



HAL
open science

Di-gluonium sum rules, $I = 0$ scalar mesons and conformal anomaly

Stephan Narison

► **To cite this version:**

Stephan Narison. Di-gluonium sum rules, $I = 0$ scalar mesons and conformal anomaly. Nucl.Phys.A, 2022, 1017, pp.122337. 10.1016/j.nuclphysa.2021.122337. hal-03341180

HAL Id: hal-03341180

<https://hal.science/hal-03341180>

Submitted on 5 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Di-Gluonium Sum Rules, $I = 0$ Scalar Mesons and Conformal Anomaly

Stephan Narison*

*Laboratoire Univers et Particules de Montpellier (LUPM), CNRS-IN2P3,
Case 070, Place Eugène Bataillon, 34095 - Montpellier, France*

*and
Institute of High-Energy Physics of Madagascar (iHEPMAD)
University of Ankatso, Antananarivo 101, Madagascar*

Abstract

We revisit, scrutinize, improve, confirm and complete our previous results [1–3] from the scalar di-gluonium sum rules within the standard SVZ-expansion at N2LO *without instantons* and *beyond the minimal duality ansatz* : “one resonance \oplus QCD continuum” parametrization of the spectral function which is necessary for a better understanding of the complex spectra of the $I = 0$ scalar mesons. We select different (un)subtracted sum rules (USR) moments of degree ≤ 4 for extracting the two lowest gluonia masses and couplings. We obtain: $[M_{\sigma_B}, f_{\sigma_B}] = [1.07(13), 0.46(16)]$, $[M_{G_1}, f_{G_1}][1.55(12), 0.37(11)]$ GeV and the corresponding masses of the radial excitations : $M_{\sigma'_B} = 1.12(12)$ and $M_{G'_1} = 1.56(14)$ GeV which are (unexpectedly) almost degenerated with the ground states. The 2nd radial excitation is found to have a much heavier mass: $M_{G_2} \simeq 2.99(22)$ GeV. Combining these results with some Low-Energy Vertex Sum Rules (LEV-SR), we predict some hadronic widths and classify them into two groups : – The σ -like (σ_B, σ'_B) which decay copiously to $\pi\pi$ from OZI-violating process and the σ'_B to $2(\pi\pi)S$ through $\sigma\sigma$. – The G -like (G_1, G'_1 and eventually G_2) which decay into $\eta'\eta, \eta\eta$ through the $U(1)_A$ gluonic vertex. Besides some eventual mixings with quarkonia states, we may expect that the observed $\sigma/f_0(500)$ and $f_0(137)$ are σ -like while the $f_0(1.5)$ and $f_0(1.7)$ are G -like gluonia. The high mass $G_2(2.99)$ can also mix with the G_1, G'_1 to bring the gluon component of the gluonia candidates above 2 GeV. We also estimate the conformal charge $\psi_G(0) = 2.09(29)$ GeV⁴ and its slope $\psi'_G(0) = 0.95(30)$ GeV². Our results are summarized in Table 1.

Keywords:

QCD Spectral Sum Rules; Perturbative and Non-perturbative QCD; Exotic hadrons; Masses and Decay constants.

*ICTP-Trieste consultant for Madagascar.
Email address: snarison@yahoo.fr (Stephan Narison)

1. Introduction

Gluonia / Glueballs bound states are expected to be a consequence of QCD [5]. However, despite considerable theoretical and experimental efforts, there are not (at present) any clear indication of their signature. The difficulty is also due to the fact that the observed candidates can be a strong mixing of the gluonia with the $\bar{q}q$ light mesons or some other exotic mesons (four-quark, hybrid states). In particular, the ($I = 0$) Isoscalar Scalar Channel channel is overpopulated beyond the conventional scalar nonet such that one suspects that some of these states can be of exotic origins and may (for instance) contain some large gluon component in their wave functions.

In some previous works [1–4, 6–13] based on QCD spectral sum rules (QSSR) à la SVZ [14–16] combined with some low-energy theorems (LET), we have tried to understand the (pseudo)scalar gluonia channels. Since then, some progresses have been accomplished from experiments and from some other approaches which we shall briefly remind below.

After this short reminder, we update and improve our previous QSSR analysis. The impact of these novel results on our present understanding of the peculiar ($I = 0$) isoscalar scalar channel is discussed.

2. $\sigma/f_0(500)$ and $f_0(980)$ from scatterings data

To quantify the observables of the low-lying scalar states, we have phenomenologically studied the $\pi\pi, \bar{K}K, \gamma\gamma$ [17, 18], Ke_4 [19], $J/\psi, \phi$ radiative and D_s semileptonic decays [20, 21] data with the aim to understand the internal substructure of the $\sigma/f_0(500)$ and $f_0(980)$ mesons,¹.

- *Masses and hadronic widths from $\pi\pi \rightarrow \pi\pi, KK$ and Ke_4 data*

◊ Using a K-matrix [26, 27] analysis of the $\pi\pi$ elastic scattering data below 0.7 GeV, we found for the complex pole mass and the residue using one bare resonance \oplus one channel [17] :

$$M_\sigma^{pole}[\text{MeV}] \simeq 422 - i290. \quad (1)$$

◊ We extend the previous analysis until 900 MeV and use two bare resonances [$\sigma(500), f_0(980)$] \oplus two open channels ($\pi\pi, \bar{K}K$) parametrization of the data [18, 19]. To the data used previously, we add the new precise measurement from NA48/2 [28] on Ke_4 ($K \rightarrow \pi\pi e\nu_e$) data for the $\pi\pi$ phase shift below 390 MeV. Then, averaging this result with the previous one in Eq.1, one obtains the final estimate :

$$M_\sigma^{pole}[\text{MeV}] \simeq 452(12) - i260(15), \quad |g_{\sigma\pi^+\pi^-}| = 1.12(31) \text{ GeV}, \quad \frac{|g_{f_0K^+K^-}|}{|g_{f_0\pi^+\pi^-}|} = 2.58(1.34), \quad (2)$$

which agrees with the ones based on the analytic continuation and unitarity properties of the amplitude to the deep imaginary region [29, 30] :

$$M_\sigma^{pole}[\text{MeV}] \simeq 444 - i272, \quad \text{and} \quad M_\sigma^{pole}[\text{MeV}] \simeq 489 - i264. \quad (3)$$

¹A more complete and comprehensive discussion on production processes and decays of scalar mesons can be found in the recent reviews of [22–25].

◇ However, in order to compare these results with the QCD spectral sum rules ones where the analysis is done in the real axis, one has to introduce the On-shell or Breit-Wigner (os) mass and width where the amplitude is purely imaginary at the phase 90^0 :

$$Re\mathcal{D}((M_\sigma^{os})^2) = 0 \quad \implies \quad (M_\sigma^{os}, \Gamma_\sigma^{os}) = (920, 700) \text{ MeV} , \quad (4)$$

where \mathcal{D} is the propagator appearing in the unitary $\pi\pi$ amplitude. Similar result has been obtained in [31, 32] where the wide resonance centered at 1 GeV has been interpreted in [31] as a resonance interfering destructively with the $f_0(980)$ and $f_0(1500)$.

◇ For the $f_0(980)$, one obtains [18, 19]:

$$M_{f_0}^{pole} [\text{MeV}] \simeq 981(34) - i18(11) , \quad |g_{f_0\pi^+\pi^-}| = 2.65(10) \text{ GeV} , \quad \frac{|g_{\sigma K^+K^-}|}{|g_{\sigma\pi^+\pi^-}|} = 0.37(6) , \quad (5)$$

and

$$M_{f_0}^{os} \simeq M_{f_0}^{pole} , \quad \Gamma_{f_0}^{os} \simeq \Gamma_{f_0}^{pole} , \quad (6)$$

where Eq. 6 comes from the fact that $f_0(980)$ is narrow.

- $\sigma/f_0(500)$ and $f_0(980)$ $\gamma\gamma$ widths from $\gamma\gamma \rightarrow \pi\pi, KK$ scatterings

◇ We proceed as above and deduce the average (in units of keV):

$$\Gamma_{\sigma_{pole}}^{dir} = 0.16(4) , \quad \Gamma_{\sigma_{pole}}^{resc} = 1.89(81) , \quad \Gamma_{\sigma_{pole}}^{tot} = 3.08(82) \quad (7)$$

for the direct, rescattering and total (direct \oplus rescattering) $\gamma\gamma$ widths. For the on-shell mass, these lead to :

$$\Gamma_{\sigma_{os}}^{dir} = 1.2(3) \text{ keV} , \quad \Gamma_{\sigma_{pole}}^{resc} = \Gamma_{\sigma_{os}}^{resc} . \quad (8)$$

◇ For the $f_0(980)$, one obtains in units of keV :

$$\Gamma_{f_0}^{dir}|_{pole} \simeq \Gamma_{f_0}^{dir}|_{os} = 0.28(1) , \quad \Gamma_{f_0}^{res} = 0.85(5) , \quad \Gamma_{f_0}^{tot} = 0.16(1) . \quad (9)$$

- *Production of $\sigma/f_0(500)$ and $f_0(980)$ from some other channels*

Isoscalar Scalar production in some other channels have been reviewed in [22–25], where the structures:

$$\sigma/f_0(500) , \quad f_0(980) , \quad f_0(1370) , \quad f_0(1500) \quad (10)$$

have been also found in different processes (J/ψ -decays, central production in pp and e^+e^- , $\bar{p}p$ and $\bar{p}n$ annihilation at rest, πp and Kp).

◇ We notice the production of the $\sigma/f_0(500)$ from $J/\psi \rightarrow \omega\pi^+\pi^-$ [34] with the extracted complex pole mass and width:

$$M_\sigma^{pole} = 541(39) - i252(42) , \quad (11)$$

and of the $f_0(980)$ from $J/\psi \rightarrow \gamma\pi^0\pi^0$ by BESIII [33] and from $J/\psi \rightarrow \phi\pi^+\pi^-$ and $J/\psi \rightarrow \phi K^+K^-$ [34] with a ratio of coupling :

$$g_{f_0K^+K^-}/g_{f_0\pi^+\pi^-} = 4.21 \pm 0.32 . \quad (12)$$

◊ The $f_0(980)$ is also produced from $\phi \rightarrow (\pi^+\pi^-, K^+K^-) + \gamma$, a glue filter channel at KLOE [35] with the couplings :

$$g_{f_0K^+K^-} = (3.97 \sim 4.74) \text{ GeV} , \quad g_{f_0\pi^+\pi^-} = -(2.22 \sim 1.82) \text{ GeV} , \quad (13)$$

where in these last two experiments the presence of the $\sigma/f_0(500)$ improves the quality of the fit. The non-vanishing of $g_{f_0\pi^+\pi^-}$ indicates that the $f_0(980)$ cannot be a pure $\bar{s}s$ state.

3. Production of some other ($I = 0$) isoscalar scalar mesons

To these data, we add the production of the new structures which can have gluon component [25] :

$$f_0(1765) , \quad f_1(2100) , \quad f_0(2020), \quad f_0(2200) . \quad (14)$$

observed from $J/\psi \rightarrow \gamma\pi^+\pi^-, K^+k^-, 4\pi$ radiative decays, $\pi p, \bar{p}p, pp \rightarrow \eta\eta', \eta\eta, 4\pi$ [25] and recently from $\eta_c \rightarrow [f_0(1.7) \rightarrow K^+K^-], f_0(2.1) \rightarrow \pi^+\pi^-]\eta'$ by BABAR [36].

4. Impacts of the previous phenomenological analysis and data

One can notice that the $I = 0$ scalar channel is overpopulated and goes beyond the usual nonet expectations. From the previous phenomenological analysis, one may expect that:

- $\sigma/f_0(500)$ meson

◊ It cannot be a pure four-quark ($\bar{u}u + \bar{d}d$) or $/$ and $\pi\pi$ molecule state as it has non-vanishing coupling to $\bar{K}K$.

◊ However, from the large value of its $\gamma\gamma$ rescattering width (see e.g. [17]), one may be tempted to interpret it as $\pi\pi$ or/and $\bar{K}K$ molecule state but the $\bar{K}K$ molecule mass is expected to be around 1 GeV due to $SU(3)$ breakings.

◊ From the size of its direct on-shell width of about 1 keV to $\gamma\gamma$ [17], it cannot be a pure ($\bar{u}u + \bar{d}d$) state which has a $\gamma\gamma$ -width expected to be about 4 keV from QCD spectral sum rules [11] and quark model [37] but larger than for a pure gluonium state of about (0.2-0.6) KeV [2, 17]. It cannot also be a four-quark state as the model predicts much smaller (0.00..keV) direct $\gamma\gamma$ widths [10, 38].

- $f_0(980)$ meson

◊ A large $\bar{s}s$ component would lead to a direct $\gamma\gamma$ width of 0.4 keV which are comparable with the data but its non-zero coupling to $\pi\pi$ rules out this possibility. An estimate of a mass of the $\bar{s}s$ state from QSSR leads to a value around 1.4 GeV [11] which is (relatively) too high for the $f_0(980)$.

- $f_0(1.37)$ meson

The large decay width of this meson to $2(\pi\pi)$ in a S-wave may signal a large gluon component in its wave functions [1, 2].

- $f_0(1.37)$ and $f_0(1.5)$ mesons

Its decay into $\eta'\eta$ and $\eta\eta$ and the presence of a $2(\pi\pi)_S$ indicate a large gluon component in its wave function.

5. Theoretical expectations

- In the $I = 0$ isoscalar scalar channel, we have [1, 2] explored the possibility that the $\sigma/f_0(500)$ meson can be the lowest scalar gluonium state σ_B (hereafter, the subindex B refers to unmixed glyonium state) having a mass around 1 GeV. From our approach the σ is composed mainly with gluons and it couples strongly and universally to $\pi\pi$ and $\bar{K}K$ explaining its large width while its “direct” decay to $\gamma\gamma$ is expected to be about (0.2-0.6) keV .

- A maximal mixing of this gluonium with a $(\bar{u}u + \bar{d}d)$ state may explain the feature of the observed $\sigma/f_0(500)$ and $f_0(980)$: narrow width in $\pi\pi$ but strong coupling to $\bar{K}K$ [4]².

- We have also argued [1, 2] that the contribution of the σ_B to the subtracted (SSR) and unsubtracted (USR) sum rules is necessary for resolving the apparent inconsistency between these two sum rules where, in addition, a higher mass gluonium $G(1.5)$ is needed which we have identified [1] with the $G(1.6)$ found from the GAMS data [39]. The contribution of σ_B to the SSR compensates the large contribution of the two-point subtraction constant to the SSR without appealing to more speculative direct instanton non-perturbative contributions.

- Another striking feature of the σ_B is its analogy with its chiral partner the η' which plays a crucial role for the $U(1)_A$ anomaly by its contribution to the topological charge (subtraction constant of the $U(1)_A$ two-point correlator) [6, 42–45]. The σ_B , being the dilaton particle of the conformal anomaly [1, 46, 47, 67, 68], is expected to be associated to the trace of the energy-momentum tensor θ_μ^μ :

$$\theta_\mu^\mu = \frac{1}{4}\beta(\alpha_s)G_a^{\mu\nu}G_{\mu\nu}^a + (1 + \gamma_m(\alpha_s)) \sum_{u,d,s} m_i \bar{\psi}_i \psi_i, \quad (15)$$

with : $\gamma_m = 2a_s + \dots$ is the quark mass anomalous dimension and $a_s \equiv \alpha_s/\pi$. $\beta(\alpha_s)$ is the β -function normalized as :

$$\beta(\alpha_s) = \beta_1 a_s + \beta_2 a_s^2 + \beta_3 a_s^3 + \dots \quad (16)$$

with:

$$\beta_1 = -\frac{11}{2} + \frac{n_f}{3}, \quad \beta_2 = -\frac{51}{4} + \frac{19}{12}n_f, \quad \beta_3 = \frac{1}{64} \left(-2857 + \frac{5033}{9}n_f - \frac{325}{27}n_f^2 \right) \quad (17)$$

for n_f number of quark flavours. For $n_f = 3$ used in this paper, one has :

$$\beta_1 = -9/2, \quad \beta_2 = -8, \quad \beta_3 = -20.1198 . \quad (18)$$

²An estimate of this mixing angle using Gaussian sum rules favours a maximal mixing [40].

