every q-cell σ is the domain of a section s_σ . On the other hand, the induction hypothesis gives a section s over the $(q-1)$-skeletton. Over the boundary of each q-cell σ , the two sections $s|_{\partial\sigma}$ and $s_\sigma|_{\partial\sigma}$ are homotopic. Indeed, consider E' the space of vertical paths $c : [0, 1] \to E_b$ such that $b \in \partial\sigma$, $c(0) = s_\sigma(b)$ and $c(1) = s(b)$. Obviously the map $c \mapsto b$ is a homotopic submersion of E' onto $\partial\sigma$ with weakly contractible fibres, so by the induction hypothesis it admits a section, i.e. a homotopy between $s|_{\partial\sigma}$ and $s_\sigma|_{\partial\sigma}$. Thus the section pasting property of the homotopic submersion π gives an extension of s over σ . •

I.2.3 : About the sections of a homotopic submersion over the interval.

In this subparagraph, consider a surjective homotopic submersion whose base is the interval :

$$\pi : E \to B = [0, 1]$$

Fix a basepoint $x_0 \in E_0$. To be short, call a section *pointed* if $s(0) = x_0$; write $\Gamma(Y)$ (resp. $\Gamma_0(Y)$) the space of (pointed) sections over $Y \subset I$; and call $s \in \Gamma(Y)$, $s' \in \Gamma(Y')$ *compatible* at point $b \in Y \cap Y'$ if $s(b)$ and $s'(b)$ belong to the same arcwise connected component of E_b .

LEMMA 9 — *Assume that all 0-vanishing cycles are trivial. If s, $s' \in \Gamma([0, 1])$ are compatible at some point, then they are compatible at every point.*

Proof. Recall that if $K \subset [0, 1]$ is a compact subset, we call $t \in \mathbf{K}$ a *first-species* point of K if t is an extremity of a nonempty open interval disjoint from K .

The set $U \subset [0, 1]$ of points where s and s' are compatible is open since π is a homotopic submersion, and its complement has no first-species point since 0-vanishing cycles are trivial. Thus $U = [0, 1]$. •

LEMMA 10 — *Assume that all 0-vanishing cycles and 0-emerging cycles are trivial. Then $\Gamma_0([0, 1])$ is not empty.*

Proof. Consider $\mathbf{T}$ the set of t's such that there exists a pointed section over $[0, t]$. It is open since π is a homotopic submersion. Thus by contradiction it would be of the form $\mathbf{T} = [0, T[$ with $T \in]0, 1]$. Let $s_n \in \Gamma([0, T - 1/n])$ be pointed. By the previous lemma, since $s_n(0) = s_{n+1}(0)$, the sections s_n and s_{n+1} are also compatible at point $T - 1/n$. Thus the sections pasting property gives a pointed $s \in \Gamma([0, T[)$. Let $s' \in \Gamma(]T - \epsilon, T])$ be any section over a left neighborhood of T . Then s and s' together form a 0-emerging cycle at point T . Since it must be trivial, there is a $s'' \in \Gamma([T - \epsilon', T])$ compatible with s at $t - \epsilon'$. The sections pasting property gives a pointed section over $[0, T]$, contradiction. •

LEMMA 11 — *any surjective homotopic submersion all of whose vanishing 0-cycles and emerging 0-cycles are trivial, is a 0-fibration.*

Proof : immediate by lemma 10 and pullbacks. •

LEMMA 12 — *Assume that all vanishing cycles and emerging cycles of all dimensions are trivial. Then* $\Gamma_0([0,1])$ *is weakly contractible.*

Proof. For every polytope X, recall $V(X)$, the space of vertical maps $X \to E$. The canonical map $\pi_X : V(X) \to [0,1]$ is a homotopic submersion, as well as the restriction map :

$$\rho^p : V(\mathbf{B}^{p+1}) \to V(\mathbf{S}^p)$$

Since we assumed that all vanishing (resp. emerging) cycles of π of dimension $p+1$ are trivial, it follows that all 0-dimensionnal vanishing (resp. emerging) cycles of ρ^p are also trivial.

Indeed, let $f : \mathbf{S}^0 \times [0,1] \to V(\mathbf{B}^{p+1})$ (resp. $f : \mathbf{S}^0 \times]0,1] \to V(\mathbf{B}^{p+1})$) be a vanishing (resp. emerging) 0-cycle for ρ^p . Gluing, for each $t \in [0,1]$ (resp. each $t \in]0,1]$) , $f(-1,t)$ and $f(+1,t)$ along their common boundary, one gets a vanishing (resp. emerging) $(q+1)$-cycle for π , the triviality of which implies that f is also trivial.

Thus after lemma 11, ρ^p is a 0-fibration *as a map* $V(\mathbf{B}^{p+1}) \to V^0(\mathbf{S}^p)$, where $V^0(\mathbf{S}^p)$ is the image of ρ^p , i.e. the space of vertical p-cycles null-homotopic in their fibre.

Every given map $g : \mathbf{S}^p \to \Gamma_0([0,1])$, may as well be considered as a section $s : [0,1] \to V(\mathbf{S}^p)$ (set $s(t)(x) = g(x)(t)$). The set of t's such that $s(t) \in V^0(\mathbf{S}^p)$ is open since π is a homotopic submersion, and has no first-species boundary point since p-vanishing cycles are trivial. Thus s is a path in $V^0(\mathbf{S}^p)$, and thus lifts to $V(\mathbf{B}^{p+1})$, i.e. g extends to $\mathbf{B}^{p+1}$. •

I.2.4 : End of the proof of theorem A .

Given a map $\pi : E \to \mathbf{B}$ verifying the three conditions of the theorem, a map $f : X \to E$ and a homotopy F for $\pi \circ f$, consider E' the space of of pairs (x,c) where $x \in X$ and where $c : [0,1] \to E$ verifies $c(0) = f(x)$ and $\pi(c(t)) = F(x,t)$ for every $t \in [0,1]$. Then :

$$\pi' : E' \to X : (x,c) \mapsto x$$

is a homotopic submersion, as follows from lemma 5. It is surjective by lemma 10, and its fibres are weakly contractible by lemma 12. After lemma 8, this map π' admits a section; in other words F lifts. •

I.3 — Corollaries. The two first ones immediately follow from theorem A.

