Convex Difference Algorithm and Applications to Some Mechanical Problems

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Abstract. The splitting of functional into two convex functions provides a specific algorithm to determine the local minima. This approach is applied to two mechanical problems with non convex potentials: the modeling of shape memory alloys and the buckling of thin beams.

1 Introduction

In non linear mechanics, the equilibrium states minimize the potential energy which is not necessary convex. Consequently solve such a mechanical problem consists in finding the local minima of the potential energy. The functions which may be written as the difference of two convex functions constitute a first set of non convex functions. Some properties associated with the convexity may be used in order to formulate some extremality characterizations [9] and to define appropriate solution algorithms. Such an approach is not very easy to handle in infinite spaces because the convexity is a power tool to get compactness theorems in such a way to prove existence of solutions of a minimization problems. Then the numerical results presented in the following have to be considered with caution; they may depend on the mesh and not necessary converge towards a continuous solution when the discretisation step tends to zero. But we have to try new algorithms to reach possible equilibrium states in non linear mechanics often characterized by instability; these possible solutions have to be discussed from a mechanical point of view a posteriori; these paper dedicated to two applications is a first attempt in this way.

2 CD formulation and CD algorithm

Let consider a boundary problem on an open domain Ω of the space. The potential energy Π associated with this mechanical problem is written,

$$\Pi(\mathbf{v}) = \Phi_1(\mathbf{v}) - (\Phi_2 \circ \mathbf{D})(\mathbf{v}), \tag{1}$$

where $\Phi_1(\mathbf{v}) = \displaystyle\int_\Omega \phi_1(\mathbf{v}, \boldsymbol{\nabla}\mathbf{v})\, d\mathbf{x} - l(\mathbf{v})$, $\Phi_2(\mathbf{D}\mathbf{v}) = \displaystyle\int_\Omega \phi_2(\mathbf{D}\mathbf{v})\, d\mathbf{x}$ and $\mathbf{D}$ is a linear partial differential operator.

A *critical point* $\mathbf{u}$ of the potential energy Π is defined by,

$$\mathbf{0} \in \bar{\partial}\Pi(\mathbf{u}) = \partial\Phi_1(\mathbf{u}) - \partial(\Phi_2 \circ \mathbf{D})(\mathbf{u}). \tag{2}$$

where $\bar{\partial}$ denotes the generalized gradient introduced by Clarke [2] for the locally Lipschitz functions and ∂ denotes the subdifferential of a convex function. Since Φ_1 is a strictly differentiable function [2] the set equality in Eq. 2 holds and the previous inclusion is equivalent to,

$$\partial(\Phi_2 \circ \mathbf{D})(\mathbf{u}) \cap \partial\Phi_1(\mathbf{u}) \neq \emptyset.$$

Among the critical points, a *local minimum* $\mathbf{u}$ satisfies,

$$\partial(\Phi_2 \circ \mathbf{D})(\mathbf{u}) \subset \partial\Phi_1(\mathbf{u}). \tag{3}$$

But this relation characterizes a local minimum only if $\Phi_2 \circ \mathbf{D}$ is a piecewise affine function [4,5]. This last property is not true in general for the continuum problem but can be verified in finite dimensional cases, i.e. after finite element approximation. This decomposition leads to define a type II Lagrangian [1,5,9] depending on two fields,

$$\mathrm{L}_{\mathrm{II}}(\mathbf{v}, \boldsymbol{\tau}_2) = \Phi_1(\mathbf{v}) + \Phi_2^*(\boldsymbol{\tau}_2) - \int_\Omega \mathbf{D}\mathbf{v} : \boldsymbol{\tau}_2 d\mathbf{x}, \tag{4}$$

where the density $\phi_2^*(\mathbf{v}, \boldsymbol{\tau}_2) = \sup_{\mathbf{e}} \{\mathbf{e} : \boldsymbol{\tau}_2 - \phi_2(\mathbf{e})\}$ is the classical Fenchel conjugate function. This Lagrangian is separately convex in each of the dual variables, but the convexity of L_{II} is not guaranteed in general; a very simple example is illustrated in the Fig. 1. Consequently a min-min problem is naturally related to this Lagrangian whose the arguments are called ∂-critical points (cf. Table 2).

