



HAL
open science

Anonymous, neutral, and resolute social choice revisited

Ali Ozkes, M. Remzi Sanver

► **To cite this version:**

Ali Ozkes, M. Remzi Sanver. Anonymous, neutral, and resolute social choice revisited. *Social Choice and Welfare*, 2021, 57 (1), pp.97-113. 10.1007/s00355-020-01308-5 . hal-03341695

HAL Id: hal-03341695

<https://hal.science/hal-03341695>

Submitted on 17 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Anonymous, neutral, and resolute social choice revisited*

Ali I. Ozkes[†] M. Remzi Sanver[‡]

September 17, 2020

Abstract

We revisit the incompatibility of anonymity and neutrality in single-valued social choice. We first analyze the irresoluteness outlook these two axioms together with Pareto efficiency impose on social choice rules and deliver a method to refine irresolute rules without violating anonymity, neutrality, and efficiency. Next, we propose a weakening of neutrality called *consequential neutrality* that requires resolute social choice rules to assign each alternative to the same number of profiles. We explore social choice problems in which consequential neutrality resolves impossibilities that stem from the fundamental tension between anonymity, neutrality, and resoluteness.

Keywords: anonymity, efficiency, neutrality, resoluteness

JEL Codes: D71, D72, D82

We are grateful to Walter Bossert, Denis Cornaz, Fatih Demirkale, Lars Ehlers, Ayça Ebru Giritligil, Jeffrey Hatley, Sean Horan, Hervé Moulin, Jean Lainé, Clemens Puppe, Yves Sprumont, and William Zwicker for helpful discussions. The paper extensively benefited from the thoughtful comments of three anonymous referees to whom we are grateful. Our work is partly supported by the projects ANR-14-CE24-0007-01, CoCoRICo-CoDec, and IDEX ANR-10-IDEX-0001-02 PSL MIFID.

[†]Wirtschaftsuniversität Wien, Institute for Markets and Strategy, Welthandelsplatz 1, 1020, Vienna, Austria. E-mail: ali.ozkes@wu.ac.at

[‡]Université Paris-Dauphine, PSL Research University, CNRS, UMR [7243], LAMSADE, 75016 Paris, France. E-mail: remzi.sanver@lamsade.dauphine.fr

1 Introduction

Equal treatment of individuals as well as of alternatives are among the core principles of democratic decision-making. Equal treatment of individuals is usually ensured by the *anonymity* condition, which requires the social choice to be invariant under renaming individuals. The typical condition to ensure equal treatment of alternatives, on the other hand, is *neutrality*, which requires the social choice to change in compliance with renaming of alternatives.

The logical incompatibility between anonymity and neutrality while ensuring an untied outcome is among the most well-known results in social choice theory. Moulin (1980, 1991) characterizes the sizes of social choice problems that admit anonymous and neutral social choice rules (SCRs) that are resolute, *i.e.*, that choose a unique alternative at any profile. More precisely, a social choice problem with n individuals and m alternatives admits an anonymous, neutral, and resolute SCR if and only if m cannot be written as the sum of some divisors of n that exceed 1 (Moulin, 1991). When (Pareto) efficiency is imposed together with anonymity and neutrality, this requirement is strengthened to “ n not having a prime divisor less than or equal to m ” (Moulin, 1980).¹

How severe is this tension between anonymity and neutrality? Campbell and Kelly (2015) show the rarity of cases where anonymous, neutral, and resolute SCRs exist: when the number of individuals is divisible by at least two distinct primes, only a finite number of social choice problems admit anonymous, neutral, and resolute SCRs. Also, when the number of alternatives exceeds the smallest prime dividing the number of individuals, a resolute SCR is anonymous and neutral only if it chooses alternatives that are in the bottom half of preferences of all individuals. Adding efficiency to anonymity and neutrality restricts the sizes of social choice problems that admit anonymous, neutral, and resolute SCRs even further.

Do these results leave any hope for guaranteeing equal treatment of voters and alternatives for untied collective choice? We reject pessimism by identifying a weakening of neutrality which allows a vast range of possibilities while pan-

¹Zwicker (2016) delivers an introduction to the theory of voting where major results regarding anonymity and neutrality are included.

dering to a very significant aspect of equal treatment of alternatives. This new condition that we call *consequential neutrality* requires that all alternatives are chosen at the same number of preference profiles.

For example, with an odd number of individuals, fixing two alternatives x and y and two individuals i and j , one can define the resolute SCR that picks the best alternative of individual i when a majority of individuals prefer x to y and otherwise, picks the best alternative of individual j . This SCR presents a procedure that depends on the names of the alternatives, hence failing neutrality. On the other hand, being consequentially neutral, it ensures that every alternative is chosen at the same number of preference profiles, thus putting forward an ex-ante fairness property that is more outcome-oriented compared to the classical neutrality approach that entails a more procedure-oriented equal treatment of alternatives.

Recent research on anonymous and neutral social choice mostly focus on rules that assign orderings (instead of alternatives) to preference profiles. Bubboloni and Gori (2014) and Doğan and Giritligil (2015) characterize the sizes of social choice problems for which anonymous and neutral rules that are resolute, *i.e.*, that assign strict social welfare orderings to profiles, exist. Building on the algebraic approach in Bubboloni and Gori (2014), Bubboloni and Gori (2015) propose a weakening of neutrality by assuming that alternatives are divided into sub-classes and requiring equal treatment of alternatives within each sub-class, while alternatives in different sub-classes may be differently treated. They provide necessary and sufficient conditions for the existence of (reversal) symmetric majority rules that satisfy this weakening of neutrality together with a weaker version of anonymity.² Bubboloni and Gori (2016) adopt the aforementioned versions of anonymity and neutrality together with efficiency and characterize when rules that assign orderings to profiles admit a resolute refinement that preserves these properties. Their characterization identifies the necessary and sufficient arithmetical conditions on the sizes of sub-committees and sub-classes.

²They propose a weaker version of anonymity in a similar way by assuming that individuals are divided into sub-committees and requiring that, within each sub-committee, individuals have equal influence on the collective decision, while people in different sub-committees may enjoy different levels of influence.

King and Powers (2018), on the other hand, dispense with neutrality altogether and identify a characterization of rules that satisfy anonymity, monotonicity, and cancellation, in the case of two alternatives.

We start by analyzing the structure of irresoluteness imposed by anonymity, neutrality, and efficiency, a previously overlooked matter. We generalize the characterization of Moulin (1980) by completely describing the sizes of unavoidable ties under these conditions (Theorem 2). This generalization paves the way to identifying a method to refine SCRs that are “more irresolute than necessary,” while anonymity, efficiency, and neutrality are preserved (Theorem 3).

We then turn to our analysis of consequential neutrality for resolute SCRs. We start with counting the number of resolute SCRs that are neutral and those that are consequentially neutral (CN) as a function of the size of the social choice problem (Theorem 4). An analytical comparison of these two numbers seems beyond reach, so we take a computational approach where we compute the numbers of resolute SCRs in each class for a small set of values of the size of the social choice problem. These numerical exercises that we report on show strong tendencies in the comparison of the numbers of CN and neutral SCRs, hence we conjecture that the class of resolute SCRs that are CN is considerably larger than those that are neutral.

Thereafter, we discuss the possibility of refining anonymous, efficient, and neutral SCRs by replacing neutrality with consequential neutrality and deliver a possibility result under certain conditions (Theorem 5). We also identify some cases where these conditions are not satisfied but there exist anonymous, CN, and resolute SCRs (Theorem 6). These positive results do not hold over all conceivable social choice problems: we point to instances where anonymity, consequential neutrality, efficiency, and resoluteness turn out to be incompatible (Theorem 7). These are instances where the incompatibility prevails even without efficiency. However, even in those cases, anonymous, CN, and resolute SCRs exist. In fact, we are able to identify a large class of social choice problems, namely those where the number of alternatives exceeds the number of individuals, for which anonymity, consequential neutrality, and resoluteness are compatible (Theorem 9).

