On the uniqueness of multi-breathers of the modified Korteweg-de Vries equation

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Abstract. We consider the modified Korteweg-de Vries equation (mKdV) and prove that given any sum P of solitons and breathers of (mKdV) (with distinct velocities), there exists a solution p of (mKdV) such that $p(t) - P(t) \to 0$ when $t \to +\infty$, which we call multi-breather. In order to do this, we work at the H^2 level (even if usually solitons are considered at the H^1 level). We will show that this convergence takes place in any H^s space and that this convergence is exponentially fast in time.

We also show that the constructed multi-breather is unique in two cases: in the class of solutions which converge to the profile P faster than the inverse of a polynomial of a large enough degree in time (we will call this a super polynomial convergence), or (without hypothesis on the convergence rate), when all the velocities, except possibly one, are positive.

1. Introduction

1.1. **Setting of the problem.** We consider the modified Korteweg-de Vries equation on $\mathbb{R}$:

$$\text{(mKdV)} \qquad \begin{cases} u_t + (u_{xx} + u^3)_x = 0 & (t,x) \in \mathbb{R}^2 \\ u(0) = u_0 & u(t,x) \in \mathbb{R} \end{cases}$$

The (mKdV) equation appears as a model of some physical problems as plasma physics [39, 9], electrodynamics [38], fluid mechanics [22], ferromagnetic vortices [46], and more.

In [24], Kenig, Ponce and Vega established local well-posedness in H^s, for $s \geq \frac{1}{4}$, of the Cauchy problem for (mKdV), by fixed point argument in $L^p_x L^q_t$ type spaces. Moreover, if $s > \frac{1}{4}$, the Cauchy problem is globally well posed [12]. Recently, Harrop-Griffiths, Killip and Visan [21] proved local well-posedness in H^s for $s > -1/2$. However, in this paper, we will only use the global well-posedness in H^2.

(mKdV) is an integrable equation (like the original Korteweg-de Vries equation) and thus it has an infinity of conservation laws, see [37, 1]. We will use three of them (the first two of them are called *mass* and *energy*; the third is sometimes called *second energy*):

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)\,dx, \tag{1.1.1}$$

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \frac{1}{4}\int_{\mathbb{R}} u^4(t,x)\,dx, \quad \text{and} \tag{1.1.2}$$

$$F[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_{xx}^2(t,x)\,dx - \frac{5}{2}\int_{\mathbb{R}} u^2(t,x)u_x^2(t,x)\,dx + \frac{1}{4}\int_{\mathbb{R}} u^6(t,x)\,dx. \tag{1.1.3}$$

Observe that if u is a solution of (mKdV) then $-u$ and, for any $x_0 \in \mathbb{R}$, $(t,x) \mapsto u(t, x - x_0)$ are solutions of (mKdV) too.

(mKdV) is a dispersive nonlinear equation that is a special case of a more general class of equations: the general Korteweg-de Vries equation (gKdV), where the nonlinearity u^3 is replaced by $f(u)$ for some real valued function f. The particularity of (mKdV) in comparison to other (gKdV) equation is that it admits special non linear solutions, namely breather solutions.

The most simple nonlinear solutions of (mKdV) are solitons, i.e. a bump of a constant shape that translates with a constant velocity without deformation, that is, solutions of the form $u(t,x) = Q_c(x - ct)$, where c is the velocity and Q_c is the profile function that depends only on one variable. $Q_c \in H^1(\mathbb{R})$ should solve the elliptic equation:

$$Q_c'' = cQ_c - Q_c^3. \tag{1.1.4}$$

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We can show that necessarily $c > 0$ and that, if $c > 0$, (1.1.4) has a unique solution in $H^1(\mathbb{R})$, up to translations and reflexion with respect to the x-axis. Actually, one has the explicit formula:

$$Q_c(x) := \left(\frac{2c}{\cosh^2\left(c^{1/2}x\right)}\right)^{\frac{1}{2}}. \tag{1.1.5}$$

Observe that we chose Q_c so that it is even and positive.

A *soliton* is a solution of (mKdV), parameterized by a velocity parameter $c > 0$, a sign parameter $\kappa \in \{-1, 1\}$ and a translation parameter $x_0 \in \mathbb{R}$ (it corresponds to the initial position of the soliton) that has the following expression:

$$R_{c,\kappa}(t, x; x_0) := \kappa Q_c(x - x_0 - ct). \tag{1.1.6}$$

When $\kappa = -1$, this object is sometimes called *antisoliton*. Notice that solitons are smooth and decaying. The generalized Korteweg-de Vries equation (gKdV) also admit soliton type solutions, and the focusing nonlinear Schrödinger equation (NLS) as well. Solitons have been extensively studied, in particular their stability. Cazenave, Lions and Weinstein in [45, 7, 8, 44] were interested in orbital stability of (gKdV) and (NLS) solitons in H^1. A soliton of (mKdV) is indeed orbitally stable, i.e. if a solution is initially close to a soliton in $H^1(\mathbb{R})$, then it stays close to the soliton, up to a space translation defined for any time, in $H^1(\mathbb{R})$. General results about orbital stability of nonlinear dispersive solitons are presented by Grillakis, Shatah and Strauss in [20]. The result about orbital stability of a soliton can be improved in a result of asymptotic stability, as it was done in the works by Martel and Merle [29, 33, 31], see also [17].

A *breather* is a solution of (mKdV), parameterized by $\alpha, \beta > 0$, $x_1, x_2 \in \mathbb{R}$ that has the following expression:

$$B_{\alpha,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x\left[\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\right)\right], \tag{1.1.7}$$

where

$$y_1 := x + \delta t + x_1 \quad \text{and} \quad y_2 := x + \gamma t + x_2,$$
$$\text{with} \quad \delta := \alpha^2 - 3\beta^2 \quad \text{and} \quad \gamma := 3\alpha^2 - \beta^2.$$

It corresponds to a localized periodic in time function (with frequency α, and exponential localization with decay rate β) that propagates at a constant velocity $-\gamma$ in time. Like solitons, breathers are smooth and decaying in space. Unlike solitons, breather's velocities can be positive, zero or negative. α, β are the shape parameters and x_1, x_2 are the translation parameters of a breather. Note that if we replace the parameter x_1 by $x_1 + \frac{\pi}{\alpha}$, we transform $B_{\alpha,\beta}(\cdot, \cdot; x_1, x_2)$ in $-B_{\alpha,\beta}(\cdot, \cdot; x_1, x_2)$ (therefore, we do not need to talk about "antibreathers").

Breathers were first introduced by Wadati in [42], and they were already used by Kenig, Ponce and Vega in [25] to prove that the flowmap associated to (mKdV) equation is *not* uniformly continuous in H^s for $s < \frac{1}{4}$: the point is that two breathers with close velocities can be very close at $t = 0$ and can separate as fast as we want in H^s with $s < \frac{1}{4}$, if α is taken large enough.

(mKdV) breathers and their properties, as well as breathers for other equations, are well studied by Alejo and Muñoz and co-authors in [3, 2, 5, 6, 4].

Let us singularize a result of H^2 orbital stability for breathers established in [3], and improved to H^1 orbital stability in [4]. In this last paper, a partial result of asymptotic stability is also given, for breathers traveling to the right only, with positive velocity $-\gamma > 0$; asymptotic stability for breathers in full generality is still an open problem.

When $\alpha \to 0$, $B_{\alpha,\beta}$ tends to a solution of (mKdV) called *double-pole solution* [43], the methods employed in this article as well as the proof of orbital stability made by Alejo and Muñoz seem not to apply for this limit, which is expected to be unstable according to the numerical computations in [18].

An important result regarding the long time dynamics of (mKdV) is the soliton-breather resolution [10]: it asserts that any generic solution can be approached by a sum of solitons and breathers when $t \to +\infty$ (up to a dispersive and a self-similar term). Together with their stability properties, the soliton-breather resolution shows why solitons and breathers are essential objects to study. This resolution was established for initial conditions in a weighted Sobolev space in [10] (see also Schuur

[40]) by inverse scattering method; see also [40] for the soliton resolution for (KdV). Observe that (mKdV) breathers do not decouple into simple solitons for large time (it is a *fully bounded state* as it is called in [3]); therefore, it must appear in the resolution. The soliton-breather resolution is one of the motivations of the study of multi-breathers, which we define below.

There are works in the literature about a more complicated object obtained from several solitons: a *multi-soliton*. A multi-soliton is a solution $r(t)$ of (mKdV) such that there exists $0 < c_1 < c_2 < ... < c_N$, $\kappa_1, ..., \kappa_N \in \{-1, 1\}$ and $x_1, ..., x_N \in \mathbb{R}$, such that

$$\lim_{t \to +\infty} \left\| r(t) - \sum_{j=1}^{N} R_{c_j,\kappa_j}(t, \cdot; x_j) \right\|_{H^1(\mathbb{R})} = 0. \tag{1.1.8}$$

This definition is not specific to (mKdV) and makes sense for many other nonlinear dispersive PDEs as soon as they admit solitons. This object is introduced by Schuur [40] and Lamb [26], see also Miura [36], where explicit formulas are given: these were obtained by inverse scattering method thanks to the integrability of the equation. Multi-solitons were first constructed in a non integrable context by Merle [34] for the mass critical (NLS). Martel [28] constructed multi-solitons for mass-subcritical and critical (gKdV) equations and proved that they are unique in $H^1(\mathbb{R})$, smooth and converge exponentially fast to their profile in any Sobolev space H^s. Similar studies were done for other nonlinear dispersive PDEs. Martel and Merle [32] have proved the existence of multi-solitons for (NLS) in H^1, Côte, Martel and Merle extended this construction to mass supercritical (gKdV) and (NLS) in [15]. Friederich and Côte in [14] proved smoothness, and uniqueness in a class of algebraic convergence. Côte and Muñoz constructed in [16] multi-solitons for the nonlinear Klein-Gordon equation. Ming, Rousset and Tzvetkov have constructed multi-solitons for the water-waves systems in [35]. Valet has proved in [41] the existence and uniqueness of multi-solitons in H^1 for the Zakharov-Kuznetsov equation, which generalizes (gKdV) to higher dimension.

1.2. **Main results.** We prove in this article that given any sum of solitons and breathers with distinct velocities, there exists a solution of (mKdV) whose difference with this sum tends to zero when time goes to infinity. This solution will be called a multi-breather. Let us make the definition more precise.

Let $J \in \mathbb{N}$ and $K, L \in \mathbb{N}$ such that $J = K + L$. We will consider a set of L solitons and K breathers:

- the breather parameters are $\alpha_k > 0$, $\beta_k > 0$, $x^0_{1,k} \in \mathbb{R}$ and $x^0_{2,k} \in \mathbb{R}$ for $1 \leq k \leq K$.
- the solitons parameters are $c_l > 0$, $\kappa_l \in \{-1, 1\}$ and $x^0_{0,l} \in \mathbb{R}$ for $1 \leq l \leq L$.

We define for $1 \leq k \leq K$, the kth breather:

$$B_k(t, x) := B_{\alpha_k,\beta_k}(t, x; x^0_{1,k}, x^0_{2,k}); \tag{1.2.1}$$

and for $1 \leq l \leq L$, the lth soliton:

$$R_l(t, x) := R_{c_l,\kappa_l}(t, x; x^0_{0,l}). \tag{1.2.2}$$

We now define the *velocity* of our objects. Recall that for $1 \leq k \leq K$, the velocity of B_k is

$$v^b_k := -\gamma_k = \beta_k^2 - 3\alpha_k^2, \tag{1.2.3}$$

and for $1 \leq l \leq L$, the velocity of R_l is

$$v^s_l := c_l. \tag{1.2.4}$$

The most important assumption we make is that all these velocities are distinct:

$$\forall k \neq k' \quad v^b_k \neq v^b_{k'}, \qquad \forall l \neq l' \quad v^s_l \neq v^s_{l'}, \qquad \forall k, l \quad v^b_k \neq v^s_l. \tag{1.2.5}$$

These implies for any two of these objects to be far from each other when time is large, and this assumption is essential in our analysis.

It will be useful to order our breathers and solitons by increasing velocities. As these are distinct, we can define an increasing function:

$$\underline{v} : \{1, ..., J\} \longrightarrow \{v^b_k, 1 \leq k \leq K\} \cup \{v^s_l, 1 \leq l \leq L\}. \tag{1.2.6}$$

The set $\{v_1, ..., v_J\}$ is thus the (ordered) set of all possible velocities of our objects. We define P_j, for $1 \leq j \leq J$, as the object (either a soliton R_l or a breather B_k) that corresponds to the velocity v_j. Hence, $P_1, ..., P_J$ are the considered objects ordered by increasing velocity.

We will need both notations: the indexation by k and l, and the indexation by j, and we will keep these notations to avoid ambiguity.

We will denote by x_j the center of mass of P_j, that is

- if $P_j = B_k$ is a breather, we set $x_j(t) := -x^0_{2,k} + v_j t$;
- if $P_j = R_l$ is a soliton, we set $x_j(t) := x^0_{0,l} + v_j t$.

We denote:

$$R = \sum_{l=1}^{L} R_l, \quad B = \sum_{k=1}^{K} B_k, \quad P = R + B = \sum_{j=1}^{J} P_j. \tag{1.2.7}$$

We can now define a multi-breather: as solitons are objects which can be studied naturally in $H^1(\mathbb{R})$, it turns out that breathers are best studied in $H^2(\mathbb{R})$; therefore, it is in this latter space that we develop our analysis.

Definition 1.1. A *multi-breather* associated to the sum P given in (1.2.7) of solitons and breathers is a solution $p \in C([T^*, +\infty), H^2(\mathbb{R}))$, for a constant $T^* > 0$, of (mKdV) such that

$$\lim_{t \to +\infty} \|p(t) - P(t)\|_{H^2} = 0. \tag{1.2.8}$$

We will prove two results in this article. The first one is the existence and the regularity of a multi-breather, the second one is the uniqueness of a multi-breather. The uniqueness is established in two settings: in the case when all velocities (except possibly one) are positive, and without any assumption on the sign of the considered velocities. However, in the last case, the uniqueness is obtained in a narrower class of functions.

Theorem 1.2. *Given solitons and breathers (1.2.1), (1.2.2) whose velocities (1.2.3) and (1.2.4) satisfy (1.2.5), there exists a multi-breather p associated to P given in (1.2.7). Moreover,*

$$p \in C^\infty(\mathbb{R} \times \mathbb{R}) \cap C^\infty(\mathbb{R}, H^s(\mathbb{R}))$$

for any $s \geq 0$ and there exists $\theta > 0$ such that for any $s \geq 0$, there exists $A_s \geq 1$ and $T^* > 0$ such that

$$\forall t \geq T^*, \quad \|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \tag{1.2.9}$$

Remark 1.3. We will also show that θ does only depend on the shape parameters of our objects: α_k, β_k, c_l. Moreover, if there exists $D > 0$ such that for all $j \geq 2$, $x_j(0) \geq x_{j-1}(0) + D$, then A_s and T^* do not depend on $x^0_{1,k}, x^0_{2,k}, x^0_{0,l}$ but only on α_k, β_k, c_l and D. Finally, if $D > 0$ is large enough with respect to the problem data, then (1.2.9) is true for $T^* = 0$. See Section 3.2 for further details.

Theorem 1.4. *Given the same set of solitons and breathers as in Theorem 1.2 whose velocities satisfy (1.2.5) and $v_2 > 0$ (so that all the velocities, except possibly one, are positive), the multi-breather p associated to P by Theorem 1.2, in the sense of Definition 1.1, is unique.*

Proposition 1.5. *Given the same set of solitons and breathers as in Theorem 1.2 whose velocities satisfy (1.2.5), there exists $N > 0$ large enough such that the multi-breather p associated to P by Theorem 1.2 is the unique solution $u \in C([T_0, +\infty), H^2(\mathbb{R}))$ of* (mKdV) *such that*

$$\|u(t) - P(t)\|_{H^2} = O\left(\frac{1}{t^N}\right), \qquad \text{as } t \to +\infty. \tag{1.2.10}$$

In [43], there exists a formula for a multi-breather, obtained by inverse scattering method, that in some sense already gives the existence of a multi-breather. However, the proof of the theorem 1.2 from this formula is rather involved.

In this paper, we give here a different approach to prove the existence of a multi-breather and we clearly show that we have convergence of the constructed multi-breather to the corresponding sum of solitons and breathers in H^s, that this convergence is exponentially fast in time and that the constructed multi-breather is smooth. To do this, we use the variational structure of solitons and breathers. This is why, we give a proof that is potentially generalizable to non integrable equations, and that uses similar type of techniques as in the proof of the uniqueness (the latter cannot be deduced from the formula). In any case, uniqueness of multi-breathers is new.

In this paper, we adapt the arguments given by Martel and Merle [32], by Martel [28] and by Côte and Friederich [14] to the context of breathers. To do so, one needs to understand the variational

structure of breathers, in the same fashion as Weinstein did in [45] for (NLS) solitons. Such results were obtained by Alejo and Muñoz in [3]: a breather is a critical point of a Lyapunov functional at the H^2 level, whose Hessian is coercive up to several (but finitely many) orthogonal conditions, see Section 2 for details. As we see from [3], the H^2 regularity level is the most natural setting to study breathers, and the H^1 regularity level is natural for the study of solitons (as we see in [28, 32]). One important issue we face is therefore to understand soliton variational structure at H^2 level, and to adapt the Lyapunov functional in [3] to accommodate for a sum of breathers (and solitons). Notice that arguments based on monotonicity may be adapted only if we suppose that all the considered velocities, except possibly one, are positive. Because [32, 14] are not based on monotonicity (these are results for (NLS) which is not well suited for monotonicity), we can adapt their arguments to obtain existence and uniqueness results for our case without any condition on the sign of velocities. The uniqueness result obtained in this setting is however weaker than the one that is obtained with monotonicity arguments.

1.3. **Outline of the proof.** The proof of Theorem 1.2 (the existence of multi-breathers) is split into two main parts: the construction of an H^2 multi-breather and the proof that this multi-breather is smooth.

1.3.1. *An H^2 multi-breather.* Let us start with the first part. We consider an increasing sequence (T_n) of $\mathbb{R}_+$ with $T_n \to +\infty$, and for $n \in \mathbb{N}$, let p_n the unique global H^2 solution of (mKdV) such that $p_n(T_n) = P(T_n)$ (recall that the Cauchy problem for (mKdV) is globally well-posed in H^2).

We will prove the following *uniform estimate*:

Proposition 1.6. *There exists $T^* > 0$, $A > 0$, $\theta > 0$ such that, for any $n \in \mathbb{N}$ such that $T_n \geq T^*$,*

$$\forall t \in [T^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \tag{1.3.1}$$

With this proposition in hand, we can construct an H^2 multi-breather which converges exponentially fast to its profile, which is the first part of Theorem 1.2, as stated below.

Proposition 1.7. *There exists $T^* \in \mathbb{R}$, $A > 0$, $\theta > 0$ and a solution $p \in C([T^*, +\infty), H^2(\mathbb{R}))$ of* (mKdV) *such that*

$$\forall t \geq T^*, \quad \|p(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \tag{1.3.2}$$

Proof of Proposition 1.7 assuming Proposition 1.6. We show that the sequence $\big(p_n(T^*)\big)$ is L^2-compact, in the following sense:

Lemma 1.8. *For any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\forall n \in \mathbb{N} \quad \int_{|x|>R} p_n^2(T^*, x)\, dx < \varepsilon. \tag{1.3.3}$$

An analogous lemma has already been proved on p. 1111 of [28], which is the proof of formula (14) (and can also be found in [32]). The same proof works here. We need to use Proposition 1.6 for T_n large enough and then make a time variation to obtain the result in T^*. We can first find R that works for $P^2(t_0)$ instead of $p_n^2(T^*)$ for a fixed $t_0 > T^*$ large enough. From Proposition 1.6, we see that if we take t_0 large enough, we obtain the desired lemma for $p_n^2(t_0)$ instead of $p_n^2(T^*)$. To finish, with the help of a cut-off function, we control time variations of $\int_{|x|>R} p_n^2(t)\, dx$, where R is taken larger if needed. This is why, we obtain the result at $t = T^*$.

As a consequence of the Proposition 1.6 above, $(\|p_n(T^*)\|_{H^2})$ is a bounded sequence. Thus, there exists $p^* \in H^2(\mathbb{R})$ such that, up to a subsequence,

$$p_n(T^*) \rightharpoonup p^* \quad \text{in } H^2. \tag{1.3.4}$$

Thus, from Lemma 1.8, there holds the strong convergence:

$$p_n(T^*) \to p^* \quad \text{in } L^2. \tag{1.3.5}$$

Therefore, we obtain by interpolation:

$$p_n(T^*) \to p^* \quad \text{in } H^1. \tag{1.3.6}$$

Now, let us consider the global H^1 (even H^2) solution p of (mKdV) such that $p(T^*) = p^*$. As shown in [28], the Cauchy problem for (mKdV) has a continuous dependence in H^1 on compact sets

of time. Let $t \geq T^*$. By continuous dependence, we deduce that $p_n(t) \to p(t)$ in H^1. $(p_n(t) - P(t))$ is a bounded sequence in H^2, which admits a unique weak limit and so

$$p_n(t) - P(t) \rightharpoonup p(t) - P(t) \quad \text{in } H^2. \tag{1.3.7}$$

By weak convergence and from Proposition 1.6, we obtain:

$$\|p(t) - P(t)\|_{H^2} \leq \liminf_{n \to +\infty} \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \tag{1.3.8}$$

As this is true for any $t \geq T^*$. This ends the proof of the Proposition 1.7. □

It remains to prove Proposition 1.6, for which we rest on a bootstrap argument. More precisely, we will reduce the proof to the following proposition:

Proposition 1.9. *There exists $T^* > 0$, $A > 0$, $\theta > 0$, such that for any $n \in \mathbb{N}$ such that $T_n \geq T^*$, for any $t^* \in [T^*, T_n]$, if*

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}, \tag{1.3.9}$$

then

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq \frac{A}{2} e^{-\theta t}. \tag{1.3.10}$$

The proof of Proposotion 1.6 then follows from a simple continuity argument.

Proof of Proposition 1.6 assuming Proposition 1.9. We define t_n^* in the following way:

$$t_n^* := \inf\{t^* \in [T^*, T_n), \quad \forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}\}. \tag{1.3.11}$$

The map $t \mapsto \|p_n(t) - P(t)\|_{H^2}$ is a continuous function and $\|p_n(T_n) - P(T_n)\|_{H^2} = 0$. This means that there exists $T^* \leq t^* < T_n$ such that

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}. \tag{1.3.12}$$

Therefore, we have that

$$T^* \leq t_n^* < T_n. \tag{1.3.13}$$

We would like to prove that $t_n^* = T^*$. Let us argue by contradiction and assume that $t_n^* > T^*$. The Proposition 1.9 allows us to deduce that

$$\forall t \in [t_n^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leq \frac{A}{2} e^{-\theta t}. \tag{1.3.14}$$

This means that

$$\|p_n(t_n^*) - P(t_n^*)\|_{H^2} \leq \frac{A}{2} e^{-\theta t_n^*}, \tag{1.3.15}$$

which means that t_n^* could be chosen smaller, by continuity. This is a contradiction. □

Hence, we are left to prove Proposition 1.9, which will be done in Section 2.

1.3.2. *The H^2 multi-breather is smooth.* We now turn to the second part of Theorem 1.2, which is strongly adapted from [28]. The heart of this part is to prove uniform estimates in H^s for $p_n - P$, for any $s \geq 0$:

Proposition 1.10. *There exists $T^* > 0$, $\theta > 0$, such that for any $s \geq 0$, there exists $A_s \geq 1$ such that for any $n \in \mathbb{N}$ such that $T_n \geq T^*$,*

$$\forall t \in [T^*, T_n], \quad \|p_n(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \tag{1.3.16}$$

With this improved version of Proposition 1.6, one can prove by the same reasonning as in the proof of the Proposition 1.7, that for any $s \geq 0$, p actually belongs to $L^\infty([T^*, +\infty), H^s(\mathbb{R}))$ and that the convergence of $p(t) - P(t)$ occurs in H^s with an exponential decay rate. More precisely,

Theorem 1.11. *For any $s \geq 2$, we have that $p \in C([T^*, +\infty), H^s(\mathbb{R}))$, and furthermore,*

$$\forall t \geq T^*, \quad \|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \tag{1.3.17}$$

It remains to prove Proposition 1.10, which will be done in Section 3.

1.3.3. *The uniqueness result.* We denote p the multi-breather constructed in the previous sections, the existence of which is established. Let u be a solution of (mKdV) such that

$$\|u - P\|_{H^2} \to_{t\to+\infty} 0. \tag{1.3.18}$$

Equivalently, there holds:

$$\|u - p\|_{H^2} \to_{t\to+\infty} 0. \tag{1.3.19}$$

We denote

$$z := u - p. \tag{1.3.20}$$

The goal is to prove that $z = 0$. We prove it in two configurations: when all the velocities, except possibly one, are positive (Theorem 1.4), and without any assumption on velocities (Proposition 1.5), but in this last case we need to assume a stronger convergence than given in (1.3.18).

The proof of Theorem 1.4 will be carried out in two steps.

We start with Proposition 1.5, which is adapted from [14]. For this, we do not study $u - P$ anymore, we deal only with $z = u - p$. z is the difference of two solutions of (mKdV), which is much more precise than $u - P$. Thus, we do not modulate parameters of the solitons, as it is needed in other parts of the proof in order to deal with the soliton part of the linear part of the Lyapunov functional, and we avoid some difficulty. In order to prove our inequalities, we need again to use coercivity of the same type of quadratic forms. In order to do this, we replace z by $\widetilde{z} = z + \sum_{j=1}^{J} c_j K_j$, where K_j, $j = 1, ..., J$ is a well chosen basis of the kernel of the quadratic form, in order to have $\widetilde{z}$ orthogonal to any K_j. A important idea is to use slow variations of localized functionals with adapted cut-off functions of the form $\varphi\left(\frac{x-vt}{\delta t}\right)$, which provides an extra $O(1/t)$ decay when derivatives fall on the cut-off, and ultimately explain why algebraic decay comes into play.

In the context of Theorem 1.4, we actually prove that

$$v := u - P \tag{1.3.21}$$

converges exponentially fast to 0: this is the purpose of Proposition 4.10, which uses some ideas of [28]. Due to Proposition 1.5, we deduce immediately from there that an exponential convergence is trivial, that is $z = 0$.

To prove Proposition 4.10, we use monotonicity properties combined with coercivity of an energy type functional very similar to that used for the existence result. This is why, we also need to modulate, and the choice of the orthogonality condition is essential: it allows to bound linear terms in w that appear in the computations. An issue of the mixed breathers/solitons context is that one cannot build a functional adapted to all the nonlinear objects at once, as it is done in [28]. Instead, we carry out an induction and we argue successively around each object, soliton or breather, separately.

1.3.4. *Organisation of the paper.* Sections 2 and 3 are devoted to the proof of the existence of a multi-breather: Proposition 1.9 is proved in Section 2, Proposition 1.10 is proved in Section 3. Section 4 gathers the proofs of the uniqueness results: Section 4.1 is devoted to the proof of Proposition 1.5, and Sections 4.2 and 4.3 are devoted to the proof of Theorem 1.4.

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2. Construction of a multi-breather in $H^2(\mathbb{R})$

We set

$$\begin{aligned} \beta &:= \min\{\beta_k, 1 \le k \le K\} \cup \{\sqrt{c_l}, 1 \le l \le L\}, \\ \tau &:= \min\{v_{j+1} - v_j, 1 \le j \le J-1\}. \end{aligned} \tag{2.0.1}$$

Our goal in this section is to prove Proposition 1.9.

2.1. **Elementary results.** Let us first collect a few basic facts that will be used throughout the article. One may check an exponential decay result for any of our objects:

Proposition 2.1. *Let $j = 1, ..., J$, $n, m \in \mathbb{N}$. Then, there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \partial_t^m P_j(t,x)| \leq Ce^{-\beta|x-v_j t|}. \tag{2.1.1}$$

Corollary 2.2. *Let $r > 0$. For t, x such that $v_j t + r < x < v_{j+1} t - r$, we have that*

$$|P(t,x)| \leq Ce^{-\beta r}. \tag{2.1.2}$$

The same is true for any space or time derivative of P.

We will also use the following cross-product result:

Proposition 2.3. *Let $i \neq j \in \{1, ..., J\}$ and $m, n \in \mathbb{N}$. There exists a constant C that depends only on P, such that for any $t \in \mathbb{R}$,*

$$\left| \int \partial_x^m P_i \partial_x^n P_j \right| \leq Ce^{-\beta\tau t/2}. \tag{2.1.3}$$

There is also an orthogonality result for breathers that will be useful:

Lemma 2.4. *Let $B := B_{\alpha,\beta}$ be a breather. We denote $B_1 := \partial_{x_1} B$ and $B_2 := \partial_{x_2} B$. Then,*

$$\int BB_1 = \int BB_2 = 0. \tag{2.1.4}$$

Proof. Note that $\mathrm{Span}(B_1, B_2) = \mathrm{Span}(B_x, B_t)$. Therefore, it is enough to prove that

$$\int BB_x = \int BB_t = 0. \tag{2.1.5}$$

Firstly,

$$\int BB_x = \frac{1}{2} \int \left(B^2\right)_x = 0. \tag{2.1.6}$$

Secondly,

$$\int BB_t = \frac{1}{2} \int \left(B^2\right)_t = \frac{1}{2} \frac{d}{dt} \int B^2 = 0, \tag{2.1.7}$$

by mass conservation and because a breather is a solution of (mKdV). □

2.2. **Almost-conservation of localized conservation laws.** From now on, we will fix $n \in \mathbb{N}$. This is why, for the simplicity of notations, we can write T for T_n, and p for p_n. The goal will be to find constants $T^*, A > 1, \theta$ that do not depend on n, nor on the translation parameters of the given objects, and that will be chosen later (T^* will depend on A and θ), such that Proposition 1.9 is verified. We will take $t^* \in [T^*, T]$, and we will make the following bootstrap assumption for the remaining of the article:

$$\forall t \in [t^*, T], \quad \|p(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}, \tag{2.2.1}$$

where $p(T) = P(T)$.

Remark 2.5. We have the following property for solutions of (mKdV): there exists $C_0 > 0$ such that for any solution w of (mKdV), w is global and

$$\forall t \in \mathbb{R}, \quad \|w(t)\|_{H^2} \leq C_0 \|w(T)\|_{H^2}. \tag{2.2.2}$$

Therefore,

$$\forall t \in \mathbb{R}, \quad \|p(t)\|_{H^2} \leq C_0 \|P(T)\|_{H^2} \leq C_0 \sum_{j=1}^{J} \|P_j(T)\|_{H^2} \leq C_0 C, \tag{2.2.3}$$

where C is a constant that depends only on the problem data (because the H^s-norm of solitons or breathers can be easily bounded).

Let $\theta := \frac{\beta\tau}{32}$. Let $\min(1, \frac{\tau}{4}) > \delta > 0$ be a constant to be chosen later.

This part of the proof is adapted from [32].

Let $\psi(x)$ be a C^3 function such that

$$\begin{aligned} &0 \leq \psi \leq 1 \quad \text{on } \mathbb{R}, \quad \psi' \geq 0 \quad \text{on } \mathbb{R}, \\ &\psi(x) = 0 \quad \text{for } x \leq -1, \quad \psi(x) = 1 \quad \text{for } x \geq 1, \end{aligned} \tag{2.2.4}$$

and satisfying, for a constant $C > 0$, for any $x \in \mathbb{R}$,

$$(\psi'(x))^{4/3} \leq C\psi(x), \quad (\psi'(x))^{4/3} \leq C(1 - \psi(x)), \quad |\psi''(x)|^{\frac{3}{2}} \leq C\psi'(x). \tag{2.2.5}$$

Note that it is enough to take ψ that is equal to $(1 + x)^4$ on a neighbourhood of -1 and equal to $1 - (-1 + x)^4$ on a neighbourhood of 1.

These conditions on ψ will be needed for the proof of Proposition 2.19.

For any $j = 2, ..., J$, let

$$\sigma_j := \frac{1}{2}(v_{j-1} + v_j). \tag{2.2.6}$$

For any $j = 2, ..., J - 1$, let

$$\varphi_j(t, x) := \psi\left(\frac{x - \sigma_j t}{\delta t}\right) - \psi\left(\frac{x - \sigma_{j+1} t}{\delta t}\right), \tag{2.2.7}$$

$$\varphi_1(t, x) := 1 - \psi\left(\frac{x - \sigma_2 t}{\delta t}\right), \qquad \varphi_J(t, x) := \psi\left(\frac{x - \sigma_J t}{\delta t}\right), \tag{2.2.8}$$

so that the function φ_j corresponds obviously to the object P_j. We will also use notations φ_l^s and φ_k^b, which represent the same functions, and where φ_l^s corresponds to the soliton R_l and φ_k^b corresponds to the breather B_k.

We will also denote, for $j = 2, ..., J - 1$,

$$\varphi_{1,j}(t, x) := \psi'\left(\frac{x - \sigma_j t}{\delta t}\right) - \psi'\left(\frac{x - \sigma_{j+1} t}{\delta t}\right), \tag{2.2.9}$$

$$\varphi_{1,1}(t, x) := -\psi'\left(\frac{x - \sigma_2 t}{\delta t}\right), \qquad \varphi_{1,J}(t, x) := \psi'\left(\frac{x - \sigma_J t}{\delta t}\right). \tag{2.2.10}$$

Of course, notations $\varphi_{1,k}^b$, $\varphi_{1,l}^s$ or $\varphi_{2,j}$ will be used, with similar obvious definitions.