6. Prospects

In this paper, we shall :

- Use selected low and high degree moments of QCD spectral sum rules (QSSR) within the *standard SVZ expansion without instantons* and parametrize the spectral function beyond the minimal duality ansatz “*one resonance \oplus QCD continuum*” in order to extract the masses and decay constants of the scalar gluonia.
- Extract the value of the conformal charge $\psi_G(0)$ (value of the two-point correlator at zero momentum) in order to test the Low Energy Theorem (LET) [68] estimate.
- Discuss some phenomenological implications of our results for an attempt to understand the complex spectrum of the $I = 0$ observed scalar mesons.

7. The QCD anatomy of the two-point correlator

We shall work with the gluonium two-point correlator :

$$\psi_G(q^2) = 16i \int d^4x e^{iqx} \langle 0 | (\theta_\mu^\mu)_G(x) (\theta_\mu^\mu)_G^\dagger(0) | 0 \rangle \quad (19)$$

built from the gluon component of the trace of the energy-momentum tensor in Eq. 15.

- *The standard SVZ-expansion*

Using the Operator Product Expansion (OPE) à la SVZ , its QCD expression can be written as :

$$\psi_G(q^2) = \beta^2(\alpha_s) \left(\frac{2}{\pi^2} \right) \sum_{0,1,2,\dots} C_{2n} \langle \mathcal{O}_{2n} \rangle . \quad (20)$$

where C_{2n} is the Wilson coefficients calculable perturbatively while $\langle \mathcal{O}_{2n} \rangle$ is a short-hand notation for the non-perturbative vacuum condensates $\langle 0 | \mathcal{O}_{2n} | 0 \rangle$ of dimension $2n$.

- ◇ *The unit perturbative operator ($n = 0$)*

Its contribution reads :

$$\begin{aligned} C_0 &\equiv -Q^4 L_\mu \left[C_{00} + C_{01} L_\mu + C_{02} L_\mu^2 \right] \text{ with :} \\ C_{00} &= 1 + \frac{659}{36} a_s + 247.480 a_s^2, \quad C_{01} = -a_s \left(\frac{9}{4} + 65.781 a_s \right), \quad C_{02} = 5.0625 a_s^2, \end{aligned} \quad (21)$$

where the NLO (resp. N2LO) contributions have been obtained in [52] (resp. [53]) and has been adapted for a sum rule use in [54]. $L_\mu \equiv \text{Log}(Q^2/\mu^2)$ where μ is the subtraction point. We shall use for 3 flavours :

$$\Lambda = 340(28) \text{ MeV} \quad (22)$$

deduced from $\alpha_s(M_Z) = 0.1182(19)$ from $M_{\chi_{c0,b0}} - M_{\eta_{c,\eta_b}}$ mass-splittings [61, 62], τ -decays [55, 56] and the world average [25, 57]. We shall use the running QCD coupling to order α_s^2 :

$$a_s(\mu) = a_s^{(0)} \left\{ 1 - a_s^{(0)} \frac{\beta_2}{\beta_1} L L_\mu + \left(a_s^{(0)} \right)^2 \left[\left(\frac{\beta_2}{\beta_1} \right)^2 (L L_\mu^2 - L L_\mu - 1) + \frac{\beta_3}{\beta_1} \right] \right\}, \quad (23)$$

where :

$$a_s^{(0)} \equiv \frac{1}{-\beta_1 \text{Log}(\mu/\Lambda)} \quad \text{and} \quad LL_\mu \equiv \text{Log} \left[2 \text{Log}(\mu/\Lambda) \right]. \quad (24)$$

◇ *The dimension-four gluon condensate ($n = 2$)*

Its contribution reads :

$$C_4 \langle \mathcal{O}_4 \rangle \equiv (C_{40} + L_\mu C_{41}) \langle \alpha_s G^2 \rangle \quad : \quad C_{40} = 2\pi a_s \left(1 + \frac{175}{36} a_s \right), \quad C_{41} = -\frac{9}{2} \pi a_s^2. \quad (25)$$

We shall use the value :

$$\langle \alpha_s G^2 \rangle = (6.35 \pm 0.35) \times 10^{-2} \text{ GeV}^4 \quad (26)$$

determined from light and heavy quark systems [60–62].

◇ *The dimension-six ($n = 3$) gluon condensate*

Its contribution reads (see [63] for the α_s correction for $n_f = 0$ and where a $1/\pi$ misprint is corrected):

$$C_6 \langle \mathcal{O}_6 \rangle = \langle C_{60} + L_\mu C_{61} \rangle \langle g^3 f_{abc} G^a G^b G^c \rangle / Q^2 \quad : \quad C_{60} = a_s, \quad C_{61} = -\frac{29}{4} a_s^2 \quad (27)$$

with [64–66] :

$$\langle g^3 f_{abc} G^a G^b G^c \rangle = (8.2 \pm 1.0) \text{ GeV}^2 \langle \alpha_s G^2 \rangle, \quad (28)$$

which notably differs from the instanton liquid model estimate $\langle g^3 f_{abc} G^a G^b G^c \rangle \approx (1.5 \pm 0.5) \text{ GeV}^2 \langle \alpha_s G^2 \rangle$ [14, 67–69] used in [2].

◇ *The dimension-eight ($n = 4$) gluon condensate*

Its contribution reads :

$$C_8 \langle \mathcal{O}_8 \rangle = C_{80} \langle \alpha_s^2 G^4 \rangle / Q^4 \quad : \quad C_{80} = 4\pi \alpha_s \quad (29)$$

with :

$$\langle \alpha_s^2 G^4 \rangle \equiv \left[14 \langle (\alpha_s f_{abc} G_{\mu\rho}^a G_{\nu\rho}^b)^2 \rangle - \langle (\alpha_s f_{abc} G_{\mu\nu}^a G_{\rho\lambda}^b)^2 \rangle \right] \simeq (0.55 \pm 0.01) \langle \alpha_s G^2 \rangle^2 \quad (30)$$

from a modified factorization with $1/N_c^2$ corrections (13/24) [70] and factorization (9/16) [67]. (0.55 ± 0.01) measures the deviation from factorization which has been found to be largely violated for the four-quark condensates [41, 56, 71].

• *Beyond the standard SVZ-expansion*

◇ *The tachyonic gluon mass ($n = 1$)*

To these standard contributions in the OPE, we can consider the one from a dimension-two tachyonic gluon mass contribution introduced by ³ which phenomenologically parametrizes the large order terms of the

³For reviews, see e.g. [72, 73].

perturbative QCD series [74]. Its existence is supported by some AdS approaches [75–77]. This effect has been calculated explicitly in [78]:

$$C_2(O_2) = C_{21} L_\mu \lambda^2 Q^2 \quad : \quad C_{21} = \frac{3}{a_s} \quad (31)$$

where λ^2 is the tachyonic gluon mass determined from $e^+e^- \rightarrow$ hadrons data and the pion channel [78–80]:

$$a_s \lambda^2 \simeq -(6.0 \pm 0.5) \times 10^{-2} \text{ GeV}^2 . \quad (32)$$

◇ *The direct instanton* ($n \geq 5/2$)

In an instanton liquid model [68, 69], the direct instanton contribution is assumed to be dominated by the single instanton-anti-instanton contribution via a non-perturbative contribution to the perturbative Wilson coefficient [81, 82]:

$$\psi(Q^2)|_{\bar{I}-I} = 32\beta^2 Q^4 \int \rho^4 \left[K_2 \left(\rho\sqrt{Q^2} \right) \right]^2 dn(\rho) \quad (33)$$

where $K_2(x)$ is the modified Bessel function of the second kind. At this stage this classical field effect is beyond the SVZ expansion where the later assumes that one can separate without any ambiguity the perturbative Wilson coefficients from the non-perturbative condensate contributions.

Besides the fact that it contributes in the OPE as $1/Q^5$, i.e it acts like other high-dimension condensates not taken into account in the OPE, the above instanton effect depends crucially on the (model-dependent) overall density $\bar{n} = \int_0^\infty d\rho n(\rho)$ and on its average size $\bar{\rho} = (1/\bar{n}) \int_0^\infty d\rho \rho n(\rho)$ which (unfortunately) are not quantitatively under a good control ($\bar{\rho}$ ranges from 5 [68] to 1.94,[65, 66] and 1.65 GeV⁻¹[69], while $\bar{n} \approx (0.5 \sim 1.2) \text{ fm}^{-4}$). They contribute with a high power in ρ to the spectral function $\text{Im}\psi(t)$, which can be found explicitly in [81, 82], and behave as :

$$\text{Im}\psi_G(t)|_{\bar{I}-I} \quad \begin{array}{l} \xrightarrow{t \rightarrow \infty} n \left(\rho\sqrt{t} \right)^{-5} \\ \xrightarrow{t \rightarrow 0} n \left(\rho\sqrt{t} \right)^4 . \end{array} \quad (34)$$

As such effects are quite inaccurate and model-dependent, we shall not consider them explicitly in the analysis. Instead, an eventual deviation of our results within the standard SVZ expansion from some experimental data or/and or some other alternative estimates (Low-Energy Theorems (LET), Lattice calculations,...) may signal the need of such (beyond the standard OPE) effects in the analysis. One should mention that the approach within the standard SVZ OPE and without a direct instanton effect used in the :

- $U(1)_A$ channel has predicted successfully the value of the topological charge, its slope and the η' -mass and decay constant [6, 13].

- Pseudoscalar pion and kaon channels have reproduced successfully the value of the light quark masses where we have also explicitly shown [83] that the direct instanton effect induces a relatively small correction contrary to some other strong claims [84] in the literature.

8. The Inverse Laplace transform sum rules

From its QCD asymptotic behaviour $\sim (-q^2)^2 \text{Log}(-q^2/\mu^2)$ one can write a twice subtracted dispersion relation:

$$\psi_G(q^2) = \psi_G(0) + q^2 \psi'_G(0) + \frac{q^4}{\pi} \int_0^\infty \frac{dt}{t^2} \frac{\text{Im}\psi_G(t)}{(t - q^2 - i\epsilon)}. \quad (35)$$

• Following standard QSSR techniques [14, 16], one can derive from it different form of the sum rules. In this paper, we shall work with the Exponential or Borel [14, 85, 86] or Inverse Laplace transform [87] finite energy sum rule(LSR)⁴:

$$\mathcal{L}_n^c(\tau) = \int_0^{t_c} dt t^n e^{-t\tau} \frac{1}{\pi} \text{Im}\psi_G(t) \quad : n = -1, 0, 1, 2, 3 \quad (36)$$

and the corresponding ratios of sum rules :

$$\mathcal{R}_{n+l}^c(\tau) \equiv \frac{\mathcal{L}_{n+l}^c(\tau)}{\mathcal{L}_n^c(\tau)}, \quad (37)$$

where τ is the Laplace sum rule variable. In the duality ansatz :

$$\frac{1}{\pi} \text{Im}\psi_G(t) = 2 \sum_G f_G^2 M_G^4 \delta(t - M_G^2) + \theta(t - t_c) \text{ "QCD continuum" }, \quad (38)$$

where the f_G are the lowest resonances couplings normalized as $f_\pi = 93$ MeV and M_G their masses while the "QCD continuum" comes from the discontinuity $\text{Im}\psi_G(t)|_{QCD}$ of the QCD diagrams from the continuum threshold t_c . In the "one narrow resonance \oplus QCD continuum" parametrization of the spectral function:

$$\mathcal{R}_{n+l}^c(\tau) \simeq M_G^2. \quad (39)$$

• To get \mathcal{L}_{-1} , we find convenient to take the Inverse Laplace transform of the first superconvergent 2nd derivative of the correlator. It reads explicitly:

$$\mathcal{L}_{-1}^c(\tau) = \beta^2(\alpha_s) \left(\frac{2}{\pi^2} \right) \tau^{-2} \sum_{n=0,2,\dots} D_n^{-1} + \psi_G(0), \quad (40)$$

with:

$$\begin{aligned} D_0^{-1} &= \left[C_{00} + 2C_{01}(1 - \gamma_E - L_\tau) + 3C_{02}[1 - \pi^2/6 + (-1 + \gamma_E + L_\tau)^2] \right] (1 - \rho_1) \\ D_2^{-1} &= C_{21} \lambda^2 \tau (1 - \rho_0), \\ D_4^{-1} &= - \left[C_{40} - \frac{C_{41}}{36} [55 + 6(\gamma_E + L_\tau)] \right] \langle \alpha_s G^2 \rangle \tau^2, \\ D_6^{-1} &= - [C_{60} - C_{61}(-1 + \gamma_E + L_\tau)] \langle g^3 f_{abc} G^a G^b G^c \rangle \tau^3, \\ D_8^{-1} &= \frac{C_{80}}{2} (1.1 \pm 0.5) \langle \alpha_s G^2 \rangle^2 \tau^4, \end{aligned} \quad (41)$$

with :

$$L_\tau \equiv -\text{Log}(\tau\mu^2), \quad \rho_n \equiv e^{-t_c\tau} \left(1 + t_c\tau + \dots + \frac{(t_c\tau)^n}{n!} \right) \quad (42)$$

⁴The name inverse Laplace transform has been attributed due to the fact that perturbative radiative corrections have this property.

is the QCD continuum contribution from the discontinuity of the QCD diagrams to the sum rule. We omit some eventual continuum contributions from the non-perturbative condensates which are numerically tiny.

$\psi_G(0)$ is the value of the two-point correlator at zero momentum and can be fixed by a LET as [67, 68]:

$$\psi_G(0) \simeq -\frac{16}{\pi}\beta_1\langle\alpha_s G^2\rangle = (1.46 \pm 0.08) \text{ GeV}^4, \quad (43)$$

which we shall test later on.

- To get \mathcal{L}_0 , we take the Inverse Laplace transform of the 1st superconvergent 3rd derivative of the two-point correlator. In this way, we obtain :

$$\mathcal{L}_0^c(\tau) = \beta^2(\alpha_s) \left(\frac{2}{\pi^2}\right) \tau^{-2} \sum_{n=0,2,\dots} D_n^0, \quad (44)$$

with:

$$\begin{aligned} D_0^0 &= \left[2C_{00} - 2C_{01}(3 - 2\gamma_E - 2L_\tau) - 6C_{02}\left[1 - 3\gamma_E + \gamma^2 - \pi^2/6 + (-3 + 2\gamma_E)L_\tau + L_\tau^2\right]\right](1 - \rho_2) \\ D_2^0 &= -\frac{C_{21}}{2}\lambda^2\tau(1 - \rho_1), \\ D_4^0 &= -C_{41}\langle\alpha_s G^2\rangle\tau^2, \\ D_6^0 &= [C_{60} - C_{61}(\gamma_E - L_\tau)]\langle g^3 f_{abc}G^a G^b G^c\rangle\tau^3, \\ D_8^0 &= C_{80}(0.55 \pm 0.01)\langle\alpha_s G^2\rangle^2\tau^4, \end{aligned} \quad (45)$$

- The other higher degrees sum rules $\mathcal{L}_n^c(\tau)$ for $n \geq 1$ can be deduced from the n^{th} τ -derivative of $\mathcal{L}_0^c(\tau)$:

$$\mathcal{L}_n^c(\tau) = (-1)^n \frac{d^n}{d\tau^n} \mathcal{L}_0^c(\tau). \quad (46)$$

- These superconvergent sum rules obey the homogeneous renormalization group equation (RGE):

$$\left\{ -\frac{\partial}{\partial t} + \beta(\alpha_s)\alpha_s \frac{\partial}{\partial \alpha_s} \right\} \mathcal{L}_n^c(e^t\tau, \alpha_s) = 0, \quad (47)$$

with $t \equiv (1/2)L_\tau$ with the renormalization group improved (RGI) solution :

$$\mathcal{L}_n^c(e^t\tau, \alpha_s) = \mathcal{L}_n^c(t=0, \bar{\alpha}_s(\tau)), \quad (48)$$

where $\bar{\alpha}_s$ is the QCD running coupling.