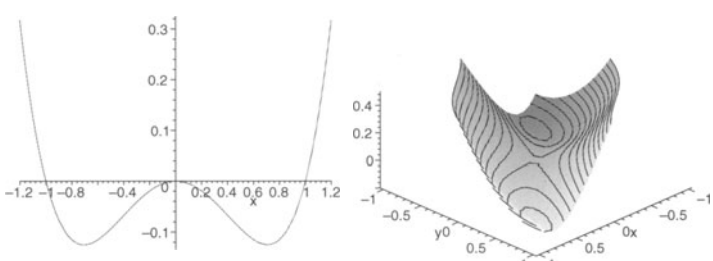

Fig. 1. $\Pi(\mathbf{v}) = \frac{\mathbf{v}^4}{2} - \frac{\mathbf{v}^2}{2}$ and $\mathrm{L}_{\mathrm{II}}(\mathbf{v}, \boldsymbol{\tau}) = \frac{\mathbf{v}^4}{2} + \frac{\boldsymbol{\tau}^2}{2} - \mathbf{v}\boldsymbol{\tau}$

CD Algorithm

- Initialization of the algorithm with $(\mathbf{u}^0, \boldsymbol{\sigma}_2^0)$,
- with $(\mathbf{u}^{n-1}, \boldsymbol{\sigma}_2^{n-1})$ known, we have to determine $\mathbf{u}^n, \boldsymbol{\sigma}_2^n$ as follows,

$$\text{step 1 : } \mathbf{u}^n = \arg\min \mathrm{L}_{\mathrm{II}}(., \boldsymbol{\sigma}_2^{n-1})$$

$$\text{step 2 : } \boldsymbol{\sigma}_2^n \in \arg\min \mathrm{L}_{\mathrm{II}}(\mathbf{u}^n, .)$$

Since D is a partial differentiable operator, by duality the field $\boldsymbol{\tau}_2$ belongs to a less regular space than $\mathbf{u}$. Then the *second minimization*, involving eventually a non differentiable ϕ_2 function, may be performed *locally*, i.e. in each finite element for instance. On the contrary, the *first minimization* always stays *global* on the structure and may take advantage to be quadratic in order to restrict this step to a linear solution. This situation (global but linear first step / non linear but local second step) compares the CD algorithm to augmented Lagrangian techniques based on this splitting.

3 Two applications

3.1 A set of models for Shape Memory Alloys (SMA)

In the first example we use the model of Shape Memory Alloys (SMA) given in [3,5,7] in an isothermal case. To do this, let us consider a decomposition into the difference between two convex functions Φ_1 and $\Phi_{2\mathcal{G}}$ of the potential energy Π associated with the mechanical problem,

$$\Pi(\mathbf{v}) = \Phi_1(\mathbf{v}) - (\Phi_{2\mathcal{G}} \circ \mathbf{e})(\mathbf{v}), \tag{5}$$

where $\Phi_1(\mathbf{v}) = \displaystyle\int_\Omega \phi_1(\mathbf{e}(\mathbf{v}))\, d\mathbf{x} - l(\mathbf{v})$ and $\Phi_{2\mathcal{G}}(\mathbf{e}) = \displaystyle\int_\Omega \phi_{2\mathcal{G}}(\mathbf{e})\, d\mathbf{x}$. According to the temperature range ($c = c(T)$) we have two behaviours; a pseudo-elastic behaviour with five regimes and a quasi-plastic one with three regimes,
if $c + \frac{\mathcal{G}}{2} > 0$,