We have that, for $j = 1, ..., J$,

$$|\varphi_{1,j}| \leq C\varphi_j^{\frac{3}{4}}. \tag{2.2.11}$$

Remark 2.6. If $\delta \leq \frac{\tau}{4}$,

$$\begin{aligned} \int_{-\infty}^{\sigma_j t + \delta t} e^{-2\beta|x - v_j t|}\, dx &= e^{-2\beta v_j t} \int_{-\infty}^{\sigma_j t + \delta t} e^{2\beta x}\, dx \\ &= \frac{1}{2\beta} e^{-2\beta v_j t} e^{\beta(v_j + v_{j-1})t} e^{2\beta\delta t} \\ &\leq C e^{-\beta\tau t} e^{2\beta\delta t} \leq C e^{-\beta\tau t/2}, \end{aligned} \tag{2.2.12}$$

and

$$\int_{\sigma_{j+1} t - \delta t}^{+\infty} e^{-2\beta|x - v_j t|}\, dx \leq C e^{-\beta\tau t/2}, \tag{2.2.13}$$

for the same reason, and if $i \neq j$, e.g. $j > i$,

$$\begin{aligned} \int_{\sigma_j t - \delta t}^{\sigma_{j+1} t + \delta t} e^{-2\beta|x - v_i t|}\, dx &= e^{2\beta v_i t} \int_{\sigma_j t - \delta t}^{\sigma_{j+1} t + \delta t} e^{-2\beta x}\, dx \\ &\leq \frac{1}{2\beta} e^{2\beta v_i t} e^{-\beta(v_j + v_{j-1})t} e^{2\beta\delta t} \\ &\leq C e^{-\beta\tau t} e^{2\beta\delta t} \leq C e^{-\beta\tau t/2}. \end{aligned} \tag{2.2.14}$$

And finally, we set for all $j = 1, ..., J$:

$$\begin{aligned} M_j(t) &:= \int \frac{1}{2} p^2(t,x)\varphi_j(t,x)\,dx =: M_j[p](t), \\ E_j(t) &:= \int \left(\frac{1}{2}p_x^2(t,x) - \frac{1}{4}p^4(t,x)\right)\varphi_j(t,x)\,dx =: E_j[p](t). \end{aligned} \tag{2.2.15}$$

Notations $M_l^s, M_k^b, E_l^s, E_k^b$ will also be used.

These are local versions of the mass and the energy of the solution p considered (localized around each breather or soliton). We will prove the following result for the localized mass and energy:

Lemma 2.7. *There exists $C > 0$ and $T_1^* := T_1^*(A)$ such that, if $T^* \geq T_1^*$, then for any $j = 1, ..., J$, for any $t \in [t^*, T]$,*

$$|M_j(T) - M_j(t)| + |E_j(T) - E_j(t)| \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}. \tag{2.2.16}$$

Proof. We will use the results of the computations made on the bottom of page 1115 and on the bottom of page 1116 of [28] to claim the following facts:

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int p^2 f &= \int \left(-\frac{3}{2}p_x^2 + \frac{3}{4}p^4\right) f' - \int p_x p f'', \\ \frac{d}{dt}\int \left[\frac{1}{2}p_x^2 - \frac{1}{4}p^4\right] f &= \int \left[-\frac{1}{2}(p_{xx} + p^3)^2 - p_{xx}^2 + 3p_x^2 p^2\right] f' \\ &\quad - \int p_{xx} p_x f'', \end{aligned} \tag{2.2.17}$$

where f is a C^2 function that does not depend on time.

$M_j(t)$ is a sum of quantities of the form $\frac{1}{2}\int p^2 \psi(\frac{x-\sigma_j t}{\delta t})$. This is why, we compute:

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int p^2 \psi\left(\frac{x-\sigma_j t}{\delta t}\right) &= \frac{1}{\delta t}\int \left(-\frac{3}{2}p_x^2 + \frac{3}{4}p^4\right)\psi'\left(\frac{x-\sigma_j t}{\delta t}\right) \\ &\quad - \frac{1}{(\delta t)^2}\int p_x p \psi''\left(\frac{x-\sigma_j t}{\delta t}\right) - \frac{1}{2}\int p^2 \frac{x}{\delta t^2}\psi'\left(\frac{x-\sigma_j t}{\delta t}\right). \end{aligned} \tag{2.2.18}$$

$\psi'(\frac{x-\sigma_j t}{\delta t})$ is zero outside of $\Omega_j(t) := (-\delta t + \sigma_j t, \delta t + \sigma_j t)$. Thus, for $x \in \Omega_j(t)$, $\left|\frac{x}{t}\right| \leq |\sigma_j| + |\delta| \leq |\sigma_j| + 1$, this means that $|\frac{x}{t}|$ is bounded by a constant (that depends only on the given parameters). We can deduce that

$$\left|\frac{d}{dt}\frac{1}{2}\int p^2 \psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right| \leq \frac{C}{\delta^2 t}\left(\int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p^4 + \int_{\Omega_j(t)} p^2\right). \tag{2.2.19}$$

We bound $\int_{\Omega_j(t)} p^4$ by the following way:

$$\begin{aligned} \int_{\Omega_j(t)} p^4 &\leq \|p\|_{L^\infty}^2 \int_{\Omega_j(t)} p^2 \\ &\leq C\|p\|_{H^1}^2 \int_{\Omega_j(t)} p^2 \qquad \text{by Sobolev embedding} \\ &\leq C\int_{\Omega_j(t)} p^2 \qquad \text{by Remark 2.5.} \end{aligned} \tag{2.2.20}$$

Thus, we have for any $t \in [t^*, T]$,

$$\left|\frac{d}{dt}\frac{1}{2}\int p^2 \psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right| \leq \frac{C}{\delta^2 t}\left(\int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p^2\right). \tag{2.2.21}$$

$E_j(t)$ is a sum of quantities of the form $\int[\frac{1}{2}p_x^2 - \frac{1}{4}p^4]\psi(\frac{x-\sigma_j t}{\delta t})$. So, we compute:

$$\begin{aligned}
&\frac{d}{dt}\int\left[\frac{1}{2}p_x^2-\frac{1}{4}p^4\right]\psi\left(\frac{x-\sigma_j t}{\delta t}\right)\\
&\quad=\frac{1}{\delta t}\int\left[-\frac{1}{2}(p_{xx}+p^3)^2-p_{xx}^2+3p_x^2p\right]\psi'\left(\frac{x-\sigma_j t}{\delta t}\right)\\
&\quad\quad-\frac{1}{(\delta t)^2}\int p_{xx}p_x\psi''\left(\frac{x-\sigma_j t}{\delta t}\right)-\int\left[\frac{1}{2}p_x^2-\frac{1}{4}p^4\right]\frac{x}{\delta t^2}\psi'\left(\frac{x-\sigma_j t}{\delta t}\right).
\end{aligned}\tag{2.2.22}$$

We deduce from this, by using similar arguments as for the mass, that for any $t\in[t^*,T]$,

$$\left|\frac{d}{dt}\int\left[\frac{1}{2}p_x^2-\frac{1}{4}p^4\right]\psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right|\le\frac{C}{\delta^2 t}\left(\int_{\Omega_j(t)}p^2+\int_{\Omega_j(t)}p_x^2+\int_{\Omega_j(t)}p_{xx}^2\right).\tag{2.2.23}$$

Now, we write $p(t)=P(t)+(p(t)-P(t))$, and we use the triangular inequality:

$$\int_{\Omega_j(t)}\left(p^2+p_x^2+p_{xx}^2\right)\le 2\int_{\Omega_j(t)}\left(P^2+P_x^2+P_{xx}^2\right)+2\|p-P\|_{H^2}^2.\tag{2.2.24}$$

We have assumed that $\|p-P\|_{H^2}^2\le A^2e^{-2\theta t}$, so we need to study P on $\Omega_j(t)$. The following computations work also for the derivatives of P:

$$\begin{aligned}
\int_{\Omega_j(t)}P^2&=\int_{\Omega_j(t)}\left(\sum_{m=1}^{J}P_m(t,x)\right)^2dx=\sum_{1\le m,l\le J}\int_{\Omega_j(t)}P_m(t,x)P_l(t,x)\,dx\\
&\le C\sum_{1\le m,l\le J}\int_{\Omega_j(t)}e^{-\beta|x-v_mt|}e^{-\beta|x-v_lt|}\,dx,
\end{aligned}\tag{2.2.25}$$

where we use the Proposition 2.1.

We assume that $m\ge j$ (we argue similarly if $m\le j-1$). Then,

$$\begin{aligned}
x\in\Omega_j(t)&\Leftrightarrow-\delta t+\sigma_j t\le x\le\delta t+\sigma_j t\\
&\Leftrightarrow-\delta t+(\sigma_j-v_m)t\le x-v_mt\le\delta t+(\sigma_j-v_m)t.
\end{aligned}\tag{2.2.26}$$

We note that $\sigma_j-v_m\le-\frac{1}{2}\tau<0$, we can thus deduce from the condition on δ that $\sigma_j-v_m+\delta\le-\frac{1}{4}\tau<0$. We deduce that $x-v_mt$ is negative for $x\in\Omega_j(t)$. Similarly, if $m\le j-1$, $x-v_mt$ is positive for $x\in\Omega_j(t)$. We will now make calculations for different cases. If $m,l\le j-1$,

$$\begin{aligned}
\int_{\Omega_j(t)}e^{-\beta|x-v_mt|}e^{-\beta|x-v_lt|}\,dx&\le\int_{\Omega_j(t)}e^{-\beta(x-v_mt)}e^{-\beta(x-v_lt)}\,dx\\
&=\frac{1}{2\beta}e^{\beta t(-v_j-v_{j-1}+v_m+v_l)}(e^{2\beta\delta t}-e^{-2\beta\delta t})\\
&\le Ce^{\beta t(-v_j-v_{j-1}+v_m+v_l+2\delta)}\le Ce^{-\beta\tau t/2}.
\end{aligned}\tag{2.2.27}$$

Similarly, if $m,l\ge j$,

$$\int_{\Omega_j(t)}e^{-\beta|x-v_mt|}e^{-\beta|x-v_lt|}\,dx\le Ce^{-\beta\tau t/2}.\tag{2.2.28}$$

And, if $m\le j-1, l\ge j$,

$$\begin{aligned}
\int_{\Omega_j(t)}e^{-\beta|x-v_mt|}e^{-\beta|x-v_lt|}\,dx&\le\int_{\Omega_j(t)}e^{-\beta(x-v_mt)}e^{\beta(x-v_lt)}\,dx\\
&\le 2\delta te^{\beta t(v_m-v_l)}\le Ce^{-\frac{\beta\tau t}{2}}.
\end{aligned}\tag{2.2.29}$$

Thus,

$$\int_{\Omega_j(t)}P^2\le Ce^{-\frac{\beta\tau t}{2}},\tag{2.2.30}$$

and the same is valid for the derivatives of P.

Thus, for $t \in [t^*, T]$,

$$\sum_{j=1}^{J} \left| \frac{d}{dt} \frac{1}{2} \int p^2 \psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right| + \left| \frac{d}{dt} \int \left[\frac{1}{2} p_x^2 - \frac{1}{4} p^4\right] \psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right| \tag{2.2.31}$$
$$\leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t} + \frac{C}{\delta^2 t} e^{-\frac{\beta \tau t}{2}} \leq \frac{C}{\delta^2 t}(A^2 + e^{-2\theta t}) e^{-2\theta t} \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}.$$

Thus, for $j = 1, ..., J$, $t \in [t^*, T]$,

$$\begin{aligned} &|M_j(T) - M_j(t)| + |E_j(T) - E_j(t)| \\ &\leq \int_t^T \frac{C}{\delta^2 s} A^2 e^{-2\theta s}\, ds \leq \frac{C}{\delta^2 t} A^2 \int_t^T e^{-2\theta s}\, ds \\ &= \frac{C}{\delta^2 t} A^2 \frac{1}{2\theta}(e^{-2\theta t} - e^{-2\theta T}) \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}. \end{aligned} \tag{2.2.32}$$

□

2.3. **Modulation.**

Lemma 2.8. *There exists $C > 0$, $T_2^* = T_2^*(A)$ such that, if $T^* > T_2^*$, then there exist unique C^1 functions $x_{1,k} : [t^*, T] \to \mathbb{R}$, $x_{2,k} : [t^*, T] \to \mathbb{R}$ for $1 \leq k \leq K$ and $x_{0,l} : [t^*, T] \to \mathbb{R}$, $c_{0,l} : [t^*, T] \to \mathbb{R}$, such that if we set*

$$\varepsilon(t,x) = p(t,x) - \widetilde{B}(t,x) - \widetilde{R}(t,x) = p(t,x) - \widetilde{P}(t,x), \tag{2.3.1}$$

where, for $1 \leq k \leq K$,

$$\widetilde{B}(t,x) = \sum_{k=1}^{K} \widetilde{B_k}(t), \quad \widetilde{B_k}(t,x) = B_{\alpha_k,\beta_k}(t, x; x_{1,k}^0 + x_{1,k}(t), x_{2,k}^0 + x_{2,k}(t)), \tag{2.3.2}$$

for $1 \leq l \leq L$,

$$\widetilde{R}(t,x) := \sum_{l=1}^{L} \widetilde{R}_l(t), \quad \widetilde{R}_l(t,x) := \kappa_l Q_{c_l + c_{0,l}(t)}(x - x_{0,l}^0 + x_{0,l}(t) - c_l t), \tag{2.3.3}$$

$$\widetilde{P}(t) := \widetilde{R}(t) + \widetilde{B}(t), \tag{2.3.4}$$

and

$$\widetilde{P}(t) := \sum_{j=1}^{J} \widetilde{P}_j(t), \tag{2.3.5}$$

where there is the usual correspondence between $\widetilde{P}_j$ and $\widetilde{B_k}$ or $\widetilde{R}_l$,

then, $\varepsilon(t)$ satisfies, for any $k = 1, ..., K$, for any $l = 1, ..., L$ and for any $t \in [t^*, T]$,

$$\begin{aligned} \int \widetilde{R}_l(t)\varepsilon(t)\sqrt{\varphi_l^s(t)} &= \int \partial_x \widetilde{R}_l(t)\varepsilon(t)\sqrt{\varphi_l^s(t)} = 0, \\ \int \partial_{x_1} \widetilde{B_k}(t)\varepsilon(t)\sqrt{\varphi_k^b(t)} &= \int \partial_{x_2} \widetilde{B_k}(t)\varepsilon(t)\sqrt{\varphi_k^b(t)} = 0. \end{aligned} \tag{2.3.6}$$

Moreover, for any $t \in [t^*, T]$,

$$\|\varepsilon(t)\|_{H^2} + \sum_{k=1}^{K}(|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^{L}(|x_{0,l}(t)| + |c_{0,l}(t)|) \leq CAe^{-\theta t}, \tag{2.3.7}$$

and

$$\sum_{k=1}^{K}(|x_{1,k}'(t)| + |x_{2,k}'(t)|) + \sum_{l=1}^{L}(|x_{0,l}'(t)| + |c_{0,l}'(t)|) \leq C\|\varepsilon(t)\|_{L^2} + Ce^{-\theta t}. \tag{2.3.8}$$

Finally, $p(T) = P(T) = \widetilde{P}(T)$ and $\varepsilon(T) = x_{0,l}(T) = x_{1,k}(T) = x_{2,k}(T) = c_{0,l}(T) = 0$.

Proof: see for example [13] *for reference.* Let, for $t \in [t^*, T]$,

$$F_t : L^2(\mathbb{R}) \times \mathbb{R}^{2K} \times \mathbb{R}^{2L} \to \mathbb{R}^{2K+2L}, \tag{2.3.9}$$

such that

$$\begin{aligned}
&(w, x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) \\
&\longmapsto \Bigg(\int \sqrt{\varphi_k^b(t,x)}\partial_{x_1} B_{\alpha_k,\beta_k}(t,x; x_{1,k}^0 + x_{1,k}, x_{2,k}^0 + x_{2,k})\epsilon, \\
&\int \sqrt{\varphi_k^b(t,x)}\partial_{x_2} B_{\alpha_k,\beta_k}(t,x; x_{1,k}^0 + x_{1,k}, x_{2,k}^0 + x_{2,k})\epsilon, \\
&\int \sqrt{\varphi_l^s(t,x)}\partial_x \kappa_l Q_{c_l+c_{0,l}}(x - x_{0,l}^0 + x_{0,l} - c_l t)\epsilon, \\
&\int \sqrt{\varphi_l^s(t,x)}\kappa_l Q_{c_l+c_{0,l}}(x - x_{0,l}^0 + x_{0,l} - c_l t)\epsilon \Bigg),
\end{aligned} \tag{2.3.10}$$

where

$$\begin{aligned}
\epsilon := w &- \sum_{m=1}^{K} B_{\alpha_m,\beta_m}(t,x; x_{1,m}^0 + x_{1,m}, x_{2,m}^0 + x_{2,m}) \\
&- \sum_{n=1}^{L} \kappa_n Q_{c_n+c_{0,n}}(x - x_{0,n}^0 + x_{0,n} - c_n t).
\end{aligned} \tag{2.3.11}$$

We observe that F_t is a C^1 function and that $F_t(P(t), 0, 0, 0, 0) = 0$. Now, let us consider the matrix which gives the differential of F_t (with respect to $x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}$) in $(P(t), 0, 0, 0, 0)$ (we consider diagonal and extra-diagonal terms for each bloc):

$$DF_t = \begin{pmatrix}
B_{k,k}^1 & B_{k,k}^3 & \times & \times & \times & \times & \times & \times \\
B_{k,k}^3 & B_{k,k}^2 & \times & \times & \times & \times & \times & \times \\
\times & \times & B_{k',k'}^1 & B_{k',k'}^3 & \times & \times & \times & \times \\
\times & \times & B_{k',k'}^3 & B_{k',k'}^2 & \times & \times & \times & \times \\
\times & \times & \times & \times & R_{l,l}^1 & R_{l,l}^4 & \times & \times \\
\times & \times & \times & \times & R_{l,l}^3 & R_{l,l}^2 & \times & \times \\
\times & \times & \times & \times & \times & \times & R_{l',l'}^1 & R_{l',l'}^4 \\
\times & \times & \times & \times & \times & \times & R_{l',l'}^3 & R_{l',l'}^2
\end{pmatrix}, \tag{2.3.12}$$

where

$$\begin{aligned}
B_{k,k}^1 &:= -\int \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 \sqrt{\varphi_k^b}, \quad B_{k,k}^2 := -\int \left(\partial_{x_2} B_{\alpha_k,\beta_k}\right)^2 \sqrt{\varphi_k^b}, \\
B_{k,k}^3 &:= -\int \partial_{x_1} B_{\alpha_k,\beta_k} \partial_{x_2} B_{\alpha_k,\beta_k} \sqrt{\varphi_k^b},
\end{aligned} \tag{2.3.13}$$

$$\begin{aligned}
R_{l,l}^1 &:= -\int \left(\partial_x Q_{c_l}(y_{0,l}^0)\right)^2 \sqrt{\varphi_l^s}, \quad R_{l,l}^3 := -\int Q_{c_l}(y_{0,l}^0)\partial_x Q_{c_l}(y_{0,l}^0)\sqrt{\varphi_l^s}, \\
R_{l,l}^2 &:= -\frac{1}{2c_l}\int Q_{c_l}(y_{0,l}^0)\Big(Q_{c_l}(y_{0,l}^0) + y_{0,l}^0 \partial_x Q_{c_l}(y_{0,l}^0)\Big)\sqrt{\varphi_l^s}, \\
R_{l,l}^4 &:= -\frac{1}{2c_l}\int \partial_x Q_{c_l}(y_{0,l}^0)\Big(Q_{c_l}(y_{0,l}^0) + y_{0,l}^0 \partial_x Q_{c_l}(y_{0,l}^0)\Big)\sqrt{\varphi_l^s},
\end{aligned} \tag{2.3.14}$$

denoting $y_{0,l}^0 := x - x_{0,l}^0 - c_l t$, and crosses stand for exponentially decaying terms when $t \to +\infty$, and where we consider variables in the following order: $x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}$, ..., $x_{1,K}, x_{2,K}$, $x_{0,1}, c_{0,1}$, ..., $x_{0,L}, c_{0,L}$ and we order the coefficients of the function in the similar way. This is a matrix with dominant diagonal blocs.

Note that $B^1_{k,k}$ is exponentially close to $-\int \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2$, because if $P_j = B_k$ is a breather,

$$\begin{aligned}\int \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 \left(1-\sqrt{\varphi_k^b}\right) &\leq \int_{-\infty}^{\sigma_j t+\delta t} \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 + \int_{\sigma_{j+1} t+\delta t}^{+\infty} \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 \\ &\leq C\int_{-\infty}^{\sigma_j t+\delta t} e^{-2\beta|x-v_j t|} + \int_{\sigma_{j+1} t+\delta t}^{+\infty} e^{-2\beta|x-v_j t|} \\ &\leq C e^{-\frac{\beta\tau}{2} t},\end{aligned} \tag{2.3.15}$$

and the same is true for the other dominant diagonal terms of the matrix (we can get rid of φs).

Therefore, the determinant of the matrix is exponentially close to:

$$\begin{aligned}&\det(DF_t) \\ &= \prod_{k=1}^{K} \left(\int \left(\partial_{x_1} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k})\right)^2 \int \left(\partial_{x_2} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k})\right)^2 \right. \\ &\qquad \left. - \left(\int \partial_{x_1} B_{\alpha_k,\beta_k} \partial_{x_2} B_{\alpha_k,\beta_k} \right)^2 \right) \\ &\cdot \prod_{l=1}^{L} \left(\frac{1}{2c_l} \int Q_{c_l}(y^0_{0,l}) \left(Q_{c_l}(y^0_{0,l}) + y^0_{0,l} \partial_x Q_{c_l}(y^0_{0,l}) \right) \int \left(\partial_x Q_{c_l}(y^0_{0,l})\right)^2 \right),\end{aligned} \tag{2.3.16}$$

because $\int Q_{c_l}(y^0_{0,l}) \partial_x Q_{c_l}(y^0_{0,l})\, dx = 0$.

By Cauchy-Schwarz inequality and the fact that

$$\partial_{x_1} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k}) \quad \text{and} \quad \partial_{x_2} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k})$$

are linearly independent as functions of the x variable, for any time t fixed, we see that the first product is positive. Since each member of the product is periodic in time, then the first product is bounded below by a positive constant independent from time and translation parameters.

For the second product, by translation of the variable in the integrations, for any time t fixed, we see that we can replace $y^0_{0,l}$ by x. Then, by integration by parts,

$$\int x Q_{c_l}(x) \partial_x Q_{c_l}(x)\, dx = -\frac{1}{2} \int Q_{c_l}(x)^2\, dx. \tag{2.3.17}$$

By scaling, if q denotes the soliton with $c = 1$, i.e. $q = Q_1$,

$$\int Q^2_{c_l} = \sqrt{c_l} \int q^2, \quad \int \partial_x Q^2_{c_l} = c_l^{\frac{3}{2}} \int q_x^2. \tag{2.3.18}$$

Therefore,

$$\begin{aligned}&\frac{1}{2c_l} \int Q_{c_l}(y^0_{0,l}) \left(Q_{c_l}(y^0_{0,l}) + y^0_{0,l} \partial_x Q_{c_l}(y^0_{0,l}) \right) \int (\partial_x Q_{c_l}(y^0_{0,l}))^2 \\ &\qquad = \frac{1}{4} c_l \int q^2 \int (q_x)^2 \\ &\qquad \geq \frac{1}{4} \min\{c_n, 1 \leq n \leq L\} \int q^2 \int q_x^2.\end{aligned} \tag{2.3.19}$$

This means that the second product is bounded below by a positive constant independent from time and translation parameters.

This means that if T_2^* is large enough, the considered matrix is invertible.

Now, we may use the implicit function theorem (actually, we use a quantitative version of the implicit function theorem, see [11, Section 2.2] for a precise statement). If w is close enough to $P(t)$, then there exists

$$(x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) \tag{2.3.20}$$

such that

$$F_t(w, x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) = 0, \tag{2.3.21}$$

where (2.3.20) depends in a regular C^1 way on w. It is possible to show that the "close enough" in the previous sentence does not depend on t; for this, it is required to use a uniform implicit function theorem. This means that for T_2^* large enough (depending on A), $Ae^{-\theta t}$ is small enough for $t \in [t^*, T]$, thus for $t \in [t^*, T]$, $p(t)$ is close enough to $P(t)$ in order to apply the implicit function theorem. Therefore, we have for $t \in [t^*, T]$, the existence of $x_{1,k}(t)$, $x_{2,k}(t)$, $x_{0,l}(t)$ and $c_{0,l}(t)$. It is possible to show that these functions are C^1 in time. Basically, this comes from the fact that they are C^1 in $p(t)$ and that $p(t)$ has a similar regularity in time (see [13] for more details).

Now, we prove the inequalities (2.3.7) and (2.3.8). We can take the differential of the implicit functions with respect to $p(t)$ for $t \in [t^*, T]$. For this, we differentiate the following equation with respect to $p(t)$:

$$F_t\Big(p(t), x_{1,k}\big(p(t)\big), x_{2,k}\big(p(t)\big), x_{0,l}\big(p(t)\big), c_{0,l}\big(p(t)\big)\Big) = 0. \tag{2.3.22}$$

We know that the matrix that gives the differential of F_t (with respect to $x_{1,k}$, $x_{2,k}$, $x_{0,l}$, $c_{0,l}$) in

$$\Big(p(t), x_{1,k}\big(p(t)\big), x_{2,k}\big(p(t)\big), x_{0,l}\big(p(t)\big), c_{0,l}\big(p(t)\big)\Big) \tag{2.3.23}$$

is invertible and its inverse is bounded in time (from the formula giving the inverse of a matrix from the comatrix and the determinant). The differential of F_t with respect to the first variable is also bounded. Thus, by the mean-value theorem:

$$|x_{1,k}| \leq C\|p - P\| \leq CAe^{-\theta t}. \tag{2.3.24}$$

The same is true for $x_{2,k}$, $x_{0,l}$ and $c_{0,l}$.

By applying the mean-value theorem (inequality) for Q_{c_l} with respect to $x_{0,l}$ and $c_{0,l}$ or for B_{α_k,β_k} with respect to $x_{1,k}$ and $x_{2,k}$, we deduce that

$$\|P_j(t) - \widetilde{P}_j(t)\|_{H^2} \leq C\big(|x_{1,k}(t)| + |x_{2,k}(t)|\big), \tag{2.3.25}$$

if $P_j = B_k$ is a breather, and

$$\|P_j(t) - \widetilde{P}_j(t)\|_{H^2} \leq C\big(|x_{0,l}(t)| + |c_{0,l}(t)|\big), \tag{2.3.26}$$

if $P_j = R_l$ is a soliton.

Finally, by triangular inequality,

$$\begin{aligned}\|\varepsilon(t)\|_{H^2} &\leq \|p(t) - P(t)\|_{H^2} + \|P(t) - \widetilde{P}(t)\|_{H^2} \leq \|p(t) - P(t)\|_{H^2} \\ &\quad + C\left(\sum_{k=1}^{K}\big(|x_{1,k}(t)| + |x_{2,k}(t)|\big) + \sum_{l=1}^{L}\big(|x_{0,l}(t)| + |c_{0,l}(t)|\big)\right) \\ &\leq C\|p(t) - P(t)\|_{H^2} \leq CAe^{-\theta t}.\end{aligned} \tag{2.3.27}$$

This completes the proof of (2.3.7).

For (2.3.8), we will take time derivatives of the equations (2.3.6). From now on, we write $\widetilde{B_{k1}}$ for $\partial_{x_1}\widetilde{B_k}$ and $\widetilde{B_{k2}}$ for $\partial_{x_2}\widetilde{B_k}$. Firstly, we write the PDE verified by ε (knowing that $p, B_1, ..., B_K, R_1, ..., R_L$ are solutions of (mKdV)):

$$\begin{aligned}\partial_t\varepsilon &= -\varepsilon_{xxx} - \left[\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{j=1}^{J}\widetilde{P}_j + 3\sum_{i,j=1}^{J}\widetilde{P}_i\widetilde{P}_j\right)\right]_x \\ &\quad - \sum_{k=1}^{K}x'_{1,k}(t)\widetilde{B_{k1}} - \sum_{k=1}^{K}x'_{2,k}(t)\widetilde{B_{k2}} - \sum_{l=1}^{L}x'_{0,l}(t)\widetilde{R}_{lx} \\ &\quad - \sum_{l=1}^{L}\frac{c'_{0,l}(t)}{2(c_l + c_{0,l}(t))}\Big(\widetilde{R}_l + y_{0,l}(t)\widetilde{R}_{lx}\Big) - \sum_{h\neq i \text{ or } i\neq j}\Big(\widetilde{P_h}\widetilde{P_i}\widetilde{P_j}\Big)_x,\end{aligned} \tag{2.3.28}$$

where $y_{0,l}(t) := x - x^0_{0,l} + x_{0,l}(t) - c_l t$. Now, we will take the time derivative of the equation $\int \widetilde{B_{k1}}\varepsilon\sqrt{\varphi^b_k} = 0$ (and perform an integration by parts):

$$
\begin{aligned}
&- \int \left(\widetilde{B_k}^3\right)_{1x}\varepsilon\sqrt{\varphi^b_k} - \int \widetilde{B_{k1}} \sum_{h\neq i \text{ or } g\neq h} \left(\widetilde{P_h}\widetilde{P_i}\widetilde{P_g}\right)_x \sqrt{\varphi^b_k} \\
&+ x'_{2,k}(t)\int \widetilde{B_{k12}}\varepsilon\sqrt{\varphi^b_k} + \frac{1}{2\delta t}\int \widetilde{B_{k1}}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P_i} + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\
&+ \int \widetilde{B_{k1x}}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P_i} + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\sqrt{\varphi^b_k} - \frac{1}{2\delta t^2}\int \widetilde{B_{k1}}\varepsilon x\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\
&+ \frac{1}{2\delta t}\int \widetilde{B_{k1}}\varepsilon_{xx}\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} - \frac{1}{2\delta t}\int \widetilde{B_{k1x}}\varepsilon_x\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} + \frac{1}{2\delta t}\int \widetilde{B_{k1xx}}\varepsilon\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\
&+ x'_{1,k}(t)\int \widetilde{B_{k11}}\varepsilon\sqrt{\varphi^b_k} = \sum_{m=1}^{K} x'_{1,m}(t)\int \widetilde{B_{k1}}\widetilde{B_{m1}}\sqrt{\varphi^b_k} \\
&+ \sum_{m=1}^{K} x'_{2,m}(t)\int \widetilde{B_{k1}}\widetilde{B_{m2}}\sqrt{\varphi^b_k} + \sum_{n=1}^{L} x'_{0,n}(t)\int \widetilde{B_{k1}}\widetilde{R_{nx}}\sqrt{\varphi^b_k} \\
&+ \sum_{n=1}^{L}\frac{c'_{0,n}(t)}{2\big(c_n + c_{0,n}(t)\big)}\int \widetilde{B_{k1}}\Big(\widetilde{R_n} + y_{0,n}(t)\widetilde{R_{nx}}\Big)\sqrt{\varphi^b_k}.
\end{aligned}
\tag{2.3.29}
$$

Similarly, taking the time derivative of $\int \widetilde{B_{k2}}\varepsilon\sqrt{\varphi^b_k} = 0$:

$$
\begin{aligned}
&- \int \left(\widetilde{B_k}^3\right)_{2x}\varepsilon\sqrt{\varphi^b_k} - \int \widetilde{B_{k2}} \sum_{h\neq i \text{ or } g\neq h} \left(\widetilde{P_h}\widetilde{P_i}\widetilde{P_g}\right)_x \sqrt{\varphi^b_k} \\
&+ x'_{2,k}(t)\int \widetilde{B_{k22}}\varepsilon\sqrt{\varphi^b_k} + \frac{1}{2\delta t}\int \widetilde{B_{k2}}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P_i} + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\
&+ \int \widetilde{B_{k2x}}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P_i} + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\sqrt{\varphi^b_k} + \frac{1}{2\delta t}\int \widetilde{B_{k2}}\varepsilon_{xx}\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\
&- \frac{1}{2\delta t}\int \widetilde{B_{k2x}}\varepsilon_x\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} + \frac{1}{2\delta t}\int \widetilde{B_{k2xx}}\varepsilon\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} - \frac{1}{2\delta t^2}\int \widetilde{B_{k2}}\varepsilon x\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\
&+ x'_{1,k}(t)\int \widetilde{B_{k12}}\varepsilon\sqrt{\varphi^b_k} = \sum_{m=1}^{K} x'_{1,m}(t)\int \widetilde{B_{k2}}\widetilde{B_{m1}}\sqrt{\varphi^b_k} \\
&+ \sum_{m=1}^{K} x'_{2,m}(t)\int \widetilde{B_{k2}}\widetilde{B_{m2}}\sqrt{\varphi^b_k} + \sum_{n=1}^{L} x'_{0,n}(t)\int \widetilde{B_{k2}}\widetilde{R_{nx}}\sqrt{\varphi^b_k} \\
&+ \sum_{n=1}^{L}\frac{c'_{0,n}(t)}{2\big(c_n + c_{0,n}(t)\big)}\int \widetilde{B_{k2}}\Big(\widetilde{R_n} + y_{0,n}(t)\widetilde{R_{nx}}\Big)\sqrt{\varphi^b_k}.
\end{aligned}
\tag{2.3.30}
$$

Similarly, taking the time derivative of $\int \widetilde{R}_{lx}(t)\varepsilon(t)\sqrt{\varphi_l^s} = 0$:

$$
\begin{aligned}
&-\int \left(\widetilde{R_l}^3\right)_{xx} \varepsilon\sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2\big(c_l + c_{0,l}(t)\big)} \int \Big(\widetilde{R}_{lx} + y_{0,l}(t)\widetilde{R}_{lxx}\Big)\varepsilon\sqrt{\varphi_l^s} \\
&+ x'_{0,l}(t)\int \widetilde{R}_{lxx}\varepsilon\sqrt{\varphi_l^s} + \frac{1}{2\delta t}\int \widetilde{R}_{lx}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P}_i\right)\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \int \widetilde{R}_{lxx}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P}_i\right)\sqrt{\varphi_l^s} - \frac{1}{2\delta t^2}\int \widetilde{R}_{lx}\varepsilon x\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \frac{1}{2\delta t}\int \widetilde{R}_{lx}\varepsilon_{xx}\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t}\int \widetilde{R}_{lxx}\varepsilon_x\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} + \frac{1}{2\delta t}\int \widetilde{R}_{lxxx}\varepsilon\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&- \int \widetilde{R}_{lx}\sum_{h\neq i \text{ or } g\neq h}\Big(\widetilde{P_h}\widetilde{P_i}\widetilde{P_g}\Big)_x\sqrt{\varphi_l^s} = \sum_{n=1}^{L} x'_{0,n}(t)\int \widetilde{R}_{lx}\widetilde{R_n}_x\sqrt{\varphi_l^s} \\
&+ \sum_{n=1}^{L}\frac{c'_{0,n}(t)}{2\big(c_n + c_{0,n}(t)\big)}\int \widetilde{R}_{lx}\Big(\widetilde{R_n} + y_{0,n}(t)\widetilde{R_n}_x\Big)\sqrt{\varphi_l^s} \\
&+ \sum_{m=1}^{K} x'_{1,m}(t)\int \widetilde{R}_{lx}\widetilde{B_m}_1\sqrt{\varphi_l^s} + \sum_{m=1}^{K} x'_{2,m}(t)\int \widetilde{R}_{lx}\widetilde{B_m}_2\sqrt{\varphi_l^s}.
\end{aligned}
\tag{2.3.31}
$$

Finally, taking the time derivative of $\int \widetilde{R}_l\varepsilon\sqrt{\varphi_l^s} = 0$:

$$
\begin{aligned}
&-\int \left(\widetilde{R_l}^3\right)_{x} \varepsilon\sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2\big(c_l + c_{0,l}(t)\big)} \int \Big(\widetilde{R}_l + y_{0,l}(t)\widetilde{R}_{lx}\Big)\varepsilon\sqrt{\varphi_l^s} \\
&+ x'_{0,l}(t)\int \widetilde{R}_{lx}\varepsilon\sqrt{\varphi_l^s} + \frac{1}{2\delta t}\int \widetilde{R}_{l}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P}_i\right)\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \int \widetilde{R}_{lx}\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P}_i\right)\sqrt{\varphi_l^s} - \frac{1}{2\delta t^2}\int \widetilde{R}_{l}\varepsilon x\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \frac{1}{2\delta t}\int \widetilde{R}_{l}\varepsilon_{xx}\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t}\int \widetilde{R}_{lx}\varepsilon_x\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} + \frac{1}{2\delta t}\int \widetilde{R}_{lxx}\varepsilon\frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&- \int \widetilde{R}_{l}\sum_{h\neq i \text{ or } g\neq h}\Big(\widetilde{P_h}\widetilde{P_i}\widetilde{P_g}\Big)_x\sqrt{\varphi_l^s} = \sum_{n=1}^{L} x'_{0,n}(t)\int \widetilde{R}_{l}\widetilde{R_n}_x\sqrt{\varphi_l^s} \\
&+ \sum_{n=1}^{L}\frac{c'_{0,n}(t)}{2\big(c_n + c_{0,n}(t)\big)}\int \widetilde{R}_{l}\Big(\widetilde{R_n} + y_{0,n}(t)\widetilde{R_n}_x\Big)\sqrt{\varphi_l^s} \\
&+ \sum_{m=1}^{K} x'_{1,m}(t)\int \widetilde{R}_{l}\widetilde{B_m}_1\sqrt{\varphi_l^s} + \sum_{m=1}^{K} x'_{2,m}(t)\int \widetilde{R}_{l}\widetilde{B_m}_2\sqrt{\varphi_l^s}.
\end{aligned}
\tag{2.3.32}
$$

By the Proposition 2.10 below (that follows from the first part of the lemma we prove) and its corollary, several terms of the equalities (2.3.29), (2.3.30), (2.3.31) and (2.3.32) are bounded by $Ce^{-\theta t}$; other terms are $O(\|\varepsilon\|_{L^2})$. We remind that $O(\|\varepsilon\|_{L^2}) \leq CAe^{-\theta t}$. From the basic properties of φ_j (see Section 2.2), $\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}$ is bounded. Because of the compact support of φ_j, $\frac{x}{t}\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}$ is bounded independently on x and t. Using these bounds, and after several linear combinations, we obtain the desired inequalities. □

Remark 2.9. As a consequence of Lemma 2.8, there exists a constant $C > 0$ such that

$$\forall t \in [t^*, T] \quad \sum_{k=1}^{K} \big(|x_{1,k}(t)| + |x_{2,k}(t)|\big) + \sum_{l=1}^{L} \big(|x_{0,l}(t)| + |c_{0,l}(t)|\big) \leq CAe^{-\theta T^*}. \tag{2.3.33}$$

This means, that if we take T_2^* eventually larger (which we will assume in the following of the article), we may extend Proposition 2.1 to $\widetilde{P_j}$ in the following way, by integration of the bounds given by modulation (the constant C is a bit larger in a controled way, we write $\frac{\beta}{2}$ because the shape of the solitons is a bit modified in a controled way):

Proposition 2.10. *Let $j = 1, ..., J$, $n \in \mathbb{N}$. If $T^* > T_2^*$, then there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \widetilde{P_j}(t, x)| \leq Ce^{-\frac{\beta}{2}|x - v_j t|}. \tag{2.3.34}$$

We will also use that any $\|\partial_x^n \widetilde{P_j}\|_{H^2}$ is bounded by C.