- In the following analysis, we shall work with the family of sum rules having degrees less or equal to 4. Then, we shall select the sum rules which present stability (minimum or inflexion point) in the sum rule variable τ and in the continuum threshold t_c such that we can extract optimal information from the analysis.

9. τ and t_c behaviours of the QCD side of the LSR \mathcal{L}_n^c

Before doing the phenomenological analysis of the LSR, we study the τ and t_c behaviour of their QCD expressions. For instance, we show the subtracted (SSR) \mathcal{L}_{-1}^c and unsubtracted (USR) \mathcal{L}_0^c sum rules in Figs1 and 2, where we notice that the subtraction constant shifts the USR minimum (optimization point)

at lower values of τ . This feature has led to the inconsistencies of the gluonium mass from these sum rules within a “*one resonance \oplus QCD continuum*” parametrization of the spectral function as noted in [1, 2] which can be cured by working instead with “*two resonances \oplus QCD continuum*”

- The other USR moments \mathcal{L}_2^c and \mathcal{L}_4^c stabilize for τ of about 1.5 GeV^{-2} while \mathcal{L}_1^c and \mathcal{L}_3^c do not present τ -stability. To understand these τ -behaviours :

- ◊ We write explicitly the OPE expression of \mathcal{L}_0 and \mathcal{L}_1 to lowest order but includes the α_s correction of the $D = 4$ contribution as this contribution vanishes to LO. Normalized to $(2/\pi^2)\beta^2$, they read:

$$\begin{aligned}\mathcal{L}_0 &= \tau^{-3} \left[1 + \frac{9}{2} \pi a_s^2 \langle \alpha_s G^2 \rangle \tau^2 + a_s \langle g^3 f_{abc} G^a G^b G^c \rangle \tau^3 + 4\pi \alpha_s \langle \alpha_s^2 G^4 \rangle \tau^4 \right], \\ \mathcal{L}_1 &= 3\tau^{-4} \left[1 - \frac{9}{6} \pi a_s^2 \langle \alpha_s G^2 \rangle \tau^2 - 4\pi \alpha_s \langle \alpha_s^2 G^4 \rangle \tau^4 \right],\end{aligned}\tag{49}$$

where the changes of sign of the condensate contributions destabilize \mathcal{L}_1 .

- ◊ The inclusion of the huge PT radiative corrections shows that they also participate to the τ -behaviour of the QCD expression due to the non-trivial reorganization of these PT terms in each Laplace transformed moment.

- From this explicit analysis, one expects that, a priori:

- ◊ One may extract reliably the decay constant of the gluonium from \mathcal{L}_1^c . However, we shall see later on that the extraction of the decay constant from \mathcal{L}_3^c can present τ -stability due to the exponential weight $e^{M_G^2 \tau}$ brought by the resonance contributions which compensates the decrease of the QCD expression [$\mathcal{L}_n^c \sim \tau^{-(n+3)}$] for increasing τ . In the specific example of \mathcal{L}_2^c , we shall that the exponential factor shifts the τ -minimum to lower τ -values (Fig. 10) at which the PT corrections are much smaller than in Fig. 4, while for \mathcal{L}_3^c , there appears a plateau inflexion point (Fig. 11) which render reliable the extraction of f_{G_1} .

- ◊ The mass of the gluonium can be reliably obtained from the ratios \mathcal{R}_{nl}^c of the USR $\mathcal{L}_{n,l}^c$ which optimize at about the same value of τ and where the PT corrections tend to compensate.

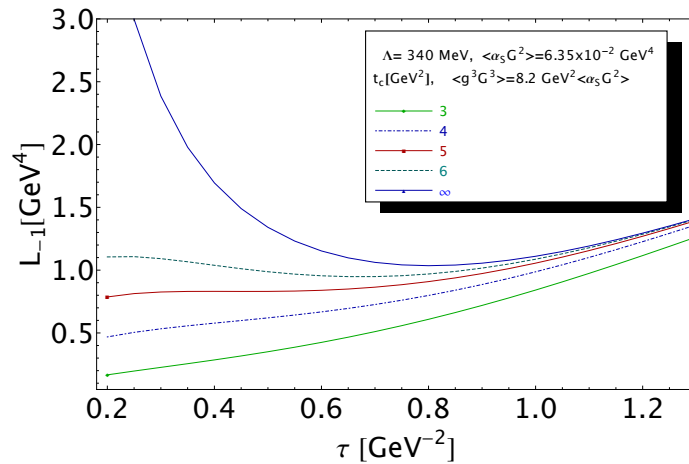


Figure 1: \mathcal{L}_{-1}^c as a function of τ at N2LO for different values of t_c .

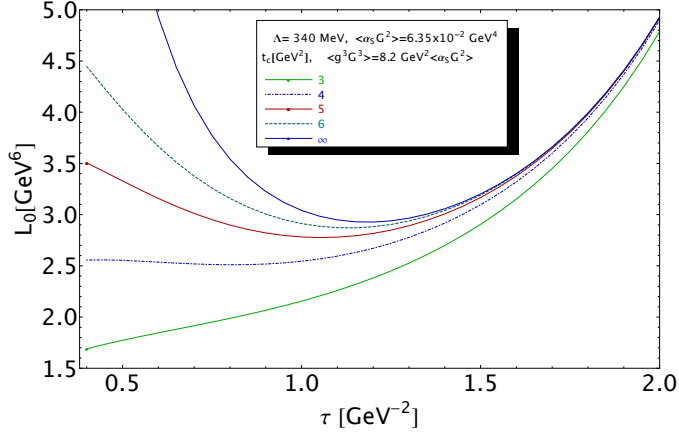


Figure 2: \mathcal{L}_0^c as a function of τ at N2LO for different values of t_c .

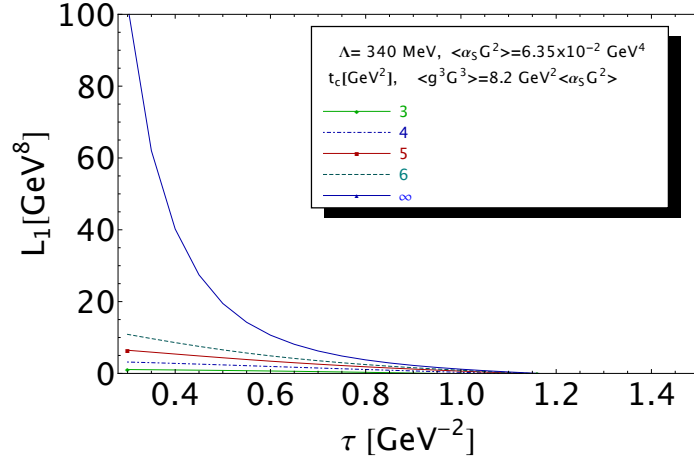


Figure 3: \mathcal{L}_1^c as a function of τ at N2LO for different values of t_c .

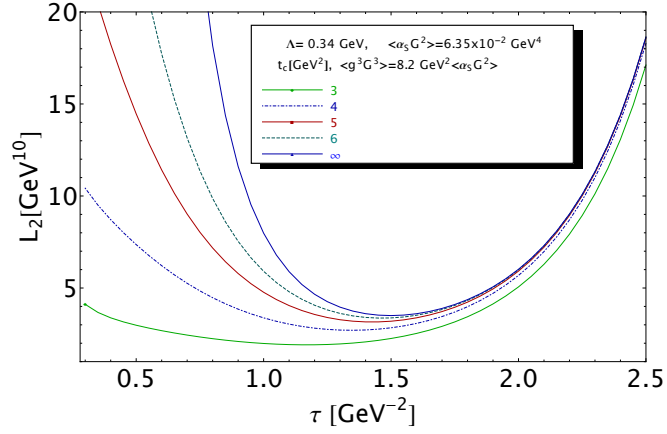


Figure 4: \mathcal{L}_2^c as a function of τ at N2LO for different values of t_c .

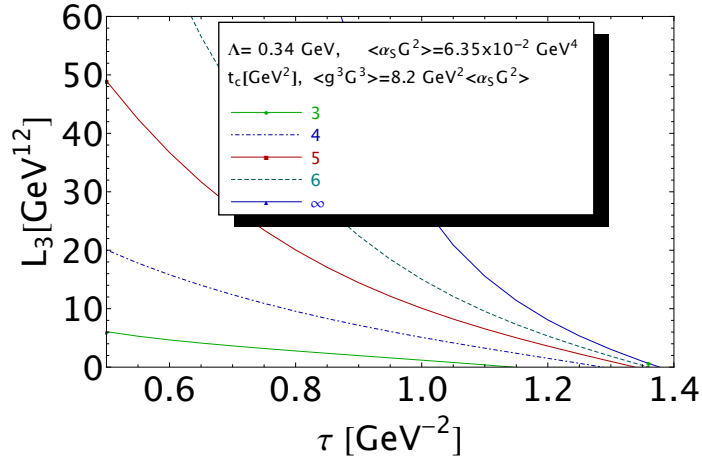


Figure 5: \mathcal{L}_3^c as a function of τ at N2LO for different values of t_c .

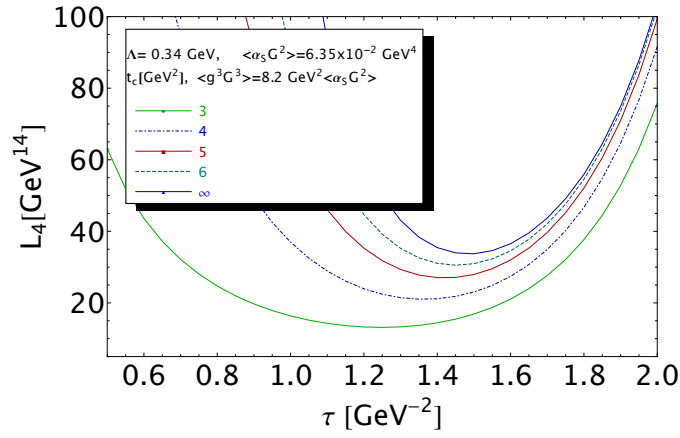


Figure 6: \mathcal{L}_4^c as a function of τ at N2LO for different values of t_c .

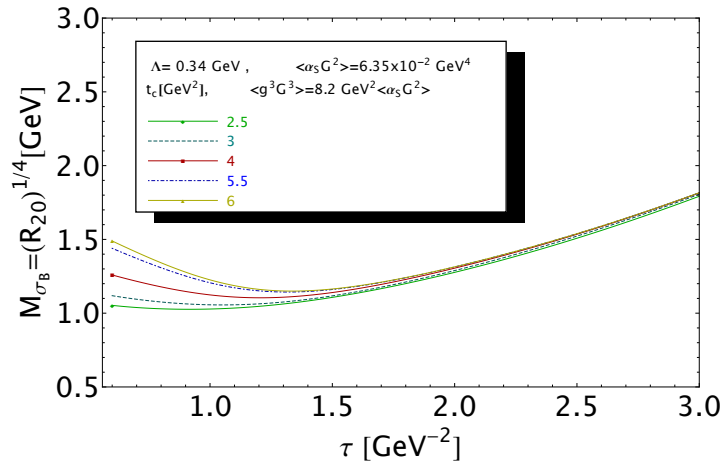


Figure 7: M_{σ_B} from \mathcal{R}_{20}^c as a function of τ at N2LO for different values of t_c within a one resonance parametrization.

10. The masses of the two lightest gluonia σ_B and G_1

In order to get the lowest ground state gluonia masses, we shall work with different forms of USR. We shall not use for this case the observables involving \mathcal{L}_{-1} in order to avoid the dependence of the result on the input value of $\psi_G(0)$ which we shall test later on. Among the possible ratios of sum rules, we have selected \mathcal{R}_{20}^c and \mathcal{R}_{42}^c which appear to give the most stable results versus the variation of different parameters.

- *Mass of the lightest ground state (hereafter named σ_B) from \mathcal{R}_{20}^c*

In so doing, we work with the ratio of sum rules \mathcal{R}_{20}^c . We show its τ -behaviour for different values of t_c in Fig. 8.

- ◇ *The case : one resonance \oplus QCD continuum*

We show in Fig.7 the result with one resonance \oplus QCD continuum parametrization of the spectral function.

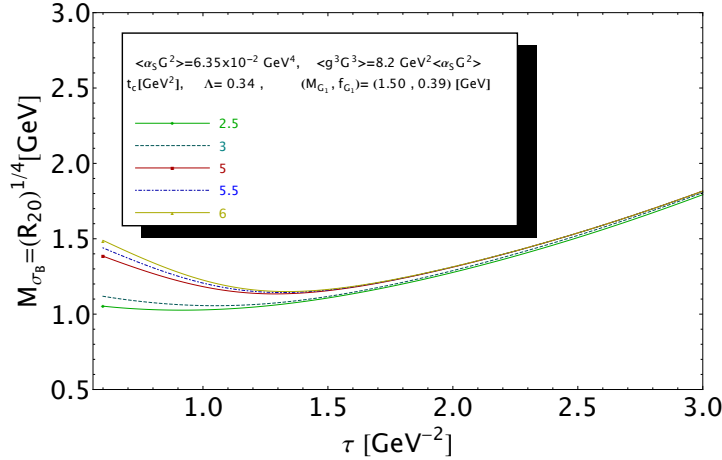


Figure 8: M_{σ_B} from \mathcal{R}_{20}^c as a function of τ at N2LO for different values of t_c including the effect of the 2nd resonance with $(M_{G_1}, f_{G_1}) = (1.50, 0.39)$ GeV [2].

The τ -stability starts from $t_c = 2.5$ GeV² while t_c -stability is reached around 5.5 GeV². We deduce in this range:

$$M_{\sigma_B} = 1088(2)_\tau(54)_{t_c}(57)_\Lambda(1)_{G^2}(2)_{G^3} = 1088(79) \text{ MeV}, \quad (50)$$

where the errors are mainly due to Λ and t_c .

- ◇ *The case : two resonances \oplus QCD continuum*

We show in Fig. 8 the effect of a 2nd resonance with the parameters in [2] :

$$M_{G_1} = 1.5 \text{ GeV} \quad \text{and} \quad f_{G_1} = 0.39 \text{ GeV}. \quad (51)$$

It affects the σ_B mass result by a negligible amount of about 22 MeV. We deduce the conservative result from $t_c = 3$ to 5.5 GeV^2 and use an input error of 0.1 GeV^{-2} for the localization of τ :

$$M_{\sigma_B} = 1085(5)_{\tau}(59)_{t_c}(89)_{\Lambda}(1)_{G^2}(2)_{G^3}(1)_{\lambda^2}(39)_{M_G}(53)_{f_G} = 1085(126) \text{ MeV}, \quad (52)$$

where the errors due to the non-perturbative condensates are negligible.

- *Mass of the medium ground state gluonium G_1 from \mathcal{R}_{42}^c*

Among the different combinations of sum rules, we find that \mathcal{R}_{42}^c can give a reliable prediction of the medium ground state G_1 because \mathcal{L}_2^c and \mathcal{L}_4^c stabilize at about the same value of $\tau \simeq (1.2 \sim 1.5) \text{ GeV}^{-2}$ (see Figs 4 and 6). We show in Fig. 9 the τ behaviour of \mathcal{R}_{42} for different values of t_c .

