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An assessment of hierarchicalness in renormalizable network models

Aurélien Hazan¹

Université Paris-Est Créteil Val-de-Marne
LISSI, IUTSF, F-77567 Lieusaint, France
aurelien.hazan@u-pec.fr,
WWW home page: <http://hazan.xyz>

Abstract. In order to clarify the relationship between renormalizable network models and hierarchicalness, we elaborate on a recent binary fitness-based renormalizable network model and evaluate how it can cope with hierarchicalness, that was characterized in several ways in the literature, and remains an elusive concept.

To do so a benchmark hierarchical network is fit to a renormalizable network recently proposed. We report that under specific conditions the latter can accommodate hierarchicalness, examine various extensions, and discuss future applications to real hierarchical networks.

Keywords: multi-scale, hierarchical, nested, clustering, community, fitness

1 Introduction

In the present article, the relationship between renormalizable network models and hierarchicalness is studied, for random graphs. Let G be such a graph, composed of vertices V and edges $E \in V \times V$ with a nested community structure. Let A be its adjacency matrix with elements a_{ij} .

Renormalizable network models in this article refer to the existence of a unique expression for the edge probabilities p_{ij}^l , verified at several hierarchical levels l , where i, j index the nodes. The availability of such an expression is important: firstly it gives a simplified, coarse-grained representation of a network. Secondly, it might shed light on scale-dependence issues, for example in exponents of gravity-like networks models [1].

Hierarchicalness is often associated in the literature to a modular structure with modules arranged in a specific, nested way. Modules, or communities are usually thought of as locally densely connected subnetworks. Similarly to renormalizable network models, hierarchicalness allows a simplified coarse-grained and multi-scale representation of networks that favors interpretability. In the complex networks literature, several coarse-graining schemes are used, with the

aim of understanding the existence of self-similarity¹ since this concept turned out to be of paramount importance in modern Physics².

To make the comparison concrete we elaborate on a recently proposed fitness-based renormalizable network model by Garuccio et al. [8] on one hand, and on previously published definitions of hierarchicalness in networks that are summarized below, on the other hand.

The renormalizable network model in [8], henceforth referred to as GLG, aims at describing the connectivity of undirected unweighted networks displaying nested community structure with a single probability p_{ij}^l .

The GLG model aims at preserving p_{ij} under coarse-graining. Interestingly the authors show that under the hypotheses below, there is a unique functional form for p_{ij} that ensures renormalizability:

- a hierarchical partition must be known a priori, for example in the form of a dendrogram.
- nodes are associated with known fitnesses x_i , that must sum across aggregation.

The GLG model, described in more detail in the appendix, is analytically tractable with an edge probability p_{ij}^l a level l , that depends on just one parameter δ to tune the network density at level $l = 0$:

$$p_{i,j}(\delta) = 1 - e^{-\delta x_i x_j} \quad (1)$$

$$x_{i_{l+1}} = \sum_{i_i \in i_{l+1}} x_{i_i} \quad (2)$$

where x_{i_i} are node-specific predefined parameters, the *fitnesses* that must sum across aggregation.

Importantly, it is independent of geometry, and can accomodate constant or random fitnesses. Optionnally it can take dyadic information (such as distances between nodes, or group membership) as an input. Recent works started tackling the weighted case but the topic is still open [11].

The GLG model relies on the notion of hierarchical levels. Our first contribution in this article is to remark that the definition of hierarchy in the GLG model is implicit, and could be made explicit, and compared to existing definitions of hierarchicalness available in the field.

Secondly, we start reviewing useful definitions of hierarchy, as well as statistical tests to assess hierarchicalness, and related synthetic benchmark models.

¹ for example geographical coarsening [10] considers graphs embedded in a 2d space used to perform box-covering, while fractal-oriented studies [6] use a shortest-path-distance on a graph to perform renormalization without needing an embedding space.

² for example the Renormalization Group (RG) theory uses coarse-graining extensively and allows to classify lattice systems in universality classes independently of the implementation details. Such universality classes results not available for complex nets, but results exist for hierarchical networks.

Lastly, given that the GLG model aims at preserving p_{ij} under coarse-graining, we first ask if hierarchicalness in benchmark model is preserved under fitting by the GLG model.