Corollary 2.11. *Let $i \neq j \in \{1, ..., J\}$ and $m, n \in \mathbb{N}$. If $T^* > T_2^*$, then there exists a constant C that depends only on P, such that for any $t \in \mathbb{R}$,*

$$\left| \int \partial_x^m \widetilde{P_i} \partial_x^n \widetilde{P_j} \right| \leq Ce^{-\beta\tau t/8}. \tag{2.3.35}$$

2.4. **Study of coercivity.** In [3], the Lyapunov functional that was introduced to study the orbital stability of a breather was the following conserved-in-time functional:

$$F[p](t) + 2\big(\beta^2 - \alpha^2\big)E[p](t) + \big(\alpha^2 + \beta^2\big)^2 M[p](t). \tag{2.4.1}$$

The functional that we will consider here is adapted from the latter. For $t \in [t^*, T]$, we set

$$\begin{aligned} \mathcal{H}[p](t) := F[p](t) &+ \sum_{k=1}^{K} \Big(2\big(\beta_k^2 - \alpha_k^2\big)E_k^b[p](t) + \big(\alpha_k^2 + \beta_k^2\big)^2 M_k^b[p](t)\Big) \\ &+ \sum_{l=1}^{L} \Big(2c_l E_l^s[p](t) + c_l^2 M_l^s[p](t)\Big). \end{aligned} \tag{2.4.2}$$

For the simplicity of notations, for $j \in \{1, ..., J\}$, a_j will denote α_k if P_j is the breather B_k or 0 if P_j is a soliton, and b_j will denote β_k if P_j is the breather B_k or $c_l^{1/2}$ if P_j is the soliton R_l. With these notations, we may write:

$$\mathcal{H}[p](t) = F[p](t) + \sum_{j=1}^{J} \Big(2\big(b_j^2 - a_j^2\big)E_j[p](t) + \big(a_j^2 + b_j^2\big)^2 M_j[p](t)\Big). \tag{2.4.3}$$

We would like to study locally this functional around the considered sum of breathers and solitons. The aim of this section will be to prove two following propositions:

Proposition 2.12 (Expansion of H^2 conserved quantity)**.** *There exists $T_4^* > 0$ such that if $T^* \geq T_4^*$, for all $t \in [t^*, T]$, we have that*

$$\begin{aligned} \mathcal{H}[p](t) = &\sum_{j=1}^{J} \Big(F\big[\widetilde{P_j}\big](t) + 2\big(b_j^2 - a_j^2\big)E\big[\widetilde{P_j}\big](t) + \big(a_j^2 + b_j^2\big)^2 M\big[\widetilde{P_j}\big](t)\Big) \\ &+ H_2[\varepsilon](t) + O\big(\|\varepsilon(t)\|_{H^2}^3\big) + O\big(e^{-2\theta t}\|\varepsilon(t)\|_{H^2}\big) + O\big(e^{-2\theta t}\big), \end{aligned} \tag{2.4.4}$$

where

$$\begin{aligned} H_2[\varepsilon](t) := &\frac{1}{2}\int \varepsilon_{xx}^2 - \frac{5}{2}\int \widetilde{P}^2\varepsilon_x^2 + \frac{5}{2}\int \widetilde{P}_x^2\varepsilon^2 + 5\int \widetilde{P}\widetilde{P}_{xx}\varepsilon^2 + \frac{15}{4}\int \widetilde{P}^4\varepsilon^2 \\ &+ \sum_{j=1}^{J} \big(b_j^2 - a_j^2\big)\Big(\int \varepsilon_x^2\varphi_j - 3\int \widetilde{P}^2\varepsilon^2\varphi_j\Big) + \sum_{j=1}^{J} \big(a_j^2 + b_j^2\big)^2 \frac{1}{2}\int \varepsilon^2\varphi_j. \end{aligned} \tag{2.4.5}$$

Proposition 2.13 (Coercivity of H_2)**.** *There exists $\mu > 0$, $T_3^* = T_3^*(A)$ such that, if $T^* \geq T_3^*$, we have for any $t \in [t^*, T]$,*

$$H_2[\varepsilon](t) \geq \mu \|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu} \sum_{k=1}^{K} \left(\int \varepsilon \widetilde{B_k} \sqrt{\varphi_k^b} \right)^2. \tag{2.4.6}$$

The Propositions 2.12 and 2.13 will be used in the next concluding subsection to prove the Proposition 1.9.

Firstly, let us prove the Proposition 2.12.

Proof of Proposition 2.12. We would like to compare $\mathcal{H}[\widetilde{P} + \varepsilon](t)$ and $\mathcal{H}[\widetilde{P}](t)$ (recall that $p = \widetilde{P} + \varepsilon$) by studying the difference asymptotically when ε is small. Firstly, let us see how we could simplify the expression of $\mathcal{H}[\widetilde{P}](t)$.

Step 1:

Claim 2.14. If T^* is large enough, for all $t \in [t^*, T]$, we have that

$$\begin{aligned} \mathcal{H}[\widetilde{P}](t) = \sum_{j=1}^{J} \Big(F\big[\widetilde{P}_j\big](t) + 2\big(b_j^2 - a_j^2\big) E\big[\widetilde{P}_j\big](t) + \big(a_j^2 + b_j^2\big)^2 M\big[\widetilde{P}_j\big](t) \Big) \\ + O\big(e^{-2\theta t}\big). \end{aligned} \tag{2.4.7}$$

Proof. We prove that, for $t \in [t^*, T]$,

$$\left| \mathcal{H}[\widetilde{P}] - \sum_{j=1}^{J} \Big(F\big[\widetilde{P}_j\big] + 2\big(b_j^2 - a_j^2\big) E\big[\widetilde{P}_j\big] + \big(a_j^2 + b_j^2\big)^2 M\big[\widetilde{P}_j\big] \Big) \right| \leq C e^{-2\theta t}. \tag{2.4.8}$$

Let us compare $F_j[\widetilde{P}]$ and $F[\widetilde{P}_j]$:

$$F_j[\widetilde{P}] = \int \left(\frac{1}{2} \widetilde{P}_{xx}^2 - \frac{5}{2} \widetilde{P}^2 \widetilde{P}_x^2 + \frac{1}{4} \widetilde{P}^6 \right) \varphi_j(t,x)\, dx, \tag{2.4.9}$$

$$F[\widetilde{P}_j] = \int \left(\frac{1}{2} \widetilde{P}_{jxx}^{\,2} - \frac{5}{2} \widetilde{P}_j^{\,2} \widetilde{P}_{jx}^{\,2} + \frac{1}{4} \widetilde{P}_j^{\,6} \right) dx. \tag{2.4.10}$$

We compare the corresponding terms of these equalities. Let us start by the first one:

$$\begin{aligned} &\left| \int \left(\widetilde{P}_{xx}^2 \varphi_j(t,x) - \widetilde{P}_{jxx}^{\,2} \right) \right| \\ &\quad \leq \int \widetilde{P}_{jxx}^{\,2} \big| 1 - \varphi_j(t,x) \big| + \sum_{(r,s) \neq (j,j)} \int \left| \widetilde{P}_{rxx} \widetilde{P}_{sxx} \right| \varphi_j(t,x) \\ &\quad \leq C \int e^{-\frac{\beta}{2}|x - v_j t|} e^{\frac{\beta\tau}{32} t} \big| 1 - \varphi_j(t,x) \big|\, dx \\ &\qquad + C \sum_{i \neq j} \int e^{-\frac{\beta}{2}|x - v_i t|} e^{\frac{\beta\tau}{32} t} \varphi_j(t,x)\, dx \\ &\quad \leq C e^{\frac{\beta\tau}{32} t} \left[\left(\int_{-\infty}^{\sigma_j t + \delta t} + \int_{\sigma_{j+1} t - \delta t}^{+\infty} \right) e^{-\frac{\beta}{2}|x - v_j t|}\, dx \right. \\ &\qquad \left. + \sum_{i \neq j} \int_{\sigma_j t - \delta t}^{\sigma_{j+1} t + \delta t} e^{-\frac{\beta}{2}|x - v_i t|}\, dx \right] \leq C e^{-\beta\tau t/16}, \end{aligned} \tag{2.4.11}$$

by Proposition 2.10 and Remark 2.6. For the other terms of the difference to be bounded, we reason in a similar way. This completes the proof of the claim. □

Step 2:

Therefore, when we manage to compare $\mathcal{H}[p](t)$ and $\mathcal{H}[\widetilde{P}](t)$, we are also able to compare $\mathcal{H}[p](t)$ and

$$\sum_{j=1}^{J}\Big(F\big[\widetilde{P}_j\big](t)+2\big(b_j^2-a_j^2\big)E\big[\widetilde{P}_j\big](t)+\big(a_j^2+b_j^2\big)^2M\big[\widetilde{P}_j\big](t)\Big). \tag{2.4.12}$$

We compute the Taylor expansion of $\mathcal{H}[p]=\mathcal{H}[\widetilde{P}+\varepsilon]$:

$$\begin{aligned}
\mathcal{H}\big[\widetilde{P}+\varepsilon\big] &= \frac{1}{2}\int\big(\widetilde{P}+\varepsilon\big)_{xx}^2-\frac{5}{2}\int\big(\widetilde{P}+\varepsilon\big)^2\big(\widetilde{P}+\varepsilon\big)_x^2+\frac{1}{4}\int\big(\widetilde{P}+\varepsilon\big)^6\\
&\quad+\sum_{j=1}^{J}\left[\big(b_j^2-a_j^2\big)\left(\int\big(\widetilde{P}+\varepsilon\big)_x^2\varphi_j-\frac{1}{2}\int\big(\widetilde{P}+\varepsilon\big)^4\varphi_j\right)\right]\\
&\quad+\sum_{j=1}^{J}\left[\big(a_j^2+b_j^2\big)^2\frac{1}{2}\int\big(\widetilde{P}+\varepsilon\big)^2\varphi_j\right]\\
&=\frac{1}{2}\int\widetilde{P}_{xx}^2-\frac{5}{2}\int\widetilde{P}^2\widetilde{P}_x^2+\frac{1}{4}\int\widetilde{P}^6+\int\widetilde{P}_{(4x)}\varepsilon+5\int\widetilde{P}\widetilde{P}_x^2\varepsilon\\
&\quad+5\int\widetilde{P}^2\widetilde{P}_{xx}\varepsilon+\frac{3}{2}\int\widetilde{P}^5\varepsilon+\frac{1}{2}\int\varepsilon_{xx}^2-\frac{5}{2}\int\widetilde{P}^2\varepsilon_x^2\\
&\quad+\frac{5}{2}\int\widetilde{P}_x^2\varepsilon^2+5\int\widetilde{P}\widetilde{P}_{xx}\varepsilon^2+\frac{15}{4}\int\widetilde{P}^4\varepsilon^2+O\big(\|\varepsilon(t)\|_{H^2}^3\big)\\
&\quad+\sum_{j=1}^{J}\big(b_j^2-a_j^2\big)\left(\int\widetilde{P}_x^2\varphi_j-\frac{1}{2}\int\widetilde{P}^4\varphi_j-2\int\widetilde{P}_{xx}\varepsilon\varphi_j\right.\\
&\quad\left.-2\int\widetilde{P}_x\varepsilon\varphi_{j,x}-2\int\widetilde{P}^3\varepsilon\varphi_j+\int\varepsilon_x^2\varphi_j-3\int\widetilde{P}^2\varepsilon^2\varphi_j\right)\\
&\quad+\sum_{j=1}^{J}\big(a_j^2+b_j^2\big)^2\frac{1}{2}\left(\int\widetilde{P}^2\varphi_j+2\int\widetilde{P}\varepsilon\varphi_j+\int\varepsilon^2\varphi_j\right).
\end{aligned}\tag{2.4.13}$$

We can observe that the sum (2.4.13) is composed of 0-order terms in ε, of 1^{st}-order terms in ε, of 2^{nd}-order terms in ε; 3^{rd} and larger-order terms in ε are contained in $O(\|\varepsilon(t)\|_{H^2}^3)$. The sum of the 0-order terms is actually $\mathcal{H}[\widetilde{P}]$. The sum of 2^{nd}-order terms in ε is $H_2[\varepsilon](t)$.

Let us study more closely the 1^{st}-order terms:

$$\begin{aligned}
H_1 &= \int\widetilde{P}_{(4x)}\varepsilon+5\int\widetilde{P}\widetilde{P}_x^2\varepsilon+5\int\widetilde{P}^2\widetilde{P}_{xx}\varepsilon+\frac{3}{2}\int\widetilde{P}^5\varepsilon\\
&\quad+\sum_{j=1}^{J}\big(b_j^2-a_j^2\big)\left(2\int\widetilde{P}_x\varepsilon_x\varphi_j-2\int\widetilde{P}^3\varepsilon\varphi_j\right)+\sum_{j=1}^{J}\big(a_j^2+b_j^2\big)^2\int\widetilde{P}\varepsilon\varphi_j.
\end{aligned}\tag{2.4.14}$$

From [3], we know that a breather $A=A_{\alpha,\beta}$ satisfies for any fixed $t\in\mathbb{R}$, the following nonlinear equation:

$$A_{(4x)}-2\big(\beta^2-\alpha^2\big)\big(A_{xx}+A^3\big)+\big(\alpha^2+\beta^2\big)^2A+5AA_x^2+5A^2A_{xx}+\frac{3}{2}A^5=0. \tag{2.4.15}$$

This equation is also satisfied for $A=\widetilde{B}_k$ with $\alpha=\alpha_k$ and $\beta=\beta_k$ for any $k=1,...,K$ (the shape parameters of a breather are not changed by modulation).

For a soliton $Q=R_{c,\kappa}$, we know from $Q_{xx}=cQ-Q^3$ that Q satisfies for any fixed $t\in\mathbb{R}$, the following nonlinear equation (see Section 5.1 (Appendix)):

$$Q_{(4x)}-2c\big(Q_{xx}+Q^3\big)+c^2Q+5QQ_x^2+5Q^2Q_{xx}+\frac{3}{2}Q^5=0. \tag{2.4.16}$$

This equation is not exactly satisfied for $Q = \widetilde{R}_l$ for any $l = 1, ..., L$ (the shape parameters of a soliton are changed by modulation). The exact equation satisfied by $Q = \widetilde{R}_l$ is:

$$\begin{aligned} &Q_{(4x)} - 2c_l\big(Q_{xx} + Q^3\big) + c_l^2 Q + 5QQ_x^2 + 5Q^2Q_{xx} + \frac{3}{2}Q^5 \\ &\qquad = 2c_{0,l}(t)\big(Q_{xx} + Q^3\big) - 2c_l c_{0,l}(t)Q - c_{0,l}(t)^2 Q. \end{aligned} \tag{2.4.17}$$

We will compare H_1 and

$$\begin{aligned} H_1' := &\int \widetilde{P}_{(4x)}\varepsilon + 5\sum_{j=1}^{J}\int \widetilde{P}_j\widetilde{P}_{j_x}^2\varepsilon + 5\sum_{j=1}^{J}\int \widetilde{P}_{j_x}^2\widetilde{P}_{j_{xx}}\varepsilon + \frac{3}{2}\sum_{j=1}^{J}\int \widetilde{P}_j^5\varepsilon \\ &- 2\sum_{j=1}^{J}\big(b_j^2 - a_j^2\big)\bigg(\int \widetilde{P}_{j_{xx}}\varepsilon + \int \widetilde{P}_j^3\varepsilon\bigg) + \sum_{j=1}^{J}\big(a_j^2 + b_j^2\big)^2\int \widetilde{P}_j\varepsilon. \end{aligned} \tag{2.4.18}$$

Firstly, let us compare $\int \widetilde{P}\widetilde{P}_x^2\varepsilon$ and $\sum_{j=1}^{J}\int \widetilde{P}_j\widetilde{P}_{j_x}^2\varepsilon$:

$$\begin{aligned} \int \widetilde{P}\widetilde{P}_x^2\varepsilon &= \int \bigg(\sum_{j=1}^{J}\widetilde{P}_j\bigg)\bigg(\sum_{j=1}^{J}\widetilde{P}_{j_x}\bigg)^2\varepsilon \\ &= \sum_{j=1}^{J}\int \widetilde{P}_j\widetilde{P}_{j_x}^2\varepsilon + \sum_{h\neq i \text{ or } i\neq j}\int \widetilde{P_h}\widetilde{P}_{i_x}\widetilde{P}_{j_x}\varepsilon. \end{aligned} \tag{2.4.19}$$

To succeed, we need to find a bound for a term of the type $\int \widetilde{P_h}\widetilde{P}_{i_x}\widetilde{P}_{j_x}\varepsilon$ where $h \neq i$ or $i \neq j$. We can perform the following upper bounding (where without loss of generality, we suppose that $i \neq j$):

$$\begin{aligned} \bigg|\int \widetilde{P_h}\widetilde{P}_{i_x}\widetilde{P}_{j_x}\varepsilon\bigg| &\leq Ce^{\frac{\beta\tau}{16}t}\int e^{-\frac{\beta}{2}|x-v_it|}e^{-\frac{\beta}{2}|x-v_jt|}|\varepsilon| \\ &\leq C\|\varepsilon\|_{L^\infty}e^{\frac{\beta\tau}{16}t}\int e^{-\frac{\beta}{2}|x-v_it|}e^{-\frac{\beta}{2}|x-v_jt|} \\ &\leq C\|\varepsilon\|_{H^2}e^{-\beta\tau t/8}, \end{aligned} \tag{2.4.20}$$

by Sobolev embeddings and Proposition 2.3.

The bounding is quite similar for $\int \widetilde{P}^2\widetilde{P}_{xx}\varepsilon$ and $\int \widetilde{P}^5\varepsilon$. We observe that $-\int \widetilde{P}_{j_{xx}}\varepsilon = \int \widetilde{P}_{j_x}\varepsilon_x$. To compare $\int \widetilde{P}_x\varepsilon_x\varphi_j$ and $\int_{\mathbb{R}} \widetilde{P}_{j_x}\varepsilon_x$, and for similar terms, we can use computations that we have already performed at the beginning of this proof. Therefore,

$$\bigg|\int \widetilde{P}_x\varepsilon_x\varphi_j - \int_{\mathbb{R}}\widetilde{P}_{j_x}\varepsilon_x\bigg| \leq C\|\varepsilon\|_{H^2}e^{-\frac{\beta\tau t}{16}}. \tag{2.4.21}$$

This enables us to bound the difference between H_1 and H_1':

$$\big|H_1 - H_1'\big| \leq C\|\varepsilon(t)\|_{H^2}e^{-\frac{\beta\tau t}{16}}. \tag{2.4.22}$$

Now, because our objects are not only breathers, H_1' is not equal to 0. Actually, we have that

$$\begin{aligned} H_1' = &2\sum_{l=1}^{L}c_{0,l}(t)\bigg(\int \widetilde{R}_{l_{xx}}\varepsilon + \int \widetilde{R}_l^3\varepsilon\bigg) \\ &- 2\sum_{l=1}^{L}c_l c_{0,l}(t)\int \widetilde{R}_l\varepsilon - \sum_{l=1}^{L}c_{0,l}(t)^2\int \widetilde{R}_l\varepsilon. \end{aligned} \tag{2.4.23}$$

Now, we introduce:

$$\begin{aligned} H_1'' = 2\sum_{l=1}^{L} c_{0,l}(t)\Big(\int \widetilde{R}_{lxx}\varepsilon\sqrt{\varphi_l^s} + \int \widetilde{R}_l^{\,3}\varepsilon\sqrt{\varphi_l^s}\Big) \\ -2\sum_{l=1}^{L} c_l c_{0,l}(t)\int \widetilde{R}_l\varepsilon\sqrt{\varphi_l^s} - \sum_{l=1}^{L} c_{0,l}(t)^2\int \widetilde{R}_l\varepsilon\sqrt{\varphi_l^s}. \end{aligned} \tag{2.4.24}$$

By reasonning the same way as for H_1 and H_1', we see that

$$\left|H_1' - H_1''\right| \leq C\|\varepsilon(t)\|_{H^2}e^{-2\theta t}. \tag{2.4.25}$$

Because of (2.3.6) and because of the elliptic equation satisfied by a soliton, we have that

$$H_1'' = 0. \tag{2.4.26}$$

Thus,

$$|H_1| = |H_1 - H_1'| + |H_1' - H_1''| + |H_1''| \leq C\|\varepsilon(t)\|_{H^2}e^{-2\theta t}. \tag{2.4.27}$$

The proof of Proposition 2.12 is now completed. □

Now, we would like to study the quadratic terms in ε of the development of $\mathcal{H}[\widetilde{P}+\varepsilon]$. They are contained in $H_2[\varepsilon](t)$.

Let $A = B_{\alpha,\beta}$ be a breather (we note $A_1 := \partial_{x_1}A$ and $A_2 := \partial_{x_2}A$). We define a quadratic form associated to this breather:

$$\begin{aligned} \mathcal{Q}^b_{\alpha,\beta}[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int A^2\epsilon_x^2 + \frac{5}{2}\int A_x^2\epsilon^2 + 5\int AA_{xx}\epsilon^2 + \frac{15}{4}\int A^4\epsilon^2 \\ + \left(\beta^2-\alpha^2\right)\Big(\int \epsilon_x^2 - 3\int A^2\epsilon^2\Big) + \left(\alpha^2+\beta^2\right)^2\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{\alpha,\beta}[\epsilon]. \end{aligned} \tag{2.4.28}$$

From [3], we know that the kernel of this quadratic form is of dimension 2 and is spanned by $\partial_{x_1}B_{\alpha,\beta}$ and $\partial_{x_2}B_{\alpha,\beta}$, and that this quadratic form has only one negative eigenvalue that is of multiplicity 1:

Proposition 2.15 (Proposition 4.11, [32])**.** *There exists $\mu^b_{\alpha,\beta} > 0$ that depends only on α and β (and does not depend on time), such that if $\epsilon \in H^2(\mathbb{R})$ is such that*

$$\int A_1\epsilon = \int A_2\epsilon = 0, \tag{2.4.29}$$

then

$$\mathcal{Q}^b_{\alpha,\beta}[\epsilon] \geq \mu^b_{\alpha,\beta}\|\epsilon\|^2_{H^2} - \frac{1}{\mu^b_{\alpha,\beta}}\Big(\int \epsilon A\Big)^2. \tag{2.4.30}$$

Remark 2.16. $\mu^b_{\alpha,\beta}$ is continuous in α,β. Note that translation parameters are implicit in $\mathcal{Q}^b_{\alpha,\beta}$.

Let $Q = R_{c,\kappa}$ be a soliton. We define a quadratic form associated to this soliton:

$$\begin{aligned} \mathcal{Q}^s_c[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ + c\Big(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\Big) + c^2\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{0,\sqrt{c}}[\epsilon]. \end{aligned} \tag{2.4.31}$$

By the same techniques, such as those presented in [3], adapted to the quadratic form of a soliton, we may establish that the kernel of this quadratic form is of dimension 2, and is spanned by $\partial_x Q$ and $\partial_c Q$, and that this quadratic form does not have any negative eigenvalue (see Section 5.2 (Appendix)). After that, from Section 5.3 (Appendix), we deduce that the coercivity still works when ϵ is orthogonal to Q and $\partial_x Q$. More precisely:

Proposition 2.17. *There exists $\mu^s_c > 0$ that depends only on c (and does not depend on time), such that if $\epsilon \in H^2(\mathbb{R})$ is such that*

$$\int Q\epsilon = \int Q_x\epsilon = 0, \tag{2.4.32}$$

then

$$Q_c^s[\epsilon] \geq \mu_c^s \|\epsilon\|_{H^2}^2. \tag{2.4.33}$$

Remark 2.18. μ_c^s is continuous in c. Note that translation and sign parameters are implicit in the notation Q_c^s.

We would like to find a similar minoration for H_2 (which is a generalization of Q).

For $j = 1, ..., J$, let us define for $\epsilon \in H^2$,

$$\begin{aligned} Q_j[\epsilon] := & \frac{1}{2}\int \epsilon_{xx}^2\varphi_j - \frac{5}{2}\int \widetilde{P_j}^2\epsilon_x^2\varphi_j + \frac{5}{2}\int \widetilde{P_j}_x^2\epsilon^2\varphi_j \\ & + 5\int \widetilde{P_j}\widetilde{P}_{j_{xx}}\epsilon^2\varphi_j + \frac{15}{4}\int \widetilde{P_j}^4\epsilon^2\varphi_j \\ & + \left(b_j^2 - a_j^2\right)\left(\int \epsilon_x^2\varphi_j - 3\int \widetilde{P_j}^2\epsilon^2\varphi_j\right) + \left(a_j^2 + b_j^2\right)^2\frac{1}{2}\int \epsilon^2\varphi_j, \end{aligned} \tag{2.4.34}$$

and

$$\begin{aligned} Q_j'[\epsilon] := & \frac{1}{2}\int \epsilon_{xx}^2\varphi_j - \frac{5}{2}\int \widetilde{P}^2\epsilon_x^2\varphi_j + \frac{5}{2}\int \widetilde{P}_x^2\epsilon^2\varphi_j \\ & + 5\int \widetilde{P}\widetilde{P}_{xx}\epsilon^2\varphi_j + \frac{15}{4}\int \widetilde{P}^4\epsilon^2\varphi_j \\ & + \left(b_j^2 - a_j^2\right)\left(\int \epsilon_x^2\varphi_j - 3\int \widetilde{P}^2\epsilon^2\varphi_j\right) + \left(a_j^2 + b_j^2\right)^2\frac{1}{2}\int \epsilon^2\varphi_j. \end{aligned} \tag{2.4.35}$$

We have that

$$H_2[\varepsilon(t)] = \sum_{j=1}^{J} Q_j'[\varepsilon(t)]. \tag{2.4.36}$$

Notations Q_k^b, $(Q_k^b)'$, Q_l^s and $(Q_l^s)'$ will also be used.

We note that the support of φ_j increases with time, so that Q_j is near a $Q^b_{\alpha_k,\beta_k}$ or a $Q^s_{c_l}$ when time is large (note that $Q^b_{\alpha_k,\beta_k}$ is the canonical quadratic form associated to the breather $\widetilde{B_k}$, but the canonical quadratic form associated to the soliton $\widetilde{R_c}$ is $Q^s_{c_l+c_{0,l}(t)}$). However, firstly, let us study the difference between Q_j and Q_j'. Using the computations carried out at the beginning of this part (those done for the linear part) and Sobolev inequalities, we obtain:

$$\left|Q_j[\epsilon] - Q_j'[\epsilon]\right| \leq Ce^{-2\theta t}\|\epsilon\|_{H^2(\mathbb{R})}^2. \tag{2.4.37}$$

Lemma 2.19. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that, if $T^* \geq T_3^*$, for any $\epsilon \in H^2(\mathbb{R})$, for any $t \in [t^*, T]$,*

if

$$\int \widetilde{B_{k1}}(t)\epsilon\sqrt{\varphi_k^b(t)} = \int \widetilde{B_{k2}}(t)\epsilon\sqrt{\varphi_k^b(t)} = 0, \tag{2.4.38}$$

then

$$Q_k^b[\epsilon] \geq \mu\int \left(\epsilon^2 + \epsilon_x^2 + \epsilon_{xx}^2\right)\varphi_k^b(t) - \frac{1}{\mu}\left(\int \epsilon\widetilde{B_k}(t)\sqrt{\varphi_k^b(t)}\right)^2 - \rho\|\epsilon\|_{H^2}^2. \tag{2.4.39}$$

Proof of Lemma 2.19. The idea is to write $Q_k^b[\epsilon]$ as $Q_{\alpha_k,\beta_k}[\epsilon\sqrt{\varphi_k^b}]$ plus several error terms. Let j such that $\widetilde{P_j} = \widetilde{B_k}$. We will denote $\varphi_{1,j} := \psi'(\frac{x-\sigma_j t}{\delta t}) - \psi'(\frac{x-\sigma_{j+1}t}{\delta t})$ and $\varphi_{2,j} := \psi''(\frac{x-\sigma_j t}{\delta t}) - \psi''(\frac{x-\sigma_{j+1}t}{\delta t})$, as defined by (2.2.9) and (2.2.10), which will be useful to write the derivatives of φ_j. We recall that

they have the same support and bounding properties as φ_j. We have that

$$\int \left(\epsilon\sqrt{\varphi_j}\right)_{xx}^2 = \int \epsilon_{xx}^2\varphi_j + \int \frac{\epsilon_x^2}{(\delta t)^2}\frac{\varphi_{1,j}^2}{\varphi_j} + \frac{1}{4}\int \frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{2,j}^2}{\varphi_j} + \frac{1}{16}\int \frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{1,j}^4}{\varphi_j^3}$$

$$- \frac{1}{4}\int \frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{2,j}\varphi_{1,j}^2}{\varphi_j^2} + 2\int \frac{\epsilon_{xx}\epsilon_x}{\delta t}\varphi_{1,j} + \int \frac{\epsilon_{xx}\epsilon}{(\delta t)^2}\varphi_{2,j} \tag{2.4.40}$$

$$- \frac{1}{2}\int \frac{\epsilon_{xx}\epsilon}{(\delta t)^2}\frac{\varphi_{1,j}^2}{\varphi_j} + \int \frac{\epsilon_x\epsilon}{(\delta t)^3}\frac{\varphi_{1,j}\varphi_{2,j}}{\varphi_j} - \frac{1}{2}\int \frac{\epsilon_x\epsilon}{(\delta t)^3}\frac{\varphi_{1,j}^3}{\varphi_j^2}.$$

We observe that, for T_3^* large enough, and by using the inequalities that define ψ, the error terms can be bounded by $\frac{C}{\delta t}\|\epsilon\|_{H^2}^2 \leq \frac{\rho}{100}\|\epsilon\|_{H^2}^2$. The computation for the other terms is similar and the same bound can be used for the error terms.

