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Extension of the Hoff solutions framework to cover compressible Navier-Stokes equations with possible anisotropic viscous tensor

D. Bresch*, C. Burtea †

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Abstract

In this paper, we construct global weak solutions à la Hoff (i.e. intermediate regularity) for the compressible Navier-Stokes system governing a barotropic fluid with a pressure law $p(\rho) = a\rho^\gamma$ where $a > 0$ and $\gamma \geq d/(4-d)$) and with an anisotropic fourth order symmetric viscous tensor with smooth coefficients under the assumption that the norms of the initial data $(\rho_0 - M, u_0) \in L^{2\gamma}\left(\mathbb{T}^d\right) \times (H^1(T^d))^d$ are sufficiently small, where M denotes the total mass of the fluid. We consider periodic boundary conditions for simplicity i.e. a periodic box $\Omega = T^d$ with $d = 2, 3$ with $|\Omega| = 1$. The main technical contribution of our paper is the extension of the Hoff solutions framework by relaxing the integrability needed for the initial density which is usually assumed to be $L^\infty(T^d)$. In this way, we are able to cover the case of viscous tensors that depend on the time and space variables. Moreover, when comparing to the results known for the global weak solutions à la Leray (i.e. obtained assuming only the basic energy bounds), we obtain a relaxed condition on the range of admissible adiabatic coefficients γ.

Keywords: Compressible fluids, Navier–Stokes Equations, Anisotropic Viscous Tensor, Hoff solutions, Intermediate regularity

MSC: 35Q35, 35B25, 76T20.

1 Introduction and main result

In this paper, we study the problem of existence of global solutions in the spirit of Hoff (intermediate regularity) for the compressible Navier-Stokes equations in a periodic domain $\mathbb{T}^d$, $d = 2, 3$ with a general strain tensor given by a fourth order symmetric tensor and a pressure law given by $p(\rho) = a\rho^\gamma$ with $a > 0$ given and $\gamma \geq d/(4-d)$. More precisely we consider the following system

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla(a\rho^\gamma) = \operatorname{div}(\tilde{\mathcal{E}}(\nabla u)). \end{cases} \tag{1.1}$$

The viscous tensor is a fourth order tensor

$$\tilde{\mathcal{E}} = \left(\tilde{\varepsilon}_{ijk\ell}\right)_{i,j,k,\ell \in \overline{1,d}}$$

where we use the notations

$$\tilde{\mathcal{E}}\left(\nabla u\right) = \tilde{\varepsilon}_{ijk\ell}\partial_\ell u^k, \quad \operatorname{div}\left(\tilde{\mathcal{E}}\,\nabla u\right) = \partial_j\left(\tilde{\varepsilon}_{ijk\ell}\partial_\ell u^k\right).$$

The system is completed with the initial data

$$\rho|_{t=0} = \rho_0 \geq 0, \qquad \rho u|_{t=0} = m_0. \tag{1.2}$$

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Most of the literature concerning compressible fluid mechanics deals with the classical isotropic tensor

$$\mathcal{I} = \left(\varepsilon^{iso}_{ijk\ell}\right)_{i,j,k,\ell\in\overline{1,d}} \tag{1.3}$$

which is given by

$$\varepsilon^{iso}_{ijk\ell} = \varepsilon^{iso}_{k\ell ij} = \begin{cases} \mu \text{ if } (i,j) = (k,\ell) \text{ and } i \neq j, \\ 2\mu + \lambda \text{ if } (i,j) = (k,\ell) \text{ and } i = j, \\ 0 \text{ otherwise,} \end{cases} \tag{1.4}$$

and $\mu, \lambda > 0$ are given constants. This implies in particular that one has

$$-\operatorname{div}(\mathcal{I}\nabla u) = -\mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u.$$

For the reader's convenience, let us recall some well known results concerning the existence of solutions for compressible Navier-Stokes equations for isotropic viscous tensor and then for anisotropic viscous tensors. We will discuss different notions of solutions (strong solutions, critical spaces, global weak solutions à la Leray, intermediate regularity à la Hoff) in order to understand why we have chosen the Hoff-solutions framework and why it is necessary to extend the method to the L^p framework on the density to obtain our result. In this paper, we do not discuss density dependent viscosity for compressible Navier-Stokes equations.

1) *A short review of know results for isotropic stress tensors.* The study of system $(1.1)-(1.2)$ with a given pressure law $s \mapsto p(s)$ in the case of isotropic stress tensors goes back to the work of J. Nash [Nas62] where the author shows the existence of local-in-time strong solutions in Hölder spaces. Then local strong existence for initial data in Sobolev spaces was investigated by Solonnikov [Sol80] in the $80's$ while the first global result is due to Matsumura and Nishida [MN$^+$80] where they prove the existence of global-in-time solutions in $3D$ if the initial data are sufficiently close to equilibrium in H^3. In the $2000s$, R. Danchin [Dan00] constructed global small solutions in the so-called critical spaces. Very recently, [MRRS19] F. Merle, P. Raphaël, I. Rodnianski and J.Szeftel prove that in $3D$, for small $\gamma \leq 1 + 2/\sqrt{3}$ there exists local smooth solutions which explode in finite time : the L^∞-norms of the density and the velocity blow-up. Thus, in some sense, the smallness condition, which express the fact that the initial configuration is sufficiently close to an constant equilibrium state, is necessary in order to insure global well-posedness. The question of global well-posedness in $2D$ remains open without assumption on the size of the data compared to equilibrium, the most advanced result in this direction being the one by R. Danchin and P.B. Mucha [DM19] who, although working with rough initial densities, require only that the divergence of the velocity field should be small.

Another category of results regarding the solvability of $(1.1)-(1.2)$ concerns the so-called weak-solutions à la Leray: solutions in the sense of distributions satisfying the energy inequality for which one can guarantee their global existence for arbitrary large initial data. Of course, few things are known regarding the uniqueness of these solutions. We mention the, by now classical results of P.L. Lions [Lio96], E. Feireisl et al. [FNP01]. Recently, the first author and P.–E. Jabin extended these two results in order to cover on one hand some anisotropic stress tensors [BJ18] and on the other hand more general pressure functions [BJ18, BJW21] that could not be treated with the Lions-Feireisl theory. We will return back and comment a bit more on these results in the context of anisotropy.

A third category of results concerns an intermediate regularity functional framework which was pioneered in the works of D. Hoff [Hof95a, Hof95b, Hof02, HS08] (that we will called solutions à la Hoff) and B. Desjardins [Des97]. By intermediate regularity we mean of course between the regularity needed to construct strong solution and weak solutions à la Leray (see [Lio96]). *These solutions are interesting since they allow to work with discontinuous densities while granting some extra regularity for the velocity field which turns out to be sufficiently regular in order to generate a log-Lipschitz flow.* These solutions were used by D. Hoff to study the dynamics of a surface of discontinuities initially present in the density, see [Hof02, HS08] and found applications in the context of multifluids, see the work of the first author and X. Huang [BH11]. Since the present work deals with these kind of solutions, we will take the time to give more details. In [Hof95a, Hof95b], for the case of isotropic stress tensors, D. Hoff introduced and studied the properties of two energy-type functionals

$$A_1(t) = \frac{\mu\sigma(t)}{2}\int \left|\partial_k u^i(t)\right|^2 + \frac{(\mu+\lambda)\,\sigma(t)}{2}\int \left|\operatorname{div} u(t)\right|^2 + \int_0^t \int \sigma\rho\,|\dot{u}|^2 \tag{1.5}$$

and

$$A_2(t) = \sigma^{1+d}(t) \int \frac{\rho(t)|\dot{u}(t)|^2}{2} + \mu \int_0^t \int \sigma^{1+d} \left|\partial_k \dot{u}^i(t)\right|^2 + (\mu+\lambda) \int_0^t \int \sigma^{1+d} |\mathrm{div}\, \dot{u}|^2 \tag{1.6}$$

where

$$\dot{u} = u_t + u \cdot \nabla u \text{ and } \sigma(t) = \min\{1, t\}.$$

These functionals naturally appear: The first one when multiplying the momentum equation with $\sigma \dot{u}$ and integrating while the other one appears when applying $\partial_t + \mathrm{div}(u\cdot)$ to the momentum equation and multiplying with $\sigma^{1+d}\dot{u}$. D. Hoff shows that A_1, A_2 can be controlled globally in time if the initial data have suitably small energy and ρ_0 *is close to a constant in* L^∞. The fact that these two functionals can be controlled translate some fine smoothing properties due to the diffusion: it turns out that u is Hölder continuous in time-space, far from $t = 0$ and that $\mathrm{curl}\, u$ and the effective flux

$$F = (2\mu + \lambda)\, \mathrm{div}\, u - p(\rho)$$

are H^1 in space for a.e. $t > 0$. In particular, this later properties render mathematically clear the fact that discontinuities in the density are advected by the flow but in such a way that the so called effective flux, i.e. F stays fairly smooth. This property enjoyed by the effective flux, known and exploited in the $1d$ case in [HS85, Hof87], also turns out to be crucial when showing the stability of sequences of weak-solutions. In order to give a meaning to A_1, A_2 very little extra information is need when comparing to the energy level $\rho_0 u_0^2 \in L^1$, $\rho_0 \in L^\gamma$ which essentially is that $\rho_0 \in L^\infty$ and $u_0 \in L^{2^n}$ (in the whole space case). If more information is available for the initial data, modified versions of the two functionals can be used: for instance if $u_0 \in H^1$, one can control

$$\tilde{A}_1(t) = \frac{\mu}{2} \int \left|\partial_k u^i(t)\right|^2 + \frac{\mu+\lambda}{2} \int |\mathrm{div}\, u(t)|^2 + \int_0^t \int \rho |\dot{u}|^2, \tag{1.7}$$

respectively

$$\tilde{A}_2(t) = \sigma(t) \int \frac{\rho(t)|\dot{u}(t)|^2}{2} + \mu \int_0^t \int \sigma \left|\partial_k \dot{u}^i(t)\right|^2 + (\mu+\lambda) \int_0^t \int \sigma |\mathrm{div}\, \dot{u}|^2, \tag{1.8}$$

which of course express the fact that due to the extra information the solution is better behaved close to the initial time layer $t = 0$. We also mention the related but independent work of B. Desjardins [Des97] where the author obtains local in time results showing that is possible to control a function which is essentially equivalent to $\tilde{A}_1$. *We mention that in all the above cited papers, the assumption that* $\rho_0 \in L^\infty$ *turns out to be crucial. The fact that one can propagate control of the* L^∞*-norm of the density heavily depends on the algebraic structure of the isotropic Navier-Stokes system throughout the so called-effective flux* $F = (2\mu + \lambda)\, \mathrm{div}\, u - p(\rho)$ *defined above.*

2) *The case of anisotropic stress tensors.* In this case, the mathematical results are in short supply. Let us mention that in the context of strong solutions [MN$^+$80], [Dan00] where the results are proved by maximum regularity results, at least if the stress tensor is "close enough to the isotropic tensor" then there should virtually be little change needed in order to accommodate these kind of solutions. However, as explained above, when dealing with classical solutions the density is a continuous function thus excluding many interesting situations in applications (for example, mixtures of fluids).

The first paper providing a result in this direction has been obtained by the first author and P.–E. Jabin in [BJ18] and concerns the existence of global weak solutions à la Leray with an anisotropic diffusion of the form:

$$-\mathrm{div}(A(t)\nabla u) - (\mu+\lambda)\nabla \mathrm{div} u \tag{1.9}$$

where

$$A(t) = \mu \mathrm{Id} + \delta A(t) \text{ with } \mu > 0.$$

The result proved in [BJ18] states that there exists an universal constant $c > 0$ such that if[1]

$$\|\delta A(t)\|_{L^\infty} \leq c\left(\frac{2\mu}{d} + \lambda\right)$$

and if

$$\gamma > \frac{d}{2}\left[\left(1 + \frac{1}{d}\right) + \sqrt{1 + \frac{1}{d^2}}\right] \tag{1.10}$$

then, there exists global weak solutions à la Leray for the Navier-Stokes system $(1.1) - (1.2)$ with the strain tensor given by (1.9). This result extended to the anisotropic case the global existence of weak solution à la Leray obtained for the isotropic case in [Lio96], [FNP01]. The result of the first author and P.-E. Jabin is based on new estimates for the transport equation. This result requires in a crucial manner some form of compactness in space for

$$(2\mu + \lambda)\operatorname{div} u - L(a\rho^\gamma)$$

where L is a non-local operator of order 0. *It is at this level that the authors use the fact that A depends only on time has been used by the authors. The extension of this result to space dependent strain tensors represents a serious difficulty that remains an open problem.* Moreover, the restriction for the adiabatic coefficient γ given by (1.10) excludes most of the physically realistic values : monoatomic gases 5/3, ideal diatomic gases 7/5, viscous shallow–water $\gamma = 2$.

Let us also mention our results concerning global weak solutions à la Leray for the quasi-stationary compressible Stokes in [BB20] where an anisotropic diffusion $-\operatorname{div}(AD(u))$ is considered with no smallness assumption on the anisotropic amplitude needed and for the stationary compressible Navier-Stokes equations in [BB21] with a viscous diffusion operator given by $-\mathcal{A}u$ (under some constraints) where $\mathcal{A}$ is composed by a classical constant viscous part plus an anisotropic contribution and a possible nonlocal contribution.

3) *Motivation to extend Hoff solution framework and description of our main result.* When dealing with weak solutions for non-linear PDE systems, one of the most delicate aspects is the stability analysis: given a sequence of weak solutions for some well-chosen approximated systems, show that this sequence converges to a solution for the initial system. The key ingredient in [BB20] and [BB21] is an identity that we found when comparing on the one hand, the limiting energy equation and on the other hand, the equation of the energy associated to the limit system. In order to justify such an identity, a crucial assumption seems to be the fact that the pressure is L^2, an apriori estimate which is ensured by basic a-priori estimates in the case of the Stokes system or for the stationary Navier-Stokes system. However, in the case of system $(1.1) - (1.2)$ in the isotropic case, the best estimate for the density is due to P. Lions who showed for global weak solutions à la Leray that $\rho \in L^{5\gamma/3-1}_{t,x}$. This makes it impossible to write the energy equation because, loosely speaking, the velocity cannot be used as a test function in a weak-formulation of (1.1). Thus, it seems hopeless to justify the limiting passage as in [BB20] and [BB21] in the most general setting of weak-solutions à la Leray. Obviously, one may ask if we can work in an intermediate regularity setting. However, one learns fast that we are faced with a serious problem when trying to propagate the L^∞-information for the density. In the isotropic case it is based on the fact that the effective viscous flux

$$(2\mu + \lambda)\operatorname{div} u - \left(a\rho^\gamma - \int_{\mathbb{T}^d} a\rho^\gamma\right) = \Delta^{-1}\operatorname{div}(\rho\dot{u}) \tag{1.11}$$

which, granted we have the control of the second Hoff functional, namely (1.6), can be shown to belong to L^∞, at least far from the initial time layer $t = 0$ in the most general case. Of course, the situation is not the same in the anisotropic case, where

$$(2\mu + \lambda)\operatorname{div} u - \left(a\rho^\gamma - \int_{\mathbb{T}^d} a\rho^\gamma\right) = \Delta^{-1}\operatorname{div}(\rho\dot{u}) + \Delta^{-1}\operatorname{div}\operatorname{div}\left(\left(\tilde{\mathcal{E}} - \mathcal{I}\right)(\nabla u)\right) \tag{1.12}$$

and the term $\Delta^{-1}\operatorname{div}\operatorname{div}\left(\left(\tilde{\mathcal{E}} - \mathcal{I}\right)(\nabla u)\right)$, being of the same order as ∇u we cannot expect it to be L^∞. Because of the lack of algebraic structure we are led to abandon any hope of propagating an L^∞ bound for

[1] d stands for the space dimmension.

the density. *A natural question then appears: is it possible to bound the Hoff functionals without working in an L^∞ framework for the density? The main contribution of this paper is to show that this is indeed the case. Of course, this fact makes it possible to construct global weak solutions close to equilibrium for the Navier–Stokes system in the anisotropic case in an intermediate regularity setting.* This program requires establishing L^p-estimates for the density that are compatible with the Hoff functionals. In order to avoid further technical difficulties, we will assume the best information possible for the velocity, namely $u_0 \in \left(H^1\right)^d$ such that we will rather work with the anisotropic equivalent of the functionals defined in (1.7) – (1.8).

