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# Application of the Wavelet-element Method to Linear Random Materials

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**Abstract.** This paper is devoted to the statistical study of the effective linear properties of random materials, that is to say microstructures are random lattices given by a stochastic process. The local numerical procedure associated to homogenization techniques is based on a wavelet-element method. The numerical results are compared with classical theories. A new approach is obtained in order to determine these effective properties if the details of the microstructure are not well-known.

## 1 Introduction

The purpose of this paper is to determine and to compute the effective properties of linear random heterogeneous media. While natural geomaterials are heterogeneous, they are regarded as homogeneous with effective properties which manipulate the overall behavior. Grounds, for example, are usually modeled as a body consisting of homogeneous materials with such effective properties. Behavior of these grounds cannot be predicted if the effective properties are poorly determined. The necessity to incorporate more detailed informations on the microstructure is clearly recognized. In the last years, a large part of literature has presented measurements, empirical and theoretical techniques in order to determine these effective properties. Clearly, the numerical cost to attack this problem directly for a given material sample is prohibitive. In former papers, the authors have presented numerical scheme associated to homogenization techniques in order to take into account the details of the microstructure [4-6,8,11]. These algorithms are based on wavelet basis and permit to predict the effective properties from the knowledge of the microstructure. Thus, it is possible to optimally design linear composite materials from real images of the sample.

In most cases, unfortunately, the details of the microstructure are not absolutely known, and leads us to estimate the effective properties from statistical studies. In a famous paper [14], Torquato has presented results concerned with the case where a partial statistical information, as correlation functions, is given. Improved bounds on effectives properties of two-phase

random heterogeneous media are obtained. In this paper, we suppose that only the volume fraction of each phase is known, and we intend to find the averaged effective properties of the composite. In addition, we suppose that the microstructure is given pixel by pixel i.e. the characteristic shape of the microstructure components is a small square and its characteristic length is the length of the pixel.

In section 1, we present the notations and the mechanical problem. Section 2 is devoted to the numerical algorithm. In section 3, we present a statistical study for a AlSiC composite. The results obtained are compared with Hashin and Shtrickman bounds [9] and with the self-consistent schema [10,13]. At the end of the paper, perspectives to this work are presented.

## 2 The mechanical problem

We consider a two phase isotropic elastic composite and we intend to study the behaviour of this heterogeneous media. A step of homogenization consists in determining the effective properties which characterize the equivalent homogeneous medium. Let us consider a plane periodic composite  $\Omega$  (macro-scale, variable  $x$ ) and a rectangular periodicity cell  $Y$  (micro-scale, variable  $y$ ). We assume that the bonds at all interfaces of the composite are perfect. The elasticity coefficients  $C_{ijkl}^\varepsilon$  are supposed to be periodic on the period  $Y$ . In elastostatics, the problem ( $P_\varepsilon$ ) is written

*Problem*( $P_\varepsilon$ )

Find  $u^\varepsilon \in V$  such that  $a_\varepsilon(u^\varepsilon, v) = L(v), \forall v \in V,$

where  $a_\varepsilon(u, v) = \int_{\Omega} C_{ijkl}^\varepsilon e_{kl}(u) e_{ij}(v) dx,$

$e_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k}),$

$L(v) = \int_{\Omega} f v dx + \int_{\partial_1 \Omega} F v dl$

and  $V = \{v \in (H^1(\Omega))^2, v = 0 \text{ on } \partial_2 \Omega\}.$

The solution of the previous problem  $u^\varepsilon$  tends to  $u$  in  $V$ , the space of admissible global displacements,  $u$  is the solution of problem  $P$  [1]

*Problem*( $P$ )

Find  $u \in V$  such that  $a(u, v) = L(v), \forall v \in V,$

where  $a(u, v) = \int_{\Omega} A_{ijkl} e_{kl}(u) e_{ij}(v) dx$

and  $A_{ijkl} = \frac{1}{\text{meas}(Y)} \int_Y (C_{ijkl} + C_{ijpq} e_{pq}(u^{kl})) dy.$