COROLLARY 13 — *Let $\pi : E \to B$ be a surjective homotopic submersion. Under either of the following assumptions, π is a fibration :*

1. Each fibre is weakly contractible;

2. The inclusion of each fibre into E is a weak homotopy equivalence.

Surprisingly, point 1, essentially identic to Haefliger's lemma, seems among our various corollaries, to be the only one to have been previously known.

COROLLARY 14 — *Let $\pi : E \to B$ be a surjective homotopic submersion. The following properties are equivalent :*

1. The map π is a fibration :

2. For each b in B, the map π induces a weak homotopy equivalence of pairs $(E, E_b) \to (B, b)$;

3. (dim-1 base principle for fibrations) For each path $c : [0,1] \to B$, the pullback c^*E is a fibration over $[0,1]$.

The dim-1 base principle allows us, whenever useful, to restrict our attention to the case $B = [0,1]$.

COROLLARY 15 — *In a commutative diagram of maps :*

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ & \pi \searrow \quad \swarrow \pi' & \\ & B & \end{array}$$

assume that π and π' are surjective homotopic submersions. For each $b \in B$, consider $E_b = \pi^{-1}(b)$, $E'_b = \pi'^{-1}(b)$, and the restriction $h_b = h|_{E_b} : E_b \to E'_b$.

(i) (Fibrewise fibration principle) If π , π' and every h_b are fibrations and if h is a surjective homotopic submersion, then h is a fibration;

(ii) If π is a fibration and if every h_b is a weak homotopy equivalence, then π' is a fibration;

(iii) If π' is a fibration and if every h_b is a weak homotopy equivalence, then π is a fibration.

Proof. (i) We can assume, to simplify, that B is contractible (for example using the dim-1 base principle). Fix $x' \in E'$, write $b = \pi'(x')$, $E_{x'} = h^{-1}(x')$, and consider the inclusions $j : E_b \to E$, $j' : E'_b \to E'$. We have a commutative diagram of pairs :

$$\begin{array}{ccc} (E_b, E_{x'}) & \xrightarrow{j_*} & (E, E_{x'}) \\ (h_b)_* \downarrow & & \downarrow h_* \\ (E'_b, x') & \xrightarrow{j'_*} & (E', x') \end{array}$$

Since π, π' and h_b are fibrations, j_*, j'_* and $(h_b)_*$ are weak homotopy equivalences, and thus so is h_* . Corollary 14 concludes.

(ii) Using the dim-1 base principle, we can assume, to fix ideas, that $B = [0,1]$. Then we have a section-lifting lemma : *for every subinterval $I \subset [0,1]$ and every section $s' : I \to E'$, there is a section $s : I \to E$ compatible with s',* i.e. such that for every $t \in I$, the points $h(s(t))$ and $s'(t)$ lie in the same arcwise connected component of E'_t .

Proof of this lemma : since every h_b is a 0-homotopy equivalence, there exists a set-theoric (nonnecessarily continuous) section $\sigma : I \to E$ compatible with s' . Since π is a homotopic submersion, each $t \in I$ has a neighborhood $U(t)$ in I with a (continuous) section $s_t : U(t) \to E$ such that $s_t(t) = \sigma(t)$. Since π' is a homotopic submersion, s_t is compatible with s' over some smaller neighborhood of t . Thus I can be divided into subintervals $I_i = [t_i, t_{i+1}]$, domains of sections $s_i : I_i \to E$ compatible with s' . In particular, since h_{t_i} is a 0-homotopy equivalence, $s_{i-1}(t_i)$ and $s_i(t_i)$ lie in the same arcwise connected component of E_{t_i} . Thus the section pasting tool (lemma 7) allows us to build a section $s : I \to E$ compatible with s' . The lemma is proved.

Since 0-vanishing and 0-emerging cycles of π are trivial, and since every h_b is a 0-homotopy equivalence, the section-lifting lemma gives straightforwardly that 0-vanishing and 0-emerging cycles of π' are trivial.

Changing E for $V(\mathbf{S}^p)$, the space of vertical maps $\mathbf{S}^p \to E$, and changing E' for $V'(\mathbf{S}^p)$, the space of vertical maps $\mathbf{S}^p \to E'$, we get that p-vanishing and p-emerging cycles of π' are trivial, for every $p \geq 0$. Theorem A concludes.

(iii) Much like (ii), but simpler. •

COROLLARY 17 — *Let $\pi : E \to B$ be a surjective homotopic submersion. The map π is a fibration if and only if the canonical map from each fibre into the homotopy-theoric fibre of π is a weak homotopy equivalence.*

Follows immediately from the preceding one, (iii).

We can also deduce two characterizations for product maps. Of course, we call $\pi : E \to B$ a fibration *over* a given subset $B' \subset B$ if the restriction

$$\pi|_{\pi^{-1}(B')} : \pi^{-1}(B') \to B'$$

is a fibration.

COROLLARY 18 — *Assume that $\pi = (\pi_1, \pi_2) : E \to B = B_1 \times B_2$ is a surjective homotopic submersion. The three following properties are equivalent :*

(i) The map π is a fibration;

(ii) the map π is, for each $b_1 \in B_1$, a fibration over $\{b_1\} \times B_2$; and π_1 is a fibration;

(iii) the map π is, for each $b_1 \in B_1$, a fibration over $\{b_1\} \times B_2$; and, for each $b_2 \in B_2$, a fibration over $B_1 \times \{b_2\}$.

Proof. The equivalence of (i) and (ii) is a special case of the fibrewise fibration principle (corollary 15.) It is obvious that (i) implies (iii).

Let us assume that (iii) is true. To prove (i),we can, using the dim-1 base principle, assume that $B_2 = [0,1]$. For every $b_1 \in B_1$, since π is a fibration over $\{b_1\} \times B_2$ which is contractible, $\pi^{-1}(b_1, 0)$ is a deformation retract of $\pi^{-1}(\{b_1\} \times B_2) = {\pi_1}^{-1}(b_1)$.