The paper is organized as follows. Section 2 gives basic notation and notions. Section 3 delivers a generalization of the classical result on incompatibility of anonymity, neutrality, and efficiency with resoluteness and proposes a refinement method towards resoluteness. Section 4 introduces consequential neutrality and presents our more permissive results when it replaces neutrality. Section 5 concludes.

2 Basic notions and notation

Writing \mathbb{N} for the set of natural numbers and picking $m, n \in \mathbb{N} \setminus \{1\}$, we conceive a *social choice problem* as a set A of alternatives with $\#A = m$ and a set N of individuals with $\#N = n$. We refer to (m, n) as the *size* of the social choice problem (A, N) . Writing $\mathcal{L}(X)$ for the set of linear orders, *i.e.*, complete, asymmetric, and transitive binary relations on a given set X , let $P_i \in \mathcal{L}(A)$ denote the *preference* of $i \in N$.³ An n -tuple of such individual preferences indicates a (*preference*) *profile* $P_N \in \mathcal{L}(A)^N$. A *social choice rule* (SCR) is a mapping $f : \mathcal{L}(A)^N \rightarrow \mathcal{A}$, where $\mathcal{A} = 2^A \setminus \{\emptyset\}$ is the set of non-empty subsets of A .

Given any two sets S and T , we write $S \subseteq T$ whenever S is a subset of T and $S \subset T$ whenever S is a proper subset of T . We let $P_N|_B$ denote the restriction of $P_N \in \mathcal{L}(A)^N$ to those alternatives in $B \in \mathcal{A}$ so that $P_N|_B \in \mathcal{L}(B)^N$ and $xP_iy \iff xP_i|_By$ for all $x, y \in B$ and $i \in N$. Given any two SCRs f_1 and f_2 , we say that f_2 *refines* f_1 iff $f_2(P_N) \subseteq f_1(P_N) \forall P_N \in \mathcal{L}(A)^N$ and $f_2(P'_N) \subset f_1(P'_N)$ for some $P'_N \in \mathcal{L}(A)^N$. An SCR f is *resolute* whenever $\#f(P_N) = 1 \forall P_N \in \mathcal{L}(A)^N$. For a resolute SCR f , we write $f(P_N) = x$ in place of $f(P_N) = \{x\}$.

We now define two equal treatment conditions that are at the core of our analysis. For any non-empty finite set X , a *permutation* on X is a bijection $\sigma : X \leftrightarrow X$. Let Σ_X be the set of all permutations on X . We write, by a slight abuse of notation, $\sigma(P_N) = (P_{\sigma(i)})_{i \in N}$ for the profile obtained from $P_N \in \mathcal{L}(A)^N$ by a permutation $\sigma \in \Sigma_N$. An SCR is *anonymous* iff $f(P_N) = f(\sigma(P_N))$

³So, given any distinct $x, y \in A$ and $P_i \in \mathcal{L}(A)$, precisely one of xP_iy and yP_ix holds. Moreover, xP_iy and yP_iz implies xP_iz for all $x, y, z \in A$ and $P_i \in \mathcal{L}(A)$. Finally, xP_ix does not hold for any $x \in A$.

$\forall P_N \in \mathcal{L}(A)^N \forall \sigma \in \Sigma_N$. Again, by an abuse of notation, we write $\sigma(P_i)$ for the preference obtained from $P_i \in \mathcal{L}(A)$ by a permutation $\sigma \in \Sigma_A$ on A , *i.e.*, $x P_i y \iff \sigma(x) \sigma(P_i) \sigma(y) \forall x, y \in A$.

Moreover, we set $\sigma(P_N) = (\sigma(P_i))_{i \in N} \forall P_N \in \mathcal{L}(A)^N$. An SCR is *neutral* iff $f(\sigma(P_N)) = \sigma(f(P_N)) \forall P_N \in \mathcal{L}(A)^N, \forall x \in A$, and $\forall \sigma \in \Sigma_A$.

We close the section by noting that an SCR f is *efficient* iff given any $P_N \in \mathcal{L}(A)^N$ and any $x \in f(P_N)$, $\nexists y \in A \setminus \{x\}$ with $y P_i x \forall i \in N$.

3 Anonymous, neutral, and efficient social choice

Given $k, l \in \mathbb{N}$, we write $k \mid l$ whenever k divides l , *i.e.*, $\frac{l}{k} \in \mathbb{N}$, and $k \nmid l$ otherwise. Let $\mathcal{D}(n) = \{k \in \mathbb{N} : k \mid n\}$ be the set of divisors of $n \in \mathbb{N}$ and $\mathcal{D}^*(n) = \{k \text{ is a prime} : k \mid n\} \cup \{1\}$ be the set consisting of prime divisors of n as well as 1. Thus, $\mathcal{D}^*(n) \subseteq \mathcal{D}(n)$ for all $n \in \mathbb{N}$. For any $m, n \in \mathbb{N}$, the set of divisors of n that do not exceed m is denoted $\mathcal{D}_m(n) = \{d \in \mathcal{D}(n) : d \leq m\}$ and similarly $\mathcal{D}_m^*(n) = \{d \in \mathcal{D}^*(n) : d \leq m\}$. Again, $\mathcal{D}_m^*(n) \subseteq \mathcal{D}_m(n)$ for all $m, n \in \mathbb{N}$. Imposing $\mathcal{D}_m^*(n) = \{1\}$ is shown by Moulin (1980) to be a necessary and sufficient condition for the size of a social choice problem to admit an anonymous, neutral, efficient, and resolute SCR.

Theorem 1 (Moulin (1980)). *A social choice problem (A, N) with size (m, n) admits an anonymous, efficient, neutral, and resolute SCR f if and only if $\mathcal{D}_m^*(n) = \{1\}$.*

The condition $\mathcal{D}_m^*(n) = \{1\}$ in Theorem 1 can be replaced by $\mathcal{D}_m(n) = \{1\}$.⁴ We will refer to $\mathcal{D}_m(n) = \{1\}$ as Condition $\mu(m, n)$.⁵

Theorem 1 gives a complete picture of the sizes of social choice problems where irresoluteness is inevitable but is silent about the structure of irresolute-

⁴Having already noted $\mathcal{D}_m^*(n) \subseteq \mathcal{D}_m(n)$, we now remark that $\mathcal{D}_m^*(n) = \{1\} \implies \mathcal{D}_m(n) = \{1\}$ for all $m, n \in \mathbb{N}$. To prove this by contradiction, we first let $k \in \mathcal{D}_m(n) \setminus \{1\}$. Thus, we have $k \in \mathcal{D}(n)$ and $k \leq m$. Due to the fundamental theorem of arithmetic, k has a prime divisor k^* , which divides n as well, hence $k^* \in \mathcal{D}_m^*(n)$, implying $\mathcal{D}_m^*(n) \neq \{1\}$.

⁵This condition is equivalent to asking the greatest common divisor of $m!$ and n to be 1, as mentioned by Doğan and Girtlilgil (2015), who reconsider the problem through a group theoretic approach. Interestingly, as Doğan and Girtlilgil (2015) as well as Bubboloni and Gori (2014) show, $\gcd(m!, n) = 1$ turns out to be necessary and sufficient for the existence of anonymous and neutral social welfare functions (*i.e.*, functions which assign to every preference profile a strict ranking of alternatives).

ness in such cases. To analyze this, we define $K_f = \{\#f(P_N) : P_N \in \mathcal{L}(A)^N\}$ as the *irresoluteness outlook* of SCR f . So for any natural number $k \leq m$, we have $k \in K_f$ if and only if there exists a profile to which f assigns a set of k alternatives.⁶

Theorem 2. *Take any social choice problem (A, N) with size (m, n) .*

- i.* *An SCR f is anonymous, efficient, and neutral only if $K_f \supseteq \mathcal{D}_m(n)$.*
- ii.* *There exists an anonymous, efficient, and neutral SCR f with $K_f = \mathcal{D}_m(n)$.*

Proof.

