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Splittable Routing Games in Ring Topology with Losses

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Abstract. We consider a splittable atomic game with lossy links on a ring in which the cost that each player i minimizes is their own loss rate of packets. The costs are therefore non-additive (unlike costs based on delays or tolls) and moreover, there is no flow conservation (total flow entering a link is greater than the flow leaving it). We derive a closed-form for the equilibrium, which allows us to obtain insight on the structure of the equilibrium. We also derive the globally optimal solution and obtain conditions for the equilibrium to coincide with the globally optimal solution.

Keywords: Routing games · Loss probabilities · Ring topology

1 Introduction

We study routing on a ring network in which traffic originates from nodes on the ring and is destined to the center node. Each node has two possible paths: either a direct path from the node to the center node or a two-hop path in which the packet is first relayed to the next node on the ring and then takes the direct link from that next node to the common destination. The traffic originating from a given node is assumed to form an independent Poisson process with some intensity (which we call the demand). Beyond forwarding the traffic that arrives from the previous node, each node has to decide what fraction of its own traffic would be routed on each one of the two possible paths to the destination.

Routing games of this type have been intensively studied both in the road traffic community [13] as well as in the community of telecommunications network [12] under additive costs (such as delays or tolls) and conservation constraints (at each node, the sum of incoming flow equals the sum of outgoing flows). In this paper, we depart from these assumptions by considering loss networks in which losses may occur at all links: there are links with i.i.d. losses

(relay links) and collision losses (on direct links between source and the common destination node).

Two levels of system modeling are presented here: a flow level in which routing decisions are taken, and a more detailed packet-level modeling that determines the losses and thus the interference between flows from different sources. The decisions of a node concern only the fraction of packets originated in that node which will be routed on each of the two paths available to traffic from that node. Then the actual packets to be transmitted over each one of the paths are selected according to an i.i.d. Bernoulli thinning process. The decisions are thus the Bernoulli thinning parameters.

The ring topology has received much less attention than the parallel-link topology as well as the load-balancing triangular topology introduced by Prof Kameda and his students. Although in practice, the ring topology may seem to be a toy problem, we do encounter ring networks quite often in practice, mostly in runabouts. Ring topologies can also be found in access to communication networks, both in local area networks – see IEEE 802.5 token ring standard and the metropolitan area network FDDI [4].

Related work Previous studies of routing games with circular topology have appeared in citecircle1,burra,chen. [7, 8] consider linear costs, and none of these references consider non-conservation of flows. We note also that in [1], there are either bi-directional roads or two rings, one in each of two directions (clockwise and anti-clockwise), and cars have to decide the direction to drive. In [3] other non-additive cost criteria have been introduced in a context of load-balancing games (triangle topology) where their performance measure is related to blocking probabilities. See also [2] and [5] for other related work.

Focusing on symmetric ring networks, our main contribution is to obtain closed-form expressions with the help of Maple. This includes best response functions, derivatives of the costs that are used to compute the best response, and the symmetric equilibrium. We derive the globally optimal solution as well as the equilibrium solution.

2 The model

We consider K nodes on a circle, indexed by $0, 1, \dots, K - 1$, see Figure 1.

Each node k is connected to a set N_k containing N players. Each player $(n, k) \in N_k$ has to ship a strictly positive demand ϕ_k to a destination Δ common to all players. Each of the Nk players generates packets following an independent Poisson process. The player decides with what probability to send an arrival that it generates over one of two possible paths to Δ ; the packet can use a direct transmission link $D(k)$ or an indirect path consisting of first relaying the packet to node $k+1$ and only then transmit it to Δ through $D(k+1)$ (note that node indices are modulo K .) Let x_k^n and α_k^n denote the amount and fraction, respectively, of class k flow originating from player (n, k) through the direct path, i.e., through link $D(k)$. We call