Selected definitions of hierarchicalness are reviewed in sec. 2. The representation by the GLG model of hierarchical networks is dealt with in sec. 3. Sec. 4 concludes.

2 Selected definitions of hierarchicalness

The seminal work of Simon [18] characterizes hierarchical systems as being composed of interactive systems arranged in a nested way³ and has inspired many complex networks works.

Furthermore Simon mentions the fact that, although the etymology of the term hierarchic refers to some notion of subordination or directedness, the definition should remain more general. We shall also adopt this point of view, and consequently let aside the works that rely on directedness [12, 4, 5], which will also ensure comparability with GLG.

In the field of complex networks, the work in [15] proposed hierarchical structures as a solution to the problem of exhibiting a synthetic network model displaying several properties simultaneously (scale-freeness, high and size-independent average clustering), as was observed in some empirical networks. A solution to this problem involved nested assortative communities, in which nodes are densely connected with each other, and weakly connected to other communities. The deterministic fractal-like Ravasz-Barabási (RB) algorithm was proposed, and features a $C(k) \sim k^{-1}$ scaling of the local clustering $C(k) = \frac{\#\text{triangles}}{k_i(k_i-1)/2}$ with respect to the local degree $k_i = \sum_{j \neq i} a_{ij}$. In conjunction with global clustering C being independent on the network size, this is considered by the authors as a signature of hierarchicalness. This algorithm was recently extended to generate hierarchical benchmark models for community detection algorithms

In [3] the authors define hierarchical networks as being formed of assortative communities, organized in a nested way, that is communities dividing recursively⁴. The formal representation of a hierarchy is a dendrogram, with node weights encoding the probability of connection among leaf nodes. The authors use MCMC to learn the parameters of the model, and consensus clustering to clean a population of dendrograms and retain dominant features. The representation can express assortative communities only, but has been extended to handle also disassortative communities [13]. However in both cases, the learning

³ "By a hierarchic system, or hierarchy, I mean a system that is composed of inter-related subsystems, each of the latter being, in turn, hierarchic in structure until we reach some lowest level of elementary subsystem." [18], this type of structure is supposed to allow a quicker evolution.

⁴ "a fractal-like structure in which vertices cluster together into groups that then join to form groups of groups, and so forth, from the lowest levels of organization up to the level of the entire network" in [3]

algorithm will always find a set of locally optimal parameters, and it is not clear how to test if a given empirical network is hierarchical or not.

In the community finding literature, several popular algorithms use nested communities [2, 16, 14], and model selection algorithms to learn their parameter. The average depth associated to nodes is used as an indicator of hierarchicalness of the graph [16]. But it was also reported [17] that related model selection algorithms would select hierarchies with a significant depth, even though the inspected graph contained none⁵.

In [17] a formal definition of hierarchicalness based on block models generalizes the notions of structural equivalence and equitable partitions while accounting for the stochastic nature of connections. A spectral detection algorithm is proposed as well. The definition can handle disassortative communities, but is not able so far to reflect degree corrections in a way similar to the degree-corrected stochastic block model (dc-SBM) [9], as discussed by the authors. Such a generalization seems necessary in order to apply the model to real networks.

In the GLG model, the notion of hierarchy refers implicitly to a nested partition of the node set. Assortative and disassortative communities can be encoded using the optional dyadic interaction term that can be tuned to mimic an SBM model in its standard, non degree-corrected, version. In contrast to other works, for example [17], the nested partition in the GLG model is extrinsic, since it constitutes an input to the model, and is not necessarily a direct consequence of the sole topology of the network.

In the next section we shall proceed to assessing how the hierarchical structure in a network is preserved by representing it with the GLG model.

3 Conservation of hierarchicalness

In this section the expressivity of hierarchicalness by the GLG model is assessed, using the RB model and definition of hierarchy as a benchmark.

