On the uniqueness of multi-breathers of the modified Korteweg-de Vries equation

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Abstract. We consider the modified Korteweg-de Vries equation (mKdV) and prove that given any sum P of solitons and breathers of (mKdV) (with distinct velocities), there exists a solution p of (mKdV) such that $p(t) - P(t) \to 0$ when $t \to +\infty$, which we call multi-breather. In order to do this, we work at the H^2 level (even if usually solitons are considered at the H^1 level). We will show that this convergence takes place in any H^s space and that this convergence is exponentially fast in time.

We also show that the constructed multi-breather is unique in two cases: in the class of solutions which converge to the profile P faster than the inverse of a polynomial of a large enough degree in time (we will call this a super polynomial convergence), or (without hypothesis on the convergence rate), when all the velocities are positive.

1. Introduction

1.1. **Setting of the problem.** We consider the modified Korteweg-de Vries equation on $\mathbb{R}$:

$$\text{(mKdV)} \qquad \begin{cases} u_t + (u_{xx} + u^3)_x = 0 & (t,x) \in \mathbb{R}^2 \\ u(0) = u_0 & u(t,x) \in \mathbb{R} \end{cases}$$

The (mKdV) equation appears as a model of some physical problems as plasma physics [41, 9], electrodynamics [40], fluid mechanics [23], ferromagnetic vortices [48], and more.

In [26], Kenig, Ponce and Vega established local well-posedness in H^s, for $s \geqslant \frac{1}{4}$, of the Cauchy problem for (mKdV), by fixed point argument in $L^p_x L^q_t$ type spaces. Moreover, if $s > \frac{1}{4}$, the Cauchy problem is globally well posed [12]. Recently, Harrop-Griffiths, Killip and Visan [22] proved local well-posedness in H^s for $s > -1/2$. However, in this paper, we will only use the global well-posedness in H^2.

(mKdV) is an integrable equation (like the original Korteweg-de Vries equation) and thus it has an infinity of conservation laws, see [39, 1]. We will use three of them (the first two of them are called *mass* and *energy*):

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)dx, \tag{1}$$

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)dx - \frac{1}{4}\int_{\mathbb{R}} u^4(t,x)dx, \quad \text{and} \tag{2}$$

$$F[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_{xx}^2(t,x)dx - \frac{5}{2}\int_{\mathbb{R}} u^2(t,x)u_x^2(t,x)dx + \frac{1}{4}\int_{\mathbb{R}} u^6(t,x)dx. \tag{3}$$

Observe that if u is a solution of (mKdV) then $-u$ and, for any $x_0 \in \mathbb{R}$, $(t,x) \mapsto u(t, x - x_0)$ are solutions of (mKdV) too.

(mKdV) is a dispersive nonlinear equation that is a special case of a more general class of equations: the general Korteweg-de Vries equation (gKdV), where the nonlinearity u^3 is replaced by $f(u)$ for some real valued function f. The particularity of (mKdV) in comparison to other (gKdV) equation is that it admits special non linear solutions, namely breather solutions.

The most simple nonlinear solutions of (mKdV) are solitons, i.e. a bump of a constant shape that translates with a constant velocity without deformation, that is, solutions of the form $u(t,x) =$

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$Q_c(x-ct)$, where c is the velocity and Q_c is the profile function that depends only on one variable. $Q_c \in H^1(\mathbb{R})$ should solve the elliptic equation:

$$Q_c'' = cQ_c - Q_c^3. \tag{4}$$

We can show that necessarily $c > 0$ and that, if $c > 0$, (4) has a unique solution in $H^1(\mathbb{R})$, up to translations and reflexion with respect to the x-axis. Actually, one has the explicit fomula

$$Q_c(x) := \left(\frac{2c}{\cosh^2\left(c^{1/2}x\right)}\right)^{\frac{1}{2}}. \tag{5}$$

Observe that we chose Q_c so that it is even and positive.

