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Occupancy grids are common tools used in robotics to represent the robot environment, and that may be used to plan trajectories, select additional measurements to acquire, etc. However, deriving information about those occupancy grids from sensor measurements often induce a lot of uncertainty, especially for grid elements that correspond to occluded or far away area from the robot. This means that occupancy information may be quite uncertain and imprecise at some places, while being very accurate at others. Modelling finely this occupancy information is essential to decide the optimal action the robot should take, but a refined modelling of uncertainty often implies a higher computational cost, a prohibitive feature for real-time applications. In this paper, we introduce the notion of credal occupancy grids, using the very general theory of imprecise probabilities to model occupancy uncertainty. We also show how one can perform efficient, real-time inferences with such a model, and show a use-case applying the model to an autonomous vehicle trajectory planning problem.

Keywords: Occupancy grids and Trajectory planning and Imprecision and Imprecise probabilities

1. Introduction

Occupancy grids are commonly used tools to represent a mobile robot (cars, drones, etc.) environment, especially autonomous ones. Based on a local perception of the

vehicle, they are instrumental in solving issues such as obstacle avoidance and short-term trajectory planning, and are particularly suitable in dynamic environments, for instance environments discovered on the fly by the robots (such as in SLAM problems [27]) or environments in which moving obstacles evolves [19]. Such grids are composed of multiple cells, which in the most usual settings can be either free or occupied. Those grids are most of the time obtained from sensor readings associated to processing steps that may be peculiar to each sensor, according to their nature (cameras, lasers, ...). Such occupancy information is therefore prone to be affected by many sources of uncertainties, such as imprecise sensor readings (camera occlusion, numerical precision, ...,), noise due to environmental conditions (rain, ...) or faulty sensors. Different frameworks have been proposed to model and reason from such uncertainties, for instance by using probabilistic [21,6], fuzzy [28,24,23] or evidential [20,33,26,30] occupancy grids.

In this paper, we explore how another uncertainty framework, namely imprecise probabilities, can be used to model uncertainty in occupancy grids. To our knowledge, it is the very first attempt to do so. Imprecise probability consists in extending the classical probabilistic model by considering convex sets of probabilities or equivalent models such as lower expectation. This representation formally includes all the other aforementioned representations [10] (possibilities, probabilities, belief functions), and has the benefit to be fully consistent with classical Bayesian probabilistic modelling, both axiomatically (as it relaxes Bayesian axioms) and formally speaking. It can therefore be interpreted either as an uncertainty theory of its own, or as a robustification of the classical probabilistic models. We refer to [2] for an introduction to various aspects of the framework, and to Walley's book [32] for a full theoretical account. In the case of cells taking values in the binary space $\mathcal{X} = \{x, \neg x\}$ composed of the states occupied (x in our notation) and free ($\neg x$ in our notation), imprecise probabilistic models encode uncertainty by explicitly encoding imprecision through the use of two boundary measures, therefore allowing to differentiate between what is commonly called aleatoric uncertainty (due to a possible ambiguity and/or random phenomena) and epistemic uncertainty (due to an absence of information).

The concept of imprecise probabilistic occupancy grids is introduced in Section 2, where we also discuss the associated challenges in terms of inference and decision. We then consider in Section 3 the specific problem of making inferences for local trajectory planning, proposing a computationally efficient and simple solution to do so. Each step of the method is illustrated using simple examples. Then, in Section 4, for illustrative purposes, we present some results obtained using real occupancy grids coming from PACPUS [17], an experimental platform developed in our laboratory, dedicated to researches on autonomous vehicles.

2. Introducing credal occupancy grids

In this section, we will introduce the notion of credal occupancy grids, and will discuss the problem of making general inferences with them. Roughly speaking, such inferences come down to make computations over (pseudo-)Boolean functions where the probabilities of literals are imprecisely specified. We will however not treat this very general issue that is encountered in many fields such as data-bases [13], game-theory [25], but will focus on the most common issue in applications using occupancy grids, that is guiding a moving object through it.

For this reason, we will consider from now that our problem concerns a moving robot, and that the robot environment is modelled by a large grid of cells, and that each cell X can take a binary value, occupied (x) or unoccupied ($\neg x$). We assume that when a robot perceives its environment through measurement devices (sensors, cameras), it receives uncertain information about whether a given cell is occupied. Imprecise probabilities [32] offer a very generic way to model uncertainty, including most known models of uncertainty such as precise probabilities, belief functions or possibility distributions [10]. It also comes with a fully-fledged theory to reason with such uncertainty and make decisions [32].

In a binary setting such as the occupancy of cell X, any imprecise probabilistic model can be reduced to an interval $[\underline{p}(x), \overline{p}(x)]$ of the probability to be occupied, as we have that $\underline{p}(\neg x) = 1 - \overline{p}(x)$ and similarly for the upper bound. Note that the fact that any convex probability set can be reduced to such intervals on singletons is only true in the binary case, and is used here as one of the key feature that makes our approach tractable. We can in particular model the following extreme situations:

- certain occupancy: $[\underline{p}(x), \overline{p}(x)] = [1, 1]$,
- certain inoccupancy: $[\underline{p}(x), \overline{p}(x)] = [0, 0]$,
- unknown occupancy: $[\underline{p}(x), \overline{p}(x)] = [0, 1]$,

as well as all the intermediate ones, with precise probabilities being retrieved when $\underline{p}(x) = \overline{p}(x)$. A credal occupancy grid is pictured in Figure 1, where cells are voluntary of very large size for illustrative purposes. Green cells are likely to be unoccupied, red ones occupied, and grey/dark ones are in between (i.e., $0.5 \in [\underline{p}(x), \overline{p}(x)]$). The darker the shades, the wider are the corresponding intervals.

