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► **To cite this version:**

Melanie Bertelson, Gael Meigniez. Conformal Symplectic Structures, Foliations and Contact Structures. 2021. hal-03341749

HAL Id: hal-03341749

<https://hal.science/hal-03341749>

Preprint submitted on 12 Sep 2021

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CONFORMAL SYMPLECTIC STRUCTURES, FOLIATIONS AND CONTACT STRUCTURES

MÉLANIE BERTELSON, GAËL MEIGNIEZ

ABSTRACT. This paper presents two existence h-principles, the first for conformal symplectic structures on closed manifolds, and the second for leafwise conformal symplectic structures on foliated manifolds with non empty boundary. The latter h-principle allows to linearly deform certain codimension-1 foliations to contact structures. These results are essentially applications of the Borman-Eliashberg-Murphy h-principle for overtwisted contact structures and of the Eliashberg-Murphy symplectization of cobordisms, together with tools pertaining to foliated Morse theory, which are elaborated here.

1. INTRODUCTION

In high dimensions, symplectic and contact topology require modern methods different from those effective in dimension three. In the present paper, we essentially explore some consequences of the Eliashberg-Murphy symplectization of cobordisms [EM] together with the Borman-Eliashberg-Murphy h-principle for overtwisted contact structures [BEM]. Also, one needs some tools falling to the Morse theory of codimension-one foliations, which we elaborate.

1.1. Existence of conformal symplectic structures. On a manifold M , a *conformal symplectic structure* is a conformal class of nondegenerate 2-forms, admitting a local symplectic representant in a neighborhood of every point. This generalization of symplectic geometry is classical (see for example [V85], [Ba02] and [CM]). To such a structure, one associates its *Lee class*: a real cohomology class of degree 1 on M , which is the obstruction to finding a global symplectic representant.

Theorem A in Section 3 is an existence h-principle for conformal symplectic structures on closed manifolds whose Lee class is any nonzero de Rham cohomology class of degree one. The only case excluded is

Date: September 7, 2021.

2010 Mathematics Subject Classification. 57R30, 57R17, 53D05, 53D10.

Key words and phrases. conformal symplectic, symplectic cobordism, foliation, contact, h-principle.

the one that would yield a genuine symplectic structure. We thus generalize a result obtained by Eliashberg-Murphy for Lee classes of rank one over \mathbb{Z} (Theorem 1.8 in [EM]).

1.2. Making foliated cobordisms conformally symplectic. The data are a cobordism $(W, \partial_-W, \partial_+W)$ endowed with a codimension-1 coorientable taut foliation \mathcal{F} , whose leaves all meet both ∂_-W and ∂_+W transversely, and with a leafwise closed 1-form η . Theorem *B* in Section 5 is an existence h-principle for leafwise conformal symplectic structures whose Lee class in every leaf is the cohomology class of η ; the boundary component ∂_-W (respectively ∂_+W) being leafwise convex (resp. concave) in a certain sense. In particular, the leaves of $\mathcal{F}|_{\partial_-W}$ (respectively $\mathcal{F}|_{\partial_+W}$) are positive (resp. negative) contact submanifolds of W .

Two tools belonging to the Morse theory of codimension-1 foliations are essential here. A real function f on W is called *leafwise Morse* if its restriction to every leaf of \mathcal{F} is a Morse function in the leaf. The tools in question are the construction of *ordered* leafwise Morse functions, and a cancellation method for leafwise local extrema, analogous to the Cerf cancellation of local extrema in 1-parametric families of functions. The first tool also reproves the existence of faithful submanifolds in taut codimension-1 foliations [M16]. Also, these tools show that for any taut codimension-1 foliation on a compact manifold with boundary, if all the leaves meet the boundary transversely and non-trivially, then this foliation is uniformly open in the sense of [Be02]; hence, the h-principle for open leafwise invariant differential relations is verified for such foliations.

1.3. Deforming foliations to contact structures. Given a codimension-1 foliation \mathcal{F} on a manifold M , defined by a non-vanishing 1-form α , a *linear contactizing deformation* is a 1-parameter family $\alpha_t = \alpha + t\lambda$ of 1-forms such that α_t is contact for all sufficiently small positive t . These deformations are completely understood in dimension 3 (see [ET]), but not much is known in higher dimensions.

A linear contactizing deformation is provided by any *exact* leafwise conformal symplectic structure whose Lee class is precisely the linear holonomy of the foliation. Here, “exact” refers to the Lichnerowicz differential in the leaves. Indeed, the deformation $\alpha + t\lambda$ is contact for all sufficiently small positive t 's if $d\lambda - \eta \wedge \lambda$ is leafwise nondegenerate. Moreover, such leafwise conformal structure exist (Theorem *C* in Section 6) for a large class of foliations on cobordisms, which we call *holonomous*. Although the leafwise *convex* boundary may be empty,

the leafwise concave one cannot. Unfortunately, it seems that our construction is not able to produce contactizing deformations on closed manifolds.

Acknowledgments This work was carried out partly while the second author was the host of the Mathematics Research Center in Stanford, and partly while both authors were hosts of the *Forschungsinstitut für Mathematik* in ETH Zürich. We thank both institutions. The first author is supported by the *Excellence of Sciences* project *Symplectic Techniques in Differential Geometry* jointly funded by the F.R.S.-FNRS and the FWO. It is a pleasure to thank Yasha Eliashberg for his observations and support.

NOTATION 1.1. One denotes hereafter by \mathbf{I} the compact interval $[0, 1]$, by $\mathbf{D}^d \subset \mathbb{R}^d$ the unit compact ball, and by \mathbf{S}^{d-1} the sphere $\partial\mathbf{D}^d$. Every foliation \mathcal{F} on a manifold M with boundary is assumed to be transverse to the boundary, inducing therefore a foliation $\partial\mathcal{F} = \mathcal{F}|_{\partial M}$ on ∂M . The notation $\mathcal{O}_{p_X}(Y)$, or $\mathcal{O}_p(Y)$ if X is understood, holds for “some open neighborhood of Y in X ”.

2. ELEMENTS OF CONFORMAL SYMPLECTIC GEOMETRY

Let η be a closed 1-form on a manifold M of dimension $2n \geq 2$. The *Lichnerowicz* (also known as *Novikov*) *differential with respect to* η , denoted by d_η , is the ordinary Cartan differential somehow twisted by η ; precisely, for every differential form $\theta \in \Omega^*(M)$:

$$d_\eta\theta = d\theta - \eta \wedge \theta$$

It is immediately verified that $d_\eta^2 \equiv 0$. The cohomology of the differential operator d_η is called the *Novikov cohomology of M with respect to η* and denoted by $H_\eta^*(M)$.

REMARK 2.1. The Novikov cohomology only depends, up to isomorphism, on the de Rham cohomology class of η . Precisely, when $\eta' - \eta = dF$ for some smooth function F on M , the differential complexes $(\Omega^*(M), d_{\eta'})$ and $(\Omega^*(M), d_\eta)$ are isomorphic through a conformal rescaling of the differential forms :

$$d_{\eta'}\theta = e^F d_\eta(e^{-F}\theta).$$

(Beware that the resulting isomorphism between $H_\eta^*(M)$ and $H_{\eta'}^*(M)$ is not canonical, depending on the choice of F .)

We have not found in the litterature the following generalization, which we shall need further down, of the Poincaré Lemma.

LEMMA 2.2. *Let M' be a manifold, $h : M' \times \mathbf{I} \rightarrow M$ be a smooth map. Define*

$$F : M' \times \mathbf{I} \rightarrow \mathbb{R} : (x, t) \mapsto \int_0^t \eta\left(\frac{\partial h}{\partial t}(x, \tau)\right) d\tau$$

$$\mathcal{H} : \Omega^*(M) \rightarrow \Omega^{*-1}(M') : \theta \mapsto \int_0^1 e^{-F} \iota_{\partial/\partial t}(h^*(\theta)) dt$$

Consider on M' the function $F_1 : x \mapsto F(x, 1)$, and for every $t \in \mathbf{I}$ the map $h_t : x \mapsto h(x, t)$, and the 1-form $\eta'_0 = h_0^(\eta)$.*

Then:

i) $e^{-F_1} h_1^(\theta) - h_0^*(\theta) = d_{\eta'_0} \mathcal{H}(\theta) + \mathcal{H}(d_\eta \theta)$;*

ii) The morphisms $\theta \mapsto h_0^(\theta)$ and $\theta \mapsto e^{-F_1} h_1^*(\theta)$ induce the same morphism in Novikov cohomology*

$$H_\eta^*(M) \rightarrow H_{\eta'_0}^*(M')$$

Proof of Lemma 2.2. Let $\theta \in \Omega^*(M)$. On $M' \times \mathbf{I}$, consider the forms $\eta' = h^*(\eta)$ and $\theta' = e^{-F} h^*(\theta)$. Writing for short ι_t instead of $\iota_{\partial/\partial t}$, developpe the Cartan formula:

$$\begin{aligned} \mathcal{L}_{\partial/\partial t}(\theta') &= d\iota_t(\theta') + \iota_t(d\theta') \\ &= e^{-F} (d\iota_t(h^*\theta) - dF \wedge \iota_t(h^*\theta) + \iota_t(d(h^*\theta)) - \iota_t(dF \wedge h^*\theta)) \end{aligned}$$

Since moreover

$$\iota_t(dF) = \frac{\partial F}{\partial t} = \eta\left(\frac{\partial h}{\partial t}\right) = \iota_t(\eta')$$

defining $H(\theta) = e^{-F} \iota_t(h^*\theta)$, one gets

$$\mathcal{L}_{\partial/\partial t}(\theta') = d_{\eta'} H(\theta) + H(d_\eta \theta)$$

Of course, (i) follows by restriction to the slices $M' \times t$ and integration with respect to t ; then, (ii) follows immediately from (i). \square

If M has a boundary, the *relative* Novikov cohomology $H_\eta^*(M, \partial M)$ with respect to η is defined, as usual, as the cohomology of

$$\Omega^*(M) \times \Omega^{*-1}(\partial M)$$

under the differential operator

$$(1) \quad D_\eta : (\theta, \theta') \mapsto (d_\eta \theta, \theta|_{\partial M} - d_\eta \theta')$$

A number of standard notions in symplectic geometry admit obvious generalizations to conformal symplectic geometry: one simply replaces the Cartan operator d by its twisted version d_η .

DEFINITION 2.3. A 2-form ω on M is η -*symplectic* if ω is d_η -closed and nondegenerate.

Every η -symplectic form ω defines a conformal symplectic structure on M . Indeed, after Remark 2.1, for every open subset U of M on which η admits a primitive F , the 2-form $e^{-F}\omega$ is closed, and hence genuinely symplectic on U .

Conversely, for every nondegenerate 2-form ω representing a conformal symplectic structure, one has a closed 1-form η on M whose local primitives F satisfy $d(e^{-F}\omega) = 0$. In other words, ω is d_η -closed. If $n \geq 2$, then η is uniquely defined by ω , and called *the Lee form of ω* .

DEFINITION 2.4. A 1-form λ on M is η -Liouville if $d_\eta\lambda$ is nondegenerate. Its η -Liouville vector field, or η -dual, $Z = Z_\lambda$, is defined by the relation

$$\lambda = \iota_Z(d_\eta\lambda).$$

If moreover M is oriented, λ is of course called *positive* if the volume form $(d_\eta\lambda)^n$ defines the given orientation; and *negative* otherwise.

Notice that, in any open subset of M where η admits a primitive F , the η -Liouville vector field Z is nothing but the ordinary Liouville vector field of the ordinary Liouville form $e^{-F}\lambda$. In other words, the dual Liouville vector field is invariant by conformal equivalence of conformally Liouville forms. Besides, after Cartan's formula :

$$(2) \quad \begin{aligned} \mathcal{L}_Z\lambda &= \iota_Z(d\lambda) = \iota_Z(d_\eta\lambda + \eta \wedge \lambda) = (1 + \eta(Z))\lambda \\ \mathcal{L}_Z(d_\eta\lambda) &= d\mathcal{L}_Z\lambda - (\mathcal{L}_Z\eta) \wedge \lambda - \eta \wedge \mathcal{L}_Z\lambda = (1 + \eta(Z))d_\eta\lambda. \end{aligned}$$

Actually, the second equation is equivalent to Z being a η -Liouville vector field for ω ; and the first one too, under the hypothesis $\lambda(Z) = 0$.

REMARK 2.5. After Remark 2.1, the existence of a η -Liouville 1-form (respectively η -symplectic 2-form) does not depend on the choice of the form η in its de Rham cohomology class. Precisely, if λ (respectively ω) is a η -Liouville 1-form (respectively η -symplectic 2-form), then for any function F on M , the form $e^F\lambda$ (respectively $e^F\omega$) is $(\eta + dF)$ -Liouville (respectively $(\eta + dF)$ -symplectic).

REMARK 2.6. One can alternatively interpret a η -symplectic (respectively η -Liouville) form on M as a genuinely symplectic (respectively Liouville) and *equivariant* form on the abelian covering of M defined by η , or on the universal cover. This alternative viewpoint does not seem to be the most efficient for the present paper.

DEFINITION 2.7. Let ω be a η -symplectic form on M and $H \subset M$ be a cooriented hypersurface. As usual, one orients H by the volume form $\iota_X(\omega^n)$, where X is transverse to H and positive (with respect to the coorientation). One calls H of *convex* (respectively *concave*) *contact*

type with respect to ω if H is transverse to a positive (resp. negative) η -Liouville vector field Z defined near H .

Even contact structures will play a crucial role in our construction. Recall that they are defined as maximally non-integrable hyperplane fields on even-dimensional manifolds. We will assume that the ambient manifold M is $2n$ -dimensional ($n \geq 1$) and oriented, and that the even contact structure ε is cooriented, so that $\varepsilon = \ker \lambda$ for some non-vanishing 1-form λ defining the coorientation (*even contact form*). The maximal non-integrability of ε is equivalent to the non-vanishing of $\lambda \wedge d\lambda^{n-1}$. The dimension-1-foliation \mathcal{Z} spanned by the rank-1 distribution $T\mathcal{Z} = \ker(d\lambda|_\varepsilon)$ is called the *characteristic foliation* of ε . The foliation \mathcal{Z} is transversely contact: ε is invariant by any vector field tangential to \mathcal{Z} , and $\lambda \wedge d\lambda^{n-1}$ defines a volume form on $TM/T\mathcal{Z}$. We fix an orientation for \mathcal{Z} , namely, a section Z of $T\mathcal{Z}$ is positive if Z followed by a basis of $TM/T\mathcal{Z}$ which is positive with respect to $\lambda \wedge d\lambda^{n-1}$, makes a positive basis of M .