$$\phi_{2\mathcal{G}}(\mathbf{e}) = \begin{cases} -\alpha\boldsymbol{\tau} : \mathbf{e} - c & \text{if } \alpha\boldsymbol{\tau} : \mathbf{e} \le -c + \frac{\mathcal{G}}{2} \\ \frac{-1}{2\mathcal{G}}(\alpha\boldsymbol{\tau} : \mathbf{e} + c + \frac{\mathcal{G}}{2})^2 & \text{if } |\alpha\boldsymbol{\tau} : \mathbf{e} + c| \le -\frac{\mathcal{G}}{2} \\ 0 & \text{if } \alpha|\boldsymbol{\tau} : \mathbf{e}| \le c + \frac{\mathcal{G}}{2} \\ \frac{-1}{2\mathcal{G}}(\alpha\boldsymbol{\tau} : \mathbf{e} - c - \frac{\mathcal{G}}{2})^2 & \text{if } |\alpha\boldsymbol{\tau} : \mathbf{e} - c| \le -\frac{\mathcal{G}}{2} \\ \alpha\boldsymbol{\tau} : \mathbf{e} - c & \text{if } \alpha\boldsymbol{\tau} : \mathbf{e} \ge c - \frac{\mathcal{G}}{2} \end{cases} \tag{6}$$

if $c + \frac{\mathcal{G}}{2} \le 0$,

$$\phi_{2\mathcal{G}}(\mathbf{e}) = \begin{cases} -\alpha\boldsymbol{\tau} : \mathbf{e} - c & \text{if } \alpha\boldsymbol{\tau} : \mathbf{e} \le \mathcal{G} \\ \frac{-1}{2\mathcal{G}}(\alpha\boldsymbol{\tau} : \mathbf{e})^2 - c - \frac{\mathcal{G}}{2} & \text{if } \alpha|\boldsymbol{\tau} : \mathbf{e}| \le -\mathcal{G} \\ \alpha\boldsymbol{\tau} : \mathbf{e} - c & \text{if } \alpha\boldsymbol{\tau} : \mathbf{e} \ge -\mathcal{G} \end{cases} \tag{7}$$

where $l(\mathbf{v})$ is the work of the external loading, α, $\boldsymbol{\tau}$, c are thermo-mechanical constants, $\mathbf{e}$ is the small strain tensor and $\mathcal{G}$ is a parameter characterizing the scale the phase transition is modelized ($\mathcal{G} = 0$ for the microscopic scale and $\mathcal{G} = \mathcal{G}_c$ for a macroscopic scale [5,6]). This decomposition leads us quite naturally to introduce a type II Lagrangian which depends on the displacement field $\mathbf{u}$ and on the "stress" phase transformation field $\boldsymbol{\sigma}_2$; in that case $\Phi^*_{2\mathcal{G}}$ is easy to compute and is given in [5,7]. The CD algorithm may catch the local minima and so reproduce hysteretic processes and phase propagations [6] (an augmented Lagrangian technique finds a global minimum which is more homogeneous).

3.2 Buckling of rods by the Elastica theory

We are interested to modelize the buckling of flexible and inextensible rods. It is well known that a force applied to the ends of the rod and oriented according to the rod may lead to different equilibrium states if the magnitude of the force is large enough. The critical values and buckling modes around the trivial solution may be obtained by a stability theory using a linear modeling of the rod; but the deformed shape may be reached only by a non linear modeling. Moreover the possible buckling shapes, if unilateral conditions have to be considered (cf. Fig. 5), are an open problem (cf. [8]). The existence of local minima in buckling problems constitutes a motivation to try a CD approach. In a first approach we use the Elastica theory where a single scalar field θ must be found. We have to minimize the following potential where P is an horizontal force, θ the angle between the deformed rods and horizontal axe, E the young modulus and I the inertial momentum. We remark that the non quadratic part is due to the work of the external force since the bulk energy is still a quadratic form,

$$\inf_{\theta} \int_0^{\ell} \frac{EI}{2}\, \theta'^2 \; - P \, \cos\theta \; d\mathbf{x}, \quad P < 0. \tag{8}$$