i. Take $d \in \mathcal{D}_m(n)$. As $d \in \mathcal{D}(n)$, there exists $t \in \mathbb{N}$ such that $n = dt$. Take any set of alternatives $\{x_1, \dots, x_d\} \subseteq A$ and the partition $\{S_1, \dots, S_d\}$ of N with $S_1 = \{1, \dots, t\}$, $S_2 = \{t+1, \dots, 2t\}$, $S_3 = \{2t+1, \dots, 3t\}$, and so on. Let $X = A \setminus \{x_1, \dots, x_d\}$. Note that X may be empty. Now construct a profile P_N as depicted below:

S_1	S_2	S_3	\dots	S_d
x_1	x_2	x_3	\dots	x_d
\cdot	\cdot	\cdot	\dots	x_1
\cdot	\cdot	\cdot	\dots	\cdot
\cdot	\cdot	\cdot	\dots	\cdot
\cdot	\cdot	x_d	\dots	x_{d-3}
\cdot	x_d	x_1	\dots	x_{d-2}
x_d	x_1	x_2	\dots	x_{d-1}
X	X	X	\dots	X

As f is efficient, $f(P_N) \subseteq \{x_1, \dots, x_d\}$. Note that when $d = 1$, this implies $\#f(P_N) = 1$, hence $1 \in K_f$. Now let $d > 1$. Consider the permutation σ of A such that

$$\sigma(x_i) = x_{i+1} \quad \forall i \in \{1, \dots, d-1\},$$

$$\sigma(x_d) = x_1, \text{ and}$$

$$\sigma(\bar{x}) = \bar{x} \quad \forall \bar{x} \in X.$$

⁶Note that the irresoluteness outlook of an SCR f does not specify to how many profiles f assigns k alternatives when $k \in K_f$.

Next, let $\sigma' \in \Sigma_N$ be such that $\sigma'(i) = i - t$ for all $i \in \{t + 1, \dots, dt = n\}$ and $\sigma'(i) = (d - 1)t + i$ for all $i \in \{1, \dots, t\}$. Note that $\sigma(P_N) = (\sigma')^{-1}(P_N)$.

Since $f(P_N)$ is a nonempty subset of A , there exists $x_i \in f(P_N)$ for some $i \in \{1, \dots, m\}$. By neutrality, $x_i \in f(P_N)$ implies $\sigma(x_i) \in f(\sigma(P_N))$. By anonymity, $x_i \in f(P_N)$ implies $x_i \in f(\sigma(P_N))$. This is only possible when $f(P_N) = \{x_1, \dots, x_d\}$, as $x_i \in f(P_N)$ implies both $x_i \in f(\sigma(P_N))$ and $x_{i+1} \in f(\sigma(P_N))$ for all $i \in \{1, \dots, d - 1\}$ and $x_d \in f(P_N)$ implies both $x_d \in f(\sigma(P_N))$ and $x_1 \in f(\sigma(P_N))$. Hence, $\#f(\sigma(P_N)) = d$, thus $d \in K_f$.

ii. Let $\tau(x, P_N) = \#\{i \in N : xP_i y \forall y \in A \setminus \{x\}\}$ denote the number of individuals that rank x on top of their preferences in the profile P_N . Define the *plurality rule* $\Upsilon : \mathcal{L}(A)^N \rightarrow \mathcal{A}$ so that

$$\Upsilon(P_N) = \{x \in A : \tau(x, P_N) \geq \tau(y, P_N) \forall y \in A\}.$$

We now define the *iterative plurality rule* $v : \bigcup_{B \in \mathcal{A}} \mathcal{L}(B)^N \rightarrow \mathcal{A}$ such that $v(P_N) \subseteq B$ for all $P_N \in \mathcal{L}(B)^N$ and $B \in \mathcal{A}$, which selects the plurality winners after successive restriction of profiles to plurality winners.⁷ Let $\Upsilon^{i+1}(P_N) = \Upsilon(P_N|_{\Upsilon^i(P_N)})$ for all $i \geq 1$ and $\Upsilon^1(P_N) = \Upsilon(P_N)$. Define

$$v(P_N) = \Upsilon^k(P_N),$$

where k is the minimal integer that satisfies $\Upsilon^k(P_N) = \Upsilon^{k+1}(P_N)$. Such an integer always exists given the finiteness of A . It is easily checked that v is anonymous, efficient, and neutral. We will show that $K_v = \mathcal{D}_m(n)$ for all $n, m \in \mathbb{N}$. By definition, for any $x, y \in A$ and any $P_N \in \mathcal{L}(A)^N$, $x, y \in v(P_N)$ implies $\tau(x, P_N|_{\Upsilon^k(P_N)}) = \tau(y, P_N|_{\Upsilon^k(P_N)}) = t$ for some $t \in \mathbb{N}$. Furthermore, for any $z \notin v(P_N)$, $\tau(z, P_N|_{\Upsilon^k(P_N)}) = 0$. Thus, we have $(\#v(P_N))t = n$, implying $K_v \subseteq \mathcal{D}(n)$. As $\#v(P_N) \leq m$, in fact, $K_v \subseteq \mathcal{D}_m(n)$. Given part *i* of the theorem, we have $K_v = \mathcal{D}_m(n)$. \square

Theorem 2 generalizes Theorem 1, which now comes as a corollary; when Condition $\mu(m, n)$ fails, by Theorem 2.*i*, every anonymous, efficient, and neutral SCR has $\#f(P_N) > 1$ for some $P_N \in \mathcal{L}(A)^N$ and when Condition $\mu(m, n)$ is

⁷Note that the iterative plurality rule presents a generalization of our concept of an SCR in the sense that it is defined on the domain of preference profiles over every $B \in \mathcal{A}$.

satisfied, the iterative plurality rule v ensures the existence of an anonymous, efficient, and neutral f with $\#f(P_N) = 1$ for all $P_N \in \mathcal{L}(A)^N$.⁸

As a matter of fact, Theorem 2 establishes that the irresoluteness outlook K_v of the iterative plurality rule is the best that an anonymous, efficient, and neutral SCR can achieve. To be sure, this does not mean that v cannot be refined while preserving anonymity, neutrality, and efficiency. But surely, any anonymous, efficient, and neutral refinement of v will have the same irresoluteness outlook as v itself. Moreover, as we formally state in the next theorem, any anonymous, efficient, and neutral SCR whose irresoluteness outlook is a proper superset of K_v can be refined while anonymity, neutrality, and efficiency are preserved. Before, we define, for any SCR $f : \mathcal{L}(A)^N \rightarrow \mathcal{A}$, the composite rule $vf : \mathcal{L}(A)^N \rightarrow \mathcal{A}$ by $vf(P_N) = v(P_N|_{f(P_N)}) \forall P_N \in \mathcal{L}(A)^N$.

Theorem 3. *Given a social choice problem (A, N) , an anonymous, efficient, and neutral SCR f with $K_f \supset K_v$ admits the anonymous, efficient, and neutral refinement $vf : \mathcal{L}(A)^N \rightarrow \mathcal{A}$ with $K_{vf} = K_v$.*

Proof. Take any social choice problem (A, N) and any anonymous, efficient, and neutral f with $K_f \supset K_v$. By definition of vf , we have $vf(P_N) \subseteq f(P_N)$ for all $P_N \in \mathcal{L}(A)^N$. Take any $P'_N \in \mathcal{L}(A)^N$. First, let $\#f(P'_N) \in K_v$. As $K_v = \mathcal{D}_m(n)$ by Theorem 2, we have $\#f(P'_N) \in \mathcal{D}_m(n)$ which implies $\#vf(P'_N) \in K_v$ as well. Now, let $\#f(P'_N) \in K_f \setminus K_v$. As $\#vf(P'_N) \in \mathcal{D}(n)$ by definition of v , we cannot have $vf(P'_N) = f(P'_N)$. Thus, $vf(P'_N) \subset f(P'_N)$. Furthermore, as $\#vf(P'_N) \leq m$, we have $\#vf(P'_N) \in \mathcal{D}_m(n) = K_v$. Noting that vf is anonymous, efficient, and neutral concludes the proof. \square

As an instance of Theorem 3, consider the social choice problem $A = \{x, y, z\}$, $N = \{1, 2\}$, and $P_N \in \mathcal{L}(A)^N$ with xP_1yP_1z and zP_2yP_2x . For the Borda rule β , which chooses the alternatives that have the minimal sum of ranks over individuals, we have $\beta(P_N) = \{x, y, z\}$, thus $3 \in K_\beta$, while one can check that $K_v = \{1, 2\}$, and, indeed, the composite rule $v\beta$, which gives $v\beta(P_N) = \{x, z\} \subset \beta(P_N)$, refines β and is anonymous, efficient, and neutral.