Let R be any vector field on M such that $\lambda(R) \equiv 1$, let Z be any positive section of $T\mathcal{Z}$; choose any 1-form θ on M such that the function $\theta(Z) + d\lambda(Z, R)$ is positive on M . Equivalently, the 1-form $\theta - \iota_R(d\lambda)$ is positive on \mathcal{Z} . Then, the 2-form

$$(3) \quad \omega_\theta = \theta \wedge \lambda + d\lambda$$

is positive nondegenerate. Indeed,

$$\omega_\theta^n = (\theta \wedge \lambda + d\lambda)^n = n\theta \wedge \lambda \wedge d\lambda^{n-1} + d\lambda^n$$

is positive, since

$$\iota_R(\iota_Z(\omega_\theta^n)) = n(\theta(Z) + d\lambda(Z, R))d\lambda^{n-1}$$

induces a positive form on $\varepsilon/T\mathcal{Z}$, as a consequence of our choice of orientation for \mathcal{Z} . The homotopy class of nondegenerate 2-forms represented by ω_θ depends only on ε and of its coorientation, and not on the choices of the forms λ nor θ . Call this class the *almost symplectic class associated to ε* .

Finally, fix a positively oriented nonsingular section Z of $T\mathcal{Z}$. Since ε is Z -invariant, for every defining form λ , one has a unique function χ_λ on M such that

$$\mathcal{L}_Z \lambda = \chi_\lambda \lambda.$$

If one likes better,

$$(4) \quad \chi_\lambda = d\lambda(Z, R) = -\lambda([Z, R])$$

where R is any vector field on M such that $\lambda(R) = 1$. Clearly, for every function u on M :

$$(5) \quad \chi_{e^{u\lambda}} = \chi_\lambda + Z(u).$$

The method used in the present paper to build an η -Liouville form consists of two steps, whose details will be given in Lemmas 2.11 and 2.12. We use an auxiliary ambient codimension-1 foliation \mathcal{F} . Essentially, first, the h-principle for overtwisted contact structures yields an even contact structure ε whose characteristic foliation is transverse to \mathcal{F} . Second, a η -Liouville form is found among the 1-forms representing ε .

REMARK 2.8. McDuff's early h-principle for even contact structures ([McD87], Proposition 7.2) does unfortunately not seem to allow such control on the characteristic foliation.

DEFINITION 2.9. Given a codimension-1 foliation \mathcal{F} on a manifold M , recall the *leafwise* (also known as “foliated”, or “tangential”) differential forms: $\Omega^k(\mathcal{F})$ stands for the collection of the smooth sections of $\Lambda^k(T\mathcal{F})$. For $\theta \in \Omega^*(M)$, one has the restriction $\theta|_{\mathcal{F}} \in \Omega^*(\mathcal{F})$. For $\theta \in \Omega^*(\mathcal{F})$, one has the *leafwise Cartan differential* $d_{\mathcal{F}}\theta \in \Omega^{*+1}(\mathcal{F})$, such that $d_{\mathcal{F}}(\theta|_{\mathcal{F}}) = (d\theta)|_{\mathcal{F}}$. The differential operator $d_{\mathcal{F}}$ on $\Omega^*(\mathcal{F})$ defines the leafwise (also known as “foliated”) cohomology

$$H^*(\mathcal{F}) = \ker(d_{\mathcal{F}})/\text{Im}(d_{\mathcal{F}})$$

A leafwise 1-form $\alpha \in \Omega^1(\mathcal{F})$ is *contact* if the leafwise $(2n-1)$ -form $\alpha \wedge (d_{\mathcal{F}}\alpha)^{n-1}$ does not vanish, where $\dim(M) = 2n$.

An *almost contact structure* on a manifold Σ of dimension $2n-1$ is a pair $(\alpha, \varpi) \in \Omega^1(\Sigma) \times \Omega^2(\Sigma)$ such that $\alpha \wedge \varpi^{n-1}$ does not vanish.

In the same way, a *leafwise almost contact structure* on \mathcal{F} is a pair $(\alpha, \varpi) \in \Omega^1(\mathcal{F}) \times \Omega^2(\mathcal{F})$ such that the leafwise $(2n-1)$ -form $\alpha \wedge \varpi^{n-1}$ does not vanish.

REMARK 2.10. In a real vector space E of dimension $2n$, given a codimension-1 vector subspace $\Sigma \subset E$ and a vector $Z \in E$ not in Σ , let θ be the linear form of kernel Σ such that $\theta(Z) = 1$, and let $\pi : E \rightarrow \Sigma$ be the linear projection parallel to Z . Then, the linear mappings

$$\Lambda^2(E) \rightarrow \Lambda^1(\Sigma) \times \Lambda^2(\Sigma) : \omega \mapsto (\iota_Z(\omega)|_{\Sigma}, \omega|_{\Sigma})$$

$$\Lambda^1(\Sigma) \times \Lambda^2(\Sigma) \rightarrow \Lambda^2(E) : (\alpha, \varpi) \mapsto \theta \wedge \pi^*(\alpha) + \pi^*(\varpi)$$

are reciprocal isomorphisms. Moreover, $\omega^n \neq 0$ if and only if the corresponding pair (α, ϖ) satisfies $\alpha \wedge \varpi^{n-1} \neq 0$.

Consequently, given a cooriented codimension-1 foliation \mathcal{F} on the $(2n)$ -manifold M , choose a nonvanishing 1-form θ on M defining \mathcal{F} and

a vector field Z such that $\theta(Z) = 1$; and denote by π the projection $TM \rightarrow T\mathcal{F}$ parallel to Z . Then, the linear mappings

$$\Omega^2(M) \rightarrow \Omega^1(\mathcal{F}) \times \Omega^2(\mathcal{F}) : \omega \mapsto (\iota_Z(\omega)|_{\mathcal{F}}, \omega|_{\mathcal{F}})$$

$$\Omega^1(\mathcal{F}) \times \Omega^2(\mathcal{F}) \rightarrow \Omega^2(M) : (\alpha, \varpi) \mapsto \theta \wedge \pi^*(\alpha) + \pi^*(\varpi)$$

are reciprocal isomorphisms. Moreover, ω is an almost symplectic structure on M if and only if the corresponding pair (α, ϖ) is a leafwise almost contact structure on \mathcal{F} .

LEMMA 2.11. (i) *On a manifold M , let ω be a nondegenerate 2-form, and \mathcal{F} be a cooriented codimension-1 foliation.*

Then, M admits an even contact structure ε such that

- ω lies in the almost symplectic class associated with ε ;
- The characteristic foliation of ε is positively transverse to \mathcal{F} ;
- On every leaf L of \mathcal{F} , the contact structure $\varepsilon \cap L$ is overtwisted.

(ii) *Moreover, given a closed subset $A \subset M$, assume that an even contact structure ε_A is already given on some open neighborhood U of A , such that*

- $\omega|_U$ lies in the almost symplectic class associated with ε_A ;
- The characteristic foliation of ε_A is positively transverse to $\mathcal{F}|_U$;
- On every leaf L of \mathcal{F} which is entirely contained in U , if any, the contact structure $\varepsilon_A \cap L$ is overtwisted.

Then, one can choose ε to coincide with ε_A on some smaller neighborhood of A .

Note — For the case $\dim(M) = 2$: one agrees that any nonsingular 1-form on a 1-manifold is an *overtwisted* contact form on this manifold.

Proof of Lemma 2.12. (i) Choose a vector field Z over M , positively transverse to \mathcal{F} . The ambient nondegenerate 2-form ω induces on \mathcal{F} a leafwise almost contact structure $(\iota_Z(\omega)|_{\mathcal{F}}, \omega|_{\mathcal{F}})$ (Remark 2.10). After the h-principle for overtwisted contact structures on foliations ([BEM], Theorem 1.5), M carries a leafwise contact structure $\xi \subset T\mathcal{F}$ (a cooriented $(2n - 2)$ -plane field defining on every leaf a contact structure) which is overtwisted in every leaf, and homotopic to $(\iota_Z(\omega)|_{\mathcal{F}}, \omega|_{\mathcal{F}})$ as a leafwise almost contact structure. Let ε be on M the hyperplane field spanned by ξ and Z , and λ be a nonvanishing 1-form defining ε and its coorientation.

We claim that there is a unique vector field X over M , contained in ξ , and such that the flow (ϕ^t) of $Z' = Z + X$ preserves ε .

Indeed, this foliated version of the Gray stability theorem can be proved, like the classical one, by a Moser-type argument:

The condition $(\phi_t)^*(\varepsilon) = \varepsilon$ amounts to $\varphi_t^*\lambda = g_t\lambda$, for some smooth family of positive functions (g_t) . Derivating with respect to t yields :

$$\phi_t^*(\mathcal{L}_{Z'}\lambda) = \dot{g}_t\lambda,$$

which implies that

$$(6) \quad \iota_{Z'}(d\lambda) = G_t\lambda$$

where $G_t = (\dot{g}_t/g_t) \circ \phi_t^{-1}$. Let R be the vector field on M tangential to \mathcal{F} , and which is, in every leaf L of \mathcal{F} , the Reeb vector field of the contact form $\lambda|_L$. Since $Z' = Z + X$, we can rewrite (6) as a system of two equations :

$$\begin{cases} \iota_X(d\lambda)|_\xi = -\iota_Z(d\lambda)|_\xi \\ d\lambda(Z, R) = G \end{cases}$$

The first one uniquely determines X since λ is leafwise contact; the second one determines G . The claim is proved.

Clearly, ε is an even contact structure whose characteristic foliation is positively spanned by Z' . The almost symplectic class associated with ε does coincides with that of ω , thanks to Equation (3) and Remark 2.10.

(ii) The plane field $\xi_A = \varepsilon_A \cap T\mathcal{F}$ is contact in every leaf of $\mathcal{F}|_U$. After the unicity up to isotopy of overtwisted contact structures on foliations ([BEM], Theorem 1.6), after pushing ξ by an isotopy of M tangential to \mathcal{F} , we can arrange that $\xi = \xi_A$ on $\mathcal{O}p_M(A)$. Then, choose Z so that it spans positively the characteristic foliation of ε_A on $\mathcal{O}p_M(A)$. Hence, $\varepsilon = \varepsilon_A$ on $\mathcal{O}p_M(A)$. After the local unicity property of X , one has $X = 0$ on $\mathcal{O}p_M(A)$. Hence, $Z' = Z$ on $\mathcal{O}p_M(A)$. \square

LEMMA 2.12. *On the even-dimensional oriented manifold M , let η be a closed 1-form, and let λ be a non-vanishing 1-form.*

i) Assume that λ is η -Liouville. Then, λ yields an even contact structure $\ker \lambda$, whose characteristic foliation is positively spanned by the η -Liouville vector field Z_λ , and whose associated almost symplectic class is represented by $d_\eta\lambda$.

ii) Conversely, assume that λ defines an even contact structure. Fix a positive section Z of the characteristic foliation. Then, λ is η -Liouville positive if and only if $\chi_\lambda > \eta(Z)$. Moreover, if so, the η -Liouville vector field Z_λ η -dual to λ coincides with $(\chi_\lambda - \eta(Z))^{-1}Z$.

Proof. Let R be a vector field on M such that $\lambda(R) \equiv 1$.

(i) Since λ is η -Liouville, the $(2n - 1)$ -form

$$n\lambda \wedge (d\lambda)^{n-1} = n\lambda \wedge (d_\eta\lambda)^{n-1} = \iota_{Z_\lambda}(d_\eta\lambda)^n$$

does not vanish, hence $\ker \lambda$ is an even contact structure. Clearly, Z_λ positively spans the characteristic foliation. Put $\theta = -\eta$. Then,

$$d_\eta \lambda(Z_\lambda, R) = \iota_{Z_\lambda}(d_\eta \lambda)(R) = \lambda(R) = 1$$

$$(\eta \wedge \lambda)(Z_\lambda, R) = \eta(Z_\lambda)$$

Hence, $\theta(Z_\lambda) + d\lambda(Z_\lambda, R) = 1$ is positive; while $\omega_\theta = d_\eta \lambda$ (recall Equation (3)).

(ii) The vector field Z spanning positively the characteristic foliation \mathcal{L} , the 1-form λ is η -Liouville positive if and only if $\iota_R(\iota_Z((d_\eta \lambda)^n))$ induces a positive form on ε/\mathcal{L} . This condition amounts to the positivity of the function $\chi_\lambda - \eta(Z)$, since

$$\iota_R(\iota_Z((d_\eta \lambda)^n)) = \iota_R(\iota_Z(d\lambda^n - n\eta \wedge \lambda \wedge d\lambda^{n-1})) = n(\chi_\lambda - \eta(Z)) \wedge d\lambda^{n-1}$$

and since $d\lambda^{n-1}$ induces a positive volume form on ε/\mathcal{L} . Finally, the value of Z_λ results from the computation:

$$\iota_Z(d_\eta \lambda) = \mathcal{L}_Z \lambda - \iota_Z(\eta \wedge \lambda) = (\chi_\lambda - \eta(Z)) \lambda.$$

□

As a first application of Lemma 2.12 (ii), one has for conformal symplectic structures an obvious cut-and-paste method along hypersurfaces of contact type (Definition 2.7), generalizing the classical method for genuine symplectic structures. The following lemma gives some precisions about these hypersurfaces.

LEMMA 2.13. *Let ω be a η -symplectic form on M , and $H \subset M$ be a cooriented hypersurface.*

i) If H is transverse to a η -Liouville vector field Z for ω , then $\lambda = \iota_Z(\omega)$ is a (d_η) -primitive of ω near H , and $\alpha = \lambda|_H$ is a contact form on H .

ii) Conversely, if ω admits in restriction to H a (d_η) -primitive α which is contact, then α extends to a (d_η) -primitive λ of ω on $\mathcal{O}p_M(H)$ whose η -dual vector field Z is transverse to H .

iii) Moreover, in (i) and (ii), Z is positively (resp. negatively) transverse to H iff α is positive (resp. negative) as a contact form on H .