On the other hand, since π is a fibration over ${\pi_2}^{-1}(0)$, equivalently π_1 restricted to ${\pi_2}^{-1}(0)$ is a fibration.

It follows from corollary 15 (ii) that π_1 is also a fibration. Thus (ii) is verified. Thus (i) is also. •

The next and last corollary, two refined forms of the dim-1 base principle, will be a crucial tool in part II :

COROLLARY 19 — *Let $\pi : E \to B$ be a surjective homotopic submersion.*

a) Assume that $B = \mathbf{R}^q$. Then π is a fibration if and only if it is a fibration over every straight line in the base parallel to one of the axis;

b) Assume that B is a C^∞ (resp. real analytic) (resp. PL) manifold. Then π is a fibration if and only if its pullback by every C^∞ (resp. real analytic) (resp. PL) path in B is a fibration over $[0,1]$.

Proof. a): immediate by the preceding corollary and an induction on q ; b): immediate by a). •

EXAMPLE 20 — in which $\pi = (\pi_1, \pi_2)$ is not a fibration (its fibres don't have the same homotopy type) although it is a surjective submersion and π_1 , π_2 are both fibrations (their fibres are contractible) :

$$E = \{\ (x,y,z) \in \mathbf{R}^3\ /\ z \neq 0 \text{ or } x > y\ \}$$

and $\pi : E \to B = \mathbf{R}^2 : (x,y,z) \mapsto (x,y)$.

II — Submersions & Bundles.

In this second part we turn to submersions between manifolds, establish a necessary and sufficient condition for such a map to be a (locally trivial fibre) bundle (theorem B below), and apply it to several typical situations. The main tool is the analytic understanding of fibrations we got in §*I* . It is applied to spaces of embeddings of compact domains into the fibres.

The first question that raises is probably : for such a submersion, is it the same to be a fibration or to be a bundle ? It was considered by Ferry [3], in a framework slightly different from ours; he built counterexamples. On the other hand, Haefliger's lemma allows much simple ones, e.g. :

EXAMPLE 21 — Let $W \subset \mathbf{R}^3$ be the Whitehead manifold — thus an open subset which is contractible, but has some π_1 at infinity, and in particular is not homeomorphic to $\mathbf{R}^3$. Let $E \subset \mathbf{R}^4$ be the set of quadruples (x,y,z,t) such that $(x,y,z) \in W$ or $t \neq 0$. Let

$$\pi : E \to B = \mathbf{R} : (x,y,z,t) \mapsto t$$

Then p is a smooth submersion (since E is open in $\mathbf{R}^4$) and a fibration (since all fibres are contractible, see §*I* .) But it is certainly *not* a locally trivial fibre bundle, since one of the fibres is not homeomorphic to the neighbouring ones.

II.1. — *Definitions.* To fix ideas, work in the smooth (C^∞ real) differentiability class. Let E^{p+q}, B^q be manifolds — this means paracompact, not necessarily compact, real differential manifolds. For simplicity, assume that E, B are without boundary. Let $\pi : E \to B$ be a surjective smooth submersion — that is, its differential at every point of E is onto. Then each "fibre" $E_b = \pi^{-1}(b)$ is a p-manifold.

Call a submanifold of E a *product* if it is the image of a fibred (see §*I*) smooth embedding $X \times Y \to E$. Recall π is a (locally trivial, fibre) *bundle* if every $b \in B$ has a neighborhood Y such that $\pi^{-1}(Y)$ is a product.

A popular sufficient condition for π to be a bundle is that π be a *riemannian* submersion [5], [10]. Recall that a riemannian metric on E is called *bundle-like* if there exists a riemannian metric on the base B such that, for each $b \in B$ and each $x \in E_b$, the differential $D_x\pi$ establishes a linear isometry from the normal vector space $\nu_x E_b$ onto the tangent vector space $T_b B$. It is easy, using a partition of the unity, to make a (maybe noncomplete) bundle-like metric. The 'orthogonal geodesic lemma' asserts that if any geodesic line is once normal to the fibre it crosses, then it is forever. It follows easily that if $X \subset E_b$ is an open subset (to fix ideas) such that the exponential $\exp(v_x)$ is defined for all $x \in X$ and all $v_x \in \nu_x E_b$ with norm less than a uniform positive constant, then the set of all these $\exp(v_x)$'s is a product. In particular we get the fundamental :

LEMMA 22 — *Every compact subset of every fibre has a product neighborhood.*

One calls π *riemannian* if E admits a *complete* bundle-like metric, and, taking $X = E_b$ above, one sees that every riemannian submersion is a bundle. There are many generalizations.

Our viewpoint is different: we look for purely differential-topologic conditions in terms of the topology of the fibres. In this direction, very few seems to be known, namely :

1. If π is proper, then it is a bundle (Ehresmann). This follows at once from lemma 22.

2. If all the leaves are compact and have the same number of connected components, then π is a bundle. Also obvious by lemma 22.

3. If each fibre is diffeomorphic to $\mathbf{R}$, then π is a bundle. Proof: one may assume that B is orientable. Using a partition of the unity, make a nonsingular vector field tangent to the fibres, make it integrable, and integrate it.

4. Much less elementary is Palmeira's lemma [9]: if each fibre is diffeomorphic to $\mathbf{R}^p$, p a nonnegative integer, and if the base B has dimension $q = 1$, then π is a bundle.

A fibred embedding of $X \times Y$ into E can be considered as a section, over some subset of B , with values in the space $VE(X, E)$ of vertical embeddings of X into E . Thus an approach could be to start from a large compact domain X in a fibre and from a ball $Y \subset B$; to use our knowledge of the topology of the fibres to compute as much as possible the homotopy type of the space of embeddings of X into each fibre, and to use $\S I$ to get such sections. Of course there would remain the problem to engulf arbitrary large compact subsets of $\pi^{-1}(Y)$. For instance, in example 21, the canonical map $VE(\mathbf{B}^3, E) \to B$ is a fibration (see the proof of corollary 31 below) and nevertheless the vertical embeddings of $\mathbf{B}^3$ exhaust all fibres but E_0 . This leads us to the following notions.