A *soliton* is a solution of (mKdV), parameterized by a velocity parameter $c > 0$, a sign parameter $\kappa \in \{-1, 1\}$ and a translation parameter $x_0 \in \mathbb{R}$ (it corresponds to the initial position of the soliton) that has the following expression:

$$R_{c,\kappa}(t, x; x_0) := \kappa Q_c\left(x - x_0 - ct\right). \tag{6}$$

When $\kappa = -1$, this object is sometimes called *antisoliton*. Notice that solitons are smooth and decaying. The generalized Korteweg-de Vries equation also admit soliton type solutions, and the focusing nonlinear Schrödinger equation (NLS) as well. Solitons have been extensively studied, in particular their stability. Cazenave, Lions and Weinstein in [46, 7, 8, 47] were interested in orbital stability of (gKdV) and (NLS) solitons in H^1. A soliton of (mKdV) is indeed orbitally stable, i.e. if a solution is initially close to a soliton in $H^1(\mathbb{R})$, then it stays close to the soliton up to a translation for all times in $H^1(\mathbb{R})$. General results about orbital stability of nonlinear dispersive solitons are presented by Grillakis, Shatah and Strauss in [21]. The result about orbital stability of a soliton can be improved in a result of asymptotic stability, as it was done in the works by Martel and Merle [30, 32, 31], see also [18].

A *breather* is a solution of (mKdV), parameterized by $\alpha, \beta > 0$, $x_1, x_2 \in \mathbb{R}$ that has the following expression:

$$B_{\alpha,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x\left[\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\right)\right], \tag{7}$$

where

$$y_1 := x + \delta t + x_1 \quad \text{and} \quad y_2 := x + \gamma t + x_2,$$
$$\text{with} \quad \delta := \alpha^2 - 3\beta^2 \quad \text{and} \quad \gamma := 3\alpha^2 - \beta^2.$$

It corresponds to a localized periodic in time function (with frequency α, and exponential localization with decay rate β) that propagates at a constant velocity $-\gamma$ in time. Like solitons, breathers are smooth and decaying in space. Unlike solitons, breathers' velocities can be positive, zero or negative. α, β are the shape parameters and x_1, x_2 are the translation parameters of a breather. Note that if we replace the parameter x_1 by $x_1 + \frac{\pi}{\alpha}$, we transform $B_{\alpha,\beta}(\cdot, \cdot; x_1, x_2)$ in $-B_{\alpha,\beta}(\cdot, \cdot; x_1, x_2)$ (therefore, we do not need to talk about "antibreathers").

Breathers were first introduced by Wadati in [44], and they were already used by Kenig, Ponce and Vega in [25] to prove that the flowmap associated to (mKdV) equation is *not* uniformly continuous in H^s for $s < \frac{1}{4}$: the point is that two breathers with close velocities can be very close at $t = 0$ and can separate as fast as we want in H^s with $s < \frac{1}{4}$, if α is taken large enough.

(mKdV) breathers and their properties, as well as breathers for other equations, are well studied by Alejo and Muñoz and co authors in [5, 3, 6, 4, 2].

Let us singularize a result of H^2 orbital stability for breathers established in [5], and improved to H^1 orbital stability in [2]. In this last paper, a partial result of asymptotic stability is also given, for breathers traveling to the right only, with positive velocity $-\gamma > 0$; asymptotic stability for breathers in full generality is still an open problem.

When $\alpha \to 0$, $B_{\alpha,\beta}$ tends to a solution of (mKdV) called *double-pole solution* [45], the methods employed in this article as well as the proof of orbital stability made by Alejo and Muñoz seem not to apply for this limit, which is expected to be unstable according to the numerical computations in [19].