Because $\epsilon\sqrt{\varphi_k^b}$ satisfies the orthogonality conditions, we can apply Proposition 2.15, and obtain that

$$\mathcal{Q}_{\alpha_k,\beta_k}\left[\epsilon\sqrt{\varphi_k^b}\right] \geq \mu_k^b\left\|\epsilon\sqrt{\varphi_k^b}\right\|_{H^2}^2 - \frac{1}{\mu_k^b}\left(\int \epsilon\sqrt{\varphi_k^b}\widetilde{B_k}\right)^2. \tag{2.4.41}$$

To finish, $\|\epsilon\sqrt{\varphi_k^b}\|_{H^2}^2$ is $\int(\epsilon^2+\epsilon_x^2+\epsilon_{xx}^2)\varphi_k^b(t)$ plus several error terms as in (2.4.40). □

Lemma 2.20. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that if $T^* \geq T_3^*$, then for any $\epsilon \in H^2(\mathbb{R})$, for any $t \in [t^*, T]$, we have that*
if

$$\int \widetilde{R_l}(t)\epsilon\sqrt{\varphi_l^s(t)} = \int \widetilde{R_l}_x(t)\epsilon\sqrt{\varphi_l^s(t)} = 0, \tag{2.4.42}$$

then

$$\mathcal{Q}_l^s[\epsilon] \geq \mu\int\left(\epsilon^2+\epsilon_x^2+\epsilon_{xx}^2\right)\varphi_l^s(t) - \rho\|\epsilon\|_{H^2}^2. \tag{2.4.43}$$

Proof. As in the previous proof, we write $\mathcal{Q}_l^s[\epsilon]$ as $\mathcal{Q}_{c_l}^s[\epsilon\sqrt{\varphi_l^s}]$ (with $Q = \widetilde{R_l}$) plus several error terms, that are all bounded by $\rho\|\epsilon\|_{H^2}^2$ if T_3^* is chosen large enough. However, $\mathcal{Q}_{c_l}^s[\epsilon\sqrt{\varphi_l^s}]$ is not appropriate in order to have coercivity, the appropriate quadratic form is $\mathcal{Q}_{c_l+c_{0,l}(t)}^s[\epsilon\sqrt{\varphi_l^s}]$. This is why, we need to bound the difference between $\mathcal{Q}_{c_l}^s[\epsilon\sqrt{\varphi_l^s}]$ and $\mathcal{Q}_{c_l+c_{0,l}(t)}^s[\epsilon\sqrt{\varphi_l^s}]$. This difference is

$$c_{0,l}(t)\left(\int \epsilon_x^2\varphi_j - 3\int \widetilde{R_l}^2\epsilon^2\varphi_j\right) + c_l c_{0,l}(t)\int \epsilon^2\varphi_j + c_{0,l}(t)^2\frac{1}{2}\int \epsilon^2\varphi_j, \tag{2.4.44}$$

which can, because of the bound for $c_{0,l}(t)$, for T_3^* large enough (depending on A), be bounded by $\rho\|\epsilon\|_{H^2}^2$.

Now, $\epsilon\sqrt{\varphi_l^s}$ satisfies the orthogonality conditions we need, and as in the previous proof we may apply coercivity. □

Proof of Proposition 2.13. We will now use the Lemma 2.19 and its version for solitons (Lemma 2.20) for $\epsilon = \varepsilon(t)$. From this, we deduce that for $\rho > 0$ small enough, we have that

$$\sum_{j=1}^{J}\mathcal{Q}_j[\varepsilon(t)] \geq \mu\|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu}\sum_{k=1}^{K}\left(\int \varepsilon(t)\widetilde{B_k}\sqrt{\varphi_k^b}\right)^2, \tag{2.4.45}$$

for a suitable constant $\mu > 0$. This means that for T_3^* large enough, by taking, if needed, a smaller constant μ,

$$H_2[\varepsilon(t)] \geq \mu\|\varepsilon\|_{H^2}^2 - \frac{1}{\mu}\sum_{k=1}^{K}\left(\int \varepsilon\widetilde{B_k}\sqrt{\varphi_k^b}\right)^2. \tag{2.4.46}$$

The proof of Proposition 2.13 is now completed. □

2.5. **Proof of Proposition 1.9 (Bootstrap).** We recall that p_n from Proposition 1.9 is denoted by p and T_n is denoted by T in what follows, in order to simplify the notations. We do the proof that follows under the assumption (2.2.1), so that the Propositions proved above are true for $t \in [t^*, T]$.

The aim of this subsection is to complete the proof of Proposition 1.9 by using the Propositions 2.12 and 2.13.

We note that by Lemma 2.7, the conservation of $F[p](t)$ and the definition of $\mathcal{H}[p]$, we have for any $t \in [t^*, T]$, that

$$\left|\mathcal{H}[p](T) - \mathcal{H}[p](t)\right| \leq \frac{CA^2}{\delta^2 t} e^{-2\theta t}. \tag{2.5.1}$$

Thus, for any $t \in [t^*, T]$,

$$\mathcal{H}[p](t) \leq \mathcal{H}[p](T) + \frac{CA^2}{\delta^2 t} e^{-2\theta t}. \tag{2.5.2}$$

From Proposition 2.12,

$$\begin{aligned} &\Big|\mathcal{H}\big[\widetilde{P} + \varepsilon\big](t) - H_2[\varepsilon](t) \\ &\qquad - \sum_{j=1}^{J} \Big(F\big[\widetilde{P}_j\big](t) + 2\big(b_j^2 - a_j^2\big)E\big[\widetilde{P}_j\big](t) + \big(a_j^2 + b_j^2\big)^2 M\big[\widetilde{P}_j\big](t)\Big)\Big| \\ &\qquad \leq Ce^{-2\theta t} + C\|\varepsilon\|_{H^2} e^{-2\theta t} + C\|\varepsilon\|_{H^2}^3 \leq Ce^{-2\theta t} + \frac{\mu}{100}\|\varepsilon\|_{H^2}^2. \end{aligned} \tag{2.5.3}$$

In order to obtain the last line, we use the fact that $\|\varepsilon(t)\|_{H^2} \leq CAe^{-\theta t}$, and we take $T^* \geq T_5^*$ for T_5^* large enough (depending on A) so that $\|\varepsilon\|_{H^2} \leq C$ and $C\|\varepsilon(t)\|_{H^2} \leq \frac{\mu}{100}$, and thus $C\|\varepsilon(t)\|_{H^2}^3 \leq \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2$.

We remark that if $P_j = B_k$ is a breather, then $F[\widetilde{P}_j]$, $E[\widetilde{P}_j]$ and $M[\widetilde{P}_j]$ are all constants in time. If $P_j = R_l$ is a soliton and we denote q the basic ground state (i.e. the ground state for $c = 1$), we have the following:

$$M[\widetilde{R}_l](t) = \big(c_l + c_{0,l}(t)\big)^{1/2} M[q], \tag{2.5.4}$$

$$E[\widetilde{R}_l](t) = \big(c_l + c_{0,l}(t)\big)^{\frac{3}{2}} E[q], \tag{2.5.5}$$

$$F[\widetilde{R}_l](t) = \big(c_l + c_{0,l}(t)\big)^{5/2} F[q]. \tag{2.5.6}$$

Using that, we can simplify $\mathcal{R}_l(t) := F[\widetilde{R}_l](t) + 2c_l E[\widetilde{R}_l](t) + c_l^2 M[\widetilde{R}_l](t)$ as follows:

$$\begin{aligned} \mathcal{R}_l(t) &= \big(c_l + c_{0,l}(t)\big)^{5/2} F[q] + 2c_l\big(c_l + c_{0,l}(t)\big)^{\frac{3}{2}} E[q] \\ &\quad + c_l^2\big(c_l + c_{0,l}(t)\big)^{1/2} M[q] \\ &= c_l^{5/2}\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{5/2} F[q] + 2c_l^{5/2}\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{\frac{3}{2}} E[q] \\ &\quad + c_l^{5/2}\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{1/2} M[q]. \end{aligned} \tag{2.5.7}$$

Note that from Lemma 2.8, $|c_{0,l}(t)|^3 \leq CA^3 e^{-\theta t} e^{-2\theta t}$. That is why, if we take T_5^* eventually larger, $|c_{0,l}(t)|^3 \leq Ce^{-2\theta t}$. For this reason, we will do Taylor expansions of order 2 of (2.5.7):

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{5/2} = 1 + \frac{5}{2}\frac{c_{0,l}(t)}{c_l} + \frac{15}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O\big(e^{-2\theta t}\big), \tag{2.5.8}$$

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{\frac{3}{2}} = 1 + \frac{3}{2}\frac{c_{0,l}(t)}{c_l} + \frac{3}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O\big(e^{-2\theta t}\big), \tag{2.5.9}$$

$$\left(1+\frac{c_{0,l}(t)}{c_l}\right)^{1/2}=1+\frac{1}{2}\frac{c_{0,l}(t)}{c_l}-\frac{1}{8}\frac{c_{0,l}(t)^2}{c_l^2}+O\left(e^{-2\theta t}\right). \tag{2.5.10}$$

This allows us to write:

$$\begin{aligned}\mathcal{R}_l(t)=c_l^{5/2}\big(F[q]+2E[q]+M[q]\big)+c_l^{\frac{3}{2}}c_{0,l}(t)\left(\frac{5}{2}F[q]+3E[q]+\frac{1}{2}M[q]\right)\\ +c_l^{1/2}c_{0,l}(t)^2\left(\frac{15}{8}F[q]+\frac{3}{4}E[q]-\frac{1}{8}M[q]\right)+O\left(e^{-2\theta t}\right).\end{aligned} \tag{2.5.11}$$

Now, $c_l^{5/2}(F[q]+2E[q]+M[q])$ is constant in time. For both other terms, we use that $M[q]=2$, $E[q]=-\frac{2}{3}$ and $F[q]=\frac{2}{5}$, and we see that $\frac{5}{2}F[q]+3E[q]+\frac{1}{2}M[q]=0$ and $\frac{15}{8}F[q]+\frac{3}{4}E[q]-\frac{1}{8}M[q]=0$. This allows us to write:

$$\mathcal{R}_l(t)=\frac{16}{15}c_l^{5/2}+O\left(e^{-2\theta t}\right). \tag{2.5.12}$$

From this, we deduce that

$$\mathcal{R}_l(t)-\mathcal{R}_l(T)=O\left(e^{-2\theta t}\right). \tag{2.5.13}$$

By using that $\mathcal{H}[p](T)=\mathcal{H}[P](T)=\mathcal{H}[\widetilde{P}](T)$, the equations (2.5.3) and (2.5.2), Claim 2.14, and the fact that for $t\geq T_4^*$, $O(\|\varepsilon(t)\|_{H^2}^3)\leq\frac{\mu}{100}\|\varepsilon\|_{H^2}^2$, we have that

$$\begin{aligned}&H_2[\varepsilon](t)\\ &\leq\mathcal{H}[p](t)+Ce^{-2\theta t}+\frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2\\ &\quad-\sum_{j=1}^{J}\left(F\left[\widetilde{P}_j\right](t)+2\big(b_j^2-a_j^2\big)E\left[\widetilde{P}_j\right](t)+\big(a_j^2+b_j^2\big)^2M\left[\widetilde{P}_j\right](t)\right)\\ &\leq\mathcal{H}[\widetilde{P}](T)+C\left(\frac{A^2}{\delta^2t}+1\right)e^{-2\theta t}+\frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2\\ &\quad-\sum_{j=1}^{J}\left(F\left[\widetilde{P}_j\right](t)+2\big(b_j^2-a_j^2\big)E\left[\widetilde{P}_j\right](t)+\big(a_j^2+b_j^2\big)^2M\left[\widetilde{P}_j\right](t)\right)\\ &\leq\mathcal{H}\left[\widetilde{P}\right](T)+C\left(\frac{A^2}{\delta^2t}+1\right)e^{-2\theta t}+\frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2+\sum_{l=1}^{L}\big(\mathcal{R}_l(T)-\mathcal{R}_l(t)\big)\\ &\quad-\sum_{j=1}^{J}\left(F\left[\widetilde{P}_j\right](T)+2\big(b_j^2-a_j^2\big)E\left[\widetilde{P}_j\right](T)+\big(a_j^2+b_j^2\big)^2M\left[\widetilde{P}_j\right](T)\right)\\ &\leq C\left(\frac{A^2}{\delta^2t}+1\right)e^{-2\theta t}+\frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2.\end{aligned} \tag{2.5.14}$$

From Proposition 2.13, we deduce (by taking a smaller constant μ) that

$$\mu\|\varepsilon\|_{H^2}^2\leq C\left(\frac{A^2}{\delta^2t}+1\right)e^{-2\theta t}+\frac{1}{\mu}\sum_{k=1}^{K}\left(\int\varepsilon\widetilde{B_k}\sqrt{\varphi_k^b}\right)^2. \tag{2.5.15}$$

We will now need to establish a result close to Lemma 2.7. We set for any $j=1,...,J$:

$$m_j(t):=\int\frac{1}{2}p^2(t,x)\sqrt{\varphi_j(t,x)}\,dx:=m_j[p](t). \tag{2.5.16}$$

Lemma 2.21. *There exists $C>0$, $T_6^*=T_6^*(A)$ such that, if $T^*\geq T_6^*$, for any $j=1,...,J$, for any $t\in[t^*,T]$,*

$$\big|m_j(T)-m_j(t)\big|\leq\frac{C}{\delta^2t}A^2e^{-2\theta t}. \tag{2.5.17}$$

Proof. We compute:

$$\begin{aligned}\frac{d}{dt}\int \frac{1}{2}p^2(t,x)\sqrt{\varphi_j(t,x)}\,dx \\ = \frac{1}{2\delta t}\int\left(-\frac{3}{2}p_x^2+\frac{3}{4}p^4\right)\frac{\varphi_{1,j}}{\sqrt{\varphi_j}} - \frac{1}{2(\delta t)^2}\int p_x p\frac{\varphi_{2,j}}{\sqrt{\varphi_j}} \\ + \frac{1}{4(\delta t)^2}\int p_x p\frac{\varphi_{1,j}^2}{\varphi_j^{\frac{3}{2}}} - \frac{1}{4}\int p^2\frac{x}{\delta t^2}\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}.\end{aligned} \tag{2.5.18}$$

From the inequalities that define ψ, we find that

$$\left|\frac{d}{dt}\int \frac{1}{2}p^2(t,x)\sqrt{\varphi_j(t,x)}\,dx\right| \leq \frac{C}{\delta^2 t}\int_{\Omega_j(t)\cup\Omega_{j+1}(t)}\left(p_x^2+p^2+p^4\right). \tag{2.5.19}$$

From now on, we can follow the proof of Lemma 2.7. □

Now, we observe the following:

$$\int\left(\widetilde{P}+\varepsilon\right)^2\sqrt{\varphi_k^b} = \int\widetilde{B_k}^2 + 2\int\widetilde{B_k}\varepsilon\sqrt{\varphi_k^b} + \int\varepsilon^2\sqrt{\varphi_k^b} + \mathrm{Err}, \tag{2.5.20}$$

where Err stands for the other terms of the sum, which we consider as error terms, and we will show that they are bounded by $Ce^{-\theta t}$.

For $i \neq j$ and any h (if $P_j = B_k$ is a breather),

$$\left|\int\widetilde{P_i}\widetilde{P_h}\sqrt{\varphi_j}\right| \leq C\int_{-\delta t+\sigma_j t}^{\delta t+\sigma_{j+1}t} e^{-\frac{\beta}{2}|x-v_i t|}\,dx \leq Ce^{-\theta t}, \tag{2.5.21}$$

and

$$\begin{aligned}\left|\int\widetilde{P_i}\varepsilon\sqrt{\varphi_j}\right| &\leq \sqrt{\left(\int\widetilde{P_i}^2\varphi_j\right)\left(\int\varepsilon^2\right)} \\ &\leq Ce^{-\frac{\theta}{2}t}\|\varepsilon\|_{H^2} \leq CAe^{-\theta t}e^{-\frac{\theta}{2}t} \leq Ce^{-\theta t},\end{aligned} \tag{2.5.22}$$

where $T^* \geq T_7^*$ with T_7^* being large enough depending on A. If we use the calculations we have made in the proof of Claim 2.14, we see that

$$\left|\int\widetilde{P_j}^2 - \int\widetilde{P_j}^2\sqrt{\varphi_j}\right| \leq Ce^{-\theta t}. \tag{2.5.23}$$

This proves the bound for the error terms.

Now, we study the variations of (2.5.20). We know that $\int\widetilde{P_j}^2 = \int\widetilde{B_k}^2$ has no variations. We can apply Lemma 2.21 for $\int(\widetilde{P}+\varepsilon)^2\sqrt{\varphi_j}$. By writing the difference of the equation (2.5.20) between t and T, and using that $\varepsilon(T) = 0$, we deduce, for $T^* \geq \max(T_6^*, T_7^*)$, that

$$\begin{aligned}\left|\int\widetilde{P_j}\varepsilon\sqrt{\varphi_j}(t)\right| &\leq C\left(\frac{A^2}{\delta^2 t}+1\right)e^{-\theta t} + \|\varepsilon\|_{H^2}^2 \\ &\leq C\left(\frac{A^2}{\delta^2 t}+1\right)e^{-\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}.\end{aligned} \tag{2.5.24}$$

Thus,

$$\begin{aligned}\mu\|\varepsilon\|_{H^2}^2 &\leq C\left(\frac{A^2}{\delta^2 t}+1\right)e^{-2\theta t} + \frac{1}{\mu}\sum_{j=1}^{J}\left(\int\varepsilon\widetilde{P_j}\sqrt{\varphi_j}\right)^2 \\ &\leq C\left(\frac{A^4}{\delta^4 t}+1\right)e^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2.\end{aligned} \tag{2.5.25}$$

Therefore,

$$\|\varepsilon(t)\|_{H^2}^2 \leq C\left(\frac{A^4}{\delta^4 t}+1\right)e^{-2\theta t}. \tag{2.5.26}$$

By using (2.5.26), the mean-value theorem and Lemma 2.8, we deduce that for $t \in [t^*, T]$,

$$
\begin{aligned}
\left\|p(t) - P(t)\right\|_{H^2} &\le \left\|\varepsilon(t)\right\|_{H^2} + \left\|\widetilde{P}(t) - P(t)\right\|_{H^2} \\
&\le C\left(\sqrt{\frac{A^4}{\delta^4 t} + 1}\right) e^{-\theta t} \\
&\quad + C\left(\sum_{k=1}^{K} \left(|x_{1,k}(t)| + |x_{2,k}(t)|\right) + \sum_{l=1}^{L} \left(|x_{0,l}(t)| + |c_{0,l}(t)|\right)\right) \\
&\le C\left(\sqrt{\frac{A^4}{\delta^4 t} + 1}\right) e^{-\theta t} + C\sum_{k=1}^{K} \left(\left|\int_t^T x'_{1,k}(s)\, ds\right| + \left|\int_t^T x'_{2,k}(s)\, ds\right|\right) \\
&\quad + C\sum_{l=1}^{L} \left(\left|\int_t^T x'_{0,l}(s)\, ds\right| + \left|\int_t^T c'_{0,l}(s)\, ds\right|\right) \\
&\le C\left(\frac{A^4}{\delta^4 t} + 1\right) e^{-\theta t} + C\left(\int_t^T \left\|\varepsilon(s)\right\|_{H^2} ds + \int_t^T e^{-\theta s}\, ds\right) \\
&\le C\left(\frac{A^4}{\delta^4 t} + 1\right) e^{-\theta t}.
\end{aligned}
\tag{2.5.27}
$$

We take $A = 4C$ (where C is a constant that can be used anywhere in the proof above) and

$$
T^* := \max\left(T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, T_6^*, T_7^*, T_8^*\right) \tag{2.5.28}
$$

(depending on A), where $T_8^* := T_8^*(A)$ is such that for $t \ge T_8^*$, $\frac{A^4}{\delta^4 t} \le 1$. And thus, for any $t \in [t^*, T]$,

$$
C\left(\frac{A^4}{\delta^4 t} + 1\right) \le 2C = \frac{A}{2}, \tag{2.5.29}
$$

which is exactly what we wanted to prove.

3. p is a smooth multi-breather

Our goal here is to prove Proposition 1.10.

3.1. **Estimates in higher order Sobolev norms.** Firstly, we notice that the proposition is already established for $s = 2$. We note also that if this proposition is proved for an $s \ge 2$ with a corresponding constant A_s, then this proposition is also valid for any $s' \le s$ with the same constant A_s. This means that A_s can possibly increase with s and that this proposition is already established for $0 \le s \le 2$. From now on, we will denote (as before) p_n by p, T_n by T and $p_n - P$ by v, and make sure that the constant A_s that we will obtain in the proof does not depend on n (although it will depend on s). For the constant θ, we will take the usual value: $\theta := \frac{\beta\tau}{32}$. For the constant T^*, we will also take the value that works for Proposition 1.6.

We will prove the proposition by induction on s (it is sufficient to prove the proposition for any integer s). Let $s \ge 3$. We will prove the proposition for s, assuming that the proposition is true for any $0 \le s' \le s - 1$.

Let us deduce from the (mKdV) equation the equation satisfied by v:

$$\begin{aligned} v_t &= p_t - \sum_{j=1}^{J} P_{jt} \\ &= -\left(p_{xx} + p^3 - \sum_{j=1}^{J} P_{jxx} - \sum_{j=1}^{J} P_j^3 \right)_x \\ &= -\left(v_{xx} + (v+P)^3 - \sum_{j=1}^{J} P_j^3 \right)_x \\ &= -\left(v_{xx} + v^3 + 3v^2P + 3vP^2 + P^3 - \sum_{j=1}^{J} P_j^3 \right)_x . \end{aligned} \tag{3.1.1}$$

Firstly, we compute $\frac{d}{dt}\int (\partial_x^s v)^2$ by integration by parts:

$$\begin{aligned} \frac{d}{dt}\int \left(\partial_x^s v\right)^2 &= 2\int \left(\partial_x^s v_t\right)\left(\partial_x^s v\right) \\ &= -2\int \partial_x^{s+1}\left(v_{xx} + v^3 + 3v^2P + 3vP^2 + P^3 - \sum_{j=1}^{J} P_j^3 \right)\left(\partial_x^s v\right) \\ &= 2(-1)^{s+1}\int \partial_x^{2s+1}\left(P^3 - \sum_{j=1}^{J} P_j^3 \right) v - 2\int \partial_x^{s+1}\left(v^3\right)\left(\partial_x^s v\right) \\ &\quad - 6\int \partial_x^{s+1}\left(v^2P\right)\left(\partial_x^s v\right) - 6\int \partial_x^{s+1}\left(vP^2\right)\left(\partial_x^s v\right), \end{aligned} \tag{3.1.2}$$

because $\int (\partial_x^{s+3} v)(\partial_x^s v) = -\int (\partial_x^{s+2} v)(\partial_x^{s+1} v) = 0$.

We will now bound above each of the terms of the obtained sum. By Sobolev embedding, Proposition 2.3 and Proposition 1.6,

$$\begin{aligned} \left| \int \partial_x^{2s+1}\left(P^3 - \sum_{j=1}^{J} P_j^3 \right) v \right| &\le \|v\|_{L^\infty} \int \left| \partial_x^{2s+1}\left(P^3 - \sum_{j=1}^{J} P_j^3 \right) \right| \\ &\le C\|v\|_{H^1} e^{-\beta\tau t/2} \\ &\le CAe^{-\theta t} e^{-\beta\tau t/2} \\ &\le CAe^{-2\theta t} \le CA_{s-1}^2 e^{-2\theta t}, \end{aligned} \tag{3.1.3}$$

where $C \ge 0$ is a constant that depends only on s.

We observe that

$$\begin{aligned} \partial_x^{s+1}\left(v^3\right) &= 3\left(\partial_x^{s+1} v\right)v^2 + 6(s+1)\left(\partial_x^s v\right)v_x v + Z_1\left(v, v_x, ..., \partial_x^{s-1} v\right), \\ \partial_x^{s+1}\left(v^2P\right) &= 2\left(\partial_x^{s+1} v\right)vP + 2(s+1)\left(\partial_x^s v\right)(vP)_x \\ &\quad + Z_2\left(v, v_x, ..., \partial_x^{s-1} v, P, P_x, ..., \partial_x^{s+1} P\right), \end{aligned} \tag{3.1.4}$$

where Z_1 and Z_2 are homogeneous polynomials of degree 3 with constant coefficients.

Now, let us look for a bound for $\int \partial_x^{s+1}(v^3)(\partial_x^s v)$. Firstly, by integration by parts,

$$\begin{aligned} \int \partial_x^{s+1}\left(v^3\right)\left(\partial_x^s v\right) &= \frac{3}{2}\int \left(\left(\partial_x^s v\right)^2\right)_x v^2 + 3(s+1)\int \left(\partial_x^s v\right)^2\left(v^2\right)_x + \int \left(\partial_x^s v\right) Z_1 \\ &= \frac{6(s+1)-3}{2}\int \left(\partial_x^s v\right)^2\left(v^2\right)_x + \int \left(\partial_x^s v\right) Z_1. \end{aligned} \tag{3.1.5}$$

Then, we bound above each of the terms of the obtained sum:

$$\begin{aligned}\left|\int (\partial_x^s v)^2 (v^2)_x\right| &\le C\|v\|_{L^\infty}\|v_x\|_{L^\infty}\int \left(\partial_x^s v\right)^2\\ &\le C\|v\|_{H^2}^2\int \left(\partial_x^s v\right)^2\\ &\le C\left(\|p\|_{H^2}+\|P\|_{H^2}\right)Ae^{-\theta t}\int \left(\partial_x^s v\right)^2\\ &\le CC_0Ae^{-\theta t}\int \left(\partial_x^s v\right)^2 \le CA_{s-1}e^{-\theta t}\int \left(\partial_x^s v\right)^2.\end{aligned} \tag{3.1.6}$$

We have actually shown in the computation above that $\|v\|_{H^2}^2$ can be bounded above by $\|v\|_{H^2}$ (with a constant that depends only on problem data), and therefore the degree of $\|v\|_{H^2}$ can be lowered without harm in the upper bound. We will use this fact again for the rest of the proof. In fact, all what it means is that, for several terms, what we have is more than what we need.

By the Cauchy-Schwarz and Gagliardo-Nirenberg-Sobolev inequalities,

$$\begin{aligned}\left|\int \left(\partial_x^s v\right) Z_1\right| &\le C\int \left|\partial_x^s v\right|\left(\sum_{s'=0}^{s-1}\left|\partial_x^{s'} v\right|^3\right)\\ &\le C\left(\int \left|\partial_x^s v\right|^2\right)^{1/2}\sum_{s'=0}^{s-1}\left(\int \left|\partial_x^{s'} v\right|^6\right)^{1/2}\\ &\le C\left(\int \left|\partial_x^s v\right|^2\right)^{1/2}\sum_{s'=0}^{s-1}\left(\int \left|\partial_x^{s'} v\right|^2\right)\left(\int \left|\partial_x^{s'+1} v\right|^2\right)^{1/2}\\ &\le C\sum_{s'=0}^{s-1}\left(\int \left|\partial_x^{s'} v\right|^2\right)\left(\int \left|\partial_x^s v\right|^2+\int \left|\partial_x^{s'+1} v\right|^2\right)\\ &\le CA_{s-1}^2e^{-2\theta t}+CA_{s-1}e^{-\theta t}\int \left|\partial_x^s v\right|^2.\end{aligned} \tag{3.1.7}$$

Similarly, we bound $\int \partial_x^{s+1}(v^2P)(\partial_x^s v)$. By integration by parts,

$$\begin{aligned}\int \partial_x^{s+1}\left(v^2P\right)\left(\partial_x^s v\right) &= \int \left(\left(\partial_x^s v\right)^2\right)_x vP + 2(s+1)\int \left(\partial_x^s v\right)^2\left(vP\right)_x + \int \left(\partial_x^s v\right) Z_2\\ &= (2s+1)\int \left(\partial_x^s v\right)^2\left(vP\right)_x + \int \left(\partial_x^s v\right) Z_2.\end{aligned} \tag{3.1.8}$$

We bound above each of the terms of the obtained sum, starting by

$$\begin{aligned}\left|\int \left(\partial_x^s v\right)^2\left(vP\right)_x\right| &\le C(\|v\|_{L^\infty}+\|v_x\|_{L^\infty})\int \left(\partial_x^s v\right)^2\\ &\le CAe^{-\theta t}\int \left(\partial_x^s v\right)^2.\end{aligned} \tag{3.1.9}$$

The upper bound of $\left|\int (\partial_x^s v) Z_2\right|$ is similar to (3.1.7) above:

$$\left|\int \left(\partial_x^s v\right) Z_2\right| \le CA_{s-1}^2e^{-2\theta t}+CA_{s-1}e^{-\theta t}\int \left|\partial_x^s v\right|^2. \tag{3.1.10}$$

$\int \partial_x^{s+1}(vP^2)(\partial_x^s v)$ remains to be bounded. By integration by parts,

$$\begin{aligned}\int \partial_x^{s+1}\big(vP^2\big)\big(\partial_x^s v\big) &= -\int \partial_x^{s+2}\big(vP^2\big)\big(\partial_x^{s-1}v\big)\\ &= -\int \big(\partial_x^{s+2}v\big)\big(\partial_x^{s-1}v\big)P^2 - (s+2)\int \big(\partial_x^{s+1}v\big)\big(\partial_x^{s-1}v\big)\big(P^2\big)_x\\ &- \frac{(s+2)(s+1)}{2}\int \big(\partial_x^{s}v\big)\big(\partial_x^{s-1}v\big)\big(P^2\big)_{xx} + \int \big(\partial_x^{s-1}v\big)Z_3^0(v,v_x,...,\partial_x^{s-1}v)\\ &= \frac{1}{2}\int \Big(\big(\partial_x^s v\big)^2\Big)_x P^2 + (s+1)\int \big(\partial_x^s v\big)^2\big(P^2\big)_x\\ &- \frac{s(s+1)}{4}\int \Big(\big(\partial_x^{s-1}v\big)^2\Big)_x\big(P^2\big)_{xx} + \int \big(\partial_x^{s-1}v\big)Z_3^0(v,v_x,...,\partial_x^{s-1}v)\\ &= \frac{2s+1}{2}\int \big(\partial_x^s v\big)^2\big(P^2\big)_x + \int \big(\partial_x^{s-1}v\big)Z_3(v,v_x,...,\partial_x^{s-1}v),\end{aligned} \tag{3.1.11}$$

where Z_3^0 and Z_3 are homogeneous polynomials of degree 1 whose coefficients are polynomials in P and its space derivatives. We have that $|Z_3| \le C(\sum_{s'=0}^{s-1}|\partial_x^{s'}v|)$. Therefore,

$$\left|\int \big(\partial_x^{s-1}v\big)Z_3\right| \le CA_{s-1}^2 e^{-2\theta t}. \tag{3.1.12}$$

Thus, by taking the sum of all those inequalities, we obtain:

$$\left|\frac{d}{dt}\int \big(\partial_x^s v\big)^2 + 3(2s+1)\int \big(\partial_x^s v\big)^2\big(P^2\big)_x\right| \le CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}e^{-\theta t}\int \big|\partial_x^s v\big|^2. \tag{3.1.13}$$

Next, we perform similar computations for $\frac{d}{dt}\int(\partial_x^{s-1}v)^2P^2$:

$$\begin{aligned}\frac{d}{dt}\int(\partial_x^{s-1}v)^2P^2 &= 2\int \big(\partial_x^{s-1}v_t\big)\big(\partial_x^{s-1}v\big)P^2 + 2\int \big(\partial_x^{s-1}v\big)^2P_tP\\ &= -2\int \partial_x^s\left(v_{xx}+v^3+3v^2P+3vP^2+P^3-\sum_{j=1}^J P_j^3\right)\big(\partial_x^{s-1}v\big)P^2\\ &- 2\int \big(\partial_x^{s-1}v\big)^2\left(P_{xx}+\sum_{j=1}^J P_j^3\right)_x P.\end{aligned} \tag{3.1.14}$$

Let us study each of the obtained terms.

Firstly,

$$\begin{aligned}-2\int \big(\partial_x^{s+2}v\big)\big(\partial_x^{s-1}v\big)P^2 &= 2\int \big(\partial_x^{s+1}v\big)\big(\partial_x^{s}v\big)P^2 + 2\int \big(\partial_x^{s+1}v\big)\big(\partial_x^{s-1}v\big)\big(P^2\big)_x\\ &= -3\int \big(\partial_x^s v\big)^2\big(P^2\big)_x - 2\int \big(\partial_x^s v\big)\big(\partial_x^{s-1}v\big)\big(P^2\big)_{xx}\\ &= -3\int \big(\partial_x^s v\big)^2\big(P^2\big)_x + \int \big(\partial_x^{s-1}v\big)^2\big(P^2\big)_{xxx}.\end{aligned} \tag{3.1.15}$$

Indeed,

$$\left|\int \big(\partial_x^{s-1}v\big)^2\big(P^2\big)_{xxx}\right| \le CA^2e^{-2\theta t}. \tag{3.1.16}$$

Secondly,

$$\left|\int \partial_x^s\left(P^3-\sum_{j=1}^J P_j^3\right)\big(\partial_x^{s-1}v\big)P^2\right| \le CA_{s-1}^2e^{-2\theta t} \tag{3.1.17}$$

can be obtained similarly to the first part of the proof (starting by an integration by parts to have $\partial_x^{s-2}v$ at the place of $\partial_x^{s-1}v$).

Thirdly,

$$\begin{aligned}\int \partial_x^s\big(v^3\big)\big(\partial_x^{s-1}v\big)P^2 &= 3\int \big(\partial_x^s v\big)\big(\partial_x^{s-1}v\big)v^2P^2 + \int Z_4(v,v_x,...,\partial_x^{s-1}v)P^2\\ &= -\frac{3}{2}\int \big(\partial_x^{s-1}v\big)^2\big(v^2P^2\big)_x + \int Z_4P^2,\end{aligned} \tag{3.1.18}$$

where Z_4 is a homogeneous polynomial of degree 4 with constant coefficients. Both terms are easily bounded by $CA_{s-1}^2e^{-2\theta t}$.

Fourthly, for $\int \partial_x^s(v^2P)(\partial_x^{s-1}v)P^2$ and $\int \partial_x^s(vP^2)(\partial_x^{s-1}v)P^2$, we reason similarly.

Fifthly,

$$\left|\int \big(\partial_x^{s-1}v\big)^2\left(P_{xx}+\sum_{j=1}^{J}P_j^3\right)_x P\right| \le CA_{s-1}^2e^{-2\theta t} \tag{3.1.19}$$

is clear.