By a *vertical domain* we mean a p-dimensionnal compact submanifold of a fibre, $X \subset E_b$, with a smooth boundary. Endow $VE(X, E)$, the set of vertical embeddings of X into E , with the topology of smooth uniform convergence. Let also $VE^0(X, E)$ be the connected component of $VE(X, E)$ containing the original inclusion $X \to E_b$.

DEFINITIONS 23 — *Let*

$$\mathrm{VD} = \coprod_{b \in B} \mathrm{VD}_b$$

be a family of vertical domains. Call it :

a) exhaustive *if every compact subset of every fibre is contained in some* $X \in \mathrm{VD}$ *;*

b) isotopy invariant *if for every* $X \in \mathrm{VD}$ *and every* $\phi \in VE^0(X, E)$ *, we have* $\phi(X) \in \mathrm{VD}$ *;*

c) r-fibred *if, for every two domains* X, $X' \in \mathrm{VD}_b$ *such that* $X \subset \mathrm{Int}(X')$ *, the restriction map*

$$\rho_{X,X'} : VE^0(X', E) \to VE^0(X, E)$$

is a r*-fibration (i.e. has the homotopy extension property for polytopes of dimension at most* r *, see* $\S$ *I.)*

II.2. — *Characterizations and criteria for bundles.*

THEOREM B — *A surjective smooth submersion* $\pi : E \to B^q$ *is a (locally trivial fibre) bundle if and only if it admits an exhaustive, isotopy invariant,* $(q-1)$*-fibred family of vertical domains.*

Before the proof, let us precise definitions and establish preliminary lemmas.

LEMMA 24 — *For every pair of domains* $X \subset \mathrm{Int}(X')$ *, the restriction map* $\rho_{X,X'}$ *is a topological submersion.*

This follows at once from lemma 22.

For every vertical domain $X \in \mathrm{VD}_b$ and every subset $Y \subset B$ homeomorphic to a contractible polytope (in fact it will allways be homeomorphic to $\mathbf{B}^k$ for some $0 \leq k \leq q$), write $\Gamma(X, Y)$ for the space of *continuous* sections $\gamma : Y \to VE^0(X, E)$. Observe in particular that for $\gamma \in \Gamma(X, Y)$, the variable point $\gamma(x)(y)$ admits continuous partial derivates of all order with respect to x , bur not necessarily with respect to y . Since we

shall perform non strictly differentiable operations on the base, it is more convenient to consider all those sections than merely smooth ones. Write :

$$\mathrm{Im}\,\gamma = \{\ \gamma(y)(x)\ /\ x \in X, y \in Y\ \}$$

Call γ a *parametrization* of the VD-*box* $\mathrm{Im}\,\gamma$.

Recall that $X \subset E_b$. Call γ *pointed* (at point b) if $b \in Y$ and if $\gamma(b) = \mathrm{id}_X$.

For every compact subset $C \subset E$, write $C < \mathrm{Im}\,\gamma$ if $C \subset \mathrm{Im}\,\gamma$ and if $C \cap E_y \subset \mathrm{Int}(\mathrm{Im}\,\gamma \cap E_y)$ for every $y \in Y$.

Given two sections $\gamma \in \Gamma(X, Y)$, $\gamma' \in \Gamma(X', Y')$, say of course that γ' *extends* γ if $X \subset X'$, $Y \subset Y'$, and if $\gamma(y)(x) = \gamma'(y)(x)$ for every $x \in X$, $y \in Y$.

LEMMA 25 — *If γ is pointed and if $\mathrm{Im}\,\gamma < \mathrm{Im}\,\gamma'$, then there exists a pointed reparametrization of $\mathrm{Im}\,\gamma'$ extending γ .*

This is an exercise using the classical [1] :

PROPOSITION 26 — *Let X, X' be compact manifolds with $X \subset \mathrm{Int}(X')$. Let $Diff^0(X')$ denote the group of isotopies of X' and $E^0(X, X')$ denote the canonical connected component of the space of embeddings of $X \to X'$. Then the restriction map $Diff^0(X') \to E^0(X, X')$ is a fibration (and even a principal bundle.)*

Here is the main tool to engulf large compact subsets.

LEMMA 27 — *Let $K_i \subset E$ (i = 1,2,3) be three* VD*-box with base $Y_i = \pi(K_i)$. Assume that $Y_1 \cap Y_2 = Y_3$, that $K_i \cap \pi^{-1}(Y_3) < K_3$ for $i = 1, 2$, and that* VD *is ($\dim Y_3$)-fibred. Then there is a* VD*-box K_4 such that $K_i < K_4$ for $i = 1, 2$.*

Proof. Fix a basepoint $b \in Y_3$. For i = 1, 2, 3, let $X_i = K_i \cap E_b$ and let $\gamma_i \in \Gamma(X_i, Y_i)$ be a pointed parametrization of K_i . According to lemma 25, we can assume that γ_3 extends $\gamma_1|_{Y_3}$.

Since VD is ($\dim Y_3$)-fibred, there exists a pointed $\gamma_1' \in \Gamma(X_3, Y_1)$, extending γ_1 , and which coincides with γ_3 over Y_3 . According to lemma 25, there is a reparametrization γ_1'' of γ_1' such that $\gamma_1''|_{Y_3}$ extends $\gamma_2|_{Y_3}$.

In the same way, since VD is ($\dim Y_3$)-fibred, there exists a $\gamma_2' \in \Gamma(X_3, Y_2)$ extending γ_2 and which coincides with γ_1'' over Y_3 .

Since γ_1'' and γ_2' coincide over Y_3, they define together an element of $\Gamma(X_3, Y_1 \cup Y_2)$, whose image K_4 is obviously $> K_1$ and $> K_2$. •

Proof of theorem B. "Only if" is trivial: just take for VD the set of all vertical domains. Thus reciprocally we assume that $\pi : E \to B$ is a smooth surjective submersion which admits an exhaustive, isotopy invariant, $(q-1)$-fibred set VD of vertical domains, and let us prove that it is a fibration. For every $b \in B$, let Y be a neighborhood of b diffeomorphic to $\mathbf{B}^q$. We first claim :

Every compact subset $C \subset \pi^{-1}(Y)$ is $<$ some VD*-box.*

The proof is by induction on k , the smallest integer such that there exists Y^k homeomorphic to $\mathbf{B}^k$ and verifying $\pi(C) \subset Y^k \subset Y$.