We call H of *overtwisted* contact type if moreover $\ker \alpha$ is overtwisted on each connected component of H .

Proof. (i): after Lemma 2.12 (i).

(ii): To get the extension λ , consider a tubular neighborhood T of H in M and a deformation retraction $h = (h_t) : T \times \mathbf{I} \rightarrow T$ such

that $h_0 = \text{id}_T$ and $h_1(T) = H$ and $h_t|_H = \text{id}_H$. After the generalized Poincaré lemma 2.2 applied to ω ,

$$e^{-F_1} h_1^*(\omega) - \omega = d_\eta \mathcal{H}(\omega)$$

On the other hand, after the ordinary Poincaré lemma applied to η ,

$$h_1^*(\eta) - \eta = dF_1$$

so, using Remark 2.1:

$$\begin{aligned} e^{-F_1} h_1^*(\omega) &= e^{-F_1} h_1^*(d\alpha - \eta \wedge \alpha) = e^{-F_1} (dh_1^*(\alpha) - h_1^*(\eta) \wedge h_1^*(\alpha)) \\ &= e^{-F_1} d_{h_1^*(\eta)} h_1^*(\alpha) = d_\eta (e^{-F_1} h_1^*(\alpha)) \end{aligned}$$

Hence,

$$\lambda = e^{-F_1} h_1^*(\alpha) - \mathcal{H}(\omega)$$

is a (d_η) -primitive of ω on T ; and coincides with α in restriction to H , since F_1 and $\mathcal{H}(\omega)$ vanish identically on H .

Finally, $\lambda|_H = \alpha$ being positive (resp. negative) contact, after Lemma 2.12 (i), Z is transverse to H .

(iii) Obvious from Lemma 2.12 (i). \square

3. EXISTENCE OF CONFORMAL SYMPLECTIC STRUCTURES

THEOREM A. *On a closed connected manifold M of dimension $2n \geq 2$, let η be a closed, non-exact 1-form; and let ω be a nondegenerate 2-form.*

Then, ω is homotopic to a η -symplectic form, whose Novikov cohomology class in $H_\eta^2(M)$ may be prescribed.

When moreover the cohomology class of η is integral: $[\eta] \in H^1(M; \mathbb{Z})$, Eliashberg-Murphy have already obtained [EM] that, for some constant $c \neq 0$, the manifold M admits a $(c\eta)$ -symplectic form homotopic to ω as a nondegenerate 2-form.

We choose to prove Theorem A with by means of an auxiliary Morse function. One could, alternatively, use a handle decomposition; but the Morse function method adapts painlessly to the foliated framework (see Section 5).

One begins by endowing the compact solid torus

$$T^{2n} = \mathbf{S}^1 \times \mathbf{D}^{2n-1}$$

with a conformal symplectic structure inducing an overtwisted contact structure on its boundary, either concave or convex. These conformal symplectic tori will further down somehow play the role of ‘‘symplectic cap’’ and ‘‘symplectic cup’’ in the proof of Theorem A.

Denote by θ the pullback to T^{2n} of the volume form on \mathbf{S}^1 with total volume one, and endow T^{2n} with an arbitrary orientation.

LEMMA 3.1. *For every integer $n \geq 1$, there exist a real constant c_n , and on T^{2n} a 1-form λ and a vector field Z such that*

- (1) $\theta(Z) = 1$, and Z exits or enters (at choice) T^{2n} transversely through ∂T^{2n} ;
- (2) λ is, for every real $c > c_n$, a positive $(-c\theta)$ -Liouville form, whose $(-c\theta)$ -dual vector field is positively colinear to Z ;
- (3) λ restricts on ∂T^{2n} to an overtwisted contact form.

Proof. Fix a compact collar neighborhood A of the boundary ∂T^{2n} in T^{2n} ; and a vector field Z on A such that $\theta(Z) = 1$, and which exits or enters T^{2n} transversely through ∂T^{2n} , at choice. The parallelizable solid torus T^{2n} bears a positive nondegenerate 2-form ω . Consider on ∂T^{2n} the induced almost contact structure $(\iota_Z(\omega)|_{\partial T^{2n}}, \omega|_{\partial T^{2n}})$ (Remark 2.10). After the h-principle for overtwisted contact structures [BEM], ∂T^{2n} admits an overtwisted contact structure ξ in the same formal homotopy class as $(\iota_Z(\omega)|_{\partial T^{2n}}, \omega|_{\partial T^{2n}})$.

Shrinking A if necessary, let ε be the even contact structure on A pullback of ξ under the projection $A \rightarrow \partial T^{2n}$ along the flow lines of Z . Applying Lemma 2.11 to the slice foliation of T^{2n} by the disks $t \times \mathbf{D}^{2n-1}$ ($t \in \mathbf{S}^1$), to A , and to ω , extend ε to a global even contact structure on T^{2n} , still called ε , whose characteristic foliation \mathcal{F} is transverse to the disks, and spanned by Z on A . Extend Z to a global vector field on T^{2n} , still called Z , spanning \mathcal{F} , and such that $\theta(Z) = 1$.

Choose any 1-form λ on T^{2n} representing ε . Let c_n be the maximum of the function $-\chi_\lambda$ on T^{2n} . For $c > c_n$, one has $(-c\theta)Z < \chi_\lambda$. After Lemma 2.12 (ii), the 1-form λ is $(-c\theta)$ -Liouville positive on T^{2n} , and its $(-c\theta)$ -dual vector field is positively colinear to Z . \square

REMARK 3.2. For $n > 1$, the constant c_n given by Lemma 3.1 is necessarily non-negative. Indeed, assume by contradiction that $c_n < 0$. Fix a negative $c > c_n$. Then, after properties (1) and (2) and after the proof of Lemma 2.12, (i), the function

$$d\lambda(Z_\lambda, R) = 1 - c\theta(Z_\lambda)$$

is > 1 on T^{2n} . After Equation (4), the function χ_λ is positive on T^{2n} . So, λ is a positive genuinely Liouville form on T^{2n} , whose corresponding Liouville vector field is positively colinear to Z . In particular, Z has to exit the solid torus; and the genuinely symplectic form $d\lambda$ fills the overtwisted contact structure ξ : a contradiction.

Consequently, Lemma 3.1 lacks some symmetry: it would not hold if one changed $-c\theta$ to $c\theta$ in property (2). Equivalently, one cannot

change $\theta(Z) = 1$ to $\theta(Z) = -1$ in property (1). This lack of symmetry is linked to the inability of our method to deform foliations to contact structures on *closed* manifolds, see Remark 6.5.

REMARK 3.3. In Lemma 3.1, there is no need to explicitly prescribe the homotopy class of $d_\eta\lambda$ among the positive nondegenerate 2-forms on T^{2n} ; indeed, there is only one such class, $\mathrm{SO}(2n)/\mathrm{U}(n)$ being simply connected.

Proof of Theorem A. The closed 1-form η being not exact, M contains two disjoint, embedded circles on which the integral of η is less than minus the constant c_n of Lemma 3.1. Thicken them into two disjoint $(2n)$ -dimensional compact solid tori $T_-, T_+ \subset M$. After Remark 2.1, without loss of generality, we can change η on M to any cohomologous closed 1-form; in particular, we can arrange that on $T_\pm \cong \mathbf{S}^1 \times \mathbf{D}^{2n-1}$, the form η is proportional to θ .

One endows M with the orientation defined by ω^n . The lemma 3.1 provides, on T_- (resp. T_+), a positive η -Liouville form λ_- (resp. λ_+) whose η -dual vector field Z_- (resp. Z_+) enters (resp. exits) the torus transversely through its boundary, on which the contact form $\lambda_-|_{\partial T_-}$ (resp. $\lambda_+|_{\partial T_+}$) is overtwisted.

We extend λ_- and Z_- (resp. λ_+ and Z_+) to a small open neighborhood of T_- (resp. T_+) in W .

After Remark 3.3, $d_\eta\lambda_-$ (resp. $d_\eta\lambda_+$) is homotopic to ω over T_- (resp. T_+) as a nondegenerate 2-form. After a homotopy of ω , we arrange without loss of generality that $\omega = d_\eta\lambda_-$ (resp. $d_\eta\lambda_+$) over some neighborhood of T_- (resp. T_+).

On the complement $W = M \setminus \mathrm{Int}(T_- \cup T_+)$, fix a Morse function f such that $f^{-1}(0) = \partial T_-$ and $f^{-1}(1) = \partial T_+$, and without local extrema in the interior of W . Choose a descending pseudo-gradient Z for f on W , coinciding with Z_- (resp. Z_+) close to ∂T_- (resp. ∂T_+).

Every critical point c of f of index $1 \leq i \leq 2n - 1$ admits in W a small compact neighborhood H_c with (convex) cornered boundary, as follows. H_c is diffeomorphic to $\mathbf{D}^i \times \mathbf{D}^{2n-i}$ minus a small open tubular neighborhood of the corner $\mathbf{S}^{i-1} \times \mathbf{S}^{2n-i-1}$; and the boundary splits as

$$\partial H_c = \partial_+ H_c \cup \partial_0 H_c \cup \partial_- H_c$$

where

- f is constant on $\partial_+ H_c$ and on $\partial_- H_c$;
- Z enters (resp. exits) H_c transversely through $\partial_+ H_c$ (resp. $\partial_- H_c$);
- Z is tangential to the \mathbf{I} factor on $\partial_0 H_c \cong \mathbf{S}^{i-1} \times \mathbf{S}^{2n-i-1} \times \mathbf{I}$.

Since H_c is simply connected, we can without loss of generality arrange that $\eta = 0$ on $\mathcal{O}p_W(H_c)$ (Remark 2.1). Hence, in H_c , we actually look for a *genuine* Liouville form. After the symplectization of cobordisms [EM], there is a positive Liouville form λ_c on $\mathcal{O}p_W(H_c)$ whose dual Liouville vector field is positively colinear to Z on $\mathcal{O}p_W(\partial H_c)$; and such that λ_c restricts to an overtwisted contact structure on every connected component of $\partial_{\pm}H_c$.

Put for short $H = \cup_c H_c$, where c runs over the critical points of f . After Lemma 2.11 (ii) applied in $W' = W \setminus \text{Int}(H)$ foliated by the level hypersurfaces of f , and $A = \partial W'$, there is an even contact structure ε on W' such that

- The almost symplectic class associated with ε contains ω ;
- The characteristic foliation \mathcal{Z} of ε is transversal to the level hypersurfaces of f ;
- ε coincides respectively with the kernels of λ_- , λ_+ and λ_c on neighborhoods of ∂T_- , ∂T_+ and ∂H_c , for every critical point c .

Changing the pseudo-gradient Z in $\text{Int}(W')$, one moreover arranges that Z spans \mathcal{Z} positively on W' .

By means of a partition of the unity, make a 1-form λ on M representing ε on W' , matching λ_+ on $\mathcal{O}p_W(T_+)$, matching λ_- on $\mathcal{O}p_W(T_-)$; and, for each critical point c , matching λ_c on $\mathcal{O}p_W(H_c)$.

In particular, λ is η -Liouville on some small open neighborhood V of $T_- \cup T_+ \cup H$ in M .

Claim 1: there is a smooth real function g on M , locally constant on T_- , T_+ and H , such that $Z \cdot g < 0$ on $\text{Int}(W')$.

Indeed, such a function g will be obtained from f by a modification in arbitrarily small neighborhoods of T_- , T_+ and H . The modification is obvious close to T_- and T_+ . Now, consider a critical point c . Write $t_- = f(\partial_- H_c)$ and $t_+ = f(\partial_+ H_c)$. On a small enough open neighborhood Ω of H_c in W , one easily builds a smooth plateau function $\phi : \Omega \rightarrow [0, 1]$ such that

- ϕ is compactly supported in Ω , while $\phi^{-1}(1) = H_c$;
- $Z \cdot \phi(x) \geq 0$ (resp. $= 0$) (resp. ≤ 0) at every point x of Ω such that $f(x) \geq t_+$ (resp. $t_- \leq f(x) \leq t_+$) (resp. $f(x) \leq t_-$).

Then, $g = (1 - \phi)f + \phi f(c)$ works on Ω . The claim 1 is proved.

After multiplying g by a large enough positive constant, one arranges moreover that $Z \cdot g$ is less than $\chi_\lambda - \eta Z$ on $M \setminus V$.

Claim 2: the 1-form $\mu = e^{-g}\lambda$ is η -Liouville on M .

This holds of course on T_- , on T_+ and on each H_c , since on these domains, μ is a constant multiple of λ_- , λ_+ and λ_c , respectively. On

W' , in view of Lemma 2.12, there remains to verify that $\chi_\mu = \chi_\lambda - Z \cdot g$ is more than ηZ . But this inequality does hold on $M \setminus V$ by choice of g ; while on $W' \cap V$, the function $\chi_\lambda - Z \cdot g$ is not less than χ_λ , which is more than ηZ by Lemma 2.12. The claim 2 is proved.

By construction, $d_\eta \mu$ lies over W' in the almost symplectic class associated with ε . Globally, $d_\eta \mu$ is homotopic to ω among the nondegenerate 2-forms on M .

Finally, in order to obtain a η -symplectic form ω' in any prescribed cohomology class $a \in H_\eta^2(M)$: first, fix an arbitrary (d_η) -closed 2-form ϖ on M representing a . Second, define

$$\omega' = \varpi + K d_\eta \mu$$

for some large positive real constant K . Provided that K is large enough, ω' is nondegenerate; and homotopic to $d_\eta \mu$, among the nondegenerate 2-forms, through the homotopy $(1-t)\varpi + K d_\eta \mu$ ($t \in \mathbf{I}$) followed by the homotopy $((1-t)K + t)d_\eta \mu$ ($t \in \mathbf{I}$). \square

Theorem A admits the following (easy) generalization, allowing a smooth boundary for the ambient manifold, and prescribing a natural boundary condition. Let M be an oriented compact $(2n)$ -manifold whose nonempty smooth boundary is splitted into two disjoint compact subsets $\partial_\pm M$, each of which may be empty.