Let us now consider a set $X_1, \ldots, X_m$ of cells. As they are equivalent to Boolean variables, any event A concerning these cells (e.g., "at least one cell is occupied","k consecutive cells are occupied") can be expressed as a Boolean formula, and in particular can be put in Disjunctive Normal Form (DNF). This means that we can express any event in the form of a disjunction $A = \vee_{i=1}^{j} A_i$, where each $A_i = \wedge_{k \in A_i^+} x_k \bigwedge \wedge_{k \in A_i^-} \neg x_k$ is a conjunction of values for the cells (A_i^+ denotes the set of indices of positive variable, A_i^- the indices of negative one). Such sets are sometimes called orthopairs [5] in the literature. When the uncertainty over cells is expressed by a precise probability, assessing $P(A)$ then amounts to evaluate $P(\cup_{i=1}^{j} A_i)$, a

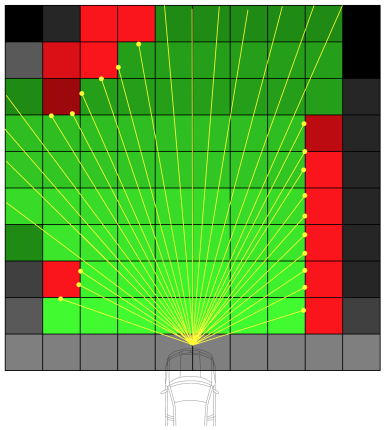

Fig. 1. Credal occupancy grid build from sensor data. Certain occupancy: $[\underline{p}(x), \overline{p}(x)] = [1, 1]$ in red, certain inoccupancy: $[\underline{p}(x), \overline{p}(x)] = [0, 0]$ in green, unknown occupancy: $[\underline{p}(x), \overline{p}(x)] = [0, 1]$ in black. The intermediate states are represented with lower intensity values.

potentially difficult task. One common way to do it is to use the following formula[a]

$$P(A) = P(\cup_{i=1}^{j} A_i) = \sum_{\mathcal{A} \subseteq \{A_1, \ldots, A_j\}} -1^{|\mathcal{A}|} P(\cap_{A \in \mathcal{A}} A_i) \tag{1}$$

where the second equality simply applies the inclusion/exclusion principle. The probability of a conjunctive event A_i is

$$P(A_i) = \prod_{k \in A_i^+} p(x_k) \prod_{k \in A_i^-} p(\neg x_k). \tag{2}$$

and likewise for intersection of such events, that remain of the same form. Solving Equation (1), hence making inferences over a probabilistic grid, is then relatively straightforward when probabilities are precise, as assessing the probabilities of conjunctions is easy. It should however be noted that there are multiple ways to decompose a given event A into a DNF, such as the use of Binary Decision Diagrams [4] or of minimal cuts in reliability theory [16], that can lead to more or less computationally friendly decompositions, in the sense that the number of terms in the right-hand side of (2) can grow exponentially. Obtaining compact Boolean representations of an event A is therefore a whole area research, that is outside of the scope of the present paper, which is why we will focus on the second part on specific cases for which efficient decompositions do exist.

However, even when one has obtained a decomposition through some means, the task of solving Equation (2) becomes much more difficult when probabilities

[a]Note that in this paper, we will use interchangeably $\vee$ and $\cup$, or $\wedge$ and $\cap$, as they have the same meaning.

become imprecise, as we then have to solve the equation

$$\underline{P}(A) = \inf_{p(x_i)\in\{\underline{p}(x_i),\overline{p}(x_i)\}} P(A) \tag{3}$$

to get the lower bound of the probability of an event A, which is NP-hard in general, as Equation 1 is a multi-linear form of the $p(x_i)$, each constrained to be in an interval (see for instance [3]). Indeed, if it is known [12] that the bound given by Equation 3 is obtained in such a case on one of the vertices of the hyper-cube $\times_i[\underline{p}(x_i), \overline{p}(x_i)]$, that is $p(x_i) \in \{\underline{p}(x_i), \overline{p}(x_i)\}$, it remains to find which one, hence to check for 2^m combinations if m is the number of cells. The upper bound can be found similarly, replacing inf by sup, with the same difficulties. This means, in particular, that reasoning over an entire imprecise probabilistic grid and for arbitrary events will be computationally prohibitive and infeasible for real-time applications.

Example 1. Let us consider two cells X_1, X_2, and the event "one and only one cell is occupied", that corresponds to a XOR logical gate, that is

$$A = (x_1 \wedge \neg x_2) \vee (x_2 \wedge \neg x_1) = A_1 \cup A_2$$

with $A_1 \cap A_2 = \emptyset$, as each contains opposite values (e.g., x_1 for A_1, and $\neg x_1$ for A_2). Let us now consider that $p(x_1) \in [0.6, 0.8]$ and $p(x_2) \in [0.3, 0.6]$, we have that $\underline{P}(A)$ is obtained on a combination of extreme points, i.e.,

$$\begin{aligned}\underline{P}(A) &= \inf_{p(x_i)\in\{\underline{p}(x_i),\overline{p}(x_i)\}} p(x_1)\,(1-p(x_2)) + (1-p(x_1))\,p(x_2)\\ &= 0.8 \cdot 0.4 + 0.2 \cdot 0.6 = 0.44\end{aligned}$$

obtained for $p(x_1) = 0.8$ and $p(x_2) = 0.6$ (the upper bounds of $p(x_1)$ and $p(x_2)$).