⁸Part (ii) of Theorem 2 can be proven with rules other than the iterative plurality rule, such as Coombs rule, as in Moulin (1983) p. 24. We are thankful to a referee for pointing out this fact.

Theorem 3 points to the possibility of shrinking the irresoluteness outlook of an anonymous, efficient, and neutral SCR f down to K_v by composing f with v , while vf preserves all three properties. However, this does not mean that vf can refine f at every $P_N \in \mathcal{L}(A)^N$ with $f(P_N) \supset v(P_N)$. To see this, let $m = 5$ and $n = 6$, and consider the following profile P_N .

P_1	P_2	P_3	P_4	P_5	P_6
x	x	t	r	y	y
y	y	z	z	x	x
z	z	x	y	r	z
r	t	y	x	z	t
t	r	r	t	t	r

We have $v(P_N) = \{x, y\}$ and $\varphi(P_N) = \{x, y, z\}$, where φ denotes the *fallback bargaining rule*.⁹ However, the composite rule $v\varphi$ does not refine φ at this profile, *i.e.*, $v\varphi(P_N) = \{x, y, z\}$.

4 Consequential neutrality

Let $W_f(S) = \{P_N \in \mathcal{L}(A)^N : f(P_N) = S\}$ for any $S \in \mathcal{A}$ be the set of profiles to which an SCR f assigns S . We write $W_f(x)$ in place of $W_f(\{x\})$ in case of singletons. Define an SCR $f : \mathcal{L}(A)^N \rightarrow \mathcal{A}$ to be *consequentially neutral* (CN) iff $\#W_f(S) = \#W_f(S')$ for all $S, S' \in \mathcal{A}$ with $\#S = \#S'$. Thus, when f is resolute, CN requires that any two alternatives are assigned to the same number of profiles.

Proposition 1. *An SCR f is neutral only if f is CN.*

Proof. Take any neutral SCR f . Let $S, S' \subseteq A$ be such that $S = \{x_1, \dots, x_h\}$ and $S' = \{x'_1, \dots, x'_h\}$ for some $h \in \{1, \dots, m\}$. Take any $P_N \in W_f(S)$. Thus, we have $f(P_N) = S$. Take any $\sigma : A \leftrightarrow A$ such that $\sigma(x_i) = x'_i$ for all $i \in \{1, \dots, h\}$. By neutrality, $P_N \in W_f(S)$ implies $f(\sigma(P_N)) = S'$, or $\sigma(P_N) \in$

⁹Define, for any $P_N \in \mathcal{L}(A)^N$, $\alpha(P_N, x, k) = \#\{i \in N : \#\{y \in A \setminus \{x\} : xP_i y\} \geq k\}$, which gives the number of individuals that rank x higher than at least k alternatives. Now, define the fallback bargaining rule $\varphi : \mathcal{L}(A)^N \rightarrow \mathcal{A}$ so that, $\forall x \in A$, $x \in \varphi(P_N)$ iff

$$\max_{k \in \{0, \dots, m-1\}} \{k \in \mathbb{N} : \alpha(P_N, x, k) = n\} \geq \max_{k \in \{0, \dots, m-1\}} \{k \in \mathbb{N} : \alpha(P_N, y, k) = n\}$$

for all $y \in A \setminus \{x\}$.

$W_f(S')$. Thus, we have $\#W_f(S) \leq \#W_f(S')$. As $\#W_f(S') \leq \#W_f(S)$ holds as well, the proof is completed. \square

Now, we focus on resolute SCRs. Given $m, n \in \mathbb{N} \setminus \{1\}$ with $\#A = m$ and $N = \{1, \dots, n\}$, we write $\mathcal{F}_{m,n}^{CN}$ for the set of resolute SCRs that are CN; $\mathcal{F}_{m,n}^{NEUTRAL}$ for the set of resolute and neutral SCRs; and $\mathcal{F}_{m,n}^{RESOLUTE}$ for the set of resolute SCRs.

We know from Proposition 1 that $\mathcal{F}_{m,n}^{NEUTRAL} \subseteq \mathcal{F}_{m,n}^{CN}$ for all $m, n \in \mathbb{N} \setminus \{1\}$. The inclusion is indeed strict. To see this, fix a profile $\bar{P}_N \in \mathcal{L}(A)^N$. Take the permutation $\sigma \in \Sigma_A$ such that $\sigma(x_1) = x_2$, $\sigma(x_2) = x_1$, and if $m \geq 3$, let $\sigma(x_j) = x_j$ for all $j \in \{3, \dots, m\}$. Take any partition $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_m\}$ of $\mathcal{L}(A)^N$ such that $\#\mathcal{P}_i = \#\mathcal{P}_j$ for all $i, j \in \{1, \dots, m\}$ and $\bar{P}_N, \sigma(\bar{P}_N) \in \mathcal{P}_1$. Define $f : \mathcal{L}(A)^N \rightarrow A$ such that $f(P_N) = x_i$ for all $P_N \in \mathcal{P}_i$ and $i \in \{1, \dots, m\}$. By construction, $\#\{P_N \in \mathcal{L}(A)^N : f(P_N) = x_i\} = \#\{P_N \in \mathcal{L}(A)^N : f(P_N) = x_j\}$ for all $i, j \in \{1, \dots, m\}$, hence f is CN. However, it fails neutrality as $f(\bar{P}_N) = x_1 = f(\sigma(\bar{P}_N))$.

This raises the following two issues: How large is $\mathcal{F}_{m,n}^{CN}$ compared to $\mathcal{F}_{m,n}^{NEUTRAL}$ and which interesting resolute SCRs, if any, does it contain? We address the first question through a counting approach.¹⁰

Theorem 4. *The following equalities hold.*

- i.* $\#\mathcal{F}_{m,n}^{RESOLUTE} = m^{(m!^n)}$.
- ii.* $\#\mathcal{F}_{m,n}^{NEUTRAL} = m^{(m!^{n-1})}$.
- iii.* $\#\mathcal{F}_{m,n}^{CN} = \frac{(m!^n)!m!}{(m!^{n-1}(m-1)!)^m}$.

Proof.

- i.* This is straightforward, as we noted before that $\#\mathcal{L}(A)^N = m!^n$.
- ii.* We say that P'_N is a *renaming* (for alternatives) of P_N iff there exists $\sigma \in \Sigma_A$ such that $P'_N = \sigma(P_N)$. We write $P_N \rho P'_N$ when P'_N is a renaming of P_N . Noting that $\rho \subseteq \mathcal{L}(A)^N \times \mathcal{L}(A)^N$ is an equivalence relation, we write

¹⁰Perry and Powers (2008) propose a method to count, in the case of two alternatives, the number of aggregation rules that satisfy anonymity and neutrality. Bubboloni and Gori (2016) give a method to perform these counts for any number of alternatives.

\mathcal{E} for the partition of $\mathcal{L}(A)^N$ provided by ρ . Note that each profile P_N admits $m!$ renamings and that $\#\mathcal{L}(A)^N = m!^n$. Thus, \mathcal{E} admits $m!^{n-1}$ equivalence classes, each of which contains $m!$ profiles. We write $\mathcal{E} = \{\mathcal{E}_i\}_{i \in \{1, \dots, m!^{n-1}\}}$ with $\#\mathcal{E}_i = m!$ for all $i \in \{1, \dots, m!^{n-1}\}$.