THEOREM 3.4. *Assume that one is given on M*

- *A nonexact closed 1-form η ;*
- *A relative Novikov cohomology class $a \in H_\eta^2(M, \partial M)$;*
- *A positive nondegenerate 2-form ω .*

Then, there exist $\varpi \in \Omega^2(M)$ and $\alpha \in \Omega^1(\partial M)$ such that

- *The pair (ϖ, α) is D_η -closed and represents the cohomology class a (recall Equation 1);*
- *ϖ is nondegenerate, and homotopic to ω among the nondegenerate 2-forms on M ;*
- *α is an overtwisted contact form on every connected component of ∂M , positive on $\partial_- M$ and negative on $\partial_+ M$.*

In particular, ϖ is η -symplectic, and $\partial_+ M$ (resp. $\partial_- M$) is of concave (resp. convex) overtwisted contact type (Definition 2.7 and Lemma 2.13) with respect to ϖ . (Our choice of signs, seemingly unnatural, is coherent with the pseudogradients being descendant in section 4).

Proof of Theorem 3.4. As in the proof of Theorem A, one chooses two disjoint solid tori T_\pm embedded in the interior of M , on the cores of

which the integral of η is less than minus the constant c_n of Lemma 3.1. Then, T_- (resp. T_+) bears a positive η -Liouville form λ_- (resp. λ_+) whose η -dual vector field enters (resp. exits) the torus transversely through its boundary, on which the contact form $\lambda_-|_{\partial T_-}$ (resp. $\lambda_+|_{\partial T_+}$) is overtwisted.

Fix on $\mathcal{O}p(\partial M)$ a vector field Z transverse to ∂M , entering (resp. exiting) M through $\partial_+ M$ (resp. $\partial_- M$). The h-principle for overtwisted contact structures [BEM] provides on ∂M an overtwisted contact form β in the same almost contact class as $(\iota_Z(\omega)|_{\partial M}, \omega|_{\partial M})$ (Definition 2.9 and Remark 2.10). By means of Lemma 2.12, extend β over $\mathcal{O}p(\partial M)$ to a η -Liouville form λ , η -dual to Z .

On the complement $W = M \setminus \text{Int}(T_- \cup T_+)$, fix a Morse function $f : W \rightarrow [0, 1]$ such that

$$f^{-1}(0) = \partial T_- \cup \partial_- M$$

$$f^{-1}(1) = \partial T_+ \cup \partial_+ M$$

and without local extrema in the interior of W . The same construction as in the proof of Theorem A yields on M a η -Liouville form μ which is on $\mathcal{O}p(\partial M)$ a positive locally constant multiple of λ ; and $d_\eta \mu$ is homotopic to ω as a nondegenerate 2-form on M .

The relative Novikov cohomology class a is represented by a pair $(\omega', 0) \in \Omega^2(M) \times \Omega^1(\partial M)$ such that $d_\eta \omega' = 0$. For a large enough positive real K , the pair (ϖ, α) obviously works, where

$$\varpi = \omega' + K d_\eta \mu$$

$$\alpha = K \mu|_{\partial M}$$

□

4. MORSE THEORY FOR CODIMENSION-1 TAUT FOLIATIONS

See e.g. [CC1] for the elements on foliations. Given a taut codimension-1 foliation, the existence of functions which are Morse in restriction to every leaf is classical [FW]. The leafwise pseudo-gradients and their dynamics appeared in [Be02], for foliations of arbitrary codimensions, in order to construct some leafwise geometric structures; and in [GL], in order to study the contact forms carried by open book decompositions on 3-manifolds. Apart from these works, the ‘‘Morse theory of foliations’’ seems not to have met the attention that it deserves.

In the present section, we elaborate the tools that we need for Section 5: essentially, the construction of *ordered* leafwise Morse functions (Definition 4.4 below), and the cancellation of leafwise local extrema.

In this section, we consider a compact manifold M of dimension $m \geq 2$, maybe with a smooth boundary ∂M , endowed with a codimension-1 foliation \mathcal{F} , coorientable to fix ideas, and transverse to ∂M .

For a smooth real function f on M , a point $c \in M$ is *leafwise critical* if the differential of f vanishes on the leaf L_c of \mathcal{F} through c . The *critical locus* $\text{Crit}(f, \mathcal{F}) \subset M$ is the set of the leafwise critical points.

DEFINITION 4.1. We call f *leafwise Morse* if f restricts to a Morse function on every leaf of \mathcal{F} , and if f is locally constant on ∂M . In particular, the critical locus is interior to M ; while ∂M splits as the disjoint union of local minima $\partial_-(M, f)$ and of local maxima $\partial_+(M, f)$.

Informally, a leafwise Morse function locally amounts to a 1-parameter family of Morse functions in dimension $m - 1$.

At every $c \in \text{Crit}(f, \mathcal{F})$, the Morse index $\text{ind}_c(f, \mathcal{F})$ of $f|_{L_c}$ lies between 0 and $m - 1$. Clearly, $\text{Crit}(f, \mathcal{F})$ is a disjoint union of circles transverse to \mathcal{F} , and the index is constant on each circle. One denotes by $\text{Crit}^i(f, \mathcal{F})$ the set of the index- i leafwise critical points. Of course, in general f is not locally constant on $\text{Crit}(f, \mathcal{F})$. Recall that

DEFINITION 4.2. \mathcal{F} is called *taut* if every leaf meets a transverse loop.

If \mathcal{F} admits a leafwise Morse function, then \mathcal{F} must be taut, since every minimal set has to meet the index-0 critical locus and the index- $(m - 1)$ critical locus. Conversely:

PROPOSITION 4.3 (Ferry-Wasserman [FW]). *Let M be a compact manifold with smooth boundary, splitted into two compact subsets $\partial_{\pm}M$ (one of which may be empty, or both). Let \mathcal{F} be on M a coorientable, taut codimension-1 foliation, transverse to ∂M .*

Then, \mathcal{F} admits a leafwise Morse function f such that $\partial_-(M, f) = \partial_-M$ and $\partial_+(M, f) = \partial_+M$.

4.1. Ordering a leafwise Morse function. Let M be as before a compact manifold of dimension $m \geq 2$ with smooth boundary (maybe empty), endowed with a coorientable codimension-1, taut foliation \mathcal{F} transverse to ∂M . Let f be a leafwise Morse function on M .

DEFINITION 4.4. The leafwise Morse function f is *ordered* if for every two leafwise critical points c, c' , the inequality

$$\text{ind}_c(f, \mathcal{F}) < \text{ind}_{c'}(f, \mathcal{F})$$

implies $f(c) < f(c')$.

PROPOSITION 4.5. *The foliation \mathcal{F} admits an ordered leafwise Morse function which has the same critical locus as f , with the same indices, and the same splitting of ∂M into local minima and local maxima.*

This will result from the generic dynamical properties of the leafwise pseudo-gradients. As is usual in Morse theory, one considers *descending* pseudo-gradients.

DEFINITION 4.6. A vector field ∇ on M , tangential to \mathcal{F} , is a *leafwise pseudo-gradient* for f if, in every leaf L , the restriction $\nabla|_L$ is a descending pseudo-gradient for the Morse function $f|_L$. In other words:

- The function $\nabla \cdot f$ is negative but at the leafwise critical points;
- The Hessian of $\nabla \cdot f$ at every leafwise critical point c is negative definite in $T_c L_c$.

The construction of such a vector field is straightforward by means of a partition of the unity. Clearly, ∇ enters M through $\partial_+(M, f)$ and exits M through $\partial_-(M, f)$. Write $\nabla^t(x)$ for the image of $x \in M$ under the flow of ∇ at the time $t \in \mathbb{R}$, whenever defined.

LEMMA 4.7. (i) For every $x \in M$, the orbit $\nabla^t(x)$ descends from a point $\alpha(x) \in \text{Crit}(f, \mathcal{F}) \cup \partial_+(M, f)$ to a point $\omega(x) \in \text{Crit}(f, \mathcal{F}) \cup \partial_-(M, f)$.

(ii) Moreover, the lengths of the orbits have an upper bound not depending on x .

Proof. If not, some orbit would have infinite length; hence f would not be bounded on M , a contradiction. \square

DEFINITION 4.8. For every subset $X \subset \partial_-(M, f) \cup \partial_+(M, f) \cup \text{Crit}(f, \mathcal{F})$, define the stable and the unstable manifold of X with respect to ∇ as

$$\begin{aligned} \mathcal{W}^s(\nabla, X) &= \omega^{-1}(X) \\ \mathcal{W}^u(\nabla, X) &= \alpha^{-1}(X) \end{aligned}$$

In particular, for each connected component C of the index- i critical locus $\text{Crit}^i(f, \mathcal{F})$ ($0 \leq i \leq m-1$), the stable (resp. unstable) manifold $\mathcal{W}^s(\nabla, C)$ (resp. $\mathcal{W}^u(\nabla, C)$) is a submanifold of M transverse to \mathcal{F} and to ∂M ; and its interior is a bundle of fibre \mathbb{R}^{m-1-i} (resp. \mathbb{R}^i) over the circle C .

DEFINITION 4.9. We say that a leafwise pseudogradient is *globally Kupka-Smale* if, for every two connected components C, C' of $\text{Crit}(f, \mathcal{F})$, the manifolds $\mathcal{W}^s(\nabla, C)$ and $\mathcal{W}^u(\nabla, C')$ are transverse in M .

The global Kupka-Smale property is generic among the leafwise pseudo-gradients for f . Indeed, this genericity is well-known in the framework of 1-parameter families of functions; a similar argument holds for leafwise Morse functions. *From now on in this subsection 4.1, we assume that ∇ is a globally Kupka-Smale leafwise pseudo-gradient.*

It is convenient to extend the index function by defining $\text{ind}_x(f, \mathcal{F}) = -\infty$ for $x \in \partial_-(M, f)$ and $\text{ind}_x(f, \mathcal{F}) = +\infty$ for $x \in \partial_+(M, f)$.

LEMMA 4.10. *For every $x \in M$, one has $\text{ind}_{\omega(x)}(f, \mathcal{F}) \leq \text{ind}_{\alpha(x)}(f, \mathcal{F})$.*

Proof. The only case to consider is when x is not leafwise critical, but $c = \omega(x)$ and $c' = \alpha(x)$ are leafwise critical. Let i (resp. i') be the index of c (resp. c'). Since $\mathcal{W}^s(\nabla, c)$ and $\mathcal{W}^u(\nabla, c')$ are transverse in M , and of respective dimensions $m - i$ and $i' + 1$, the dimension of their intersection is $i - i' + 1$. On the other hand, the intersection containing the orbit through x , its dimension is at least 1. \square

REMARK 4.11. At this point, the Morse theory of foliations shows its limitations, which are well-known in the frame of the 1-parametric families of Morse functions. Call an orbit *exceptional* if its extremities are two leafwise critical points of the same index. Such an orbit corresponds to a handle sliding, in Smale's sense, in a 1-parametric family of Morse functions. After the global Kupka-Smale property, ∇ admits at most a finite number of exceptional orbits.

The classical Kupka-Smale property does not in general hold in restriction to the leaves. In every leaf L , for every pair of critical points in L but maybe a finite number, their stable and unstable manifolds for the pseudo-gradient $\nabla|_L$ are transverse in L .

In general, no choice of the pseudo-gradient can avoid the existence of exceptional orbits. The extremities of an exceptional orbit can belong to the same connected component of the critical locus. Also, given two components C, C' of the critical locus of the same index, there may exist two exceptional orbits, the one from C to C' and the other from C' to C . Then, C and C' cannot be separated in M by any hypersurface transverse to ∇ .

However, if C and C' have distinct indices, such a separating hypersurface does exist.

LEMMA 4.12. *For each $1 \leq i \leq m - 1$, there is a closed hypersurface H in M , transverse to ∇ and splitting M into two domains M_-, M_+ , such that*

- $\partial_-(M, f)$ and the leafwise critical points of indices at most $i - 1$ lie in M_- ;
- $\partial_+(M, f)$ and the leafwise critical points of indices at least i lie in M_+ .

Proof. The proof belongs to elementary general topology. For short, put

$$M' = M \setminus \text{Crit}(f, \mathcal{F})$$

Consider the space \mathcal{O} of the orbits of ∇ which are regular (in other words, not reduced to a single leafwise critical point), endowed with

the quotient topology; and the projection $\pi : M' \rightarrow \mathcal{O}$. For short, write $\text{ind}(x)$ instead of $\text{ind}_x(f, \mathcal{F})$.

Claim 0 — *The map*

$$M \rightarrow \{0, 1, \dots, m-1, +\infty\} : x \mapsto \text{ind}(\alpha(x))$$

is lower semicontinuous on M . The map

$$M \rightarrow \{-\infty, 0, 1, \dots, m-1\} : x \mapsto \text{ind}(\omega(x))$$

is upper semicontinuous on M .

This follows at once from Lemma 4.10.

Claim 1 — \mathcal{O} *is a smooth $(m-1)$ -manifold in general not Hausdorff, compact (in the sense that \mathcal{O} has the usual open cover property) and without boundary.*

Indeed, for every $t \in \mathbb{R}$ and every embedding ϕ of the open $(m-1)$ -disk D^{m-1} into the level set $M' \cap f^{-1}(t)$, the image $\phi(D^{m-1})$ meets every orbit at most once (since $\nabla \cdot f \leq 0$); hence $\pi \circ \phi$ is a local coordinate chart for X . One thus gets an atlas whose changes of coordinates are obviously smooth. Clearly, there is a small open neighborhood of $\text{Crit}(f, \mathcal{F})$ in M whose complement meets every regular orbit; hence \mathcal{O} is compact. The claim 1 is proved.

Let us understand the lack of Hausdorff separation in \mathcal{O} . By an *orbit chain*, we mean a finite sequence of regular orbits $\pi(x_1), \dots, \pi(x_\ell)$ ($\ell \geq 1$) such that $\omega(x_{j-1}) = \alpha(x_j)$ for each $2 \leq j \leq \ell$. The *endpoints* of the chain are the pair $(\alpha(x_1), \omega(x_\ell))$. After Lemma 4.10, the indices of the critical points of the chain form a nonincreasing sequence (*monotony property*).

Claim 2 — *From any sequence of regular orbits, one can extract a subsequence Hausdorff-converging towards an orbit chain.*

This follows easily from Lemma 4.7 (ii).

Claim 3 — *If two distinct regular orbits $\pi(x), \pi(y)$ are not separated in \mathcal{O} , then they belong to a same orbit chain.*

This follows at once from Claim 2 applied to a sequence of regular orbits $\pi(x_k)$ which converges both to $\pi(x)$ and to $\pi(y)$ in \mathcal{O} .