More generally, one may be interested not only in computing the lower and upper bounds of some event probabilities, but to compute the lower and upper expected values of some reward/penalty function $f : \times_{k=1}^{m}\{x_k, \neg x_k\} \to \mathbb{R}$. Since it is defined on a discrete space, we can always define a partition $B_1, \ldots, B_k$ such that f is constant over elements of this partition, i.e. if we denote by $\mathbf{x}$ an assignment vector over cells X_i, $f(\mathbf{x}) = \mathbf{f}(\mathbf{x}')$ whenever $\mathbf{x}, \mathbf{x}' \in B_i$ for some $i \in \{1, \ldots, k\}$. It then makes sense to denote by f_{B_i} the value of f for all elements included in B_i. Computing the lower expectation of f than amounts to solve the equation

$$\underline{\mathbb{E}}(f) = \inf_{p(x_i)\in\{\underline{p}(x_i),\overline{p}(x_i)\}} \sum_{j=1}^{k} f_{B_j} P(B_j) \tag{4}$$

with each B_j that can be decomposed using (2). As (4) remains a multi-linear form, the problem for solving it is in general not more difficult than the one of finding a lower probability, and its tractability will depend on both the form of f and of the events B_i.

Example 2. Let us pursue Example 1, by considering a simple game: we get a reward of 2 if the two variables end up with the same value, and suffer a penalty of

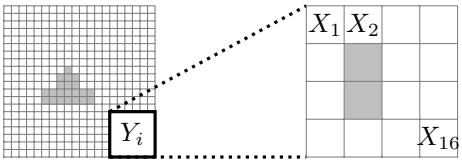

Fig. 2. The metagrid Y_i composed of 16 cells of the credal occupancy grid

1 (equivalently, a reward of -1) if they have different values. Our knowledge about the variables X_1, X_2 remains the same. One may think, for instance, of a game with two coins with ill-known probabilities of landing head and tail.

We now want to know our lower expected reward, should we play this game. We have $B_1 = (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)$ and B_2 its complement, with $f_{B_1} = 2$ and $f_{B_2} = -1$. We must now minimize

$$2(p(x_1) \cdot p(x_2) + p(\neg x_1) \cdot p(\neg x_2)) - (p(\neg x_1) \cdot p(x_2) + p(x_1) \cdot p(\neg x_2))$$

which gives $\underline{\mathbb{E}}(f) = -0.86$, obtained for the values $p(x_1) = 0.8$ and $p(x_2) = 0.6$.

In the rest of the paper, we will propose a decision model that focuses on particular types of events, that are of practical interest in robot guidance problems and are computationally tractable, in the sense that bound (3) can be computed in a time that is linear in the number of considered cells (or Boolean variables).

3. Moving in credal occupancy grid: model, inference and decision

Now that we have introduced credal occupancy grids and discussed their main aspects regarding inference, let us focus on the problem of guiding a moving robot within such an occupancy grid.

From now on, we will also be interested in particular states of subgrids of the whole occupancy grid, that we will call metagrids. Such metagrids will describe the successive locations of a robot along a chosen path. Indeed, usually the space occupied by a robot (be it a vehicle, a drone, etc.) is much bigger than a single cell, whose area is usually related to the smallest zone a robot can perceive with sufficient reliability. As an example, cells of grids for autonomous vehicles are typically squares of a few decimetres A metagrid Y_i will be composed of n cells denoted $X_1, \ldots, X_n$, while a possible path for the robot will always be composed of k metagrids $Y_1, \ldots, Y_k$, representing its successive positions. The concept of metagrid within a grid is shown in the Figure 2 (with a metagrid being made of 16 elementary cells).

Given this, our decision model will be based on the following elements, that we will detail in different subsections:

- For each metagrid Y_i, assess the uncertainty over the fact that at least one cell of the metagrid is occupied: this provides lower/upper bounds $[\underline{p}(y_i), \overline{p}(y_i)]$ over a new Boolean state (either at least one cell is occupied,

or none), that we consider as probability bounds of the fact that the robot can move over Y_i.
- Assess the lower/upper probability of the k mutually exclusive events $F_1, \ldots, F_k$ that the robot can move up to the ith metagrid, that is $F_i = \wedge_{j=1}^{i-1} \neg y_j \wedge y_i$, corresponding to the event that Y_i is the first occupied metagrid.
- Pick the trajectory that provides the best expected reward as the robot future path, unless stopping is required because the first metagrids are too likely to be occupied.

For each of those steps, we will provide inference tools whose computational complexity is fixed and linear with the number of states, hence ensuring that real-time applications of the proposed method is doable. Since they also rely on fixed formulas, they can also be easily implemented as hardware components, possibly speeding up computations even further.

3.1. *Inference about individual metagrids*

A first inference we want to do is to know, for each metagrid, what is the likelihood of having at least one cell X_i that is occupied, as having one such cell is judged sufficient for the metagrid to be unreachable. In practice, this comes down to assess the probability bounds for the event

$$Y = x_1 \vee x_2 \vee \ldots \vee x_n.$$

One advantage of this event is that it satisfies a monotonicity constraint, i.e., it cannot become false if one variable X_i goes from the value $\neg x_i$ to x_i. It results that the corresponding multi-linear form given by Equation (2) is increasing[b] in each variable $p(x_i)$. Previous results about such multi-linear forms [12], applied for instance in [1] to the case of reliability functions, then indicate that $\underline{P}(Y)$ is obtained by computing $P(Y)$ when $p(x_i) = \underline{p}(x_i)$. Yet it should be noticed that the form $x_1 \vee x_2 \vee \ldots \vee x_n$ is not very computationally friendly, as every pair of elements of the disjunction have a non-empty intersection, since no pair of formulas contains opposite values of variables (each formula is only made of positive values). This means that Equation (1) contains a number of terms that factorially increases with n. However, we can re-express Y as a disjunction of non-intersecting events (using, e.g., ordered binary decision diagrams [4])

$$Y = x_1 \vee (\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3) \vee \ldots \vee (\wedge_{i=1}^{n-1} \neg x_i \bigwedge x_n)$$

that counts n different conjunctions that have the advantage to be pairwise disjoint, meaning that we finally get

[b] One can simply consider the partial derivatives $\partial P(A)/\partial p_i$ to check it.