$$\alpha = (\alpha_k^n, k = 0, \dots, K - 1, n = 1, \dots, N)$$

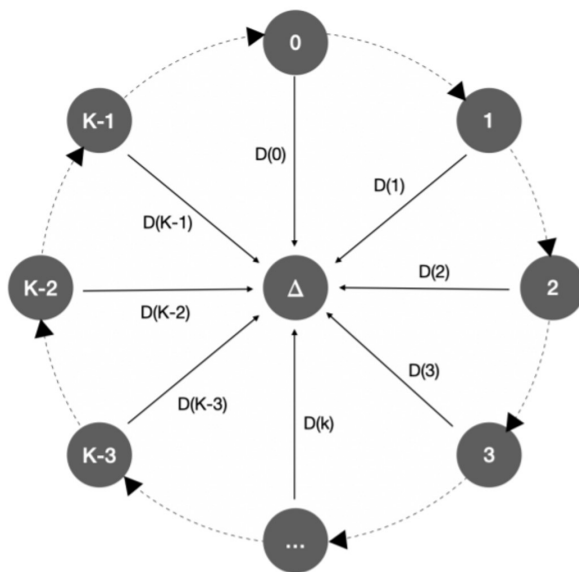


Fig. 1. Ring Network Topologu

the assignment or th action vector. For a given demand vector $\phi = (\phi_k^n, k = 0, \dots, K - 1, n = 1, \dots, N)$, the assignment vector α completely specifies the set of flows

$$\mathbf{x} = (x_k^n, k = 0, \dots, K - 1, n = 1, \dots, N) = (\alpha_k^n \phi_k^n, k = 0, \dots, K - 1, n = 1, \dots, N).$$

With probability α_j^n , a packet originating from player n in N_j takes a direct path to the center, and otherwise it takes an indirect path: it is first relayed to node $j + 1$ and then forwarded to the destination through link $D(k + 1)$.

Loss probabilities We consider two types of losses: (1) i.i.d. losses at the relay: a packet originating from node k is lost if relayed to node $k + 1$ with probability q_k . (2) collision losses on the links D_k : whenever an arrival occurs while there is another packet in service then there is a loss. The transmission duration of a packet in link $D(k)$ is exponentially distributed with parameter μ_k .

The total flow sent to the link $D(k)$ consists of the superposition of (1) the Poisson flows that arrive at node k and are transmitted over $D(k)$, and (2) the Poisson flow originated in node $k - 1$ consisting of the packets that were not lost in the relay to node k .

Thus the total rate of arrivals to $D(k)$ is

$$R(k) = \sum_{n=1}^N \phi_k^n \alpha_k^n + \phi_{k-1}^n (1 - \alpha_{k-1}^n) (1 - q_k)$$

The loss probability of packets at $D(k)$ is

$$P(k) = \frac{R(k)}{R(k) + \mu_k} \quad (1)$$

By the same arguments, the loss probability of packets at $D(k+1)$ is

$$P(k+1) = \frac{R(k+1)}{R(k+1) + \mu_{k+1}} \quad (2)$$

The total rate of losses of packets of player (n, k) is

$$J_k^n(\alpha) = \phi_k^n \left(\alpha_k^n P(k) + (1 - \alpha_k^n) [q_k + (1 - q_k) P(k+1)] \right) \quad (3)$$

In the rest of the paper we assume that ϕ_k^n , μ_k and q_k are constant, independent of k and n .

3 The globally optimal solution and the equilibrium

3.1 Minimizing average loss probability

Consider a symmetric multi-strategy α for all players, i.e. in which α_k^n are the same for all players (n, k) . Without loss of generality, let $\phi = 1/N$. Then the rate of arrival of packets at the links $D(k)$ is $R = 1 - (1 - \alpha)q$. The loss probability on link $D(k)$ is

$$P_l = \frac{R}{R + \mu} = 1 - \frac{\mu}{1 - (1 - \alpha)q + \mu}$$

so the global loss probability is

$$\pi = P_l + (1 - \alpha)q.$$

It is minimized at $\alpha = 1$, which means that all players take direct path to the destination.