Consider the subset $U \subset M'$ of the noncritical points x such that

$$(7) \quad \text{ind}(\alpha(x)) \geq i \text{ and } \text{ind}(\omega(x)) \leq i-1$$

Since U is open in M' (Claim 0), $\pi(U)$ is open in \mathcal{O} . By Claim 3 and the monotony property, $\pi(U)$ is Hausdorff.

Claim 4 — $\pi(U)$ is compact.

Indeed, let (u_k) be a sequence in U . Following Claim 2, after passing to a subsequence, the orbits through u_k Hausdorff-converge in M to an orbit chain. The index function being locally constant on $\partial M \cup \text{Crit}(f, \mathcal{F})$, the endpoints (c, c') of this chain satisfy $\text{ind}(c) \geq i$ and $\text{ind}(c') \leq i - 1$. By the monotony property, one of the orbits $\pi(x)$ composing the chain satisfies the inequalities (7). We have thus found an accumulation point $\pi(x)$ in $\pi(U)$ for the sequence $\pi(u_k)$: the claim 4 is proved.

To sum up, $\pi(U)$ is a Hausdorff closed $(m - 1)$ -manifold. The restricted projection $\pi|_U$ is a submersion of U onto $\pi(U)$ whose fibres are all diffeomorphic to \mathbb{R} . Such a projection necessarily admits a section s : one can solve this elementary exercise, or alternatively apply a more general lemma due to Haefliger, see e.g. [M02]. The image $H = s(\pi(U)) \subset U$ is a closed hypersurface transverse to ∇ , and separating U into two domains U_- , U_+ such that ∇ enters U_- and exits U_+ along H . Let M_- (resp. M_+) be the topological closure of U_- (resp. U_+) in W .

Claim 5 — For every $x \in M \setminus U$, one has

- $x \in M_-$ iff $\text{ind}(\alpha(x)) \leq i - 1$;
- $x \in M_+$ iff $\text{ind}(\omega(x)) \geq i$.

Indeed, the points $y \in M$ such that

$$\text{ind}(\alpha(y)) \geq m - 1 \text{ and } \text{ind}(\omega(y)) \leq 0$$

form an open (after Claim 0) and dense subset in M . Hence, x is the limit of a sequence (y_p) of such points. For p large enough, $y_p \in U$, hence its orbit $\pi(y_p)$ intersects transversely H in a unique point $h_p = s(\pi(y_p))$. The point h_p splits the orbit $\pi(y_p)$ into two subintervals $\pi(y_p) \cap U_-$ and $\pi(y_p) \cap U_+$. By Claim 2, after passing to a subsequence, the orbits $\pi(y_p)$ Hausdorff-converge in M to an orbit chain $\pi(x_1), \dots, \pi(x_\ell)$, such that $x = x_j$ for some $1 \leq j \leq \ell$. The index function being locally constant on $\partial M \cup \text{Crit}(f, \mathcal{F})$, the endpoints of the chain satisfy

$$\text{ind}(\alpha(x_1)) \geq m - 1 \text{ and } \text{ind}(\omega(x_\ell)) \leq 0$$

By the monotony property, one and only one of the orbits $\pi(x_k)$ in the chain lies in U . We can choose $x_k \in H$. The point x_k splits the orbit chain into two subintervals which are the Hausdorff limits of $\pi(y_p) \cap U_-$ and $\pi(y_p) \cap U_+$; the first is thus contained in M_- , the second in M_+ . Since $x \notin U$, one has $j \neq k$. If $j < k$, then $\text{ind}(\omega(x)) \geq i$ (monotony property) and $x \in M_+$. If $j > k$, then $\text{ind}(\alpha(x)) \leq i - 1$ (monotony

property) and $x \in M_-$. The proofs of Claim 5 and of Lemma 4.12 are complete. \square

Proof of Proposition 4.5. For each $1 \leq i \leq m - 1$, after Lemma 4.12, there is a smooth plateau function ϕ_i on M such that

- $\nabla \cdot \phi_i \leq 0$;
- $\phi_i = 0$ on a neighborhood of $\partial_-(M, f)$ and of $\text{Crit}^{\leq i-1}(f, \mathcal{F})$;
- $\phi_i = 1$ on a neighborhood of $\partial_+(M, f)$ and of $\text{Crit}^{\geq i}(f, \mathcal{F})$.

For every $\epsilon > 0$, consider on M the function

$$g_\epsilon = \phi_1 + \cdots + \phi_{m-1} + \epsilon f$$

Obviously, every point of ∂M is a local extremum for g_ϵ ; one has $\partial_\pm(M, g_\epsilon) = \partial_\pm(M, f)$; and g_ϵ coincides, for each i , with $i + \epsilon f$ on some neighborhood of $\text{Crit}^i(f, \mathcal{F})$. Moreover, $\nabla \cdot g_\epsilon < 0$ on $M \setminus \text{Crit}(f, \mathcal{F})$. Hence, g_ϵ is leafwise Morse with the same critical locus and the same indices as f . Clearly, g_ϵ is ordered provided that $\epsilon|f| < 1/2$ on M . \square

We end this subsection with a corollary of Proposition 4.5. Let f be an ordered leafwise Morse function on M .

DEFINITION 4.13. We call f *nearly self-indexing* if

- $|f(c) - \text{ind}(c)| < 1/2$ at every leafwise critical point c ;
- $f^{-1}(-1/2) = \partial_-(M, f)$ and $f^{-1}(m - 1/2) = \partial_+(M, f)$.

Recall that $\partial_-(M, f)$ and/or $\partial_+(M, f)$ can be empty. It is convenient, after reparametrizing the values of f , to arrange that f is nearly self-indexing. We have thus decomposed M into m compact domains

$$M_i = f^{-1}[i - 1/2, i + 1/2]$$

($0 \leq i \leq m - 1$) with boundaries transverse to \mathcal{F} . Write $M_{\leq i}$ for $f^{-1}([-1/2, i + 1/2])$ and $M_{\geq i}$ for $f^{-1}([i - 1/2, m - 1/2])$.

REMARK 4.14. This is not quite a handle decomposition. In every leaf L , and for each i , the generally noncompact manifold $L \cap M_i$ decomposes into a countable, locally finite family of index- i compact handles. When one moves continuously from one leaf to another, at each exceptional orbit, one of these handles slides on another.

DEFINITION 4.15. A closed submanifold $S \subset M$ is said to be *faithful* to \mathcal{F} if S is transverse to \mathcal{F} , meets every leaf L of \mathcal{F} , and if the intersection $L \cap S$ is connected (hence a single leaf of $\mathcal{F}|_S$).

In other words, the embedding $S \subset M$ induces a Haefliger equivalence between the holonomy pseudogroups of the foliations $\mathcal{F}|_S$ and \mathcal{F} .

COROLLARY 4.16. *Let f be a nearly self-indexing leafwise Morse function on the m -dimensional foliated manifold (M, \mathcal{F}) . If $m \geq 4$, then for each $1 \leq i \leq m - 3$, the level set $f^{-1}(i + \frac{1}{2})$ is faithful to \mathcal{F} .*

Proof. Consider a leaf L of \mathcal{F} and, on this connected $(m - 1)$ -manifold maybe with boundary, the genuinely Morse function $g = f|_L$. In \mathbb{R} , the value $i + \frac{1}{2}$ does not lie in the closure of $g(\partial L)$, and separates the critical values of g of indices 0 and 1 from the critical values of g of indices $m - 2$ and $m - 1$. Hence, $g^{-1}(i + \frac{1}{2})$ is connected. \square

By induction on m , there also exists a closed 3-dimensional submanifold of M faithful to \mathcal{F} . The existence of faithful hypersurfaces in every taut codimension-1 foliated manifold of dimension at least 4 is already known [M16], but we feel that the present construction, by means of the Morse theory of foliations, is more natural and clearer. See also [MT14] for some particular cases of faithful submanifolds, obtained by different means.

4.2. Cancelling leafwise local extrema. Let, as before, M be a compact m -dimensional manifold with smooth boundary, endowed with a codimension-1 coorientable taut foliation \mathcal{F} , transverse to ∂M . Let $\partial_- M, \partial_+ M$ be a partition of ∂M into two open subsets (perhaps empty).

PROPOSITION 4.17. *Assume that $m \geq 5$ and that every leaf of \mathcal{F} meets $\partial_- M$ (respectively both $\partial_- M$ and $\partial_+ M$).*

Then, M admits a nearly self-indexing leafwise Morse function f without leafwise local minima (respectively extrema) in the interior of M and such that $\partial_{\pm}(M, f) = \partial_{\pm} M$.

REMARK 4.18. Conversely, the existence of a function f such that $\partial_{\pm}(M, f) = \partial_{\pm} M$ and without leafwise local minima (resp. extrema) in the interior implies that every leaf L meets $\partial_- M$ (resp. both $\partial_- M$ and $\partial_+ M$), since the topological closure \bar{L} of L in M being compact and saturated, the restricted function $f|_{\bar{L}}$ must reach a minimum and a maximum.

REMARK 4.19. We do not know if Proposition 4.17 holds as well for $m = 3$ nor 4.

REMARK 4.20. When $\partial_- M = \emptyset$, the conclusion of Proposition 4.17 amounts to say that \mathcal{F} is, in the interior of M , “uniformly open” in the sense of [Be02]. Hence, any leafwise open invariant differential relation abides the h-principle. In particular, the parametric h-principle holds for leafwise symplectic structures on such manifolds. Unfortunately,

that h -principle does not allow any control over the structure at the boundary of M , unlike Theorem C .

In order to prove Proposition 4.17, we start with a nearly self-indexing leafwise Morse function f on M such that $\partial_{\pm}(M, f) = \partial_{\pm}M$ (Propositions 4.3 and 4.5). We assume that every leaf meets ∂_-M , and we shall cancel the interior leafwise local minima of f . Of course, if moreover every leaf meets also ∂_+M , a symmetric method also cancels the interior leafwise local maxima.

The cancellation method is inspired by Laudenbach's reproof [L14] of the classical Cerf cancellation lemma for local extrema in 1-parameter families of functions. We refer to Laudenbach for some details. However, our foliated framework also calls for some specific arguments.

We shall see how to cancel one connected component C of the index-0 critical locus. Repeating this argument removes all the components. After each step, f is not ordered any more, but we apply Proposition 4.5 and reorder f . The steps of the cancellation of C are represented on the (somehow round) Cerf diagrams of Figure 2.

Fix for f a leafwise descending pseudo-gradient ∇ which is globally Kupka-Smale. Recall the notations $\omega(x)$, $\mathcal{W}^s(\nabla, X)$ from the above Lemma 4.7 and Definition 4.8.

The level set $S = f^{-1}(3/2)$ is important in the proof. This compact hypersurface of M is transverse to \mathcal{F} and to ∇ , and separates the leafwise critical points of f of index 0 and 1 from the leafwise critical points of indices ≥ 2 . For every connected component C' of $\text{Crit}^1(f, \mathcal{F})$, the intersection $S \cap \mathcal{W}^s(C')$ is in S a hypersurface (in fact a bundle of fibre \mathbf{S}^{m-3} over the circle) transverse to the foliation $\mathcal{F}|_S$. The intersections of S with the stable manifolds of ∂_-M and of $\text{Crit}^0(f, \mathcal{F})$ are, in S , finitely many open domains separated by these hypersurfaces.

LEMMA 4.21. *The endpoint map ω restricted to S admits over the circle C a smooth section $\sigma : C \rightarrow S$.*

Proof. The bundle map $\omega : \mathcal{W}^s(\nabla, C) \rightarrow C$ has fibre \mathbb{R}^{m-1} . Since $m \geq 4$, a generic section s of this bundle is disjoint from C and from the 2-dimensional unstable manifolds of the index-1 critical locus. For every $c \in C$, the orbit of ∇ through $s(c)$ enters $\{f \leq 3/2\}$ at a unique point $\sigma(c) \in S$. \square

Note that σ is transverse in S to the foliation $\mathcal{F}|_S$.

LEMMA 4.22. *Every leaf of $\mathcal{F}|_S$ meets the stable manifold $\mathcal{W}^s(\nabla, \partial_-M)$.*

Proof. Fix a point $x \in S$; let L_x be the leaf of \mathcal{F} through x in M . On the one hand, ∇ exits M through ∂_-M which meets L_x by hypothesis. On the other hand, in L_x which is of dimension ≥ 2 , the unstable

manifolds of the index-1 critical points of $f|_{L_x}$ form only a denumerable (and even locally finite) family of 1-dimensional orbits. Hence, $\mathcal{W}^s(\nabla, \partial_- M) \cap L_x$ intersects S in a point y . Finally, x and y both lie on $L_x \cap S$, which is a single leaf of $\mathcal{F}|_S$ since the level set S is faithful (Corollary 4.16). \square

Proof of Proposition 4.17. For every $c \in C$ and $v \in \mathbb{R}$, by a *bridge arc* for c at level v , we mean an embedding a of the interval $\mathbf{I} = [0, 1]$ into $f^{-1}(v)$, tangential to \mathcal{F} , such that $\omega(a(0)) = c$ and $\omega(a(1)) \in \partial_- M$. We say that the bridge arc a is *pointed* if moreover $v = 3/2$ and $a(1/2) = \sigma(c)$.

We are first interested in the level $v = 3/2$. For every $c \in C$, a pointed bridge arc for c exists by Lemma 4.22. Then, pushing this arc into the neighboring leaves of $\mathcal{F}|_S$, one obtains, over some small open neighborhood V_c of c in C , a choice of a pointed bridge arc for every $c' \in V_c$, depending smoothly on c' . Identifying C with \mathbb{R}/\mathbb{Z} , let $\delta > 0$ be a Lebesgue number for the open cover (V_c) . Choose a subdivision of the circle C into ℓ intervals $[c_{i-1}, c_i]$ ($1 \leq i \leq \ell$, $c_0 = c_\ell$) of length less than δ . Then, fix $\epsilon > 0$ so small that for each i :

$$2\epsilon < |c_i - c_{i-1}| < \delta - 2\epsilon$$

Put $I_i = [c_{i-1} - \epsilon, c_i + \epsilon] \subset C$. Choose for every $c \in I_i$ a pointed bridge arc $t \mapsto a_i(c, t)$, the map a_i being a smooth embedding $I_i \times \mathbf{I} \hookrightarrow S$.