Take any $\mathcal{E}_t \in \mathcal{E}$ and pick any $P_N \in \mathcal{E}_t$. Let $f(P_N) = x$ for some $x \in A$. Neutrality, together with the definition of \mathcal{E}_t , determines $f(P'_N)$ for all $P'_N \in \mathcal{E}_t \setminus \{P_N\}$. As there are m alternatives that can be assigned to P_N by f , there are m neutral and resolute ways an SCR can be defined for the profiles in \mathcal{E}_t . As $t \in \{1, \dots, m!^{n-1}\}$, there are $m^{(m!^{n-1})}$ neutral and resolute SCRs altogether.

iii. First observe that, given any two natural numbers p and q , there are $\frac{(pq)!}{q!^p}$ ways to partition a set of cardinality pq into p sets, each with cardinality q . Hence, there are $\frac{(m!^n)!}{(m!^{n-1}(m-1)!)^m}$ ways to partition $\mathcal{L}(A)^N$ with cardinality $m!^n$ into m sets, each with cardinality $m!^n/m = m!^{n-1}(m-1)!$. For each of these ways, $m!$ distinct resolute SCRs can be defined. As a result, $\frac{(m!^n)!}{(m!^{n-1}(m-1)!)^m} m!$ resolute SCRs that satisfy consequential neutrality can be constructed. \square

From Theorem 4, one can compute $\frac{\#\mathcal{F}_{m,n}^{NEUTRAL}}{\#\mathcal{F}_{m,n}^{RESOLUTE}} = m^{(-1 + \frac{1}{m})m!^n}$. As $\frac{1}{m!} < 1$ for all $m > 1$, this ratio approaches to 0 as $m \rightarrow \infty$ or $n \rightarrow \infty$. Thus, we conclude that the ratio of the number of neutral and resolute SCRs to the number of all resolute SCRs is negligible in the limit. Although we do not have analytical solutions for the comparisons regarding consequential neutrality, we obtained some numerical observations through computations for small values of m and n that are provided in Appendix A. These indicate that both $\frac{\#\mathcal{F}_{m,n}^{CN}}{\#\mathcal{F}_{m,n}^{RESOLUTE}}$ and $\frac{\#\mathcal{F}_{m,n}^{NEUTRAL}}{\#\mathcal{F}_{m,n}^{CN}}$ converge to 0, as m or n increases. Thus, although consequential neutrality and neutrality are both hard to satisfy, neutrality seems to be considerably more demanding compared to consequential neutrality.¹¹

Now we address whether $\mathcal{F}_{m,n}^{CN} \setminus \mathcal{F}_{m,n}^{NEUTRAL}$ contains interesting SCRs and the answer is affirmative, at least for certain sizes of the social choice problem.

We say that (m, n) satisfies Condition $\gamma(m, n)$ iff $m \mid \binom{m}{k}$ for all $k \in \mathcal{D}_m(n)$.

¹¹We are providing computational results for only some small values of m and n because as m and n increase, these values grow dramatically. As diminution in the ratios are also fast, these values appear to be sufficient for this conclusion. Since the aim of this counting exercise is to assess the comparison of numbers of functions that satisfy the two versions of neutrality, we leave out other axioms (such as anonymity and efficiency), although it certainly is an interesting question.

This condition ensures the existence of anonymous, CN, efficient, and resolute SCRs, as shown in Theorem 5 below.

Theorem 5. *Let (A, N) be a social choice problem with size (m, n) where Condition $\gamma(m, n)$ is satisfied. Every anonymous, efficient, irresolute, and neutral SCR f admits a resolute refinement which is anonymous, CN, and efficient.*

Proof. Fix $m, n \in \mathbb{N}$ as such, let $A = \{x_1, \dots, x_m\}$, and take any anonymous, efficient, and neutral $f : \mathcal{L}(A)^N \rightarrow \mathcal{A}$. Clearly, $\#vf(P_N) \in \mathcal{D}_m(n)$ for all $P_N \in \mathcal{L}(A)^N$ due to Theorems 2 and 3. Next, observe that $m \nmid n$, as otherwise we would have $m \in \mathcal{D}_m(n)$, while $m \nmid \binom{m}{m} = 1$. Thus, there does not exist a profile P_N with $vf(P_N) = A$. For any $k \in \mathcal{D}_m(n) \setminus \{1\}$, denote with \mathcal{A}_k the subsets of A with precisely k elements, that is, $\mathcal{A}_k = \{S \subseteq A : \#S = k\}$. Thus, we have $\#\mathcal{A}_k = \binom{m}{k} = m \binom{m-1}{k-1} / k$. Condition $\gamma(m, n)$ ensures that $\binom{m-1}{k-1} / k$ ($= t$ from now on) is a natural number. We want to show that each $x_i \in A$ can be assigned to t distinct sets of cardinality k that contains x_i . This will suffice to prove the theorem, given Proposition 1, as the number of profiles these k -sets are assigned to under a neutral SCR are equal. To do that, we introduce the following iterative approach.

First, define $\sigma : A \leftrightarrow A$ such that $\sigma(x_i) = x_{i+1}$ for all $i \in \{1, \dots, m-1\}$ and $\sigma(x_m) = x_1$. Next, let $\succ_{\mathcal{A}_k}$ denote the lexicographic order of the k -sets in \mathcal{A}_k according to the order $x_1 \succ \dots \succ x_m$. Thus, for instance, $\{x_1, \dots, x_k\} \succ_{\mathcal{A}_k} \{x_1, \dots, x_{k-1}, x_{k+1}\} \succ_{\mathcal{A}_k} S$ for all $S \in \mathcal{A}_k \setminus \{\{x_1, \dots, x_k\}, \{x_1, \dots, x_{k-1}, x_{k+1}\}\}$.

Step 1.

- Take the first set in \mathcal{A}_k according to $\succ_{\mathcal{A}_k}$, *i.e.*, $\{x_1, \dots, x_k\}$, and denote this set with $A_{k,1}$. Now, denote with $\sigma^r(A_{k,1})$, the r -time application of permutation σ , with a slight abuse of notation. Thus, for instance, $\sigma^1(A_{k,1}) = \{x_2, \dots, x_{k+1}\}$. Denote with $\mathcal{A}_{k,1}$ the set comprising of $A_{k,1}$ together with $m-1$ other k -sets in \mathcal{A}_k that are constructed from $A_{k,1}$ by applying σ iteratively $m-1$ times. Thus, $\mathcal{A}_{k,1} = \{A_{k,1}, \sigma^1(A_{k,1}), \dots, \sigma^{m-1}(A_{k,1})\}$.

Step 2.

- Take the first set in $\mathcal{A}_k \setminus \mathcal{A}_{k,1}$ according to $\succ_{\mathcal{A}_k}$ and denote this set with $A_{k,2}$. Construct $\mathcal{A}_{k,2}$ as in Step 1 so that $\mathcal{A}_{k,2} = \{A_{k,2}, \sigma^1(A_{k,2}), \dots, \sigma^{m-1}(A_{k,2})\}$.

Step j for $j \in \{3, \dots, t\}$ if $t > 2$.

- Take the first set in $\mathcal{A}_k \setminus \bigcup_{c=1}^{j-1} \mathcal{A}_{k,c}$ according to $\succ_{\mathcal{A}_k}$ and denote this set with $A_{k,j}$. Construct $\mathcal{A}_{k,j}$ as in Step 1.

The proof concludes by constructing an SCR g that refines vf such that for all $P_N \in \mathcal{L}(A)^N$, $k \in \mathcal{D}_m(n) \setminus \{1\}$, and $j \in \{1, \dots, t\}$,

- $vf(P_N) = A_{k,j} \implies g(P_N) = x_1$ and
- $vf(P_N) = \sigma^r(A_{k,j}) \implies g(P_N) = x_{r+1} \forall r \in \{1, \dots, m-1\}$.

We need to make sure that the following hold.

- (i) $x_1 \in A_{k,j}$ for all $j \in \{1, \dots, t\}$.
- (ii) $x_{r+1} \in \sigma^r(A_{k,j})$ for all $j \in \{1, \dots, t\}$ and for all $r \in \{1, \dots, m-1\}$.
- (iii) $\{\mathcal{A}_{k,j}\}_{j \in \{1, \dots, t\}}$ is a partition of \mathcal{A}_k .