We set to approximate the deterministic Ravasz-Barabási (RB) network with the GLG model, and to examine whether or not the property $C(k) \sim k^{-1}$ is retained in the obtained GLG model. The graph G_{RB} and the adjacency matrix A_{RB} of the RB network are shown in Fig. 1(a,b) for a network size $n = 125$ with a base motif of size $m = 5$. Then, a standard (non dc-corrected) SBM is obtained by block-averaging A_{RB} , which yields an average adjacency matrix P_{RB} composed of connection probabilities p_{ij} between edges (i, j) , as shown in Fig. 1(c). The conservation of hierarchicalness (in the RB sense) after block-modelling can be assessed thanks to Fig. 1(d) displaying the local clustering as a function of the degree $C(k) = f(k)$, for a random sample of the SBM fit of the RB graph, as well as for the related RB network with $n = 3125$

⁵ "However, applying the hierarchical model of Peixoto (4) to this network produces a hierarchical clustering (Fig. 1B). The reason is that the hierarchical model is designed primarily to allow for the detection of communities that are smaller than the resolution limit (31) imposed by the Bayesian model selection procedure employed" in [17]

nodes. Clearly, the $C(k) \sim k^{-1}$ scaling is observed in the case of the RB graph, which is expected. More surprising is the same scaling applying to the SBM model fitted as explained above, since the standard SBM is not able to retain some scaling properties such as scale-freeness. A hand-waiving explanation can concentrate on the central block of the SBM, corresponding to the central node of the RB network, featuring the highest-degree and the lowest local clustering. When passing from the RB network to the SBM model, the degree of each node in this block is divided by $m = 5$, being scattered over the rows of the adjacency matrix. Meanwhile, the triangles formed with other blocks (with dense inner connection pattern) are now scattered over the m nodes of this central block: their number is approximately divided by m . The numerator in the local clustering $C(k) = \frac{\#\text{triangles}}{k_i(k_i-1)/2}$ of an SBM sample is divided by m , while the denominator is divided by roughly m^2 . Wrapping it up, the degree is divided by m , so is the local clustering, but the proportional relation is approximately preserved.

Pursuing the goal to fit the GLG model to the RB network, P_{RB} is finally used as input to parametrize the dyadic term in a GLG model with nodes weights verifying $\forall i_0, x_{i_0}$, as written below at level 1:

$$p_{i_1, j_1}(\delta) = 1 - e^{-\delta p_{i_0, j_0}} \quad (3)$$

δ is found by a numerical root-finding algorithm and ensures the preservation of the network density.

Samples can now be randomly generated from this probability, and the graph $C(k) = f(k)$ can be computed. Although it is slightly perturbed with respect to the SBM step, notably in the low k zone, we find the same scaling in the GLG case as evidenced by Fig. 1(e) which displays the local clustering as a function of degree $C(k) = f(k)$ for a random sample of the GLG fit of the RB graph.

As a result of this first comparison, we can thus conclude that the GLG model at level $l = 0$ -that is before coarse graining takes place- can conserve the hierarchical property as defined by a $C(k) \sim k^{-1}$ scaling, when the initial network is RB, and the parameters of the GLG model are known and tailored to the existing block-structure (taking for reference the smallest motif with size $m = 5$).

If the parameters are partially known, the conservation is expected to degrade. This can happen for example if the set of input parameters $\{p_{i_0, j_0}\}$ that encodes the underlying block structure in eq.(3) is incorrect. The correct block structure used in Fig. 1(e) is randomly shuffled, and the resulting scatter plot is shown in Fig. 1(f). As expected, the result is significantly degraded, especially in low k zone where connection pattern is sparse. A well defined cluster persists in high k /low $C(k)$ zone since the dominant row/column is preserved under shuffling, it is preserved under block approximation, and random matrix sampling.

Further testing when the input block structure parameter of the GLG model is not aligned with ground truth, or when there is no block structure at all [?] while the $C(k) \sim k^{-1}$ scaling exists will be considered in further work.