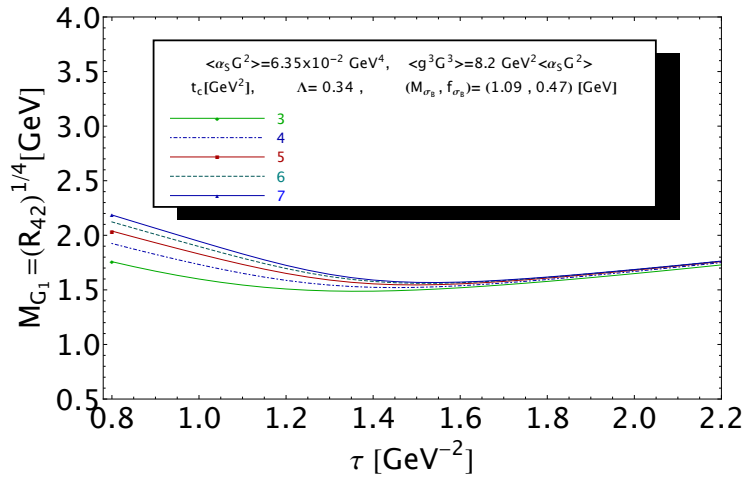


Figure 9: M_{G_1} as a function of τ at N2LO for different values of t_c from a two resonances parametrization.

- ◇ *The case : two resonances \oplus QCD continuum*

We obtain for $t_c = 3$ to 6 GeV^2 :

$$M_{G_1} = 1524(36)_{t_c}(7)_{\tau}(115)_{\Lambda}(2)_{G^2}(2)_{G^3}(0)_{\lambda^2}(6)_{M_{\sigma}}(11)_{f_{\sigma}} = 1524(121) \text{ MeV} , \quad (53)$$

where the error comes mainly from the one of Λ . We have used $f_{\sigma_B} = 0.47$ as anticipated from the next section.

- ◇ *The case : one resonance \oplus QCD continuum*

A similar analysis leads to:

$$M_{G_1} = 1515(123) \text{ MeV} \quad (54)$$

indicating that the σ_B effect to the value of the mass is quite small (about 9 MeV).

11. Decay constants f_{σ_B} and f_{G_1}

To extract these decay constants, we work simultaneously with \mathcal{L}_2^c , \mathcal{L}_3^c and \mathcal{L}_{-1}^c and use an iteration procedure. The two former are expected to be more sensitive to f_{G_1} while the third to f_{σ_B} .

◊ We shall take M_{σ_B} from Eq. 52 and $M_{G_1} = 1.548$ GeV anticipated from Eq. 53. We start the iteration by using $f_{\sigma_B} = 800$ MeV from [1, 2]. Then, we compare the value of f_{G_1} from \mathcal{L}_2^c , \mathcal{L}_3^c . Then, we use the common solution of f_{G_1} from the two sum rules into \mathcal{L}_{-1}^c to extract f_{σ_B} . We continue the iterations until we reach convergent solutions. The analysis corresponding to the final value of the decay constants are shown in Figs. 10 to 13.

– In Figs. 10 and reffig:fg12, we show the τ and t_c behaviours of f_{G_1} given the value of f_{σ_B} . – In Fig. 10, we show the results from \mathcal{L}_2^c , \mathcal{L}_3^c at the τ minimas or plateau from which we deduce the common solution for f_{G_1} for fixed f_{σ_B} which allows to fix :

$$t_c \simeq (3.3 \pm 0.3) \text{ GeV}^2 . \quad (55)$$

– In Fig. 13, we show the τ and t_c behaviours of f_{σ_B} where we take the value $\tau = 0.5 \text{ GeV}^{-2}$ at the inflexion point which is more pronounced for higher values of t_c .

Then, we obtain :

$$\begin{aligned} f_{G_1} &= 394(45)_{t_c(2)}\tau(22)\Lambda(0)_{G^2(1)}G^3(23)_{M_{\sigma_B}(9)}f_{\sigma_B}(94)_{M_{G_1}} = 394(109) \text{ MeV} \quad \text{from } \mathcal{L}_3^c \\ f_{\sigma_B} &= 563(36)_{t_c(64)}\tau(81)\Lambda(9)_{G^2(11)}G^3(10)_{M_{\sigma}(3)}M_{G_1}(107)_{f_{G_1}} = 563(154) \text{ MeV} \quad \text{from } \mathcal{L}_{-1}^c \end{aligned} \quad (56)$$

One can note that value of f_{G_1} agrees perfectly with the one in [2] while f_{σ_B} is in the range of the one obtained in [1, 2].

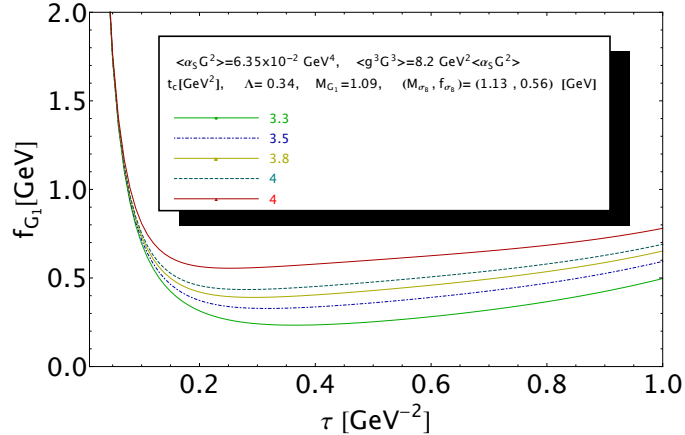


Figure 10: f_{G_1} from \mathcal{L}_2^c as a function of τ at N2LO for different values of t_c from a two resonances parametrization and for given values of $(M_{\sigma_B}, (f_{\sigma_B}))$.

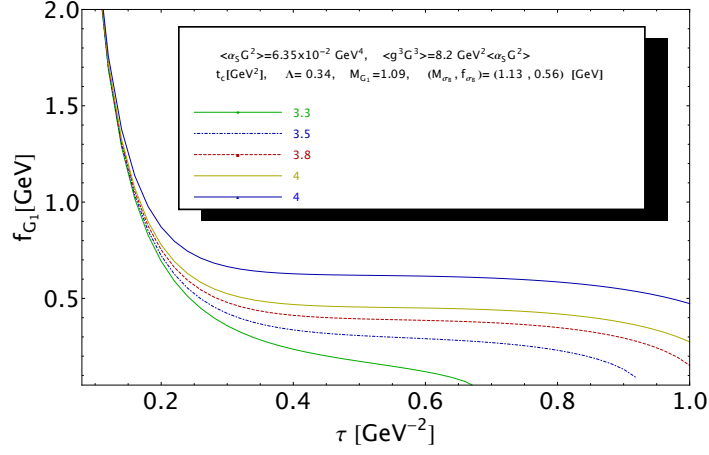


Figure 11: f_{G_1} from \mathcal{L}_3^c as a function of τ at N2LO for different values of t_c from a two resonances parametrization and for given values of $(M_{\sigma_B}, f_{\sigma_B})$.

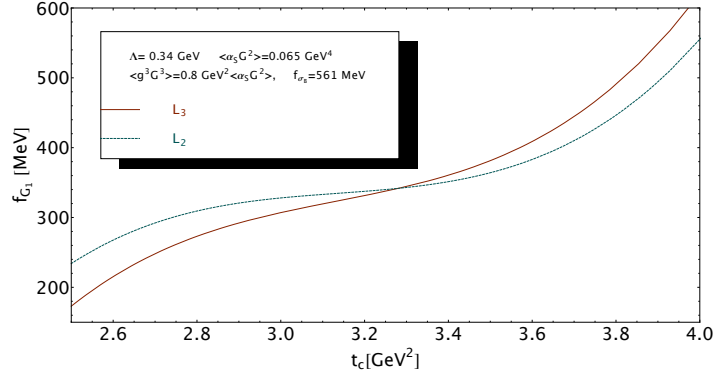


Figure 12: f_{G_1} from \mathcal{L}_2^c and \mathcal{L}_3^c from the τ -minimum $\tau \simeq 0.5 \text{ GeV}^{-2}$ from previous figures as a function of t_c for given values of $(M_{\sigma_B}, f_{\sigma_B})$.

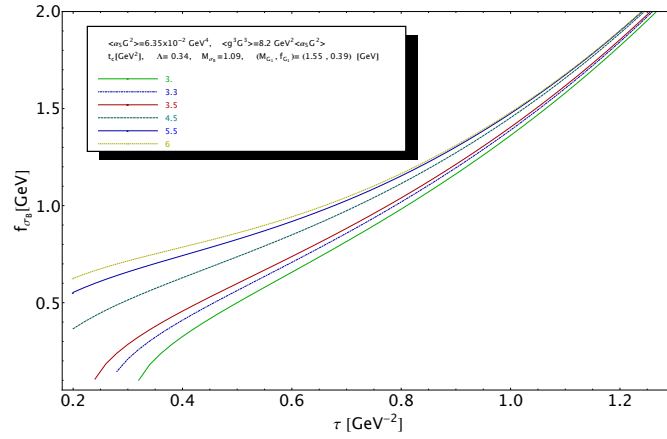


Figure 13: f_{σ_B} as a function of τ at N2LO for different values of t_c from a two resonances parametrization and for given values of (M_{G_1}, f_{G_1}) .

12. First radial excitation σ'_B of σ_B

◇ The σ'_B mass

We attempt to extract its mass from \mathcal{R}_{20}^c which we have used to estimate M_{σ_B} . In so doing, we subtract the contributions of the σ_B and G_1 at take the result for very large values of t_c i.e by replacing the QCD continuum contribution by the σ'_B . In this way, we obtain the result in Fig. 14 from \mathcal{R}_{20}^c .

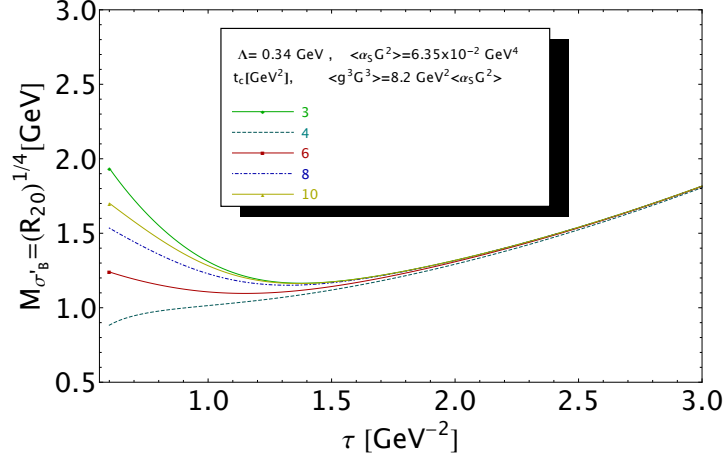


Figure 14: $M_{\sigma'_B}$ as a function of τ at N2LO for different values of t_c within a three resonances parametrization. from \mathcal{R}_{20}^c

We obtain for $t_c \simeq (6 \sim 8)$ GeV^2 : at the τ -minimum :

$$M_{\sigma'_B} = 1121(25)_{t_c(10)}\tau(110)\Lambda(0)G^2(3)G^3(1)M_{\sigma_B}(12)f_{\sigma_B}(7)M_{G_1}(23)f_{G_1} = 1124(117) \text{ MeV}, \quad (57)$$

where the contributions of σ_B and G_1 have been included after iteration from 2 ($M_{\sigma'_B} = 1124$) to 3 (value in Eq. 57) resonances values of the σ_B and G_1 masses and decay constants.

◇ The σ'_B decay constant

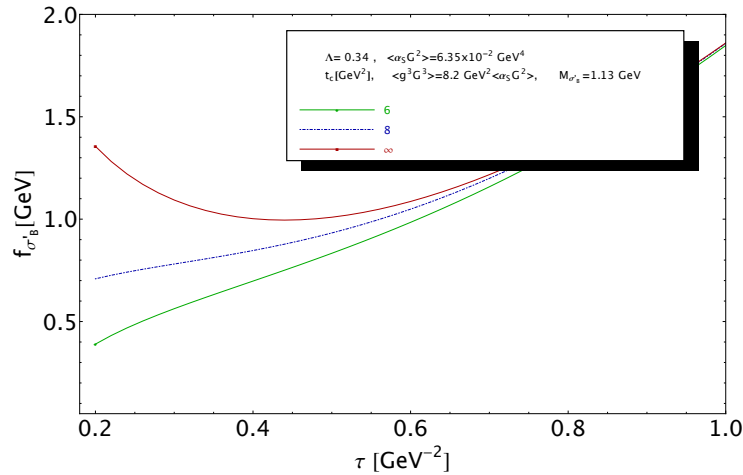


Figure 15: $f_{\sigma'_B}$ as a function of τ at N2LO for different values of t_c within a three resonances parametrization. from \mathcal{L}_{-1}^c

We extract the decay constant from \mathcal{L}_{-1}^c using the same range of t_c as for $M_{\sigma'_B}$ and taking $\tau \simeq 0.4$ GeV $^{-2}$. We obtain:

$$\begin{aligned} f_{\sigma'_B} &= 648(75)_{t_c}(43)_{\tau}(124)_{\Lambda}(20)_{G^2}(7)_{G^3}(24)_{M_{\sigma_B}}(137)_{f_{\sigma_B}}(0)_{M_{G_1}}(56)_{f_{G_1}}(33)_{M_{\sigma'_B}} \\ &= 648(216) \text{ MeV from } \mathcal{L}_{-1}^c. \end{aligned} \quad (58)$$

Anticipating the estimate in Section 20 by extracting the decay constant from $f_0(1.37) \rightarrow 2(\pi\pi)_S$ data through a LEV-SR, we shall work instead with the more precise value:

$$f_{\sigma'_B} = 329(30) \text{ MeV}. \quad (59)$$

13. Iterative improvements of the previous estimates

In the following, we study the effect of the σ'_B which is below M_{G_1} on the previous estimates

- *The mass M_{σ_B}*

We include the contribution of the σ'_B which is below the G_1 mass. Using \mathcal{R}_{20}^c as in previous section, the curves are similar to the ones in Fig. 8. The mass in Eq. 52 becomes :

$$M_{\sigma_B} = 1070(126)(1)_{f_{\sigma'_B}}(1)_{M_{\sigma'_B}} \text{ MeV}, \quad (60)$$

which is our final estimate.

- *The mass M_{G_1}*

Including the contribution of the σ'_B into \mathcal{R}_{42}^c , the mass in Eq. 53 becomes :

$$M_{G_1} = 1548(118)(15)_{f_{\sigma'_B}}(13)_{M_{\sigma'_B}} = 1548(120) \text{ MeV}, \quad (61)$$

which is our final estimate.

- *The decay constants f_{σ_B} and f_{G_1}*

We repeat the procedure used previously by extracting simultaneously f_{σ_B} and f_{G_1} from \mathcal{L}_{-1}^c and from \mathcal{L}_2^c , \mathcal{L}_3^c by including the σ'_B contribution. The value of t_c at which the solutions from \mathcal{L}_2^c , \mathcal{L}_3^c meet is slightly shifted from 3.3 to 3.2 GeV 2 . We obtain the final estimate:

$$f_{\sigma_B} = 456(154)(8)_{M_{\sigma'_B}}(12)_{f_{\sigma'_B}} = 456(157) \text{ MeV}, \quad (62)$$

and:

$$f_{G_1} = 365(109)(17)_{M_{\sigma'_B}}(1)_{f_{\sigma'_B}} = 365(110) \text{ MeV}, \quad (63)$$

14. The first radial excitation G'_1 of G_1

- *The G'_1 mass*

◊ We attempt to estimate the mass of the first radial excitation by replacing the QCD continuum by a 4th resonance in the sum rule \mathcal{R}_{42}^c which we have used to extract M_{G_1} . The analysis is shown in Fig.16

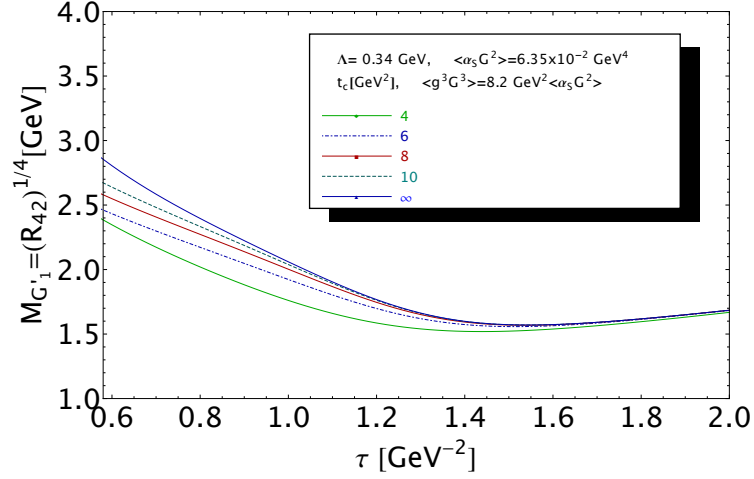


Figure 16: $M_{G'_1}$ as a function of τ at N2LO for from a four resonances parametrization.