We postulate the following CD splitting by introducing a parameter β,

$$\phi_1(\theta) = \frac{EI}{2}\, \theta'^2 \; - \beta \, P \, \frac{\theta^2}{2}, \tag{9a}$$

$$\phi_2(\theta) = -P \left(\beta \, \frac{\theta^2}{2} \; - \; \cos\theta \right). \tag{9b}$$

The first energy density is convex if $\beta \geq 0$ because $P < 0$ and the second one is convex if $\beta \geq 1$ for all θ. By restricting θ to the interval $\left]\frac{\Pi}{2}, \frac{\Pi}{2}\right[$ – which may be relevant in some cases – the convexity holds for β non negative. The previous decomposition gives an optimal CD algorithm in the following meaning : the first minimization consists of a linear global problem since the second one is local and consists only in deriving ϕ_2 to update the $\boldsymbol{\sigma}_2$ multiplier. The two numerical examples show the ability of the CD algorithm

to reach equilibrium states if they are local minima of the potential. But only a first mode is obtained (even if the load permits others modes). In the first case (cf. Fig. 2 and Table 1) the first critical value is equal to $\frac{\Pi^2 EI}{4\ell^2} \simeq 25$ and the buckling may be recovered only for β equal to 0. For P less than 50, the solution satisfies the condition $\theta \in]\frac{\Pi}{2}, \frac{\Pi}{2}[$ and the CD algorithm converges for β non negative. On the contrary, for P bigger than 50 it is necessary to choose $\beta \geq 1$. The convexity assumption for ϕ_2 is therefore essential. But the parameter β has not to be too big otherwise the convergence rate slows down.

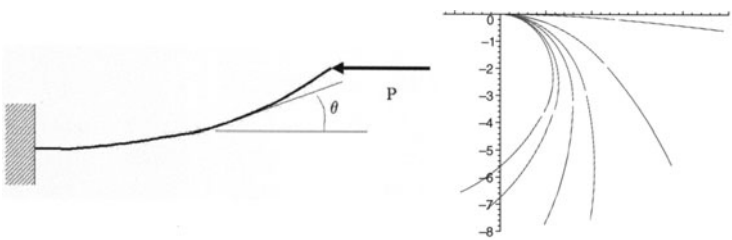

Fig. 2. The first buckling test and some deformed shapes (two elements)

The second example is quite different : a kinetic condition is imposed at the end of the beam (cf. Fig. 3) in such a way the first buckling mode may occur for P equal to $\frac{\Pi^2 EI}{\ell^2} \simeq 100$. With the Elastica theory this condition is satisfied implicitly by imposing the slope at the end opposite to the slope at the origin. This fact is a severe limit to use Elastica theory in more complex boundary problems. However the conclusions of the parametric study on β and P are the same than the previous case as illustrated in the Table 1 and in Fig. 3.

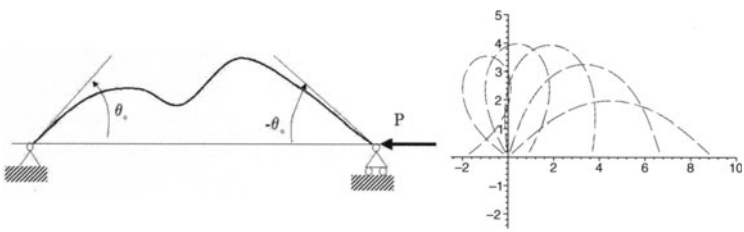

Fig. 3. The second buckling test and some deformed shapes (ten elements)

Table 1. Number of iteration for the CD algorithm. * : No buckling

P	Example 1						Example 2				
β	26	30	40	50	75	100	105	120	150	200	300
0	1053	30	9	19	∞	∞	104	34	14	*36*	∞
0.25	*	41	14	9	*25*	∞	133	46	32	13	*45*
1	*	71	31	23	16	13	219	81	42	28	22
2	*	111	54	41	32	30	334	129	72	51	41