For $k = 0$ the claim follows from the exhaustiveness of VD .

Assume That the claim is proved for $k-1$. Identify Y^k with $[0,1]^k$. For each $y \in Y$, by exhaustiveness of VD there is a vertical domain belonging to VD and containing $C \cap E_y$ in its interior. By lemma 24 this domain is the intersection of E_y with a VD-box K_y whose base $\pi(K_y)$ is a neighborhood of y . Since C is compact, y admits a smaller neighborhood W_y such that $C \cap \pi^{-1}(W_y) < K_y$. Thus dividing $Y^k = [0,1]^k$ into small enough cubes of equal size $1/N$, say $Q_1, \ldots, Q_{N^k}$, each $C \cap \pi^{-1}(Q_i)$ is $<$ some VD-box, say K_i , of base $\pi(K_i) = Q_i$.

Consider $U_i = Q_1 \cup \ldots \cup Q_i$ the union of the i first small cubes (ranged of course in the natural lexicographic order). Assume by induction on i that $C \cap \pi^{-1}(U_i)$ is $<$ some VD-box K_i' of base $\pi(K_i') = U_i$. The intersection $U_i \cap Q_{i+1}$ is obviously homeomorphic to $\mathbf{B}^{k-1}$, thus, by the induction (on k) hypothesis, $(K_i \cup K_i') \cap \pi^{-1}(U_i \cap Q_{i+1})$ is $<$ a VD-box. Since moreover U_i and Q_{i+1} are contractible and since VD is $(k-1)-fibred$, lemma 27 assures the existence of a VD-box $K_{i+1}' > C \cap \pi^{-1}(U_{i+1})$. For $i = N^k - 1$, we are done : the claim is proved.

Fix (C_n) an increasing sequence of compact sets whose union is $\pi^{-1}(Y)$. The previous claim gives a sequence of pointed sections $\xi_n \in \Gamma(X_n, Y)$, with $X_n \in \mathrm{VD}_b$, such that :

$$C_n \cup \mathrm{Im}\, \xi_{n-1} < \mathrm{Im}\, \xi_n$$

Changing if necessary ξ_n to another section close to it, we can assume that the embedding $(x, y) \mapsto \xi_n(y)(x)$ is smooth. After lemma 25, a convenient reparametrization of ξ_n gives ξ_n as an extension of ξ_{n-1} . Consider ξ the inductive limit of the ξ_n's . Its image is $\pi^{-1}(Y)$ since it contains every C_n . In other words ξ is a smooth trivialization of E over Y . •

Of course, in practice it may be hard to verify that a map such as $\rho_{X,X'}$ is $(q-1)$-fibred, since, after part I, this is something like comparing the $(q-1)$-homotopy type of the embedding spaces of domains into the different fibres. So our next tasks will be to change this condition to more handy ones.

COROLLARY 28 (dim-1 base principle for bundles) —

a) A surjective smooth submersion $\pi : E \to B = \mathbf{R}^q$ is a bundle iff it is a bundle over each straight line in the base parallel to one of the axis.

b) A surjective smooth submersion $\pi : E \to B$ is a bundle iff for every smooth path $\gamma : [0,1] \to B$, the pullback $\gamma^*\pi : \gamma^*E \to [0,1]$ is a bundle.

c) Theorem B is still true if we change the hypothesis 'VD is $(q-1)$-fibred' to the weaker one 'VD is 0-fibred'.

Proof. To prove a), we make an induction on q . Assume That $\pi : E \to \mathbf{R}^q$ is a surjective submersion and a bundle over each straight line L parallel to an axis . Let VD be the set of all vertical domains. After theorem B, it is enough to prove that $\rho : VE^0(X', E) \to VE^0(X, E)$ is a fibration for every two vertical domains X, $X' \in \mathrm{VD}$ such that the first one is contained in the interior of the second one.

By assumption, for every L, our submersion π is a bundle over L , and in particular the map $VE^0(X, \pi^{-1}(L)) \to L$ is a fibration. By corollary 19,

$$\pi_X : VE^0(X, E) \to \mathbf{R}^q$$

is a fibration. Write $\pi_X = (\pi_X^1, \ldots, \pi_X^q)$. Thus the last factor,

$$\pi_X^q : VE^0(X, E) \to \mathbf{R}$$

is also a fibration.

In the same way, $\pi_{X'}^q$ is also a fibration.

But by the induction hypothesis, ρ is a fibration over $(\pi_{X'}^q)^{-1}(t)$ for each $t \in \mathbf{R}$. The fibrewise fibration principle (corollary 15,i) concludes that ρ is actually a fibration.

Affirmation b) then follows immediately from a); and affirmation c) from theorem B and from b). •

Also, since our theorem A gave a satisfying analysis of fibrations, one can assume that π is a submersion *and a fibration,* say a "submersion-fibration," and ask for sufficient conditions which make it a bundle.

DEFINITION 29 — *Call* VD engulfing *when for every three domains X, X', $X'' \in \mathrm{VD}_b$ such that $X \subset \mathrm{Int}(X') \cap \mathrm{Int}(X'')$, if, in the ambiant space E_b , the domain X' can be pushed into X'' by a* homotopy *relative to X , then it can also by an* isotopy *relative to X .*

Observe that we only ask the homotopy, and the isotopy, to be the identity *on X ,* rather than on the whole of $X' \cap X''$.