Our result should be seen to be complementary to the work of the first author and P.–E. Jabin [BJ18]. Our extension of the Hoff solutions framework allow us to:

- treat viscous strain tensors depending on time and also on the space variable
- consider a range of adiabatic coefficient namely

$$\gamma \geq \frac{d}{4-d} \text{ for } d \in \{2,3\}.$$

  In particular, in 2D we are able to treat all coefficients that are of practical interest $\gamma \geq 1$.

- this method could be adapted to bounded domains with Dirichlet boundary conditions for which the existence result of global weak solutions (à la Leray or intermediate regularity) with anisotropic tensors remains open.

Note that the range for the coefficient γ is larger than the one in [BJ18] namely (1.10) and we cover strain tensors which may depend on the space variable. Of course, the price to pay is that the initial conditions are supposed to be close to equilibrium and that we require the initial velocity field to be in $(H^1(\mathbb{T}^d))^d$.

Assumptions and notations. We will rather write $\tilde{\mathcal{E}} = \mathcal{I} + \mathcal{E}$ with $\mathcal{I}$, the usual isotropic tensor (1.3)-(1.4) and where $\mathcal{E}$ measures in some sense the anisotropic perturbation. With this new notations the system (1.1) becomes[2]

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma = \mu \Delta u + (\mu+\lambda)\nabla \operatorname{div} u + \operatorname{div}(\mathcal{E}(\nabla u)). \end{cases} \tag{1.13}$$

We suppose that $\mu, \lambda \in \mathbb{R}$ such that

$$\mu > 0 \text{ and } \mu + \lambda \geq 0. \tag{1.14}$$

We will assume that $\mathcal{E} = (\varepsilon_{ijk\ell})_{i,j,k,\ell \in \overline{1,d}}$ verifies the following properties:

- For all $i,j,k,\ell \in \overline{1,d}$ we assume the following symmetry property:

$$\varepsilon_{ijk\ell} = \varepsilon_{k\ell ij}. \tag{H1}$$

  The later property ensures that[3]

$$\varepsilon_{ijk\ell} a_{ij} b_{k\ell} = \varepsilon_{ijk\ell} b_{ij} a_{k\ell}.$$

- Strict coercivity of the diffusive part:

$$\overline{\varepsilon} \left|a_{ij}\right|^2 \geq \varepsilon_{ijk\ell} a_{ij} a_{k\ell} \geq -\underline{\varepsilon} \left|a_{ij}\right|^2 \tag{H2}$$

  where $\underline{\varepsilon}, \overline{\varepsilon} > 0$ such that

$$0 < \mu - \underline{\varepsilon}.$$

- Regularity: for all $i,j,k,\ell \in \overline{1,d}$, $\varepsilon_{ijk\ell} \in W^{1,\infty}\left((0,\infty)\times \mathbb{T}^d\right)$ with

$$\|\partial_t \varepsilon_{ijk\ell}\|_{L^\infty((0,+\infty)\times\mathbb{T}^d)} + \|\nabla \varepsilon_{ijk\ell}\|_{L^\infty((0,+\infty)\times\mathbb{T}^d)} < \infty. \tag{H3}$$

[2] for the sake of simplicity of notations, we set the constant $a = 1$ in the pressure term.
[3] All along this paper, we will use the Einstein summation over repeated indices convention.

- We will suppose that

$$\|\mathcal{E}\|_{L^\infty((0,+\infty)\times\mathbb{T}^d)} = \sup_{i,j,k,l\in\overline{1,d}} \|\varepsilon_{ijk\ell}\|_{L^\infty((0,+\infty)\times\mathbb{T}^d)} \leq \eta \min\{\mu, 2\mu+\lambda\} \tag{H4}$$

  for a small constant η.

Main Result. Let us define the following:

$$E(\rho/M, u) = \int_{\mathbb{T}^d} \left(H_1(\rho/M) + \frac{1}{2}\rho|u|^2\right)$$

with $0 < M < +\infty$ and where

$$H_1(\rho/M) = H_1(\rho) - H_1(M) + H_1'(M)(\rho - M) \tag{1.15}$$

with

$$H_1(\rho) = \rho \int_0^\rho P(s)/s^2\, ds = \frac{\rho^\gamma}{\gamma - 1}.$$

Also, we introduce

$$H_\ell(\rho/M) = \rho \int_M^\rho \frac{|P(s) - P(M)|^{\ell-1}(P(s) - P(M))}{s^2}\, ds \text{ with } \ell \in \{2,3\}. \tag{1.16}$$

We are now in the position of stating our main result:

Theorem 1 *Let $\tilde{\mathcal{E}} = \mathcal{I} + \mathcal{E}$ with $\mathcal{I}$ the usual isotropic tensor* (1.3)–(1.4) *and $\mathcal{E} = (\varepsilon_{ijk\ell})_{i,j,k,\ell\in\overline{1,d}}$ a fourth order tensor verifying the hypothesis* (H1)–(H4)*. Consider $\mu, \lambda \in \mathbb{R}$ such that $\mu > 0, \mu + \lambda > 0$. Then, there exists a constant c_0 such that the following holds true: for any $(\rho_0, u_0) \in L^{2\gamma}(\mathbb{T}^d) \times \left(H^1(\mathbb{T}^d)\right)^d$ with*

$$\int_{\mathbb{T}^d} \rho_0 = M, \quad \int_{\mathbb{T}^d} \rho_0 u_0 = \overrightarrow{\mathrm{P}}$$

such that if

$$E(\rho_0/M, u_0) + \int_{\mathbb{T}^d} H_2(\rho_0/M) + \|u_0\|_{(H^1(\mathbb{T}^d))^d} \leq c_0,$$

then, there exist a constant (ρ, u) a global weak solution to (1.1)–(1.2) *with*

$$(\rho - M, \rho u - \overrightarrow{\mathrm{P}}) \in \mathcal{C}([0,+\infty), H^{-1}(\mathbb{T}^d)) \times \mathcal{C}([0,+\infty); (H^{-1}(\mathbb{T}^d))^d)$$

and such that:

$$E(\rho/M, u) + (\mu - \underline{\varepsilon}) \int_{\mathbb{T}^d} \left|\partial_k u^i\right|^2 + (\mu+\lambda) \int_{\mathbb{T}^d} |\mathrm{div}\, u|^2 \leq E(\rho_0/M, u_0).$$

$$\frac{1}{2}\left\{\mu \int_{\mathbb{T}^d} \left|\partial_k u^i\right|^2(t) + (\mu+\lambda) \int_{\mathbb{T}^d} |\mathrm{div}\, u|^2(t)\right\} + \int_0^t \int_{\mathbb{T}^d} \rho\, |\dot{u}|^2 \leq C c_0,$$

$$\sigma(t) \int_{\mathbb{T}^d} \frac{\rho(t)\, |\dot{u}(t)|^2}{2} + \mu \int_{\mathbb{T}^d} \sigma \left|\partial_k \dot{u}^i\right|^2 + (\mu+\lambda) \int_{\mathbb{T}^d} \sigma\, |\mathrm{div}\, \dot{u}|^2 \leq C c_0,$$

$$(2\mu+\lambda)(1+\alpha) \int_{\mathbb{T}^d} H_2(\rho(t)/M) + \sigma(t) \int_{\mathbb{T}^d} H_3(\rho(t)/M)$$
$$+ \int_0^t \int_{\mathbb{T}^d} |P(\rho) - P(M)|^3 + \frac{1}{2\mu+\lambda} \int_0^t \int_{\mathbb{T}^d} \sigma\, |P(\rho) - P(M)|^4 \leq C c_0$$

where $\sigma(t) = \min\{1, t\}$ while $C = C(\mu, \lambda, \gamma, M, E_0, c_0)$ is a constant that depends on $\mu, \lambda, \gamma, M, E_0$.

Remark 1.1 *It is important to remark that it seems a difficult open problem to propagate the L^∞-norm for the density as it has been done by D. Hoff for the isotropic compressible Navier-Stokes equations with a barotropic pressure law.*

Remark 1.2 *In order to treat the stability part of the proof, the fact that the uniform bounds announced in Theorem 1 are crucial. The fact that the pressure is $L^3_{t,x}$ allows us to justify the passage from*

$$\partial_t \rho + \operatorname{div}(\rho u) = 0,$$

to

$$\partial_t P(\rho) + \operatorname{div}(P(\rho) u) + (\rho P'(\rho) - P(\rho)) \operatorname{div} u = 0.$$

Another crucial aspect is that when considering a sequence of solutions of systems that approximate the Navier-Stokes system, controlling the second Hoff functional allows to obtain information for the time derivative of the velocities. As a consequence of the Aubin-Lions lemma, we obtain that the sequence of velocities converges strongly in $L^2_{t,x}$, at least far from $t = 0$ which is crucial in order to implement the idea from [BB20].

Main steps and organization of the paper. We detail below the main steps of the proof of Theorem 1. Inspired from the approximate system proposed by the first author and P.–E. Jabin in [BJ18], we will consider a regularized version of the Navier-Stokes system (1.1):

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla a \rho^\gamma = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + \omega_\delta * \operatorname{div}(\mathcal{E}(\nabla \omega_\delta * u)), \end{cases} \tag{1.17}$$

where

$$\omega_\delta(x) = \frac{1}{\delta^d} \omega(\frac{x}{\delta}), \tag{1.18}$$

with ω a smooth, nonnegative, radial function compactly supported in the unit ball centered at the origin and with integral equal to 1. Since system (1.17) can be seen as a regular perturbation of the Navier-Stokes system for a compressible barotropic fluid, classical results [Sol80, Des97, Dan10] can be invoked in order to ensure the existence of a *local* classical solution.

Remark 1.3 ***(Important remark on the anisotropy).*** *To simplify the writing of the paper, we will assume in the proof that*

$$\int_0^T \int_{\mathbb{T}^d} \mathcal{E} \nabla w : \nabla w \geq 0 \qquad \text{for all } T \in (0, +\infty] \text{ and } w \in L^2((0,T); H^1(\mathbb{T}^d)).$$

This assumption is needed in order to treat the stability of weak-solutions of system (1.17) part of the proof. In order to avoid this assumption and treat the general case, it is sufficient to consider an approximate system with diffusion given by

$$(\mu - \underline{\varepsilon}) \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + \underline{\varepsilon} \Delta \omega_\delta * u + \omega_\delta * \operatorname{div}(\mathcal{E}(\nabla \omega_\delta * u)).$$

change the coefficients λ and μ in the isotropic part to allow to satisfy these assumptions.

We show that these solutions have the property that the two Hoff functionals associated are bounded independently of δ. This is one of the main contributions of this paper.

Following exactly the same steps as in R. Danchin and P.B. Mucha [DM19], shows that the local solutions of (1.17) can be prolonged to global ones. The fact that the Hoff functionals are independent of δ is of course crucial in order to show that we can extract a subsequence converging to a weak-solution à la Hoff of $(1.1) - (1.2)$. Here, we are faced with, let us say the classical difficulty in compressible fluid mechanics which is to be able to identify the pressure in the limit. More precisely,

$$\lim (\rho_n)^\gamma = (\lim \rho_n)^\gamma. \tag{1.19}$$

Of course, when dealing with weak solutions, the density is just a Lebesgue function and no gain of regularity is to be expected. Since weak limits, in general, do not commute with nonlinear functions, showing (1.19) has to take into consideration some algebraic properties of solutions of the NS system. Let us recall that classical techniques due to P.L. Lions [Lio96] and E. Feireisl [Fei01] do not apply in this context, see the discussions from the introductions of [BJ18],[BB20],[BB21] for more details. Moreover, the work by the first author and P.E. Jabin requires a relative large γ and, maybe more importantly, as it was explained above, it is not straightforward to extend it to heterogenous in space anisotropic tensors (the fact that $\mathcal{E}$ can depend also on the space variable). *Here, it is crucial to extend our idea from [BB20]* that we successfully implemented in order to construct global weak solutions à la Leray for the Stokes-Brinkman system in [BB20] and for the stationary NS system in [BB21]. In these two papers, we did not need however to impose any restriction on the size of the initial data or the forcing terms. This is essentially due to the fact that, in the previous cases, the pressure turns out to be an L^2 function (if γ is large for the stationary NS system).

The rest of the paper unfolds as follows. In the second section, we prove the main result: First we recall basic mass conservation and energy estimate, secondly we extend the Hoff estimates in a L^p framework, third we construct a sequence of approximate solutions and then finally we show the stability property. In an appendix, we present a tool box with Fourier multipliers properties, Sobolev inequality and Gronwall-Bihari inequality and finally we give the detailed computations for the Hoff functionals that we strongly use.

2 Proof of the main result

2.1 Basic mass conservation and energy estimate

The conservation of mass and momentum. The simplest *a priori* estimate we have is given by the conservation of mass:

$$\int_{\mathbb{T}^d} \rho(t) = \int_{\mathbb{T}^d} \rho_0 \overset{not.}{=} M > 0, \tag{2.1}$$

$$\int_{\mathbb{T}^d} \rho(t)\, u(t) = \int_{\mathbb{T}^d} m_0 \overset{not.}{=} \overrightarrow{\mathrm{P}} \in \mathbb{R}^d. \tag{2.2}$$

The energy estimate. From the continuity equation, we can also deduce the following equation

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\, u) + (\rho b'(\rho) - b(\rho)) \operatorname{div} u = 0, \tag{2.3}$$

a priori for all b continuous. Taking $b(\rho) = \rho^\alpha$ in (2.3) yields

$$\partial_t \rho^\alpha + \operatorname{div}(\rho^\alpha u) + (\alpha - 1)\, \rho^\alpha \operatorname{div} u = 0.$$

Also, we can write that

$$\begin{aligned} u \cdot \nabla P(\rho) &= \operatorname{div}(u(P(\rho) - P(M)) - (P(\rho) - P(M)) \operatorname{div} u \\ &= \operatorname{div}(u(P(\rho) - P(M)) + \frac{d}{dt} H_1(\rho/M) + \operatorname{div}(H_1(\rho/M)\, u), \end{aligned}$$

where $H_1(\rho/M)$ has been defined in (1.15). The function $H_1(\rho/M)$ is more appropriate in order to study densities that are close to some constant state. Thus we get the following energy estimate

$$\begin{aligned} &\int_{\mathbb{T}^d} \left(H_1(\rho/M) + \rho u^2\right) + \mu \int_0^t \int_{\mathbb{T}^d} |\nabla u|^2 + (\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} |\operatorname{div} u|^2 + \int_0^t \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell u_\delta^k \partial_j u_\delta^i \\ &\leq \int_{\mathbb{T}^d} \left(H_1(\rho_0/M) + \rho_0 u_0^2\right) := E_0. \end{aligned}$$

Note that we assume E_0 to be small in the Theorem 1.