We deduce for $t_c \simeq (4 \sim 8) \text{ GeV}^2$ at the τ -minimum:

$$M_{G'_1} = 1563(23)_{t_c}(2)_{\tau}(138)_{\Lambda}(3)_{G^2}(3)_{G^3}(6)_{f_{\sigma_B}}(2)_{M_{\sigma_B}}(3)_{f_{G_1}}(9)_{M_{G_1}}(11)_{f_{\sigma'_B}}(1)_{M_{\sigma'_B}} = 1563(141) \text{ MeV} \quad (64)$$

where the error comes mainly from Λ . We have included the contributions of the σ_B, σ'_B and G_1 .

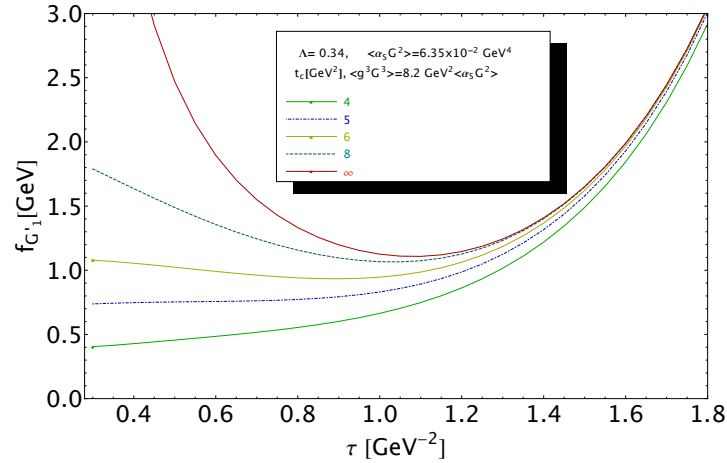


Figure 17: $f_{G'_1}$ as a function of τ at N2LO from a three resonances parametrization.

- The decay constant $f_{G'_1}$

To extract the decay constant we shall work with the sum rules \mathcal{L}_2^c or/and \mathcal{L}_3^c used previously to get f_{G_1} where \mathcal{L}_2^c presents a τ -minimum (Fig. 17) while \mathcal{L}_3^c an inflexion point. From \mathcal{L}_2^c , one deduces:

$$\begin{aligned} f_{G'_1} &= 1000(65)_{t_c}(11)_{\tau}(150)_{\Lambda}(1)_{G^2}(2)_{G^3}(15)_{f_{\sigma_B}}(58)_{M_{\sigma_B}}(53)_{f_{G_1}}(63)_{M_{G_1}}(1)_{f_{\sigma'_B}}(8)_{M_{\sigma'_B}}(125)_{M_{G'_1}} \\ &= 1000(230) \text{ MeV}. \end{aligned} \quad (65)$$

15. The second radial excitation G_2

- The mass M_{G_2}

In so doing, we work with some judicious sum rules which optimize at smaller values of τ and are more sensitive to the high-mass meson contributions. This criterion is satisfied by \mathcal{R}_{21}^c and \mathcal{R}_{31}^c which one can see in Figs 18 and 19 where the contributions of the σ_B , σ' and G_1 , G'_1 have been subtracted. \mathcal{R}_{21}^c has a (τ, t_c) minimas at (0.18,12) in units of $(\text{GeV}^{-2}, \text{GeV}^2)$ while \mathcal{R}_{31}^c has the minimas at (0.24,10).

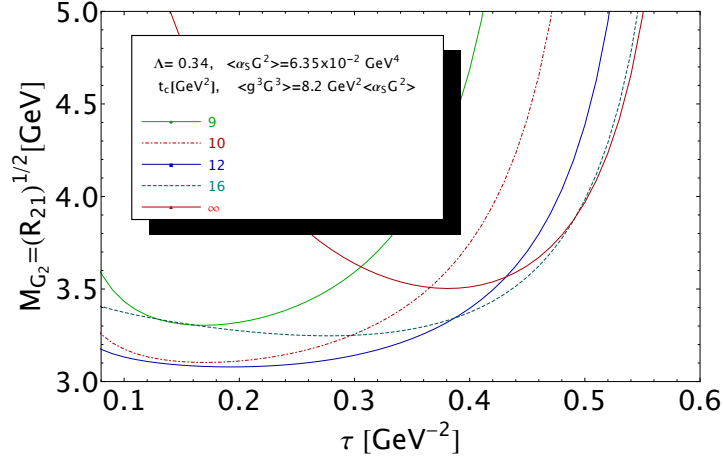


Figure 18: M_{G_2} as a function of τ at N2LO for a four resonances parametrization. from \mathcal{R}_{21}^c .

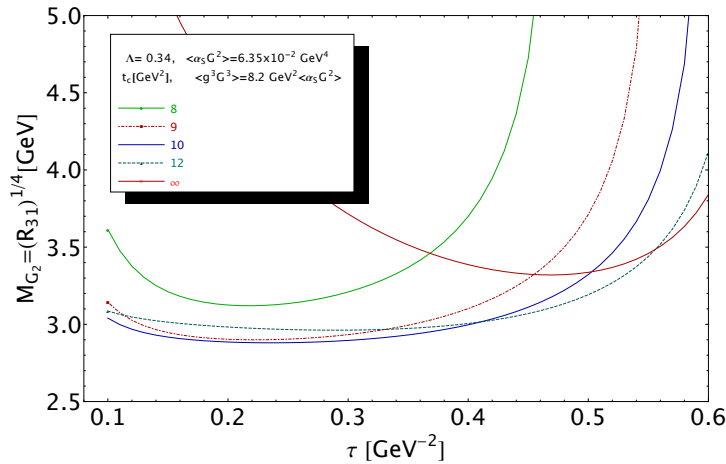


Figure 19: M_{G_2} as a function of τ at N2LO for a four resonances parametrization. from \mathcal{R}_{31}^c .

We obtain :

$$\begin{aligned}
M_{G_2} &= 3079(15)_{t_c}(10)_\tau(63)_\Lambda(1)_{G^2}(2)_{G^3}(10)_{M_{\sigma_B}}(7)_{f_{\sigma_B}}(20)_{M_{G_1}}(36)_{f_{G_1}}(6)_{M_{\sigma'}}(1)_{f_{\sigma'}}(193)_{M_{G'_1}}(207)_{f_{G'_1}} \\
&= 3079(294) \text{ MeV} \quad \text{from } \mathcal{R}_{21}^c.
\end{aligned} \tag{66}$$

and:

$$\begin{aligned}
M_{G_2} &= 2879(9)_{t_c}(3)_\tau(64)_\Lambda(1)_{G^2}(2)_{G^3}(10)_{M_{\sigma_B}}(8)_{f_{\sigma_B}}(25)_{M_{G_1}}(40)_{f_{G_1}}(6)_{M_{\sigma'}}(0)_{f_{\sigma'}}(226)_{M_{G'_1}}(233)_{f_{G'_1}} \\
&= 2879(334) \text{ MeV} \quad \text{from } \mathcal{R}_{31}^c,
\end{aligned} \tag{67}$$

from which we deduce the mean value:

$$M_{G_2} = 2992(221) \text{ MeV}. \tag{68}$$

This second radial excitation is far above the ground states and first radial excitations and may mix with the trigluonium scalar ground state [2, 12]:

$$M_{3g} \simeq 3.1 \text{ GeV}. \tag{69}$$

◇ *The decay constant f_{G_2}*

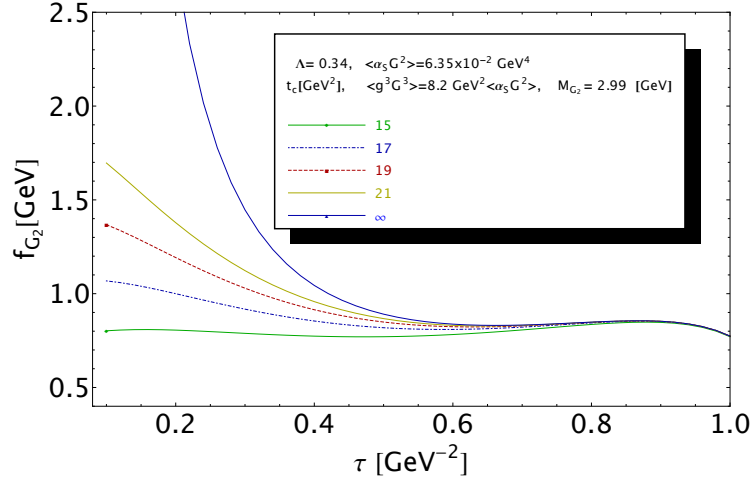


Figure 20: f_{G_2} as a function of τ at N2LO for different values of t_c for a four resonances parametrization from \mathcal{L}_3^c

We shall work with \mathcal{L}_3^c (used to get f_{G_1}) for extracting the decay constant f_{G_2} . The analysis is shown in Fig. 20 where one obtains:

$$f_{G_2} = 797(27)_{t_c}(57)_\Lambda(1)_{M_{\sigma_B}}(3)_{M_{G_1}}(3)_{f_{G_1}}(36)_{M_{G'_1}}(11)_{f_{G'_1}}(11)_{M_{G_2}} = 797(74) \text{ MeV}. \tag{70}$$

The errors due to the other terms not quoted are zero. One should note that, like in the case of $f_{\sigma'}^{\text{eff}}$ which is comparable in size, the previous coupling of the G_2 to the gluonic correlator can be interpreted as the sum of effective couplings of all higher states contributing to the spectral function.

16. Truncation of the PT series, tachyonic gluon mass and the OPE

- *Truncation of the PT series*

One can notice from the expression of the two-point correlator in Eq. 21 that the radiative corrections to the unit operator are huge.

- ◊ *M_{σ_B} and M_{G_1}*

We compare explicitly the effect of PT radiative corrections for the predictions of M_{σ_B} and M_{G_1} and \mathcal{R}_{42}^c in the case of “one resonance \oplus QCD continuum” parametrization of the spectral function. For e.g $t_c = 4$ GeV², one obtains:

$$\begin{aligned} M_{\sigma_B} &\simeq 768 \text{ (LO)} \quad \rightarrow \quad 1000 \text{ (NLO)} \quad \rightarrow \quad 1076 \text{ (N2LO)} \text{ from } \mathcal{R}_{20}^c, \\ M_{G_1} &\simeq 1492 \text{ (LO)} \quad \rightarrow \quad 1522 \text{ (NLO)} \quad \rightarrow \quad 1530 \text{ (N2LO)} \text{ from } \mathcal{R}_{42}^c, \end{aligned} \quad (71)$$

indicating that the α_s corrections increase the LO masses by about 30/% for the σ_B while the increase from NLO to N2LO is only about 8%. For M_{G_1} the effect of radiative corrections is much smaller. This relatively small effect is due to the fact that radiative PT corrections tend to compensate in the ratios of sum rules.

- ◊ *The decay constant f_{σ_B}*

Fixing e.g $t_c \simeq 4$ GeV², we study the effect of the truncation of the PT series on the estimate of the coupling from \mathcal{L}_{-1}^c . Taking $\tau \simeq 0.5$ GeV⁻² at the inflexion point, we obtain the result in units of MeV for a “one resonance” parametrization :

$$f_{\sigma_B} = 202 \text{ (LO)} \quad \rightarrow \quad 406 \text{ (NLO)} \quad \rightarrow \quad 696 \text{ (N2LO)} . \quad (72)$$

The effect of radiative corrections is huge as expected.

- ◊ *The decay constant f_{G_1}*

Here we fix $t_c = 3.75$ GeV² as obtained in the previous analysis . There is a large plateau in τ for different PT series. We obtain in units of MeV from \mathcal{L}_3^c within a “one resonance \oplus QCD continuum”:

$$f_{G_1} = 360 \text{ (LO)} \quad \rightarrow \quad 470 \text{ (NLO)} \quad \rightarrow \quad 580 \text{ (N2LO)} , \quad (73)$$

which shows relatively moderate PT corrections.

- *Gluon condensates and the OPE*

Though the presence of the gluon condensates in the OPE are crucial for the stability points, their effects are relatively small ensuring the convergence of the OPE.

- *The tachyonic gluon mass λ^2*

As emphasized in [78] this effect is expected to explain the large scale of the scalar gluonium channel compared to the ordinary ρ meson channel. It is also expected [74] to give a phenomenological estimate of the uncalculated terms of the PT series.

◇ M_{σ_B} and M_{G_1}

Including the λ^2 contribution in the OPE, we check that its effect on the mass determination is negligible which can be due to the fact that the effects tend to compensate in the ratio of moments.

◇ *The decay constants f_{σ_B} and f_{G_1}*

Within a two resonances parametrization, we find for a given value of t_c and τ that the tachyonic gluon mass decreases the value of f_{σ_B} by about 30-50 % while it increases only slightly f_{G_1} by about 5%.

• *Comparison with some other determinations*

◇ *The σ_B*

The existence of the σ_B with a mass about 1 GeV is compatible with the idea that it is the dilaton associated to conformal $U(1)_V$ invariance analogous to the case where the η' -meson is associated to the $U(1)_A$ symmetry. The dilaton has been extensively discussed in an effective Lagrangian approach (see e.g. [46, 47, 89]) including the $U(1)_V$ symmetry.

– The mass in Eq. 52 is in a good agreement with the earlier sum rule result [7]:

$$M_{\sigma_B} \simeq (0.95 \sim 1.10) \text{ GeV}, \quad (74)$$

from a least square fit of the USR ratio of sum rules. A similar result has been obtained from the analysis [40] based on Gaussian sum rules [41] including instanton effects.

– A full lattice calculation [48, 49] finds a relative suppressed mass of the lightest glueball which moves from the quenched result about 1.6 GeV to the full QCD one around 1 GeV. However, a more appropriate comparison with the lattice results requires a simulation beyond the one resonance contribution to the scalar gluonium correlator.

– A strong coupling analysis of the gluon propagator [90] finds a gluonium pole mass around M_σ and $M_{f_0(980)}$ from the mass gap [90].

◇ *The higher mass gluonium G_1*

– Its mass is given in Eqs. 54 and 53 in the case of *one or two resonances* \oplus *QCD continuum* parametrization of the spectral function. These results agree with the earlier ones in Ref. [2] from ratio of USR and can be compared to some other sum rules results [81, 91, 92]. We conclude from the paper of Ref. [93] that their result for $M_{G_1} \simeq 1.8$ GeV is obtained outside the τ minima of their sum rule which leads to an overestimate.

– Our result for M_{G_1} is slightly lower than the one from quenched and full lattice calculations [48–51]. However, a precise comparison of the results from the two approaches is delicate as the lattice situation for the 1 GeV result remains unclear [48, 49], while the lattice analysis is done by only retaining a single resonance contribution to the two-point correlator.