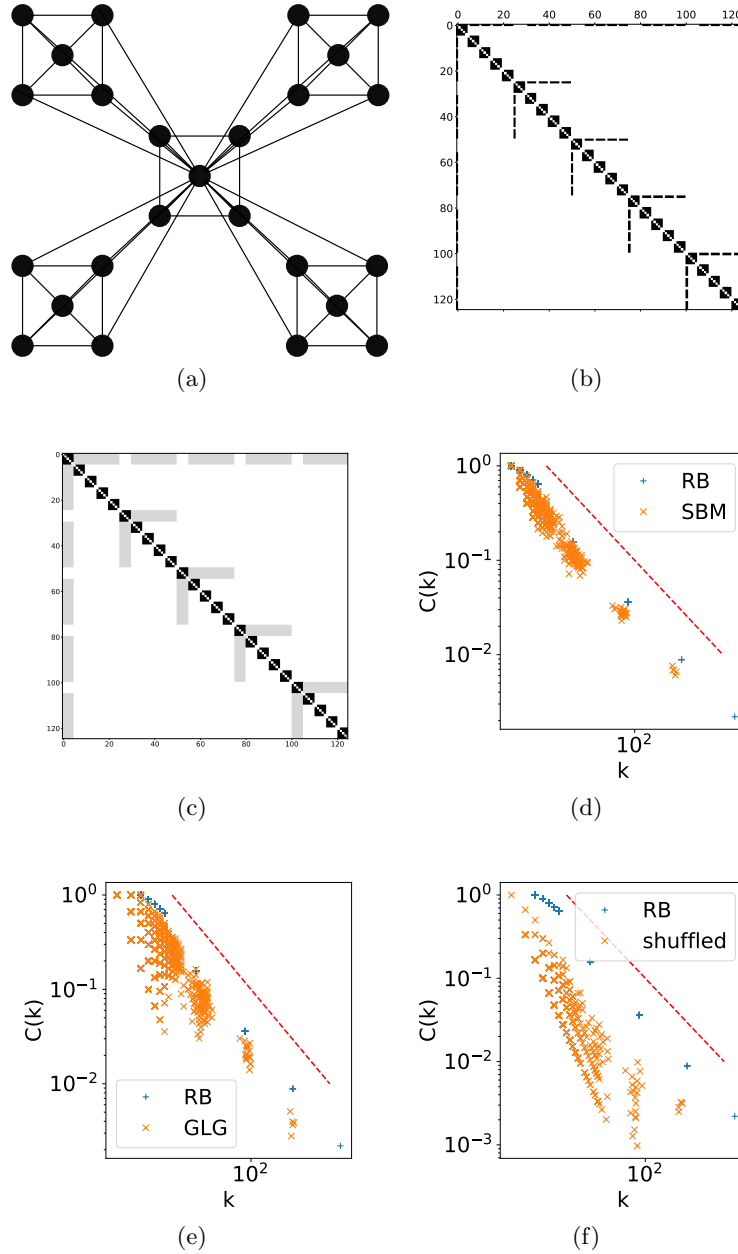


Fig. 1. Properties of random models of the RB graph (a) RB graph $n = 125$. The basic motif with 5 nodes is fully-connected; (b) adjacency matrix of RB graph; (c) probability matrix p_{ij} of the SBM fit of the RB graph; (d) local clustering wrt to degree $C(k) = f(k)$ for a random sample of the SBM fit of the RB graph with $n = 3125$ nodes, the red dashed curved gives the reference curve $C(k) = k^{-1}$; (e) local clustering wrt to degree $C(k) = f(k)$ for a random sample of the GLG fit of the RB graph with $n = 3125$ nodes; (f) $C(k) = f(k)$ for a random sample of the GLG fit of the RB graph with randomly shuffled block structure and $n = 3125$ nodes

4 Conclusion

In this article we addressed the general topic of understanding the relationship between renormalizable network models and hierarchicalness, a notion that received many definitions in the network science literature.

Our contribution here builds on a particular definition of renormalizability -the GLG model- and on a specific definition of hierarchicalness based on the scaling relation between local clustering and degree. The deterministic Ravasz-Barabási network served as a benchmark network.

We report that the GLG model can in fact conserve hierarchicalness in this specific setting. This was shown using only the dyadic term, while node-specific terms x_i were kept constant with equal value, which implied approximating the RB network by an SBM network. The observations concern only level $l = 0$, before any coarse-graining.

This observation paves the way for further studies, first investigating the effect of coarse-graining, then defining fit quality for the GLG model, as well as a systematic fitting procedure involving node-specific terms.

Other relevant definitions of hierarchicalness should be included in the comparison, such as [17]. For example, whether or not stochastic externally equitable partitions are stable after fitting the GLG model could be tested.

Applications to various real hierarchical network will be examined, first selecting real network that are hierarchical according to one or several definitions, then examining which can be modelled satisfactorily by the GLG, and finally which model conserves hierarchicalness.