2.2 Extension of the Hoff's estimates in a L^p framework

This part is the key of the paper: Assuming the initial velocity $u_0 \in H^1(\mathbb{T}^d)$ and $\rho_0 \in L^{2\gamma}(\mathbb{T}^d)$, instead of $\rho_0 \in L^\infty(\mathbb{T}^d)$ as in [Hof95a], we allow more general densities that in [Hof95a]. This L^p, $p < \infty$ framework for the density is important when considering anisotropic viscous tensors for which it is not so straightforward to propagate L^∞-information. Consider

$$A_1(t) = \frac{1}{2}\mu \int_{\mathbb{T}^d} \left|\partial_k u^i(t)\right|^2 + (\mu+\lambda)\int_{\mathbb{T}^d} |\operatorname{div} u(t)|^2 + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u^k_\delta(t)\,\partial_j u^i_\delta(t) + \int_0^t \int_{\mathbb{T}^d} \rho\,|\dot{u}|^2, \tag{2.4}$$

and

$$A_2(t) = \sigma(t)\int_{\mathbb{T}^d} \frac{\rho(t)\,|\dot{u}(t)|^2}{2} + \mu\int_0^t\int_{\mathbb{T}^d} \sigma\left|\partial_k \dot{u}^i(t)\right|^2 + (\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\,|\operatorname{div}\dot{u}|^2 + \int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell\dot{u}^k_\delta\partial_j\dot{u}^i_\delta. \tag{2.5}$$

Multiplying the momentum equation with

$$\dot{u} = u_t + u\cdot\nabla u$$

we obtain (see the detailed computations in the appendix) that

$$\begin{aligned}
A_1(t) =& \frac{1}{2}\mu\int_{\mathbb{T}^d}\left|\partial_k u^i_0\right|^2 + (\mu+\lambda)\int_{\mathbb{T}^d}|\operatorname{div} u_0|^2 + \int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_{0,\delta}\partial_j u^i_{0,\delta}(t)\\
&- \mu\int_0^t\int_{\mathbb{T}^d}\partial_k u^i\partial_k u^\ell\partial_\ell u^i + \frac{\mu}{2}\int_0^t\int_{\mathbb{T}^d}\left|\partial_k u^i\right|^2\operatorname{div} u\\
&- (\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\operatorname{div} u\,\partial_i u^\ell\partial_\ell u^i + \frac{\mu+\lambda}{2}\int_0^t\int_{\mathbb{T}^d}(\operatorname{div} u)^3\\
&+ \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\left\{\partial_t\varepsilon_{ijkl} + \partial_q(\varepsilon_{ijkl}u^q)\right\}\partial_j u^i_\delta\partial_k u^\ell_\delta - \int_0^t\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_\delta\omega_\delta * (\partial_j u^q\partial_q u^i)\\
&- \int_0^t\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_\delta\left[u^q,\omega_\delta\right]\partial^2_{qj}u^i\\
&+ \int_0^t\int_{\mathbb{T}^d}\rho P'(\rho)\,\partial_\ell u^k\partial_k u^\ell + \int_0^t\int_{\mathbb{T}^d}\rho\dot{u}f.
\end{aligned} \tag{2.6}$$

Applying the operator $\partial_t + \operatorname{div}(u\cdot)$ to the momentum equation we obtain (see the detailed computations in the appendix) that:

$$\begin{aligned}
A_2(t) =& \int_0^1\int_{\mathbb{T}^d}\sigma\frac{\rho|\dot{u}|^2}{2} + \mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_q u^i\partial_k\dot{u}^i + \mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_k u^i\partial_q\dot{u}^i - \mu\int_0^t\int_{\mathbb{T}^d}\sigma\operatorname{div} u\partial_k u^i\partial_k\dot{u}^i\\
&+ (\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_\ell u^q\partial_q u^\ell\operatorname{div}\dot{u} + (\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_i u^q\partial_q\dot{u}^i\operatorname{div} u - (\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\,|\operatorname{div} u|^2\operatorname{div}\dot{u}\\
&- \int_0^t\int_{\mathbb{T}^d}\sigma\left(\partial_t\varepsilon_{ijkl} + \partial_q\left(u^q\varepsilon_{ijkl}\right)\right)\partial_\ell u^k_\delta\partial_j\dot{u}^i_\delta - \int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}(\omega_\delta * (\partial_\ell u^q\partial_q u^k))\partial_j\dot{u}^i_\delta\\
&- \int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta\omega_\delta * (\partial_j u^q\partial_q\dot{u}^i)\\
&+ \int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}([u^q,\omega_\delta *]\,\partial^2_{\ell q}u^k)\partial_j\dot{u}^i_\delta + \int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta([u^q,\omega_\delta *]\,\partial^2_{jq}\dot{u}^i)\\
&- \int_0^t\int_{\mathbb{T}^d}\sigma\left\{P(\rho)\,\partial_j u^k\partial_k\dot{u}^j + (\rho P'(\rho) - P(\rho))\operatorname{div} u\operatorname{div}\dot{u}\right\}.
\end{aligned} \tag{2.7}$$

Let us introduce the effective flux

$$F = (2\mu+\lambda)\operatorname{div} u - (P(\rho) - P(M)).$$

The details leading to these formulae are by now classic for the isotropic case and they were used by D. Hoff in the series of works with isotropic viscosities [Hof95a, Hof95b, Hof02, HS08]. The only added value is that these estimates are adapted for the anisotropic approximate system (1.17). As mentioned before, for the reader's convenience we gather and detail them in the Appendix. One of the key difficulties is to recover informations for the gradient of the velocity. A quick analysis of A_1 and A_2 reveals that we need to control

$$\int_0^t \|\nabla u\|^3_{L^3(\mathbb{T}^d)} \text{ and } \int_0^t \sigma \|\nabla u\|^4_{L^4(\mathbb{T}^d)}.$$

in order to close the estimates. The classical Calderón-Zygmund theory ensures that for $p \in \{3,4\}$ one has

$$\|\nabla u\|^p_{L^p(\mathbb{T}^d)} \leq C \left(\|\mathrm{curl}\, u\|^p_{L^p(\mathbb{T}^d)} + \|\mathrm{div}\, u\|^p_{L^p(\mathbb{T}^d)} \right),$$

for some numerical constant C. We deduce that

$$\begin{aligned}
\|\nabla u\|^p_{L^p(\mathbb{T}^d)} &\leq C \left(\frac{1}{\mu^p} \|\mu\, \mathrm{curl}\, u\|^p_{L^p(\mathbb{T}^d)} + \frac{1}{(2\mu+\lambda)^p} \|(2\mu+\lambda)\, \mathrm{div}\, u\|^p_{L^p(\mathbb{T}^d)} \right) \\
&\leq C \left(\frac{1}{\mu^p} \|\mu\, \mathrm{curl}\, u\|^p_{L^p(\mathbb{T}^d)} + \frac{1}{(2\mu+\lambda)^p} \|F\|^p_{L^p(\mathbb{T}^d)} + \frac{1}{(2\mu+\lambda)^p} \|P(\rho) - P(M)\|^p_{L^p(\mathbb{T}^d)} \right),
\end{aligned}$$

from which we infer that

$$\begin{aligned}
&\int_0^t \|\nabla u\|^3_{L^3(\mathbb{T}^d)} + \int_0^t \sigma \|\nabla u\|^4_{L^4(\mathbb{T}^d)} \\
&\leq C \left(\frac{1}{\mu^3} \int_0^t \|\mu\, \mathrm{curl}\, u\|^3_{L^3(\mathbb{T}^d)} + \frac{1}{(2\mu+\lambda)^3} \int_0^t \|F\|^3_{L^3(\mathbb{T}^d)} + \frac{1}{(2\mu+\lambda)^3} \int_0^t \|P(\rho) - P(M)\|^3_{L^3(\mathbb{T}^d)} \right) \\
&+ C \left(\frac{1}{\mu^4} \int_0^t \sigma \|\mu\, \mathrm{curl}\, u\|^4_{L^4(\mathbb{T}^d)} + \frac{1}{(2\mu+\lambda)^4} \int_0^t \sigma \|F\|^4_{L^4(\mathbb{T}^d)} + \frac{1}{(2\mu+\lambda)^4} \int_0^t \sigma \|P(\rho) - P(M)\|^4_{L^4(\mathbb{T}^d)} \right).
\end{aligned} \tag{2.8}$$

Thus, in order to close the estimate we have, to recover a control for the density.

Remark 2.1 *This is where our approach starts to diverge from Hoff's approach. In the latter, there is an extra algebraic structure which allows to recover an L^∞-bound for the density. In the anisotropic case, we have to work with weaker norms, essentially because of the failure of homogeneous Fourier multipliers of order 0 to map L^∞ to L^∞. The idea is to try only to propagate what seem to be necessary to show that the two functionals A_1 and A_2 are bounded:*

$$\int_0^t \|P(\rho) - P(M)\|^3_{L^3(\mathbb{T}^d)}, \quad \int_0^t \sigma \|P(\rho) - P(M)\|^4_{L^4(\mathbb{T}^d)}.$$

2.2.1 Bounds for the density

In the following lines we want to obtain estimates for the density. We recall that for all functions b with sufficient regularity and control we have that

$$\partial_t b(\rho) + \mathrm{div}\,(b(\rho)u) + (\rho b'(\rho) - b(\rho))\, \mathrm{div}\, u = 0. \tag{2.9}$$

Thus, we can reformulate the above equation as

$$\begin{aligned}
&\partial_t b(\rho) + \mathrm{div}\,(b(\rho)\, u) + (\rho b'(\rho) - b(\rho)) \frac{(P(\rho) - P(M))}{2\mu+\lambda} \\
&= -\frac{1}{2\mu+\lambda} (\rho b'(\rho) - b(\rho)) \left((2\mu+\lambda)\, \mathrm{div}\, u - (P(\rho) - P(M)) \right).
\end{aligned}$$

Recall the definitions of $H_2(\cdot/M)$ and $H_3(\cdot/M)$ given in (1.16).
A L^3 control for the pressure. Let us take $b = H_2(\cdot/M)$ in (2.9) with

$$\rho H_2'(\rho/M) - H_2(\rho/M) = |P(\rho) - P(M)|(P(\rho) - P(M)),$$

in (2.9) in order to obtain

$$\begin{aligned}
&\int_{\mathbb{T}^d} H_2(\rho(t)/M) + \frac{1}{2(2\mu+\lambda)} \int_0^t \int_{\mathbb{T}^d} |P(\rho) - P(M)|^3 \\
&\leq \int_{\mathbb{T}^d} H_2(\rho_0/M) + \frac{C}{2\mu+\lambda} \int_0^t \int_{\mathbb{T}^d} |(2\mu+\lambda)\operatorname{div} u - (P(\rho) - P(M))|^3,
\end{aligned}$$

where C is a numerical constant independent of the parameters of the problem.

A L^4 control of the pressure. Finally, take $b = H_3(\cdot/M)$ with

$$\rho H_3'(\rho/M) - H_3(\rho/M) = (P(\rho) - P(M))^3$$

in order to obtain that

$$\begin{aligned}
&\sigma(t) \int_{\mathbb{T}^d} H_3(\rho(t)/M) + \frac{1}{2(2\mu+\lambda)} \int_0^t \int_{\mathbb{T}^d} \sigma |P(\rho) - P(M)|^4 \\
&\leq \int_0^1 \int_{\mathbb{T}^d} H_3(\rho(t)/M) + \frac{C}{2\mu+\lambda} \int_0^t \int_{\mathbb{T}^d} \sigma |(2\mu+\lambda)\operatorname{div} u - (P(\rho) - P(M))|^4
\end{aligned}$$

Using (3.7) from Lemma 3.4 from the appendix, we infer that:

$$\begin{aligned}
\int_0^1 \int_{\mathbb{T}^d} H_3(\rho(t)/M) &\leq \alpha \int_0^1 \int_{\mathbb{T}^d} |P(\rho) - P(M)|^3 + \kappa \int_0^1 \int_{\mathbb{T}^d} H_1(\rho/M) \\
&\leq \alpha \int_0^1 \int_{\mathbb{T}^d} |P(\rho) - P(M)|^3 + \kappa E_0,
\end{aligned} \tag{2.10}$$

for some α and κ depending on M, γ. Let us combine (3.7) with (3.8) and use (2.10), we deduce that

$$\begin{aligned}
B(t) &:= (1 + 2\alpha(2\mu+\lambda)) \int_{\mathbb{T}^d} H_2(\rho(t)/M) + \sigma(t) \int_{\mathbb{T}^d} H_3(\rho(t)/M) \\
&+ \frac{1}{2(2\mu+\lambda)} \int_0^t \int_{\mathbb{T}^d} |P(\rho) - P(M)|^3 + \frac{1}{2(2\mu+\lambda)} \int_0^t \int_{\mathbb{T}^d} \sigma |P(\rho) - P(M)|^4 \\
&\leq (1 + 2\alpha(2\mu+\lambda)) \int_{\mathbb{T}^d} H_2(\rho_0/M) + \kappa E_0 \\
&+ \frac{C(1 + 2\alpha(2\mu+\lambda))}{(2\mu+\lambda)} \int_0^t \|F\|^3_{L^3(\mathbb{T}^d)} + \frac{C}{(2\mu+\lambda)} \int_0^t \sigma \|F\|^4_{L^4(\mathbb{T}^d)}.
\end{aligned} \tag{2.11}$$

with C a numerical constant independent of the .

2.2.2 Bounds for the Hoff functionals

Using the estimates (2.8) and (2.11) we obtain that:

$$\begin{aligned}
\int_0^t \|\nabla u\|^3_{L^3(\mathbb{T}^d)} + \int_0^t \sigma \|\nabla u\|^4_{L^4(\mathbb{T}^d)} &\leq C\left(E_0 + \int_{\mathbb{T}^d} H_2(\rho_0/M)\right) \\
&+ \frac{C}{\mu^3} \int_0^t \|\mu \operatorname{curl} u\|^3_{L^3(\mathbb{T}^d)} + \frac{C}{(\mu+\lambda)^3} \int_0^t \|F\|^3_{L^3(\mathbb{T}^d)}
\end{aligned}$$

$$+\frac{C}{\mu^4}\int_0^t \sigma \left\|\mu \operatorname{curl} u\right\|_{L^4(\mathbb{T}^d)}^4+\frac{C}{(\mu+\lambda)^4}\int_0^t \sigma \left\|F\right\|_{L^4(\mathbb{T}^d)}^4 \tag{2.12}$$

where, from now on C represents a generic constant depending on

$$C=C\left(\mu,\lambda,\gamma,M\right)$$

the exact value of which can change from a line to another. Recall that

$$\begin{cases} \mu \operatorname{curl} u=\Delta^{-1}\operatorname{curl}(\rho\dot{u})+\Delta^{-1}\operatorname{curl}\left(\operatorname{div}(\omega_\delta*(\mathcal{E}\nabla(\omega_\delta*u)))\right), \\ F=\Delta^{-1}\operatorname{div}\left(\rho\dot{u}\right)+\Delta^{-1}\operatorname{div}\left(\operatorname{div}(\omega_\delta*(\mathcal{E}\nabla(\omega_\delta*u)))\right) \end{cases}$$

and therefore, we have that

$$\begin{aligned}
&\frac{C}{\mu^4}\int_0^t \sigma \left\|\mu \operatorname{curl} u\right\|_{L^4(\mathbb{T}^d)}^4+\frac{C}{(\mu+\lambda)^4}\int_0^t \sigma \left\|F\right\|_{L^4(\mathbb{T}^d)}^4\\
&\leq \frac{C}{\mu^4}\int_0^t \sigma \left\|\operatorname{curl} u\right\|_{W^{1,4d/(d+4)}(\mathbb{T}^d)}^4+\frac{C}{(\mu+\lambda)^4}\int_0^t \sigma \left\|F\right\|_{W^{1,4d/(d+4)}(\mathbb{T}^d)}^4\\
&\leq \frac{C}{\mu^4}\int_0^t \sigma \left\|\sqrt{\rho}\right\|_{L^{4d/(4-d)}(\mathbb{T}^d)}^4\left\|\sqrt{\rho}\dot{u}\right\|_{L^2(\mathbb{T}^d)}^4+C\left\|\mathcal{E}-\mathcal{I}\right\|_{L^\infty((0,t)\times\mathbb{T}^d)}^4\max\left\{\frac{1}{(\mu+\lambda)^4},\frac{1}{\mu^4}\right\}\int_0^t \sigma \left\|\nabla u\right\|_{L^4(\mathbb{T}^d)}^4\\
&\leq \frac{C}{\mu^4}\sup_t\left\|\sqrt{\rho}\right\|_{L^{4d/(4-d)}(\mathbb{T}^d)}^4\sup_t \sigma\left\|\sqrt{\rho}\dot{u}\right\|_{L^2(\mathbb{T}^d)}^2\int_0^t\left\|\sqrt{\rho}\dot{u}\right\|_{L^2(\mathbb{T}^d)}^2\\
&\qquad\qquad +C\left\|\mathcal{E}-\mathcal{I}\right\|_{L^\infty((0,t)\times\mathbb{T}^d)}^4\max\left\{\frac{1}{(\mu+\lambda)^4},\frac{1}{\mu^4}\right\}\int_0^t \sigma \left\|\nabla u\right\|_{L^4(\mathbb{T}^d)}^4\\
&\leq \frac{C}{\mu^4}\sup_t\left\|\sqrt{\rho}\right\|_{L^{4d/(4-d)}(\mathbb{T}^d)}^4 A_1\left(t\right)A_2\left(t\right)+C\left\|\mathcal{E}-\mathcal{I}\right\|_{L^\infty((0,t)\times\mathbb{T}^d)}^4\max\left\{\frac{1}{(\mu+\lambda)^4},\frac{1}{\mu^4}\right\}\int_0^t \sigma \left\|\nabla u\right\|_{L^4(\mathbb{T}^d)}^4
\end{aligned}$$