COROLLARY 30 — *Let $\pi : E \to B^q$ be a (surjective, smooth) submersion-fibration. If it admits an exhaustive, isotopy invariant, engulfing set of vertical domains, then it is a bundle.*

Proof. After the preceding corollary, we can assume that $B = [0,1]$. Given X , $X' \in \mathrm{VD}_0$ such that $X \subset \mathrm{Int}(X') \subset X'$ and $\xi \in \Gamma(X, [0,1])$ pointed, we have to build a pointed $\xi' \in \Gamma(X', [0,1])$ such that $\rho_{X,X'}(\xi') = \xi$. Consider the set $\mathbf{T}$ of t's such that there exists a pointed $\xi'_t \in \Gamma(X', [0,t])$ such that $\rho_{X,X'}(\xi') = \xi|_{[0,t]}$. Obviously $\mathbf{T}$ is an interval, open after lemma 24., and containing 0. Assume, by contradiction, that $\mathbf{T} = [0, T[$. Since π is a fibration, there is a section γ of the restriction map $V^0(X', E) \to VE^0(X, E)$, where $V^0(X', E)$ is the space of vertical *maps* $X' \to E$ homotopic to $\mathrm{id}_{X'}$. Since

VD is exhaustive, there is a $X'' \in \mathrm{VD}_T$ containing $\gamma(T)(X')$ in its interior. By lemma 24, for $t < T$ close enough to T there is a $\xi'' \in \Gamma(X'', [t, T])$ such that $\xi''(T) = \mathrm{id}_{X''}$.

In the fibre E_t, we have three domains $\xi(t)(X)$, $\xi'(t)(X')$, $\xi''(t)(X'')$; and since π is a fibration, the second one can be pushed into the third one by a homotopy relative to the first one. Since VD is engulfing, it can also by an isotopy relative to the first one. In other words $\xi'(t)$ is isotopic in E_t, relatively to $\xi(t)(X)$, to some embedding $\phi : X' \to \xi''(t)(X'')$. The section :

$$s \mapsto \xi''(s) \circ \xi''(t)^{-1} \circ \phi$$

belonging to $\Gamma(X', [t, T])$, is homotopic to ξ' at point t, thus after lemmas 7 and 24, there exists a pointed section over $[0, T]$, contradiction. •

II.3. — *Applications.* We show how the criteria established in the preceding paragraph apply in several typical situations.

COROLLARY 31 — *A surjective smooth submersion with each fibre diffeomorphic to* $\mathbf{R}^p$ *is a bundle.*

Proof. The family of all vertical domains diffeomorphic to the compact p-ball is exhaustive since each fibre is diffeomorphic to $\mathbf{R}^p$.

This family is obviously isotopy invariant.

Recall Alexander's trick : let M^p be any smooth manifold without boundary, let $E(\mathbf{B}^p, M)$ be the space of embeddings of the compact p-ball into M , and let $F(M)$ the space of frames of TM . Then the map :

$$J_0^1 : E(\mathbf{B}^p, M) \to F(M)$$

that to each embedding associates its 1-jet at the origin, is a homotopy equivalence.

It follows that the restriction map $\rho_{X,X'}$ has contractible fibres. By lemma 8 it is a fibration. Thus theorem B applies. •

COROLLARY 32 — *Let* $\pi : E^{p+q} \to B^q$ *be a (surjective, smooth)* submersion-fibration.

a) If $p = 2$ *then* π *is a bundle;*

b) If $p \geq 5$ *, if each fibre* E_b *is topologically finite, i.e. diffeomorphic to the interior of a compact* p*-manifold* M_b *with smooth boundary* ∂M_b *, and if every connected component of every* ∂M_b *is simply connected, then* π *is a bundle;*

c) (Stabilization)The map $E \times \mathbf{R}^{3p+1} \to B : (x, x') \mapsto \pi(x)$ *is a bundle.*

Point c) answers a question of [3].

Proof. a) We can assume that the fibres are connected. Then the family of all connected vertical domains is obviously exhaustive and isotopy invariant. The engulfing property is verified by connected domains in surfaces — this is an exercise, using for example the results and methods of [2]. Thus corollary 30 applies.

b) The set of all vertical domains $X \subset E_b$ such that X is a deformation retract of E_b is obviously exhaustive and isotopy invariant. After Van Kampen's and Grushko-Neumann's theorem, each component of $W = E_b \backslash \mathrm{Int}(X)$ is simply connected. After Poincaré duality and the h-cobordism theorem, $W \cong \partial X \times [0, 1($. The relative engulfing property follows immediately, and corollary 30 applies.

c) Given a domain X and a polytope $K \subset \mathrm{Int}(X)$, say that X *shrinks* to K if, for every neighborhood $U(K)$, the whole domain X can be pushed into $U(K)$ by an istopy of embeddings of X into X , relative to a neighborhood of K .

For every $b \in B$, define VD_b the set of $(4p + 1)$-dimensionnal vertical domains $X \subset E'_b = E_b \times \mathbf{R}^{3p+1}$ with the two following properties :

1. X shrinks to a $(p - 1)$-dimensionnal polytope;

2. Every $(p - 1)$-dimensionnal subpolytope in $\mathrm{Int}(X)$ is contained in a $(2p - 1)$-dimensionnal polytope to which X shrinks.

Obviously VD is isotopy invariant. To prove that it is also engulfing, let X, X', $X'' \in \mathrm{VD}_b$ be as in definition 29. Let K, K' be $(p - 1)$-dimensionnal polytopes on which X , X' respectively shrink (property 1.) Since $\dim K + \dim K' < \dim E'_b$, by a general position argument, we can chose K, K' disjoint. Since $X' \in \mathrm{VD}$, by property 2 there is a $(2p - 1)$-dimensionnal complex $L \subset \mathrm{Int}(X')$ containing $K \cup K'$ and to which X' shrinks.

By hypothesis, there is a homotopy that pushes X' into X'' and is the identity on X . Again by a general position argument, since $2(\dim L + 1) < \dim E'_b$, after a small perturbation if necessary, this homotopy induces, in restriction to L , an isotopy of embeddings of L into E'_b , and remains the identity on a neighborhood of K .

Since X' shrinks to L , we get an isotopy of embeddings of X' into E'_b , that pushes it into X'' . Moreover it is the identity on a neighborhood of K .

Since X shrinks to K , we can chose this isotopy to be identity on the whole of X , and the engulfing condition is established.