	X_1	X_2	X_3	X_4
$\underline{p}(x_i)$	0.2	0.1	0	0.6
$\overline{p}(x_i)$	0.2	1	0.1	0.7

Table 1. Example 2: a metagrid with 4 cells

$$\underline{P}(Y) = \sum_{i=1}^{n} \left(\underline{p}(x_i) \prod_{j=1}^{i-1} \overline{p}(\neg x_j) \right),$$

a sum that has a linear number of terms with respect to n. We can similarly obtain an upper bound $\overline{P}(Y)$, just inverting the bounds over $p(x_i)$ in the equation. The bounds $[\underline{P}(Y^c), \overline{P}(Y^c)]$ over the event Y^c="none of the cells of the metagrid are occupied" can then be obtained by complementation, as we have $\underline{P}(Y^c) = 1 - \overline{P}(Y)$ and $\overline{P}(Y^c) = 1 - \underline{P}(Y)$.

Example 3. Let us consider a metagrid composed of four cells with probability intervals of occupancy given in Table 1. In this case, we have:

$$\begin{aligned}\underline{P}(Y) =& \underline{p}(x_1) + \underline{p}(x_2)\overline{p}(\neg x_1) \\ &+ \underline{p}(x_3)\overline{p}(\neg x_1)\overline{p}(\neg x_2) \\ &+ \underline{p}(x_4)\overline{p}(\neg x_1)\overline{p}(\neg x_2)\overline{p}(\neg x_3).\end{aligned}$$

Using the fact that $\overline{p}(\neg x_i) = 1 - \underline{p}(x_i)$, we obtain:

$$\begin{aligned}&\underline{P}(Y) = 0.2 + 0.1(1 - 0.2) \\ &+ 0(1 - 0.2)(1 - 0.1) \\ &+ 0.6(1 - 0.2)(1 - 0.1)(1 - 0) = 0.712\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{P}(Y) =& \overline{p}(x_1) + \overline{p}(x_2)\underline{p}(\neg x_1) \\ &+ \overline{p}(x_3)\underline{p}(\neg x_1)\underline{p}(\neg x_2) \\ &+ \overline{p}(x_4)\underline{p}(\neg x_1)\underline{p}(\neg x_2)\underline{p}(\neg x_3).\end{aligned}$$

Thus, we have:

$$\begin{aligned}&\overline{P}(Y) = 0.2 + 1(1 - 0.2) \\ &+ 0.1(1 - 0.2)(1 - 1) \\ &+ 0.7(1 - 0.2)(1 - 1)(1 - 0.1) = 1,\end{aligned}$$

giving a probability interval of occupancy equal to $[0.712, 1]$, what might indicate that the metagrid is not free. Note that the upper bound is equal to 1 which is quite natural because it is totally plausible for the second cell to be occupied. Similarly, the high lower bound is induced by the fact that the fourth cell is probably occupied.

	Y_1	Y_2	Y_3	Y_4
$\underline{P}(y_i)$	0.1	0.3	0.712	0
$\overline{P}(y_i)$	0.2	0.5	1	1

Table 2. Example 3: a trajectory composed of four metagrids

3.2. *Inferences along a trajectory*

Having made inferences over the k metagrids $Y_1, \ldots, Y_k$ along a given trajectory, we are now concerned about selecting the path that the robot should take. A first step for this is to assess how far we could go along a given trajectory. Indeed, if one of the first metagrid is almost surely occupied, it is not very useful to know that the last ones are surely free, as we will have encountered an obstacle before them. Hence, what matters is how soon we are likely to encounter an obstacle on a path. With this reasoning in mind, we propose to assess the lower and upper probability bounds over consecutive events F_i="the ith metagrid is the first non-free", or in variable terms,

$$F_i = \wedge_{j=1}^{i-1} \neg y_j \wedge y_i \quad i = 1, k.$$

The event F_i being actually expressible as a Cartesian product of the subspaces $\{y_i, \neg y_i\}$, we have in the probabilistic case that

$$P(F_i) = p(y_i) \prod_{j=1}^{i-1} p(\neg y_j).$$

Getting bounds over this probability is then very easy, since if all the terms are independent, it is enough to just minimize each of them. We thus end up with

$$\underline{P}(F_i) = \underline{p}(y_i) \prod_{j=1}^{i-1} \underline{p}(\neg y_j) \tag{5}$$

and similarly for $\overline{P}(F_i)$, just switching the bounds. Computing this bound involves at most k products, and is therefore quite efficient again. Note that we compute also the bounds for an additional event $F_{k+1} = \wedge_{j=1}^{k} \neg y_j$ which represents the event "all metagrids are free".

Example 4. Let us consider a trajectory composed of 4 metagrids which probabilities are given in Table 2 (in which the third metagrid could be the one of Example 3).

These values could correspond to the following situation: the first two metagrids are likely to be free, then the sensor detects almost surely an obstacle on the third metagrid, and does not see anything on the last metagrid because of occlusions. We have:

$$F_1 = y_1 \times \{y_2, \neg y_2\} \times \{y_3, \neg y_3\} \times \{y_4, \neg y_4\}$$

	F_1	F_2	F_3	F_4	F_5
$\underline{P}(F_i)$	0.1	0.24	0.28	0	0
$\overline{P}(F_i)$	0.2	0.45	0.63	0.18	0.18

Table 3. Example 3: bounds computation for the F_i

$$F_2 = \neg y_1 \times y_2 \times \{y_3, \neg y_3\} \times \{y_4, \neg y_4\}$$

$$F_3 = \neg y_1 \times \neg y_2 \times y_3 \times \{y_4, \neg y_4\}$$

$$F_4 = \neg y_1 \times \neg y_2 \times \neg y_3 \times y_4$$

$$F_5 = \neg y_1 \times \neg y_2 \times \neg y_3 \times \neg y_4$$

Using Esq. (5), we can compute easily the probability bounds of the five events. They are given in Table 3.