Therefore,

$$\left|\frac{d}{dt}\int \big(\partial_x^{s-1}v\big)^2P^2 + 3\int \big(\partial_x^s v\big)^2\big(P^2\big)_x\right| \le CA_{s-1}^2e^{-2\theta t}. \tag{3.1.20}$$

We set

$$F(t) := \int \big(\partial_x^s v\big)^2 - (2s+1)\int \big(\partial_x^{s-1}v\big)^2P^2. \tag{3.1.21}$$

By putting the both parts of the proof together:

$$\left|\frac{d}{dt}F(t)\right| \le CA_{s-1}^2e^{-2\theta t} + CA_{s-1}e^{-\theta t}\int \big|\partial_x^s v\big|^2. \tag{3.1.22}$$

Because $\left|\int(\partial_x^{s-1}v)^2P^2\right| \le CA^2e^{-2\theta t}$, we can write the following upper bound:

$$\int \big(\partial_x^s v\big)^2 \le |F(t)| + CA_{s-1}^2e^{-2\theta t}. \tag{3.1.23}$$

Therefore, we have, for a suitable constant $C>0$ that depends only on s,

$$\left|\frac{d}{dt}F(t)\right| \le CA_{s-1}^2e^{-2\theta t} + CA_{s-1}e^{-\theta t}|F(t)|. \tag{3.1.24}$$

For $t\in[T^*,T]$, by integration between t and T (we recall that $F(T)=0$),

$$\begin{aligned}|F(t)| = |F(T)-F(t)| &= \left|\int_t^T \frac{d}{dt}F(\sigma)\,d\sigma\right| \le \int_t^T\left|\frac{d}{dt}F(\sigma)\right|d\sigma\\ &\le CA_{s-1}^2\int_t^T e^{-2\theta\sigma}\,d\sigma + CA_{s-1}\int_t^T e^{-\theta\sigma}|F(\sigma)|\,d\sigma\\ &\le CA_{s-1}^2e^{-2\theta t} + CA_{s-1}\int_t^T e^{-\theta\sigma}|F(\sigma)|\,d\sigma.\end{aligned} \tag{3.1.25}$$

By Gronwall lemma, for all $t \in [T^*, T]$,

$$\begin{aligned}
|F(t)| &\leq CA_{s-1}^2 e^{-2\theta t} \\
&\quad + CA_{s-1}\int_t^T e^{-\theta\sigma} CA_{s-1}^2 e^{-2\theta\sigma} \exp\left(\int_t^\sigma CA_{s-1}e^{-\theta u}du\right) d\sigma \\
&\leq CA_{s-1}^2 e^{-2\theta t} \\
&\quad + CA_{s-1}^3 \exp\left(\frac{CA_{s-1}}{\theta}e^{-\theta t}\right)\int_t^T e^{-3\theta\sigma}\exp\left(-\frac{CA_{s-1}}{\theta}e^{-\theta\sigma}\right)d\sigma \\
&\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}^3 \exp\left(\frac{CA_{s-1}}{\theta}\right)\int_t^T e^{-3\theta\sigma}\, d\sigma \\
&\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}^3 \exp\left(\frac{CA_{s-1}}{\theta}\right)e^{-3\theta t} \\
&\leq CA_{s-1}^3 \exp\left(\frac{CA_{s-1}}{\theta}\right)e^{-2\theta t}.
\end{aligned} \tag{3.1.26}$$

Therefore,

$$\int \left(\partial_x^s v\right)^2 \leq A_s e^{-2\theta t}, \tag{3.1.27}$$

where $A_s := CA_{s-1}^3 \exp\left(\frac{CA_{s-1}}{\theta}\right)$ and C is a constant large enough that depends only on s. This conclude the proof of Proposition 1.10, and so of Theorem 1.2.

3.2. **Uniformity of constants.** We conclude this section with an explanation regarding Remark 1.3.

In the proof above, the constants that we obtain A, T^*, θ do depend on $P_j(0)$ $(1 \leq j \leq J)$. Actually, we may characterize this dependence. In fact, they do not depend on the initial positions of our objects in the case when our objects are initially ordered in the right order and sufficiently far from each other.

Theorem 3.1. *Given parameters (1.2.1), (1.2.2), (1.2.3) and (1.2.4) which satisfy (1.2.5), there exists $D > 0$ large enough that depends only on α_k, β_k, c_l such that if*

$$\forall j \geq 2, \quad x_j(0) \geq x_{j-1}(0) + D, \tag{3.2.1}$$

then the following holds. We set $\theta := \frac{\beta\tau}{32}$, with β and τ given by (2.0.1) and $p(t)$ the multi-breather associated to P by Proposition 1.7. There exists $A_s \geq 1$ for any $s \geq 2$ that depends only on α_k, β_k, c_l and D such that

$$\forall t \geq 0, \quad \|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}. \tag{3.2.2}$$

Firstly, we will prove that for any $D > 0$, if (3.2.1) is satisfied, then the constants A_s and T^* do only depend on α_k, β_k, c_l and D. Finally, we will prove that if $D > 0$ is large enough with respect to the given parameters, then we can take $T^* = 0$.

To establish the validity of this theorem, it is enough to read again the whole article and to make sure that on any step of the proof, there is no dependence on initial positions of our objects when our objects are initially far from each other for the constant C. This will allow to claim the same for the constants A and T^* (but, these constants may depend on D). This works, but we should change a bit the way we write our results.

For Proposition 2.1, we should write:

$$|\partial_x^n \partial_t^m P_j(t,x)| \leq Ce^{-\beta|x - v_j t - x_j(0)|}. \tag{3.2.3}$$

Therefore, in Proposition 2.3, we have nothing to change, but the constant C do depend on D. This will also be the case in the following propositions and lemmas of this proof.

We should replace $\sigma_j t$ for the definition of φ_j in (2.2.7) and (2.2.8) by $\sigma_j t + \frac{x_{j-1}(0)+x_j(0)}{2}$ to take into account of initial positions. More precisely, we will have for any $j = 2, ..., J-1$,

$$\varphi_j(t,x) := \psi\left(\frac{x - \sigma_j t - \frac{x_{j-1}(0)+x_j(0)}{2}}{\delta t}\right) - \psi\left(\frac{x - \sigma_{j+1} t - \frac{x_j(0)+x_{j+1}(0)}{2}}{\delta t}\right), \tag{3.2.4}$$

and similarly for other definitions.

After having done the modulation with C and T^* depending on D, for Proposition 2.10, we should write:

$$|\partial_x^n \widetilde{P_j}(t,x)| \leq Ce^{-\frac{\beta}{2}|x-v_jt-x_j(0)|}e^{\frac{\beta\tau}{32}t}. \tag{3.2.5}$$

Therefore, with these adaptations, the same proof works to prove that A_s and T^* do depend only on α_k, β_k, c_l and D.

Now, given α_k, β_k, c_l, we choose $D_0 > 0$ in an arbitrary maner. Therefore, we get $A_s(D_0)$ and $T^*(D_0)$ associated to D_0. Let $\Lambda := v_J - v_1$ the maximal difference between two velocities. We set $D := D_0 + \Lambda \cdot T^*(D_0)$. Therefore, if we suppose (3.2.1) in $t = 0$ for D, then we have (3.2.1) in $t = -T^*(D_0)$ for D_0. Therefore, by appliying the intermediate result for D_0, we obtain the desired conclusion with D and A_s that depend on D_0.

4. Uniqueness

p is the multi-breather constructed in the existence part. The goal here is to prove that if a solution u converges to p when $t \to +\infty$ (in some sense), then $u = p$ (under well chosen assumptions).

We prove here two propositions. For both of them, we assume that the velocities of all our objects are distinct (this was also an assumption for the existence). The first proposition does not make more assumptions on velocities of our objects, but it is a partial uniqueness result as we restrict ourselves to the class of super polynomial convergence to the multi-breather. The second proposition assumes that the velocities of all our objects, except possibly one, are positive (this is a new assumption and it is needed because this proof uses monotonicity arguments).

4.1. **A solution converging super polynomialy to a multi-breather is this multi-breather.**

The goal of this subsection is to prove Proposition 1.5.

Remark 4.1. Note that in Proposition 1.5, we don't make any assumptions on the sign of v_1 or v_2. This uniqueness proposition has the same degree of generality as Theorem 1.2.

Proof of Proposition 1.5. Let $p(t)$ be the multi-breather associated to P by Theorem 1.2. Recall that for any s,

$$\|p(t) - P(t)\|_{H^s} = O(e^{-\theta t}), \tag{4.1.1}$$

for a suitable $\theta > 0$.

Let $N > 2$ to be chosen later. We take $u(t)$ an H^2 solution of (mKdV) such that there exists $C_0 > 0$ such that for t large enough,

$$\|u(t) - P(t)\|_{H^2} \leq \frac{C_0}{t^N}. \tag{4.1.2}$$

From that, we may deduce that for t large enough (namely, $t \geq 2C_0$ along with the previous condition),

$$\|u(t) - P(t)\|_{H^2} \leq \frac{1}{2}\frac{1}{t^{N-1}}. \tag{4.1.3}$$

Our goal is to find a condition on N that do not depend on u, such that the condition (4.1.3) on u for t large enough implies that $u \equiv p$.

Because of (4.1.1), the condition (4.1.3) for t large enough is equivalent to: for t large enough,

$$\|u(t) - p(t)\|_{H^2} \leq \frac{1}{t^{N-1}}. \tag{4.1.4}$$

We denote $z(t) := u(t) - p(t)$. Our goal is to find N large enough that do not depend on z, for which we will be able to prove that $z \equiv 0$, given

$$\|z(t)\|_{H^2} \leq \frac{1}{t^{N-1}}, \tag{4.1.5}$$

for t large enough. Because z is a difference of two solutions of (mKdV), we may derive the following equation for z:

$$z_t + \left(z_{xx} + (z+p)^3 - p^3\right)_x = 0. \tag{4.1.6}$$

We divide our proof in several steps.

Step 1. Modulation on z.

For $j = 1, ..., J$, if $P_j = B_k$ is a breather, we denote

$$K_j := \begin{pmatrix} \partial_{x_1} B_k \\ \partial_{x_2} B_k \end{pmatrix}, \tag{4.1.7}$$

and if $P_j = R_l$ is a soliton, we denote:

$$K_j = \partial_x R_l. \tag{4.1.8}$$

We may derive the following equation for K_j:

$$(K_j)_t + \left((K_j)_{xx} + 3P_j^2 K_j\right)_x = 0. \tag{4.1.9}$$

For $j = 1, ..., J$, if $P_j = B_k$ is a breather, let $c_j(t) \in \mathbb{R}^2$ defined for t large enough and if $P_j = R_l$ is a soliton, let $c_j(t) \in \mathbb{R}$ defined for t large enough such that for

$$\widetilde{z}(t) := z(t) + \sum_{j=1}^{J} c_j(t) K_j(t), \tag{4.1.10}$$

where $c_j K_j$ is either a product of two numbers of $\mathbb{R}$ or a scalar product of two vectors of $\mathbb{R}^2$, the following condition is satisfied: for any $j = 1, ..., J$, for t large enough,

$$\int \widetilde{z}(t) K_j(t) \sqrt{\varphi_j(t)} = 0, \tag{4.1.11}$$

where φ_j is defined in Section 2.2 (in this proof, it is OK to take $\delta = 1$). It is possible to do so in a unique way, because the Gram matrix associated to $K_j(t)\sqrt{\varphi_j(t)}$, $1 \le j \le J$, is invertible; which is the case because $K_j(t)\sqrt{\varphi_j(t)}$, $1 \le j \le J$, are linearly independent. This is why $c_j(t)$, $1 \le j \le J$, are defined in a unique way. For the same reason, $c_j(t)$ is obtained linearly from $\int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \le k \le J$, with coefficients that depend only on K_k, $1 \le k \le J$. This is why, from Cauchy-Schwarz, we may deduce the following lemma.

Lemma 4.2. *For any $j = 1, ..., J$, for t large enough, there exists $C > 0$ that do not depend on z, such that*

$$|c_j(t)| \le C \|z(t)\|_{L^2}, \tag{4.1.12}$$

$$\|\widetilde{z}(t)\|_{H^2} \le C \|z(t)\|_{H^2}. \tag{4.1.13}$$

The Gram matrix is C^1 in time and invertible. This is why, its inverse is C^1 in time. Because $\int K_j z \sqrt{\varphi_j}$ are C^1 in time, we deduce by multiplication that $c_j(t)$ are C^1 in time.

By differentiating in time the linear relation that defines $c_j(t)$, we see that $c_j'(t)$ is obtained linearly from $\int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \le k \le J$, and from $\frac{d}{dt} \int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \le k \le J$, with coefficients that depend on K_k, $1 \le k \le J$ (and their derivatives). Because it is easy to see that $\frac{d}{dt} \int K_k(t) z(t) \sqrt{\varphi_k(t)}$ may still be bounded by $C\|z(t)\|_{L^2}$, we deduce that for any $j = 1, ..., J$, for t large enough, there exists $C > 0$ that do not depend on z, such that

$$|c_j'(t)| \le C \|z(t)\|_{L^2}. \tag{4.1.14}$$

We may derive the following equation for $\widetilde{z}$:

$$\widetilde{z}_t + \left(\widetilde{z}_{xx} + 3\widetilde{z}p^2\right)_x = -\left(3z^2 p + z^3\right)_x + \sum_{k=1}^{J} c_k'(t) K_k - 3 \sum_{k=1}^{J} c_k(t) \left((P_k^2 - p^2) K_k\right)_x. \tag{4.1.15}$$

Step 2. A bound for $|c_j'(t)|$.

The goal here is to improve (4.1.14).

Lemma 4.3. *For any $j = 1, ..., J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z, such that*

$$|c_j'(t)| \le C \|\widetilde{z}(t)\|_{H^2} + C e^{-\theta t} \|z(t)\|_{H^2} + C \|z(t)\|_{H^2}^2. \tag{4.1.16}$$

Proof. We may differentiate (4.1.11):

$$
\begin{aligned}
0 &= \frac{d}{dt}\int \widetilde{z}K_j\sqrt{\varphi_j}\\
&= \int \widetilde{z}_t K_j\sqrt{\varphi_j} + \int \widetilde{z}(K_j)_t\sqrt{\varphi_j} + \int \widetilde{z}K_j(\sqrt{\varphi_j})_t\\
&= -\int \left(\widetilde{z}_{xx} + 3\widetilde{z}p^2\right)_x K_j\sqrt{\varphi_j} - \int \left(3z^2p + z^3\right)_x K_j\sqrt{\varphi_j}\\
&+ \sum_{k=1}^{J}\int \left(c_k'(t)\cdot K_k\right)K_j\sqrt{\varphi_j} - 3\sum_{k=1}^{J} c_k(t)\int \left(c_k(t)\cdot\left((P_k^2-p^2)K_k\right)_x\right)K_j\sqrt{\varphi_j}\\
&- \int \widetilde{z}\left((K_j)_{xx} + 3K_jP_j^2\right)_x\sqrt{\varphi_j} + \int \widetilde{z}K_j(\sqrt{\varphi_j})_t.
\end{aligned}
\tag{4.1.17}
$$

We know that $(\sqrt{\varphi_j})_x$ and $(\sqrt{\varphi_j})_t$ are bounded (from inequalities established in Section 2.2). This is why, for any t large enough,

$$
\left|\int \widetilde{z}K_j(\sqrt{\varphi_j})_t\right| \le C\|\widetilde{z}(t)\|_{H^2}. \tag{4.1.18}
$$

For the same reason, after eventually doing an integration by parts, for any t large enough,

$$
\left|\int \left(\widetilde{z}_{xx} + 3\widetilde{z}p^2\right)_x K_j\sqrt{\varphi_j}\right| + \left|\int \widetilde{z}\left((K_j)_{xx} + 3K_jP_j^2\right)_x\sqrt{\varphi_j}\right| \le C\|\widetilde{z}(t)\|_{H^2}. \tag{4.1.19}
$$

$\int (3z^2p + z^3)_x K_j\sqrt{\varphi_j}$ is clearly bounded by $C\|z(t)\|_{H^2}^2$. Finally, we see that $(P_k^2 - p^2)K_k$ is exponentially bounded in time (in Sobolev or L^∞ norm), and using Lemma 4.2, we deduce that

$$
\int \left(c_k(t)\cdot\left((P_k^2-p^2)K_k\right)_x\right)K_j\sqrt{\varphi_j} \le Ce^{-\theta t}\|z(t)\|_{H^2}, \tag{4.1.20}
$$

for a suitable $\theta > 0$ that do not depend on z. This is why, we deduce that for any $j = 1, ..., J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z, such that

$$
\left|\sum_{k=1}^{J}\int \left(c_k'(t)\cdot K_k\right)K_j\sqrt{\varphi_j}\right| \le C\|\widetilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \tag{4.1.21}
$$

We recall that for any $(e_1, e_2) \in (\mathbb{R})^2$ or $(\mathbb{R}^2)^2$, $e_3 \in \mathbb{R}$ or $\mathbb{R}^2$, we have the following equality between two elements of $\mathbb{R}$ or $\mathbb{R}^2$ (where vectors are denoted as a colon)

$$
\left(e_1\cdot e_2\right)e_3 = \left(e_1^T\left(e_2e_3^T\right)\right)^T, \tag{4.1.22}
$$

where ^{T} denotes the transpose.

First of all, because $\int K_kK_j^T\sqrt{\varphi_j}$ converges exponentially to $\int K_kK_j^T$, for $k \neq j$, $\int K_kK_j^T$ is exponentially decreasing, and from (4.1.14), we may write that for any $j = 1, ..., J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z, such that

$$
\left|\left(c_j'(t)^T\int K_jK_j^T\right)^T\right| \le C\|\widetilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \tag{4.1.23}
$$

Now, in the case when $K_j \in \mathbb{R}^2$, using the fact that its components are linearly independent and Cauchy-Schwarz inequality, we deduce the desired lemma. □

Step 3. Coercivity.

We define the following functional quadratic in $\widetilde{z}$:

$$
\begin{aligned}
H(t) &= \frac{1}{2}\int \widetilde{z}_{xx}^2 - \frac{5}{2}\int p^2\widetilde{z}_x^2 + \frac{5}{2}\int p_x^2\widetilde{z}^2 + 5\int pp_{xx}\widetilde{z}^2 + \frac{15}{4}\int p^4\widetilde{z}^2\\
&+ \sum_{j=1}^{J}\left(b_j^2 - a_j^2\right)\left(\int \widetilde{z}_x^2\varphi_j - 3\int p^2\widetilde{z}^2\varphi_j\right) + \sum_{j=1}^{J}\left(a_j^2 + b_j^2\right)^2\frac{1}{2}\int \widetilde{z}^2\varphi_j.
\end{aligned}
\tag{4.1.24}
$$

We will prove the following lemma:

Lemma 4.4. *There exists $C > 0$ that do not depend on z, such that for t large enough,*

$$\|\widetilde{z}(t)\|_{H^2}^2 \leq CH(t) + C\sum_{j=1}^{J}\left(\int \widetilde{z}P_j\right)^2. \tag{4.1.25}$$

Proof. We denote Q_j the quadratic form associated to P_j. We remind that

$$\begin{aligned} Q_j[\varepsilon] := \frac{1}{2}\int \varepsilon_{xx}^2 - \frac{5}{2}\int P_j^2\varepsilon_x^2 + \frac{5}{2}\int (P_j)_x^2\varepsilon^2 + 5\int P_j(P_j)_{xx}\varepsilon^2 \\ + \frac{15}{4}\int P_j^4\varepsilon^2 + \left(b_j^2 - a_j^2\right)\left(\int \varepsilon_x^2 - 3\int P_j^2\varepsilon^2\right) + \left(a_j^2 + b_j^2\right)^2\frac{1}{2}\int \varepsilon^2. \end{aligned} \tag{4.1.26}$$

In any case, we have that for any $j = 1, ..., J$, there exists $\mu_j > 0$, such that if $\varepsilon \in H^2$ satisfies $\int K_j\varepsilon = 0$, then we have

$$Q_j[\varepsilon] \geq \mu_j\|\varepsilon\|_{H^2}^2 - \frac{1}{\mu_j}\left(\int \varepsilon P_j\right)^2. \tag{4.1.27}$$

Here, we apply this coercivity result with $\varepsilon = \widetilde{z}\sqrt{\varphi_j}$ for which the orthogonality conditions (4.1.11) are satisfied. Thus,

$$\|\widetilde{z}\sqrt{\varphi_j}\|_{H^2}^2 \leq CQ_j[\widetilde{z}\sqrt{\varphi_j}] + C\left(\int \widetilde{z}P_j\sqrt{\varphi_j}\right)^2. \tag{4.1.28}$$

We denote:

$$\begin{aligned} Q_j'[\varepsilon] := \frac{1}{2}\int \varepsilon_{xx}^2\varphi_j - \frac{5}{2}\int p^2\varepsilon_x^2\varphi_j + \frac{5}{2}\int p_x^2\varepsilon^2\varphi_j + 5\int pp_{xx}\varepsilon^2\varphi_j \\ + \frac{15}{4}\int p^4\varepsilon^2\varphi_j + \left(b_j^2 - a_j^2\right)\left(\int \varepsilon_x^2\varphi_j - 3\int p^2\varepsilon^2\varphi_j\right) + \left(a_j^2 + b_j^2\right)^2\frac{1}{2}\int \varepsilon^2\varphi_j, \end{aligned} \tag{4.1.29}$$

and we observe that

$$H(t) = \sum_{j=1}^{J} Q_j'[\widetilde{z}(t)]. \tag{4.1.30}$$

In $Q_j'[\widetilde{z}(t)]$, we may replace p by P_j with an error bounded by $Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}^2$, because of (4.1.1) mainly. After that, the expression obtained may be replaced by $Q_j[\widetilde{z}(t)\sqrt{\varphi_j(t)}]$ with an error bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$ (cf. calculations done in the proof of Lemma 2.19). For the same reason, $\|\widetilde{z}\sqrt{\varphi_j}\|_{H^2}^2$ may be replaced by $\int(\widetilde{z}^2 + \widetilde{z}_x^2 + \widetilde{z}_{xx}^2)\varphi_j$ with an error bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$. Therefore, because of

$$\|\widetilde{z}\|_{H^2}^2 = \sum_{j=1}^{J}\int \left(\widetilde{z}^2 + \widetilde{z}_x^2 + \widetilde{z}_{xx}^2\right)\varphi_j, \tag{4.1.31}$$

the fact that $P_j\sqrt{\varphi_j}$ converges exponentially to P_j, and the fact that $\frac{C}{t}$ may be as small as we want if we take t large enough, we deduce the desired lemma. □

Step 4. Modification of H for the sake of simplification.

We define:

$$\begin{aligned} \widetilde{H}(t) := \int \left[\frac{1}{2}\widetilde{z}_{xx}^2 - \frac{5}{2}\left((\widetilde{z}+p)^2(\widetilde{z}+p)_x^2 - p^2p_x^2 - 2\widetilde{z}pp_x^2 - 2\widetilde{z}_xp^2p_x\right)\right. \\ \left. + \frac{1}{4}\left((\widetilde{z}+p)^6 - p^6 - 6\widetilde{z}p^5\right)\right] + \frac{1}{2}\sum_{j=1}^{J}\left(a_j^2 + b_j^2\right)^2\int \widetilde{z}^2\varphi_j \\ + 2\sum_{j=1}^{J}\left(b_j^2 - a_j^2\right)\int\left[\frac{1}{2}\widetilde{z}_x^2 - \frac{1}{4}\left((\widetilde{z}+p)^4 - p^4 - 4\widetilde{z}p^3\right)\right]\varphi_j. \end{aligned} \tag{4.1.32}$$

We observe that the difference between H and $\widetilde{H}$ is bounded by $O\big(\|\widetilde{z}(t)\|_{H^2}^3\big)$. We can thus claim:

Lemma 4.5. *There exists $C > 0$ that do not depend on z, such that for t large enough,*

$$\|\widetilde{z}(t)\|_{H^2}^2 \leq C\widetilde{H}(t) + C\sum_{j=1}^{J}\left(\int \widetilde{z}P_j\right)^2. \tag{4.1.33}$$

Step 5. A bound for $\frac{d\widetilde{H}}{dt}$.

Lemma 4.6. *There exists $C > 0$ and $\theta > 0$ that do not depend on z, such that for t large enough,*

$$\left|\frac{d\widetilde{H}}{dt}\right| \leq \frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2 + Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2. \tag{4.1.34}$$

Proof. We develop the expression of $\widetilde{H}(t)$, we differentiate each term obtained and we use (4.1.15), the fact that p is a solution of (mKdV) and the fact that $(\varphi_j)_t = -\frac{x}{t}(\varphi_j)_x$, where $\frac{x}{t}$ is bounded independently from z because of the compact support of φ_j. We obtain several sorts of terms after doing several integrations by parts and several obvious simplifications.

Several terms are clearly bounded by one of the bounds of the lemma, because in these terms, the cumulated degree of z and $\widetilde{z}$ is larger than 2. As an example, we show how to deal with $\int z_{xxx}z\widetilde{z}_{xx}p$. We use the fact that $z = \widetilde{z} - \sum_{j=1}^{J} c_jK_j$, and we obtain the following:

$$\begin{aligned}\int z_{xxx}z\widetilde{z}_{xx}p &= \int \widetilde{z}_{xxx}\widetilde{z}\widetilde{z}_{xx}p - \int \widetilde{z}_{xxx}\left(\sum_{j=1}^{J} c_jK_j\right)\widetilde{z}_{xx}p \\ &\quad - \int\left(\sum_{j=1}^{J} c_j(K_j)_{xxx}\right)\widetilde{z}\widetilde{z}_{xx}p + \int\left(\sum_{j=1}^{J} c_j(K_j)_{xxx}\right)\left(\sum_{j=1}^{J} c_jK_j\right)\widetilde{z}_{xx}p.\end{aligned} \tag{4.1.35}$$

It is easy to see that any of these terms is bounded as we want in the lemma (several of them are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$, the last one is bounded by $C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2$), because of Lemma 4.2 and of (4.1.5).

Other terms contain $\widetilde{z}$ quadratically and contain $(\varphi_j)_x$. And, $(\varphi_j)_x$ is bounded by $\frac{C}{t}$. This is why, such terms are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$.

Several other terms can be, by doing suitable integrations by parts transformed in one of the two following expressions:

$$\begin{aligned}6\sum_{j=1}^{J}\int \widetilde{z}\widetilde{z}_x p\Big[&p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p \\ &+5pp_x^2 + 5p^2p_{xx} + \frac{3}{2}p^5\Big]\varphi_j,\end{aligned} \tag{4.1.36}$$

$$\begin{aligned}3\sum_{j=1}^{J}\int \widetilde{z}^2 p_x\Big[&p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p \\ &+5pp_x^2 + 5p^2p_{xx} + \frac{3}{2}p^5\Big]\varphi_j.\end{aligned} \tag{4.1.37}$$

To deal with these two expressions, we use the elliptic equation satisfied by P_j:

$$\begin{aligned}&(P_j)_{xxxx} - 2(b_j^2 - a_j^2)\big((P_j)_{xx} + P_j^3\big) + (a_j^2 + b_j^2)^2 P_j \\ &\qquad +5P_j(P_j)_x^2 + 5P_j^2(P_j)_{xx} + \frac{3}{2}P_j^5 = 0,\end{aligned} \tag{4.1.38}$$

and the fact that

$$\Big[p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p + 5pp_x^2 + 5p^2p_{xx} + \frac{3}{2}p^5\Big]\varphi_j \tag{4.1.39}$$

converges exponentially to

$$(P_j)_{xxxx} - 2(b_j^2 - a_j^2)\big((P_j)_{xx} + P_j^3\big) + (a_j^2 + b_j^2)^2 P_j + 5P_j(P_j)_x^2 + 5P_j^2(P_j)_{xx} + \frac{3}{2}P_j^5, \tag{4.1.40}$$

which is a direct consequence of (4.1.1). This is why, such terms are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$.

Other terms contain $(P_j^2 - p^2)K_j$, which is bounded exponentially, with c_j bounded by $\|z\|_{H^2}$. Those terms are obviously bounded by $Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$.

Other terms contain K_k (or a derivative) and φ_j with $j \neq k$. In this case, this product gives an exponential decreasing, and such a term is bounded by $Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$, using (4.1.14).

Therefore, we are left with the following terms:

$$\sum_{j=1}^{J} c_j'(t)\int \Big[(K_j)_{xx}\widetilde{z}_{xx} - 10K_j\widetilde{z}_x p p_x - 5K_j\widetilde{z}p_x^2 - 10(K_j)_x\widetilde{z}pp_x - 5(K_j)_x\widetilde{z}_x p^2 + \frac{15}{4}K_j\widetilde{z}p^4 + 2(b_j^2 - a_j^2)(K_j)_x\widetilde{z}_x - 6(b_j^2 - a_j^2)K_j\widetilde{z}p^2 + (a_j^2 + b_j^2)^2 K_j\widetilde{z}\Big]\varphi_j. \tag{4.1.41}$$

We may replace p by P_j in the preceeding expression with an error bounded by

$$Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}, \tag{4.1.42}$$

because of (4.1.14) and (4.1.1). This is acceptable, knowing the result we want to prove. By integration by parts, we obtain several terms of the form $c_j'(t)\int (K_j)_{xx}\widetilde{z}_x(\varphi_j)_x$, which are bounded by $\frac{C}{t}|c_j'(t)|\|\widetilde{z}(t)\|_{H^2}$. Now, from Lemma 4.3, we deduce that they are bounded by

$$\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2 + Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2, \tag{4.1.43}$$

which is exactly the bound that we want. And, we are left with the following terms:

$$\sum_{j=1}^{J} c_j'(t)\int \Big[(K_j)_{xxxx} + 10(K_j)_x P_j(P_j)_x + 5K_j(P_j)_x^2 + 10K_j P_j(P_j)_{xx} + 5(K_j)_{xx}P_j^2 + \frac{15}{2}K_j P_j^4 - 2(b_j^2 - a_j^2)(K_j)_{xx} - 6(b_j^2 - a_j^2)K_j P_j^2 + (a_j^2 + b_j^2)^2 K_j\Big]\widetilde{z}\varphi_j. \tag{4.1.44}$$

The last expression equals zero, because of the elliptic equation satisfied by K_j, which we may derive by differentiating (4.1.38). □

Step 6. A bound for $\frac{d}{dt}\int \widetilde{z}P_j$.

Lemma 4.7. *There exists $C > 0$ and $\theta > 0$ that do not depend on z, such that for t large enough, for any $j = 1, ..., J$,*

$$\left|\frac{d}{dt}\int \widetilde{z}P_j\right| \leq Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \tag{4.1.45}$$

Proof. We observe that

$$\int \widetilde{z}P_j = \int zP_j + \sum_{k=1}^{J} c_k(t)\int K_k P_j. \tag{4.1.46}$$

Firstly, for $k = j$,

$$\int K_j P_j = 0, \tag{4.1.47}$$

and for $k \neq j$,

$$\frac{d}{dt}\left[c_k(t)\int K_k P_j\right] = c_k'(t)\int K_k P_j + c_k(t)\int (K_k)_t P_j + c_k(t)\int K_k (P_j)_t, \tag{4.1.48}$$

and it is obvious, from Lemma 4.2 and (4.1.14), that the latter is bounded by $Ce^{-\theta t}\|z(t)\|_{H^2}$.

It is left to bound $\frac{d}{dt}\int zP_j$. We use (4.1.6) and we obtain:

$$\frac{d}{dt}\int zP_j = -\int \left(z_{xx} + (z+p)^3 - p^3\right)_x P_j - \int z\left((P_j)_{xx} + P_j^3\right)_x. \tag{4.1.49}$$

Several terms are immediately boundable by $C\|z(t)\|_{H^2}^2$, we kill several others by integration by parts and we are left with

$$\int z(p^2 - P_j^2)(P_j)_x, \tag{4.1.50}$$

which is obviously bounded by $Ce^{-\theta t}\|z(t)\|_{H^2}$, because of (4.1.1). □

By differentiation of a square, we obtain that

Lemma 4.8. *There exists $C > 0$ and $\theta > 0$ that do not depend on z, such that for t large enough, for any $j = 1, ..., J$,*

$$\left|\frac{d}{dt}\left(\int \widetilde{z}P_j\right)^2\right| \leq Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2. \tag{4.1.51}$$

Step 7. A bound for $\|z(t)\|_{H^2}$ in function of $\widetilde{z}(t)$.

Because we have chosen $N > 2$ and because of (4.1.5), we may claim that for t large enough, the integral

$$\int_t^{+\infty} \|z(s)\|_{H^2}\, ds$$

is finite.

Because of Lemma 4.2 and (4.1.5), we deduce that

$$c_j(t) \to_{t\to+\infty} 0. \tag{4.1.52}$$

Knowing this, from Lemma 4.3, we deduce by integration that

$$\begin{aligned} |c_j(t)| &\leq \int_t^{+\infty} |c_j'(s)|\, ds \\ &\leq C\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2}\, ds + C\int_t^{+\infty} e^{-\theta s}\|z(s)\|_{H^2}\, ds + \int_t^{+\infty} \|z(s)\|_{H^2}^2\, ds. \end{aligned} \tag{4.1.53}$$

Knowing this and using (4.1.10), we may deduce that

$$\begin{aligned} \|z(t)\|_{H^2} &\leq C\|\widetilde{z}(t)\|_{H^2} + C\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2}\, ds + C\int_t^{+\infty} e^{-\theta s}\|z(s)\|_{H^2}\, ds \\ &\quad + \int_t^{+\infty} \|z(s)\|_{H^2}^2\, ds \\ &\leq C\|\widetilde{z}(t)\|_{H^2} + C\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2}\, ds + C\sup_{s\geq t}\|z(s)\|_{H^2} e^{-\theta t} \\ &\quad + C\sup_{s\geq t}\|z(s)\|_{H^2}\int_t^{+\infty} \|z(s)\|_{H^2}\, ds, \end{aligned} \tag{4.1.54}$$

which implies, because

$$\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2}\, ds, \quad \sup_{s\geq t}\|z(s)\|_{H^2} e^{-\theta t}, \quad \sup_{s\geq t}\|z(s)\|_{H^2}\int_t^{+\infty} \|z(s)\|_{H^2}\, ds \tag{4.1.55}$$

are decreasing in time, that

$$\begin{aligned}\sup_{s\geq t}\|z(s)\|_{H^2} \leq C\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} + C\int_t^{+\infty}\|\widetilde{z}(s)\|_{H^2}\,ds + C\sup_{s\geq t}\|z(s)\|_{H^2}e^{-\theta t}\\ + C\sup_{s\geq t}\|z(s)\|_{H^2}\int_t^{+\infty}\|z(s)\|_{H^2}\,ds,\end{aligned} \tag{4.1.56}$$

and because $e^{-\theta t}$ and $\int_t^{+\infty}\|z(s)\|_{H^2}\,ds$ may be as small as we want for t large enough (dependent on z), we may deduce that

Lemma 4.9. *There exists $C > 0$ that do not depend on z, such that for t large enough,*

$$\|z(t)\|_{H^2} \leq \sup_{s\geq t}\|z(s)\|_{H^2} \leq C\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} + C\int_t^{+\infty}\|\widetilde{z}(s)\|_{H^2}\,ds. \tag{4.1.57}$$

Step 8. Conclusion.