The determination of the displacements  $u^{ij}$  needs to solve three local problems on  $Y$  (Problem  $P_Y$ ) in the space  $H = (H_{per}^1(Y))^2$ , the set of admissible local displacements:

*Problem( $P_Y$ )*

$E^{ij}$  be given, find  $u^{ij} \in H$  such that  $a_Y(u^{ij}, v) = l(v) \forall v \in H$ ,

where  $a_Y(u, v) = \int_Y C_{ijkl}(y) e_{ij}(u) e_{kl}(v) dy$

and  $l(v) = - \int_Y C_{ijkl}(y) E^{ij} e_{kl}(v) dy$ .

### 3 Numerical procedure

In this section, we focus on the solution of problem  $P_Y$ . This problem is solved by a wavelet-element method [15] associated to Daubechies wavelets [3]. This method is based on the standard form of the discretized operator [2]. Details concerning this method can be found in previous papers of the authors [5,8,?]. Let  $\Lambda_j = [0, 2^j - 1]$ , the local displacement  $u = (u_1, u_2)$  is approached in the space  $V_{j_{max}}$  as

$$u_i(y) = \sum_{k \in \Lambda_{j_0}^2} u_{j_0 k}^i \Psi_{j_0 k}^0(y) + \sum_{l=1}^{l=3} \sum_{j=j_0}^{j=j_{max}} \sum_{k \in \Lambda_j^2} u_{j k}^i \Psi_{j k}^l(y),$$

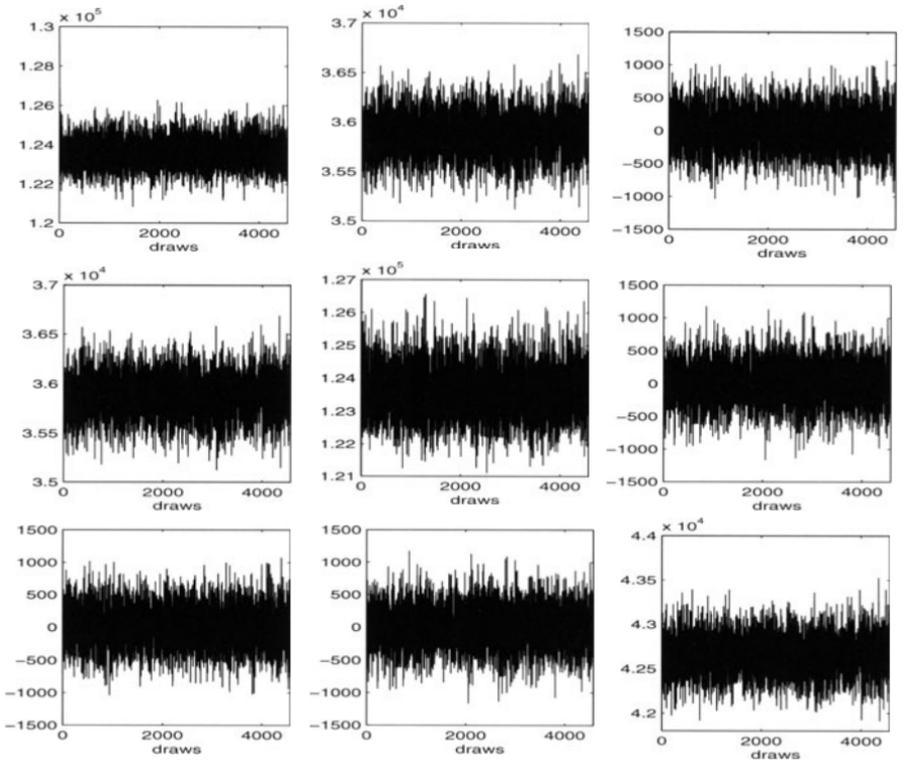
$$\Psi_{j k}^0(y) = \phi_{j k_1}(y_1) \phi_{j k_2}^l(y_2), \Psi_{j k}^1(y) = \psi_{j k_1}(y_1) \psi_{j k_2}^l(y_2),$$