It should also be noted that the events $F_1, \ldots, F_{k+1}$ actually form a partition of the space $\prod_{i=1}^{k}\{y_i, \neg y_i\}$ ($F_i \cap F_j = \emptyset$ for any i, j, and $\cup_{i=1}^{k+1} F_i = \prod_{i=1}^{k}\{y_i, \neg y_i\}$), meaning that they can be interpreted as probability bounds over atoms of a Boolean algebra. This means in particular that the set of intervals

$$[\underline{P}(F_i), \overline{P}(F_i)],\ i = 1, k$$

can be interpreted as probability intervals [9], a well known imprecise probabilistic model for which efficient inference tools do exist. We will use this fact in the next section, when deciding which path a mobile robot should follow in order to maximize its utility.

3.3. *Deciding the best trajectory*

We now consider that we have to choose between several paths τ, each moving over a fixed number of k metagrids whose states are uncertain. To choose the best trajectory, we propose to use a utility function quantifying the fact that a trajectory is usable or not. This utility is defined on the basis of the events F_i.

A first remark is that the states or atoms F_i, $i = 1, k+1$ of our decision space are naturally ordered in terms of increasing utility, in the sense that F_j (the j-th metagrid is the first occupied) happening is worse than F_{j+1} happening. This means, among other things, that the utility function associated to those states should be increasing, i.e., that $u(F_i) \leq u(F_{i+1})$ in any case.

In the case of precise probabilities (we have a precise value $P_j(F_i)$ for each element of a trajectory τ_j), the value of a trajectory would then be its expected utility $\mathbb{E}(\tau_j)$, i.e.

$$\mathbb{E}(\tau_j) = \sum_{i=1}^{k+1} P_j(F_i)u(F_i).$$

When the $P(F_j)$'s become imprecise and are only known to belong to some intervals $[\underline{P}(F_j), \overline{P}(F_j)]$, this expectation becomes interval-valued for each trajectory τ_j. Given the fact that our uncertainty model is expressed as intervals on singletons, the upper and lower probabilities are 2-monotone and thus we can compute the lower and upper expected utilities of each trajectory thanks to a Choquet integral (we refer to [9] for the details of the derivation). Moreover, as the utilities are always ordered in the same way, computing the bounds of the expected utility is simple as we have:

$$\underline{\mathbb{E}}(\tau_j) = \sum_{i=1}^{k+1} (u(F_i) - u(F_{i-1}))\, \underline{P}(\{F_i, \ldots, F_{k+1}\}),$$

and

$$\underline{P}(\{F_i, \ldots, F_{k+1}\}) = \max(\sum_{l=i}^{k+1} \underline{p}(F_l), 1 - \sum_{l=1}^{i-1} \overline{p}(F_l)),$$

with $u(F_0) = 0$. Similarly

$$\overline{\mathbb{E}}(\tau_j) = \sum_{i=1}^{k+1} (u(F_i) - u(F_{i-1}))\, \overline{P}(\{F_i, \ldots, F_{k+1}\}),$$

and

$$\overline{P}(\{F_i, \ldots, F_{k+1}\}) = \min(\sum_{l=i}^{k+1} \overline{p}(F_l), 1 - \sum_{l=1}^{i-1} \underline{p}(F_l)).$$

Again, due to our assumptions (increasing and ordered utilities along the F_i, that form a partition), these formulas are fixed and straightforward to compute.

Example 5. Let us consider the utilities given in Table 4. We have chosen to fix negative utilities for the two first metagrids, representing a kind of "security distance" of at least two metagrids, a negative expected utility then corresponding to the fact that if this path is chosen, the mobile robot should preferably brake or stop. If we compute the lower and upper bounds for the trajectory of Example 4, we have

$$\begin{aligned}\underline{\mathbb{E}}(\tau) &= (-20-0)\cdot \underline{P}(\{F_1, \ldots, F_5\}) + (-10-(-20))\cdot \underline{P}(\{F_2, \ldots, F_5\}) + \ldots \\ &\quad + (20-10)\cdot \underline{P}(\{F_5\}) \\ &= (-20-0)\cdot 1 + (-10-(-20))\cdot 0.8 + \ldots + (20-10)\cdot 0 = -8.5\end{aligned}$$

and we finally find $[-8.5, 1]$, which corresponds to the fact that the second metagrid is likely to be occupied.

As each trajectory is characterized by a lower and an upper expectation, there are several possibilities to make a decision about the best trajectory, as we must now

$u(F_0)$	$u(F_1)$	$u(F_2)$	$u(F_3)$	$u(F_4)$	$u(F_5)$
0	-20	-10	0	10	20

Table 4. Example 3 continuation: utility values

compare intervals and no longer point-valued evaluations. Some classical ways [31] to do so are the following:

- Order 1. We can say that a trajectory τ_j is better than a trajectory τ_k ($\tau_j \succ \tau_k$) if and only if $\underline{\mathbb{E}}(\tau_j) \geq \overline{\mathbb{E}}(\tau_k)$.
- Order 2. We can say that a trajectory τ_j is better than a trajectory τ_k if and only if $\underline{\mathbb{E}}(\tau_j) \geq \underline{\mathbb{E}}(\tau_k)$ and $\overline{\mathbb{E}}(\tau_j) \geq \overline{\mathbb{E}}(\tau_k)$.
- Order 3. We can say that a trajectory τ_j is better than a trajectory τ_k if and only if $\underline{\mathbb{E}}(\tau_j) \geq \underline{\mathbb{E}}(\tau_k)$.
- Order 4. We can say that a trajectory τ_j is better than a trajectory τ_k if and only if $\overline{\mathbb{E}}(\tau_j) \geq \overline{\mathbb{E}}(\tau_k)$.