◇ *The gluonium radial excitations σ' and G_2 and*

We have also attempted to extract the masses and decay constants of the gluonium first radial excitations with the result in Eqs. 57, 58, 68 and 70. In the next phenomenological analysis, we are tempted to identify the σ' (1.32) with the f_0 (1.37). The G_2 (2.5) is (within the large errors) in the range of the observed f_0 (2.02) and f_0 (2.2) [25].

◇ *Comments*

– In the example of the scalar gluonium channel, we have seen that (at least) NLO corrections are important in the determination of the scalar gluonium masses from the sum rules, while N2LO corrections are needed for a good determination of the decay constants. Therefore, one should make a great care for quoting the LO results given in the literature.

– The smallness of the tachyonic gluon mass contribution to the gluonium mass determinations may indicate that the PT series reaches its asymptotic form after the inclusion of the N2LO contribution which can be sufficient for extracting with a good accuracy these observables from the sum rules. However, we have also noticed that the decay constant of the σ_B is largely affected (-30 to -40%) by the tachyonic gluon mass which tends to decouple the σ_B from the two-point correlator.

– Some other analysis using “two resonances \oplus QCD continuum parametrization from sum rules with instantons [40] and an inverse problem dispersive approach without instantons [94] reproduce our results for M_{σ_B} and M_{G_1} . The apparent discrepancies with the lattice results requests a clarification of the full QCD results and a two resonance parametrization of the two-point correlator.

– Some phenomenological implications of our results will be discussed in the next section.

17. The conformal charge $\psi_G(0)$

• *Test of the Low Energy Theorem (LET)*

LET suggests that the value of $\psi_G(0)$ is given by Eq.43. We test the accuracy of this relation by saturating the two-point function by the two lowest ground state resonances σ_B and G_1 . In this way, we obtain :

$$\psi_G(0) = \sum_{i=\sigma, G, \dots} 2 f_G^2 M_G^2 \approx 1.95 \text{ GeV}^4 , \quad (75)$$

which is slightly higher than the LET estimate in Eq.43 :

$$\psi_G^{LET}(0) = (1.46 \pm 0.08) \text{ GeV}^2 , \quad (76)$$

without any appeal to some contributions beyond the OPE such as the instantons ones.

• *Sum rule extraction of conformal charge $\psi_G(0)$*

Here, we shall, instead, determine $\psi_G(0)$ from the sum rule within the standard SVZ expansion. A similar analysis has been already done in [7] which we shall update here. In so doing, we work with the combination

of sum rules⁵:

$$\psi_G^{\text{LSR}}(0) = \int_0^{t_c} \frac{dt}{t} e^{-t\tau} \left(1 - \frac{t\tau}{2}\right) \frac{1}{\pi} \text{Im}\psi_G(t) - \left(\mathcal{L}_{-1}^c - \frac{\tau}{2}\mathcal{L}_0^c\right)_{\text{QCD}}, \quad (77)$$

where the 1st part of the RHS is parametrized by the lowest resonances σ_B , G_1 , σ'_B and the 2nd part is the QCD expression of the moment sum rules \mathcal{L}_n^c . The other radial excitations G'_1 and G_2 having less certain values of the decay constants are included in the QCD continuum.

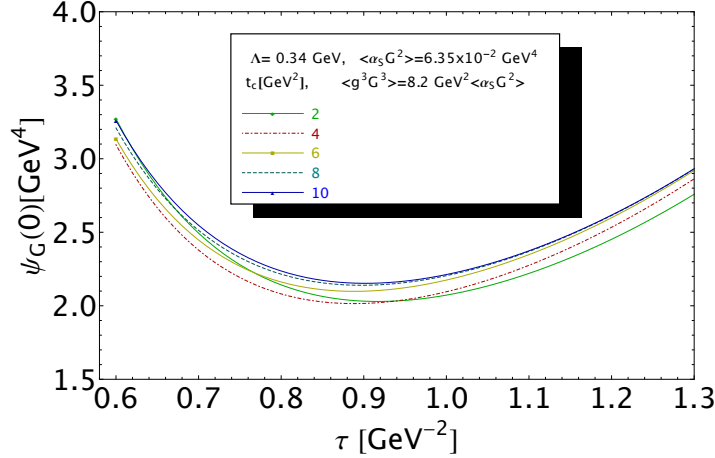


Figure 21: $\psi(0)_G$ from LSR as a function of τ for different values of t_c .

• The result of the analysis is shown in Fig. 21, where the curves present nice τ at $(1.0 \pm 0.04) \text{ GeV}^{-2}$. A t_c -minimum appears at $(2 \sim 4) \text{ GeV}^2$ and a stability from 8 GeV^2 . We obtain:

$$\begin{aligned} \psi_G^{\text{LSR}}(0) &= 2.09(2)_{t_c}(0)_{\tau}(18)_{\Lambda}(11)_{G^2}(10)_{G^3}(1)_{M_{\sigma_B}}(14)_{f_{\sigma_B}}(2)_{M_G}(7)_{f_G}(0)_{M_{\sigma'_B}}(2)_{f_{\sigma'_B}} \\ &= 2.09(29) \text{ GeV}^4, \end{aligned} \quad (78)$$

where the central value is 1.4 times the LET estimate in Eq. 43 and the one in [7] obtained within a one resonance and to LO.

◊ One can notice that adding the two radial excitation contributions changes this value to $2.23 G_2$. The contribution of σ_B is about 70% which is crucial for recovering the LET result.

• We analyze the effect of this result in the estimate of f_{σ_B} and $f_{\sigma'_B}$ obtained previously from the moment \mathcal{L}_{-1}^c where $\psi_G(0)$ contributes.

◊ We find e.g for f_{σ_B} that the inflexion point in τ moves from $\tau = 0.56 \text{ GeV}^{-2}$ (see Fig. 13) to 0.46 for e.g $t_c = 3.75 \text{ GeV}^2$ to which corresponds to an increase of about 4 MeV, which is negligible compared to the large error of f_{σ_B} .

⁵We remind that this combination of sum rules has been firstly (and successfully) introduced to quantify the deviation from kaon PCAC [95] due to $SU(3)$ breakings and later to estimate the topological charge of the $U(1)_A$ channel [7] and its slope [7, 13] within the standard SVZ expansion without any additional instanton contributions.

◊ For $f_{\sigma'_B}$, the LSR value of $\psi_G(0)$ increases the decay constant by about 100 MeV but this value is inside the range spanned by the large error of 342 MeV obtained in its determination. Therefore, we keep the result obtained in the previous section.

18. The slope $\psi'_G(0)$ of the conformal charge

The slope $\psi'_G(0)$ obeys the twice subtracted sum rule:

$$\psi'_G|_{\text{LSR}}(0) = \int_0^{t_c} \frac{dt}{t^2} e^{-t\tau} \frac{1}{\pi} \text{Im}\psi_G(t) - \mathcal{L}_{-2}^c|_{QCD} + \tau \psi(0)_G, \quad (79)$$

where :

$$\mathcal{L}_{-2}^c(\tau) = \beta^2(\alpha_s) \left(\frac{2}{\pi^2} \right) \tau^{-1} \sum_{n=0,2,\dots} D_n^{-2}, \quad (80)$$

with:

$$\begin{aligned} D_0^{-2} &= \left[C_{00} - 2C_{01}\gamma_E + C_{02} \left[\gamma_E - \frac{\pi^2}{6} \right] \right] (1 - \rho_0) \\ D_2^{-2} &= \tau C_{21} (1 - \gamma_E) \lambda^2 \\ D_4^{-2} &= -\tau^2 \left[C_{40} - C_{41} \left[\frac{1}{2} - \gamma_E \right] \right] \langle \alpha_s G^2 \rangle, \\ D_6^{-2} &= -\frac{\tau^3}{2} \left[C_{60} - C_{61} \left(\frac{1}{3} - \gamma_E \right) \right] \langle g^3 f_{abc} G^a G^b G^c \rangle, \\ D_8^{-2} &= -\frac{\tau^4}{6} C_{80} (1.1 \pm 0.5) \langle \alpha_s G^2 \rangle^2. \end{aligned} \quad (81)$$

We show the analysis in Fig. 22 from which we deduce from $t_c = 2 \sim 8 \text{ GeV}^2$ at the τ -minima $\pm 0.04 \text{ GeV}^{-2}$:

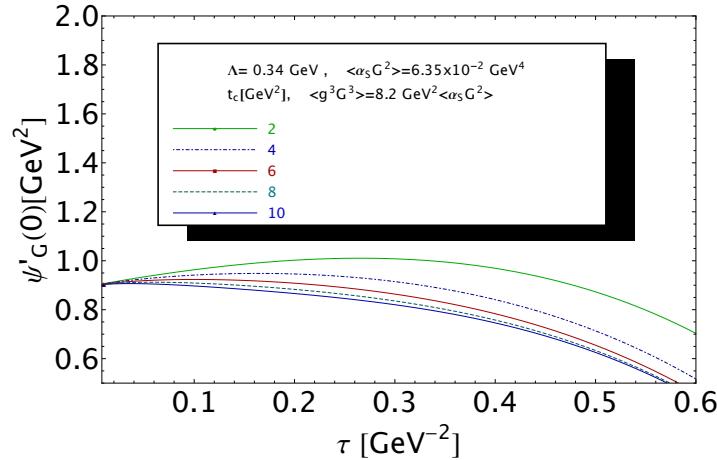


Figure 22: $\psi'_G(0)_G$ from LSR as a function of τ for different values of t_c .

$$\begin{aligned} \psi'_G|_{\text{LSR}}(0) &= 0.95(5)_{t_c} (0)_\tau (2)\Lambda(0)_{G^2} (0.1)_{G^3} (1.5)_{M_{\sigma_B}} (25)_{f_{\sigma_B}} (1.1)_{M_{G_1}} (11.4)_{f_{G_1}} (0.7)_{M_{\sigma'}} (1.7)_{f_{\sigma'}} (11)_{\psi_G(0)} \\ &= 0.95(30) \text{ GeV}^2, \end{aligned} \quad (82)$$

where the σ_B contributes to about 80%. The present result updates the earlier one in Ref. [7].

19. Hadronic couplings of the σ_B meson

- *The σ_B coupling to $\pi\pi$*

One can use the previous result of the decay constant in Eq.62 to predict the hadronic width of the σ_B . For this purpose, we consider the vertex (see e.g [1]):

$$V_\pi(q^2) = \langle \pi_1 | \theta_\mu^\mu | \pi_2 \rangle \quad : \quad q = p_1 - p_2 \quad \text{with} \quad V(0) = \mathcal{O}(m_\pi^2) . \quad (83)$$

In the chiral limit ($m_\pi^2 = 0$), one has :

$$V(q^2) = q^2 \int_{4m_\pi^2}^{\infty} \frac{dt}{t(t - q^2 - i\epsilon)} \frac{1}{\pi} \text{Im}V(t) . \quad (84)$$

Using the fact that $V'(0) = 1$ [67, 68], one can deduce the Low Energy Vertex sum rule (LEV-SR) [1]:

$$\frac{1}{4} \sum_{G=\sigma_B, \dots} g_{G\pi\pi} \frac{\sqrt{2}f_G}{M_G^2} = 1 , \quad (85)$$

where the contribution of the light σ_B is enhanced. Then, assuming (to a first approximation) that the σ_B dominates the sum rule, one can deduce :

$$g_{\sigma_B\pi^+\pi^-} \simeq (7.1 \pm 2.5) \text{ GeV} , \quad (86)$$

to which corresponds the width:

$$\Gamma[\sigma_B \rightarrow \pi^+\pi^- + 2\pi^0] = \frac{3}{2} \frac{|g_{\sigma_B\pi^+\pi^-}|^2}{16\pi M_{\sigma_B}} \left(1 - \frac{4m_\pi^2}{M_{\sigma_B}^2}\right)^{1/2} \simeq (1.4 \pm 0.8) \text{ GeV} . \quad (87)$$

To check the validity of this approximation, we use the average of the recent data from 2005 [25] which is :

$$\Gamma[f_0(1.37) \rightarrow \pi\pi] \simeq 215(14) \text{ MeV} \implies g_{\sigma'_B\pi\pi} \simeq (3.2 \pm 0.1) \text{ GeV} , \quad (88)$$

while the $\pi\pi$ width of the $f_0(1.5)$ of 39 MeV [25] can be neglected. Including the $f_0(1.37)$ contribution, the previous sum rule gives:

$$g_{\sigma_B\pi^+\pi^-} \simeq 5.7 \text{ GeV} \implies \Gamma[\sigma_B \rightarrow \pi^+\pi^- + 2\pi^0] \simeq 873 \text{ MeV} , \quad (89)$$

in agreement with previous estimates [1, 2].

- *The σ_B coupling to K^+K^-*

Up to $SU(3)$ breaking, the vertex sum rule in Eq.85 indicates that the σ_B couples (almost) universally to Goldstone boson pairs, such that in this gluonium picture, one also expects a large coupling of the σ_B to K^+K^- . This feature has been observed from the analysis of $\pi\pi \rightarrow K^+K^-$ scattering data [18–20].

- σ_B mass and width confronted to the data

It is informative to compare the previous mass and width to the one of the observed σ from scatterings data analysis :

◊ The values of the mass and width obtained in Eqs.52 and 87 for σ_B are comparable with the on-shell/Breit-Wigner mass and width obtained in Eq.4.

◊ These results together with the ones from data analysis reviewed in the first sections can indicate the presence of an important gluon component inside the observed $\sigma/f_0(500)$ meson wave function.

20. Hadronic couplings of the higher mass gluonium G

- G_1 coupling to $\eta\eta', \eta\eta$

In order to compute the $g_{G_1\eta_1\eta_1}$ coupling of the G_1 to the singlet η_1 , one starts from the [1, 42]:

$$\tilde{V}_{\mu\nu}(q_1, q_2) = i \int d^4x_1 d^4x_2 e^{i(q_1x_1 + q_2x_2)} \langle 0 | \mathcal{T} Q(x_1) Q(x_2) \theta_{\mu\nu}(0) | 0 \rangle, \quad (90)$$

where $\theta_{\mu\nu}$ is the energy-momentum tensor with 3 light quarks and :

$$Q(x) = \frac{\alpha_s}{16\pi} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a, \quad (91)$$

is the topological charge density, where one reminds that, in the large N_c limit (no quark loops), the solution to the $U(1)_A$ problem is [1, 42]:

$$\Gamma_2(q) \equiv i \int d^4x \langle 0 | \mathcal{T} Q(x) Q(0) | 0 \rangle \rightarrow \Gamma_2^{\text{YM}}(q \ll \Lambda) \rightarrow \Gamma_2^{\text{YM}}(0) \simeq (180 \text{ MeV})^4. \quad (92)$$

Using the expression of the topological charge in the presence of quark loops [44] and the consistent one for the vertex $\tilde{V}_{\mu\nu}(q_1, q_2)$, one can deduce [1] the constraint:

$$\langle \eta_1 | \theta_{\mu}^{\mu} |_{n_f=3} | \eta_1 \rangle \simeq \frac{9}{11} \frac{12}{f_{\pi}^2} \Gamma_2^{\text{YM}}(0) = 2M_{\eta_1}^2 = 1.15 \text{ GeV}^2, \quad (93)$$

where the factor (9/11) comes from the ratio of the β -function β_1 for $n_f = 3$ and $n_f = 0$ and $M_{\eta_1} \simeq 0.76$ GeV [1, 42]. Then, we deduce the LEV-SR [1]:

$$\frac{1}{4} \sum_{i=\sigma_B, G_1} g_{G\eta_1\eta_1} \sqrt{2} f_i \simeq 1.15 \text{ GeV}^2. \quad (94)$$

Picking the singlet component of the physical η, η' via the pseudoscalar mixing angle $\theta_P \simeq -(18 \pm 2)^\circ$ [96] : $\eta = \sin \theta_P \eta_1 + \dots$, $\eta' = \cos \theta_P \eta_1 + \dots$, one deduces:

$$\sum_{G=\sigma_B, G_1} g_{G\eta\eta'} f_G \simeq \frac{4}{\sqrt{2}} (\sin \theta_P) 1.15 \text{ GeV}^2 \simeq -1.0 \text{ GeV}^2 \quad \text{and} \quad g_{G_1\eta\eta'} = (\sin \theta_P) g_{G_1\eta_1\eta_1}. \quad (95)$$

Following Ref. [1] by using the constraint from $\eta' \rightarrow \eta\pi\pi$ (allowing a 1% deviation from the precision of the data [25]), and a Gell-Mann, Sharp, Wagner-type model, for the intermediate state σ_B , one can deduce the upper bound for $M_{\sigma_B} = 1.08$ GeV :

$$|g_{\sigma_B\eta\eta'}| \leq 0.34 \text{ GeV}, \quad (96)$$

which is a relatively small contribution. Neglecting this contribution and the $f_0(1.37)$ one, the LEV-SR in Eq. 95 leads to :

$$g_{G_1\eta\eta'} \simeq 2.74(78) \text{ GeV} \quad \text{and} \quad g_{G_1\eta\eta} \simeq 0.85(25) \text{ GeV} , \quad (97)$$

We deduce the width for $M_{G_1} = 1.55 \text{ GeV}$, $p_{\eta\eta'} = 20 \text{ MeV}$ and $p_{\eta\eta} = 517 \text{ MeV}$:

$$\Gamma(G_1 \rightarrow \eta\eta') = \frac{|g_{G_1\eta'\eta}|^2}{8\pi} \frac{p_{\eta\eta'}}{M_{G_1}^2} \simeq (2.5 \pm 1.4) [(2.6 \pm 0.9) \text{ Data}] \text{ MeV}, \quad (98)$$

which implies:

$$\frac{\Gamma(G \rightarrow \eta\eta)}{\Gamma(G \rightarrow \eta\eta')} = \sin^2 \theta_P \frac{p_{\eta\eta}}{p_{\eta'\eta}} \simeq (2.3 \pm 0.6) [(3.0) \text{ Data}] . \quad (99)$$

The results agree perfectly within the errors with the data.