For every subset $X \subset M$, let $\nabla^+(X)$ denote the set of the points $\nabla^t(x)$ for $x \in X$ and $t \geq 0$ (wherever defined). The cancellation method will modify f and ∇ , in M , close to the squares $a_i(I_i \times \mathbf{I})$ and $\nabla^+(a_i(I_i \times 0))$ and $\nabla^+(a_i(I_i \times 1))$ ($1 \leq i \leq \ell$). Since these squares can intersect each other for different values of i , we take some previous cautions so that the modifications don't interfere with each other. Fix a small compact interval $J \subset \mathbb{R}$ centered at $3/2$, so small that J contains no leafwise critical value of f ; fix ℓ values $v_i \in J$ ($1 \leq i \leq \ell$), two by two distinct; let $\pi_i : f^{-1}(J) \rightarrow f^{-1}(v_i)$ be the projection along the flowlines of ∇ ; let $\tilde{a}_i = \pi_i \circ a_i$. Thus, for each i :

- For every $c \in I_i$, the arc $t \mapsto \tilde{a}_i(c, t)$ is a bridge arc for c at level v_i ;
- The arc $I_i \rightarrow f^{-1}(v_i) : c \mapsto \tilde{a}_i(c, 1/2)$ extends to a global section $\sigma_i : C \rightarrow f^{-1}(v_i)$ of ω over C .

(Namely, $\sigma_i = \pi_i \circ \sigma$). These two properties are stable by any small enough isotopy of the embedding \tilde{a}_i in $f^{-1}(v_i)$ tangentially to $\mathcal{F}|_{f^{-1}(v_i)}$. Since $m \geq 5$, after a generic such perturbation for each i , we can arrange that for every $1 \leq i < j \leq \ell$ and $c \in I_i \cap I_j$, the two flow lines $\nabla^+(\tilde{a}_j(c \times \partial \mathbf{I}))$ are disjoint from the arc $\tilde{a}_i(c \times \mathbf{I})$.

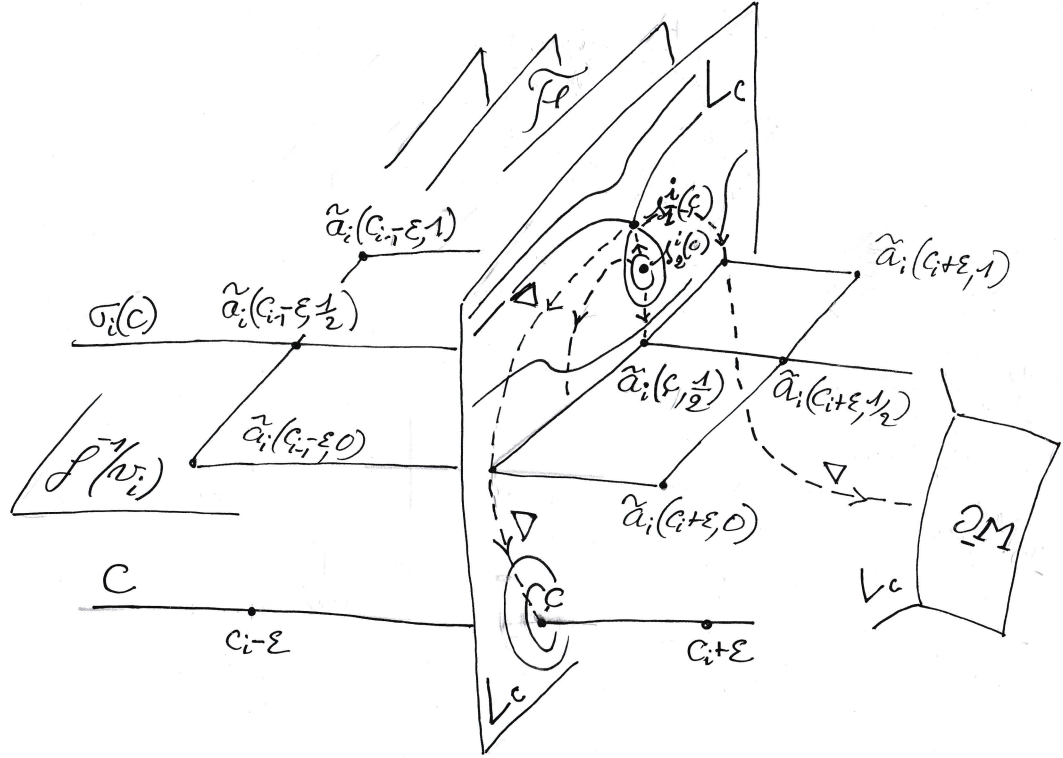


FIGURE 1. Creation of two circles of leafwise critical points of respective indices 1, 2. Beware that the ambient dimension $m = 3$, which this figure evokes, is excluded in the text. Of course, for $m \geq 4$, the point $s_2^i(c)$ is not a local extremum in the leaf L_c .

Next, for each $1 \leq i \leq \ell$, modify f in a small neighborhood of $\sigma_i(C)$ in M by introducing, for every $c \in C$, in the function $f|_{L_c}$, close to $\sigma_i(c)$ in the leaf L_c , and slightly above the level v_i , a pair of critical points $s_1^i(c)$, $s_2^i(c)$ of respective indices 1, 2, in cancellation position. (Figures 1 and 2 (a))

For a suitable new leafwise pseudo-gradient (still denoted by ∇), for every $c \in I_i$ one has (Figure 1)

$$\mathcal{W}^u(\nabla, s_1^i(c)) \cap f^{-1}(v_i) = \tilde{a}_i(c \times \partial \mathbf{I})$$

$$\mathcal{W}^u(\nabla, s_2^i(c)) \cap f^{-1}(v_i) = \tilde{a}_i(c \times \text{Int}(\mathbf{I}))$$

In particular, one of the two branches of the unstable manifold of $s_1^i(c)$ descends to c , and the other descends to $\partial_- M$.

Since moreover the values of f on $\partial_- M$ are less than the values of f on C , the *parametric Morse cancellation lemma* applies to the pairs of

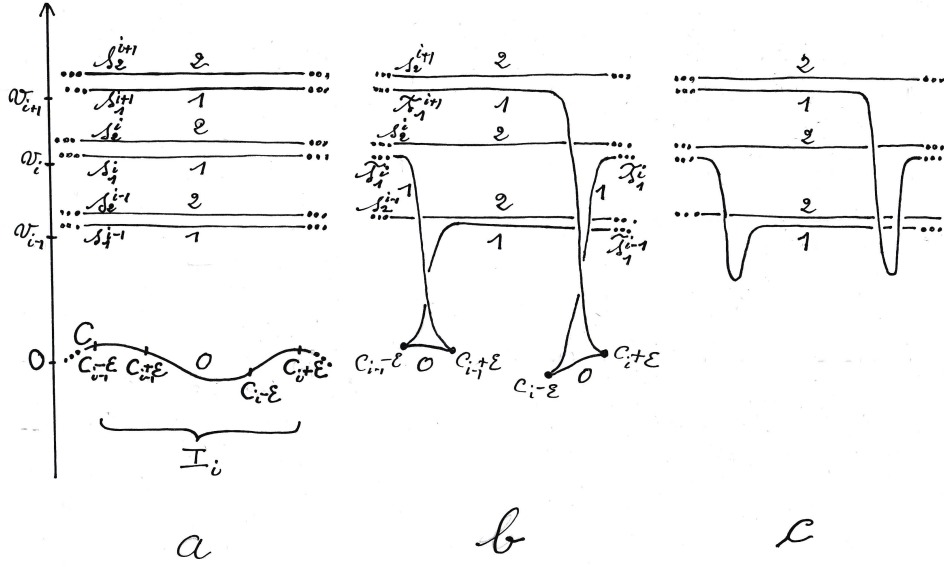


FIGURE 2. A somewhat round Cerf diagram for the cancellation of leafwise local extrema.

critical points $(c, s_1^i(c))$ for $c \in [c_{i-1} + \epsilon, c_i - \epsilon]$. As a result, the function f is modified in a small neighborhood of the square

$$\mathcal{W}^u(\nabla, s_1^i([c_{i-1} + \epsilon, c_i - \epsilon])) \cup [c_{i-1} + \epsilon, c_i - \epsilon]$$

so that the pairs $(c, s_1^i(c))$ are cancelled; but we may arrange that f remains unchanged close to $\partial_- M$.

Once these cancellations have been performed for every $1 \leq i \leq \ell$, the resulting function is of course not leafwise Morse any longer. On the way, at the point $c_i - \epsilon$ (resp. $c_{i-1} + \epsilon$), a birth (resp. death) critical point has been created in its leaf; so that, instead of the original index-0 leafwise critical circle C , one now has (see Figure 2 b) for each $i = 1, \dots, \ell$:

- An open arc $(c_i - \epsilon, c_i + \epsilon)$ of index-0 leafwise critical points;
- An open arc of leafwise index-1 critical points $\tilde{s}_1^i(c) \in L_c$ ($c \in (c_i - \epsilon, c_{i-1} + \epsilon)$), such that $\tilde{s}_1^i(c) = s_1^i(c)$ on a neighborhood of $[c_i + \epsilon, c_{i-1} - \epsilon]$;
- An index-2 leafwise critical circle $s_2^i(C)$.

The resulting Cerf diagram looks like Figure 2 (b), with ℓ swallow-tails.

Once the leafwise pseudogradient has been modified accordingly, one of the two branches of the unstable manifold of $\tilde{s}_1^i(c)$ descends to c , for every $1 \leq i \leq \ell$ and $c \in (c_i - \epsilon, c_{i-1} + \epsilon)$. Hence, for each $1 \leq i \leq \ell$, and for $c \in [c_i - \epsilon, c_i + \epsilon]$, the family of triples of leafwise critical points c , $\tilde{s}_1^i(c)$, $\tilde{s}_1^{i+1}(c)$ of respective indices 0, 1, 1 matches the hypotheses of the *elementary swallowtail lemma* (Lemma 3.5 of [L14]). We modify the function according to this lemma: the index-0 leafwise critical points vanish. (Figure 2, (c)).

Once the ℓ swallowtails have been so cancelled, the resulting function is leafwise Morse again, and has got, instead of the original circle C of index-0 leafwise critical points, ℓ more circles of index-2 leafwise critical points, and one more circle (covering ℓ times the circle C) of index-1 leafwise critical points. The proof of Proposition 4.17 is complete. \square

4.3. Orienting the stable and unstable manifolds. We shall need in Section 5 yet another normalization lemma falling to the Morse theory of foliations. Consider, on an orientable manifold M of dimension $m \geq 3$, a coorientable codimension-1 foliation \mathcal{F} , a leafwise Morse function f , and a connected component C of the critical locus of f , of some nonextremal index $1 \leq i \leq m - 2$.

There are of course, up to isomorphism, exactly two dimension- i real vector bundles over the circle : the trivial one and the nonorientable one. The unstable manifold $\mathcal{W}^u(C)$ with respect to any leafwise pseudo-gradient ∇ is isomorphic to one of them, not depending on the choice of ∇ . Since \mathcal{F} is tangentially orientable, the orientability of $\mathcal{W}^u(C)$ is equivalent to that of $\mathcal{W}^s(C)$.

PROPOSITION 4.23. *Assume that the stable and unstable manifolds of C are not orientable.*

Then, there exists a leafwise Morse function g on M , coinciding with f outside some arbitrarily small neighborhood T of C , and whose leafwise critical locus in T consists of two critical circles, one of index i and one of index $i + 1$ (or $i - 1$ if one prefers); both of which have orientable stable and unstable manifolds.

Proof. We prove the $i + 1$ case. Replacing f by $-f$ implies the $i - 1$ case.

Let $x, y, z_1, \dots, z_{m-3}$ denote the standard coordinates on \mathbb{R}^{m-1} . Consider the quadratic form $q(x, y) = (-x^2 + y^2)/2$ on \mathbb{R}^2 , and on \mathbb{R}^{m-3} a nondegenerate quadratic form Q of index $i - 1$. Let R^θ denote the rotation of angle θ in the (x, y) -plane.

Since M is orientable but the stable and unstable manifolds of C are not, the *parametric Morse lemma* implies that C admits a compact

tubular neighborhood $T \cong (\mathbb{R}/\mathbb{Z}) \times \mathbf{D}^2 \times \mathbf{D}^{m-3}$ in M on which f has the form :

$$f(t, x, y, z) = f(t, 0, 0, 0) + q(R^{t\pi}(x, y)) + Q(z).$$

Let $b : \mathbb{R} \rightarrow [0, 1]$ be a smooth even function with support $[-1, 1]$ such that $b(y) = 1 - y^2/2$ near $y = 0$ and the derivative b' is negative on $(0, 1)$ and has a unique critical point there, where it reaches its minimum. Hence, for $0 < \varepsilon < 1$, the function

$$y \mapsto y^2/2 + \varepsilon b(y/\varepsilon)$$

is Morse on \mathbb{R} with three critical points $-c_\varepsilon < 0 < c_\varepsilon$ of respective indices 0, 1, 0. Let $\rho : \mathbb{R}^+ \rightarrow [0, 1]$ be a smooth function with support in $[0, 1)$ such that $\rho = 1$ near 0. Define $\phi_\varepsilon : \mathbf{D}^2 \times \mathbf{D}^{m-3} \rightarrow \mathbb{R}$ by

$$\phi_\varepsilon(x, y, z) = q(x, y) + \varepsilon \rho(x^2 + y^2 + |z|^2) b(y/\varepsilon) + Q(z).$$

It enjoys the following properties :

- $\phi_\varepsilon(-x, -y, z) = \phi_\varepsilon(x, y, z)$.
- For $\varepsilon > 0$ sufficiently small, the function ϕ_ε has exactly three critical points : 0 of index $i + 1$, and $(0, \pm c_\varepsilon, 0, \dots, 0)$ of index i .
- $\phi_\varepsilon(x, y, z) = q(x, y) + Q(z)$ on a neighborhood of $\partial(\mathbf{D}^2 \times \mathbf{D}^{m-3})$.

Fix such an $\varepsilon > 0$ and set

$$g(t, x, y, z) = f(t, 0, 0, 0) + \phi_\varepsilon(R^{t\pi}(x, y), z).$$

The function g is leafwise Morse on T and coincides with f near ∂T . It has two leafwise critical circles in T of respective indices i (covering twice the original C) and $i + 1$, whose stable and unstable manifolds are orientable. \square

5. MAKING FOLIATED COBORDISMS CONFORMALLY SYMPLECTIC

Here, we deduce from the tools developed in the previous section and from the symplectization theorem for cobordisms ([EM]), a foliated version of the latter.