The two first rules provide a partial order on the trajectories allowing for incomparabilities, whereas the two last rules provide a linear order. Rule 3 is a pessimistic rule, it consists in retaining the best trajectory in the worst case, i.e. the trajectory associated to the highest lower expected utility. Rule 4 on the contrary gives the preference to the highest upper expected utility, adopting in this way an optimistic behaviour. When the order is partial (rules 1 and 2), one can determine a set of non-dominated trajectories:

$$T^* = \{\tau_j | \nexists \tau_k \text{ such that } \tau_k \succ \tau_j\}$$

If this set is composed of more than one trajectory and that a unique trajectory is mandatory, several strategies can then be applied. It is for example possible to postpone the decision by asking complementary information on the navigable space. It is also possible to use additional criteria in the decision process, like the proximity to a reference trajectory (using in this case what is commonly known as lexicographic orders [11]).

Example 6. Let us suppose that we have 5 possible trajectories, whose lower and upper expected utilities are given in Table 5 and represented in Figure 3. Using Rule 1, there are three maximal elements in the partial order induced by the intervals (τ_2, τ_4, and τ_5). Two maximal elements are found using Rule 2 (τ_4 and τ_5). The pessimistic rule 3 leads to the choice of τ_4, whereas Rule 4 chooses τ_5.

4. Illustrative application

As a proof of concept, we experiment our approach using real data, issued from the platform PACPUS [17] developed in our lab. The objective of PACPUS is to provide

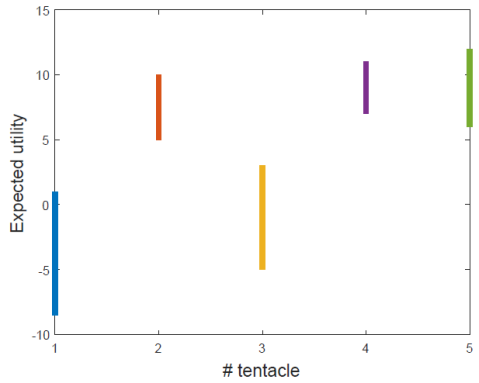

Fig. 3. Representation of the expected utilities of the trajectories

	τ_1	τ_2	τ_3	τ_4	τ_5
$\underline{\mathbb{E}}(\tau_j)$	-8.5	5	-5	7	6
$\overline{\mathbb{E}}(\tau_j)$	1	10	3	11	12

Table 5. Bounds of expected utilities of the trajectories

tools and resources for experimenting on intelligent vehicles. We use a Lidar as a perception sensor. This sensor can distinguish between free and occupied space and model it in 2D (x,y coordinates) with respect to the vehicle bodyframe. We used the credal grids generated by a C++ code [19] with data acquired on our experimental vehicle (cf Figure 4). We refer to the provided references for further details about the grid acquisition. We consider a set of 40 different grids (or images). Each grid of 20*50 meters is built with uniform cells of size 0.1*0.1 meters. Each grid is represented as an image in which one pixel corresponds to one cell, the green color shows the free cells (usable), the red one shows the occupied cells, while the black represents unexplored cells (unknown). The color intensity reflects the certainty degree. For each image (or grid), we generated n possible trajectories, as represented in Figure 5. The shapes of the trajectories are clothoids, that are generated following the approach proposed by Mouhagir *et al.* [22]. The number of generated trajectories depends on internal states of ego-vehicle (speed and yaw rate). This type of trajectory called "tentacle" is indeed well suited to local planning. The final aim is then to automatically select the best trajectory, or "tentacle". In the following we consider $n = 5$ trajectories. Generating such a finite set of pre-computed trajectories is quite common when needing real-time decision for objects having a high velocity (rockets, cars, etc.), as it speeds up the decision process. In this case, exploring all solutions

Fig. 4. Experimental vehicle

and trajectories may simply be too costly, even with efficient algorithms at hand.

Each trajectory is composed of several metagrids, each of them are defined as a square encompassing a circle of 3 meters diameter (30*30 cells). The security distance is fixed to 15 meters which is equivalent to 5 metagrids along a trajectory. The red circles indicate that the distance of the vehicule to the metagrid is less than the security distance. Each cell X of a metagrid is characterized by a basic belief mass assignment (bba) m on the frame $\mathcal{X} = \{x, \neg x\}$, where x stand for "occupied" and $\neg x$ for "free", that can be easily transformed into lower/upper probabilistic bounds[c]. This means that for each cell, we retrieve three masses $m(\{x\}), m(\{\neg x\}), m(\mathcal{X})$. Each metagrid, initially of size 30*30, was transformed into metagrids of size 6*6 by averaging the masses of adjacent cells.

To have access to a ground truth and to be able to compare methods for choosing the best trajectory, we have used two human "experts" (in this case, experienced drivers from the laboratory) to label the possible trajectories. An example of some evaluations given by one expert is given in Table 6. The instructions given to the experts were to assign a rank to the proposed trajectories, with the convention that 0 means that the trajectory is not at all acceptable (i.e., could lead to an accident), that rank 1 is the best rank, and that the same rank can be given to several trajectories if the expert is not able to distinguish them in terms of drivability. For example, in Table 6, for grid 200, trajectory τ_5 is declared unacceptable by the expert, and τ_3 is preferred to τ_2, which is in turned preferred to τ_1 and τ_4, without any preference between these two last trajectories. In image 209, τ_4 and τ_5 are unacceptable, τ_2 and τ_3 are equally preferred, and τ_1 is in the third position.

To apply our method, we convert the evidential grids (which we simply use as an estimation/learning tool) into imprecise probability grids in the following a way: each cell of a metagrid is characterized by an interval of probability $[\underline{p}(x), \overline{p}(x)]$ such that:

$$\underline{p}(x) = m(\{x\}),$$

[c]We consider in this paper only normal bbas so that $m(\emptyset) = 0$ for any cell of the grid, so we can just treat them as our way to obtain probability bounds, without considering their potential semantic within evidence theory.