(i): First, note that $x_1 \in A_{k,1}$. Next, observe that $\#\{S \in \mathcal{A}_{k,1} : x_1 \in S\} = k$ as $\#A_{k,1} = k$, $x_1 \in A_{k,1}$, and for each $x_j \in A_{k,1} \setminus \{x_1\}$, there exists a unique $r \in \{1, \dots, m-1\}$ such that $\sigma^r(x_j) = x_1$. Thus, in fact, for all $j \in \{2, \dots, t\}$, $\bigcup_{c=1}^{j-1} \mathcal{A}_{k,c}$ contains $(j-1)k$ sets that contain x_1 and there remain $(t-j+1)k$ sets in $\mathcal{A}_k \setminus \bigcup_{c=1}^{j-1} \mathcal{A}_{k,c}$ that contain x_1 .¹² As for each $j \in \{2, \dots, t\}$, the first set in $\mathcal{A}_k \setminus \bigcup_{c=1}^{j-1} \mathcal{A}_{k,c}$ according to $\succ_{\mathcal{A}_k}$ is chosen to be $A_{k,j}$, $x_1 \in A_{k,j}$ for all $j \in \{1, \dots, t\}$ is established.

(ii): As $x_1 \in A_{k,j}$ for all $j \in \{1, \dots, t\}$ and $\sigma^r(x_1) = x_{r+1}$, it is straightforward to note that $x_{r+1} \in \sigma^r(A_{k,j})$ for all $j \in \{1, \dots, t\}$ and for all $r \in \{1, \dots, m-1\}$.

(iii): First, note that $\#\mathcal{A}_{k,i} = m$ for all $i \in \{1, \dots, t\}$. We need to show that $\mathcal{A}_{k,i} \cap \mathcal{A}_{k,j} = \emptyset$ for any $i, j \in \{1, \dots, t\}$. This follows from the statement (i) as $A_{k,j} \in \mathcal{A}_k \setminus \bigcup_{c=1}^{j-1} \mathcal{A}_{k,c}$ for all $j \in \{1, \dots, t\}$. To see that, suppose for a contradiction that there exists $S \in \mathcal{A}_{k,i} \cap \mathcal{A}_{k,j}$ for some $i, j \in \{1, \dots, t\}$ with

¹²This holds for any $x_i \in A$.

$j > i$. We must have $S = \sigma^{r_i}(A_{k,i}) = \sigma^{r_j}(A_{k,j})$ for some $r_i, r_j \in \{1, \dots, m-1\}$. Note that $r_i \neq r_j$ as $A_{k,i} \neq A_{k,j}$. In case $r_i > r_j$, we have $\sigma^{r_i-r_j}(A_{k,i}) = A_{k,j}$ and in case $r_i < r_j$, we have $\sigma^{r_i+m-r_j}(A_{k,i}) = A_{k,j}$, both contradicting that $A_{k,j} \in \mathcal{A}_k \setminus \bigcup_{c=1}^{j-1} \mathcal{A}_{k,c}$.¹³

We let $g(P_N) = vf(P_N)$ for all $P_N \in \mathcal{L}(A)^N$ with $\#vf(P_N) = 1$. g is a resolute refinement of f . Naturally, g is anonymous as f is. Furthermore, by construction of the iterative process above, g is also CN. Finally, as f is efficient, g is efficient as well. \square

To see how the refinement in Theorem 5 can be constructed, consider the following example with $A = \{x_1, \dots, x_7\}$ and $n = 5$. Take any f that is anonymous, efficient, and neutral. We have $\mathcal{D}_7(5) = \{1, 5\}$, so we focus on $k = 5$ for the iterative process. We have $\binom{7}{5} = 21$ and $t = 3$, that is, there are 21 distinct sets with 5 alternatives and all profiles that result in 5 alternatives under vf should be assigned to each of the 7 alternatives so that each will be chosen within 3 different sets. So in Step 1, we set $A_{5,1} = \{x_1, x_2, x_3, x_4, x_5\}$ and construct $\mathcal{A}_{5,1}$. In Step 2, we set $A_{5,2} = \{x_1, x_2, x_3, x_4, x_6\}$ which follows $\{x_1, x_2, x_3, x_4, x_5\}$ in the lexicographic order $\succ_{\mathcal{A}_5}$ and construct $\mathcal{A}_{5,2}$. In Step 3, final step, we cannot set the next set according to $\succ_{\mathcal{A}_5}$, *i.e.*, $\{x_1, x_2, x_3, x_4, x_7\}$, as $A_{5,3}$ because it already appears in $\mathcal{A}_{5,1}$ as $\sigma^6(A_{5,1}) = \{x_1, x_2, x_3, x_4, x_7\}$. However, we can set $\{x_1, x_2, x_3, x_5, x_6\}$ as $A_{5,3}$, which follows $\{x_1, x_2, x_3, x_4, x_7\}$ in the lexicographic order $\succ_{\mathcal{A}_5}$ and construct $\mathcal{A}_{5,3}$ to complete the iterative process as depicted in the Table 1.

The refinement g is constructed so that for all $P_N \in \mathcal{L}(A)^N$ and for all $j \in \{1, 2, 3\}$

- (i) $vf(P_N) \in A_{5,j}$ implies $g(P_N) = x_1$ and
- (ii) $vf(P_N) \in \sigma^r(A_{5,j})$ implies $g(P_N) = x_{r+1}$ for all $r \in \{1, \dots, 6\}$.

Note that Condition $\gamma(m, n)$ ensures the existence of anonymous, CN, efficient, and resolute SCRs by asking $m \mid \binom{m}{k}$ to hold for every $k \in \mathcal{D}_m(n)$. On the other hand, as it follows from the proof of Theorem 5, a weaker version of Condition $\gamma(m, n)$ that asks $m \mid \binom{m}{k}$ to hold for some $k \in \mathcal{D}_m(n)$, although

¹³Note that $\sigma^{r+m}(x_i) = \sigma^r(x_i)$ for all $r \in \mathbb{N}$ and $i \in \{1, \dots, m\}$.

	$j = 1$	$j = 2$	$j = 3$
$A_{5,j}$	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_4, x_6\}$	$\{x_1, x_2, x_3, x_5, x_6\}$
$\sigma(A_{5,j})$	$\{x_2, x_3, x_4, x_5, x_6\}$	$\{x_2, x_3, x_4, x_5, x_7\}$	$\{x_2, x_3, x_4, x_6, x_7\}$
$\sigma^2(A_{5,j})$	$\{x_3, x_4, x_5, x_6, x_7\}$	$\{x_3, x_4, x_5, x_6, x_1\}$	$\{x_3, x_4, x_5, x_7, x_1\}$
$\sigma^3(A_{5,j})$	$\{x_4, x_5, x_6, x_7, x_1\}$	$\{x_4, x_5, x_6, x_7, x_2\}$	$\{x_4, x_5, x_6, x_1, x_2\}$
$\sigma^4(A_{5,j})$	$\{x_5, x_6, x_7, x_1, x_2\}$	$\{x_5, x_6, x_7, x_1, x_3\}$	$\{x_5, x_6, x_7, x_2, x_3\}$
$\sigma^5(A_{5,j})$	$\{x_6, x_7, x_1, x_2, x_3\}$	$\{x_6, x_7, x_1, x_2, x_4\}$	$\{x_6, x_7, x_1, x_3, x_4\}$
$\sigma^6(A_{5,j})$	$\{x_7, x_1, x_2, x_3, x_4\}$	$\{x_7, x_1, x_2, x_3, x_5\}$	$\{x_7, x_1, x_2, x_4, x_5\}$
	$\mathcal{A}_{5,1}$	$\mathcal{A}_{5,2}$	$\mathcal{A}_{5,3}$

Table 1: Iterative process for constructing the partitioning $\{\mathcal{A}_{5,j}\}_{j \in \{1, \dots, t\}}$ of \mathcal{A}_5 , where $t = \binom{7}{5}/7 = 3$.

does not ensure resoluteness, allows the existence of anonymous and efficient refinements that satisfy CN. We state this formally in the following remark and leave out its proof as it follows from the proof of Theorem 5. Let us say that (m, n) satisfies Condition $\gamma'(m, n)$ iff $m \mid \binom{m}{k}$ for some $k \in \mathcal{D}_m(n)$.