Let W be a compact manifold of dimension $2n + 1 \geq 5$, whose smooth boundary ∂W is splitted into two disjoint nonempty compact subsets $\partial_\pm W$. Let \mathcal{F} be on W a cooriented codimension-1 foliation, transverse to ∂W . One has the induced foliations $\partial\mathcal{F} = \mathcal{F}|_{\partial W}$ and $\partial_\pm\mathcal{F} = \mathcal{F}|_{\partial_\pm W}$. For every leaf L of \mathcal{F} , put $\partial_\pm L = L \cap \partial_\pm W$.

Recall Definition 2.9. A leafwise 2-form $\omega \in \Omega^2(\mathcal{F})$ is of course *nondegenerate* if ω^n does not vanish. Such a form defines a leafwise orientation on \mathcal{F} , hence also on W and on ∂W .

Let η be a leafwise 1-form on \mathcal{F} which is closed ($d_{\mathcal{F}}\eta = 0$). Then, every $\theta \in \Omega^*(\mathcal{F})$ has a *leafwise Lichnerowicz differential* with respect to η

$$d_{\eta}\theta = d_{\mathcal{F}}\theta - \eta \wedge \theta$$

Just as in the nonfoliated case, $d_{\eta}^2 = 0$, hence the differential operator d_{η} on $\Omega^*(\mathcal{F})$ defines some Novikov leafwise cohomology groups. We are interested in the relative ones: precisely, $H_{\eta}^*(\mathcal{F}, \partial\mathcal{F})$ is the cohomology of $\Omega^*(\mathcal{F}) \times \Omega^{*-1}(\partial\mathcal{F})$ under the differential operator

$$(\theta, \theta') \mapsto (d_{\eta}\theta, \theta|_{\partial\mathcal{F}} - d_{\eta}\theta')$$

THEOREM B. *Let $W, \partial_{\pm}W, \mathcal{F}, \omega, \eta$ be as above. Let $a \in H_{\eta}^2(\mathcal{F}, \partial\mathcal{F})$. Assume that \mathcal{F} is taut (Definition 4.2) and that every leaf of \mathcal{F} meets both $\partial_{+}W$ and $\partial_{-}W$.*

Then, there exist $\varpi \in \Omega^2(\mathcal{F})$ and $\alpha \in \Omega^1(\partial\mathcal{F})$ such that

- $d_{\eta}\varpi = 0$, and $\varpi|_{\partial\mathcal{F}} = d_{\eta}\alpha$;
- (ϖ, α) lies in the relative Novikov leafwise cohomology class a ;
- ϖ is nondegenerate and homotopic to ω among the nondegenerate 2-forms on \mathcal{F} ;
- α is a negative (resp. positive) overtwisted contact form on every leaf of $\partial_{+}\mathcal{F}$ (resp. $\partial_{-}\mathcal{F}$).

In particular, on every leaf L of \mathcal{F} , the 2-form $\varpi|_L$ is η -symplectic; and $\partial_{+}L$ (resp. $\partial_{-}L$) is of concave (resp. convex) overtwisted contact type (Definition 2.7 and Lemma 2.13) with respect to $\varpi|_L$.

REMARK 5.1. Here, η may be $d_{\mathcal{F}}$ -exact, or even vanish identically.

Proof of Theorem B. The proof is essentially a “foliated” version of parts of the above proof of Theorem A, using the tools elaborated in Section 4. We begin with the case $a = 0$. So, we are actually looking for a *leafwise η -Liouville form* on \mathcal{F} : a $\lambda \in \Omega^1(\mathcal{F})$ such that $d_{\eta}\lambda \in \Omega^2(\mathcal{F})$ is nondegenerate.

Applying Propositions 4.17, 4.23 and 4.5, one makes on W a leafwise Morse function f (Definition 4.1) such that

- $\partial_{-}(W, f) = \partial_{-}W$ and $\partial_{+}(W, f) = \partial_{+}W$;
- f has no leafwise local extrema in $\text{Int}(W)$;
- The stable and unstable manifolds of $\text{Crit}(f, \mathcal{F})$ are orientable.

Choose a leafwise pseudo-gradient Z for f on W (Definition 4.6).

Leafwise symplectization close to the critical locus — Thanks to the orientability of the (un)stable manifolds, every connected component C of the critical locus of index $1 \leq i \leq 2n - 1$ admits in W a small

compact neighborhood H_C which is a topological solid torus with (convex) cornered boundary, as follows. H_C is diffeomorphic with $\mathbf{S}^1 \times H_i$ where H_i is $\mathbf{D}^i \times \mathbf{D}^{2n-i}$ minus a small open tubular neighborhood of the corner $\mathbf{S}^{i-1} \times \mathbf{S}^{2n-i-1}$; and the boundary splits as

$$\partial H_i = \partial_+ H_i \cup \partial_0 H_i \cup \partial_- H_i$$

where (writing $\partial_+ H_C$ for $\mathbf{S}^1 \times \partial_+ H_i$ and $\partial_0 H_C$ for $\mathbf{S}^1 \times \partial_0 H_i$ and $\partial_- H_C$ for $\mathbf{S}^1 \times \partial_- H_i$):

- $\mathcal{F}|_{H_C}$ is the slice foliation parallel to the factor H_i ;
- f is leafwise constant on $\partial_+ H_C$ and on $\partial_- H_C$;
- Z enters (resp. exits) H_C transversely through $\partial_+ H_C$ (resp. $\partial_- H_C$);
- Z is tangential to the \mathbf{I} factor on

$$\partial_0 H_C \cong \mathbf{S}^1 \times \mathbf{S}^{i-1} \times \mathbf{S}^{2n-i-1} \times \mathbf{I}$$

Since the leaves of $\mathcal{F}|_{H_C}$ are topological disks, η admits a $d_{\mathcal{F}}$ -primitive u on H_C . Extending u to a smooth function over W , and changing η to $\eta - d_{\mathcal{F}}u$ on W , we can without loss of generality arrange that $\eta = 0$ on $\mathcal{O}p_W(H_C)$ (here we use an obvious foliated version of Remark 2.1). Also, $U(n)/SO(2n)$ being simply connected, after a homotopy of ω , we can arrange that $\omega|(s \times H_i)$ does not depend on $s \in \mathbf{S}^1$. Hence, in H_C , we actually look for a *genuine* Liouville form on a single slice H_i . Such a form is given by the symplectization of cobordisms [EM]. One gets a leafwise Liouville form λ_C on $\mathcal{O}p_W(H_C)$, positive with respect to the orientation of the leaves by ω^n ; and whose dual Liouville vector field is positively colinear to Z on $\mathcal{O}p_W(\partial H_C)$; moreover, λ_C induces an overtwisted contact structure on every leaf of $\mathcal{F}|_{\partial_{\pm} H_C}$.

Construction of a leafwise even contact structure away from the critical locus — For short, put $H = \cup_C H_C$ and write λ_H for the leafwise 1-form equal to λ_C on each H_C .

Consider the codimension-2 foliation \mathcal{L} of $W' = W \setminus \text{Int}(H)$ by the level hypersurfaces of f in the leaves of \mathcal{F} , cooriented by df . Rescale Z such that $Z \cdot f = -1$ on W' .

The pair $(\iota_Z \omega, \omega)$ restricts, on \mathcal{L} , to a leafwise almost contact structure (Definition 2.9). The h-principle for overtwisted contact structures on foliations ([BEM], Theorem 1.5) provides for the foliation \mathcal{L} a leafwise contact, cooriented $(2n-2)$ -plane field $\xi \subset T\mathcal{L}$ such that

- ξ lies in the leafwise almost contact class of $((\iota_Z \omega)|_{\mathcal{L}}, \omega|_{\mathcal{L}})$;
- ξ is an overtwisted contact structure in every leaf of \mathcal{L} ;
- ξ coincides with $\ker(\lambda_H) \cap T\mathcal{L}$ near H .

Just like in the proof of Lemma 2.11, after the Gray stability theorem, there is a unique vector field X on W' , tangential to ξ , such that $Z' = Z + X$ preserves ξ . In particular, X vanishes on some neighborhood of ∂H . Change Z to Z' on W' and put $\varepsilon = \mathbb{R}Z + \xi$. So, in every leaf F of $\mathcal{F}|_{W'}$, the hyperplane field ε is an even contact structure (Section 2), represented by $\lambda_H|_F$ close to $\partial H \cap F$, and whose characteristic foliation is positively spanned by Z .

Construction of a leafwise η -Liouville form — By means of a partition of the unity, make over W a leafwise 1-form $\lambda \in \Omega^1(\mathcal{F})$ such that

- λ represents ε in each leaf of $\mathcal{F}|_{W'}$;
- λ coincides with λ_H on some open neighborhood V of H in W .

By an easy modification of f in a small neighborhood of H , one gets a smooth real function g on W , constant on every leaf of $\mathcal{F}|_H$, and such that $Z \cdot g < 0$ on $W \setminus H$. After multiplying g by a large enough positive constant, one arranges moreover that on $W \setminus V$:

$$(8) \quad Z \cdot g < \chi_\lambda - \eta(Z)$$

After Lemma 2.12, $\chi(\lambda) > \eta(Z)$ on $V \setminus H$. After Equation (5) and Inequation (8), changing λ to $e^{-g}\lambda$, one can moreover arrange that $\chi(\lambda) > \eta(Z)$ on the all of $W \setminus H$. Hence, λ is leafwise η -Liouville on $W \setminus H$; and also on H , being there a leafwise locally constant multiple of λ_H . By construction, $\varpi = d_\eta \lambda$ satisfies all the properties of Theorem B in the exact case $a = 0$.

General case — In order to obtain a leafwise η -symplectic form in a given relative cohomology class a , we proceed as in the non-foliated case: a is represented by a pair $(\omega', 0)$ such that $\omega' \in \Omega^2(\mathcal{F})$ is d_η -closed. For a large enough positive real constant K , the leafwise forms

$$\begin{aligned} \varpi &= \omega' + K d_\eta \lambda \\ \alpha &= K \lambda|_{\partial \mathcal{F}} \end{aligned}$$

satisfy the required properties. □

6. DEFORMING FOLIATIONS INTO CONTACT STRUCTURES

Consider the problem of approximating a foliation by contact structures, which was solved in the 3-dimensional case by Eliashberg and Thurston in their seminal monography [ET].

On a compact oriented manifold M of dimension $2n + 1 \geq 5$, let \mathcal{F} be a cooriented codimension-1 foliation transverse to the boundary.

The simplest way to such an approximation is a so-called *linear deformation*: that is, the foliation \mathcal{F} , being cooriented, is defined by a global non-vanishing 1-form $\alpha \in \Omega^1(M)$; and one looks for a 1-form $\lambda \in \Omega^1(M)$ such that $\alpha_t = \alpha + t\lambda$ is contact for every small enough positive t .

The actual geometric nature of the problem will appear through an elementary computation.

Recall [CC1] that the integrability of α amounts to the existence of a 1-form η on M such that $d\alpha = \eta \wedge \alpha$; that η is then leafwise closed; the integral of η on every tangential loop being the logarithm of the linear holonomy of the loop. The restriction $\eta|_{\mathcal{F}}$ is uniquely determined by α . One may call η a *holonomy form* associated to α . Then, for any smooth function F on M , $\eta + dF$ is a holonomy form associated with $e^F\alpha$. The cohomology class of $\eta|_{\mathcal{F}}$ in $H^1(\mathcal{F})$ (recall Definition 2.9) thus depends only on the foliation \mathcal{F} , not on the choice of α .

LEMMA 6.1. *Let \mathcal{F} , α , η be as above. If $\lambda \in \Omega^1(M)$ and if $\lambda|_{\mathcal{F}}$ is leafwise η -Liouville, then $\alpha + t\lambda$ is contact for every small enough positive t .*

Proof of Lemma 6.1. For $\theta \in \Omega^*(M)$, we use the notation $d_\eta\theta$ for $d\theta - \eta \wedge \theta$, although η being in general not globally closed on M , the operator d_η^2 is not in general a differential operator on $\Omega^*(M)$: its square vanishes in restriction to \mathcal{F} . One gets straightforwardly

$$(d_\eta\lambda)^n = (d\lambda)^n - n\eta \wedge \lambda \wedge (d\lambda)^{n-1}$$

$$(9) \quad \alpha_t \wedge (d\alpha_t)^n = t^n \alpha \wedge (d_\eta\lambda)^n + t^{n+1} \lambda \wedge (d\lambda)^n$$

Hence, a *sufficient* condition for α_t to be contact for every small enough positive t is that $\alpha \wedge (d_\eta\lambda)^n$ be a volume form on M . \square

REMARK 6.2. The existence of such a form λ , and the leafwise conformal class of $d_\eta\lambda|_{\mathcal{F}}$, depend only on the cooriented foliation \mathcal{F} , not on the choice of α . Indeed, let λ be leafwise η -Liouville. Change α to $e^F\alpha$ for some smooth function F on M ; then, $e^F\lambda$ is leafwise $(\eta + dF)$ -Liouville (Remark 2.1).

DEFINITION 6.3. We call \mathcal{F} *holonomous* if every minimal set contains a tangential loop whose linear holonomy is nontrivial.

In dimension 3, Eliashberg and Thurston proved that the condition of being holonomy-rich is necessary and sufficient for a cooriented foliation to admit a linear deformation ([ET], Theorem 2.1.2.). In the higher dimensions, being holonomy-rich remains *necessary* for the existence

of a leafwise holonomy-Liouville form: indeed, in this direction, the arguments of [ET] hold as well in all dimensions (recently, Lauran Toussaint has given an alternative proof [T20]).