Let us now remark from (3.8) in Lemma 3.4 and using $\gamma\geq d/(4-d)$ that

$$\left\|\sqrt{\rho}\right\|_{L^{4d/((4-d)}(\mathbb{T}^d)}^4=\left\|\rho\right\|_{L^{2d/(4-d)}(\mathbb{T}^d)}^2\leq\left\|P(\rho)\right\|_{L^2(\mathbb{T}^d)}^{2/\gamma}\leq\left\{P\left(M\right)+\int_{\mathbb{T}^d}\left\{H_1\left(\rho/M\right)+H_2\left(\rho/M\right)\right\}\right\}^{\frac{1}{\gamma}}\lesssim M+B\left(t\right).$$

Similarly,

$$\begin{aligned}
&\frac{C}{\mu^3}\int_0^t\left\|\mu\operatorname{curl}u\right\|_{L^3(\mathbb{T}^d)}^3+\frac{C}{(\mu+\lambda)^3}\int_0^t\left\|F\right\|_{L^3(\mathbb{T}^d)}^3\\
&\leq\int_0^t\left\|\sqrt{\rho}\right\|_{L^{6d/(6-d)}(\mathbb{T}^d)}^3\left\|\sqrt{\rho}\dot{u}\right\|_{L^2(\mathbb{T}^d)}^3+C\left\|\mathcal{E}-\mathcal{I}\right\|_{L^\infty((0,t)\times\mathbb{T}^d)}^3\max\left\{\frac{1}{(\mu+\lambda)^3},\frac{1}{\mu^3}\right\}\int_0^t\left\|\nabla u\right\|_{L^3(\mathbb{T}^d)}^3\\
&\leq\sup_t\left\|\sqrt{\rho}\right\|_{L^{6d/(6-d)}(\mathbb{T}^d)}^3\sup_t\left\|\sqrt{\rho}\dot{u}\right\|_{L^2(\mathbb{T}^d)}\int_0^t\left\|\sqrt{\rho}\dot{u}\right\|_{L^2(\mathbb{T}^d)}^2+C\left\|\mathcal{E}-\mathcal{I}\right\|_{L^\infty((0,t)\times\mathbb{T}^d)}^3\max\left\{\frac{1}{(\mu+\lambda)^3},\frac{1}{\mu^3}\right\}\int_0^t\left\|\nabla u\right\|_{L^3(\mathbb{T}^d)}^3\\
&\leq\left(M+B\left(t\right)\right)\sqrt{A_1\left(t\right)}A_2\left(t\right)+C\left\|\mathcal{E}-\mathcal{I}\right\|_{L^\infty((0,t)\times\mathbb{T}^d)}^3\max\left\{\frac{1}{(\mu+\lambda)^3},\frac{1}{\mu^3}\right\}\int_0^t\left\|\nabla u\right\|_{L^3(\mathbb{T}^d)}^3.
\end{aligned}\tag{2.13}$$

Thus, under the hypothesis (H4) we have

$$\int_0^t\left\|\nabla u\right\|_{L^3(\mathbb{T}^d)}^3+\int_0^t\sigma\left\|\nabla u\right\|_{L^4(\mathbb{T}^d)}^4\leq C\left(E_0+\int_{\mathbb{T}^d}H_2\left(\rho_0/M\right)\right)+C\left(M+B\left(t\right)\right)\left(\sqrt{A_1\left(t\right)}A_2\left(t\right)+A_1\left(t\right)A_2\left(t\right)\right).$$

Terms appearing in the RHS of (2.6)**.** Recall that

$$
\begin{aligned}
A_1(t) = & \frac{\mu}{2}\int_{\mathbb{T}^d}\left|\partial_k u_0^i\right|^2 + \frac{\mu+\lambda}{2}\int_{\mathbb{T}^d}\left|\operatorname{div} u_0\right|^2 + \int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_{0,\delta}^k\partial_j u_{0,\delta}^i \\
& \int_{\mathbb{T}^d} P(\rho(t))\operatorname{div} u(t) - \int_{\mathbb{T}^d} P(\rho(0))\operatorname{div} u(0) \\
& -\mu\int_0^t\int_{\mathbb{T}^d}\partial_k u^i\partial_k u^\ell\partial_\ell u^i + \frac{\mu}{2}\int_0^t\int_{\mathbb{T}^d}\left|\partial_k u^i\right|^2\operatorname{div} u \\
& -(\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\operatorname{div} u\partial_i u^\ell\partial_\ell u^i + \frac{\mu+\lambda}{2}\int_0^t\int_{\mathbb{T}^d}(\operatorname{div} u)^3 \\
& +\frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\left\{\partial_t\varepsilon_{ijkl} + \partial_q(\varepsilon_{ijkl}u^q)\right\}\partial_j u_\delta^i\partial_k u_\delta^\ell \\
& -\int_0^t\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\omega_\delta * (\partial_j u^q\partial_q u^i) - \int_0^t\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\left[u^q,\omega_\delta\right]\partial_{qj}^2 u^i \\
& +\int_0^t\int_{\mathbb{T}^d}\rho P'(\rho)\partial_\ell u^k\partial_k u^\ell + \int_0^t\int_{\mathbb{T}^d}\rho\dot{u}f. \qquad (2.14)
\end{aligned}
$$

First, using (3.4) we have that

$$
\begin{aligned}
\int_{\mathbb{T}^d} P(\rho(t))\operatorname{div} u(t) &= \int_{\mathbb{T}^d}(P(\rho(t)) - P(M))\operatorname{div} u(t) \\
&\le C(\eta)\int_{\mathbb{T}^d}(P(\rho(t)) - P(M))^2 + \eta\int_{\mathbb{T}^d}\left|\operatorname{div} u\right|^2(t) \\
&\le C(\eta)\left\{\int_{\mathbb{T}^d} H_1(\rho) + H_2(\rho)\right\} + \eta\int_{\mathbb{T}^d}\left|\operatorname{div} u\right|^2(t),
\end{aligned}
$$

where η will be chosen later. Using the last estimate, we obtain that

$$
\begin{aligned}
& \int_{\mathbb{T}^d} P(\rho(t))\operatorname{div} u(t) - \int_{\mathbb{T}^d} P(\rho(0))\operatorname{div} u(0) \\
& + \frac{\mu}{2}\int_{\mathbb{T}^d}\left|\partial_k u_0^i\right|^2 + \frac{\mu+\lambda}{2}\int_{\mathbb{T}^d}\left|\operatorname{div} u_0\right|^2 + \frac{1}{2}\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_{0,\delta}^k\partial_j u_{0,\delta}^i + \frac{1}{2}\int_{\mathbb{T}^d}\partial_t\varepsilon_{ijkl}\partial_j u_\delta^i\partial_k u_\delta^\ell \\
& \le C(\eta)\left\{E_0 + \left\|\nabla u_0\right\|_{L^2(\mathbb{T}^d)} + B(t)\right\} + \eta\int_{\mathbb{T}^d}\left|\nabla u\right|^2(t) \qquad (2.15)
\end{aligned}
$$

Using (2.13) and hypothesis (H3) we infer that

$$
\begin{aligned}
& \int_0^t\int_{\mathbb{T}^d}\rho P'(\rho)\partial_\ell u^k\partial_k u^\ell = \gamma P(M)\int_0^t\int_{\mathbb{T}^d}\partial_\ell u^k\partial_k u^\ell + \gamma\int_0^t\int_{\mathbb{T}^d}(P(\rho) - P(M))\partial_\ell u^k\partial_k u^\ell \\
& \le CE_0 + \int_0^t\int_{\mathbb{T}^d}(P(\rho) - P(M))^3 + \int_0^t\int_{\mathbb{T}^d}\left|\nabla u\right|^3 \\
& \le C(E_0 + \int_{\mathbb{T}^d} H_2(\rho_0/M) + B(t)) + C(M + B(t))\left(\sqrt{A_1(t)}A_2(t) + A_1(t)A_2(t)\right). \qquad (2.16)
\end{aligned}
$$

Obviously we have that:

$$
\begin{aligned}
& -\mu\int_0^t\int_{\mathbb{T}^d}\partial_k u^i\partial_k u^\ell\partial_\ell u^i + \frac{\mu}{2}\int_0^t\int_{\mathbb{T}^d}\left|\partial_k u^i\right|^2\operatorname{div} u - (\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\operatorname{div} u\partial_i u^\ell\partial_\ell u^i + \frac{\mu+\lambda}{2}\int_0^t\int_{\mathbb{T}^d}(\operatorname{div} u)^3 \\
& -\int_0^t\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\omega_\delta * (\partial_j u^q\partial_q u^i) - \int_0^t\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\left[u^q,\omega_\delta\right]\partial_{qj}^2 u^i \\
& \le C\int_{\mathbb{T}^d}\left\|\nabla u\right\|_{L^3(\mathbb{T}^d)}^3. \qquad (2.17)
\end{aligned}
$$

Finally, using Poincaré's inequality, we obtain that

$$\frac{1}{2}\int_{\mathbb{T}^d}\left\{\partial_t\varepsilon_{ijkl}+\partial_q(\varepsilon_{ijkl}u^q)\right\}\partial_j u^i_\delta\partial_k u^\ell_\delta$$
$$\leq CE_0+C\int_0^t\int\|\nabla u\|^3_{L^3(\mathbb{T}^d)}+\int_0^t\left|\int_{\mathbb{T}^d}u^q(\tau)\right|\left|\int_{\mathbb{T}^d}\partial_q\varepsilon_{ijkl}\partial_j u^i_\delta\partial_k u^\ell_\delta\right| \tag{2.18}$$

In order to treat the mean of u^q we follow the lines by P.–L. Lions book. Let us recall, after verifying that ρ^2 is controlled by $B(t)$, that

$$\left\|\rho(t)\left(u^q(t)-\int_{\mathbb{T}^d}u^q(t)\right)\right\|_{L^1(\mathbb{T}^d)}\leq C\|\rho(t)\|_{L^2(\mathbb{T}^d)}\|\nabla u(t)\|_{L^2(\mathbb{T}^d)}.$$

Thus, we have that

$$\left|\int_{\mathbb{T}^d}\left(\rho(t,x)\int_{\mathbb{T}^d}u^q(t,y)\,dy-\rho(t,x)\,u^q(t,x)\right)dx\right|\leq\left\|\rho(t)\left(u^q(t)-\int_{\mathbb{T}^d}u^q(t)\right)\right\|_{L^1(\mathbb{T}^d)}$$

from which it follows that

$$M\left|\int_{\mathbb{T}^d}u^q(t,y)\,dy\right|\leq M^{\frac{1}{2}}E_0^{\frac{1}{2}}+C\|\rho(t)\|_{L^2(\mathbb{T}^d)}\|\nabla u(t)\|_{L^2(\mathbb{T}^d)}. \tag{2.19}$$

Consequently

$$\begin{aligned}
&\int_0^t\left|\int_{\mathbb{T}^d}u^q(\tau)\right|\left|\int_{\mathbb{T}^d}\partial_q\varepsilon_{ijkl}\partial_j u^i_\delta\partial_k u^\ell_\delta\right|\\
&\leq\|\partial_q\varepsilon_{ijkl}\|_{L^\infty((0,t)\times\mathbb{T}^d)}\left(M^{\frac{1}{2}}E_0^{\frac{3}{2}}+C\int_0^t\|\nabla u(t)\|^3_{L^2(\mathbb{T}^d)}\right)\\
&\leq\|\partial_q\varepsilon_{ijkl}\|_{L^\infty((0,t)\times\mathbb{T}^d)}\left(M^{\frac{1}{2}}E_0^{\frac{3}{2}}+\int_0^t\|\nabla u(t)\|^3_{L^3(\mathbb{T}^d)}\right)\\
&\leq\|\partial_q\varepsilon_{ijkl}\|_{L^\infty((0,t)\times\mathbb{T}^d)}\left(M^{\frac{1}{2}}E_0^{\frac{3}{2}}+E_0+\int_{\mathbb{T}^d}H_2(\rho_0)+(M+B(t))\sqrt{A_1(t)}A_2(t)\right).
\end{aligned}$$

Taking η sufficiently small and summing up (2.15), (2.16) we obtain

$$A_1(t)\leq C\left(E_0+\int_{\mathbb{T}^d}H_2(\rho_0)+\|\nabla u_0\|_{L^2(\mathbb{T}^d)}\right)+CB(t)+C(M+B(t))\sqrt{A_1(t)}A_2(t). \tag{2.20}$$

Terms appearing in the RHS of (2.7)**.** We recall the definition of $A_2(t)$ which is

$$\begin{aligned}
&\sigma(t)\int_{\mathbb{T}^d}\frac{\rho(t)|\dot u(t)|^2}{2}+\mu\int_{\mathbb{T}^d}\sigma\left|\partial_k\dot u^i\right|^2+(\mu+\lambda)\int_{\mathbb{T}^d}\sigma|\operatorname{div}\dot u|^2+\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell\dot u^k_\delta\partial_j\dot u^i_\delta\\
&=\int_0^1\int_{\mathbb{T}^d}\sigma\frac{\rho|\dot u|^2}{2}\\
&+\mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_q u^i\partial_k\dot u^i+\mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_k u^i\partial_q\dot u^i-\mu\int_0^t\int_{\mathbb{T}^d}\sigma\operatorname{div}u\partial_k u^i\partial_k\dot u^i\\
&+(\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_\ell u^q\partial_q u^\ell\operatorname{div}\dot u+(\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_i u^q\partial_q\dot u^i\operatorname{div}u-(\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma|\operatorname{div}u|^2\operatorname{div}\dot u\\
&-\int_0^t\int_{\mathbb{T}^d}\sigma\left(\partial_t\varepsilon_{ijkl}+\partial_q(u^q\varepsilon_{ijkl})\right)\partial_\ell u^k_\delta\partial_j\dot u^i_\delta-\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}(\omega_\delta*(\partial_\ell u^q\partial_q u^k))\partial_j\dot u^i_\delta-\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta\omega_\delta*(\partial_j u^q\partial_q\dot u^i)\\
&+\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}([u^q,\omega_\delta*]\,\partial^2_{\ell q}u^k)\partial_j\dot u^i_\delta+\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta([u^q,\omega_\delta*]\,\partial^2_{jq}\dot u^i)
\end{aligned}$$

$$-\int_0^t\int_{\mathbb{T}^d}\sigma\left\{P\left(\rho\right)\partial_j u^k\partial_k\dot{u}^j+\left(\rho P'\left(\rho\right)-P\left(\rho\right)\right)\operatorname{div}u\operatorname{div}\dot{u}\right\}.$$