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References

1. Barbosa, H., Barthelemy, M., Ghoshal, G., James, C.R., Lenormand, M., Louail, T., Menezes, R., Ramasco, J.J., Simini, F., Tomasini, M.: Human mobility: Models and applications. *Physics Reports* 734, 1–74 (Mar 2018), <https://linkinghub.elsevier.com/retrieve/pii/S037015731830022X>
2. Blondel, V.D., Guillaume, J.L., Lambiotte, R., Lefebvre, E.: Fast unfolding of communities in large networks. *Journal of Statistical Mechanics: Theory and Experiment* 2008(10), P10008 (Oct 2008), <https://iopscience.iop.org/article/10.1088/1742-5468/2008/10/P10008>
3. Clauset, A., Moore, C., Newman, M.E.J.: Hierarchical structure and the prediction of missing links in networks. *Nature* 453(7191), 98–101 (May 2008), <http://www.nature.com/articles/nature06830>
4. Corominas-Murtra, B., Goni, J., Sole, R.V., Rodriguez-Caso, C.: On the origins of hierarchy in complex networks. *Proceedings of the National Academy of Sciences* 110(33), 13316–13321 (Aug 2013), <http://www.pnas.org/cgi/doi/10.1073/pnas.1300832110>

5. Coscia, M.: Using arborescences to estimate hierarchicalness in directed complex networks. PLOS ONE 13(1), e0190825 (Jan 2018), <https://dx.plos.org/10.1371/journal.pone.0190825>
6. Gallos, L.K., Song, C., Makse, H.A.: A review of fractality and self-similarity in complex networks. Physica A: Statistical Mechanics and its Applications 386(2), 686–691 (Dec 2007), <https://linkinghub.elsevier.com/retrieve/pii/S0378437107007741>
7. Garlaschelli, D., Loffredo, M.I.: Fitness-Dependent Topological Properties of the World Trade Web. Physical Review Letters 93(18) (Oct 2004), <https://link.aps.org/doi/10.1103/PhysRevLett.93.188701>
8. Garuccio, E., Lalli, M., Garlaschelli, D.: Multiscale network renormalization: scale-invariance without geometry. Tech. rep. (2020), [eprint: 2009.11024](https://arxiv.org/abs/2009.11024)
9. Karrer, B., Newman, M.E.J.: Stochastic blockmodels and community structure in networks. Physical Review E 83(1), 016107 (Jan 2011), <https://link.aps.org/doi/10.1103/PhysRevE.83.016107>
10. Kim, B.J.: Geographical Coarse Graining of Complex Networks. Physical Review Letters 93(16), 168701 (Oct 2004), <https://link.aps.org/doi/10.1103/PhysRevLett.93.168701>
11. Lardy, T.D.: A Renormalizable Random Graph Model. Tech. rep. (2018)
12. Mones, E., Vicsek, L., Vicsek, T.: Hierarchy Measure for Complex Networks. PLoS ONE 7(3), e33799 (Mar 2012), <https://dx.plos.org/10.1371/journal.pone.0033799>
13. Peel, L., Clauset, A.: Detecting Change Points in the Large-Scale Structure of Evolving Networks. In: Proc. of the 29th International Conference on Artificial Intelligence (AAAI). pp. 2914–2920 (2015), <https://arxiv.org/abs/1403.0989>
14. Peixoto, T.P.: Hierarchical Block Structures and High-Resolution Model Selection in Large Networks. Physical Review X 4(1), 011047 (Mar 2014), <https://link.aps.org/doi/10.1103/PhysRevX.4.011047>
15. Ravasz, E., Barabási, A.L.: Hierarchical organization in complex networks. Physical Review E 67(2), 026112 (Feb 2003), <https://link.aps.org/doi/10.1103/PhysRevE.67.026112>
16. Rosvall, M., Bergstrom, C.T.: Multilevel Compression of Random Walks on Networks Reveals Hierarchical Organization in Large Integrated Systems. PLoS ONE 6(4), e18209 (Apr 2011), <https://dx.plos.org/10.1371/journal.pone.0018209>
17. Schaub, M.T., Peel, L.: Hierarchical community structure in networks (2020), [eprint: 2009.07196](https://arxiv.org/abs/2009.07196)
18. Simon, H.A.: The Architecture of Complexity. In: Facets of Systems Science, pp. 457–476. Springer US, Boston, MA (1991)

A The renormalizable model by Garuccio et al.