By integration, from Lemmas 4.5, 4.6 and 4.8, for t large enough (depending on z), with constants C and θ that do not depend on z,

$$\begin{aligned}\|\widetilde{z}(t)\|_{H^2}^2 &\leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}^2\,ds + C\int_t^{+\infty}e^{-\theta s}\|\widetilde{z}(s)\|_{H^2}\|z(s)\|_{H^2}\,ds\\ &\quad + C\int_t^{+\infty}\|\widetilde{z}(s)\|_{H^2}\|z(s)\|_{H^2}^2\,ds\\ &\leq C\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2}\int_t^{+\infty}\left(\frac{1}{s}\|\widetilde{z}(s)\|_{H^2} + e^{-\theta s}\|z(s)\|_{H^2} + \|z(s)\|_{H^2}^2\right)ds.\end{aligned} \tag{4.1.58}$$

Because the right-hand side of the inequality above is decreasing in time, we deduce after taking the supremum of the previous inequality and after simplification, that for t large enough,

$$\begin{aligned}\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} &\leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}\,ds + C\int_t^{+\infty}e^{-\theta s}\|z(s)\|_{H^2}\,ds\\ &\quad + C\int_t^{+\infty}\|z(s)\|_{H^2}^2\,ds\\ &\leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}\,ds + C\sup_{s\geq t}\|z(s)\|_{H^2}e^{-\theta t}\\ &\quad + C\sup_{s\geq t}\|z(s)\|_{H^2}\int_t^{+\infty}\|z(s)\|_{H^2}\,ds.\end{aligned} \tag{4.1.59}$$

And using (4.1.5), the fact that $N-1 > 1$ and the fact that $e^{-\theta t}$ is decreasing faster than $\frac{1}{t^{N-2}}$, we deduce that for t large enough,

$$\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} \leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}\,ds + C\frac{1}{t^{N-2}}\sup_{s\geq t}\|z(s)\|_{H^2}. \tag{4.1.60}$$

And using Lemma 4.9, we deduce that

$$\begin{aligned}\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} \leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}\,ds + C\frac{1}{t^{N-2}}\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2}\\ + C\frac{1}{t^{N-2}}\int_t^{+\infty}\|\widetilde{z}(s)\|_{H^2}\,ds.\end{aligned} \tag{4.1.61}$$

And because $\frac{1}{t^{N-2}}$ can be as small as we want for t large enough, we deduce that for t large enough and for a constant $C > 0$ that do not depend on z or on N,

$$\|\widetilde{z}(t)\|_{H^2} \leq \sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} \leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}\,ds + C\frac{1}{t^{N-2}}\int_t^{+\infty}\|\widetilde{z}(s)\|_{H^2}\,ds. \tag{4.1.62}$$

Let us pick $T > 0$ large enough such that for $t \geq T$, the inequality (4.1.62) works (i.e. T is large enough so that every part of the preceeding proof works). From (4.1.10) and Lemma 4.2, we know

that for $t \geq T$ (by taking T larger if needed),

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{C}{t^{N-1}}. \tag{4.1.63}$$

This is why, the following quantity is well defined:

$$A := \sup_{t \geq T} \{t^{N-1} \|\widetilde{z}(t)\|_{H^2}\}, \tag{4.1.64}$$

which means that for $t \geq T$,

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{A}{t^{N-1}}. \tag{4.1.65}$$

Now, using (4.1.63) and (4.1.65), we deduce from (4.1.62) that for $t \geq T$, with $C > 0$ that do not depend on z, on N or on A,

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{CA}{N-1} \frac{1}{t^{N-1}} + \frac{CA}{N-2} \frac{1}{t^{2N-4}} \leq \frac{CA}{N-2} \frac{1}{t^{N-1}}, \tag{4.1.66}$$

if we assume that $N > 3$. Now, from (4.1.64), we deduce that there exists $T^* > T$ such that

$$(T^*)^{N-1} \|\widetilde{z}(T^*)\|_{H^2} \geq \frac{A}{2}. \tag{4.1.67}$$

This is why, by evaluating (4.1.66) in $t = T^*$, we find that

$$\frac{A}{2(T^*)^{N-1}} \leq \frac{CA}{N-2} \frac{1}{(T^*)^{N-1}}, \tag{4.1.68}$$

which, if we assume that $A > 0$, after simplification yields:

$$N - 2 \leq 2C. \tag{4.1.69}$$

This means that if we assume that $N > 2C + 2$ and $N > 3$, the assumption $A > 0$ leads to a contradiction. Therefore, $A = 0$ under that assumption on N, which implies $\|\widetilde{z}(t)\|_{H^2} = 0$, and from Lemma 4.9, this implies that $z \equiv 0$. This means that the condition that we have established for N, namely

$$N > \max(2C + 2, 3), \tag{4.1.70}$$

do not depend on z and allows us to deduce that under (4.1.5), we may establish that $z \equiv 0$. The Proposition 1.5 is now proved. □

4.2. **A solution converging to a multi-breather converges exponentially to this multi-breather, if all the velocities (except possibly one) are positive.**

Proposition 4.10. *Let $u(t)$ be an H^2 solution of* (mKdV) *on $[T, +\infty)$, for $T \in \mathbb{R}$. We assume that*

$$\|u(t) - p(t)\|_{H^2} \to_{t \to +\infty} 0, \tag{4.2.1}$$

where p is the multi-breather constructed in Section 2. If

$$v_2 > 0, \tag{4.2.2}$$

then there exists $\omega > 0$, $T_0 \geq T$ and $C > 0$ such that for any $t \geq T_0$,

$$\|u(t) - p(t)\|_{H^2} \leq Ce^{-\omega t}. \tag{4.2.3}$$

Note that in the formulation of the Proposition above, we may replace p by P without changing its content (it is a consequence from (1.2.9)).

Proof. We set $v(t) := u(t) - P(t)$, such that $\|v(t)\|_{H^2} \to_{t \to +\infty} 0$.

We denote:

$$\Psi(x) := \frac{2}{\pi} \arctan\left(\exp(-\sqrt{\sigma}x/2)\right), \tag{4.2.4}$$

where $\sigma > 0$ is small enough (with precise conditions that will be mentioned throughout the proof). By direct calculations,

$$\Psi'(x) = \frac{-\sqrt{\sigma}}{2\pi \cosh(\sqrt{\sigma}x/2)}. \tag{4.2.5}$$

Thus,

$$|\Psi'(x)| \leq C \exp(-\sqrt{\sigma}|x|/2). \tag{4.2.6}$$

We have the following properties: $\lim_{+\infty} \Psi = 0$, $\lim_{-\infty} \Psi = 1$, for all $x \in \mathbb{R}$ $\Psi(-x) = 1 - \Psi(x)$, $\Psi'(x) < 0$, $|\Psi''(x)| \leq \frac{\sqrt{\sigma}}{2}|\Psi'(x)|$, $|\Psi'''(x)| \leq \frac{\sqrt{\sigma}}{2}|\Psi''(x)|$, $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2}\Psi$ and $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2}(1-\Psi)$.

For $j = 3, ..., J$, let m_j be such that

$$m_j := \frac{v_{j-1} + v_j}{2}. \tag{4.2.7}$$

If $v_1 \geq 0$, we may also set:

$$m_2 := \frac{v_1 + v_2}{2}, \tag{4.2.8}$$

and if $v_1 < 0$, we set:

$$m_2 := \frac{v_2}{2}, \tag{4.2.9}$$

and so, for any $j = 2, ..., J$, we have that

$$v_J > m_j > 0, \tag{4.2.10}$$

and we may consider m_j, $j = 2, ..., J$, as part of problem data from now on. Let us denote $\tau_0 > 0$ the minimal distance between a v_j and a m_j.

From this, we define for $j = 2, ..., J$,

$$\Phi_j(t, x) := \Psi(x - m_j t). \tag{4.2.11}$$

We may extend this definition to $j = 1$ and $j = J + 1$ in the following way: $\Phi_1 := 0$ and $\Phi_{J+1} := 1$. (4.2.10) allows us to claim that, for any j, Φ_j moves to the right. Thus, the function that allows us to study properties around each object P_j (for $j = 1, ..., J$) is $\chi_j := \Phi_{j+1} - \Phi_j$.

The goal is to prove that, for t large enough,

$$\|v(t)\|_{H^2} \leq Ce^{-\omega t}, \tag{4.2.12}$$

where $\omega > 0$ is a constant to be deduced from the constants of the problem. Proposition 4.10 follows from this, because of Theorem 1.2.

Let $\omega > 0$ to be deduced from the constants of the problem with respect to the needs of the following proof.

We will prove (4.2.12) by induction. Let us denote $\mathcal{P}_j$ the following assertion: $\int (v^2 + v_x^2 + v_{xx}^2)\Phi_j \leq Ce^{-2\omega t}$ for t large enough. We will prove, for $j = 2, ..., J + 1$, that we have $\mathcal{P}_j$, knowing that we have $\mathcal{P}_{j-1}$. Note that the assertion $\mathcal{P}_1$ is obviously true. This implies the desired inequality. (Note that it is OK if ω becomes smaller after a step of this induction, as long as it stays positive.)

Let us write the j-th step of our reasoning by induction (where $j \in \{2, ..., J + 1\}$). Thus, j is fixed in the rest of the proof. We assume that

$$\int \left(v^2 + v_x^2 + v_{xx}^2\right)\Phi_{j-1} \leq Ce^{-2\omega t}, \tag{4.2.13}$$

for t large enough.

We divide our proof in several steps.

Step 1. Almost-growth of localized conservation laws.

We define quantities that are similar to quantities defined in Section 2.2. We note that we localize around the first $j - 1$ objects, not only around the $(j - 1)$-th object. Notations defined in Section 2.2 should not be considered in the following proof and should be replaced by notations we define here:

$$M_j(t) := \frac{1}{2} \int u^2(t)\Phi_j(t), \tag{4.2.14}$$

$$E_j(t) := \int \left[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\right]\Phi_j(t), \tag{4.2.15}$$

$$F_j(t) := \int \left[\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\right]\Phi_j(t). \tag{4.2.16}$$

Lemma 4.11. *Let $\omega_2, \omega_6 > 0$, as small as desired. There exists $T_1 \geq T$ and $C > 0$ such that for $t \geq T_1$,*

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) \geq -Ce^{-2\omega t}, \tag{4.2.17}$$

$$\sum_{i=1}^{j-1} \big(E[P_i] + \omega_2 M[P_i]\big) - \big(E_j(t) + \omega_2 M_j(t)\big) \geq -Ce^{-2\omega t}, \tag{4.2.18}$$

$$\sum_{i=1}^{j-1} \big(F[P_i] + \omega_6 M[P_i]\big) - \big(F_j(t) + \omega_6 M_j(t)\big) \geq -Ce^{-2\omega t}. \tag{4.2.19}$$

Proof. We will use the results of the computations made at the bottom of page 1115 and at the bottom of page 1116 of [28], as well as in Section 5.5 (Appendix) to claim the three following facts:

$$\frac{d}{dt}\frac{1}{2}\int u^2 f = \int \Big(-\frac{3}{2}u_x^2 + \frac{3}{4}u^4\Big)f' + \frac{1}{2}\int u^2 f''', \tag{4.2.20}$$

$$\frac{d}{dt}\int \Big[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\Big]f = \int \Big[-\frac{1}{2}\big(u_{xx}+u^3\big)^2 - u_{xx}^2 + 3u_x^2u^2\Big]f' + \frac{1}{2}\int u_x^2 f''', \tag{4.2.21}$$

$$\begin{aligned}\frac{d}{dt}\int \Big(\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\Big)f \\ = \int \Big(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2u^2 + 15u_x^2uu_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 \\ - \frac{45}{4}u^4u_x^2\Big)f' + 5\int u^2u_xu_{xx}f'' + \frac{1}{2}\int u_{xx}^2 f'''.\end{aligned} \tag{4.2.22}$$

where f is a C^3 function that does not depend on time.

For the mass:

If $j \leq J$,

$$2\frac{d}{dt}M_j(t) = -\int \Big(3u_x^2 + m_ju^2 - \frac{3}{2}u^4\Big)\Phi_{jx}(t) + \int u^2\Phi_{jxxx}(t). \tag{4.2.23}$$

We recall that

$$|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2}|\Phi_{jx}|, \quad |\Phi_{jxxx}| \leq \frac{\sigma}{4}|\Phi_{jx}|, \quad \Phi_{jx} \leq 0, \tag{4.2.24}$$

where we can choose σ as small as desired. For this proof, we would like to ask for σ:

$$0 < \sigma \leq m_2 \leq m_j. \tag{4.2.25}$$

Thus,

$$2\frac{d}{dt}M_j(t) \geq \int \Big(3u_x^2 + \frac{3\sigma}{4}u^2 - \frac{3}{2}u^4\Big)|\Phi_{jx}(t)|. \tag{4.2.26}$$

By Corollary 2.2, for $r > 0$, if t, x satisfy $v_{j-1}t + r < x < v_jt - r$, then

$$\begin{aligned}|u(t,x)| &\leq |P(t,x)| + \|v(t)\|_{L^\infty} \\ &\leq Ce^{-\beta r} + C\|v(t)\|_{H^2},\end{aligned} \tag{4.2.27}$$

the same could be said for u_x.

We can thus deduce that for r large enough and for T_1 large enough, for $x \in (v_{j-1}t + r, v_jt - r)$, we can obtain that $|u|$ is bounded by any fixed constant, that can be taken as small as desired. Here, we will use the latter to bound $\frac{3}{2}u^2$ by $\frac{\sigma}{4}$.

For $t \geq T_1$ and $x \leq v_{j-1}t + r$ or $x \geq v_jt - r$, we have $|x - m_jt| \leq \tau_0 t - r$, and therefore for such t, x:

$$\begin{aligned}|\Phi_{jx}(t,x)| &\leq C\exp\big(-\sqrt{\sigma}|x - m_jt|/2\big) \\ &\leq C\exp\big(-\sqrt{\sigma}\tau_0 t/2\big)\exp\big(\sqrt{\sigma}r/2\big).\end{aligned} \tag{4.2.28}$$

Because $\int u^4$ is bounded by a constant for any time and $\exp(\sqrt{\sigma}r/2)$ is a fixed constant (r is already chosen), we have, for $t \geq T_1$,

$$\frac{d}{dt}M_j(t) \geq \int \Big(\frac{3}{2}u_x^2 + \frac{\sigma}{4}u^2\Big)|\Phi_{jx}(t)| - Ce^{-2\omega t} \geq -Ce^{-2\omega t}, \tag{4.2.29}$$

where ω is chosen as a suitable function of σ and τ_0.

By integration, we deduce that for any $t_1 \geq t$, with a constant $C > 0$ that does not depend on t_1, we have:

$$M_j(t_1) - M_j(t) \geq -Ce^{-2\omega t}. \tag{4.2.30}$$

We note that this conclusion is immediate when $j = J + 1$, because we have exactly the conserved quantity.

We have that

$$\begin{aligned}
&\Big|\sum_{i=1}^{j-1} M[P_i] - M_j(t_1)\Big| \\
&\leq \Big|\sum_{i=1}^{j-1}\frac{1}{2}\int P_i^2 - \frac{1}{2}\int P^2\Phi_j(t_1)\Big| + \frac{1}{2}\Big|\int P^2\Phi_j(t_1) - \int u^2\Phi_j(t_1)\Big| \\
&\leq Ce^{-\kappa(\beta,\sigma,\tau_0)t_1} + \frac{1}{2}\int |P^2 - u^2|\Phi_j(t_1) \\
&\leq Ce^{-\kappa(\beta,\sigma,\tau_0)t_1} + C\int |P^2 - u^2| \to_{t_1\to+\infty} 0.
\end{aligned} \tag{4.2.31}$$

This means that when we take the limit of (4.2.30) when $t_1 \to +\infty$, we obtain, for $t \geq T_1$,

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) \geq -Ce^{-2\omega t}, \tag{4.2.32}$$

which is exactly what we wished to prove.

For the energy:

If $j \leq J$,

$$\begin{aligned}
2\frac{d}{dt}E_j(t) &= \int \Big[-\big(u_{xx} + u^3\big)^2 - 2u_{xx}^2 + 6u_x^2u^2\Big]\Phi_{jx}(t) \\
&\quad - m_j\int \Big(u_x^2 - \frac{1}{2}u^4\Big)\Phi_{jx}(t) + \frac{1}{2}\int u_x^2\Phi_{jxxx}(t) \\
&\geq \int \Big[\big(u_{xx} + u^3\big)^2 + 2u_{xx}^2 - 6u_x^2u^2 + \frac{3\sigma}{4}u_x^2 - \frac{m_j}{2}u^4\Big]|\Phi_{jx}(t)|.
\end{aligned} \tag{4.2.33}$$

We can do the same reasoning as for the mass to bound above $\frac{m_j}{2}u^2$ by ω_1, a constant that we can choose as small as desired, and to bound above $6u^2$ by $\frac{\sigma}{4}$. We obtain that if T_1 is large enough (dependently on the chosen constant ω_1),

$$2\frac{d}{dt}E_j(t) \geq \int \Big[\big(u_{xx} + u^3\big)^2 + 2u_{xx}^2 + \frac{\sigma}{2}u_x^2 - \omega_1u^2\Big]|\Phi_{jx}(t)| - Ce^{-2\omega t}. \tag{4.2.34}$$

By using what we have performed for the mass, we have that if we take ω_1 small enough with respect to $\frac{\omega_2\sigma}{2}$,

$$\frac{d}{dt}\big(E_j + \omega_2M_j\big)(t) \geq -Ce^{-2\omega t}. \tag{4.2.35}$$

Then, by integration and similarly as for the mass, we obtain the desired conclusion that is true for any j.

For F:

If $j \leq J$,

$$
\begin{aligned}
2\frac{d}{dt}F_j(t) = \int \Big(& -3u_{xxx}^2 + 18u_{xx}^2u^2 + 30u_x^2uu_{xx} + \frac{9}{8}u^8 + \frac{1}{2}u_x^4 + 3u_{xx}u^5 \\
& - \frac{45}{2}u^4u_x^2\Big)\Phi_{jx}(t) - m_j\int\Big(u_{xx}^2 - 5u^2u_x^2 + \frac{1}{2}u^6\Big)\Phi_{jx}(t) \\
& + 10\int u^2u_{xx}u_x\Phi_{jxx}(t) + \int u_{xx}^2\Phi_{jxxx}(t) \\
\geq \int \Big(& 3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 - 18u_{xx}^2u^2 - 15u_x^2u^2 - 15u_x^2u_{xx}^2 - \frac{9}{8}u^8 \\
& - \frac{1}{2}u_x^4 - \frac{3}{2}u_{xx}^2u^4 - \frac{3}{2}u^6\Big)|\Phi_{jx}(t)| \\
& + \int\Big(\sigma u_{xx}^2 + \frac{\sigma}{2}u^6 - 5m_ju^2u_x^2\Big)|\Phi_{jx}(t)| - 5\int u^2u_x^2|\Phi_{jxx}(t)| \\
& - 5\int u^2u_{xx}^2|\Phi_{jxx}(t)| - \int u_{xx}^2|\Phi_{jxxx}(t)|.
\end{aligned}
\tag{4.2.36}
$$

By the same reasoning as for the energy and the mass, if we set $\omega_3, \omega_4, \omega_5 > 0$ constants that we can take as small as desired, and if T_1 is large enough dependently on these constants, for $t \geq T_1$, we have that

$$
\begin{aligned}
2\frac{d}{dt}F_j(t) \geq \int \Big(& 3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 + \frac{3\sigma}{4}u_{xx}^2 + \frac{\sigma}{2}u^6 - \omega_3u_{xx}^2 - \omega_4u_x^2 \\
& - \omega_5u^2\Big)|\Phi_{jx}(t)| - Ce^{-2\omega t}.
\end{aligned}
\tag{4.2.37}
$$

By using what we have carried out for the mass, we have that if we take $\omega_3, \omega_4, \omega_5$ small enough (with respect to ω_6),

$$
\frac{d}{dt}\big(F_j + \omega_6M_j\big)(t) \geq -Ce^{-2\omega t}. \tag{4.2.38}
$$

Then, by integration and similarly as before, we obtain that the desired conclusion true for any j. □

Remark 4.12. If $j = J+1$, we have that

$$
\sum_{i=1}^{J} M[P_i] - M_{J+1}(t) = 0, \tag{4.2.39}
$$

$$
\sum_{i=1}^{J} E[P_i] - E_{J+1}(t) = 0, \tag{4.2.40}
$$

$$
\sum_{i=1}^{J} F[P_i] - F_{J+1}(t) = 0. \tag{4.2.41}
$$

Step 2. Almost-growth of the Lyapunov functional.

Let $0 < \nu < 1$ be close enough to 1. We set, for $j = 2, ..., J+1$, a functional that is close to the Lyapunov functional that we will consider in the following steps of the proof:

$$
\mathcal{F}_j(t) := F_j(t) + 2(b_{j-1}^2 - a_{j-1}^2)E_j(t) + \nu(a_{j-1}^2 + b_{j-1}^2)^2M_j(t). \tag{4.2.42}
$$

The following lemma states the almost-growth of $\mathcal{F}_j$:

Lemma 4.13. *There exists $0 < \nu < 1$ close enough to 1 such that there exists $T_1 \geq T$ and $C > 0$ such that for any $t \geq T_1$,*

$$
\mathcal{F}_j(t) - F[P_{j-1}] - 2(b_{j-1}^2 - a_{j-1}^2)E[P_{j-1}] - \nu(a_{j-1}^2 + b_{j-1}^2)^2M[P_{j-1}] \leq Ce^{-2\omega t}. \tag{4.2.43}
$$

Proof. If $b_{j-1}^2 - a_{j-1}^2 \geq 0$, then the Lemma 4.13 is an immediate consequence of Lemma 4.11.

For the rest of the proof, we consider the case $b_{j-1}^2 - a_{j-1}^2 < 0$ (that can be relevant for $j = 2$). As in the proof of Lemma 4.11, we may choose $r > 0$ large enough in order to be able to bound the terms for which the degree of u is larger than 2 with a coefficient that is as small as we want that will be denoted $\omega > 0$.

We take σ small enough with respect to ω. We may deduce from Lemma 4.11 and its proof that

$$\begin{aligned}\frac{d}{dt}F_j(t) \geq -Ce^{-2\omega t} + \frac{3}{2}\int u_{xxx}^2|\Phi_{jx}| + \frac{m_j}{2}\int u_{xx}^2|\Phi_{jx}| \\ - \omega\int u_{xx}^2|\Phi_{jx}| - \omega\int u_x^2|\Phi_{jx}| - \omega\int u^2|\Phi_{jx}|,\end{aligned} \tag{4.2.44}$$

$$\begin{aligned}-\frac{d}{dt}E_j(t) \geq -Ce^{-2\omega t} - \frac{3}{2}\int u_{xx}^2|\Phi_{jx}| - \frac{m_j}{2}\int u_x^2|\Phi_{jx}| \\ - \omega\int u_x^2|\Phi_{jx}| - \omega\int u^2|\Phi_{jx}|,\end{aligned} \tag{4.2.45}$$

$$\begin{aligned}\frac{d}{dt}M_j(t) \geq -Ce^{-2\omega t} + \frac{3}{2}\int u_x^2|\Phi_{jx}| + \frac{m_j}{2}\int u^2|\Phi_{jx}| \\ - \omega\int u^2|\Phi_{jx}|.\end{aligned} \tag{4.2.46}$$

So,

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_j(t) = -Ce^{-2\omega t} + \frac{3}{2}\int u_{xxx}^2|\Phi_{jx}| \\ + \left(3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega\right)\int u_{xx}^2|\Phi_{jx}| \\ + \left(\frac{3}{2}\nu(a_{j-1}^2 + b_{j-1}^2)^2 + m_j(b_{j-1}^2 - a_{j-1}^2) - \omega\right)\int u_x^2|\Phi_{jx}| \\ + \left(\frac{m_j}{2}\nu(a_{j-1}^2 + b_{j-1}^2)^2 - \omega\right)\int u^2|\Phi_{jx}|.\end{aligned} \tag{4.2.47}$$

We may choose m_j small enough (with respect to ν) so that

$$\frac{3}{2}\nu(a_{j-1}^2 + b_{j-1}^2)^2 + m_j(b_{j-1}^2 - a_{j-1}^2) > \frac{3}{2}\nu'(a_{j-1}^2 + b_{j-1}^2)^2, \tag{4.2.48}$$

where $0 < \nu' < \nu$ that is as close to 1 as we need.

We choose ω small enough with respect to the previous choice so that

$$\frac{3}{2}\nu(a_{j-1}^2 + b_{j-1}^2)^2 + m_j(b_{j-1}^2 - a_{j-1}^2) - \omega \geq \frac{3}{2}\nu'(a_{j-1}^2 + b_{j-1}^2)^2. \tag{4.2.49}$$

We may choose ω even smaller (with respect to m_j and ν) so that

$$\frac{m_j}{2}\nu(a_{j-1}^2 + b_{j-1}^2)^2 - \omega \geq 0, \tag{4.2.50}$$

and

$$3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega \geq 3(b_{j-1}^2 - a_{j-1}^2). \tag{4.2.51}$$

In the case when with the chosen values of m_j, ν and ω, $3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega$ is positive, the desired conclusion is straightforward by integration. From now on, we place ourselves in the case when

$$3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega < 0. \tag{4.2.52}$$

Now, we want to bound above $\int u_{xx}^2|\Phi_{jx}|$. By integration by parts,

$$\begin{aligned}\int u_{xx}^2|\Phi_{jx}| &= -\int u_x u_{xxx}|\Phi_{jx}| - \int u_x u_{xx}|\Phi_{jxx}| \\ &\leq \sqrt{\int u_x^2|\Phi_{jx}| \int u_{xxx}^2|\Phi_{jx}|} + \frac{\sqrt{\sigma}}{2}\sqrt{\int u_x^2|\Phi_{jx}| \int u_{xx}^2|\Phi_{jx}|},\end{aligned} \tag{4.2.53}$$

because $|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2}|\Phi_{jx}|$.

We denote:

$$X := \sqrt{\int u_{xx}^2|\Phi_{jx}|}, \tag{4.2.54}$$

and

$$A := \sqrt{\int u_x^2|\Phi_{jx}| \int u_{xxx}^2|\Phi_{jx}|}. \tag{4.2.55}$$

So, we have that

$$X^2 \leq A + \varepsilon X, \tag{4.2.56}$$

where

$$\varepsilon := \frac{\sqrt{\sigma}}{2}\sqrt{\int u_x^2|\Phi_{jx}|} \leq \frac{\sqrt{\sigma}}{2}\|u\|_{\dot{H}^1}, \tag{4.2.57}$$

which can be as small as we want if we take σ small enough (for a given solution u).

We deduce that

$$X \leq \frac{\varepsilon + \sqrt{\varepsilon^2 + 4A}}{2} \leq \varepsilon + \sqrt{A}. \tag{4.2.58}$$

Thus,

$$\begin{aligned}\int u_{xx}^2|\Phi_{jx}| \leq \Bigg(&\frac{\sigma}{4}\sqrt{\int u_x^2|\Phi_{jx}|} + \sqrt{\sigma}\Big(\int u_x^2|\Phi_{jx}| \int u_{xxx}^2|\Phi_{jx}|\Big)^{\frac{1}{4}} \\ &+ \sqrt{\int u_{xxx}^2|\Phi_{jx}|}\Bigg)\sqrt{\int u_x^2|\Phi_{jx}|}.\end{aligned} \tag{4.2.59}$$

So,

$$\begin{aligned}\Big(3(b_{j-1}^2 - a_{j-1}^2) + \frac{m_j}{2} - \omega\Big)\int u_{xx}^2|\Phi_{jx}| &\geq 3(b_{j-1}^2 - a_{j-1}^2)\sqrt{\int u_x^2|\Phi_{jx}| \int u_{xxx}^2|\Phi_{jx}|} \\ &+ 3(b_{j-1}^2 - a_{j-1}^2)\sqrt{\sigma}\Big(\int u_x^2|\Phi_{jx}|\Big)^{\frac{3}{4}}\Big(\int u_{xxx}^2|\Phi_{jx}|\Big)^{\frac{1}{4}} \\ &+ 3(b_{j-1}^2 - a_{j-1}^2)\frac{\sigma}{4}\int u_x^2|\Phi_{jx}|.\end{aligned} \tag{4.2.60}$$

On the other hand, we have for a choice of $\nu_1, \nu_2, \nu_3 > 0$ such that $\nu_1 + \nu_2 + \nu_3 = \nu'$ (where ν_1 should be near to ν' and ν_2 and ν_3 should be near to 0) and a choice of $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 = 1$ (where ϵ_1 should be near to 1 and ϵ_2 should be near to 0) that will be made with respect to the needs

of the proof, that

$$\begin{aligned}
&\frac{3}{2}\int u_{xxx}^2|\Phi_{jx}| + \left(\frac{3}{2}\nu'(a_{j-1}^2+b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}| \\
&\quad\geq \frac{3}{2}\epsilon_1\int u_{xxx}^2|\Phi_{jx}| + \left(\frac{3}{2}\nu_1(a_{j-1}^2+b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}| \\
&\quad+ \frac{3}{2}\epsilon_2\int u_{xxx}^2|\Phi_{jx}| + \left(\frac{3}{2}\nu_2(a_{j-1}^2+b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}| \\
&\quad+ \left(\frac{3}{2}\nu_3(a_{j-1}^2+b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}| \\
&\quad\geq 2\sqrt{\frac{3}{2}\epsilon_1\left(\frac{3}{2}\nu_1(a_{j-1}^2+b_{j-1}^2)^2\right)}\sqrt{\int u_x^2|\Phi_{jx}|\int u_{xxx}^2|\Phi_{jx}|} \\
&\quad+ 4\left(\frac{3}{2}\epsilon_2\int u_{xxx}^2|\Phi_{jx}|\right)^{\frac{1}{4}}\left(\left(\frac{1}{2}\nu_2(a_{j-1}^2+b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}|\right)^{\frac{3}{4}} \\
&\quad+ \left(\frac{3}{2}\nu_3(a_{j-1}^2+b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}| \\
&\quad\geq 3\sqrt{\epsilon_1\nu_1}(a_{j-1}^2+b_{j-1}^2)\sqrt{\int u_x^2|\Phi_{jx}|\int u_{xxx}^2|\Phi_{jx}|} \\
&\quad+ 2\cdot 3^{\frac{1}{4}}\epsilon_2^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_{j-1}^2+b_{j-1}^2)^{\frac{3}{2}}\left(\int u_x^2|\Phi_{jx}|\right)^{\frac{3}{4}}\left(\int u_{xxx}^2|\Phi_{jx}|\right)^{\frac{1}{4}} \\
&\quad+ \frac{3}{2}\nu_3(a_{j-1}^2+b_{j-1}^2)^2\int u_x^2|\Phi_{jx}|.
\end{aligned} \tag{4.2.61}$$

This is why, we deduce that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_j(t) &\geq -Ce^{-2\omega t} \\
&+ \left(3(b_{j-1}^2-a_{j-1}^2)+3\sqrt{\epsilon_1\nu_1}(a_{j-1}^2+b_{j-1}^2)\right)\sqrt{\int u_x^2|\Phi_{jx}|\int u_{xxx}^2|\Phi_{jx}|} \\
&+ \left(3(b_{j-1}^2-a_{j-1}^2)\sqrt{\sigma}+2\cdot 3^{\frac{1}{4}}\epsilon_2^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_{j-1}^2+b_{j-1}^2)^{\frac{3}{2}}\right)\left(\int u_x^2|\Phi_{jx}|\right)^{\frac{3}{4}}\left(\int u_{xxx}^2|\Phi_{jx}|\right)^{\frac{1}{4}} \\
&+ \left(3(b_{j-1}^2-a_{j-1}^2)\frac{\sigma}{4}+\frac{3}{2}\nu_3(a_{j-1}^2+b_{j-1}^2)^2\right)\int u_x^2|\Phi_{jx}|.
\end{aligned} \tag{4.2.62}$$

To finish, we need to state a condition (on ν_1, ν_2, ν_3, ϵ_1, ϵ_2) that will ensure that

$$E_1 := 3(b_{j-1}^2-a_{j-1}^2)+3\sqrt{\epsilon_1\nu_1}(a_{j-1}^2+b_{j-1}^2)\geq 0, \tag{4.2.63}$$

$$E_2 := 3(b_{j-1}^2-a_{j-1}^2)\sqrt{\sigma}+2\cdot 3^{\frac{1}{4}}\epsilon_2^{\frac{1}{4}}\nu_2^{\frac{3}{4}}(a_{j-1}^2+b_{j-1}^2)^{\frac{3}{2}}\geq 0, \tag{4.2.64}$$

and

$$E_3 := 3(b_{j-1}^2-a_{j-1}^2)\frac{\sigma}{4}+\frac{3}{2}\nu_3(a_{j-1}^2+b_{j-1}^2)^2\geq 0. \tag{4.2.65}$$

We can choose ϵ_1 and ν_1 so that $\sqrt{\epsilon_1\nu_1}\geq\frac{1}{2}$. So, we have that

$$\begin{aligned}
\frac{E_1}{3} &= (\sqrt{\epsilon_1\nu_1}+1)b_{j-1}^2+(\sqrt{\epsilon_1\nu_1}-1)a_{j-1}^2 \\
&\geq \frac{3}{2}b_{j-1}^2+(\sqrt{\epsilon_1\nu_1}-1)a_{j-1}^2.
\end{aligned} \tag{4.2.66}$$

By choosing ϵ_1 and ν_1 wisely (note that we can choose ν' and ν with respect to the condition on ν_1 that we need), we may ensure that $|\sqrt{\epsilon_1\nu_1}-1|$ is as small as we want with respect to the ratio $\frac{b_{j-1}}{a_{j-1}}$, which will ensure that $E_1\geq 0$.

In order to have $E_2 > 0$ and $E_3 > 0$, the only requirement is that $\epsilon_2, \nu_2, \nu_3 > 0$. This being set, it is enough to take σ small enough in order to obtain the desired conclusion.

Thus,

$$\frac{d}{dt}\mathcal{F}_j(t) \geq -Ce^{-2\omega t}. \tag{4.2.67}$$

We obtain the desired conclusion by integration. □

Step 3. Modulation.