An important result regarding the long time dynamics of (mKdV) is the soliton-breather resolution: it asserts that any generic solution can be approached by a sum of solitons and breathers when $t \to +\infty$. Together with their stability properties, the soliton-breather resolution shows why solitons and breathers are essential objects to study. This resolution was established for initial conditions in a weighted Sobolev space in [10] (see also Schuur [42]) by inverse scattering method; see also [42] for the soliton resolution for (KdV). Observe that (mKdV) breathers do not decouple into simple solitons for large time (it is a *fully bounded state* as it is called in [5]); therefore, it must appear in the resolution. The soliton-breather resolution is one of the motivations of the study of multi-breathers, which we define below.

There are works in the literature about a more complicated object obtained from several solitons: a *multi-soliton*. A multi-soliton is a solution $r(t)$ of (mKdV) such that there exists $0 < c_1 < c_2 < ... < c_N$, $\kappa_1, ..., \kappa_N \in \{-1, 1\}$ and $x_1, ..., x_N \in \mathbb{R}$, such that

$$\lim_{t \to +\infty} \left\| r(t) - \sum_{j=1}^{N} R_{c_j,\kappa_j}(t, \cdot; x_j) \right\|_{H^1(\mathbb{R})} = 0. \tag{8}$$

This definition is not specific to (mKdV) and makes sense for many other nonlinear dispersive PDEs as soon as they admit solitons. This object is introduced by Schuur [42] and Lamb [27], see also [42, 38], where explicit formulas are given: these were obtained by inverse scattering method thanks to the integrability of the equation. Multi-soliton were first constructed in a non integrable context by Merle [36] for the mass critical (NLS). Martel in [29] constructed multi-solitons for mass-subcritical and critical (gKdV) equations and proved that they are unique in $H^1(\mathbb{R})$, smooth and converge exponentially fast to their profile in any Sobolev space H^s. Similar studies were done for other nonlinear dispersive PDEs. Martel and Merle [33] have proved the existence of multi-solitons for (NLS) in H^1, Côte, Martel and Merle extended this construction to mass supercritical (gKdV) and (NLS) in [16]. Friederich and Côte in [14] proved smoothness, and uniqueness in a class of algebraic convergence. Côte and Muñoz constructed in [17] multi-solitons for the nonlinear Klein-Gordon equation. Ming, Rousset and Tzvetkov have constructed multi-solitons for the water-waves systems in [37]. Valet has proved in [43] the existence and uniqueness of multi-solitons in H^1 for the Zakharov-Kuznetsov equation, which generalizes (gKdV) to higher dimension.

1.2. **Main results.** We prove in this article that given any sum of solitons and breathers with distinct velocities, there exists a solution of (mKdV) whose difference with this sum tends to zero when time goes to infinity. This solution will be called a multi-breather. Let us make the definition more precise.

Let $J \in \mathbb{N}$ and $K, L \in \mathbb{N}$ such that $J = K + L$. We will consider a set of L solitons and K breathers:

- the breather parameters are $\alpha_k > 0$, $\beta_k > 0$, $x^0_{1,k} \in \mathbb{R}$ and $x^0_{2,k} \in \mathbb{R}$ for $1 \leqslant k \leqslant K$.
- the solitons parameters are $c_l > 0$, $\kappa_l \in \{-1, 1\}$ and $x^0_{0,l} \in \mathbb{R}$ for $1 \leqslant l \leqslant L$.

We define for $1 \leqslant k \leqslant K$, the kth breather:

$$B_k(t, x) := B_{\alpha_k,\beta_k}(t, x; x^0_{1,k}, x^0_{2,k}); \tag{9}$$

and for $1 \leqslant l \leqslant L$, the lth soliton:

$$R_l(t, x) := R_{c_l,\kappa_l}(t, x; x^0_{0,l}). \tag{10}$$

We now define the *velocity* of our objects. Recall that for $1 \leqslant k \leqslant K$, the velocity of B_k is

$$v^b_k := -\gamma_k = \beta_k^2 - 3\alpha_k^2, \tag{11}$$

and for $1 \leqslant l \leqslant L$, the velocity of R_l is

$$v^s_l := c_l. \tag{12}$$

The most important assumption we make is that all these velocities are distinct:

$$\forall k \neq k' \quad v^b_k \neq v^b_{k'}, \qquad \forall l \neq l' \quad v^s_l \neq v^s_{l'}, \qquad \forall k, l \quad v^b_k \neq v^s_l. \tag{13}$$

These implies for any two of these objects to be far from each other when time is large, and this assumption is essential in our analysis.