We observe that

$$\begin{aligned}
&\mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_q u^i\partial_k\dot{u}^i+\mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_k u^i\partial_q\dot{u}^i-\mu\int_0^t\int_{\mathbb{T}^d}\sigma\operatorname{div}u\partial_k u^i\partial_k\dot{u}^i\\
&+(\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_\ell u^q\partial_q u^\ell\operatorname{div}\dot{u}+(\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_i u^q\partial_q\dot{u}^i\operatorname{div}u-(\mu+\lambda)\int_0^t\int_{\mathbb{T}^d}\sigma\left|\operatorname{div}u\right|^2\operatorname{div}\dot{u}\\
&+\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}(\omega_\delta*(\partial_\ell u^q\partial_q u^k))\partial_j\dot{u}^i_\delta-\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta\omega_\delta*(\partial_j u^q\partial_q\dot{u}^i)\\
&+\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}([u^q,\omega_\delta*]\,\partial^2_{\ell q}u^k)\partial_j\dot{u}^i_\delta+\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta([u^q,\omega_\delta*]\,\partial^2_{jq}\dot{u}^i)\\
&\int_0^t\int_{\mathbb{T}^d}\sigma\left\{P\left(\rho\right)\partial_j u^k\partial_k\dot{u}^j+\left(\rho P'\left(\rho\right)-P\left(\rho\right)\right)\operatorname{div}u\operatorname{div}\dot{u}\right\}\\
&\leq P\left(M\right)E_0+\frac{C\left(\eta\right)}{2}\int_0^t\int_{\mathbb{T}^d}\sigma\left|P\left(\rho\right)-P\left(M\right)\right|^4+C\left(\eta\right)\int_0^t\int_{\mathbb{T}^d}\sigma\left|\nabla u\right|^4+\eta\int_0^t\int_{\mathbb{T}^d}\sigma\left|\nabla\dot{u}\right|^2\\
&\leq C\left(\eta\right)\left(E_0+B\left(t\right)\right)+C\left(\eta\right)\left(M+B\left(t\right)\right)A_1\left(t\right)A_2\left(t\right)+\eta\int_0^t\int_{\mathbb{T}^d}\sigma\left|\nabla\dot{u}\right|^2
\end{aligned}$$

Moreover, using the Poincaré inequality along with (2.19) we get that

$$-\int_0^t\int_{\mathbb{T}^d}\sigma\left(\partial_t\varepsilon_{ijkl}+\partial_q\left(u^q\varepsilon_{ijkl}\right)\right)\partial_\ell u^k_\delta\partial_j\dot{u}^i_\delta\leq C\left(\eta\right)E_0+\eta\int_0^t\int_{\mathbb{T}^d}\sigma\left|\nabla\dot{u}\right|^2.$$

Putting all together, we end up with

$$A_2\left(t\right)C\left(\eta\right)\left(E_0+B\left(t\right)\right)+C\left(\eta\right)\left(M+B\left(t\right)\right)A_1\left(t\right)A_2\left(t\right)+\eta\int_0^t\int_{\mathbb{T}^d}\sigma\left|\nabla\dot{u}\right|^2$$

We thus see that for η sufficiently small, we obtain that

$$A_2\left(t\right)\leq C\left(E_0+B\left(t\right)\right)+C\left(M+B\left(t\right)\right)A_1\left(t\right)A_2\left(t\right).\tag{2.21}$$

The bootstrap argument. SUppose that initially, (ρ,u) are defined on a time interval $[0,T)$. Assume that

$$E\left(\rho_0/M,u_0\right)+\int_{\mathbb{T}^d}H_2\left(\rho_0/M\right)+\left\|\nabla u_0\right\|_{L^2(\mathbb{T}^d)}\leq\varepsilon$$

for some ε to be fixed later. We want to show that there exists a constant $C=C\left(\mu,\lambda,\gamma,M,E_0\right)$ depending on μ,λ,γ,M,E_0 such that for ε sufficiently small

$$\forall t\in[0,T^*):\max_{[0,T)}\left\{A_1\left(t\right)+A_2\left(t\right)+B\left(t\right)\right\}\leq C\left(\mu,\lambda,\gamma,M,E_0\right)\varepsilon.$$

In order to do that we will use a bootstrap argument. Recall that in the previous two sections, we showed, see (2.20) and (2.21), that there exists a constant $\tilde{C}$ such that

$$A_1\left(t\right)+A_2\left(t\right)\leq\tilde{C}\left(E_0+\int_{\mathbb{T}^d}H_2\left(\rho_0\right)+\left\|\nabla u_0\right\|_{L^2}\right)+\tilde{C}B\left(t\right)+\tilde{C}\left(E_0+B\left(t\right)\right)A_1\left(t\right)A_2\left(t\right)$$

Let us introduce $T^*\in(0,T]$ such that

$$\max_{t\in[0,T^*)}\left\{A_1\left(t\right)+A_2\left(t\right)+B\left(t\right)\right\}\leq 2\varepsilon.$$

We observe that

$$
\begin{aligned}
B(t) &\leq \left(E_0 + \int_{\mathbb{T}^d} H_2(\rho_0)\right) + C(M + B(t))\left(\sqrt{A_1(t)} + A_1(t)\right) A_2(t). \\
&+ \|\mathcal{E} - \mathcal{I}\|^3_{L^\infty((0,t)\times\mathbb{T}^d)} \int_0^t \|\nabla u\|^3_{L^3(\mathbb{T}^d)} + \|\mathcal{E} - \mathcal{I}\|^4_{L^\infty((0,t)\times\mathbb{T}^d)} \int_0^t \sigma \|\nabla u\|^4_{L^4(\mathbb{T}^d)} \\
&\leq \left(E_0 + \int_{\mathbb{T}^2} H_2(\rho_0)\right) + C(M + B(t))\left(\sqrt{A_1(t)} + A_1(t)\right) A_2(t)
\end{aligned}
$$

We thus obtain that

$$
A_1(t) + A_2(t) + B(t) \leq \left(1 + \tilde{C}\right)\left(E_0 + \int_{\mathbb{T}^2} H_2(\rho_0) + \|\nabla u_0\|_{L^2}\right) + \tilde{C}\varepsilon^{\frac{3}{2}}.
$$

Thus, if the initial data

$$
E_0 + \int_{\mathbb{T}^2} H_2(\rho_0) + \|\nabla u_0\|_{L^2(\mathbb{T}^d)}
$$

where $E_0 = E(\rho_0/M, u_0)$ and ε are chosen sufficiently small we may bootstrap and obtain that $T^* = T$.

2.3 Construction of a sequence of approximate solution to (1.1)

This entire part of the proof (from local to global strong solution of approximate solutions) can be done repeating the same arguments as in the work of R. Danchin and P.B. Mucha [DM19], Section 3. It is for this reason that we only briefly recall the results and comment on what is different in our case. The approximate system (1.17) inspired from the one proposed by the first author and P.–E. Jabin in [BJ18] which we recopy here for the reader's convenience reads

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho u) = 0, \\
(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + \omega_\delta * \operatorname{div}(\mathcal{E}(\nabla \omega_\delta * u)).
\end{cases}
\tag{2.22}
$$

It differs from the classical (isotropic) Navier-Stokes system by a smooth term. Therefore, we argue that the classical results regarding existence of local solutions, see for instance Theorem 3.1. from [DM19] remain true in our case with a proof that is essentially the same. Thus, we have that

Theorem 2 *Let $\rho_0 \in W^{1,p}(\mathbb{T}^d)$ and $u_0 \in W^{2-\frac{2}{p},2}(\mathbb{T}^d)$ for some $p > d$ with $d \geq 2$. Assume that $\rho_0 > 0$. Then there exists $T_* > 0$ depending only on the norms of the data and on $\inf_{\mathbb{T}^d} \rho_0$ such that (2.22) supplemented with data ρ_0 and u_0 has a unique solution (ρ, u) on the time interval $[0, T^*]$, satisfying*

$$
u \in W^{1,p}\left((0,T_*); L^p(\mathbb{T}^d)\right) \cap L^p\left((0,T_*); W^{2,p}(\mathbb{T}^d)\right).
$$

We consider

$$
\rho_0 \in L^{2\gamma}(\mathbb{T}^d) \text{ and } u_0 \in (H^1(\mathbb{T}^d))^d,
$$

with

$$
\int_{\mathbb{T}^d} \rho_0(x)\, dx = M.
$$

We recall the notation

$$
u_\delta = \omega_\delta * u
$$

where $\omega_\delta = \frac{1}{\delta^d}\omega\left(\frac{\cdot}{\delta}\right)$ with ω a smooth, nonnegative, radial function compactly supported in the unit ball centered at the origin and with integral equal to 1. To this end, we observe that for all $\delta \in (0, M)$ there exists $\xi_\delta > \delta$ such that

$$
\tilde{\rho}_0^\delta(x) = \min\{\rho_0(x) + \delta, \xi_\delta\} \text{ and } \int_{\mathbb{T}^d} \tilde{\rho}_0^\delta(x)\, dx = M.
$$

We consider

$$
\rho_0^\delta(x) = \omega_\delta * \tilde{\rho}_0 \text{ and } u_0^\delta = \omega_\delta * u_0.
$$

Observe that for all $\delta \in (0, M)$ we have that

$$\int_{\mathbb{T}^d} \rho_0^\delta(x)\, dx = M.$$

We consider $\left(\rho^\delta, u^\delta\right)_\delta$ the sequence of solutions for the Cauchy problem associated to system (1.17) with initial

$$\begin{cases} \rho_{|t=0} = \rho_0^\delta, \\ u_{|t=0} = u_0^\delta. \end{cases}$$

A priori, each of $\left(\rho^\delta, u^\delta\right)$ is defined on its own maximal time interval $[0, T_\delta)$ with $T_\delta \in (0, \infty]$. On these time intervals the solution has enough regularity such that the computations performed above make sense and, as a consequence, we have that $\left(\rho^\delta, u^\delta\right)$ have

$$E\left(\rho^\delta/M, u^\delta\right), A_1^\delta(t), A_2^\delta(t)$$

bounded independently w.r.t. δ where $A_1^\delta(t), A_2^\delta(t)$ are the expressions defined in (2.4) respectively in (2.5) with $\left(\rho^\delta, u^\delta\right)$ instead of (ρ, u). From here on the argument leading to the conclusion that $T^\delta = +\infty$ continues mutatis mutandis as in [DM19] due to the fact that the term $\omega_\delta * \operatorname{div}(\mathcal{E}(\nabla\omega_\delta * u))$ is regular.

2.4 Stability of solutions to (2.22)

In this section we show that finite energy weak solutions for which we have an appropriate information for the time derivative of the velocity are stable by weak-convergence: given a sequence of solutions satisfying a certain number of apriori estimates, one can extract a subsequence converging weakly towards a solution of the system. We recall that this is not trivial given the fact that the pressure is nonlinear function of the density. It seems that the minimum requirements are that the sequence of pressures are bounded in $L^2_{t,x}$.

The first fact regarding system (2.23) that we infer is that we can write a local energy equation. This is an immediate consequence of the renormalization property (proposition 3.2) and the fact that when the pressure is L^2, for any regular ϕ, ϕu can be used as a test function.

Proposition 2.1 *Consider $g \in L^2\left((0,T)\times\mathbb{T}^d\right)$ and (ρ, u) a finite energy weak-solution of the modified Navier-Stokes system*

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\mu+\lambda)\nabla\operatorname{div} u + \nabla\rho^\gamma = \omega_\delta * \operatorname{div}(\mathcal{E}(\nabla\omega_\delta * u)) + \nabla g. \end{cases} \tag{2.23}$$

Suppose, moreover that

$$\rho^\gamma \in L^2\left((0,T)\times\mathbb{T}^d\right).$$

Then, one has that

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left\{\rho|u|^2 + \frac{\rho^\gamma}{\gamma-1}\right\} + \operatorname{div}\left(\left(\rho|u|^2 + \frac{\gamma\rho^\gamma}{\gamma-1}\right)u\right) \\ &+\mu\nabla u : \nabla u + (\mu+\lambda)(\operatorname{div} u)^2 - \mu\Delta\frac{|u|^2}{2} - (\mu+\lambda)\operatorname{div}(u\operatorname{div} u) \\ &= \omega_\delta * \operatorname{div}(\mathcal{E}\nabla\omega_\delta * u)\, u + \operatorname{div}(ug) - g\operatorname{div} u, \end{aligned} \tag{2.24}$$

holds true in $\mathcal{D}'_{t,x}\left((0,T)\times\mathbb{R}^d_{per}\right)$.

Proof of 2.1: The proof uses the theory of renormalized solutions of DiPerna-Lions, see also Chapter 6 of the book of A. Novotny and I. Straškraba see [NS04]. Since

$$\rho \in L^\infty\left((0,+\infty); L^{2\gamma}\left(\mathbb{T}^d\right)\right)$$

according to the Proposition 3.2 we obtain that

$$\partial_t \rho^\gamma + \operatorname{div}(\rho^\gamma u) + (\gamma - 1)\rho^\gamma \operatorname{div} u = 0.$$

Now, since for all $\phi \in \mathcal{C}_c\left((0,\infty); \mathcal{C}^\infty\left(\mathbb{T}^d\right)\right)$, given the regularity of u, ϕu is a test function for the weak formulation of the momentum equation. This allows us to write the energy equation (2.24).

Theorem 3 *Consider a sequence of finite energy weak-solutions $(\rho_\delta, u_\delta)_\delta$ for the Navier Stokes system with initial data $(\rho_{0,\delta}, \rho_{0,\delta} u_{0,\delta})_{\varepsilon>0} \subset L^\gamma\left(\mathbb{T}^d\right) \times (L^2(\mathbb{T}^d))^d$, i.e.*

$$\begin{cases} \partial_t \rho_\delta + \operatorname{div}(\rho_\delta u_\delta) = 0, \\ \partial_t(\rho_\delta u_\delta) + \operatorname{div}(\rho_\delta u_\delta \otimes u_\delta) - \mu \Delta u_\delta - (\mu+\lambda)\nabla \operatorname{div} u_\delta + \nabla(\rho_\delta)^\gamma = \omega_\delta * \operatorname{div}\left(\mathcal{E}(\nabla \omega_\delta * u_\delta)\right), \\ (\rho_\delta, \rho_\delta u_\delta)_{|t=0} = (\rho_{0,\delta}, m_{0,\delta}), \end{cases} \tag{2.25}$$

and assume that there exists $(\rho_0, m_0) \in L^\gamma\left(\mathbb{T}^d\right) \times (L^2(\mathbb{T}^d))^d$ and $(\rho, u) \in L^{2\gamma}\left((0,T)\times\mathbb{T}^d\right) \times \left[L^2(0,T; H^1(\mathbb{T}^d))\right]^d$ such that

$$\begin{cases} \rho_{0,\delta} \to \rho_0 \text{ in } L^\gamma\left(\mathbb{T}^d\right), \\ \rho_\delta \rightharpoonup \rho \text{ weakly in } L^{2\gamma}\left((0,T)\times\mathbb{T}^d\right), \\ u_\delta \rightharpoonup u \text{ weakly in } L^2(0,T; H^1(\mathbb{T}^d)), \\ u_\delta \to u \text{ strongly in } L^2((\frac{1}{n}, T)\times\mathbb{T}^d)). \end{cases} \tag{2.26}$$

for all $n \in \mathbb{N}^*$. Then the pair (ρ, u) verifies

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla \rho^\gamma = \operatorname{div}\left(\mathcal{E}(\nabla u)\right), \\ (\rho, \rho u)_{|t=0} = (\rho_0, m_0). \end{cases} \tag{2.27}$$

Proof of Theorem 3 The second assumption from (2.26) allow us to conclude that there exist $\overline{\rho^\gamma} \in L^2\left((0,T)\times\mathbb{T}^3\right)$ such that

$$\rho_\delta^\gamma \rightharpoonup \overline{\rho^\gamma} \text{ weakly in } L^2\left((0,T)\times\mathbb{T}^3\right).$$

It is by now well-understood that the assumptions (2.26) are sufficient in order to conclude that

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla \overline{\rho^\gamma} = \operatorname{div}\left(\mathcal{E}(\nabla u)\right), \end{cases} \tag{2.28}$$

with $\mathcal{E}$ defined by (2.26). Of course, in order to finish the proof we must show that the function $\overline{\rho^\gamma}$ coincides with the function ρ^γ. To this end we will essentially mimic the proof from [BB20] which consists of taking the difference between the limit of the energy equations with the energy equation of the limiting system and "multiplying" it with an appropriate quantity that renders a "conservative" identity.