Notations that were defined in Section 2.3 should not be taken into consideration in the following proof and should be replaced by notations we define here.

Lemma 4.14. *There exists $C > 0$, $T_2 \geq T$, such that there exist unique C^1 functions $y_1, y_2 : [T_2, +\infty) \to \mathbb{R}$ such that if we set:*

$$w(t,x) := u - \widetilde{P}, \tag{4.2.68}$$

where

$$\widetilde{P}(t,x) := \sum_{i=1}^{J} \widetilde{P_i}(t,x), \tag{4.2.69}$$

for $i \neq j-1$,

$$\widetilde{P_i}(t,x) := P_i(t,x), \tag{4.2.70}$$

and either,

$$\widetilde{P_{j-1}}(t,x) := \kappa_l Q_{c_l+y_1(t)}(x - x_{0,l}^0 + y_2(t) - c_l t), \quad \text{if } P_{j-1} = R_l \text{ is a soliton}, \tag{4.2.71}$$

or,

$$\widetilde{P_{j-1}}(t,x) := B_{\alpha_k,\beta_k}(t,x; x_{1,k} + y_1(t), x_{2,k} + y_2(t)), \quad \text{if } P_{j-1} = B_k \text{ is a breather}, \tag{4.2.72}$$

then, $w(t)$ satisfies, for any $t \in [T_2, +\infty)$, either,

$$\int \widetilde{P_{j-1}}_1(t) w(t) = \int \widetilde{P_{j-1}}_2(t) w(t) = 0, \quad \text{if } P_{j-1} \text{ is a breather}, \tag{4.2.73}$$

or,

$$\int \widetilde{P_{j-1}}(t) w(t) = \int \widetilde{P_{j-1}}_x(t) w(t) = 0, \quad \text{if } P_{j-1} \text{ is a soliton}, \tag{4.2.74}$$

where in the case when P_{j-1} is a breather we denote:

$$\widetilde{P_{j-1}}_1(t,x) := \partial_{x_1}\widetilde{P_{j-1}}, \quad \widetilde{P_{j-1}}_2(t,x) := \partial_{x_2}\widetilde{P_{j-1}}. \tag{4.2.75}$$

Moreover, for any $t \in [T_2, +\infty)$,

$$\|w(t)\|_{H^2} + |y_1(t)| + |y_2(t)| \leq C\|v(t)\|_{H^2}, \tag{4.2.76}$$

and, if ω is small enough,

$$|y_1'(t)| + |y_2'(t)| \leq C\left(\int w(t)^2 \Phi_j\right)^{1/2} + Ce^{-\omega t}. \tag{4.2.77}$$

Proof. The proof that has to be performed is similar to the proof of Lemma 2.8, which is a consequence of a quantitative version of the implicit function theorem. See [11, Section 2.2] for a precise statement. The proof of (4.2.77) is also similar: as in the proof of Lemma 2.8, we take the time derivative of $\int \widetilde{P_{j-1}}_1(t) w(t) = \int \widetilde{P_{j-1}}_2(t) w(t) = 0$. To be complete, let us perform this proof.

For $t \in [T_2, +\infty)$, let

$$F_t : L^2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2 \tag{4.2.78}$$

be such that if $P_{j-1} = B_k$ is a breather,

$$\begin{aligned}(U, y_1, y_2) \longmapsto \Bigg(&\int \partial_{x_1} B_{\alpha_k,\beta_k}(t,x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2)\epsilon \, dx, \\ &\int \partial_{x_2} B_{\alpha_k,\beta_k}(t,x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2)\epsilon \, dx\Bigg),\end{aligned} \tag{4.2.79}$$

where

$$\epsilon := U - P + P_{j-1} - B_{\alpha_k,\beta_k}(t,x;x^0_{1,k}+y_1,x^0_{2,k}+y_2), \tag{4.2.80}$$

and if $P_{j-1} = R_l$ is a soliton,

$$\begin{aligned}(U,y_1,y_2) \longmapsto \Bigg(&\int \kappa_l Q_{c_l+y_1}(x-x^0_{0,l}+y_2-c_lt)\epsilon\,dx,\\ &\int \partial_x\kappa_l Q_{c_l+y_1}(x-x^0_{0,l}+y_2-c_lt)\epsilon\,dx\Bigg),\end{aligned} \tag{4.2.81}$$

where

$$\epsilon := U - P + P_{j-1} - \kappa_l Q_{c_l+y_1}(x-x^0_{0,l}+y_2-c_lt). \tag{4.2.82}$$

We observe that F_t is a C^1 function and that $F_t(P(t),0,0) = 0$. Now, let us consider the matrix which gives the differential of F_t (with respect to y_1, y_2) in $(P(t),0,0)$.

In the case when $P_{j-1} = B_k$ is a breather, this matrix is:

$$DF_t = \begin{pmatrix} -\int(\partial_{x_1}B_k)^2\,dx & -\int \partial_{x_1}B_k\partial_{x_2}B_k\,dx \\ -\int \partial_{x_1}B_k\partial_{x_2}B_k\,dx & -\int(\partial_{x_2}B_k)^2\,dx \end{pmatrix}, \tag{4.2.83}$$

whose determinant is:

$$\det(DF_t) = \int(\partial_{x_1}B_k)^2\,dx \int(\partial_{x_2}B_k)^2\,dx - \left(\int \partial_{x_1}B_k\partial_{x_2}B_k\,dx\right)^2. \tag{4.2.84}$$

By Cauchy-Schwarz inequality and the fact that $\partial_{x_1}B_k$ and $\partial_{x_2}B_k$ are linearly independent as functions of the x variable, for any time t fixed, we see that $\det(DF_t)$ is positive. Since each member of its expression is periodic in time, then $\det(DF_t)$ is bounded below by a positive constant independent on time and translation parameters of B_k.

In the case when $P_{j-1} = R_l$ is a soliton, let us recall, denoting $y_{0,l} := x - x^0_{0,l} + y_2 - c_lt$, that

$$\partial_{y_1}Q_{c_l+y_1}(y_{0,l}) = \frac{1}{2c_l}\big(Q_{c_l+y_1}(y_{0,l}) + y_{0,l}\partial_x Q_{c_l+y_1}(y_{0,l})\big). \tag{4.2.85}$$

Thus, denoting $Q_{c_l}(x-x^0_{0,l}-c_lt)$ by Q_{c_l} and $x-x^0_{0,l}-c_lt$ by $y^0_{0,l}$,

$$DF_t = \begin{pmatrix} -\frac{1}{2c_l}\int Q_{c_l}\big(Q_{c_l}+y^0_{0,l}\partial_xQ_{c_l}\big)\,dx & -\int Q_{c_l}\partial_xQ_{c_l}\,dx \\ -\frac{1}{2c_l}\int \partial_xQ_{c_l}\big(Q_{c_l}+y^0_{0,l}\partial_xQ_{c_l}\big)\,dx & -\int\big(\partial_xQ_{c_l}\big)^2\,dx \end{pmatrix}, \tag{4.2.86}$$

whose determinant is:

$$\det(DF_t) = \frac{1}{2c_l}\int Q_{c_l}\big(Q_{c_l}+y^0_{0,l}\partial_xQ_{c_l}\big)\,dx \int\big(\partial_xQ_{c_l}\big)^2\,dx, \tag{4.2.87}$$

because $\int Q_{c_l}\partial_xQ_{c_l}\,dx = 0$. And, from the computations made to obtain (2.3.19), we have that

$$\det(DF_t) = \frac{1}{4}c_l\int q^2\int q_x^2, \tag{4.2.88}$$

where q denotes the soliton with $c=1$, i.e. $q = Q_1$.

This means that $\det(DF_t)$ is bounded below by a positive constant independent on time and translation parameters of R_l. Thus, in any case, DF_t is invertible.

Now, we may use the implicit function theorem. If U is close enough to $P(t)$, then there exists (y_1,y_2) such that $F_t(U,y_1,y_2) = 0$, where (y_1,y_2) depends in a regular C^1 way on U. It is possible to show that the "close enough" in the previous sentence does not depend on t; for this, it is required to use a uniform implicit function theorem. This means that for T_2 large enough, $\|v(t)\|_{H^2}$ is small enough for $t \in [T_2,+\infty)$, thus for $t \geq T_2$, $u(t)$ is close enough to $P(t)$ in order to apply the implicit function theorem. Therefore, we have for $t \in [T_2,+\infty)$, the existence of $y_1(t)$ and $y_2(t)$. It is possible to show that these functions are C^1 in time. Basically, this comes from the fact that they are C^1 in $u(t)$ and that $u(t)$ has a similar regularity in time (see [13] for more details).

Now, we prove the inequalities (4.2.76) and (4.2.77). We can take the differential of the implicit functions with respect to $u(t)$ for $t \in [T_2, +\infty)$. For this, we differentiate the following equation with respect to $u(t)$:

$$F_t(u(t), y_1(u(t)), y_2(u(t))) = 0. \tag{4.2.89}$$

We know that the matrix that gives the differential of F_t (with respect to y_1, y_2) in

$$(u(t), y_1(u(t)), y_2(u(t)))$$

is invertible and that its inverse is bounded in time. The differential of F_t with respect to the first variable is also bounded (from its expression, F_t being linear in U). Thus, by the mean-value theorem (given $(y_1, y_2)(P(t)) = (0, 0)$):

$$|y_1(u(t))| + |y_2(u(t))| \leq C\|u(t) - P(t)\| \leq C\|v(t)\|_{H^2}. \tag{4.2.90}$$

By applying the mean-value theorem (inequality) for Q_{c_l} or B_{α_k,β_k} with respect to y_1 and y_2, we deduce that

$$\|P_{j-1}(t) - \widetilde{P_{j-1}}(t)\|_{H^2} \leq C(|y_1(t)| + |y_2(t)|). \tag{4.2.91}$$

Finally, by triangular inequality,

$$\begin{aligned} \|w(t)\|_{H^2} &\leq \|u(t) - P(t)\|_{H^2} + \|P(t) - \widetilde{P}(t)\|_{H^2} \\ &\leq \|u(t) - P(t)\|_{H^2} + C\big(|y_1(t)| + |y_2(t)|\big) \\ &\leq C\|v(t)\|_{H^2}. \end{aligned} \tag{4.2.92}$$

This completes the proof of (4.2.76).

For (4.2.77), we will take time derivatives of the equations (4.2.73) and (4.2.74). Firstly, we may write the PDE verified by w:

$$\begin{aligned} \partial_t w = -w_{xxx} - \left[w\left(w^2 + 3w \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{i,m=1}^{J} \widetilde{P_i}\widetilde{P_m} \right) \right]_x \\ - \sum_{h\neq i \text{ or } i\neq m} \big(\widetilde{P_h}\widetilde{P_i}\widetilde{P_m}\big)_x - E, \end{aligned} \tag{4.2.93}$$

where, if $P_{j-1} = B_k$ is a breather,

$$E := y_1'(t)\widetilde{B_{k1}} + y_2'(t)\widetilde{B_{k2}}, \tag{4.2.94}$$

and if $P_{j-1} = R_l$ is a soliton, denoting $y_{0,l}(t) := x - x_{0,l}^0 + y_2(t) - c_l t$,

$$E := \frac{y_1'(t)}{2\big(c_l + y_1(t)\big)}\big(\widetilde{R_l} + y_{0,l}(t)\widetilde{R}_{lx}\big) + y_2'(t)\widetilde{R}_{lx}. \tag{4.2.95}$$

If $P_{j-1} = B_k$, we start by taking the time derivative of $\int \widetilde{B_{k1}} w = 0$ and perform some integrations by parts to obtain:

$$\begin{aligned} &-\int \big(\widetilde{B_k}^3\big)_{1x} w + y_1'(t)\int \widetilde{B_{k11}} w + y_2'(t)\int \widetilde{B_{k12}} w \\ &+ \int \widetilde{B_{k1x}} w\left(w^2 + 3w\sum_{i=1}^{J}\widetilde{P_i} + 3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right) - \int \widetilde{B_{k1}} \sum_{h\neq i \text{ or } g\neq h} \big(\widetilde{P_h}\widetilde{P_i}\widetilde{P_g}\big)_x \\ &= y_1'(t)\int \widetilde{B_{k1}}^2 + y_2'(t)\int \widetilde{B_{k1}}\widetilde{B_{k2}}, \end{aligned} \tag{4.2.96}$$

then, we take the time derivative of $\int \widetilde{B_{k2}} w = 0$:

$$
\begin{aligned}
&- \int \big(\widetilde{B_k}^3\big)_{2x} w + y_1'(t) \int \widetilde{B_{k12}} w + y_2'(t) \int \widetilde{B_{k22}} w \\
&+ \int \widetilde{B_{k2x}} w \Big(w^2 + 3w \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{h,i=1}^{J} \widetilde{P_h} \widetilde{P_i} \Big) - \int \widetilde{B_{k2}} \sum_{h \neq i \text{ or } g \neq h} \big(\widetilde{P_h} \widetilde{P_i} \widetilde{P_g}\big)_x \\
&= y_1'(t) \int \widetilde{B_{k1}} \widetilde{B_{k2}} + y_2'(t) \int \widetilde{B_{k2}}^2.
\end{aligned} \tag{4.2.97}
$$

If $P_{j-1} = R_l$, we start by taking the time derivative of $\int \widetilde{R_l} w = 0$ and perform some integrations by parts to obtain:

$$
\begin{aligned}
&- \int \big(\widetilde{R_l}^3\big)_x w + \frac{y_1'(t)}{2c_l} \int \big(\widetilde{R_l} + y_{0,l}(t) \widetilde{R_{lx}}\big) w + y_2'(t) \int \widetilde{R_{lx}} w \\
&+ \int \widetilde{R_{lx}} w \Big(w^2 + 3w \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{h,i=1}^{J} \widetilde{P_h} \widetilde{P_i} \Big) - \int \widetilde{R_l} \sum_{h \neq i \text{ or } g \neq h} \big(\widetilde{P_h} \widetilde{P_i} \widetilde{P_g}\big)_x \\
&= \frac{y_1'(t)}{2\big(c_l + y_1(t)\big)} \int \widetilde{R_l} \big(\widetilde{R_l} + y_{0,l}(t) \widetilde{R_{lx}}\big) + y_2'(t) \int \widetilde{R_l} \widetilde{R_{lx}},
\end{aligned} \tag{4.2.98}
$$

then, we take the time derivative of $\int \widetilde{R_{lx}} w = 0$:

$$
\begin{aligned}
&- \int \big(\widetilde{R_l}^3\big)_{xx} w + \frac{y_1'(t)}{2c_l} \int \big(\widetilde{R_{lx}} + y_{0,l}(t) \widetilde{R_{lxx}}\big) w + y_2'(t) \int \widetilde{R_{lxx}} w \\
&+ \int \widetilde{R_{lxx}} w \Big(w^2 + 3w \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{h,i=1}^{J} \widetilde{P_h} \widetilde{P_i} \Big) - \int \widetilde{R_{lx}} \sum_{h \neq i \text{ or } g \neq h} \big(\widetilde{P_h} \widetilde{P_i} \widetilde{P_g}\big)_x \\
&= \frac{y_1'(t)}{2\big(c_l + y_1(t)\big)} \int \widetilde{R_{lx}} \big(\widetilde{R_l} + y_{0,l}(t) \widetilde{R_{lx}}\big) + y_2'(t) \int \big(\widetilde{R_{lx}}\big)^2.
\end{aligned} \tag{4.2.99}
$$

As a consequence of (4.2.76), we see that $|y_1(t)| + |y_2(t)|$ tends to 0 when $t \to +\infty$. This is why, we may use Proposition 2.10 and Corollary 2.11 here, if T_2 is large enough. So, several terms of the four equalities above are obviously bounded by $(w(t)^2 \Phi_j)^{1/2}$ or $e^{-\omega t}$ for $\omega > 0$, a constant chosen small enough. Using these bounds, and after several linear combinations, we obtain (4.2.77). □

Step 4. Quadratic approximations of localized conservation laws.

Lemma 4.15. *Let $\omega > 0$ as small as we want. There exists $C > 0, T_3 \geq T$ such that the following holds for $t \geq T_3$:*

$$
\Big| M_j(t) - \sum_{i=1}^{j-1} M\big[\widetilde{P_i}\big] - \sum_{i=1}^{j-1} \int \widetilde{P_i} w - \frac{1}{2} \int w^2 \Phi_j \Big| \leq C e^{-2\omega t}, \tag{4.2.100}
$$

$$
\begin{aligned}
&\Big| E_j(t) - \sum_{i=1}^{j-1} E\big[\widetilde{P_i}\big] - \sum_{i=1}^{j-1} \int \Big[\widetilde{P_{ix}} w_x - \widetilde{P_i}^3 w \Big] \\
&\qquad - \int \Big[\frac{1}{2} w_x^2 - \frac{3}{2} \widetilde{P}^2 w^2 \Big] \Phi_j \Big| \leq C e^{-2\omega t} + \omega \int w^2 \Phi_j,
\end{aligned} \tag{4.2.101}
$$

$$
\begin{aligned}
&\Big| F_j(t) - \sum_{i=1}^{j-1} F\big[\widetilde{P_i}\big] - \sum_{i=1}^{j-1} \int \Big[\widetilde{P_{ixx}} w_{xx} - 5 \widetilde{P_i} \widetilde{P_{ix}}^2 w - 5 \widetilde{P_i}^2 \widetilde{P_{ix}} w_x + \frac{3}{2} \widetilde{P_i}^5 w \Big] \\
&\qquad - \int \Big[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w^2 \widetilde{P}_x^2 - 10 \widetilde{P} w \widetilde{P}_x w_x - \frac{5}{2} \widetilde{P}^2 w_x^2 + \frac{15}{4} \widetilde{P}^4 w^2 \Big] \Phi_j(t) \Big| \\
&\qquad \leq C e^{-2\omega t} + \omega \int \big(w^2 + w_x^2\big) \Phi_j.
\end{aligned} \tag{4.2.102}
$$

Proof. For the mass:

We compute:

$$\begin{aligned} M_j(t) &= \frac{1}{2}\int \big(\widetilde{P}+w\big)^2\Phi_j \\ &= \frac{1}{2}\int \widetilde{P}^2\Phi_j + \int \widetilde{P}w\Phi_j + \frac{1}{2}\int w^2\Phi_j. \end{aligned} \tag{4.2.103}$$

As in Step 1, we can show that $\frac{1}{2}\int \widetilde{P}^2\Phi_j$ converges exponentially (we choose ω with respect to this exponential convergence) to $\sum_{i=1}^{j-1} M[\widetilde{P}_i]$. Similarly, the difference between $\int \widetilde{P}w\Phi_j$ and $\sum_{i=1}^{j-1}\widetilde{P}_i w$ converges exponentially to 0 (the velocity of a soliton is not modified a lot by modulation, this is why it works in any cases).

For E and F, we perform similar basic computations with the only difference that there will also be terms of degree 3 or more in w. We know that $\|w(t)\|_{H^2} \to_{t\to+\infty} 0$, this is the reason why for t large enough, such terms are boundable by $\omega\int w^2\Phi_j$ or $\omega\int w_x^2\Phi_j$. □

Step 5. Approximation of the Lyapunov functional.

By analogy with the existence part, we introduce the following Lyapunov functional:

$$\mathcal{H}_j(t) := F_j(t) + 2\big(b_{j-1}^2 - a_{j-1}^2\big)E_j(t) + \big(a_{j-1}^2+b_{j-1}^2\big)^2 M_j(t). \tag{4.2.104}$$

We will use the previous steps to approximate $\mathcal{H}_j(t)$.

Lemma 4.16. *There exists $T_4 \geq T$ such that the following holds for $t \geq T_4$:*

$$\begin{aligned} \mathcal{H}_j(t) = &\sum_{i=1}^{j-1} F[\widetilde{P}_i] + 2\big(b_{j-1}^2 - a_{j-1}^2\big)\sum_{i=1}^{j-1} E[\widetilde{P}_i] + \big(a_{j-1}^2+b_{j-1}^2\big)^2\sum_{i=1}^{j-1} M[\widetilde{P}_i] \\ &+ H_j(t) + O(e^{-2\omega t}) + o\bigg(\int \big(w^2+w_x^2\big)\Phi_j\bigg), \end{aligned} \tag{4.2.105}$$

where

$$\begin{aligned} H_j(t) :=& \int \Big[\frac{1}{2}w_{xx}^2 - \frac{5}{2}w_x^2\widetilde{P_{j-1}}^2 + \frac{5}{2}w^2\widetilde{P_{j-1}}_x^2 + 5w^2\widetilde{P_{j-1}}\widetilde{P_{j-1}}_{xx} \\ &+ \frac{15}{4}w^2\widetilde{P_{j-1}}^4\Big]\Phi_j(t) + \big(b_{j-1}^2 - a_{j-1}^2\big)\int\Big[w_x^2 - 3w^2\widetilde{P_{j-1}}^2\Big]\Phi_j(t) \\ &+ \frac{1}{2}\big(a_{j-1}^2+b_{j-1}^2\big)^2\int w^2\Phi_j(t). \end{aligned} \tag{4.2.106}$$

Proof. This lemma is obtained from the summation of the facts established in the previous lemma. We get rid of the linear terms in the following way, by integrations by parts:

$$\begin{aligned} &\sum_{i=1}^{j-1}\int\Big(\widetilde{P}_{ixx}w_{xx} - 5\widetilde{P}_i\widetilde{P}_{ix}^2 w - 5\widetilde{P}_i^2\widetilde{P}_{ix}w_x + \frac{3}{2}\widetilde{P}_i^5 w\Big) \\ &\quad + 2\big(b_{j-1}^2 - a_{j-1}^2\big)\sum_{i=1}^{j-1}\int\Big(\widetilde{P}_{ix}w_x - \widetilde{P}_i^3 w\Big) + \big(a_{j-1}^2+b_{j-1}^2\big)^2\sum_{i=1}^{j-1}\widetilde{P}_i w \\ &= \sum_{i=1}^{j-1}\int\Big(\widetilde{P}_{ixxxx} + 5\widetilde{P}_i\widetilde{P}_{ix}^2 + 5\int\widetilde{P}_i^2\widetilde{P}_{ixx} + \frac{3}{2}\widetilde{P}_i^5\Big)w \\ &\quad + 2\big(b_{j-1}^2 - a_{j-1}^2\big)\sum_{i=1}^{j-1}\int\Big(-\widetilde{P}_{ixx} - \widetilde{P}_i^3\Big)w + \big(a_{j-1}^2+b_{j-1}^2\big)^2\sum_{i=1}^{j-1}\int\widetilde{P}_i w. \end{aligned} \tag{4.2.107}$$

If we consider that this sum goes from $i=1$ to $j-2$, we see that for $1\leq i\leq j-2$, this sum is exponentially bounded by induction assumption (we use that for $i\leq j-2$, a polynomial in $\widetilde{P}_i$ and its derivatives is bounded by $C\Phi_{j-1}$ and that $w = v + (P_{j-1} - \widetilde{P_{j-1}})$). It is left to consider the sum of the terms with $i = j-1$.

For $i = j-1$, we have nearly the elliptic equation satisfied by $\widetilde{P_{j-1}}$. It is actually exactly this equation in the case when $\widetilde{P_{j-1}}$ is a breather. When $\widetilde{P_{j-1}}$ is a soliton, its shape parameter is modified by modulation. This is why, in this case, the sum of the terms with $i = j-1$ is equal to

$$2y_1(t)\int\Big(-\widetilde{P_{j-1}}_{xx}-\widetilde{P_{j-1}}^3\Big)w+2b_{j-1}^2y_1(t)\int\widetilde{P_{j-1}}w+y_1(t)^2\int\widetilde{P_{j-1}}w, \tag{4.2.108}$$

which vanishes because of the orthogonality condition from the modulation (Lemma 4.14) and the elliptic equation satisfied by a soliton (1.1.4).

H_j is obtained as the sum of the quadratic parts of the previous lemma on which we have performed some integrations by parts, and some simplifications based on the fact that for $i \geq j$, $\widetilde{P_i}\Phi_j(t)$ is exponentially decreasing, and the fact that for $i \leq j-2$, $\int \widetilde{P_i}w^2$ is exponentially decreasing by the induction assumption (4.2.13). Therefore, H_j corresponds to the sum of the quadratic parts of previous lemma to which we have to add $5\int w^2\widetilde{P}\widetilde{P}_x\Phi_{jx}$, which is bounded exponentially. □

Step 6. Bound from above for $H_j(t)$.

From Lemma 4.11, we know that for any $t \geq T_1$,

$$M_j(t) - M[P_{j-1}] \leq Ce^{-2\omega t}. \tag{4.2.109}$$

By summing this fact with the fact from the Lemma 4.13, we obtain the following inequality for any t large enough:

$$\begin{aligned}\mathcal{H}_j(t) - \sum_{i=1}^{j-1}F[P_i] - 2\big(b_{j-1}^2-a_{j-1}^2\big)\sum_{i=1}^{j-1}E[P_i]\\ -\big(a_{j-1}^2+b_{j-1}^2\big)^2\sum_{i=1}^{j-1}M[P_i] \leq Ce^{-2\omega t}.\end{aligned} \tag{4.2.110}$$

From Lemma 4.16, for $t \geq T_3$,

$$\begin{aligned}H_j(t) \leq F[P_{j-1}] - F[\widetilde{P_{j-1}}] + 2\big(b_{j-1}^2-a_{j-1}^2\big)\Big(E[P_{j-1}]-E[\widetilde{P_{j-1}}]\Big)\\ +\big(a_{j-1}^2+b_{j-1}^2\big)^2\Big(M[P_{j-1}]-M[\widetilde{P_{j-1}}]\Big)+Ce^{-2\omega t}+\omega\int\big(w^2+w_x^2\big)\Phi_j.\end{aligned} \tag{4.2.111}$$

In the case if P_{j-1} is a breather, we obtain immediately that

$$H_j(t) \leq Ce^{-2\omega t}+\omega\int\big(w^2+w_x^2\big)\Phi_j. \tag{4.2.112}$$

The case when P_{j-1} is a soliton needs more inspection. As in the existence part, we have the following relations:

$$M[\widetilde{P_{j-1}}](t) = \big(b_{j-1}^2+y_1(t)\big)^{1/2}M[q], \tag{4.2.113}$$

$$E[\widetilde{P_{j-1}}](t) = \big(b_{j-1}^2+y_1(t)\big)^{\frac{3}{2}}E[q], \tag{4.2.114}$$

$$F[\widetilde{P_{j-1}}](t) = \big(b_{j-1}^2+y_1(t)\big)^{5/2}F[q]. \tag{4.2.115}$$

We set $\mathcal{R}_{j-1}(t) := F[\widetilde{P_{j-1}}](t) + 2b_{j-1}^2E[\widetilde{P_{j-1}}](t) + b_{j-1}^4M[\widetilde{P_{j-1}}](t)$, and we simplify it as follows:

$$\begin{aligned}\mathcal{R}_{j-1}(t) = b_{j-1}^5\left(1+\frac{y_1(t)}{b_{j-1}^2}\right)^{5/2}F[q]+2b_{j-1}^5\left(1+\frac{y_1(t)}{b_{j-1}^2}\right)^{\frac{3}{2}}E[q]\\ +b_{j-1}^5\left(1+\frac{y_1(t)}{b_{j-1}^2}\right)^{1/2}M[q].\end{aligned} \tag{4.2.116}$$

After making a Taylor expansion as in Section 2.5,

$$\mathcal{R}_{j-1}(t) - F[P_{j-1}] - 2b_{j-1}^2E[P_{j-1}] - b_{j-1}^4M[P_{j-1}] = O(y_1(t)^3). \tag{4.2.117}$$

Therefore, if T_4 is large enough, $\|v(t)\|_{H^2}$ can be as small as we want, and for $t \geq T_4$, if P_{j-1} a soliton, we may write:

$$H_j(t) \leq Ce^{-2\omega t} + \omega \int \left(w^2 + w_x^2\right)\Phi_j + \omega y_1(t)^2. \tag{4.2.118}$$

Step 7. Coercivity.

H_j can be seen as the quadratic form associated to $\widetilde{P_{j-1}}$ and evaluated in $w\sqrt{\Phi_j}$, modulo several terms that can be bounded by $C\sqrt{\sigma}\int(w^2 + w_x^2 + w_{xx}^2)\Phi_j$ (because these terms depend on derivatives of Φ_j). Let us prove that we can apply Section 5.4 (Appendix) for $w\sqrt{\Phi_j}$.

More precisely, we need to prove that for $\nu > 0$ small enough (from Section 5.4),

$$\left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}_1\right| + \left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}_2\right| \leq \nu\|w\sqrt{\Phi_j}\|_{H^2}, \tag{4.2.119}$$

if P_{j-1} is a breather or that

$$\left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}\right| + \left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}_x\right| \leq \nu\|w\sqrt{\Phi_j}\|_{H^2}, \tag{4.2.120}$$

if P_{j-1} is a soliton. In any case, the proof is the same and let us write K at the place of $\widetilde{P_{j-1}}_1$, $\widetilde{P_{j-1}}_2$, $\widetilde{P_{j-1}}$ or $\widetilde{P_{j-1}}_x$. This means that we want to bound $\int w\sqrt{\Phi_j}K$.

From (4.2.73), (4.2.74), we see that it is enough to bound $\int w(1-\sqrt{\Phi_j})K$ by $\nu\|w\sqrt{\Phi_j}\|_{H^2}$. The reasonning that follows works for $j \leq J$, for $j = J+1$ the result is immediate because $\Phi_{J+1} = 1$. Φ_j is a translate of Ψ, and, using the fact that when $v \to 0$, $\sqrt{1+v} = 1 + O(v)$,

$$1 - \sqrt{\Psi} = 1 - \sqrt{1+\Psi-1} = 1 - \sqrt{1-\Psi(-x)} = O(\Psi(-x)), \tag{4.2.121}$$

which means that $1 - \sqrt{\Phi_j} \leq C\min(1, \exp(\sqrt{\sigma}(x - m_j t)/2))$. We may deduce now that

$$\begin{aligned}\left|\int w(1-\sqrt{\Phi_j})K\right| &= \left|\int w\sqrt{\Phi_j}\frac{1-\sqrt{\Phi_j}}{\sqrt{\Phi_j}}K\right| \\ &\leq \left\|\frac{1-\sqrt{\Phi_j}}{\sqrt{\Phi_j}}K\right\|_{L^2}\|w\sqrt{\Phi_j}\|_{L^2} \\ &\leq Ce^{\sqrt{\sigma}(m_j - v_{j-1})t}\|w\sqrt{\Phi_j}\|_{L^2},\end{aligned} \tag{4.2.122}$$

if $\frac{\sqrt{\sigma}}{4} < \frac{\beta}{2}$. And so, if t is large enough, we get the bound we want.

Thus, there exists $\mu > 0$ such that for $t \geq T_5$ (where T_5 is large enough and depends on σ),

$$\begin{aligned}\mu\|w\sqrt{\Phi_j}\|_{H^2}^2 &\leq H_j(t) + C\sqrt{\sigma}\int\left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j + \frac{1}{\mu}\left(\int\widetilde{P_{j-1}}w\sqrt{\Phi_j}\right)^2 \\ &\leq Ce^{-2\omega t} + \omega\int\left(w^2 + w_x^2\right)\Phi_j + C\sqrt{\sigma}\int\left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j \\ &\quad + \omega y_1(t)^2 + \frac{1}{\mu}\left(\int\widetilde{P_{j-1}}w\sqrt{\Phi_j}\right)^2,\end{aligned} \tag{4.2.123}$$

where the term $\frac{1}{\mu}\left(\int\widetilde{P_{j-1}}w\sqrt{\Phi_j}\right)^2$ is present only if $\widetilde{P_{j-1}}$ is a breather and the term $\omega y_1(t)^2$ is present only if $\widetilde{P_{j-1}}$ is a soliton.

For σ small enough and ω small enough, we deduce that

$$\int\left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j \leq Ce^{-2\omega t} + \omega y_1(t)^2 + C\left(\int\widetilde{P_{j-1}}w\sqrt{\Phi_j}\right)^2. \tag{4.2.124}$$

We set $T_0 := \max(T_1, T_2, T_3, T_4, T_5)$.

Step 8. Bound for $\left|\int\widetilde{P_{j-1}}w\sqrt{\Phi_j}\right|$ (to do in the case if $\widetilde{P_{j-1}}$ is a breather).

We would like to prove that $\int \widetilde{P_{j-1}} w\sqrt{\Phi_j}$ is exponentially decreasing. To do so, we would like to get rid of $\sqrt{\Phi_j}$. It is clear that $\int \widetilde{P_{j-1}} w(1-\sqrt{\Phi_j})$ is exponentially decreasing. Thus, it is enough to prove that $\int \widetilde{P_{j-1}} w$ is exponentially decreasing.

If $i \leq j-2$, we know that $\int \widetilde{P}_i w$ is exponentially decreasing by the induction assumption (4.2.13). Thus, it is enough to prove that $\sum_{i=1}^{j-1} \int \widetilde{P}_i w$ is exponentially decreasing.

From the mass approximation of Lemma 4.15 and Lemma 4.11, and from (4.2.117), we observe that, for $t \geq T_0$,

$$\begin{aligned}\sum_{i=1}^{j-1} \int \widetilde{P}_i w &= O(e^{-2\omega t}) + M_j(t) - \sum_{i=1}^{j-1} M[P_i] - \frac{1}{2}\int w^2\Phi_j + o(y_1(t)^2) \\ &\leq Ce^{-2\omega t} - \frac{1}{2}\int w^2\Phi_{j+o(y_1(t)^2)} \leq Ce^{-2\omega t} + o(y_1(t)^2),\end{aligned} \tag{4.2.125}$$

where $+o(y_1(t)^2)$ is present only of P_{j-1} is a soliton.

Now, we use the fact that the sum of the linear parts of our localized conservation laws is exponentially decreasing, which we have established in the proof of Lemma 4.16. Therefore, the linear terms of $F_j + 2(b_{j-1}^2 - a_{j-1}^2)E_j$ are equal to $O(e^{-2\omega t}) - (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \widetilde{P}_i w$.