3.2 Equilibrium.

Assume that player (n, k) deviates and plays b instead of playing α . Let u be the multi-strategy after the deviation. Then

$$\begin{aligned} R(k, u) &= [\alpha + (1 - \alpha)(1 - q)] \frac{N-1}{N} + [b + (1 - \alpha)(1 - q)] \frac{1}{N} \\ &= \frac{b - \alpha}{N} + (1 - \alpha)(1 - q) \end{aligned}$$

$$R(k+1, u) = [\alpha + (1 - \alpha)(1 - q)] \frac{N-1}{N} + [\alpha + (1 - b)(1 - q)] \frac{1}{N}$$

$$= \alpha + (1 - \alpha)(1 - q) - (1 - q)\frac{b - \alpha}{N}$$

The loss probability for a player (k, n) is given by

$$J_k^n(u) = \frac{1}{N} \left(bP(k, u) + (1 - b)[q + (1 - q)P(k + 1, u)] \right)$$

where the path loss probabilities $P(k, u)$ and $P(k + 1, u)$ are given in (1)-(2).

To obtain the equilibrium, we:

1. differentiate $J_k^n(u)$ with respect to b , and obtain (using Maple) the expression in Figure 6. Equating the expression to 0 allows us to obtain the best response action $b = f(a)$ that minimizes the loss probabilities of a player when all others use α .
2. obtain the symmetric equilibrium by computing the fixed point of the mapping $b = f(\alpha)$. This leads to

$$\alpha^* = -\frac{N\mu q - Nq^2 + Nq + q^2 - 2q + 1}{Nq^2 - q^2 + 2q - 2}$$

If $0 < \alpha^* \leq 1$, then it is a symmetric equilibrium. If the fixed point is greater than 1, then the symmetric equilibrium is the policy $\alpha = 1$ for all players. In that case, the equilibrium is globally optimal and only the direct links are used. Thus, the equilibrium coincides with the globally optimal policy for all q large enough.

3.3 Best response

As already mentioned, we are able with the help of Maple to get an explicit cumbersome expression for the best response. This allows us to obtain a much simpler expression for the equilibrium (as a function of the parameters). We present in Figure 2 the best response b as a function of q .

3.4 When is the globally optimal policy $\alpha = 1$ an equilibrium

Consider the cost $J_k^n(b, 1)$ for some player (n, k) where $(b, 1)$ is the policy where all players use $\alpha = 1$ and the deviating player (k, n) uses b .

Theorem 1. *A necessary and sufficient condition for $\alpha = 1$ to be a symmetric equilibrium is that*

$$qN > \frac{1}{1 + \mu}$$

Proof. A necessary and sufficient condition for the symmetric policy $\alpha = 1$ to be an equilibrium is that the cost for the deviating player be decreasing in b at $b = 1$. This is equivalent to the following first-order condition. The derivative

$$\begin{aligned}
& \left(-2a^2q^2 - 2a\mu q^2 + aq^3 + 4a^2q + 2a\mu q - aq^2 - 2\mu^2q + 3\mu q^2 - q^3 - 2a^2 - aq \right. \\
& \quad + 2\mu^2 - 6\mu q + 3q^2 \\
& \quad + (-a^4q^6 - a^3\mu q^6 + 6a^4q^5 + 3a^3\mu q^5 + 3a^3q^6 - 3a^2\mu^2q^5 + 3a^2\mu q^6 - 13a^4q^4 + 4a^4 \\
& \quad - 22a^3\mu q^3 + 51a^3q^4 - 18a^2\mu^2q^3 + 13a^2\mu q^4 + 22a^2q^5 + 15a\mu^3q^3 \\
& \quad - 35a\mu^2q^4 + 19a\mu q^5 + aq^6 - \mu^4q^3 + 3\mu^3q^4 - 3\mu^2q^5 + \mu q^6 - 4a^4q^2 \\
& \quad + 24a^3\mu q^2 - 62a^3q^3 - a^2\mu^2q^2 + 28a^2\mu q^3 - 64a^2q^4 - 24a\mu^3q^2 + 72a\mu^2q^3 \\
& \quad - 43a\mu q^4 - 8a^5 + 5\mu^4q^2 - 18\mu^3q^3 + 21\mu^2q^4 - 8\mu q^5 - 8a^3\mu q + 36a^3q^2 \\
& \quad + 12a^2\mu^2q - 62a^2\mu q^2 + 94a^2q^3 + 12a\mu^3q - 59a\mu^2q^2 + 37a\mu q^3 + 26a^4 \\
& \quad - 8\mu^4q + 39\mu^3q^2 - 57\mu^2q^3 + 26\mu q^4 - 8a^3q - 4a^2\mu^2 + 40a^2\mu q - 73a^2q^2 \\
& \quad + 12a\mu^2q + 2a\mu q^2 - 44a^3 + 4\mu^4 - 36\mu^3q + 75\mu^2q^2 - 44\mu q^3 - 8a^2\mu \\
& \quad + 28a^2q + 4a\mu^2 - 20a\mu q + 41a^2 + 12\mu^3 - 48\mu^2q + 41\mu q^2 - 4a^2 + 8a\mu \\
& \quad \left. - 20aq + 12\mu^2 - 20\mu q + 4a + 4\mu \right)^{12} + a + 3\mu - 3q + 1 \Big/ (aq^3 - 3aq^2 \\
& \quad + \mu q^2 - q^3 + 4aq - \mu q + 3q^2 - 2a - 3q + 1)
\end{aligned}$$