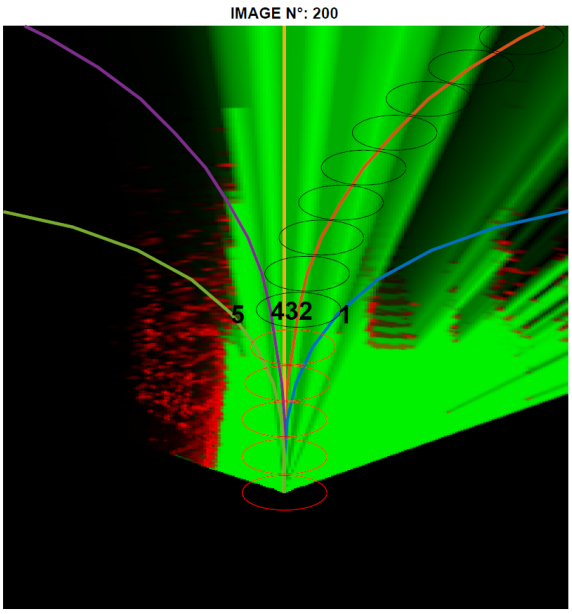

Fig. 5. A credal grid and five possible trajectories

Grid	τ_1	τ_2	τ_3	τ_4	τ_5
200	3	2	1	3	0
209	2	1	1	0	0

Table 6. Example of trajectory evaluation by an expert

and

$$\overline{p}(x) = m(\{x\}) + m(\{x, \neg x\}).$$

The decision is based on 12 metagrids per trajectory which corresponds in average to what can be visually evaluated by the expert. As it is partly outside of what can be seen by the Lidar, the first metagrid is not taken into account in the decision process. We have fixed arbitrarily the values of the utility function to -5 for the first four metagrids, and to values linearly spaced between 10 and 70 for the following ones.

As a baseline to compare our method, we also convert the evidential grids into binary grids and use them as follows:

(1) A cell is occupied if the middle of the interval $[\underline{p}(x); \overline{p}(x)]$ is greater than

0.5.

(2) A metagrid is free if none of the cells that make it up are occupied,
(3) If the rank of the first occupied metagrid is less or equal to 4, then the trajectory is declared unacceptable.
(4) The further away is the first metagrid occupied along a trajectory, the better is the trajectory.

The reason for choosing such a baseline is that it is very simple (hence easily usable by an autonomous car or any mobile robot), but retains very little of our initial, rich information. A key question is therefore to check that retaining such a rich (but computationally more demanding) information does bring some practical advantage. The methods are evaluated according to the two tasks performed during the decision: deciding if a trajectory is acceptable or not and ordering of the remaining trajectories. The methods are thus evaluated according to two different criteria.

For the first task, we use the traditional F_β measure which is defined by:

$$F_\beta = \frac{(1+\beta^2)\mathrm{TP}}{(1+\beta^2)\mathrm{TP} + \beta^2\mathrm{FN} + \mathrm{FP}},$$

where TP stands for the number of true positives, FN for false negatives and FP for false positives. In the task of recognizing acceptable trajectories, the false negative refer to acceptable trajectories which are wrongly classified as unacceptable, whereas the false positives refer to unacceptable trajectories which are wrongly declared as acceptable. The last error being much more serious and dangerous for a driver than the former, we use a value of $\beta = 0.5$ which penalizes more the false positives than the false negatives. Two possible rules have been explored in the experiments:

- Rule 1. trajectory τ_j is classified as acceptable if $\underline{\mathbb{E}}(\tau_j)>0$;
- Rule 2. trajectory τ_j is classified as acceptable if $\overline{\mathbb{E}}(\tau_j)>0$.

For the second task, we need a criterion to judge the relevance of the trajectory ordering. As explained before, the order relations on the set of trajectories obtained using our method are either total preorders (orders 3 and 4) or partial orders (orders 1 and 2). On the other hand, the evaluations of the experts, expressing their preference about the trajectories with possible indifference, lead to total preorders. A first way to compare the results is to use a general measure to evaluate the discrepancy between preorders, as partial or total orders are special cases of preorders. For this purpose, we use a distance measure between preorders proposed in [15]. Let us use the following notations: $\tau_j \succ \tau_k$ denotes the fact that τ_j is strictly preferred to τ_k, $\tau_j \approx \tau_k$ that τ_j and τ_k are indifferent, and $\tau_j ? \tau_k$ means that τ_j and τ_k are incomparable. The distance between two preorders is defined as the sum of the distances between all pairs of trajectories, using the values indicated in Table 7. These values have been arbitrarly chosen in a set of values respecting axiomatic and common sense rules.

For Orders 3 and 4, and for the binary method, as they lead to total preorders, there is a second way of comparing the results: if indecision is not allowed, each

	$\tau_j \approx \tau_k$	$\tau_j \succ \tau_k$	$\tau_j ? \tau_k$	$\tau_k \succ \tau_j$
$\tau_j \approx \tau_k$	0	1	4/3	1
$\tau_j \succ \tau_k$	1	0	4/3	5/3
$\tau_j ? \tau_k$	4/3	4/3	0	4/3
$\tau_k \succ \tau_j$	1	5/3	4/3	0

Table 7. Values of the distance between two preorders

method can be forced to select a best trajectory i.e. a trajectory which is ranked first (there may be multiple ones in case of indifference) in the preorder. We propose in this case to evaluate each method by an accuracy score computed as follows. Let $\hat{\tau}_i$ denote the prediction of a method. For each predicted trajectory, the individual accuracy is obtained as:

$$acc_i = \begin{cases} \frac{1}{r_i} \text{ if } \hat{\tau}_i \text{ is acceptable for the expert} \\ 0 \quad \text{else,} \end{cases}$$

where r_i denote the rank of $\hat{\tau}_i$ in the expert order. The total accuracy Acc is obtained as the mean of the individual accuracies.