$$\Psi_{j k}^2(y) = \psi_{j k_1}(y_1) \phi_{j k_2}^l(y_2), \Psi_{j k}^1(y) = \phi_{j k_1}(y_1) \phi_{j k_2}^l(y_2),$$

where  $\phi_{jk}(x) = 2^{j/2} \phi(2^j x - l)$  and  $\psi_{jk}(x) = 2^{j/2} \psi(2^j x - l)$ .  $\phi$  is the scale function and  $\psi$  is the associated wavelet. Problem ( $P_Y$ ) is discretized using this approximation, well-conditioned stiffness matrices are obtained [15,2].

### 4 Numerical results

In this part of the paper, numerical tests are presented. They are concerned by a AlSiC composite. The influence of two parameters is studied: the volume fraction of each component (Aluminium and Silicium) and the distribution of each material in the microstructure. For each volume fraction, we proceed to about 5000 draws (random lattices) and we study from a statistical point of view the results obtained. In the following the fourth order tensor  $A$  is represented as a matrix of order 3. For a volume fraction of 70 % the variations of each elasticity coefficient are presented in Fig. 1 for about 5000 microstructures. At first sight, the fluctuations seem very large. In the other hand, one

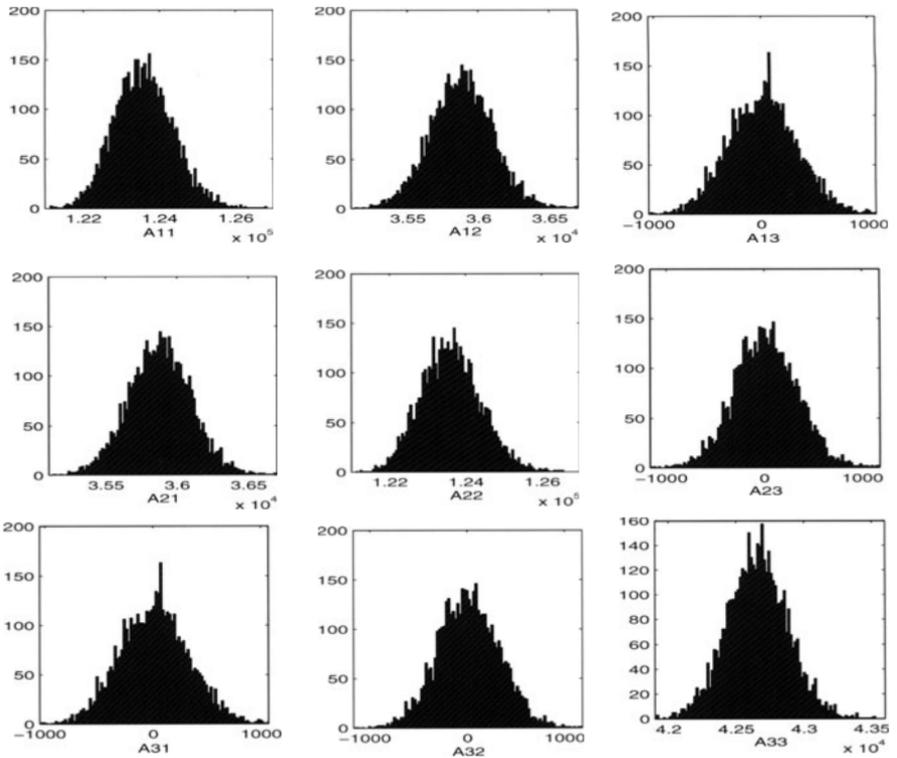


**Fig. 1.** Homogenized plane elasticity tensor (MPa) versus draws (volume fraction 70 %), line  $n$  and column  $m$  correspond to  $A_{nm}$

can present the histogram of each component of the elasticity tensor. We observe a Gaussian distribution of the values (Fig. 2).