- σ' , G_1 couplings to $\sigma_B\sigma_B$

One can also write a low-energy vertex sum rule (LEV-SR) for this coupling [1]:

$$\langle \sigma_B | \theta_\mu^\mu | \sigma_B \rangle = 2M_{\sigma_B}^2 . \quad (100)$$

Using the dispersion relation of the vertex, one can write the sum rule:

$$\frac{1}{4} \sum_{G=\sigma', G_1} g_{G\sigma_B\sigma_B} \sqrt{2} f_G \simeq 2M_{\sigma'}^2, \quad (101)$$

where here it is more appropriate to use the pole mass $M_{\sigma'} \simeq 0.5 \text{ GeV}$ for the virtual σ_B .

◊ Assuming that the σ' dominates the sum rule which is also the only one which decays copiously into $2(\pi\pi)_S$ [25], one can deduce :

$$g_{\sigma'\sigma_B\sigma_B} \simeq (2.2 \pm 0.73) \text{ GeV} , \quad (102)$$

where we have used the previous estimate of the coupling in Eq. 58.

◊ We confront this prediction with the data of $f_0(1.37, 1, 5) \rightarrow 2(\pi\pi)$. In so doing, we use the six most recent measurements of the $f_0(1.37)$ 4π width since 1995 compiled by PDG [25] from which we deduce the mean:

$$\Gamma[f_0(1.37) \rightarrow 4\pi] \simeq 365(20) \text{ MeV}, \quad (103)$$

Using the ratio [25] :

$$\frac{\Gamma[f_0(1.37) \rightarrow 2(\pi\pi)_S]}{\Gamma[f_0(1.37) \rightarrow 4\pi]} \simeq 0.51(9), \quad (104)$$

we deduce:

$$\Gamma[f_0(1.37) \rightarrow 2(\pi\pi)_S] \simeq 186(35) \text{ MeV} \implies g_{f_0\sigma_B\sigma_B} \simeq (4.33 \pm 0.39) \text{ GeV} . \quad (105)$$

where the estimate is off by about a factor two indicating that the sum rule may have overestimated the decay constant of the σ' -meson which is (presumably) the sum of effective couplings of higher state resonances.

◊ One can use instead the experimental value of $g_{f_0\sigma_B\sigma_B}$ into the LEV-SR. Neglecting the $G_1(1.55)$ contribution which has a tiny $2(\pi\pi)_S$ width of 14 MeV deduced from the data on $\Gamma_{\pi\pi} \simeq 39 \text{ MeV} \simeq 34.5\%$ of

its total width, $\Gamma_{4\pi} \simeq 48.9\%$ and $\Gamma_{2(\pi\pi)_S}/\Gamma_{4\pi} \simeq 26\%$, we deduce from $g_{f_0\sigma_B\sigma_B}$ and the LEV-SR in Eq. 101 the more accurate estimate :

$$f_{\sigma'} \simeq f_{\sigma'_B} = (329 \pm 30) \text{ MeV}. \quad (106)$$

This value is in line with the intuitive expectations from ordinary ρ and ρ' mesons.

21. Summary and conclusions

The different results obtained in this paper are summarized in Table 1 which we shall briefly comment below.

- *The $\sigma/f_0(500)$ meson*

In this paper, we have started to review our present knowledge on the nature of the σ meson from $\pi^+\pi^-$, $\gamma\gamma \rightarrow \pi^+\pi^-$, K^+K^- data and from J/ψ gluon rich channels ($\gamma\pi\pi$, $\omega\pi\pi$, $\phi\pi\pi$) and ϕ radiative decays, $D_{(s)}$ semileptonic decays and $\bar{p}p$, pp production processes. The data (a priori) favour / does not exclude a large gluon component in the σ wave function where the observed state may emerge from a maximal mixing with a $\bar{q}q$ state [4] rather than a pure four-quark ($\bar{u}u$)(dd) or a $\pi\pi$ molecule state. The four-quark or $\pi\pi$ molecule state is not favoured by the too small predicted direct $\gamma\gamma$ width of the σ [17, 19, 20] and by the large K^+K^- coupling found from scattering data [18, 19]. The assumption that the σ emerges from $\pi\pi$ rescattering is not also favoured from its large coupling to K^+K^- .

- *The $f_0(980)$ meson from the data*

◊ From the scattering data analysis, it is unlikely that the $f_0(980)$ is an $\bar{s}s$ state because it has a non negligible / large coupling to $\pi\pi$ [18–20], while the strength of its $\gamma\gamma$ coupling is relatively small for a $\bar{q}q$ state [10, 11]. Moreover, it is too light compared to the $\bar{s}s$ mass of about 1.4 GeV [11].

- *Conclusions from the data*

The previous observations may favour the interpretation that the σ and $f_0(980)$ emerge from a maximal mixing between a gluonium and a $(\bar{u}u + \bar{d}d)$ state like we have proposed in [2, 4] and what had also been expected [68] in their pioneering paper.

- *The σ_B meson from QCD spectral sum rules (QSSR)*

◊ A QSSR analysis of the lowest ratios of sum rule \mathcal{L}_{20} favours a σ_B (the subindex B refers to a pure unmixed gluonium state) having a mass $M_{\sigma_B}=1.07$ GeV (see Eq. 60) and a $\pi\pi$ width Eq. (89) compatible with the ones from scattering data for a Breit-Wigner / On-shell σ -mass.

◊ Its decay constant can be deduced from \mathcal{L}_1^c which is $f_{\sigma_B} = 456(157)$ MeV (Eq. 62).

◊ One should also note that the set $(\Gamma_{\sigma_B}, M_{\sigma_B})$ obtained from the sum rule is fairly compatible with the on-shell mass and width from $\pi^+\pi^- \rightarrow \pi^+\pi^-$, K^+K^- data .

- *The “scalar gluonium” G_1 from QSSR*

◊ An analysis of the higher weight USR \mathcal{L}_{21} and \mathcal{L}_{31} leads to a gluonium mass: $M_{G_1} = (1548 \pm 121)$ MeV (Eq. 53) for a “3 resonances” \oplus QCD continuum parametrization of the spectral function while its becomes (1515 ± 123) MeV (Eq. 54) if one uses a “one resonance” \oplus QCD continuum indicating that the presence of the σ_B does not affect notably the value of M_{G_1} . It is clear that higher moments are more appropriate in the QSSR approach to pick up the higher mass gluonium state due to the power mass suppression of the lowest mass σ_B contribution in these sum rules.

◊ The corresponding gluonium decay constant is $f_{G_1} = (365 \pm 110)$ MeV (Eq. 63) which is comparable with the one of f_{σ_B} .

◊ We have also seen that its small width into $2(\pi\pi)_S$ can be explained from the LET-V sum rule in Eq. 101 indicating that it is not a σ -like particle but instead decays into $\eta'\eta$, $\eta\eta$ through the gluon vertex $U(1)$ anomaly.

- *The radial excitations σ' , G'_1 and G_2 from QSSR*

We have attempted to extract the mass and coupling of the radial excitations σ' of σ_B , G'_1 and G_2 of G_1 .

◊ We found that $M_{\sigma'} \simeq (1121 \pm 117)$ MeV is rather low compared to the radial excitation of ordinary mesons where we notice that a similar feature has been observed in the analysis of the D^*D molecule described by high-dimension quark currents.

◊ $f_{\sigma'_B} \simeq 646(216)$ MeV can be large which we have interpreted as an effective sum of all radial excitations that the sum rule cannot disentangle. For a more phenomenological use, we identify the σ'_B with the observed $f_0(1.37)$ and extract the decay constant from its decay to $2(\pi\pi)_S \oplus$ LEV-SR analysis from which we obtain: $f_{\sigma'} \simeq (329 \pm 30)$ MeV (Eq. 101).

◊ The mass of the radial excitation G'_1 : $M_{G'_1} = 1563(141)$ MeV (Eq. 64) of G_1 is also found to be almost degenerated with the ground state mass $M_{G_1} = 1548(121)$ MeV (Eq. 61). Like in the case of σ'_B , we interpret the large coupling $f_{G'_1} = 1000(230)$ MeV (Eq. 65) as a sum of the effective couplings of all higher radial excitations entering into the spectral function.

◊ The 2nd radial excitation gluonium G_2 is found to have a relatively high mass $M_{G_2} \simeq 2.99$ GeV.

- *Phenomenology*

◊ The global picture which emerges from the approach indicates that the observed gluonia candidates can be classified into two groups :

– The σ -like gluonia [$\sigma/f_0(0.5), f_0(1.37)$] which emerge from (σ_B, σ'_B) and which decay into $\pi\pi$ via OZI violating process while the $f_0(1.37)$ decays to $2(\pi\pi)_S$ via virtual $\sigma\sigma$.

– The G -like gluonia [$f_0(1.5), f_0(1.7)$] which come from (G_1, G'_1) and which decay into $\eta'\eta$ and $\eta\eta$ through the gluon $U(1)$ anomaly vertex but not copiously to $\pi\pi$ and $2(\pi\pi)_S$.

– We may expect that high-mass 2nd radial excitation $G_2(2.99)$ has similar properties and brings some gluon component to the gluonia candidates above 2 GeV which contributes to their $\eta'\eta$ and $\eta\eta$ decays.

◊ We remind that the masses of the $S_2(\bar{u}u + \bar{d}d)$, $S_3(\bar{s}s)$ quarkonia states and their radial excitations are predicted from the sum rules to be (in units of GeV) [2] :

$$\begin{aligned} M_{S_2} &\simeq 1, & M_{S_2'} &\simeq (1.1 \sim 1.4), \\ M_{S_3} &\simeq (1.47), & M_{S_3'} &\simeq (1.7 \sim 2.4), \end{aligned} \tag{107}$$

which can mix with the previous gluonia states.

◊ A more complete analysis of the data can be done within some $\bar{q}q$ meson-gluonium mixing scheme. The case for the $\sigma(500)$ - $f_0(980)$ has been studied in [4] but may/should be extended to the higher meson masses. Some attempts in this direction have been also done in [2]⁶ but needs to be updated.

◊ We have not updated our analysis of the $\gamma\gamma$ and radiative widths obtained in [1, 2]. We expect that these results are still valid within the errors.

◊ We emphasize that a more appropriate comparison of the results obtained in this work (e.g with lattice calculations) requests an analysis beyond the "one resonance" contribution to the two-point correlator from these alternative approaches. Indeed, a simple one resonance for parametrizing the spectral function is (obviously) insufficient for describing the complex and overpopulated spectra of the $I = 0$ scalar mesons.

- *The conformal charge $\psi_G(0)$ from QSSR*

We have completed our analysis by the estimate of the conformal charge $\psi_G(0)$ and its slope $\psi'_G(0)$.

◊ Using the previous values of the σ and gluonia G masses and couplings, we have obtained $\psi_G^{LSR}(0)=2.09$ (29) GeV⁴ (Eq. 78) which can be compared with the LET prediction of about 1.46 GeV⁴ (Eq.43).

◊ The result indicates that the σ_B meson contributes predominantly in the spectral function for reproducing the LET prediction of $\psi_G(0)$. It also shows that the instanton effect in the estimate of $\psi_G(0)$ is (a priori) not necessary for a correct estimate of the conformal charge $\psi_G(0)$. A similar feature has been observed in the estimate of the topological charge and its slope in the $U(1)_A$ sector [13] .

◊ The value of the slope is $\psi'_G(0)|_{LSR} = 0.95(30)$ GeV².

- *Prospects*

◊ We plan to study in more details the mixing of the $\bar{q}q$ states with the higher gluonia masses $\sigma'(1.15)$, $G_1(1.53)$ and eventually with the $G_2(2.48)$.

◊ We plan to extend the analysis to the case of (pseudo)scalar $C = \pm$ trigluonia channels [98] which has been studied in [12] and [99].

⁶For a review prior 1996, see e.g [97].

Observables	Value	Eq.#	Source	Comments
Masses [MeV]				
<i>Ground states</i>				
M_{σ_B}	1088(78)	50	\mathcal{R}_{20}^c	1 resonance \oplus QCD continuum
	1085(126)	52	\mathcal{R}_{20}^c	2 resonances \oplus QCD continuum
	1070(126)	60	\mathcal{R}_{20}^c	3 resonances \oplus QCD continuum
M_{G_1}	1515(123)	54	\mathcal{R}_{42}^c	1 resonance \oplus QCD continuum
	1524(121)	53	\mathcal{R}_{42}^c	2 resonances \oplus QCD continuum
	1548(121)	61	\mathcal{R}_{42}^c	3 resonances \oplus QCD continuum
<i>1st Radial excitations</i>				
$M_{\sigma'_B}$	1121(117)	57	\mathcal{R}_{20}^c (after iteration)	3 resonances \oplus QCD continuum
$M_{G'_1}$	1563(141)	64	\mathcal{R}_{42}^c	4 resonances \oplus QCD continuum
M_{G_2}	2992(221)	68	mean from \mathcal{R}_{21}^c and \mathcal{R}_{31}^c	5 resonances \oplus QCD continuum
Decay constants [MeV]				
<i>Ground states</i>				
f_{σ_B}	563(156)	56	\mathcal{L}_{-1}^c	2 resonances \oplus QCD continuum
	456(157)	62	\mathcal{L}_{-1}^c	3 resonances \oplus QCD continuum
f_{G_1}	394(103)	56	$\mathcal{L}_2^c \oplus \mathcal{L}_3^c$	2 resonances \oplus QCD continuum
	365(110)	63	$\mathcal{L}_2^c \oplus \mathcal{L}_3^c$	3 resonances \oplus QCD continuum
<i>Radial excitations</i>				
$f_{\sigma'}$	329(30)	106	LEV-SR $\oplus f_0(1.37) \rightarrow 2(\pi\pi)_S$ [25]	$f_0(1.5)$ neglected
$f_{\sigma'}^{\text{eff}}$	648(216)	58	\mathcal{L}_{-1}^c	Sum of higher states effective couplings
$f_{G'_1}^{\text{eff}}$	1000(230)	65	\mathcal{L}_2^c	Sum of higher states effective couplings
$f_{G_2}^{\text{eff}}$	797(74)	70	mean from \mathcal{L}_1^c and \mathcal{L}_2^c	Sum of higher states effective couplings
Decay widths [MeV]				
$\Gamma[\sigma_B \rightarrow \pi\pi]$	873	89	LEV-SR	700 : $\pi\pi$ scattering data
$\Gamma[\sigma'(1.37) \rightarrow 2(\pi\pi)_S]$	186(35)	106	data input [25] \oplus LEV-SR \implies	$f_{\sigma'} = 329(15)$
$\Gamma[G_1(1.55) \rightarrow \eta\eta']$	(2.5 ± 1.4)	98	–	(2.6 ± 0.9) (data) [25]
$\frac{\Gamma[G_1(1.55) \rightarrow \eta\eta']}{\Gamma[G_1(1.58) \rightarrow \eta\eta']}$	(2.3 ± 0.6)	99	–	3.0 (data) [25]
Conformal Anomaly				
Charge : $\psi_G^{LSR}(0)$ [GeV ⁴]	2.09(29)	78	$\mathcal{L}_{-1}^c \oplus \mathcal{L}_0^c$	1.46(8) [LET in Eq. 43]
Slope : $\psi'_G(0) _{LSR}$ [GeV ²]	0.95(30)	82	$\mathcal{L}_{-2}^c \oplus \psi_G^{LSR}(0)$	

Table 1: Predictions from scalar di-gluonium LSR at N2LO and gluon condensates at NLO up to $D = 8$.