Fig. 2. Expression for the best response function $b = f(\alpha)$

of the cost of the deviating player evaluated at $b = 1$ when all other players use $\alpha = 1$ is non-negative. Calculation in Maple yields

$$\left. \frac{dJ_k^n(b, 1)}{db} \right|_{b=1} = -\frac{\mu(N\mu q + Nq - 1)}{(1 + \mu)^2} \quad (4)$$

This concludes the proof.

We conclude that if one invests in improving a communication channel thus decreasing the loss probabilities (the parameter q in our case), then as a result we may end up worsening the performance for all the users in the system.

4 Numerical examples

With the help of Maple, we obtained a simple expression for the equilibrium as a function of the parameters of the system. The following experiments allow us to get insight on the equilibrium behavior.

4.1 The equilibrium

We depict in Figure 3 the parameter α^* defining the symmetric equilibrium as a function of the loss rate parameter q . The following parameters are fixed: $N = 1$ and $\mu = 1$. As long as α^* is in the unit interval, it is the equilibrium. This is the case for $q \leq 0.5$. For larger q , the corresponding symmetric equilibrium is $\alpha = 1$.

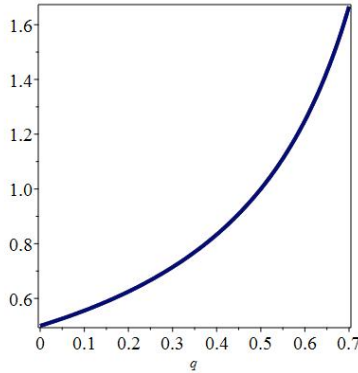


Fig. 3. The symmetric equilibrium as a function of the loss probability parameter q .

4.2 The best response

In Figure 4, we depict the loss probability of a player that plays b when the others play α . b is varied while α is held fixed. The figure thus allows us to identify the best response b . In the case that the best response $b(\alpha)$ to a given α in the unit interval satisfies $b(\alpha) = \alpha$, then α is a symmetric equilibrium.

The following parameters are fixed: $N = 1$ and $\mu = 1$. There are three curves: the first corresponds to $\alpha = 0.5012$ and $q = 0.01$, the second to $q = 0.3$ and $\alpha = 0.55$ and the third to $q = 0.9$ and $\alpha = 1$. In the two first curves, b for which the derivative of the cost is zero is within the unit interval and is thus the best response to α . In the first curve, $\alpha = 0.5012$ is a fixed point of the best response function $b(\alpha)$. This confirms that $\alpha = 0.5012$ is an equilibrium, which can be seen from Figure 3.

In the third curve, b that minimizes the cost is larger than 1 and the best response is obtained on the boundary $b = 1$ and not on the b for which the derivative w.r.t. b is zero.

4.3 Non-equilibrium of the globally optimal policy $\alpha = 1$

We depict the best response to the globally optimal policy in Figure 5, for $N = 1$ and $\mu = 1$ being held fixed, as a function of the loss parameter q . For the above parameters, the best response function is given in Figure 2.