The averaged values of each coefficient of the elasticity tensor are given in Fig. 3. We observe the convergence of the coefficients to a mean value along the draws. The values obtained in Fig. 3 show that the numerical limit is an **isotropic material**. In particular, we have  $A_{13} = A_{23} = 0$  and  $A_{11} = A_{22}$ . Thus, it is possible to compute the bulk and shear moduli of the limit material (Averaged bulk and shear moduli).

In the following we denote by L-HS and U-HS the Lower and Upper Hashin and Shtrickman bounds [9]. We denote by WAV the Averaged Value obtain by the Wavelet-element method. Fig. 6 shows that the relative behavior of the value is splitted in three phases: in the first one, for volume fraction lower than 0.25, the LHS and the WAV values are very closed. The second phase corresponds to volume fractions between 0.25 and 0.95. In this phase, the WAV value is intermediate between the two HS bounds. The third phase, for a volume fraction larger than 0.95, shows WAV values closed to the U-HS bound.

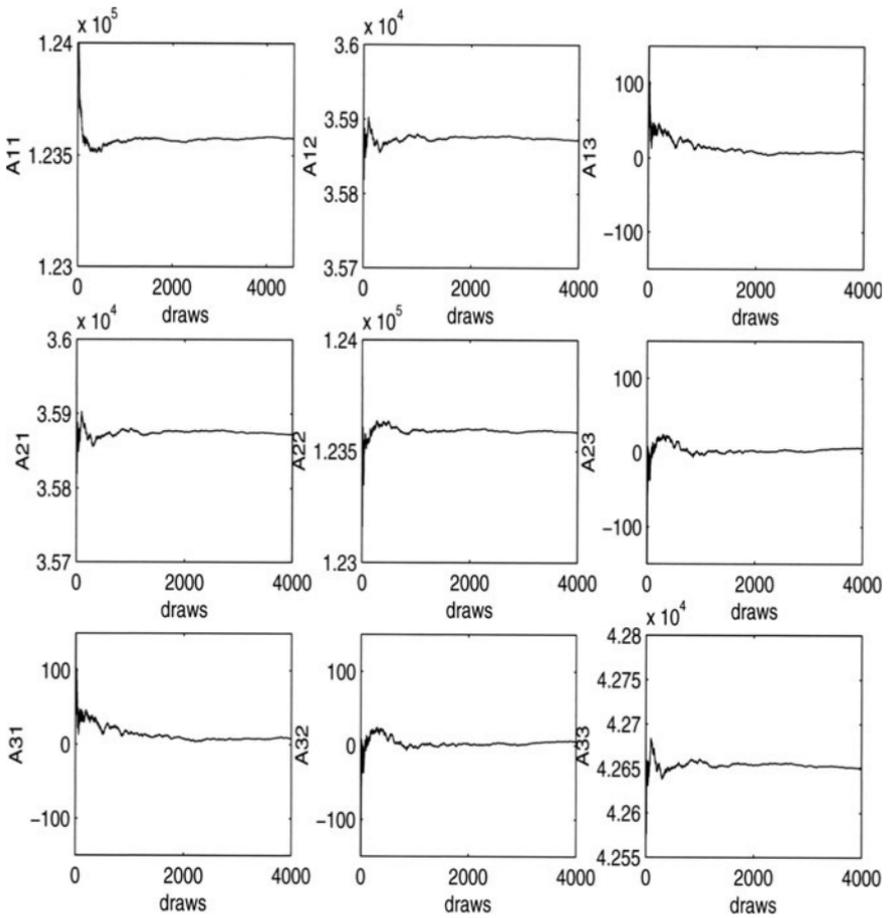


**Fig. 2.** Histogram of plane elasticity tensor (volume fraction 70 %)

We compare the results with the self-consistent schema for the shear modulus and for volume fractions lower than 0.5. We observe in Fig. 7 that for the lower value of the volume fraction (lower than 0.1) the L-HS value, the self-consistent schema (denoted SCS) and the WAV value are quite similar. For larger values from 0.1 to 0.25 the values L-HS and WAV are equal and the SCS value is lower than the two other one. For volume fractions larger than 0.25 and lower than 0.4, the L-HS and SCS values are closed, and lower than the WAV value.