THEOREM C. *On a compact manifold M of dimension $2n + 1 \geq 5$ with smooth boundary, let \mathcal{F} be a cooriented codimension-one foliation which is transverse to ∂M , taut (Definition 4.2), holonomous, and such that every leaf meets ∂M . Assume that \mathcal{F} admits a nondegenerate leafwise 2-form ω . Choose on M a nonsingular 1-form α defining \mathcal{F} , and a 1-form η such that $d\alpha = \eta \wedge \alpha$.*

Then, \mathcal{F} admits a leafwise 1-form $\lambda \in \Omega^1(\mathcal{F})$ such that

- (1) *$d_\eta \lambda$ is nondegenerate, and homotopic to ω as a nondegenerate leafwise 2-form;*
- (2) *λ restricts to a negative overtwisted contact form on every leaf of $\partial \mathcal{F} = \mathcal{F}|_{\partial M}$.*

In consequence (Lemma 6.1), \mathcal{F} admits a linear deformation into a contact structure for which the leaves of $\partial \mathcal{F}$ (cooriented by a vector tangential to \mathcal{F} and pointing outward M followed by a vector tangential to ∂M and positively transverse to \mathcal{F}) are negative contact submanifolds.

REMARK 6.4. The tautness hypothesis in Theorem C can certainly be weakened, and maybe suppressed. This hypothesis would be a serious restriction in dimension 3, since after the classical Novikov theorem, many 3-manifolds do not admit any taut foliation. But it is known that the case is different on a manifold M of higher dimension: every cooriented hyperplane field on M is homotopic to a smooth foliation, with nontrivial linear holonomy, and whose leaves are dense in M [M17]. (One may even prescribe $\mathcal{F}|_{\partial M}$.) Such a foliation is in particular taut and holonomous. Since, moreover, both properties of holonomousness and tautness are clearly open in the space of codimension-one foliations on M , we conclude that for every homotopy class of almost symplectic cooriented hyperplane fields on M , our Theorem C applies to a *nonempty open subset* of the space of foliations lying in this class.

Proof of Theorem C. The foliation being holonomous means that M contains in its interior a finite disjoint union Γ of embedded oriented circles (one in each minimal set) such that

- Each component γ of Γ is tangential to \mathcal{F} , and $h_\gamma = \int_\gamma \eta \neq 0$;
- The closure of every leaf of \mathcal{F} contains at least one component of Γ .

After changing each loop γ to a nearby small perturbation, in the same leaf, of a positive or negative multiple of γ , one can moreover

arrange that $h_\gamma < -c_n$ (the constant of Lemma 3.1). In particular, $h_\gamma < 0$ (Remark 3.2).

We now define λ close to γ .

The linear holonomy being nontrivial, γ admits a small compact tubular neighborhood $T_\gamma \cong \mathbf{S}^1 \times \mathbf{D}^{2n}$ in which \mathcal{F} coincides with a standard, somehow linear model: namely, \mathcal{F} is defined in this solid torus by the nonvanishing 1-form

$$(10) \quad \alpha = dx_{2n} - hx_{2n}\theta$$

where $h = h_\gamma$, where θ denotes the positive unit volume form on \mathbf{S}^1 , and where x_1, \dots, x_{2n} are the standard coordinates on the compact unit ball $\mathbf{D}^{2n} \subset \mathbb{R}^{2n}$. Note that

- $\eta = h\theta$ is a holonomy form associated with α in T_γ ;
- $\mathcal{F}|_{T_\gamma}$ has a unique compact leaf $L_\gamma \cong \mathbf{S}^1 \times \mathbf{D}^{2n-1}$, defined by $x_{2n} = 0$;
- \mathcal{F} is transverse to $\partial T_\gamma \cong \mathbf{S}^1 \times \mathbf{S}^{2n-1}$;
- The induced foliation $\mathcal{F}|_{\partial T_\gamma}$ consists of two $(2n)$ -dimensional Reeb components with common boundary ∂L_γ , the one on $x_{2n} \geq 0$, the other on $x_{2n} \leq 0$.

Consider in T_γ the projection $\pi : T_\gamma \rightarrow L_\gamma$ parallelly to the x_{2n} -axis, and the 1-form

$$\rho = (1 - x_1^2 - \dots - x_{2n-1}^2)\eta + x_1 dx_1 + \dots + x_{2n-1} dx_{2n-1}$$

The restriction $\rho|_{L_\gamma}$ draws on L_γ a $(2n)$ -dimensional Reeb component; while in restriction to ∂T_γ one has

$$\rho = \pi^*(\rho|_{L_\gamma}) = -x_{2n}\alpha$$

Endow the solid torus $L_\gamma \cong \mathbf{S}^1 \times \mathbf{D}^{2n-1}$, oriented by ω , with the 1-form λ and with the vector field Z given by Lemma 3.1 which *enters* L_γ transversely through ∂L_γ . Since $\theta(Z) = 1$, after pushing λ and Z by a self-diffeomorphism of $\mathbf{S}^1 \times \mathbf{D}^{2n-1}$ preserving the projection to \mathbf{S}^1 , we can arrange that moreover, $\rho(Z) < 0$ on L_γ . Extend λ over T_γ as the leafwise 1-form $\pi^*(\lambda)|_{\mathcal{F}}$ (also denoted by λ); extend Z over T_γ as the vector field that lifts Z through π tangentially to \mathcal{F} (also denoted by Z). After Lemma 3.1, λ is leafwise η -Liouville in T_γ , and its η -dual vector field is positively colinear to Z . The space $U(n)/SO(2n)$ being simply connected, $d_\eta \lambda$ is homotopic to $\omega|_{T_\gamma}$ as a nondegenerate leafwise 2-form on $\mathcal{F}|_{T_\gamma}$.

The function $\rho(Z)$ being negative on L_γ , the vector field Z enters transversely T_γ through ∂T_γ . After Lemma 3.1, the contact form $\lambda|_{\partial L_\gamma}$ is overtwisted. For every leaf L of $\mathcal{F}|_{T_\gamma}$, since ∂L accumulates on ∂L_γ in ∂T_γ , it follows that $\lambda|_{\partial L}$ is overtwisted as well (any overtwisted ball

in ∂L_γ can be pushed into ∂L by an isocontact embedding close to the identity, after Gray's stability theorem). In other words, ∂L is of overtwisted contact type and concave with respect to $d_\eta \lambda|_L$ (Definition 2.7, Lemma 2.13).

After λ has thus been constructed over the union T of the T_γ 's, Theorem *B* then allows one to complete the construction over M . Here are some precisions.

In the cobordism $W = M \setminus \text{Int}(T)$ between $\partial_- W = \partial T$ and $\partial_+ W = \partial M$, the foliation $\mathcal{F}|_W$ is transverse to ∂W , taut, and every leaf meets $\partial_- W$ and $\partial_+ W$. Extend α and η from T to M , such that α defines \mathcal{F} over M , and that η is a holonomy form associated to α over M .

After Theorem *B*, there is a η -Liouville leafwise 1-form $\lambda' \in \Omega^1(\mathcal{F}|_W)$ restricting, on every leaf ℓ of $\partial_+ \mathcal{F} = \mathcal{F}|_{\partial M}$ (resp. $\partial_- \mathcal{F} = \mathcal{F}|_{\partial T}$), to an overtwisted contact form which is negative (resp. positive) — here, ℓ is cooriented as a component of the boundary of a leaf of $\mathcal{F}|_W$ — and such that moreover, $d_\eta \lambda'$ is homotopic to ω as a nondegenerate leafwise 2-form on $\mathcal{F}|_W$.

There remains to paste the two pieces. After the h-principle for overtwisted leafwise contact structures on foliations [BEM], there is an isotopy ϕ of ∂T tangential to $\mathcal{F}|_{\partial T}$ and such that λ and $\phi^*(\lambda')$ define the same leafwise contact structure on $\mathcal{F}|_{\partial T}$.

On $\mathcal{O}p_T(\partial T)$ (resp. $\mathcal{O}p_W(\partial T)$), the leafwise 1-form λ (resp. λ') defines for \mathcal{F} a leafwise even contact structure ε (resp. ε') whose characteristic foliation \mathcal{L} (resp. \mathcal{L}') is a 1-dimensional foliation transverse to ∂T . Extend ϕ to an isotopy of W tangential to $\mathcal{F}|_W$, still denoted by ϕ , such that \mathcal{L} and $\phi^*(\mathcal{L}')$ match along ∂T , giving a global smooth 1-dimensional foliation on $\mathcal{O}p_M(\partial T)$. Thus, ε and $\phi^*(\varepsilon')$ give a global even contact structure on $\mathcal{O}p_M(\partial T)$. Then, since $\phi^*(\eta|_{\mathcal{F}})$ is cohomologous to $\eta|_{\mathcal{F}}$ in $H^1(\mathcal{F}|_W)$ (recall Definition 2.9), multiplying $\phi^*(\lambda')$ by a convenient positive function, one gets again a η -Liouville leafwise 1-form λ'' on W , see Remark 6.2. Finally, in view of Lemma 2.13, after multiplying again λ'' by a convenient positive function, the η -Liouville leafwise 1-forms λ and λ'' moreover match along ∂T , and define a global η -Liouville leafwise 1-form for \mathcal{F} over M . \square

REMARK 6.5. Our method does not seem to be able to produce contactizing linear deformations for taut foliations on closed manifolds. Precisely, Proposition 6.7 will show that starting from Theorem *B*, the concave boundary cannot be eliminated in the same way as we have eliminated the convex boundary and got Theorem *C*.

This problem raises the general question of whether the Eliashberg-Gromov tightness criterion for fillable contact structures admits the

following *foliated and conformal* analogue for leafwise contact structures in all dimensions.

QUESTION 6.6. Let M be an oriented compact manifold of dimension $2n + 1 \geq 5$ with nonempty smooth boundary, endowed with a cooriented codimension-1 foliation \mathcal{F} , transverse to ∂M . Let α , η , λ be, respectively, a defining form for \mathcal{F} , an associated holonomy 1-form, and a η -Liouville leafwise 1-form which restricts to a positive contact form on every leaf of $\partial\mathcal{F}$. Does it follow that λ is tight on every leaf of $\partial\mathcal{F}$?

PROPOSITION 6.7. *The answer to Question 6.6 is positive for the model foliation \mathcal{F} defined by Equation (10) on $T^{2n+1} = \mathbf{S}^1 \times \mathbf{D}^{2n}$, for every $n \geq 2$ and $h \neq 0$.*

Proof of Proposition 6.7. Let us begin, to fix ideas, with the case $n = 2$. In this case, Proposition 6.7 is a simple application of the Eliashberg-Gromov tightness criterion ([E91], [ET] and [G85]) in the noncompact framework: every contact 3-manifold $(\partial M, \xi)$ which bounds a symplectic 4-manifold (M, ω) with bounded symplectic geometry at infinity is tight.

Consider, on $\partial\mathcal{F}$, the leafwise contact structure

$$\xi = \ker(\lambda|_{\partial T^5}).$$

Let L_0 and ℓ_0 denote the compact leaf of \mathcal{F} and its boundary, respectively. Since every other leaf ℓ of $\partial\mathcal{F}$ accumulates on ℓ_0 , if $\xi|_{\ell_0}$ were overtwisted, then $\xi|_{\ell}$ would also be overtwisted. Hence, it suffices to prove that $\xi|_{\ell}$ is tight for every *non-compact* leaf ℓ of $\partial\mathcal{F}$.

Every noncompact leaf L of \mathcal{F} being without holonomy, the holonomy 1-form $\eta = h\theta$ is exact on L , hence (Remark 2.1) $\lambda|_L$ is conformal to a genuinely Liouville form on L . In fact, the function $\ln(|x_4|)$ is a primitive of η in restriction to every noncompact leaf. The fact that this function is bounded from above on every such leaf, which is specific to foliations of very simple dynamics like \mathcal{F} , seems to be crucial in the proof.

We now show that any non-compact leaf L has bounded symplectic geometry. The 2-form $d_\eta\lambda$ on T^5 , being leafwise nondegenerate, admits a leafwise almost complex structure J (an automorphism of the vector bundle $T\mathcal{F}$ such that $J^2 = -\text{id}$) which preserves ξ at every point of ∂T^5 , and such that

$$g = d_\eta\lambda(\cdot, J\cdot)$$

defines a Riemannian metric on every leaf L of \mathcal{F} . By compactness of T^5 , the metric $g|_L$ has *bounded geometry*, meaning that $g|_L$ is a complete Riemannian metric on L , whose injectivity radius is bounded away from zero, and whose sectional curvature is bounded.

The 1-form $\lambda' = |x_4|^{-1}\lambda|_L$ is genuinely Liouville on L ; and the exact symplectic form $d\lambda' = |x_4|^{-1}d_\eta\lambda$ *dominates* $\xi|_\ell$, in the sense that $d\lambda'$ is nondegenerate on $\xi|_\ell$ at every point of ℓ . Set

$$g' = \frac{g}{|x_4|} = d\lambda'(\cdot, J\cdot).$$

The metric g' is compatible with $d\lambda'$. It remains to verify that g' is of bounded geometry on L .

One has $g' \geq g$, so that g' is complete as well on L . It is convenient to introduce the solid cylinder

$$C = [-1, 1] \times \mathbf{D}^4$$

and, for every $s \in \mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$, the immersion

$$j_s : C \hookrightarrow T^5 : (t, x) \mapsto (s + t, x)$$

The foliation $\mathcal{C} = j_s^*(\mathcal{F})$ of C does obviously not depend on s . Consider on \mathcal{C} the smooth family of leafwise Riemannian metrics, parametrized by $s \in \mathbf{S}^1$

$$g_s = e^{-ht} j_s^*(g)$$

Let $\iota_0 > 0$ be the minimum of their injectivity radii, and $\sigma_0 < +\infty$ be the maximum of the absolute values of their sectional curvatures.

Consider the leaf L_a of \mathcal{C} through the point $(0, a)$, with $a = (a_1, \dots, a_4)$ in \mathbf{D}^4 and $a_4 \neq 0$. Since $x_4|_{L_a} = a_4 e^{ht}$, on L_a , one has $j_s^*(g') = |a_4|^{-1}g_s$; and the following bounds follow at once on L_a :

$$\iota(j_s^*(g')) = |a_4|^{-1}\iota(g_s) \geq \iota_0$$

$$|\sigma(j_s^*(g'))| = |a_4||\sigma(g_s)| \leq \sigma_0.$$

The proof of Proposition 6.7 in the higher dimensions is much alike, but instead of the original Eliashberg-Gromov tightness criterion, one applies Niederkrüger's tightness criterion [N06] (see also [BEM] paragraph 10), in the noncompact framework, under the hypothesis of bounded geometry (this noncompact version of the criterion does not appear in the litterature, but the generalization is straightforward). \square

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