Remark 1. *Let (A, N) be a social choice problem with size (m, n) where Condition $\gamma'(m, n)$ is satisfied. Every anonymous, efficient, and neutral SCR f admits a refinement which is anonymous, efficient, and CN.*

How restrictive is Condition $\gamma(m, n)$? Note that when m is a prime, Condition $\gamma(m, n)$ is satisfied.¹⁴ Thus, anonymous, CN, efficient, and resolute SCRs exist when m is prime and does not divide n . Campbell and Kelly (2015) show that when n has at least two distinct prime factors, there can only be finitely many values of m for which there are anonymous, neutral, and resolute SCRs. Our result implies, for instance, that for such n , there are infinitely many values of m (such as all primes that are greater than n) for which there are anonymous, CN, and resolute SCRs. Furthermore, these SCRs can be efficient.

To expand the picture drawn by Theorem 5, we show that Condition $\gamma(m, n)$ is not necessary for the existence of anonymous, CN, efficient, and resolute SCRs. As a matter of fact, the theorem below spans some instances where Condition $\gamma(m, n)$ fails, *e.g.*, $m = 4$ and $n = 2$ (note that $2 \in \mathcal{D}_4(2)$ and $4 \nmid \binom{4}{2}$).

¹⁴To see this, note that the numerator in $\binom{m}{k} = \frac{m \cdot \dots \cdot (m-k+1)}{k!}$ is divisible by m whereas none of $\{2, \dots, k\}$ divides m .

Theorem 6. *Any social choice problem (A, N) with $m \geq 4$ and $n \in \{2, 3\}$ admits an anonymous, CN, efficient, and resolute SCR.*

Proof. Let $m \geq 4$ and $N = \{1, 2\}$. For any $x, y \in A$, let $T_{xy} \subset \mathcal{L}(A)^N$ denote the set of profiles where individual 1 ranks x first and individual 2 ranks y first. Hence $\{T_{xy}\}_{x \neq y}$ partitions the set of profiles where there is no unanimously top ranked alternative. Given any $x, y \in A$, note that

$$T_{yx} = \{(Q, P) \in \mathcal{L}(A)^N : (P, Q) \in T_{xy}\},$$

and hence, $\#T_{xy} = \#T_{yx}$. Now, let $A = \{x_1, \dots, x_m\}$ and define for any distinct $i, j \in \{1, \dots, m\}$,

$$D_m(i, j) = \{k, l \in \{1, \dots, m\} \setminus \{i, j\} : k < l < t \ \forall t \in \{1, \dots, m\} \setminus \{i, j, k, l\}\},$$

as the doubleton that contains the lowest two indices in A excluding i and j . Let $g : \mathcal{L}(A)^N \rightarrow A$ be a resolute SCR that assigns to any profile the alternative that is ranked first by both individuals, if exists. Note that $\#\{P_N \in \mathcal{L}(A)^N : xP_i z \ \forall z \in A \ \forall i \in N\} = \#\{P_N \in \mathcal{L}(A)^N : yP_i z \ \forall z \in A \ \forall i \in N\}$ for all $x, y \in A$. Furthermore, for all $P_N \in T_{x_i x_j}$ with $i < j$ and $D_m(i, j) = \{k, l\}$, let $g(P_N) = x_i$ when $x_k P_1 x_l \iff x_k P_2 x_l$ and $g(P_N) = x_j$ otherwise. Hence, we have $\#\{P_N \in T_{x_i x_j} : g(P_N) = x_i\} = \#\{P_N \in T_{x_i x_j} : g(P_N) = x_j\}$ for all $x_i, x_j \in A$ such that $i < j$. Furthermore, let $g(P, Q) = g(Q, P) \ \forall (P, Q) \in \mathcal{L}(A)^N$. By construction, g is CN and anonymous. Moreover, as g picks an alternative only if it is ranked first by an individual, it is also efficient.

Now let $N = \{1, 2, 3\}$, hence $n = 3$. Let $\mathcal{T} \subset \mathcal{L}(A)^N$ denote the subset of profiles where each individual has a distinct alternative as most preferred. That is, $P_N \in \mathcal{T}$ iff $x_1 P_1 y$ for all $y \in A \setminus \{x_1\}$, $x_2 P_2 y$ for all $y \in A \setminus \{x_2\}$, and $x_3 P_3 y$ for all $y \in A \setminus \{x_3\}$ implies x_1, x_2 , and x_3 are all distinct. Note that under the plurality rule Υ , $\#\Upsilon(P_N) = 1$ for all $P_N \in \mathcal{L}(A)^N \setminus \mathcal{T}$ and $\#\Upsilon(P_N) = 3$ for all $P_N \in \mathcal{T}$.

Let $A = \{x_1, \dots, x_m\}$. For any $q, r, t \in \{1, \dots, m\}$ with $q < r < t$, let $M(\{q, r, t\})$ denote the minimal element in $\{1, \dots, m\} \setminus \{q, r, t\}$. So for instance, when $m = 5$, $M(\{2, 3, 4\}) = 1$ and $M(\{1, 2, 4\}) = 3$. Let $\mathcal{T}_{\{x_q, x_r, x_t\}} \subset \mathcal{T}$ denote the set of profiles where each of x_q, x_r , and x_t appears on top. We

now define an SCR h . We let $h(P_N) = \Upsilon(P_N)$ for all $P_N \in \mathcal{L}(A)^N \setminus \mathcal{T}$. For any $P_N \in \mathcal{T}_{\{x_q, x_r, x_t\}}$, let $j \in N$ denote the individual such that $x_q P_j x_w$ for all $w \in \{1, \dots, m\} \setminus \{q\}$. We let $h(P_N) = x_q$ if $x_M(\{q, r, t\}) P_j x_z$ for all $z \in \{r, t\}$, $h(P_N) = x_r$ if $x_z P_j x_M(\{q, r, t\})$ for all $z \in \{r, t\}$, and $h(P_N) = x_t$ if either $x_r P_j x_M(\{q, r, t\}) P_j x_t$ or $x_t P_j x_M(\{q, r, t\}) P_j x_r$.

Note that h is resolute by construction and is efficient as it refines the plurality rule. Moreover, it is anonymous, as $h(P_N) = h(P'_N)$ whenever P'_N is anonymously equivalent to P_N . Since we have

$$\begin{aligned} & \#\{P_N \in \mathcal{T}_{\{x_q, x_r, x_t\}} \mid x_M(\{q, r, t\}) P_j x_z \ \forall z \in \{r, t\}\} \\ &= \#\{P_N \in \mathcal{T}_{\{x_q, x_r, x_t\}} \mid x_z P_j x_M(\{q, r, t\}) \ \forall z \in \{r, t\}\} \\ &= \#\{P_N \in \mathcal{T}_{\{x_q, x_r, x_t\}} \mid x_r P_j x_M(\{q, r, t\}) P_j x_t \text{ or } x_t P_j x_M(\{t, q, r\}) P_j x_r\}, \end{aligned}$$

h is also CN. □

At this stage, one may be tempted to ask whether one can find an anonymous, CN, efficient, and resolute SCR at any (m, n) . The following theorem advises caution on this.

Theorem 7. *There exists a social choice problem which admits no anonymous, CN, efficient, and resolute SCR.*

Proof. Let $A = \{x, y\}$ and $N = \{1, 2\}$. We have four possible profiles, P_N, P'_N, P''_N, P'''_N as shown below.

$$\begin{array}{cccccc} \frac{P_1}{x} & \frac{P_2}{x} & \frac{P'_1}{y} & \frac{P'_2}{y} & \frac{P''_1}{x} & \frac{P''_2}{y} & \frac{P'''_1}{y} & \frac{P'''_2}{x} \\ \frac{y}{y} & \frac{y}{y} & \frac{x}{x} & \frac{x}{x} & \frac{y}{y} & \frac{x}{x} & \frac{x}{x} & \frac{y}{y} \end{array}$$

Efficiency implies choosing x at P_N and y at P'_N . Moreover, $f(P''_N) = f(P'''_N)$ by anonymity. Hence, $\#\{P_N \in \mathcal{L}(A)^N : f(P_N) = x\} \neq \#\{P_N \in \mathcal{L}(A)^N : f(P_N) = y\}$, a failure of consequential neutrality. □

Nevertheless, the social choice problem in the proof of Theorem 7 admits an anonymous, CN, and resolute SCR.¹⁵ This raises the question of how general

¹⁵ To see this, consider $g : \mathcal{L}(A)^N \rightarrow \mathcal{A}$ such that $g(P_N) = g(P'_N) = x$ and $g(P''_N) = g(P'''_N) = y$, which is both anonymous and CN while not efficient.

is the compatibility between anonymity and consequential neutrality when we dispense with the efficiency condition.