## 5 Concluding remarks and perspectives

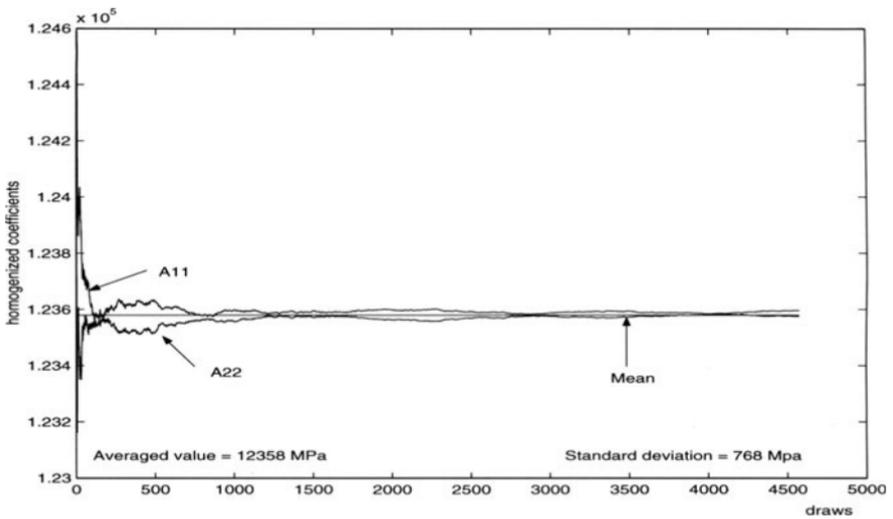
Generally, it is impossible to determine exactly the effective properties of random heterogeneous media. In the literature, rigorous statements about these effective properties take the form of bounds. In this paper, we obtain another kind of bounds as the form of an average and a standard deviation. An important result is the numerical convergence of the stochastic process of homogenization to an averaged solution, that is to say that the limit, when the number of draws tends to infinity, of the potential energy corresponding



**Fig. 3.** Averaged homogenized elasticity tensor versus draws (volume fraction 70 %)

to a 'random lattice' is the potential energy of an isotropic material. This material has effective properties depending on the volume fraction of each component. For lower volume fraction, the material has properties closed to the lower Hashin and Shtrickman bounds and to the self-consistent schema. A 'numerical theorem' is obtained. Similar results are obtained for different kinds of materials (Resin-Glass mixture for example).

Two ways to this work are proposed : the first one is a study of the influence of the shape of the lattices. We propose to study the same problem with different morphologies : n-pixels in one direction, association of n-pixels in a neighbourough of a point, ... Another perspective is to study the effective properties for more complex behavior: piezoelectricity couplings, thermome-



**Fig. 4.** Averaged homogenized elasticity coefficients A11 and A22 (volume fraction 70 %). If we study in details the variations of the averaged coefficients, we observe that the mean is obtained for a small number of draws and that the standard deviation is small compared to the coefficients. In the case presented in Fig. 4, i.e. for a volume fraction equal to 70 %, the mean is equal to 12358 Mpa and the standard deviation to 768 Mpa ( $\pm$  %). The values are included in the interval [12083,12682].

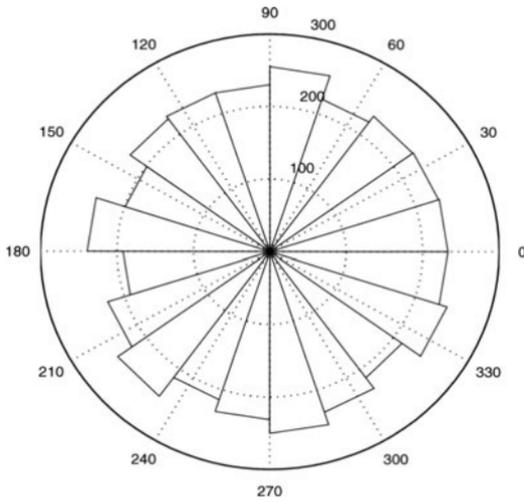
chanical couplings, plasticity [12], ... The efficiency of our method permits us to envisage to treat this kind of problems.