The assumptions (2.26) allow us to conclude the existence of positive measures $\overline{\nabla u : \nabla u}, \overline{(\operatorname{div} u)^2}, \overline{\mathcal{E}(\nabla u) : \nabla u} \in \mathcal{M}\left((0,T)\times\mathbb{T}^d\right)$ such that up to a subsequence we have

$$\begin{cases} \nabla u^\delta : \nabla u^\delta \rightharpoonup \overline{\nabla u : \nabla u} \text{ in } \mathcal{M}\left((0,T)\times\mathbb{T}^d\right) \text{ and } \nabla u : \nabla u \le \overline{\nabla u : \nabla u}, \\ (\operatorname{div} u^\delta)^2 \rightharpoonup \overline{(\operatorname{div} u)^2} \text{ in } \mathcal{M}\left((0,T)\times\mathbb{T}^d\right) \text{ and } (\operatorname{div} u)^2 \le \overline{(\operatorname{div} u)^2}, \\ \mathcal{E}(\nabla(\omega_\delta * u^\delta)) : \nabla(\omega_\delta * u^\delta) \rightharpoonup \overline{\mathcal{E}(\nabla u) : \nabla u} \text{ in } \mathcal{M}\left((0,T)\times\mathbb{T}^d\right) \\ \qquad\qquad \text{and } \mathcal{E}(\nabla u) : \nabla u \le \overline{\mathcal{E}(\nabla u) : \nabla u}. \end{cases} \tag{2.29}$$

It is in the proof of the last property, that the we need to regularize a positive definite operator and the assumption made in Remark 1.3. See the remark to see that simple change of shear viscosity may be done to satisfy such property starting with a viscosity tensor satisfying Hypothesis (H1)–(H4).

Lower semi-continuity. Indeed, for any $\phi \in \mathcal{C}([0,T]\times\mathbb{T}^d)$ with $\phi \ge 0$, we have that

$$0 \le \int_0^T \int_{\mathbb{T}^d} \mathcal{E}(\nabla(\omega_\delta * u_\delta) - \nabla u) : (\nabla(\omega_\delta * u_\delta) - \nabla u)\phi$$

$$= \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} (\partial_\ell \omega_\delta * u_\delta^k - \partial_\ell u^k)(\partial_j \omega_\delta * u_\delta^i - \partial_j u^i)\phi$$
$$= \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell \omega_\delta * u_\delta^k \partial_j \omega_\delta * u_\delta^i \phi - \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell \omega_\delta * u_\delta^k \partial_j u^i \phi - \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell u^k \partial_j \omega_\delta * u_\delta^i \phi$$
$$+ \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell u^k \partial_j u^i \phi.$$

We obviously have

$$\lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^d} \omega_\delta * (\varepsilon_{ijkl} \partial_j u^i \phi) \partial_\ell u_\delta^k = \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell u^k \partial_j u^i \phi$$

and the same for the other similar term. Thus we obtain that

$$0 \leq \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell \omega_\delta * u_\delta^k \partial_j \omega_\delta * u_\delta^i \phi - \int_0^T \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_\ell u^k \partial_j u^i \phi$$
$$= \left\langle \overline{\mathcal{E}(\nabla u) : \nabla u} - \mathcal{E}(\nabla u) : \nabla u, \phi \right\rangle_{M((0,T)\times\mathbb{T}^d); C([0,T]\times\mathbb{T}^d)}$$

Energy identities and conclusion. On the one hand, for any $\delta > 0$, $\left(\rho^\delta, u^\delta\right)$ verifies the hypothesis of Proposition 2.1 and thus we infer that

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\rho_\delta |u_\delta|^2 + \frac{\rho_\delta^\gamma}{\gamma - 1}\right\} + \operatorname{div}\left(\left(\rho_\delta |u_\delta|^2 + \frac{\gamma \rho_\delta^\gamma}{\gamma - 1}\right) u_\delta\right) + \mu \nabla u_\delta : \nabla u_\delta + (\mu + \lambda)(\operatorname{div} u_\delta)^2$$
$$-\mu \Delta \frac{|u_\delta|^2}{2} - (\mu + \lambda) \operatorname{div}(u_\delta \operatorname{div} u_\delta) - \omega_\delta * \operatorname{div}(\mathcal{E} \nabla \omega_\delta * u_\delta) u_\delta = 0, \tag{2.30}$$

Let us observe that for all $\phi \in \mathcal{C}_c\left((0,\infty); \mathcal{C}_{per}^\infty\left(\mathbb{R}^d\right)\right)$ we have that

$$-\int_{\mathbb{T}^d} \omega_\delta * \operatorname{div}(\mathcal{E} \nabla \omega_\delta * u_\delta) u_\delta \phi = \int_{\mathbb{T}^d} \mathcal{E} \nabla(\omega_\delta * u_\delta) : \omega_\delta * \nabla(u_\delta \phi)$$
$$= \int_{\mathbb{T}^d} \mathcal{E} \nabla(\omega_\delta * u_\delta) : \omega_\delta * (\nabla u_\delta \phi) + \int_{\mathbb{T}^d} \mathcal{E} \nabla(\omega_\delta * u_\delta) : \omega_\delta * (u_\delta \otimes \nabla \phi).$$

Owing to the fact that there exits some n such that

$$\operatorname{Supp} \phi(\cdot, \cdot) \subset (1/n, T) \times \mathbb{R}^d$$

that $u_\delta \to u$ strongly in $L^2((1/n, T) \times \mathbb{T}^d))^d$ and that $\nabla(\omega_\delta * u_\delta) \to \nabla u$ weakly in $L^2((0,T) \times \mathbb{T}^d))^{d\times d}$ we obtain

$$\lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^d} \mathcal{E}(\nabla(\omega_\delta * u_\delta)) : \omega_\delta * (u_\delta \otimes \nabla \phi) = \int_0^T \int_{\mathbb{T}^d} \mathcal{E}(\nabla u) : (u \otimes \nabla \phi).$$

Next, we observe that

$$\int_0^T \int_{\mathbb{T}^d} \mathcal{E} \nabla(\omega_\delta * u_\delta) : \omega_\delta * (\nabla u_\delta \phi)$$
$$= \int_0^T \int_{\mathbb{T}^d} \mathcal{E} \nabla(\omega_\delta * u_\delta) : \nabla(\omega_\delta * u_\delta) \phi + \int_0^T \int_{\mathbb{T}^d} \mathcal{E} \nabla(\omega_\delta * u_\delta) : [\omega_\delta *, \phi] \nabla u_\delta.$$

Now, for any $j, q \in \overline{1, d}$ one has

$$[\omega_\delta *, \phi] \partial_j u_\delta^q(x) = \omega_\delta * (\phi \partial_j u_\delta^q) - \phi \omega_\delta * \partial_j u_\delta^q$$
$$= \int_{\mathbb{T}^d} (\phi(x - y) - \phi(x)) \partial_j u_\delta^q(x - y) \omega_\delta(y) \, dy$$

$$= \int_{\mathbb{T}^d} \left(\phi\left(x-\delta z\right)-\phi\left(x\right)\right)\partial_j u_\delta^q\left(x-\delta z\right)\omega\left(z\right)dz.$$

Thus

$$\left|\int_0^T\int_{\mathbb{T}^d}\mathcal{E}\nabla(\omega_\delta * u_\delta):\nabla(\omega_\delta * u_\delta)\phi\right| \leq \delta \max_{i,j,k,l}\left\|\varepsilon_{ijkl}\right\|_{L^\infty(\mathbb{T}^d)}\left\|\nabla u_\delta\right\|^2_{L^2(\mathbb{T}^d)}\left\|\nabla\phi\right\|_{L^\infty} \underset{\delta\to 0}{\to} 0.$$

Moreover, using the information of relation (2.29) we may pass to the limit in (2.30) such as to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\rho_\delta\left|u_\delta\right|^2+\frac{\rho_\delta^\gamma}{\gamma-1}\right\}+\operatorname{div}\left(\left(\rho_\delta\left|u_\delta\right|^2+\frac{\gamma\rho_\delta^\gamma}{\gamma-1}\right)u_\delta\right)+\mu\nabla u_\delta:\nabla u_\delta+(\mu+\lambda)\left(\operatorname{div}u_\delta\right)^2$$
$$-\mu\Delta\frac{\left|u_\delta\right|^2}{2}-(\mu+\lambda)\operatorname{div}\left(u_\delta\operatorname{div}u_\delta\right)-\omega_\delta*\operatorname{div}\left(\mathcal{E}\nabla\omega_\delta*u_\delta\right)u_\delta=0, \tag{2.31}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\rho\left|u\right|^2+\frac{\bar{\rho}^\gamma}{\gamma-1}\right\}+\operatorname{div}\left(\left(\rho\left|u\right|^2+\frac{\gamma\bar{\rho}^\gamma}{\gamma-1}\right)u\right)+\mu\overline{\nabla u:\nabla u}+(\mu+\lambda)\overline{(\operatorname{div}u)^2}+\overline{\mathcal{E}\left(\nabla u\right):\nabla u} \tag{2.32}$$
$$-\mu\Delta\frac{\left|u\right|^2}{2}-(\mu+\lambda)\operatorname{div}\left(u\operatorname{div}u\right)-\operatorname{div}\left(u\mathcal{E}\nabla u\right)=0. \tag{2.33}$$

On the other hand, let us observe that system (2.28) can be put under the form

$$\begin{cases}\partial_t\rho+\operatorname{div}\left(\rho u\right)=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)-\operatorname{div}\left(\mathcal{E}\nabla u\right)+\nabla\rho^\gamma=\nabla(\rho^\gamma-\overline{\rho^\gamma})\end{cases} \tag{2.34}$$

such that using again Proposition 2.1 we write that

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\rho\left|u\right|^2+\frac{\rho^\gamma}{\gamma-1}\right\}+\operatorname{div}\left(\left(\rho\left|u\right|^2+\frac{\gamma\rho^\gamma}{\gamma-1}\right)u\right)+\mu\nabla u:\nabla u+(\mu+\lambda)\left(\operatorname{div}u\right)^2+\mathcal{E}\left(\nabla u\right):\nabla u$$
$$-\mu\Delta\frac{\left|u\right|^2}{2}-(\mu+\lambda)\operatorname{div}\left(u\operatorname{div}u\right)-\operatorname{div}\left(u\mathcal{E}(\nabla u)\right)=\operatorname{div}\left(u\left(\rho^\gamma-\overline{\rho^\gamma}\right)\right)-\left(\rho^\gamma-\overline{\rho^\gamma}\right)\operatorname{div}u. \tag{2.35}$$

Next, we take the difference between (2.35) and (2.32), we multiply it with $\gamma-1$ in order to obtain that

$$\partial_t\Theta+\operatorname{div}\left(\Theta u\right)+(\gamma-1)\Theta\operatorname{div}u=-(\gamma-1)\Xi \text{ in } \mathcal{D}'_{t,x}\left((0,T)\times\mathbb{R}^d_{per}\right), \tag{2.36}$$

where

$$\Theta\overset{not.}{=}\overline{\rho^\gamma}-\rho^\gamma,$$
$$\Xi\overset{not.}{=}\left(\mu\overline{\nabla u:\nabla u}+(\mu+\lambda)\overline{(\operatorname{div}u)^2}+\overline{\mathcal{E}\left(\nabla u\right):\nabla u}\right)$$
$$-\left(\mu\nabla u:\nabla u+(\mu+\lambda)\left(\operatorname{div}u\right)^2+\mathcal{E}\left(\nabla u\right):\nabla u\right).$$

Obviously,

$$\Theta,\Xi\geq 0,$$

in the sense of L^2 function respectively in the sense of measures. We regularize the previous equation with the help of a sequence of approximations of the identity ω_ε :

$$\partial_t\omega_\varepsilon*\Theta+\operatorname{div}\left(\omega_\varepsilon*\Theta u\right)+(\gamma-1)\omega_\varepsilon*\left(\Theta\operatorname{div}u\right)=r_\varepsilon\left(\Theta,u\right)-(\gamma-1)\omega_\varepsilon*\Xi, \tag{2.37}$$

see the notations introduced in (3.1) and (3.2). Multiply relation (2.37) with $\frac{1}{\gamma}(h+\omega_\varepsilon*\Theta)^{\frac{1}{\gamma}-1}$ where $h>0$ is a fixed positive constant. We end up with

$$\partial_t\left(h+\omega_\varepsilon*\Theta\right)^{\frac{1}{\gamma}}+\operatorname{div}\left(\left(h+\omega_\varepsilon*\Theta\right)^{\frac{1}{\gamma}}u\right)+\left(h+\omega_\varepsilon*\Theta\right)^{\frac{1}{\gamma}-1}[\left(\frac{1}{\gamma}-1\right)\omega_\varepsilon*\Theta-h]\operatorname{div}u$$

$$
\begin{aligned}
&+\left(1-\frac{1}{\gamma}\right)(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1}\omega_\varepsilon * (\Theta \operatorname{div} u) \\
&=\frac{1}{\gamma}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1} r_\varepsilon(\Theta, u)-\frac{1}{\gamma}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1}(\gamma-1)\,\omega_\varepsilon * \Xi, \qquad (2.38)
\end{aligned}
$$

Integrating first w.r.t. space, we obtain that the distributional time derivative of $\int_{\mathbb{T}^d}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}}$ namely

$$
\frac{d}{dt}\int_{\mathbb{T}^d}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}}
$$

belongs to some Lebesgue space. Classical results then yield that

$$
t \to \int_{\mathbb{T}^d}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}}(t)
$$

is absolutely and that the distributional derivative coincides with the derivative a.e.. We may thus write that for any $t \in (0,T)$ we have

$$
\begin{aligned}
&\int_{\mathbb{T}^d}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}}(t) \\
&=\int_{\mathbb{T}^d}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}}(0)-\int_0^t\int_{\mathbb{T}^d}\left(\frac{1}{\gamma}-1\right)(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1}[\omega_\varepsilon, \operatorname{div} u]\,\Theta+\int_0^t\int_{\mathbb{T}^d}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1} h \operatorname{div} u \\
&+\int_0^t\int_{\mathbb{T}^d}\left[\frac{1}{\gamma}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1} r_\varepsilon(\Theta, u)-\frac{1}{\gamma}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1}(\gamma-1)\,\omega_\varepsilon * \Xi\right] \\
&\leq \int_{\mathbb{T}^d}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}}(s)-\int_0^t\left(\frac{1}{\gamma}-1\right)(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1}[\omega_\varepsilon, \operatorname{div} u]\,\Theta \\
&+\int_0^t\int_{\mathbb{T}^d}\frac{1}{\gamma}(h+\omega_\varepsilon * \Theta)^{\frac{1}{\gamma}-1} r_\varepsilon(\Theta, u),
\end{aligned}
$$

where we used the positivity of Ξ. Using Proposition 3.1, we obtain that

$$
[\omega_\varepsilon, \operatorname{div} u]\,\Theta \text{ and } r_\varepsilon(\Theta, u) \to 0 \text{ in } L^1\left((0,T)\times \mathbb{T}^d\right).
$$

Notice that since $\gamma > 1$ along with $\omega_\varepsilon * \Theta \geq 0$, we also have that

$$
(h+\omega_\varepsilon * \Theta)^{1/\gamma-1} \leq h^{1/\gamma-1}.
$$

Taking in account the last observations, by making $\varepsilon \to 0$ we get that

$$
\int_{\mathbb{T}^d}(h+\Theta)^{\frac{1}{\gamma}}(t) \leq \int_{\mathbb{T}^d}(h+\Theta)^{\frac{1}{\gamma}}(0)+h^{1/\gamma}\int_0^t\int_{\mathbb{T}^d}|\operatorname{div}\, u|
$$

Letting h go to zero and using the strong convergence at initial time shows that the term in the RHS of the above equation is 0 and the conclusion is that

$$
\overline{\rho^\gamma} = \rho^\gamma \text{ a.e. on } (0,T)\times \mathbb{T}^d.
$$

This ends the proof of Theorem 3.