It must be noticed that, in contrast with the IP method, the binary method often provides several trajectories equally ranked at the first place because the order depends only on the number of metagrids on the trajectories supposed to be free. In case of a tie, we propose thus two strategies: either to retain the best rank among the set of best trajectories (which gives clearly an advantage to the binary method) or to break the tie by randomly choosing a trajectory in the best trajectories, which seems fairer for the comparison, as in principle we do not have access to the expert "ground truth".

Results are presented in Table 8,9, and 10. In Table 8 is presented the value of F_β averaged on the 40 images for the two rules and the binary approach. We can see that the best score for F_β is achieved using Rule 2. This rule consists in eliminating trajectories for which the utility is for sure less than zero. This confirms that using negative utilities for the first metagrids is a useful feature, as it is an efficient filter. Note that results of our method and the binary one are not very different. As future works, we could try to optimize/change utilities so that they provide optimal F_β on a training set, yet this would significantly increase the method complexity, with an unclear benefit.

Much more differences are found in the second task. Table 9 presents the results obtained for the ordering task (after using Rule 2 for selecting the acceptable trajectories). It gives the average distance between the expert orders and the automatic orders. It can be seen that, for both experts, the order obtained using Order 4 leads to a smallest distance with respect to the expert order and that it outperforms the

simple binary method. Interestingly, this indicates that once the undesirable (unacceptable) trajectories have been removed from the possible choices, the optimistic behaviour seems to be in very good accordance with the observed expert behaviours. Order 1 seem to be too conservative, as could be reasonably expected, while orders 2 and 3 end up in similar distances. It should however be emphasized that we are systematically comparing all orders to total preorders, meaning that it is hard, from our experimental setting, to assess the utility of the incomparabilities induced by orders 1 and 2, in particular the refine an imprecise selection. Exploring this aspect is the matter of future methodological and experimental work, for example to allow experts to express such incomparabilities.

In Table 10, we compare the results of the task consisting in choosing only one trajectory. We report three results for the simple binary method: the maximum accuracy which corresponds to breaking ties in choosing systematically the trajectory associated to the best result (the one ranked highest by the expert), the minimum accuracy which corresponds to choosing systematically the worst result and, finally, a random guess among the possible best trajectories. Without surprise, the results of the binary method in which the maximum accuracy is retained gives the best results but it may be seen that Order 4 outperforms the random guess and provides results which are close to maximum accuracy. Another thing is that the span between the minimum and maximum accuracies can be quite large, thus indicating that a refined modelling of uncertainty may be helpful ot better discriminate the trajectories.

Expert	F_β (rule 1)	F_β (rule 2)	F_β (binary)
1	0,8934	**0,9118**	0,9034
2	0,8768	**0,9460**	0,9429

Table 8. Classification unacceptable/acceptable: average F_β for the three methods

Expert	Order 1	Order 2	Order 3	Order 4	Binary
1	9,7	6,31	6,85	**2,81**	6,84
2	9,86	6,4	6,2	**3,61**	7,32

Table 9. Ordering results: mean distance between the expert and the proposed order

Expert	Order 3	Order 4	Binary (min)	Binary (random)	Binary (max)
1	0,6995	0.9125	0,5372	0.7855	0.9251
2	0,6663	0,8792	0,4422	0,6446	0,9625

Table 10. Best trajectory results: accuracy of the methods

5. Conclusion

In this article, we have introduced the notion of credal occupancy grids, that to the best of our knowledge was never studied before. The advantage of such grids is their flexibility in terms of uncertainty modelling, as well as their consistency with traditional probabilistic models, that they extend. To show how such grids could be used in practice, we have proposed a model to make cautious inferences for robot trajectory planning from uncertain information provided in the form of such credal occupancy grids. The proposed model allows one to accurately represent the available information while controlling the computational complexity by the use of particular events of the decision space. In particular, the proposed method is relatively simple and makes very clear assumptions about the uncertainty modelling and reasoning parts, as well as about the decision process. This has in our opinion two key advantages: it makes the whole method more acceptable and explainable to third parties, and it is easy to revisit, question, or modify the different components of the method.

We have illustrated the behaviour of the proposed approach using simple examples, and have tested it using real data, in the context of autonomous vehicles, comparing it favourably with a simple baseline approach. These first results indeed show that the approach is interesting thanks to a greater flexibility in the decision process. To completely validate the approach, a greater number of experts and a larger number of grids and situations in the experimental part would be needed. Future research could include also the automatic learning of the utility function and the integration of other criteria in the decision process like the similarity to a reference trajectory.

We also plan to augment the expressivity of the method, for example by better differentiating cell occupancy information, distinguishing between completely usable cells, not-fully usable cells (e.g. for vehicles, side-walks, emergency lines) and completely unusable cells (with obstacles). Such distinctions could be important elements to deal with complex situations possibly leading to accidents, but would increase the method complexity. Another important avenue of research is how dependence information could be embedded in the inferences. Indeed, so far we have considered the cells and their uncertainty model to be independent, which is an acceptable simplifying assumption in first approximation, but is not true in general.

Finally, while we considered here the problem of making real-time inferences on a pre-selected number of trajectories by using DNF Boolean formulas, it would be interesting to investigate other ways to handle either the planning problem or to obtain the normal forms of the formulas, that could prove useful in those cases where computational time should remain acceptable but is not constrained by real-time and by physical constraints induced by velocity and limiting the number of trajectories one could consider (e.g., slow moving robots in risky environments). For instance, one could consider finding an arbitrary optimal sequence of cells as a trajectory from point A to point B, taking inspiration from recent works on inferences in HMM [8] or from works extending PODMP [14] to the imprecise framework. Concerning Boolean formulas, we mainly used DNF formulas and mentioned BDDs as another way to obtain compact representations of Boolean functions, but it would interesting to make a general exploration of the interplay between different ways of compiling semantically equivalent Boolean formulas [7] and their corresponding computational complexity, notably those compilations that have an efficient imprecise counterpart such as the recent Sentential Decision Diagrams [18].

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