4 CD algorithm and coupling

It may be interesting to compare a non convex optimization problem whose the objective function may be splitted into the difference of convex functions and a convex optimization problem with constraints (cf. Table 2). Indeed the two formulations underline a part of the objective function as a convex perturbation which is either added or subtracted. Moreover this perturbation may often involve a linear operator by composition. Consequently a Lagrangian approach leads to similar problems, the first as a Inf Sup problem and the other as a Inf Inf problem. The arguments of such problems are either saddle points or ∂-critical points according to the definition of Auchmuty [1]. The algorithms associated with these formulations (Uzawa algorithm and CD algorithm) consist of two steps: the first step is a classical minimization problem according to the primal variable, the second one is either an updating of the Lagrange multipliers or a minimization problem according to the multipliers. A last common point concerns the formulation of the critical points of the augmented Lagrangian of a convex constrained problem and of critical points of the initial CD function; but this fact is out of our purpose here.

Such a analysis leads to consider a set of algorithms to solve the problems which couples a CD formulation with convex constraints, by combining the previous CD and Uzawa algorithms. The following example concerns the buckling of rods submitted to unilateral contact conditions. A previous study about coupling sleeves using SMA structure with contact was based on a similar approach. We consider a rod submitted to a concentrated force at the end and an obstacle. A distance e separates the obstacle from the initial configuration of the rod. To account for the eventual contact of the rod with the obstacle, concentrated reactions are introduced at the nodes of the finite elements. The problem consists then in minimizing the new functional in (10) under the constraints (11) which are written as complementarity conditions

on the gap $h_i(\theta)$ at each node and the contact reaction R_i.

$$\inf_{\theta} \int_0^{\ell} \left(\frac{EI}{2}\, \theta'^2 \; - P \, \cos\theta \right) d\mathbf{x} \; + \sum_{i=1}^{n_p} \int_0^{x_i} R_i \, \sin\theta d\mathbf{x}, \tag{10}$$

$$h_i(\theta) \; = \; e \; - \int_0^{x_i} \sin\theta d\mathbf{x} \geq 0; \quad R_i \; \leq \; 0; \quad h_i(\theta) R_i \; = \; 0. \tag{11}$$

The CD splitting has to be completed as follows with the same conditions for $\beta_i, \; i = 1, .., n$ as for β,

$$\phi_1(\theta) = \frac{EI}{2}\, \theta'^2 \; - \beta \; P \; \frac{\theta^2}{2} \; - \sum_{i=1}^{n_p} R_i \; \beta_i \; \frac{\theta^2}{2} \chi_{[0,x_i]}, \tag{12}$$

$$\phi_2(\theta) = -P \left(\beta \; \frac{\theta^2}{2} \; - \; \cos\theta \right) \; - \sum_{i=1}^{n_p} R_i \left(\beta_i \; \frac{\theta^2}{2} - \sin\theta \right) \chi_{[0,x_i]}. \tag{13}$$

where x_i is the abscisse of the i^{th} node and $\chi_{[0,x_i]}$ is the characteristic function of the interval $[0, x_i]$. The coupled algorithm consists then of overlapped loops, the Uzawa loop containing the CD algorithm as summarized in Fig. 4.