Finally, we prove that VD is exhaustive. It is enough to prove that for every domain $D^p \subset E_b$, the product $D^p \times \mathbf{B}^{3p+1}$ is contained in a domain $X \in \mathrm{VD}$. Let X be a regular neighborhood of $D^p \times \mathbf{B}^{3p+1}$. In particular X shrinks to $D^p \times \mathbf{B}^{3p+1}$.

If E_b is compact, then, since π is assumed a fibration, all fibres are compact with the same π_0 , thus, as previously mentionned, π is already a bundle.

Thus we can assume that E_b is not compact, thus ∂D^p is not empty, and in particular D^p shrinks to some $(p-1)$-subpolytope K . It follows obviously that X also shrinks to K .

Finally, let $K \subset \mathrm{Int}(X)$ be any $(p-1)$-polytope. We have to find a larger, $(2p-1)$-dimensionnal polytope L to which X shrinks. After a first isotopy if necessary, we can assume that K is contained in $D^p \times \mathbf{B}^{3p+1}$. Decompose $E'_b = E_b \times \mathbf{R}^{3p+1}$ as $E_b \times \mathbf{R}^{p-1} \times \mathbf{R}^{2p+2}$, and consider the canonical projection of K into the factor $E_b \times \mathbf{R}^{p-1}$. Since $2 \dim K < 2p-1$, after a small perturbation this projection become an embedding of K into $D^p \times \mathbf{B}^{p-1}$. Thus we have an isotopy of embeddings pushing K into $D^p \times \mathbf{B}^{p-1}$. Thus we can assume that $K \subset D^p \times \mathbf{B}^{p-1}$. Then $L = D^p \times \mathbf{B}^{p-1}$ fits. In conclusion VD is exhaustive.

Thus corollary 30 applies. •

Question Assume That $\pi : E \to B$ is a submersion-fibration and that a group acts freely, cocompactly and properly discontinuously on E and permutes the fibres. Does it follow that π is a bundle ?

As a last application, in corollary 31, $\mathbf{R}^p$ can be replaced by any manifold bounded by the sphere and of large enough dimension :

COROLLARY 33 — *Let M be a compact manifold of dimension $p \geq 5$ and with smooth boundary homeomorphic to $\mathbf{S}^{p-1}$. Then any (surjective, smooth) submersion all of whose fibres are diffeomorphic to $Int(M)$, is a bundle.*

Observe that here we don't assume any more the submersion to be a fibration.

Proof. We first claim that for every submanifold $X \subset Int(M)$ abstractly diffeomorphic with M, the pair (M, X) is contractible.

Indeed, after Van Kampen's and Grushko-Neumann's theorems, W is simply connected. Also, the Mayer-Vietoris sequence for $M = X \cup W$ gives, for each $2 \leq k \leq p-2$, that $H^k(X; \mathbf{Z}) \oplus H^k(W; \mathbf{Z})$ is isomorphic to $H^k(M; \mathbf{Z})$. Since $H^k(X; \mathbf{Z})$ is (abstractly) isomorphic to $H^k(M; \mathbf{Z})$, it follows from the elementary theory of finitely generated abelian groups that W is $(p-2)$-connected. Thus the pair $(W, \partial X)$ is $(p-2)$-connected. On the other hand, by Poincaré duality, $H_k(W, \partial X; \mathbf{Z})$ is isomorphic to $H^{n-k}(W, \partial M; \mathbf{Z})$, thus null for $k \geq n-1$ since W is 1-connected and ∂M is 2-connected.

To sum up, $(W, \partial X)$ has no homology. By excision principle, (M, X) doesn't either. The claim is proved.

Fix $b \in B$. Chose $X \subset E_b$ a core. By lemma 24, there is a neighborhood Y of b in B and a section $\gamma \in \Gamma(X, B)$. By the claim, for every $y \in Y$, the vertical domain $\gamma(y)(X) = \mathrm{Im}\,\gamma \cup E_y$ is a deformation retract of E_y . After corollary 15, π is a fibration over Y . Corollary 32, b) concludes. •

It follows that Palmeira's conjugation theorem for open manifolds foliated by $\mathbf{R}^p$ extends to more general kind of leaves; the (long) proof is exactly as in [9] :

COROLLARY 34 — *Let M^p be as in the preceding corollary. Let (V, F) and (V', F') be foliated, simply connected, $(p+1)$-dimensionnal, open manifolds whose leaves are all diffeomorphic to M . Assume that the (in general non-Hausdorff) leaf spaces V/F , V'/F' are diffeomorphic. Then this diffeomorphism can be realized by a smooth conjugation between the foliations.*

II.4. — *Ends of deformation-equivalent manifolds.* What do two different fibres E_b , $E_{b'}$ of a fibration-submersion $\pi : E \to B$ must have in common, assuming of course that B is connected ?

Call two such open manifolds, say, *deformation-equivalent.* Compare [3].

Of course they have the same homotopy type, but not necessarily the same *proper* homotopy type (example 21.) In fact, every contractible p-manifold U is deformation-equivalent to $\mathbf{R}^p$: proceed as in example 21, but change $\mathbf{R}^p$ to U , and W to a small open p-ball in U .

Nevertheless, in dimension 2, deformation-equivalent manifolds are necessarily diffeomorphic (corollary 32.)

The following proposition answers a question of [3], where some particular cases were obtained.

PROPOSITION 35 — *Two deformation-equivalent manifolds necessarily : a) are diffeomorphic if one of them is compact; b) have the same* tangential *homotopy type; c) have the same orientability property w.r.t. every ring R ; d) have isomorphic* algebras *of compactly supported cohomology with coefficients in every ring R that makes them orientable; e) have homeomorphic ends spaces.*

Proof. a) If E_b is compact, then all fibres, having the same dimension and homotopy type as E_b , are compact; moreover they have the same π_0 ; and we have seen as a corollary of lemma 22 that π is necessarily a bundle.

b) Because the tangent vector spaces TE_b and $TE_{b'}$ are two pullbacks of a single p-dimensionnal vector space over E : the kernel of $d\pi$.

c) Follows from b).

d) and e) One can assume that $B = \mathbf{B}^q$ and that b , $b' \in Int(B)$. Thus E has a boundary $\partial E = \pi^{-1}(\partial B)$. We shall prove that E_b has the same cohomology algebra and the same ends space as E ; of course the same will be true for $E_{b'}$.