We observe that for all $q < 0.5$, the best response to other players sending their traffic through the direct path (i.e. $\alpha = 1$) is to play $b < 1$. We conclude that for these q , $\alpha = 1$ is not an equilibrium. In contrast, for all $q \geq 0.5$, the best response to $\alpha = 1$ is also $\alpha = 1$ and hence the global optimal policy is an equilibrium policy.

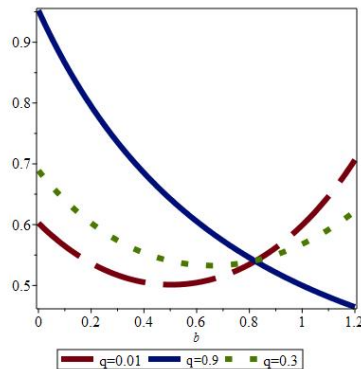


Fig. 4. The symmetric equilibrium as a function of the loss probability parameter q .

4.4 The derivative of the cost of a player

In order to provide an expression for the equilibrium, we had to differentiate the cost of the player that uses b while others use a . Below is long expression obtained by Maple for the derivative w.r.t. b of the cost for the player, i.e. of the loss probability of packets of player (k,n) who plays b when all others play a .

5 Conclusions

We have seen that for any choice of system parameters, the equilibrium performance improves when increasing the loss probability parameter q on the relay links. The equilibrium loss probability of a player is thus decreasing in q . This is a Braess-type paradox.

Moreover, for any parameters of the system, if the number N of players at each node is large enough, then the globally optimal policy is an equilibrium; this means that in the regime of a large number of players (called a Wardrop equilibrium), the above type of paradox does not occur.

The original Braess paradox [6] was shown to hold in a framework of a very large number of players (Wardrop equilibrium). Later on it was shown to occur also in the case of any number $N > 1$ players in [11]. The paradox we introduced, known as the Kameda paradox, does not occur in the case of a very large number of players. This was shown for standard delay-type cost functions in [10] for a triangular network topology.

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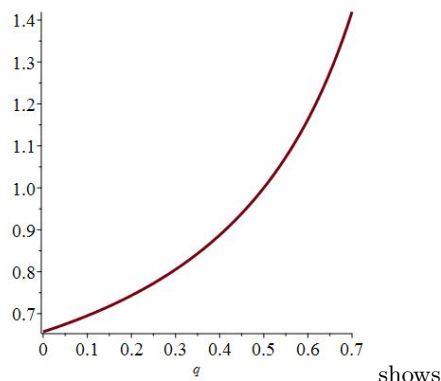


Fig. 5. Sending all the traffic through direct paths is not an equilibrium. The Figure shows the best response of a player when the others use $\alpha = 1$ as a function of the loss parameter q .

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$$\begin{aligned}
dJ_{ba}N := & \frac{(a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{b + (1-a)(1-q)}{N}}{(a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{b + (1-a)(1-q)}{N} + \mu} \\
& + \frac{b}{N \left((a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{b + (1-a)(1-q)}{N} + \mu \right)} \\
& - \frac{b \left((a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{b + (1-a)(1-q)}{N} \right)}{\left((a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{b + (1-a)(1-q)}{N} + \mu \right)^2 N} - q \\
& - \frac{(1-q) \left((a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{a + (1-b)(1-q)}{N} \right)}{(a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{a + (1-b)(1-q)}{N} + \mu} + (1 \\
& - b) \left(\frac{(1-q)(-1+q)}{N \left((a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{a + (1-b)(1-q)}{N} + \mu \right)} \right. \\
& \left. - \frac{(1-q) \left((a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{a + (1-b)(1-q)}{N} \right) (-1+q)}{\left((a + (1-a)(1-q)) \left(1 - \frac{1}{N}\right) + \frac{a + (1-b)(1-q)}{N} + \mu \right)^2 N} \right)
\end{aligned}$$

Fig. 6. Expression for the derivative of the cost of the player (k,n) with respect to b .

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