Strong convergence on u^δ. The only thing left to justify in order to conclude the proof of Theorem 1 is that the uniform bounds verified by the solutions constructed in Section (2.3) imply that up to a subsequence

$$
\lim_{\delta\to 0} u^\delta = u \text{ strongly in } L^2\left(\left(\frac{1}{n}, T\right)\times \mathbb{T}^d\right)
$$

for all n. But this is just a consequence of the fact that the second Hoff functional is uniformly bounded w.r.t. $\delta > 0$ which implies that for all $T > 0$ and all $\delta > 0$

$$\int_0^T \int_{\mathbb{T}^d} \sigma(t) \|\nabla \dot{u}_\delta\|_{L^2}^2 \leq c$$

for some c. This implies that for any $n \in \mathbb{N}$

$$\int_{1/n}^T \int_{\mathbb{T}^d} \|\nabla \dot{u}_\delta\|_{L^2}^2 \leq nc,$$

and since

$$\partial_t u_\delta = \dot{u}_\delta - u_\delta \cdot \nabla u_\delta$$

and we have

$$\|u_\delta\|_{L^2((1/n,T);L^6(\mathbb{T}^d))} + \|\nabla u_\delta\|_{L^3((1/n,T)\times\mathbb{T}^d)}$$

is uniformly bounded, we obtain that

$$\partial_t u_\delta \text{ is uniformly bounded in } L^{\frac{6}{5}}(1/n,T); L^2(\mathbb{T}^d).$$

For any n, by the Aubin-Lions theorem $\left(u^\delta\right)_{\delta>0}$ converges strongly in $L^2\left(\left(\frac{1}{n},T\right)\times\mathbb{T}^d\right)$ while applying a Cantor's diagonal type process provides us with a subsequence $\left(u^\delta\right)_{\delta>0}$ converging for any n in $L^2\left(\left(\frac{1}{n},T\right)\times\mathbb{T}^d\right)$.

3 Appendix

3.1 Appendix A: tool box

Lemma 3.1 (Fourier Multipliers) *Consider $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ a function verifying*

$$|\partial^\alpha m(\xi)| \leq c_\alpha |\xi|^{-\alpha}$$

for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d+1$. Then, for all $p \in (1,\infty)$, there exists C_p such that for any $u \in L^p$

$$\left\|\mathcal{F}^{-1}\left(m(\xi)\mathcal{F}\left(u - \int_{\mathbb{T}^d} u\right)\right)\right\|_{L^p} \leq C_p \|u\|_{L^p}.$$

Lemma 3.2 (Sobolev's Inequality) *For all $p \in (1,d)$ and $u \in L^p \cap D^{1,p^*}$ we have that*

$$\left\|u - \int_{\mathbb{T}^d} u\right\|_{L^p} \leq \|\nabla u\|_{L^{p^*}},$$

where $1/p + 1/d = 1/p^*$.

Lemma 3.3 (Grönwall-Bihari) *Consider two continuous functions $u, w : [0,T) \to [0,+\infty)$ satisfying*

$$u(t) + w(t) \leq u(0) + \int_0^t u(s)^\theta f(s)\, ds.$$

Then

$$u(t) + w(t) \leq \left[u(0)^{1-\theta} + (1-\theta)\int_0^t f(s)\, ds\right]^{\frac{1}{1-\theta}}$$

Let $g \in L^q(0,T;L^p(\mathbb{T}^d))$ with $p,q \geq 1$, introduce a new function

$$g_\delta(x) = g * \omega_\delta(x) \qquad \text{with} \qquad \omega_\delta(x) = \frac{1}{\delta^d}\omega(\frac{x}{\delta}) \tag{3.1}$$

with ω a smooth, nonnegative, even function compactly supported in the unit ball centered at the origin and with integral equal to 1. We recall the following classical analysis result

$$\lim_{\delta\to 0}\|g_\delta - g\|_{L^q(0,T;L^p(\mathbb{T}^d))} = 0.$$

Next let us recall the following commutator estimate which was obtained for the first time by DiPerna and Lions:

Proposition 3.1 *Consider $\beta \in (1,\infty)$ and (a,b) such that $a \in L^\beta\left((0,T)\times\mathbb{T}^d\right)$ and $b, \nabla b \in L^p\left((0,T)\times\mathbb{T}^d\right)$ where $\frac{1}{s} = \frac{1}{\beta} + \frac{1}{p} \le 1$. Then, we have*

$$\lim_{\delta\to 0} r_\delta^k(a,b) = 0 \text{ in } L^s\left((0,T)\times\mathbb{T}^d\right),$$

for $k \in \{1,2\}$ where

$$r_\delta^1(a,b) = b\partial_i a_\delta - (b\partial_i a)_\delta \text{ and } r_\delta^2(a,b) = \partial_i(a_\delta b) - \partial_i((ab)_\delta). \tag{3.2}$$

Moreover, the following commutator estimates hold true

$$\|b\partial_i a_\delta - (b\partial_i a)_\delta\|_{L_t^s L_x^s} \le \|\nabla b\|_{L_t^p L_x^p}\|a\|_{L_t^\beta L_x^\beta} \tag{3.3}$$

$$\|\partial_i(a_\delta b) - \partial_i((ab)_\delta)\|_{L_t^s L_x^s} \le \|\nabla b\|_{L_t^p L_x^p}\|a\|_{L_t^\beta L_x^\beta} \tag{3.4}$$

where $b\partial_i a$ should be understood as

$$b\partial_i a = \partial_i(ab) - a\partial_i b.$$

Whenever we have a *regular solution* for the transport equation

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{3.5}$$

then, multiplying the former equation with $b'(\rho)$ gives

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + \{\rho b'(\rho) - b(\rho)\}\operatorname{div} u = 0. \tag{3.6}$$

The following proposition gives us a framework for justifying this computations when ρ is just a Lebesgue function.

Proposition 3.2 *Consider $2 \le \beta < \infty$ and λ_0, λ_1 such that $\lambda_0 < 1$ and $-1 \le \lambda_1 \le \beta/2 - 1$. Also, consider $\rho \in L^\beta\left((0,T)\times\mathbb{T}^3\right)$, $\rho \ge 0$ a.e. and $u, \nabla u \in L^2\left((0,T)\times\mathbb{T}^3\right)$ verifying the transport equation (3.5) in the sense of distributions. Then, for any function $b \in C^0([0,\infty)) \cap C^1((0,\infty))$ such that*

$$\begin{cases} b'(t) \le ct^{-\lambda_0} \text{ for } t \in (0,1], \\ |b'(t)| \le ct^{\lambda_1} \text{ for } t \ge 1. \end{cases}$$

Then, equation (3.6) holds in the sense of distributions.

The proof of the above results follow by adapting in a straightforward manner lemmas 6.7. and 6.9 from the book of A. Novotný- I.Straškraba [NS04] pages 304–308.

We recall that

$$\rho H_i'(\rho) - H_i(\rho) = |\rho^\gamma - M^\gamma|^i(\rho^\gamma - M^\gamma).$$

We then have that

Lemma 3.4 *There exists $\alpha, \beta, \kappa \ge 0$ depending only on M and γ, such that for all $\rho \ge 0$ we have that:*

$$H_3(\rho) \le \alpha|P(\rho) - P(M)|^3 + \kappa H_1(\rho) \tag{3.7}$$

$$(P(\rho) - P(M))^2 \le \alpha H_1(\rho) + \beta H_2(\rho). \tag{3.8}$$

Proof of Lemma 3.4: We first prove (3.7). Let $g :]0,\infty[\to \mathbb{R}$ given by

$$g(\rho) = \frac{\alpha}{\rho} |P(\rho) - P(M)|^3 + \kappa \frac{H_1(\rho)}{\rho} - \frac{H_3(\rho)}{\rho},$$

from which we deduce that

$$\begin{aligned} g'(\rho) = &-\frac{\alpha}{\rho^2} |P(\rho) - P(M)|^3 + \frac{3\alpha\gamma\rho^{\gamma-1}}{\rho} |P(\rho) - P(M)| (P(\rho) - P(M)) \\ &+ \kappa \frac{(P(\rho) - P(M))}{\rho^2} - \frac{|P(\rho) - P(M)|^2 (P(\rho) - P(M))}{\rho^2}. \end{aligned}$$

Denote

$$x = P(\rho) - P(M) \geq -M^\gamma$$

Then

$$\begin{aligned} \rho^2 g'(\rho) &= -\alpha |x|^3 + 3\alpha\gamma (x + M^\gamma) |x| x + \kappa x - x^3 \\ &= -\alpha |x|^3 + 3\alpha\gamma |x|^3 - x^3 + 3\alpha\gamma M^\gamma |x| x + \kappa x \\ &= x \left((3\alpha(\gamma - 1)) |x| x - x^2 + 3\alpha\gamma M^\gamma |x| + \kappa \right) \end{aligned}$$

and we want to have

$$\begin{aligned} &\forall x \geq 0 : 3\alpha(\gamma - 1) |x| x - x^2 + 3\alpha\gamma M^\gamma |x| + \kappa \geq 0, \\ &\forall x \in [-M^\gamma, 0] : 3\alpha(\gamma - 1) |x| x - x^2 + 3\alpha\gamma M^\gamma |x| + \kappa \geq 0 \end{aligned}$$

The first inequality is equivalent to

$$\begin{aligned} &\forall x \geq 0 : 3\alpha(\gamma - 1) |x| x - x^2 + 3\alpha\gamma M^\gamma |x| + \kappa \geq 0, \\ &\quad \Rightarrow \forall x \geq 0 : (3\alpha(\gamma - 1) - 1)x^2 + 3\alpha\gamma M^\gamma x + \kappa \geq 0 \end{aligned}$$

Thus, if we chose

$$3\alpha(\gamma - 1) \geq 1,$$

then the first inequality is verified. Let us treat the second one:

$$\begin{aligned} &\forall x \in [-M^\gamma, 0] : (3\alpha\gamma - 1) |x| x - x^2 + 3\alpha\gamma M^\gamma |x| + \kappa \geq 0, \\ &\quad \Rightarrow \forall x \in [-M^\gamma, 0] : -(3\alpha\gamma - 1) x^2 - x^2 - 3\alpha\gamma M^\gamma x + \kappa \geq 0, \\ &\quad \Rightarrow \forall x \in [-M^\gamma, 0] : -3\alpha\gamma x^2 - 3\alpha\gamma M^\gamma x + \kappa \geq 0. \end{aligned}$$

which is true as soon as we fix α and we take κ sufficiently large.

Let us turn attention towards (3.8). Consider

$$g(\rho) = \alpha \frac{H_1(\rho)}{\rho} + \beta \frac{H_2(\rho)}{\rho} - \frac{(P(\rho) - P(M))^2}{\rho}$$

and observe that

$$g(M) = 0.$$

We have that

$$\rho^2 g'(\rho) = (\alpha - \gamma M^\gamma)(\rho^\gamma - M^\gamma) + \beta |\rho^\gamma - M^\gamma| (\rho^\gamma - M^\gamma) - (\gamma - 1)(\rho^\gamma - M^\gamma)^2.$$

For $\alpha = \gamma M^\gamma$ and $\beta = \gamma - 1$ we see that

$$g'(\rho) \leq 0 \text{ on } \rho \in [0, M] \text{ and } g'(\rho) \geq 0 \text{ if } \rho \geq M.$$

3.2 Appendix B: detailed computations for the Hoff functionals

3.2.1 Hoff's first energy functional

We put the second equation under the form

$$\rho\dot{u} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u - \operatorname{div} \omega_\delta * \mathcal{E}\left(\nabla u_\delta\right) + \nabla P\left(\rho\right) = \rho f.$$

where

$$\dot{u} = \partial_t u + u\nabla u.$$

We multiply the above equation with $\dot{u}$ and integrate. Owing to the hypothesis

$$\varepsilon_{ijkl} a_{ij} b_{kl} = \varepsilon_{ijkl} a_{kl} b_{ij}$$

we can write that

$$\begin{aligned}
\left\langle \operatorname{div} \omega_\delta * \mathcal{E}\left(\omega_\delta * \nabla u\right), \dot{u}\right\rangle &= -\int_{\mathbb{T}^d} \partial_j(\varepsilon_{ijkl}\partial_\ell u_\delta^k)\dot{u}_\delta^i \\
&= \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \partial_j\partial_t u_\delta^i + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \omega_\delta * (\partial_j u^q \partial_q u^i) + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \omega_\delta * (u^q \partial_{qj}^2 u^i) \\
&= \frac{1}{2}\left\{\int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \partial_j\partial_t u_\delta^i + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_t\partial_\ell u_\delta^k \partial_j u_\delta^i\right\} + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \omega_\delta * (\partial_j u^q \partial_q u^i) \\
&+ \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k u^q \partial_{qj}^2 u_\delta^i + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \left[u^q, \omega_\delta\right]\partial_{qj}^2 u^i \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \partial_j u_\delta^i - \frac{1}{2}\int_{\mathbb{T}^d} \partial_t\varepsilon_{ijkl}\partial_\ell u_\delta^k \partial_j u_\delta^i + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \omega_\delta * (\partial_j u^q \partial_q u^i) \\
&+ \frac{1}{2}\left\{\int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_j u_\delta^i u^q \partial_{qk}^2 u_\delta^\ell + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u^k u^q \partial_{qj}^2 u_\delta^i\right\} + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \left[u^q, \omega_\delta\right]\partial_{qj}^2 u^i \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u^k \partial_j u^i + \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \omega_\delta * (\partial_j u^q \partial_q u^i) - \frac{1}{2}\int_{\mathbb{T}^d} \left\{\partial_t\varepsilon_{ijkl} + \partial_q(\varepsilon_{ijkl}u^q)\right\}\partial_j u_\delta^i \partial_k u_\delta^\ell \\
&+ \int_{\mathbb{T}^d} \varepsilon_{ijkl}\partial_\ell u_\delta^k \left[u^q, \omega_\delta\right]\partial_{qj}^2 u^i.
\end{aligned}$$

Similar computations show that

$$\begin{aligned}
-\left\langle\left(\mu\Delta + (\mu+\lambda)\nabla \operatorname{div}\right)u, \dot{u}\right\rangle &= \frac{1}{2}\frac{d}{dt}\left\{\mu\int_{\mathbb{T}^d}\left|\partial_k u^i\right|^2 + (\mu+\lambda)\int_{\mathbb{T}^d}\left|\operatorname{div} u\right|^2\right\} \\
&+ \mu\int_{\mathbb{T}^d}\partial_k u^i \partial_k u^\ell \partial_\ell u^i - \frac{\mu}{2}\int_{\mathbb{T}^d}\left|\partial_k u^i\right|^2 \operatorname{div} u \\
&+ (\mu+\lambda)\int_{\mathbb{T}^d}\operatorname{div} u \partial_i u^\ell \partial_\ell u^i - \frac{\mu+\lambda}{2}\int_{\mathbb{T}^d}(\operatorname{div} u)^3
\end{aligned}$$