It will useful to order our breathers and solitons by increasing velocities. As these are distinct, we can define an increasing function

$$v : \{1, ..., J\} \longrightarrow \{v_k^b, 1 \leqslant k \leqslant K\} \cup \{v_l^s, 1 \leqslant l \leqslant L\}. \tag{14}$$

The set $\{v_1, ..., v_J\}$ is thus the (ordered) set of all possible velocities of our objects. We define P_j, for $1 \leqslant j \leqslant J$, as the object (either a soliton R_l or a breather B_k) that corresponds to the velocity v_j. Hence, $P_1, ..., P_J$ are the considered objects ordered by increasing velocity.

We will need both notations: the indexation by k and l, and the indexation by j, and we will keep these notations to avoid ambiguity.

We will denote by x_j the center of mass of P_j, that is

- if $P_j = B_k$ is a breather, we set $x_j(t) := -x_{2,k}^0 + v_j t$;
- if $P_j = R_l$ is a soliton, we set $x_j(t) := x_{0,l}^0 + v_j t$.

We denote:

$$R = \sum_{l=1}^{L} R_l, \quad B = \sum_{k=1}^{K} B_k, \quad P = R + B = \sum_{j=1}^{J} P_j. \tag{15}$$

We can now define a multi-breather: as solitons are objects which can be studied naturally in $H^1(\mathbb{R})$, it turns out that breathers are best studied in $H^2(\mathbb{R})$; therefore, it is in this latter space that we develop our analysis.

Definition 1. A *multi-breather* associated to the sum P given in (15) of solitons and breathers is a solution $p \in C([T^*, +\infty), H^2(\mathbb{R}))$, for a constant $T^* > 0$, of (mKdV) such that

$$\lim_{t \to +\infty} \left\| p(t) - P(t) \right\|_{H^2} = 0. \tag{16}$$

We will prove two results in this article. The first one is the existence and the regularity of a multi-breather, the second one is the uniqueness of a multi-breather in the case when all velocities are positive.

Theorem 2. *Given solitons and breathers (9), (10) whose velocities (11) and (12) satisfy (13), there exists a multi-breather p associated to P given in (15). Moreover, $p \in C^\infty(\mathbb{R} \times \mathbb{R}) \cap C^\infty(\mathbb{R}, H^s(\mathbb{R}))$ for any $s \geqslant 0$ and there exists $\theta > 0$ such that for any $s \geqslant 0$, there exists $A_s \geqslant 1$ and $T^* > 0$ such that,*

$$\forall t \geqslant T^*, \quad \|p(t) - P(t)\|_{H^s} \leqslant A_s e^{-\theta t}. \tag{17}$$

Remark 3. We will also show that θ does only depend on the shape parameters of our objects: α_k, β_k, c_l. Moreover, if there exists $D > 0$ such that for all $j \geqslant 2$, $x_j(0) \geqslant x_{j-1}(0) + D$, then A_s and T^* do not depend on $x_{1,k}^0, x_{2,k}^0, x_{0,l}^0$ but only on α_k, β_k, c_l and D. Finally, if $D > 0$ is large enough with respect to the problem data, then (17) is true for $T^* = 0$. See Section 3.2 for further details.

Theorem 4. *Given the same set of solitons and breathers as in Theorem 2 whose velocities satisfy (13) and $v_1 > 0$ (so that all velocities are positive), the multi-breather p associated to P by Theorem 2, in the sense of Definition 1, is unique.*

We will also prove a version of Theorem 4 that does not have any assumption on the sign of the considered velocities. However, the uniqueness is obtained in a narrower class.