## Acknowledgments

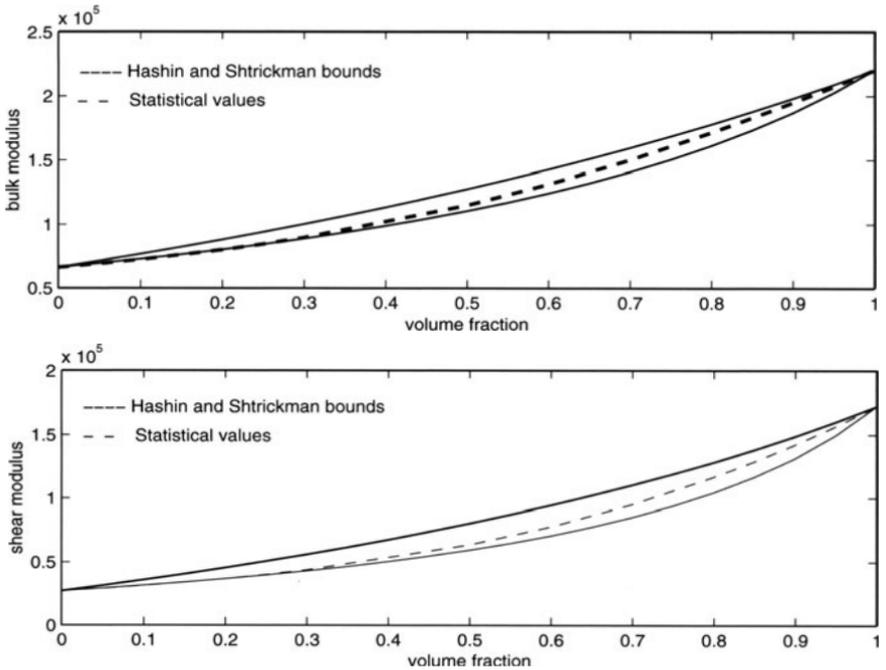
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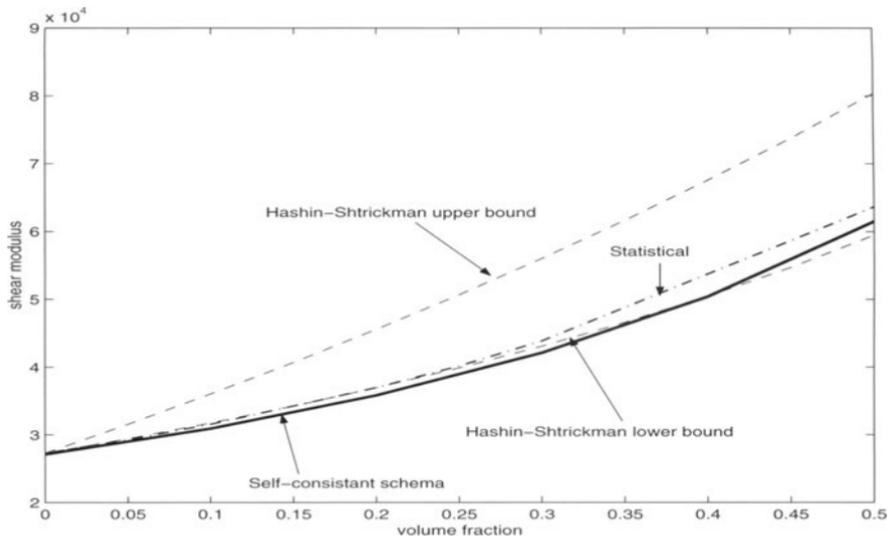
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**Fig. 5.** Histogram in polar coordinate system (volume fraction 70 %. The histogram in polar coordinate system shows the good distribution of the results according to the average value.)



**Fig. 6.** Bulk and shear moduli and Hashin-Shtrickmann bounds



**Fig. 7.** Shear modulus, Hashin-Shtrickmann bounds and self-consistent schema

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