Next, we treat the pressure term

$$\begin{aligned}
\int_{\mathbb{T}^d}\dot{u}\nabla P\left(\rho\right) &= -\int_{\mathbb{T}^d} P\left(\rho\right)\operatorname{div}\dot{u} = -\frac{d}{dt}\left\{\int_{\mathbb{T}^d} P\left(\rho\right)\operatorname{div} u\right\} + \int_{\mathbb{T}^d}\partial_t P\left(\rho\right)\operatorname{div} u - \int_{\mathbb{T}^d} P\left(\rho\right)\operatorname{div}\left(u\nabla u\right) \\
&= -\frac{d}{dt}\left\{\int_{\mathbb{T}^d} P\left(\rho\right)\operatorname{div} u\right\} + \int_{\mathbb{T}^d}\partial_t P\left(\rho\right)\operatorname{div} u - \int_{\mathbb{T}^d} P\left(\rho\right)\partial_\ell u^k \partial_k u^\ell - \int_{\mathbb{T}^d} P\left(\rho\right)u^k \partial_{k\ell}^2 u^\ell \\
&= -\frac{d}{dt}\left\{\int_{\mathbb{T}^d} P\left(\rho\right)\operatorname{div} u\right\} + \int_{\mathbb{T}^d}\partial_t P\left(\rho\right)\operatorname{div} u - \int_{\mathbb{T}^d} P\left(\rho\right)\partial_\ell u^k \partial_k u^\ell + \int_{\mathbb{T}^d}\partial_k(P\left(\rho\right)u^k)\partial_\ell u^\ell \\
&= -\frac{d}{dt}\left\{\int_{\mathbb{T}^d} P\left(\rho\right)\operatorname{div} u\right\} + \int_{\mathbb{T}^d}\left(\partial_t P\left(\rho\right) + \operatorname{div}\left(P(\rho)u\right)\right)\operatorname{div} u - \int_{\mathbb{T}^d} P\left(\rho\right)\partial_\ell u^k \partial_k u^\ell
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{d}{dt}\left\{\int_{\mathbb{T}^d} P(\rho)\operatorname{div} u\right\} + \int_{\mathbb{T}^d} \left(P(\rho) - \rho P'(\rho)\right)(\operatorname{div} u)^2 - \int_{\mathbb{T}^d} P(\rho)\,\partial_\ell u^k \partial_k u^\ell \\
&= -\frac{d}{dt}\left\{\int_{\mathbb{T}^d} P(\rho)\operatorname{div} u\right\} - \int_{\mathbb{T}^d} \rho P'(\rho)(\operatorname{div} u)^2.
\end{aligned}
$$

Putting together all the above computations, we end up with

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\mu\int_{\mathbb{T}^d}\left|\partial_k u^i\right|^2 + (\mu+\lambda)\int_{\mathbb{T}^d}|\operatorname{div} u|^2 + \int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_\delta \partial_j u^i_\delta - \int_{\mathbb{T}^d} P(\rho)\operatorname{div} u\right\} + \int_{\mathbb{T}^d}\rho\,|\dot{u}|^2 \\
&= -\mu\int_{\mathbb{T}^d}\partial_k u^i \partial_k u^\ell \partial_\ell u^i + \frac{\mu}{2}\int_{\mathbb{T}^d}\left|\partial_k u^i\right|^2 \operatorname{div} u \\
&- (\mu+\lambda)\int_{\mathbb{T}^d}\operatorname{div} u\,\partial_i u^\ell \partial_\ell u^i + \frac{\mu+\lambda}{2}\int_{\mathbb{T}^d}(\operatorname{div} u)^3 \\
&+ \frac{1}{2}\int_{\mathbb{T}^d}\left\{\partial_t\varepsilon_{ijkl} + \partial_q(\varepsilon_{ijkl}u^q)\right\}\partial_j u^i_\delta \partial_k u^\ell_\delta - \int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_\delta \omega_\delta * (\partial_j u^q \partial_q u^i) - \int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_\delta\left[u^q, \omega_\delta\right]\partial^2_{qj} u^i. \\
&+ \int_0^t\int_{\mathbb{T}^d}\rho P'(\rho)\,\partial_\ell u^k \partial_k u^\ell + \int_0^t\int_{\mathbb{T}^d}\rho\dot{u}f.
\end{aligned}
\tag{3.9}
$$

3.2.2 Hoff's second energy functional

The idea leading to the construction of this second functional is to apply to the momentum equation the material time derivative $\partial_t\cdot + \operatorname{div}(u\cdot)$, multiply with $\dot{u}$ and integrate. The detailed computations are presented below. First, we obviously have that

$$
\int_{\mathbb{T}^d}\left(\partial_t(\rho\dot{u}^j) + \partial_k(\rho u^k \dot{u}^j)\right)\dot{u}^j = \frac{d}{dt}\int_{\mathbb{T}^d}\frac{\rho\,|\dot{u}|^2}{2}
$$

Next, let us deal with the pressure term. First of all, owing to the density equation we write that

$$
\partial_t P(\rho) + \operatorname{div}(P(\rho)\,u) + (\rho P'(\rho) - P(\rho))\operatorname{div} u = 0
$$

which implies that for all $j \in \overline{1,d}$ it holds true that

$$
\partial_t\partial_j P(\rho) + \operatorname{div}(\partial_j P(\rho)\,u) + \operatorname{div}(P(\rho)\,\partial_j u) + \partial_j\left\{(\rho P'(\rho) - P(\rho))\operatorname{div} u\right\} = 0.
$$

We use this relation in order to infer

$$
\begin{aligned}
\int_{\mathbb{T}^d}\left(\partial_t\partial_j P(\rho) + \partial_k(u^k\partial_j P(\rho))\right)\dot{u}^j &= -\int_{\mathbb{T}^d}\left\{\operatorname{div}(P(\rho)\,\partial_j u) + \partial_j\left\{(\rho P'(\rho) - P(\rho))\operatorname{div} u\right\}\right\}\dot{u}^j \\
&= \int_{\mathbb{T}^d}\left\{P(\rho)\,\partial_j u^k \partial_k \dot{u}^j + (\rho P'(\rho) - P(\rho))\operatorname{div} u \operatorname{div}\dot{u}\right\}.
\end{aligned}
$$

Finally, let us treat the dissipative term. We observe that

$$
\begin{aligned}
&-\left\langle \partial_t \operatorname{div}\omega_\delta * \mathcal{E}(\nabla u_\delta) + \operatorname{div}(u\operatorname{div}\omega_\delta * \mathcal{E}(\nabla u_\delta)), \dot{u}\right\rangle \\
&-\int_{\mathbb{T}^d}\partial_j\left(\partial_t\varepsilon_{ijkl}\partial_\ell u^k_\delta\right)\dot{u}^i_\delta - \int_{\mathbb{T}^d}\partial_j\left(\varepsilon_{ijkl}\partial_t\partial_\ell u^k_\delta\right)\dot{u}^i_\delta - \int_{\mathbb{T}^d}\partial_q\left(u^q\omega_\delta * \partial_j\left(\varepsilon_{ijkl}\partial_\ell u^k_\delta\right)\right)\dot{u}^i \\
&= \int_{\mathbb{T}^d}\partial_t\varepsilon_{ijkl}\partial_\ell u^k_\delta \partial_j \dot{u}^i_\delta + \int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_t\partial_\ell u^k_\delta \partial_j \dot{u}^i_\delta + \int_{\mathbb{T}^d}\partial_j\left(\varepsilon_{ijkl}\partial_\ell u^k_\delta\right)\omega_\delta * (u^q\partial_q \dot{u}^i) \\
&= \int_{\mathbb{T}^d}\partial_t\varepsilon_{ijkl}\partial_\ell u^k_\delta \partial_j \dot{u}^i_\delta
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell(\partial_t u_\delta^k+\omega_\delta*(u^q\partial_q u^k))\partial_j\dot{u}_\delta^i-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell(\omega_\delta*(u^q\partial_q u^k))\partial_j\dot{u}_\delta^i\\
&+\int_{\mathbb{T}^d}\partial_j\left(\varepsilon_{ijkl}\partial_\ell u_\delta^k\right)\omega_\delta*(u^q\partial_q\dot{u}^i)\\
&=\int_{\mathbb{T}^d}\partial_t\varepsilon_{ijkl}\partial_\ell u_\delta^k\partial_j\dot{u}_\delta^i+\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell\dot{u}_\delta^k\partial_j\dot{u}_\delta^i\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}(\omega_\delta*(\partial_\ell u^q\partial_q u^k))\partial_j\dot{u}_\delta^i-\int_{\mathbb{T}^d}\varepsilon_{ijkl}(\omega_\delta*(u^q\partial^2_{\ell q}u^k))\partial_j\dot{u}_\delta^i\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\omega_\delta*(\partial_j u^q\partial_q\dot{u}^i)-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\omega_\delta*(u^q\partial^2_{jq}\dot{u}^i).
\end{aligned}
$$

Again, integrating by parts leads to the following identity:

$$
\begin{aligned}
&-\langle\partial_t\operatorname{div}\omega_\delta*\mathcal{E}(\nabla u_\delta)+\operatorname{div}(u\operatorname{div}\omega_\delta*\mathcal{E}(\nabla u_\delta)),\dot{u}\rangle\\
&=\int_{\mathbb{T}^d}\partial_t\varepsilon_{ijkl}\partial_\ell u_\delta^k\partial_j\dot{u}_\delta^i+\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell\dot{u}_\delta^k\partial_j\dot{u}_\delta^i\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}(\omega_\delta*(\partial_\ell u^q\partial_q u^k))\partial_j\dot{u}_\delta^i-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\omega_\delta*(\partial_j u^q\partial_q\dot{u}^i)\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}([u^q,\omega_\delta*]\,\partial^2_{\ell q}u^k)\partial_j\dot{u}_\delta^i-\int_{\mathbb{T}^d}\varepsilon_{ijkl}u^q\partial^2_{\ell q}u_\delta^k\partial_j\dot{u}_\delta^i\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k([u^q,\omega_\delta*]\,\partial^2_{jq}\dot{u}^i)-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k u^q\partial^2_{jq}\dot{u}_\delta^i\\
&=\int_{\mathbb{T}^d}\left(\partial_t\varepsilon_{ijkl}+\partial_q\left(u^q\varepsilon_{ijkl}\right)\right)\partial_\ell u_\delta^k\partial_j\dot{u}_\delta^i+\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell\dot{u}_\delta^k\partial_j\dot{u}_\delta^i\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}(\omega_\delta*(\partial_\ell u^q\partial_q u^k))\partial_j\dot{u}_\delta^i-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k\omega_\delta*(\partial_j u^q\partial_q\dot{u}^i)\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}([u^q,\omega_\delta*]\,\partial^2_{\ell q}u^k)\partial_j\dot{u}_\delta^i-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u_\delta^k([u^q,\omega_\delta*]\,\partial^2_{jq}\dot{u}^i)
\end{aligned}
$$

Similar computations lead to the identity

$$
\begin{aligned}
&-\langle\partial_t\left(\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u\right)+\operatorname{div}\left(u\left(\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u\right)\right),\dot{u}\rangle\\
&=\mu\int_{\mathbb{T}^d}\left|\partial_k\dot{u}^i\right|^2+(\mu+\lambda)\int_{\mathbb{T}^d}|\operatorname{div}\dot{u}|^2\\
&-\mu\int_{\mathbb{T}^d}\partial_k u^q\partial_q u^i\partial_k\dot{u}^i-\mu\int_{\mathbb{T}^d}\partial_k u^q\partial_k u^i\partial_q\dot{u}^i+\mu\int_{\mathbb{T}^d}\operatorname{div}u\partial_k u^i\partial_k\dot{u}^i\\
&-(\mu+\lambda)\int_{\mathbb{T}^d}\partial_\ell u^q\partial_q u^\ell\operatorname{div}\dot{u}-(\mu+\lambda)\int_{\mathbb{T}^d}\partial_i u^q\partial_q\dot{u}^i\operatorname{div}u+(\mu+\lambda)\int_{\mathbb{T}^d}|\operatorname{div}u|^2\operatorname{div}\dot{u}
\end{aligned}
$$

Putting together all the above computations, we end up with

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d}\frac{\rho\left|\dot{u}\right|^2}{2}+\mu\int_{\mathbb{T}^d}\left|\partial_k\dot{u}^i\right|^2+(\mu+\lambda)\int_{\mathbb{T}^d}|\operatorname{div}\dot{u}|^2+\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell\dot{u}_\delta^k\partial_j\dot{u}_\delta^i\\
&=\mu\int_{\mathbb{T}^d}\partial_k u^q\partial_q u^i\partial_k\dot{u}^i+\mu\int_{\mathbb{T}^d}\partial_k u^q\partial_k u^i\partial_q\dot{u}^i-\mu\int_{\mathbb{T}^d}\operatorname{div}u\partial_k u^i\partial_k\dot{u}^i\\
&+(\mu+\lambda)\int_{\mathbb{T}^d}\partial_\ell u^q\partial_q u^\ell\operatorname{div}\dot{u}+(\mu+\lambda)\int_{\mathbb{T}^d}\partial_i u^q\partial_q\dot{u}^i\operatorname{div}u-(\mu+\lambda)\int_{\mathbb{T}^d}|\operatorname{div}u|^2\operatorname{div}\dot{u}
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{T}^d}\left(\partial_t\varepsilon_{ijkl}+\partial_q\left(u^q\varepsilon_{ijkl}\right)\right)\partial_\ell u^k_\delta\partial_j\dot{u}^i_\delta\\
&-\int_{\mathbb{T}^d}\varepsilon_{ijkl}(\omega_\delta*(\partial_\ell u^q\partial_q u^k))\partial_j\dot{u}^i_\delta-\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_\delta\omega_\delta*(\partial_j u^q\partial_q\dot{u}^i)\\
&+\int_{\mathbb{T}^d}\varepsilon_{ijkl}([u^q,\omega_\delta*]\,\partial^2_{\ell q}u^k)\partial_j\dot{u}^i_\delta+\int_{\mathbb{T}^d}\varepsilon_{ijkl}\partial_\ell u^k_\delta([u^q,\omega_\delta*]\,\partial^2_{jq}\dot{u}^i)\\
&-\int_{\mathbb{T}^d}\left\{P\left(\rho\right)\partial_j u^k\partial_k\dot{u}^j+\left(\rho P'\left(\rho\right)-P\left(\rho\right)\right)\operatorname{div}u\operatorname{div}\dot{u}\right\}.
\end{aligned}
$$

We multiply the above with $\sigma\left(t\right)$ such that we obtain

$$
\begin{aligned}
&\sigma\left(t\right)\int_{\mathbb{T}^d}\frac{\rho\left(t\right)\left|\dot{u}\left(t\right)\right|^2}{2}+\mu\int_{\mathbb{T}^d}\sigma\left|\partial_k\dot{u}^i\right|^2+\left(\mu+\lambda\right)\int_{\mathbb{T}^d}\sigma\left|\operatorname{div}\dot{u}\right|^2+\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell\dot{u}^k_\delta\partial_j\dot{u}^i_\delta\\
&=\int_0^1\int_{\mathbb{T}^d}\sigma\frac{\rho\left|\dot{u}\right|^2}{2}\\
&+\mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_q u^i\partial_k\dot{u}^i+\mu\int_0^t\int_{\mathbb{T}^d}\sigma\partial_k u^q\partial_k u^i\partial_q\dot{u}^i-\mu\int_0^t\int_{\mathbb{T}^d}\sigma\operatorname{div}u\partial_k u^i\partial_k\dot{u}^i\\
&+\left(\mu+\lambda\right)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_\ell u^q\partial_q u^\ell\operatorname{div}\dot{u}+\left(\mu+\lambda\right)\int_0^t\int_{\mathbb{T}^d}\sigma\partial_i u^q\partial_q\dot{u}^i\operatorname{div}u-\left(\mu+\lambda\right)\int_0^t\int_{\mathbb{T}^d}\sigma\left|\operatorname{div}u\right|^2\operatorname{div}\dot{u}\\
&-\int_0^t\int_{\mathbb{T}^d}\sigma\left(\partial_t\varepsilon_{ijkl}+\partial_q\left(u^q\varepsilon_{ijkl}\right)\right)\partial_\ell u^k_\delta\partial_j\dot{u}^i_\delta-\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}(\omega_\delta*(\partial_\ell u^q\partial_q u^k))\partial_j\dot{u}^i_\delta-\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta\omega_\delta*(\partial_j u^q\partial_q\dot{u}^i)\\
&+\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}([u^q,\omega_\delta*]\,\partial^2_{\ell q}u^k)\partial_j\dot{u}^i_\delta+\int_0^t\int_{\mathbb{T}^d}\sigma\varepsilon_{ijkl}\partial_\ell u^k_\delta([u^q,\omega_\delta*]\,\partial^2_{jq}\dot{u}^i)\\
&-\int_0^t\int_{\mathbb{T}^d}\sigma\left\{P\left(\rho\right)\partial_j u^k\partial_k\dot{u}^j+\left(\rho P'\left(\rho\right)-P\left(\rho\right)\right)\operatorname{div}u\operatorname{div}\dot{u}\right\}.
\end{aligned}
$$

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