Moulin (1991) introduces the following condition that we call $\psi(m, n)$. Let $\mathcal{D}(n) \setminus \{1\} = \{d_1, \dots, d_K\}$ for some $K \in \mathbb{N}$.

Condition $\psi(m, n)$: $\nexists (a_1, \dots, a_K) \in (\mathbb{N} \cup \{0\})^K$ such that $\sum_{i=1}^K a_i d_i = m$.

Remark 2. $\mu(m, n) \implies \psi(m, n)$ for all $n, m \geq 2$.

The following theorem states the cases of incompatibility of anonymity and neutrality in resolute social choice.

Theorem 8 (Moulin (1991)). *There exists an anonymous, neutral, and resolute SCR if and only if $\psi(m, n)$ holds.*

We are now ready to state and prove our final theorem, which shows that if $m > n$, there exist anonymous, CN, and resolute SCRs for any social choice problem with size (m, n) .

Theorem 9. *For all social choice problems with $n < m$, there exists an anonymous, CN, and resolute SCR.*

Proof. Given any preference profile, we observe k distinct preferences for some $k \in \{1, \dots, \min\{m!, n\}\}$. There are $\binom{m!}{k}$ ways to choose k preferences from $\mathcal{L}(A)$. Let $\mathbf{P} = \{p^1, \dots, p^k\}$ be a set of k distinct preferences. Write V^k for the set of vectors $v = (v_1, \dots, v_k)$ with $v_i \geq 1$ for all $i \in \{1, \dots, k\}$ and $\sum_{i=1}^k v_i = n$. Each $v \in V^k$, combined with \mathbf{P} , induces a set of profiles $E_v^{\mathbf{P}}$ that consists of all profiles where p^i appears v_i times for all $i \in \{1, \dots, k\}$. Let $E^{\mathbf{P}} = \bigcup_{v \in V^k} E_v^{\mathbf{P}}$. Three remarks are in order. First, $\#E^{\mathbf{P}}$ depends on k and not on the preferences in \mathbf{P} . Second, an SCR f that satisfies at any given $k \in \{1, \dots, \min\{m!, n\}\}$ and \mathbf{P} the invariance $f(P_N) = f(P'_N)$ for all $P_N, P'_N \in E^{\mathbf{P}}$ is anonymous. Third, $m \mid \binom{m!}{k}$ if $k < m$, which is ensured when $n < m$. Now, let $t_k = \binom{m!}{k}/m$. Write $A = \{x_1, \dots, x_m\}$ and at each $k \in \{1, \dots, \min\{m!, n\}\}$ assign to every x_i t_k distinct sets $\mathbf{P} = \{p^1, \dots, p^k\}$ and let $f(P_N) = x_i$ for all $P_N \in E^{\mathbf{P}}$ at every \mathbf{P} assigned to x_i for all $i \in \{1, \dots, m\}$. By the three remarks, f is anonymous, CN, and resolute. \square

5 Conclusion

Using an irresolute SCR, one cannot reach a collective choice without referring to an additional mechanism that is external to the SCR. Therefore, Theorems 1 and 8 reflect the impossibility of making a collective choice by being confined to anonymous and neutral (efficient) SCRs as collective choice procedures. We take two different but related approaches to address how severe this impossibility is.

First, we identify the minimal irresoluteness outlook that would arise when anonymity, efficiency, and neutrality make ties inevitable. Based on this analysis, we deliver a method which, while preserving anonymity, efficiency, and neutrality, refines SCRs that deliver more ties than necessary.

Next, we introduce consequential neutrality as a weakening of neutrality. As expected, we obtain results that are more permissive than the (im-)possibilities announced by Theorems 1 and 8. We identify a large class of social choice problems where resoluteness becomes possible just because consequential neutrality replaces neutrality. Nevertheless, when efficiency is preserved, we know that this possibility does not hold for every social choice problem.

Dispensing with efficiency presents a case of interest. We show that anonymous, CN, and resolute social choice is possible when $m > n$. Although this condition is logically independent of the necessary and sufficient condition of Theorem 8, it opens the door of resoluteness to a large class of social choice problems that are doomed to irresoluteness by Theorem 8. Moreover, we are not able to find any social choice problem where anonymity, consequential neutrality, and resoluteness are incompatible. This provokes to ask whether these three conditions are compatible for any size of the social choice problem, which we leave as a –combinatorically difficult– open question.

References

- Bubboloni, D. and Gori, M. (2014). Anonymous and neutral majority rules. *Social Choice and Welfare*, 43(2):377–401.
- Bubboloni, D. and Gori, M. (2015). Symmetric majority rules. *Mathematical Social Sciences*, 76:73–86.
- Bubboloni, D. and Gori, M. (2016). Resolute refinements of social choice correspondences. *Mathematical Social Sciences*, 84:37–49.
- Campbell, D. E. and Kelly, J. S. (2015). The finer structure of resolute, neutral, and anonymous social choice correspondences. *Economics Letters*, 132:109–111.
- Doğan, O. and Giritligil, A. E. (2015). Anonymous and neutral social choice: Existence results on resoluteness. Technical report.
- King, S. S. and Powers, R. C. (2018). Beyond neutrality: Extended difference of votes rules. *Mathematical Social Sciences*, 93:146–152.
- Moulin, H. (1980). Implementing efficient, anonymous and neutral social choice functions. *Journal of Mathematical Economics*, 7(3):249–269.
- Moulin, H. (1983). *The strategy of social choice*. Number 18 in Advanced Textbooks in Economics. North-Holland Pub. Co.
- Moulin, H. (1991). *Axioms of Cooperative Decision Making*. Number 15 in Econometric Society Monographs. Cambridge University Press.
- Perry, J. and Powers, R. C. (2008). Aggregation rules that satisfy anonymity and neutrality. *Economics Letters*, 100(1):108–110.
- Zwicker, W. (2016). Introduction to the theory of voting. In Brandt, F., Conitzer, V., Endriss, U., Lang, J., and Procaccia, A. D., editors, *Handbook of Computational Social Choice*, pages 23–56. Cambridge University Press.

A Appendix: Observations on the numbers of CN, neutral, and resolute SCRs

Tables 1 and 2 below show ratios the observations made in Section 4 are based on. In both tables, $\mathbf{0}$ represents numbers smaller than 10^{-10mn} .

$\#\mathcal{F}_{m,n}^{NEUTRAL}/\#\mathcal{F}_{m,n}^{CN}$	$n = 2$	$n = 3$	$n = 4$
$m = 2$	0.333333	0.114286	0.00994561
$m = 3$	3.58965×10^{-14}	5.73212×10^{-85}	$\mathbf{0}$
$m = 4$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

Table 2: The ratio of $\#\mathcal{F}_{m,n}^{NEUTRAL}/\#\mathcal{F}_{m,n}^{CN}$ for different values of (m, n) .

$\#\mathcal{F}_{m,n}^{CN}/\#\mathcal{F}_{m,n}^{RESOLUTE}$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$m = 2$	0.75	0.546875	0.392761	0.2799
$m = 3$	0.135304	0.0229012	0.0038267	0.000638057
$m = 4$	0.00175989	0.0000149993	1.27583×10^{-7}	1.08512×10^{-9}
$m = 5$	8.19334×10^{-7}	5.69061×10^{-11}	3.95181×10^{-15}	$\mathbf{0}$
$m = 6$	8.12216×10^{-12}	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

Table 3: The ratio of $\#\mathcal{F}_{m,n}^{CN}/\#\mathcal{F}_{m,n}^{RESOLUTE}$ for different values of (m, n) .