From Lemma 4.11 and from (4.2.117), we observe that, for $t \geq T_0$,

$$\begin{aligned}-(1-\nu)(a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \widetilde{P}_i w &= O(e^{-2\omega t}) + O\left(\int (w^2 + w_x^2)\Phi_2\right) + o(y_1(t)^2) \\ &+ \mathcal{F}_j(t) - \sum_{i=1}^{j-1} \left(F[P_i] + 2(b_{j-1}^2 - a_{j-1}^2)E[P_i] + \nu(a_{j-1}^2 + b_{j-1}^2)^2 M[P_i]\right),\end{aligned} \tag{4.2.126}$$

where $o(y_1(t)^2)$ is present only if P_{j-1} is a soliton. And, from Lemma 4.13, we obtain that

$$-(1-\nu)(a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \widetilde{P}_i w \leq Ce^{-2\omega t} + C\int (w^2 + w_x^2)\Phi_2 + o(y_1(t)^2), \tag{4.2.127}$$

where $o(y_1(t)^2)$ is present only if P_{j-1} is a soliton. Therefore, by taking the constant $C > 0$ larger if necessarily (in function of ν), we obtain that

$$-\sum_{i=1}^{j-1} \int \widetilde{P}_i w \leq Ce^{-2\omega t} + C\int \left(w^2 + w_x^2\right)\Phi_j + o(y_1(t)^2), \tag{4.2.128}$$

where $o(y_1(t)^2)$ is present only if P_{j-1} is a soliton.

Thus, we deduce the following bound:

$$\left|\int \widetilde{P_{j-1}} w\sqrt{\Phi_j}\right| \leq Ce^{-2\omega t} + C\int \left(w^2 + w_x^2\right)\Phi_j + o(y_1(t)^2). \tag{4.2.129}$$

Because $\|w(t)\|_{H^2} \to_{t\to+\infty} 0$, we deduce that

$$\left(\int \widetilde{P_{j-1}} w\sqrt{\Phi_j}\right)^2 = o(e^{-2\omega t}) + o\left(\int \left(w^2 + w_x^2\right)\Phi_j\right) + o(y_1(t)^2). \tag{4.2.130}$$

Step 9. Conclusion.

From (4.2.124) and (4.2.130), we deduce for $t \geq T_0$, that

$$\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j = O(e^{-2\omega t}) + o\left(y_1(t)^2\right) + o\left(\int \left(w^2 + w_x^2\right)\Phi_j\right). \tag{4.2.131}$$

This means that if we take T_0 large enough, we have:

$$\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j = o\left(y_1(t)^2\right) + O(e^{-2\omega t}), \tag{4.2.132}$$

where the term $o(y_1(t)^2)$ is present only if P_{j-1} is a soliton.

Before finishing the proof, we need to find a better bound for $y_1(t)$ than just a convergence to 0 given by the modulation (in the case when P_{j-1} is a soliton). For this, we study $M_j(t)$:

$$\begin{aligned} M_j(t) &= \frac{1}{2}\int u^2(t)\Phi_j(t) = \frac{1}{2}\int \big(\widetilde{P}(t)+w(t)\big)^2\Phi_j(t)\\ &= \frac{1}{2}\int \widetilde{P}(t)^2\Phi_j(t) + \int \widetilde{P}(t)w(t)\Phi_j(t) + \frac{1}{2}\int w(t)^2\Phi_j(t)\\ &= \frac{1}{2}\sum_{i=1}^{j-1}\int \widetilde{P_i}(t)^2 + \sum_{i=1}^{j-1}\int \widetilde{P_i}(t)w(t) + O\big(e^{-2\omega t}\big) + \frac{1}{2}\int w(t)^2\Phi_j(t)\\ &= \frac{1}{2}\int \widetilde{P_{j-1}}(t)^2 + \int \widetilde{P_{j-1}}(t)w(t) + O\big(e^{-2\omega t}\big)\\ &\quad + \frac{1}{2}\int w(t)^2\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2, \end{aligned} \tag{4.2.133}$$

by the induction assumption (4.2.13), then

$$M_j(t) = \frac{1}{2}\int \widetilde{P_{j-1}}(t)^2 + O\big(e^{-2\omega t}\big) + \frac{1}{2}\int w(t)^2\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2, \tag{4.2.134}$$

by the orthogonality condition from the modulation (Lemma 4.14). Therefore,

$$\begin{aligned} M_j(t) &= \big(b_{j-1}^2 + y_1(t)\big)^{1/2}M[q] + O\big(e^{-2\omega t}\big) + \frac{1}{2}\int w(t)^2\Phi_j(t)\\ &\quad + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2. \end{aligned} \tag{4.2.135}$$

Now, if we take $t_1 \geq t$, we obtain from (4.2.132) that

$$\begin{aligned} M_j(t_1) - M_j(t) &= \Big[\big(b_{j-1}^2 + y_1(t_1)\big)^{1/2} - \big(b_{j-1}^2 + y_1(t)\big)^{1/2}\Big]M[q]\\ &\quad + O\big(e^{-2\omega t}\big) + o\big(y_1(t)^2\big) + o\big(y_1(t_1)^2\big). \end{aligned} \tag{4.2.136}$$

By doing a Taylor expansion of order 1, as in the existence part, we obtain:

$$\big(b_{j-1}^2 + y_1(t_1)\big)^{1/2} = b_{j-1}\left(1 + \frac{1}{2}\frac{y_1(t_1)}{b_{j-1}^2} + O\big(y_1(t_1)^2\big)\right). \tag{4.2.137}$$

Therefore,

$$\begin{aligned} &\big(b_{j-1}^2 + y_1(t_1)\big)^{1/2} - \big(b_{j-1}^2 + y_1(t)\big)^{1/2}\\ &\quad = \frac{1}{2b_{j-1}}\big(y_1(t_1) - y_1(t)\big) + O\big(y_1(t_1)^2\big) + O\big(y_1(t)^2\big). \end{aligned} \tag{4.2.138}$$

Now, we recall that when $t_1 \to +\infty$, we have $y_1(t_1) \to 0$. Therefore, by taking the limit of the previous formula when $t_1 \to +\infty$, we obtain:

$$b_{j-1} - \big(b_{j-1}^2 + y_1(t)\big)^{1/2} = -\frac{y_1(t)}{2b_{j-1}} + O\big(y_1(t)^2\big). \tag{4.2.139}$$

Therefore, from (4.2.136), with $t_1 \to +\infty$,

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) = -\frac{y_1(t)}{2b_{j-1}}M[q] + O\big(e^{-2\omega t}\big) + O\big(y_1(t)^2\big). \tag{4.2.140}$$

The second step is to study $E_j(t)$ (we do the same reasonning as for M_j):

$$\begin{aligned} E_j(t) &= \int \left[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\right]\Phi_j(t)\\ &= \int \left[\frac{1}{2}\widetilde{P}_x^2 - \frac{1}{4}\widetilde{P}^4\right]\Phi_j(t) + \int \left[\widetilde{P}_x w_x - \widetilde{P}^3 w\right]\Phi_j(t) + O\left(\int w^2\Phi_j(t)\right), \end{aligned} \tag{4.2.141}$$

and after simplications by Φ_j due to exponential convergences, induction assumption (4.2.13) and orthogonality conditions (Lemma 4.14),

$$\begin{aligned} E_j(t) &= E[\widetilde{P_{j-1}}(t)] + \sum_{i=1}^{j-2} E[P_i] + O\big(e^{-2\omega t}\big) + O\bigg(\int w^2\Phi_j(t)\bigg) \\ &= \big(b_{j-1}^2 + y_1(t)\big)^{\frac{3}{2}} E[q] + \sum_{i=1}^{j-2} E[P_i] + O\big(e^{-2\omega t}\big) + O\bigg(\int w^2\Phi_j(t)\bigg) \\ &= \big(b_{j-1}^2 + y_1(t)\big)^{\frac{3}{2}} E[q] + \sum_{i=1}^{j-2} E[P_i] + O\big(e^{-2\omega t}\big) + o\big(y_1(t)^2\big), \end{aligned} \tag{4.2.142}$$

by (4.2.132). And then, by taking the difference for $t_1 \geq t$,

$$\begin{aligned} E_j(t_1) - E_j(t) = &\Big[\big(b_{j-1}^2 + y_1(t_1)\big)^{\frac{3}{2}} - \big(b_{j-1}^2 + y_1(t)\big)^{\frac{3}{2}}\Big] E[q] \\ &+ O\big(e^{-2\omega t}\big) + o\big(y_1(t_1)^2\big) + o\big(y_1(t)^2\big). \end{aligned} \tag{4.2.143}$$

By taking a Taylor expansion of order 1, we obtain:

$$\big(b_{j-1}^2 + y_1(t_1)\big)^{\frac{3}{2}} = b_{j-1}^3\Big(1 + \frac{3}{2}\frac{y_1(t_1)}{b_{j-1}^2} + O\big(y_1(t_1)^2\big)\Big). \tag{4.2.144}$$

Therefore, after taking $t_1 \to +\infty$, we obtain:

$$\sum_{i=1}^{j-1} E[P_i] - E_j(t) = -\frac{3}{2} b_{j-1} y_1(t) E[q] + O\big(e^{-2\omega t}\big) + O\big(y_1(t)^2\big). \tag{4.2.145}$$

This is why, from (4.2.140), (4.2.145) and Lemma 4.11, we obtain:

$$-\frac{y_1(t)}{2b_{j-1}} M[q] + O\big(e^{-2\omega t}\big) + O\big(y_1(t)^2\big) \geq -Ce^{-2\omega t}, \tag{4.2.146}$$

and

$$-\frac{3}{2} b_{j-1} y_1(t) E[q] + O\big(e^{-2\omega t}\big) + O\big(y_1(t)^2\big) \geq -Ce^{-2\omega t}. \tag{4.2.147}$$

Because $M[q] = 2$ and $E[q] = -\frac{2}{3}$, we rewrite both previous inequalities (4.2.146) and (4.2.147) in the following way (and we pass $O(e^{-2\omega t})$ on the other side of each inequality):

$$-\frac{y_1(t)}{b_{j-1}} + O\big(y_1(t)^2\big) \geq -Ce^{-2\omega t}, \tag{4.2.148}$$

and

$$b_{j-1} y_1(t) + O\big(y_1(t)^2\big) \geq -Ce^{-2\omega t}. \tag{4.2.149}$$

Because $y_1(t) \to +\infty$, by taking T_0 larger if needed, $O(y_1(t)^2)$ can be bounded above by any positive constant multiplied by $|y_1(t)|$, so by taking this constant small enough (by taking T_0 large enough) and combining both previous inequalities (4.2.148) and (4.2.149), we obtain:

$$|y_1(t)| \leq Ce^{-2\omega t}. \tag{4.2.150}$$

Therefore, we have obtained a better bound for $y_1(t)$ in the case when P_{j-1} is a soliton. Therefore, we may conclude that in any case, for $t \geq T_0$, for T_0 large enough,

$$\int \big(w^2 + w_x^2 + w_{xx}^2\big)\Phi_j(t) = O\big(e^{-2\omega t}\big). \tag{4.2.151}$$

Then, we deduce from (4.2.77) that

$$|y_1'(t)| + |y_2'(t)| = O(e^{-\omega t}). \tag{4.2.152}$$

Because $|y_1(t)| + |y_2(t)| \to_{t\to+\infty} 0$, we obtain by integration:

$$|y_1(t)| + |y_2(t)| = O(e^{-\omega t}). \tag{4.2.153}$$

And, so, by the mean-value theorem,

$$\left\|\widetilde{P_{j-1}} - P_{j-1}\right\|_{H^2} \leq C\left(|y_1(t)| + |y_2(t)|\right) \leq Ce^{-\omega t}. \tag{4.2.154}$$

From $v = w + \widetilde{P_{j-1}} - P_{j-1}$, we deduce:

$$\begin{aligned}\int \left(v^2 + v_x^2 + v_{xx}^2\right)\Phi_j &\leq C\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j \\ &+ C\int \left[\left(\widetilde{P_{j-1}} - P_{j-1}\right)^2 + \left(\widetilde{P_{j-1}} - P_{j-1}\right)_x^2 + \left(\widetilde{P_{j-1}} - P_{j-1}\right)_{xx}^2\right]\Phi_j \\ &\leq Ce^{-2\omega t},\end{aligned} \tag{4.2.155}$$

and this finishes the induction.

□

4.3. **Proof of Theorem 1.4.**

Proof of Theorem 1.4. We suppose that $v_2 > 0$. Let p be the associated multi-breather given by Theorem 1.2. Let u be a solution of (mKdV) such that

$$\|u(t) - p(t)\|_{H^2} \to_{t\to+\infty} 0. \tag{4.3.1}$$

From Proposition 4.10, we deduce that there exists a constant $C > 0$ and a constant $\omega > 0$ such that for t large enough

$$\|u(t) - p(t)\|_{H^2} \leq Ce^{-\omega t}. \tag{4.3.2}$$

This implies that u satisfies the assumptions of Proposition 1.5. Thus, $u = p$ and Theorem 1.4 is proved.

□

5. Appendix

The first two subsections of the Appendix show that a soliton has similar properties as a "limit breather" of parameter $\alpha = 0$. Firstly, the corresponding elliptic equation is satisfied by a soliton. Secondly, the corresponding quadratic form is coercive for a soliton, and we see that its kernel is spanned by $\partial_x Q$ and $\partial_c Q$. In the third subsection, we prove that it is possible for ϵ to be orthogonal to Q and $\partial_x Q$ (instead of $\partial_x Q$ and $\partial_c Q$) in order to satisfy a coercivity for the quadratic form. We will use this fact for the proof of the existence, as well as for the first part of the proof of the uniqueness. In the fourth subsection, we prove that we can have coercivity for quadratic forms when the orthogonality condition is not exactly satisfied. We will use this result for the proof of the uniqueness. The last subsection is about computations for the third conservation law. It will be useful for the monotonicity property for localized F that we need in the proof of the uniqueness.

5.1. **Elliptic equation satisfied by a soliton.**

Lemma 5.1. *A soliton $Q = R_{c,\kappa}$ satisfies for any time $t \in \mathbb{R}$, the following nonlinear elliptic equation:*

$$Q_{(4x)} - 2c(Q_{xx} + Q^3) + c^2 Q + 5QQ_x^2 + 5Q^2 Q_{xx} + \frac{3}{2}Q^5 = 0. \tag{5.1.1}$$

Proof. In order to derive this equation, we will use the equation that defines a soliton (and that is satisfied by Q at any time):

$$Q_{xx} = cQ - Q^3. \tag{5.1.2}$$

We will also need the following equation:

$$Q_x^2 = cQ^2 - \frac{1}{2}Q^4, \tag{5.1.3}$$

that can be derived by taking the space derivative of $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$, and by showing that this derivative is zero. From this, we deduce that $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$ is constant, and by taking its limit when $x \to \pm\infty$, we see that this constant is zero. More precisely, the derivative of $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$ is:

$$2Q_xQ_{xx} - 2cQQ_x + 2Q^3Q_x = 2Q_x(Q_{xx} - cQ + Q^3) = 0. \tag{5.1.4}$$

From now on, the derivation of (5.1.1) is straightforward. It is sufficient to take space derivatives of $Q_{xx} = cQ - Q^3$ and to inject them into the right hand side of the equation (5.1.1), which we want to prove to be equal to zero. By doing this, we make the maximal order of a derivative of Q present in the right hand side equation lower. At the end, we have only, zero and first order derivatives. To have only a polynomial in Q, we have to use $Q_x^2 = cQ^2 - \frac{1}{2}Q^4$, and the calculations show that this polynomial is zero. □

5.2. **Study of coercivity of the quadratic form associated to a soliton.** In this article, we adapt the argument for the breathers in [3] to the soliton case. We consider:

$$\begin{aligned} \mathcal{Q}_c^s[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ + c\left(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\right) + c^2\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{0,\sqrt{c}}[\epsilon]. \end{aligned} \tag{5.2.1}$$

Firstly, we prove, by simple calculations, as in the previous section, that Q_x and $Q + xQ_x$ are in the kernel of this quadratic form. It is easy to see, by asymptotic study that these two functions are linearly independent.

The self-adjoint linear operator associated to this quadratic form is:

$$\begin{aligned} \mathcal{L}_c^s[\epsilon] := \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon + 5Q^2\epsilon_{xx} + 10QQ_x\epsilon_x \\ + \left(5Q_x^2 + 10QQ_{xx} + \frac{15}{2}Q^4 - 6cQ^2\right)\epsilon, \end{aligned} \tag{5.2.2}$$

so that $\mathcal{Q}_c^s[\epsilon] = \int \epsilon\mathcal{L}_c^s[\epsilon]$. $\mathcal{L}_c^s$ is a compact perturbation of the constant coefficients operator:

$$\mathcal{M}[\epsilon] := \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon. \tag{5.2.3}$$

A direct analysis involving ODE shows that the null space of $\mathcal{M}$ is spawned by four linearly independent functions:

$$e^{\pm\sqrt{c}x}, \quad xe^{\pm\sqrt{c}x}. \tag{5.2.4}$$

Among these four functions, there are only two L^2-integrable ones in the semi-infinite line $[0, +\infty)$. Therefore, the null space of $\mathcal{L}_c^s|_{H^4(\mathbb{R})}$ is spanned by at most two L^2-functions. Therefore,

$$\ker(\mathcal{L}_c^s) = \mathrm{Span}(\partial_x Q, Q + x\partial_x Q). \tag{5.2.5}$$

Lemma 5.2. *The operator $\mathcal{L}_c^s$ does not have any negative eigenvalue.*

Proof. $\mathcal{L}_c^s$ has

$$\sum_{x\in\mathbb{R}} \dim\ker W[Q_x, Q + xQ_x](t, x) \tag{5.2.6}$$

negative eigenvalues, counting multiplicity, where W is the Wronskian matrix:

$$W[Q_x, Q + xQ_x](t, x) := \begin{bmatrix} Q_x & Q + xQ_x \\ Q_{xx} & (Q + xQ_x)_x \end{bmatrix}. \tag{5.2.7}$$

For this result, see [19], where the finite interval case was considered. As shown in several articles [23, 27], the extension to the real line is direct.

Thus, it is sufficient to see that $\det W[Q_x, Q + xQ_x](t, x)$ is never zero. For this, let us simply calculate this determinant:

$$\begin{aligned} Q_x(2Q_x + xQ_{xx}) - (Q + xQ_x)Q_{xx} &= 2Q_x^2 - QQ_{xx} \\ &= 2cQ^2 - Q^4 - Q(cQ - Q^3) \\ &= cQ^2 > 0. \end{aligned} \tag{5.2.8}$$

□

5.3. **Coercivity of the quadratic form associated to a soliton.** For $Q = R_{c,\kappa}$, let

$$\begin{aligned} \mathcal{Q}_c^s[\epsilon] :&= \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ &+ c\left(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\right) + c^2\frac{1}{2}\int \epsilon^2. \end{aligned} \tag{5.3.1}$$

Lemma 5.3. *There exists $\mu_c > 0$ such that for any $\epsilon \in H^2$ such that $\int \epsilon Q = \int \epsilon Q_x = 0$, we have that*

$$\mathcal{Q}_c^s[\epsilon] \geq \mu_c \|\epsilon\|_{H^2}^2. \tag{5.3.2}$$

Proof. From Section 5.2, we know that if $\int \epsilon \partial_x Q = \int \epsilon \partial_c Q = 0$, then, for a constant $\nu_c > 0$, we have that

$$\mathcal{Q}_c^s[\epsilon] \geq \nu_c \|\epsilon\|_{H^2}^2. \tag{5.3.3}$$

Let $\epsilon \in H^2$ be such that $\int \epsilon Q = \int \epsilon \partial_x Q = 0$. There exists $a \in \mathbb{R}$ and $\epsilon_\perp \in \mathrm{Span}(\partial_x Q, \partial_c Q)^\perp$ such that

$$\epsilon = a\partial_c Q + \epsilon_\perp. \tag{5.3.4}$$

From $\int \epsilon Q = 0$, we have that

$$a\int \partial_c Q \cdot Q + \int \epsilon_\perp Q = 0, \tag{5.3.5}$$

thus,

$$\frac{a}{2}\int Q^2 + \int \epsilon_\perp Q = 0, \tag{5.3.6}$$

which allows us to derive:

$$a = -2\frac{\int \epsilon_\perp Q}{\int Q^2}. \tag{5.3.7}$$

Because $\partial_c Q$ is in the kernel of $\mathcal{Q}_c^s$, we have that

$$\mathcal{Q}_c^s[\epsilon] = \mathcal{Q}_c^s[\epsilon_\perp] \geq \nu_c \|\epsilon_\perp\|_{H^2}^2. \tag{5.3.8}$$

Now, from

$$\epsilon = -2\frac{\int \epsilon_\perp Q}{\int Q^2}\partial_c Q + \epsilon_\perp, \tag{5.3.9}$$

we have by triangular and Cauchy-Schwarz inequalities that

$$\begin{aligned} \|\epsilon\|_{H^2} &\leq \|\epsilon_\perp\|_{H^2} + 2\frac{\left|\int \epsilon_\perp Q\right|}{\|Q\|_{L^2}^2}\|\partial_c Q\|_{H^2} \\ &\leq \|\epsilon_\perp\|_{H^2} + 2\frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}}\|\epsilon_\perp\|_{L^2} \\ &\leq \left(1 + 2\frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}}\right)\|\epsilon_\perp\|_{H^2}. \end{aligned} \tag{5.3.10}$$

Therefore, we may derive a constant μ_c (independent on ϵ) such that

$$\mathcal{Q}_c^s[\epsilon] \geq \mu_c \|\epsilon\|_{H^2}^2. \tag{5.3.11}$$

□

5.4. **Coercivity with almost orthogonality conditions (to be used for the uniqueness).** For $B := B_{\alpha,\beta}$ or any of its translations, we define the canonical quadratic form associated to B:

$$\begin{aligned} Q^b_{\alpha,\beta}[\epsilon] :&= \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int B^2\epsilon_x^2 + \frac{5}{2}\int B_x^2\epsilon^2 + 5\int BB_{xx}\epsilon^2 + \frac{15}{4}\int B^4\epsilon^2 \\ &+ (\beta^2-\alpha^2)\left(\int \epsilon_x^2 - 3\int B^2\epsilon^2\right) + (\alpha^2+\beta^2)^2\frac{1}{2}\int \epsilon^2, \end{aligned} \tag{5.4.1}$$

and we know that $\partial_{x_1}B$ and $\partial_{x_2}B$ span the kernel of $Q^b_{\alpha,\beta}$. More precisely, there exists $\mu^b_{\alpha,\beta} > 0$ such that if ϵ is orthogonal to $\partial_{x_1}B$ and $\partial_{x_2}B$, we have that

$$Q^b_{\alpha,\beta}[\epsilon] \geq \mu^b_{\alpha,\beta}\|\epsilon\|^2_{H^2} - \frac{1}{\mu^b_{\alpha,\beta}}\left(\int \epsilon B\right)^2. \tag{5.4.2}$$

We would like to prove the following lemma (adapted from the Appendix A of [30]):

Lemma 5.4. *There exists $\nu := \nu^b_{\alpha,\beta} > 0$ such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\left|\int (\partial_{x_1}B_{\alpha,\beta})\epsilon\right| + \left|\int (\partial_{x_2}B_{\alpha,\beta})\epsilon\right| < \nu\|\epsilon\|_{H^2}, \tag{5.4.3}$$

then

$$Q^b_{\alpha,\beta}[\epsilon] \geq \frac{\mu^b_{\alpha,\beta}}{4}\|\epsilon\|^2_{H^2} - \frac{4}{\mu^b_{\alpha,\beta}}\left(\int \epsilon B_{\alpha,\beta}\right)^2, \tag{5.4.4}$$

where $B_{\alpha,\beta}$ denotes the breather of parameters α and β or any of its translations (in space or in time).

Proof. Take $\nu > 0$ (we will find a condition on ν later in the proof) and take ϵ satisfying the assumption of the lemma. Then (denoting $B = B_{\alpha,\beta}$) ,

$$\epsilon = \epsilon_1 + aB_1 + bB_2 = \epsilon_1 + \epsilon_2, \tag{5.4.5}$$

where $\int \epsilon_1 B_1 = \int \epsilon_1 B_2 = \int \epsilon_1\epsilon_2 = 0$.

By performing a L^2-scalar product of (5.4.5) with B_1 and B_2, we obtain, by assumption, that

$$\left|a\int B_1^2 + b\int B_1B_2\right| \leq \nu\|\epsilon\|_{H^2}, \tag{5.4.6}$$

$$\left|a\int B_1B_2 + b\int B_2^2\right| \leq \nu\|\epsilon\|_{H^2}. \tag{5.4.7}$$

Therefore, by making linear combinations of these two inequalities, using triangular and Cauchy-Schwarz inequalities, we obtain that

$$|a| + |b| \leq C\nu\|\epsilon\|_{H^2}. \tag{5.4.8}$$

We can take space derivatives of (5.4.5). And thus, we obtain, for ν small enough, that

$$\frac{1}{2}\|\epsilon\|^2_{H^2} \leq \|\epsilon_1\|^2_{H^2} \leq 2\|\epsilon\|^2_{H^2}. \tag{5.4.9}$$

Because of $\int BB_1 = \int BB_2 = 0$,

$$\int \epsilon B = \int \epsilon_1 B. \tag{5.4.10}$$

By bilinearity,

$$\begin{aligned} Q^b_{\alpha,\beta}[\epsilon] = Q^b_{\alpha,\beta}[\epsilon_1] + Q^b_{\alpha,\beta}[\epsilon_2] &+ \int \epsilon_{1,xx}\epsilon_{2,xx} - 5\int B^2\epsilon_{1,x}\epsilon_{2,x} \\ &+ 5\int B_x^2\epsilon_1\epsilon_2 + 10\int BB_{xx}\epsilon_1\epsilon_2 + \frac{15}{2}\int B^4\epsilon_1\epsilon_2 \\ &+ (\beta^2-\alpha^2)\left(2\int \epsilon_{1,x}\epsilon_{2,x} - 6\int B^2\epsilon_1\epsilon_2\right) + (\alpha^2+\beta^2)^2\int \epsilon_1\epsilon_2 \end{aligned} \tag{5.4.11}$$

We know from the coercivity of $Q^b_{\alpha,\beta}$ that

$$\begin{aligned} Q^b_{\alpha,\beta}[\epsilon_1] &\geq \mu^b_{\alpha,\beta}\|\epsilon_1\|^2_{H^2} - \frac{1}{\mu^b_{\alpha,\beta}}\Big(\int \epsilon_1 B\Big)^2 \\ &\geq \frac{\mu^b_{\alpha,\beta}}{2}\|\epsilon\|^2_{H^2} - \frac{2}{\mu^b_{\alpha,\beta}}\Big(\int \epsilon B\Big)^2. \end{aligned} \tag{5.4.12}$$

Also, if we denote by $\mathcal{L}^b_{\alpha,\beta}$ the self-adjoint operator associated to the quadratic form $Q^b_{\alpha,\beta}$,

$$Q^b_{\alpha,\beta}[\epsilon_2] = a^2 Q^b_{\alpha,\beta}[B_1] + b^2 Q^b_{\alpha,\beta}[B_2] + 2ab\int \mathcal{L}^b_{\alpha,\beta}[B_1]B_2 \leq C\nu^2\|\epsilon\|^2_{H^2}. \tag{5.4.13}$$

Actually, in this case, $Q^b_{\alpha,\beta}[\epsilon_2] = 0$, because ϵ_2 is in the kernel of $Q^b_{\alpha,\beta}$ (but, when we adapt this proof for solitons, we can only write the bound).

Now, we recall that $\int \epsilon_1\epsilon_2 = 0$, and study the other terms by using Cauchy-Schwarz:

$$\begin{aligned} &\Big|\int \epsilon_{1,xx}\epsilon_{2,xx} - 5\int B^2\epsilon_{1,x}\epsilon_{2,x} + 5\int B_x^2\epsilon_1\epsilon_2 + 10\int BB_{xx}\epsilon_1\epsilon_2 \\ &\qquad + \frac{15}{2}\int B^4\epsilon_1\epsilon_2 + \big(\beta^2-\alpha^2\big)\Big(2\int \epsilon_{1,x}\epsilon_{2,x} - 6\int B^2\epsilon_1\epsilon_2\Big)\Big| \\ &\qquad \leq C\big(|a|+|b|\big)\|\epsilon_1\|_{H^2} \leq C\nu\|\epsilon\|^2_{H^2(\mathbb{R})}. \end{aligned} \tag{5.4.14}$$

We observe that if we take ν small enough, the claim of the lemma is proved. □

We prove in the same way that we have a similar lemma for solitons:

Lemma 5.5. *There exists $\nu := \nu^s_c > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\Big|\int (\partial_c R_{c,\kappa})\epsilon\Big| + \Big|\int (\partial_x R_{c,\kappa})\epsilon\Big| \leq \nu\|\epsilon\|_{H^2}, \tag{5.4.15}$$

then

$$Q^s_c[\epsilon] \geq \frac{\mu^s_c}{4}\|\epsilon\|^2_{H^2}, \tag{5.4.16}$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

And even,

Lemma 5.6. *There exists $\nu := \nu^s_c > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\Big|\int R_{c,\kappa}\epsilon\Big| + \Big|\int (\partial_x R_{c,\kappa})\epsilon\Big| \leq \nu\|\epsilon\|_{H^2}, \tag{5.4.17}$$

then

$$Q^s_c[\epsilon] \geq \frac{\mu^s_c}{4}\|\epsilon\|^2_{H^2}, \tag{5.4.18}$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

5.5. **Computations for the third localized integral (to be used for the uniqueness).**

Lemma 5.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 function that do not depend on time and u a solution of* (mKdV). *Then,*

$$\begin{aligned} &\frac{d}{dt}\int \Big(\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\Big)f \\ &\qquad = \int \Big(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2u^2 + 15u_x^2uu_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 \\ &\qquad\quad - \frac{45}{4}u^4u_x^2\Big)f' + 5\int u^2u_xu_{xx}f'' + \frac{1}{2}\int u_{xx}^2f'''. \end{aligned} \tag{5.5.1}$$

Proof. We perform by doing integrations by parts when needed and basic calculations:

(5.5.2)

$$
\begin{aligned}
&\frac{d}{dt}\int \Big(\frac{1}{2}u_{xx}^2-\frac{5}{2}u^2u_x^2+\frac{1}{4}u^6\Big)f\\
&\quad=\int u_{txx}u_{xx}f-5\int u_tuu_x^2f-5\int u^2u_{tx}u_xf+\frac{3}{2}\int u_tu^5f\\
&\quad=-\int \big(u_{xx}+u^3\big)_{xxx}u_{xx}f+5\int \big(u_{xx}+u^3\big)_xuu_x^2f\\
&\quad+5\int u^2\big(u_{xx}+u^3\big)_{xx}u_xf-\frac{3}{2}\int \big(u_{xx}+u^3\big)_xu^5f\\
&\quad=\int \big(u_{xx}+u^3\big)_{xx}u_{xxx}f+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+5\int \big(u_{xx}+u^3\big)_xuu_x^2f\\
&\qquad+5\int u^2\big(u_{xx}+u^3\big)_{xx}u_xf-\frac{3}{2}\int \big(u_{xx}+u^3\big)_xu^5f\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u^3\big)_{xx}u_{xxx}f+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+5\int u_{xxx}uu_x^2f+5\int \big(u^3\big)_xuu_x^2f\\
&\qquad+5\int u^2u_{xxxx}u_xf+5\int u^2\big(u^3\big)_{xx}u_xf-\frac{3}{2}\int u_{xxx}u^5f-\frac{3}{2}\int \big(u^3\big)_xu^5f\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+\int \big(3u_{xx}u^2+6u_x^2u\big)u_{xxx}f+5\int u_{xxx}uu_x^2f\\
&\qquad+15\int u_x^3u^3f+5\int u^2u_{xxxx}u_xf+5\int u^2\big(3u_{xx}u^2+6u_x^2u\big)u_xf-\frac{3}{2}\int u_{xxx}u^5f-\frac{9}{2}\int u_xu^7f\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+3\int u^2u_{xx}u_{xxx}f+5\int u^2u_{xxxx}u_xf\\
&\qquad+11\int uu_x^2u_{xxx}f+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u_{xxx}u^5f+\frac{9}{16}\int u^8f'\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-2\int u^2u_{xx}u_{xxx}f\\
&\qquad+\int uu_x^2u_{xxx}f-5\int u^2u_xu_{xxx}f'+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u^5u_{xxx}f\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'-\int u^2\big(u_{xx}^2\big)_xf\\
&\qquad+\int uu_x^2u_{xxx}f+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u^5u_{xxx}f\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'\\
&\quad+\int u^2u_{xx}^2f'+2\int uu_xu_{xx}^2f-\int u_x^3u_{xx}f-2\int uu_xu_{xx}^2f\\
&\qquad-\int uu_x^2u_{xx}f'+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u^5u_{xxx}f\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'+\int u^2u_{xx}^2f'\\
&\qquad-\int uu_x^2u_{xx}f'-\frac{1}{4}\int \big(u_x^4\big)_xf+45\int u^3u_x^3f+\frac{45}{4}\int u^4\big(u_x^2\big)_xf+\frac{3}{2}\int u^5u_{xx}f'\\
&\quad=-\frac{1}{2}\int u_{xxx}^2f'+\int \big(u_{xx}+u^3\big)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'+\int u^2u_{xx}^2f'\\
&\qquad-\int uu_x^2u_{xx}f'+\frac{1}{4}\int u_x^4f'+\frac{3}{2}\int u^5u_{xx}f'+45\int u^3u_x^3f-45\int u^3u_x^3f-\frac{45}{4}\int u^4u_x^2f'
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{3}{2}\int u_{xxx}^2 f' - \int u_{xxx}u_{xx}f'' + 4\int u_{xx}^2 u^2 f' + 5\int u_x^2 u u_{xx} f' - 5\int u^2 u_x u_{xxx} f' \\
&\quad + \frac{9}{16}\int u^8 f' + \frac{1}{4}\int u_x^4 f' + \frac{3}{2}\int u^5 u_{xx} f' - \frac{45}{4}\int u^4 u_x^2 f' \\
&= -\frac{3}{2}\int u_{xxx}^2 f' + 9\int u_{xx}^2 u^2 f' + 15\int u_x^2 u u_{xx} f' + \frac{9}{16}\int u^8 f' + \frac{1}{4}\int u_x^4 f' \\
&\quad + \frac{3}{2}\int u^5 u_{xx} f' - \frac{45}{4}\int u^4 u_x^2 f' - \int u_{xxx}u_{xx}f'' + 5\int u^2 u_x u_{xx} f'' \\
&= \int \Big(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2 u^2 + 15 u_x^2 u u_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 - \frac{45}{4}u^4 u_x^2 \Big) f' \\
&\quad + 5\int u^2 u_x u_{xx} f'' + \frac{1}{2}\int u_{xx}^2 f'''.
\end{aligned}
$$

which is exactly the desired expression. □

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