In this section, a simplified version of the model by Garuccio et al. [8] is presented, keeping their notations: $\mathbf{A}^{(l)}$ is the adjacency matrix of a binary undirected graph, that results from the iterated coarse-graining of an original matrix $\mathbf{A}^{(l)}$, with dimension $N_0 \times N_0$. Non-overlapping partition functions Ω_l define partitions of the node set at level l . The partitions defined at all levels are considered as known *a priori*, for example in the form of a dendrogram. The chosen coarse-graining rule is simple as illustrated by Fig.2: if there is at least an edge between two nodes i_l, j_l belonging respectively to super-nodes

i_{l+1} and j_{l+1} , then the super-nodes are connected as well. This can be written: $a_{i_{l+1},j_{l+1}} = 1 - \prod_{i_l \in i_{l+1}} \prod_{j_l \in j_{l+1}} (1 - a_{i_l,j_l})$, where $i_l \in i_{l+1}$ means that node i_l at level l belongs to super-node i_{l+1} at level $l+1$. Random adjacency matrix are then considered, and associated with the probability $P_l(\mathbf{A}^{(l)})$. Hypothesizing independent links leads to the expression $\prod_{i_l=1}^{N_l} \prod_{j_l=1}^{i_l} [p_{i_l,j_l}^{(l)}]^{a_{i_l,j_l}^{(l)}} [1-p_{i_l,j_l}^{(l)}]^{1-a_{i_l,j_l}^{(l)}}$. Renormalizability of the model is expressed by the scale-invariance property, for any $l \geq m \geq 0$:

$$P_l(\mathbf{A}^{(l)}|\Theta_l) = \sum_{\{\mathbf{A}^{(m)}\} \rightarrow \mathbf{A}^{(l)}} P_m(\mathbf{A}^{(m)}|\Theta_m) \quad (4)$$

where the sum is on the set $\{\mathbf{A}^{(m)}\}$ of matrices that lead to $\mathbf{A}^{(l)}$ under successive application of the renormalization rule, and Θ_l is a parameter depending only on Θ_m and the partitions.

It is shown by the authors that there is a unique solution to the problem under scale-invariance when only node-dependent fitnesses are included:

$$p_{i_l,j_l}(\delta) = 1 - e^{-\delta x_{i_l} x_{j_l}} \quad (5)$$

$$x_{i_{l+1}} = \sum_{i_l \in i_{l+1}} x_{i_l} \quad (6)$$

where x_{i_l} are node-specific predefined parameters, the *fitnesses*. A dyadic term can be inserted to express specific pairwise interactions. In this case the unique solutions is:

$$p_{i_l,j_l}(\delta) = 1 - e^{-\delta x_{i_l} x_{j_l} f(d_{i_l,j_l})} \quad (7)$$

where x_{i_l} and $f(d_{i_l,j_l})$ are subject to renormalization rules, and f is a positive function. The interested reader is referred to Garuccio et al. [8] for more detail.

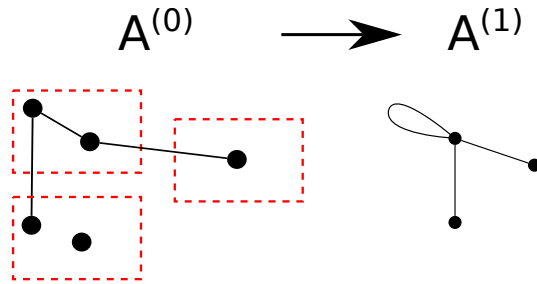


Fig. 2. Coarse-graining rule. The original graph on the left is coarse-grained, using the partition $\Omega^{(l)}$ represented by red squares.

This model is probabilistic, generative in the sense that one can sample from the set of adjacency matrices $\{A^{(l)}\}$ at all levels, and has an analytical expression

(in contrast to MCMC-based methods) which functional form is preserved across renormalization. Only one graph sample is required to fit the model. At any level $p_{i,j_l(\delta)}$ can be compared to other fitness-based models such as the FiCM model [7].