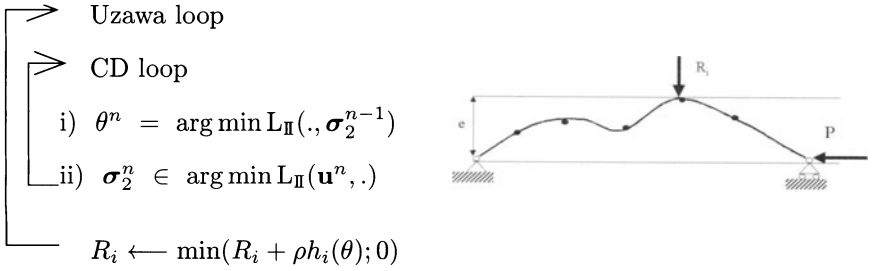

Fig. 4. Coupled Uzawa-CD algorithm and discrete nodal contact reactions R_i

The figure 5 shows three situations for different positions of the obstacle. For the lowest position no contact occurs and the low mode of buckling is obtained. For an higher position contact occurs only at the end of the rod and for the highest position the algorithm converges to the upper fundamental mode of buckling without contact because the solution with contact is unstable. The numerical experiments show that such an algorithm is unstable (because the process is unstable too). Moreover the contact treatment is not easy to handle in the context of the Elastica theory which is essentially dedicated to situations with concentrated forces. An other approach is performed today based on the modeling of inextensible rods by the bidimensional displacement field and a non convex constraint imposing the inextensibility condition.

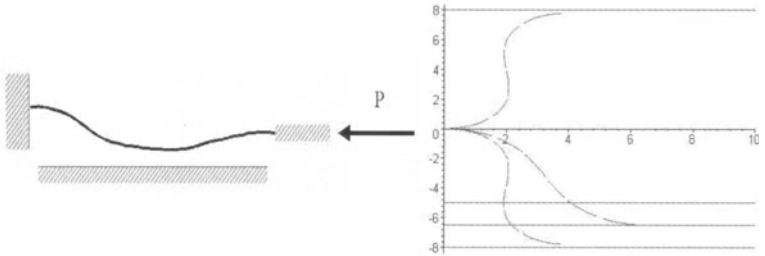

Fig. 5. A buckling test with contact and some deformed shapes (six elements)

Table 2. Comparison between a non convex problem and a convex problem with constraints. N is the trace operator on the contact boundary Γ_c, C^+ is the admissible displacements convex set and C^- is the polar cone set for the multiplier

Non smooth convex problem (contact),	Non convex problem (SMA),
$\inf_{\mathbf{v}} \ \Phi_1(\mathbf{v}) + I_{C^+}(\mathbf{N}\mathbf{v})$.	$\inf_{\mathbf{v}}(loc)\ \Phi_1(\mathbf{v}) - \Phi_2(\mathbf{D}\mathbf{v})$
Lagrangian,	Type II Lagrangian,
$\mathrm{L}(\mathbf{v},\boldsymbol{\gamma}) = \Phi_1(\mathbf{v}) - I_{C^-}(\boldsymbol{\gamma}) + < \mathbf{N}\mathbf{v}, \boldsymbol{\gamma} > .$	$\mathrm{L}_{\mathrm{II}}(\mathbf{v},\boldsymbol{\tau}_2) = \Phi_1(\mathbf{v}) + \Phi_2^*(\boldsymbol{\tau}_2) - < \mathbf{D}\mathbf{v}, \boldsymbol{\tau}_2 >$
Saddle point of L,	Critical point of L_{II},
$\inf_v \sup_{\boldsymbol{\gamma}} \mathrm{L}(\mathbf{v},\boldsymbol{\gamma})$	$\inf_v \inf_{\boldsymbol{\tau}_2} \mathrm{L}_{\mathrm{II}}(\mathbf{v},\boldsymbol{\tau}_2)$
Uzawa algorithm,	CD algorithm,
i) $\mathbf{u}^{k+1} = \arg\min \mathrm{L}(.,\boldsymbol{\lambda}^k)$,	i) $\mathbf{u}^{k+1} = \arg\min \mathrm{L}_{\mathrm{II}}(.,\boldsymbol{\sigma}_2^k)$,
ii) $\boldsymbol{\lambda}^{k+1} \in \arg\min -\mathrm{L}(\mathbf{u}^{k+1},.)$	ii) $\boldsymbol{\sigma}_2^{k+1} \in \arg\min \mathrm{L}_{\mathrm{II}}(\mathbf{u}^{k+1},.)$

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