References

- [1] S. Narison and G. Veneziano, *Int. J. Mod. Phys. A* **4** (1989) 2751.
- [2] S. Narison, *Nucl. Phys. B* **509** (1998) 312/; *ibid*, *Nucl. Phys. Proc. Suppl.* **64** (1998) 210.
- [3] For reviews, see : S. Narison, *Nucl. Phys. Proc. Suppl.* **96** (2001) 244; *Nucl. Phys. A* **675** (2000) 54; *Nucl. Phys. Proc. Suppl.* **121** (2003) 131; *Nucl. Phys. Proc. Suppl.* **186** (2009) 306.
- [4] A. Bramon and S. Narison, *Mod. Phys. Lett. A* **4** (1989) 1113.
- [5] H. Fritzsch, P. Minkowski, *Nuovo Cimento A* **30** (1975) 393.
- [6] S. Narison, *Phys. Lett. B* **125** (1983) 501.
- [7] S. Narison, *Z. Phys. C* **22** (1984) 161.
- [8] S. Narison, N. Pak, N. Paver, *Phys. Lett. B* **147** (1984) 162.
- [9] G. Mennessier, S. Narison, N. Paver, *Phys. Lett. B* **158** (1985) 153.
- [10] S. Narison, *Phys. Lett. B* **175** (1986) 88.
- [11] S. Narison, *Phys. Rev. D* **73** (2006) 114024.
- [12] J.I. Latorre, S. Narison, S. Paban, *Phys. Lett. B* **191** (1987) 437.
- [13] S. Narison, G.M. Shore, G. Veneziano, *Nucl. Phys. B* **433** (1995) 209.
- [14] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys. B* **147** (1979) 385, 448.
- [15] V.I. Zakharov, *Int. J. Mod. Phys. A* **14**, 4865 (1999).
- [16] For reviews, see e.g.: S. Narison, *Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol.* **17** (2004) 1-778 [hep-ph/0205006]; *World Sci. Lect. Notes Phys.* **26** (1989) 1-527; *Riv. Nuov. Cim.* **10 N2** (1987) 1; *Phys. Rep.* **84** (1982) .
- [17] G. Mennessier, S. Narison, W. Ochs, *Phys. Lett. B* **665** (2008) 205; *Nucl. Phys. Proc. Suppl.* **238** (2008) 181.
- [18] R. Kaminski, G. Mennessier, S. Narison, *Phys. Lett. B* **680** (2009) 148.
- [19] G. Mennessier, S. Narison, X.G Wang, *Phys. Lett. B* **688** (2010) 59.
- [20] G. Mennessier, S. Narison, X.G Wang, *Phys. Lett. B* **696** (2011) 40.
- [21] H.G. Dosch and S. Narison, *Nucl. Phys. Proc. Suppl.* **121** (2003) 114.
- [22] W. Ochs, *J. Phys. G* **40** (2013) 043001.

- [23] U. Gastaldi, *Nucl. Phys. Proc. Suppl.* **96** (2000) 234; *Nucl. and Part. Phys. Proc.* **300-302** (2018) 113.
- [24] C. Amsler et al., PDG Review on scalar mesons below 2 GeV and Review on non $\bar{q}q$ mesons in [25] and private communications from C. Amsler and C. Hanhart.
- [25] P.A. Zyla et al. (Particle Data Group), *Prog. Theor. Exp. Phys.* **2020** (2020) 083C01.
- [26] G. Mennessier, *Z. Phys.* **C 16** (1983) 241; O. Babelon et al., *Nucl. Phys.* **B 113** (1976) 445.
- [27] G. Mennessier, T.N. Truong, *Phys. Lett.* **B 177** (1986) 195; A Pean, Thèse Montpellier (1992) (unpublished).
- [28] The NA48/2 collaboration: B. Bloch-Devaux, *PoS Confinement* **8** (2008) 029; J.R. Batley et al., *Eur. Phys. J.* **C 52** (2007) 875.
- [29] I. Caprini, G. Colangelo and H. Leutwyler, *Phys. Rev. Lett.* **96** (2006) 132001.
- [30] F.J Yndurain, R. Garcia-Martin, J. R. Pelaez, *Phys. Rev.* **D 76** (2007) 074034.
- [31] P. Minkowski and W. Ochs, *Eur. Phys. J.* **C 9** (1999) 283.
- [32] K. L. Au, D. Morgan and M. R. Pennington, *Phys. Rev.* **D 35** (1987) 1633.
- [33] The BES III collaboration : L. Xu , *Nucl. and Part. Phys. Proc.* **282-284** (2017) 121.
- [34] The BES III collaboration : S. Fang, *Nucl. Phys. Proc. Suppl.* **164** (2007) 135.
- [35] The KLOE collaboration : F. Ambrosino et al. , *Nucl. Phys. Proc. Suppl.* **186** (2009) 290.
- [36] The BABAR collaboration: A. Palano, talk presented at QCD 21 (5-9 july 2021, Montpellier-FR)
- [37] J. Babcock and J.L. Rosner, *Phys. Rev.* **D 14** (1976) 1286.
- [38] N. N. Achasov, *Nucl. Phys. Proc. Suppl.* **186** (2009) 283.
- [39] The GAMS collaboration, F. Binon et al, *Nuovo Cimento* **A 78** (1983) 13; D. Alde et al., *Phys. Lett.* **B 201** (1988) 160; D. Alde et al., *Phys. Lett.* **B 269** (1988) 485.
- [40] D. Harnett, T.G. Steele, R.T. Kleiv and K. Moats *Nucl.Phys.B Proc.Suppl.* **234**(2013) 257; *ibid Nucl. Phys. A* **850** (2011) 110.
- [41] R.A. Bertlmann, G. Launer and E. de Rafael, *Nucl. Phys.* **B250**, (1985) 61.
- [42] E. Witten, *Nucl. Phys* **B156** (1979) 269;
- [43] G. Veneziano, *Nucl. Phys.* **B159** (1979) 213.
- [44] P. Di Vecchia and G. Veneziano, *Nucl. Phys.* **B171** (1980) 253

- [45] A. Di Giacomo, *Nucl. Phys. Proc. Suppl.* **B 23** (1991) 191;
- [46] R.J. Crewther, *Phys. Rev. Lett.* **28** (1972) 1421; J. Ellis and M.S. Chanowitz, *Phys. Lett.* **B 40** (1972) 397; *Phys. Rev.* **D 7** (1973) 2490; P. Di Vecchia, G. Veneziano, *Nucl. Phys.* **B 171** (1981) 253.
- [47] J. Ellis and J. Lanik, *Phys. Lett.* **B 150** (1985) 289; *Phys. Lett.* **B 175** (1986) 83; J.Lanik, *Z. Phys.* **C 39** (1988) 143.
- [48] A. Hart et al. [UKQCD Collaboration], *Phys. Rev.* **D 74** (2006) 114504
- [49] M.C. Neile, *Nucl. Phys. Proc. Suppl.* **186** (2009) 264.
- [50] E. Gregory et al, *JHEP* **10** (2012) 170; A. Rago, contribution at QCD 21 (4-9 july 2021, Montpellier - FR) and private communications.
- [51] V. Mathieu , N. Kochelev, V. Vento, *Int. J. Mod. Phys.* **E 18** (2009) 1.
- [52] A.L. Kataev, N.V. Krasnikov and A.A. Pivovarov, *Nucl. Phys.* **B 198** (1982) 508; *ibid*, *Nucl. Phys.* **B 490** (1997) 505 (erratum).
- [53] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, *Phys. Rev. Lett.* **79** (1997) 353.
- [54] D. Harnett, T.G. Steele and V. Elias, *Nucl.Phys.* **A 686** (2001) 393 and private communication with T.G. Steele.
- [55] A. Pich and A. Rodriguez-Sanchez, *Phys.Rev.* **D94** (2016) no.3, 034027
- [56] S. Narison, *Phys. Lett.* **B673** (2009) 30.
- [57] S. Bethke, *Nucl. Part. Phys. Proc.* **282-284** (2017)149.
- [58] S. Narison, *Int. J. Mod. Phys.* **A 33** 1850045 (2018); **33**, 1850045(E) (2018), and references therein.
- [59] S. Narison, *Phys. Lett.* **B 693**, 559 (2010) ; **705**, 544(E) (2011) ; *Phys. Lett.* **B 706**, 412 (2011) ; *Phys. Lett.* **B 707**, 259 (2012).
- [60] S. Narison, *Nucl. Part. Phys. Proc.* **B 207-208** (2010) 315; *ibid*, *Nucl. Part. Phys. Proc.* **258-259** (2015) 189; *ibid*, *Nucl. Part. Phys. Proc.* **300-302** (2018) 153; *ibid*, *Nucl. Part. Phys. Proc.* **309-311** (2020) 135.
- [61] S. Narison, *Int. J. Mod. Phys.* **A33** (2018) no. 10, 1850045 [arXiv:1801.00592 [hep-ph]].
- [62] S. Narison, Addendum: *Int. J. Mod. Phys.* **A33** (2018) no.10, 1850045 [arXiv:1812.09360 [hep-ph]].
- [63] E. Bagan and T.G Steele, *Phys. Lett.* **B 243** (1990) 413 and private communication from T.G Steele.
- [64] S. Narison, *Phys. Lett.* **B693** (2010) 559, erratum *ibid*, *Phys. Lett.* **705** (2011) 544.

- [65] S. Narison, *Phys. Lett.* **B706** (2012) 412.
- [66] S. Narison, *Phys. Lett.* **B707** (2012) 259.
- [67] V.A. Novikov et al., *Nucl. Phys.* **B 165** (1980) 67.
- [68] V.A. Novikov et al., *Nucl. Phys.* **B 191** (1981) 301.
- [69] E. V. Shuryak, *Nucl. Phys.* **B 203** (1982) 116; T. Schaefer and E. V. Shuryak, *Rev. Mod. Phys* **70** (1998) 323.
- [70] E. Bagan et al., *Nucl. Phys.* **B254**, (1985) 55; *ibid*, *Z. Phys.* **C32** (1986) 43.
- [71] G. Launer, S. Narison and R. Tarrach, *Z. Phys.* **C26** (1984) 433.
- [72] V.I. Zakharov, *Nucl. Phys. Proc. Suppl.* **164** (2007) 240.
- [73] S. Narison, *Nucl. Phys. Proc. Suppl.* **164** (2007) 225.
- [74] S. Narison and V.I. Zakharov, *Phys. Lett.* **B522** (2001) 266.
- [75] O. Andreev, *Phys. Rev.* **D73** (2006) 107901.
- [76] O. Andreev and V.I. Zakharov, *Phys. Rev.* **D74** (2006) 025023; *ibid*, **D76** (2007) 047705.
- [77] F. Jugeau, S. Narison, H. Ratsimbarison, *Phys. Lett.* **B722** (2013) 111.
- [78] K.G. Chetyrkin, S. Narison and V.I. Zakharov, *Nucl. Phys.* **B550** (1999) 353.
- [79] S. Narison, *Phys. Lett.* **B300** (1993) 293; *ibid*, *Phys. Lett.* **B361** (1995) 121.
- [80] M. Kozhevnikova, A. Oganesian and O. Terayev, *EPJ Web of Conferences* **204** (2019) 02005.
- [81] H. Forkel, *Phys.Rev.* **D 71** (2005) 054008; *Phys. Rev.* **D 64** (2001) 034015.
- [82] D. Harnett and T.G. Steele, *Nucl. Phys.* **A 695** (2001) 205; G. Orlandini, T. G. Steele and D. Harnett, *Nucl.Phys.* **A 686** (2001) 261.
- [83] S. Narison, *Phys. Lett.* **B738** (2014) 346.
- [84]] B. V. Geshkeinben and B. L. Ioffe, *Nucl. Phys.* **B166** (1980) 340; B.L. Ioffe, K.N. Zyablyuk, *Eur. Phys. J.* **C27** (2003) 229; B. L. Ioffe, *Prog. Part. Nucl. Phys.* **56** (2006) 232 and references therein.
- [85] J. S. Bell and R. A. Bertlmann, *Nucl. Phys.* **B177**, (1981) 218; *ibid*, *Nucl. Phys.* **B187**, (1981) 285.
- [86] R. A. Bertlmann, *Acta Phys. Austriaca* **53**, (1981) 305.
- [87] S. Narison and E. de Rafael, *Phys. Lett.* **B 522**, (2001) 266.
- [88] R.M. Albuquerque, S. Narison, D. Rabetiarivony *Phys. Rev.* **D 103** (2021) 7, 074015.

- [89] A.H. Fariborz and R. Jora, *Phys. Lett.* **B 790** (2019) 410.
- [90] M. Frasca, *Nucl. Phys. Proc. Suppl.* **207-208** (2010) 196 and references therein.
- [91] L.S.Kisslinger, J.Gardner and C.Vanderstraeten, *Phys. Lett.* **B 410** (1997) 1.
- [92] T. Huang, H.-Y. Jin and A.-L. Zhang, *Phys. Rev.* **D 59** (1999) 034026.
- [93] H.-X. Chen, W. Chen and S.-L. Zhu, arXiv 2107.05271 [hep-ph] (2021).
- [94] H.n. Li, arXiv 2109.0456 [hep-ph] (2021).
- [95] S. Narison, *Phys. Lett.* **B 104** (1981) 485.
- [96] L. Montanet, Non-perturbative methods Conf. (Montpellier 1985) ed. S. Narison, world scientific co.;
F. Gilman and R. Kauffman, *Phys. Rev.* **D 36** (1987) 2761; T. N. Pham, *Phys. Lett.* **B 246** (1990)
175 ; P. Ball, J.M. Frere and M. Tytgat, *Phys. Lett.* **B 365** (1996) 367.
- [97] F. Close, *Nucl. Phys. Proc. Suppl.* **56A** (1997) 248.
- [98] R. Albuquerque, S. Narison, A. Pimikov and D. Rabetiarivony (work in progress).
- [99] A. Pimikov et al., *Phys. Rev.* **D 96** (2017) 11, 114024; *Phys. Rev. Lett.* **119** (2017) 7, 079101; S.
Narison, contribution at QCD 21 (5-9 july 2021, Montpellier-FR).

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

□