We have a commutative diagram :

$$\begin{array}{ccc} H_c^*(E;R) & \xrightarrow{i^*} & H_c^*(E_b;R) \\ \downarrow & & \downarrow \\ H_{p+q-*}(E,\partial E;R) & \xrightarrow{j_*} & H_{p+q-*}(E,E\setminus E_b;R) \end{array}$$

Vertical arrows are Poincaré and Alexander duality and they are one-to-one; i^* is the morphism of restriction to the properly embedded submanifold E_b ; and j_* is the inclusion morphism, one-to-one because ∂B is a deformation retract of $B \setminus \{b\}$ and because π is a fibration. Thus i^* is one-to-one; and in the same time an algebra morphism. This proves d).

For every locally compact space S, its space of ends can be defined as the spectrum of an algebra, namely the algebra $B(S)$ of germs, at the neighborhood of infinity, of locally constant functions with value in $R = \mathbf{Z}/2\mathbf{Z}$. On the other hand, we have a commutative diagram whose lines are exact :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & B(E) & \longrightarrow & H_c^1(E;R) & \longrightarrow & H^1(E;R) \\ & & =\downarrow & & f\downarrow & & i^*\downarrow & & g\downarrow \\ 0 & \longrightarrow & R & \longrightarrow & B(E_b) & \longrightarrow & H_c^1(E_b;R) & \longrightarrow & H^1(E_b;R) \end{array}$$

where f , i^* and g are restriction morphisms. Since π is a fibration, g is one-to-one. By d), i^* is one-to-one. By the five lemma, f is also one-to-one. Thus the algebras $B(E)$, $B(E_b)$ are isomorphic, thus they have homeomorphic spectra. •

II.5. — *More examples and questions.* We end with a few (pleasant) monsters.

EXAMPLE 36. — Let again W be the Whitehead manifold, let $V \subset W$ be an open 3-ball, and let $U \subset V$ be an open subset diffeomorphic to W . Play the same game as in example 21, but with W instead of $\mathbf{R}^3$ and with U instead of W . Then again π is a fibration of $\mathbf{R}^4$ onto $\mathbf{R}$, but this time all fibres are diffeomorphic to W . If it were a bundle, then there would be a 1-parameter family of embeddings $i_t : W \to W$ such that $i_0 = \mathrm{id}_W$ and $i_1(W) = U$. But this would imply that every compact subset of W could be engulfed by the 3-ball V, and W would be diffeomorphic to $\mathbf{R}^3$, a contradiction. Thus π is not a bundle.

EXAMPLE 37. — Let $V \subset \mathbf{R}^4$ an open subset homeomorphic but not diffeomorphic to $\mathbf{R}^4$ (Casson-Friedman, see for example [6].) Play the same game as in example 21, but with $\mathbf{R}^4$ instead of $\mathbf{R}^3$ and with V instead of W . The total space E is diffeomorphic to $\mathbf{R}^5$ since it is 5-dimensionnal, contractible and simply connected at infinity [12]. This (real-analytic !) submersion-fibration of $\mathbf{R}^5$ onto $\mathbf{R}$ is not a bundle in the C^∞ (or even C^1) category, since one of the fibres is not diffeomorphic to the other ones. On the other hand, all our work extends to the C^r categories ($r \in \mathbf{N}$) . The reader will provide himself a proof of lemma

22 in the C^0 and the C^1 differentiability class. In particular, corollary 31 is valid in class C^0 ; thus our submersion-fibration is a bundle in the C^0 sense.

EXAMPLE 38 — Let :

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ & {\scriptstyle\pi'}\searrow\swarrow{\scriptstyle\pi} & \\ & B & \end{array}$$

be a commutative diagram of (smooth, surjective) submersions. If π' and h are fibrations, then π is also — obvious, since to be a fibration it is enough to have the homotopy lifting property *for simplicies.* Jean Pradines asked if we can change "fibration" to "bundle". The answer is negative, always with the same counterexample: let E, B, π be as in the example 21, let $E' = E \times \mathbf{R}$ and let h be the first projection. Then h is a bundle. Also π' is a bundle by corollary 31, since all its fibres are $\mathbf{R}^3 \times \mathbf{R}$ or $W \times \mathbf{R}$, thus diffeomorphic to $\mathbf{R}^4$. But π is not. This phenomenon was already observed in [3].

EXAMPLE 39 — We can also answer negatively Pradines' question with h a *normal, infinite cyclic covering.* Let E^*, B, π^* be as in example 21, let $E = E^* \times \mathbf{S}^1$, let $\pi(x, y) = \pi^*(x)$, let E' be the universal covering of E and let $h : E' \to E$ be the canonical projection. Then h is a normal covering and again π' is a bundle by corollary 31, since $W \times \mathbf{R}$ is diffeomorphic to $\mathbf{R}^4$. But π is not, since $W \times \mathbf{S}^1$ does not have the same proper homotopy type as $\mathbf{R}^3 \times \mathbf{S}^1$.

EXAMPLE 40 — As Alan Weinstein points out [13], our results also allow to answer negatively Pradines' question with E the quotient of E' by the free, fibrewise, action of a compact group. Actually, let E, B, π be as in the example 37 (thus with a fibre being an exotic $\mathbf{R}^4$), let $E' = E \times \mathbf{S}^3$ and let h be the first projection. Then h is a bundle. Also, each fibre of π' is $\mathbf{S}^3\times$ a topological $\mathbf{R}^4$, thus the interior of a 7-dimensionnal compact manifold with (simply connected) boundary, by [11]. By corollary 32 b), π' is a bundle. But π is not.

Question (Weinstein) What about Pradines' question with h a *finite covering ?* Or equivalently, if one likes better, a normal one, i.e. the quotient by the free fibrewise action of a finite group ?

Some of these results have been announced or conjectured in [**7**] *and* [**8**].

[**1**] CERF J., *Topologie de certains espaces de plongements,* Bull. Soc. Mat. France 89 (1961), 227–380.

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