Proposition 5. *Given the same set of solitons and breathers as in Theorem 2 whose velocities satisfy (13), there exists $N > 0$ large enough such that the multi-breather p associated to P by Theorem 2 is the unique solution $u \in C([T_0, +\infty), H^2(\mathbb{R}))$ of* (mKdV) *such that*

$$\|u(t) - P(t)\|_{H^2} = O\left(\frac{1}{t^N}\right), \qquad as\ t \to +\infty. \tag{18}$$

In [45], there exists a formula for a multi-breather, obtained by inverse scattering method, that in some sense already gives the existence of a multi-breather. However, it is seems unclear how to derive from this formula the convergence to a sum of solitons and breathers uniformly in space and how to show that this convergence is exponential (local convergence around the center of mass of each object is already rather involved).

In this paper, we give here a different approach to prove the existence of a multi-breather and we clearly show that we have convergence of the constructed multi-breather to the corresponding sum of solitons and breathers in H^s and that this convergence is exponentially fast in time, and that the constructed multi-breather is smooth. In any case, uniqueness of multi-breathers is new.

In this paper, we adapt the arguments given by Martel and Merle [33], by Martel [29] and by Côte and Friederich [14] to the context of breathers. To do so, one needs to understand the variational structure of breathers, in the same fashion as Weinstein did in [46] for (NLS) solitons. Such results were obtained by Alejo and Muñoz in [5]: a breather is a critical point of a Lyapunov functional at the H^2 level, whose Hessian is coercive up to several (but finitely many) orthogonal conditions, see Section 2 for details. As we see from [5], the H^2 regularity level is the most natural setting to study breathers, and the H^1 regularity level is natural for the study of solitons, as we see in [29, 33]. One important issue we face is therefore to understand soliton variationnal structure at H^2 level, and to adapt the Lyapunov functional in [5] to accommodate for a sum of breathers (and solitons). Notice that arguments based on monotonicity may be adapted only if we suppose that all the considered velocities are positive. Because [33, 14] are not based on monotonicity (these are results for (NLS) which is not well suited for monotonicity), we can adapt their arguments to obtain existence and uniqueness results for our case without any condition on the sign of velocities. The uniqueness result obtained in this setting is however weaker than the one that is obtained with monotonicity arguments.

1.3. **Outline of the proof.** The proof of Theorem 2 (the existence of multi-breathers) is split into two main parts: the construction of an H^2 multi-breather and the proof that this multi-breather is smooth.

1.3.1. *An H^2 multi-breather.* Let us start with the first part. We consider (T_n) an increasing sequence of $\mathbb{R}_+$ with $T_n \to +\infty$, and for $n \in \mathbb{N}$, let p_n the unique global H^2 solution of (mKdV) such that $p_n(T_n) = P(T_n)$ (recall that the Cauchy problem for (mKdV) is globally well-posed in H^2).

We will prove the following *uniform estimate*:

Proposition 6. *There exists $T^* > 0$, $A > 0$, $\theta > 0$ such that, for all $n \in \mathbb{N}$ such that $T_n \geqslant T^*$,*

$$\forall t \in [T^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leqslant Ae^{-\theta t}. \tag{19}$$

With this proposition in hand, we can construct an H^2 multi-breather which converges exponentially fast to its profile, which is the first part of Theorem 2, as stated below.

Proposition 7. *There exists $T^* \in \mathbb{R}$, $A > 0$, $\theta > 0$ and a solution $p \in C([T^*, +\infty), H^2(\mathbb{R}))$ of* (mKdV) *such that,*

$$\forall t \geqslant T^*, \quad \|p(t) - P(t)\|_{H^2} \leqslant Ae^{-\theta t}. \tag{20}$$

Proof of Proposition 7 assuming Proposition 6. The sequence $(p_n(T^*))_n$ is L^2-compact, in the following sense:

Lemma 8. $\forall \varepsilon > 0\, \exists R > 0$ *such that*

$$\forall n \in \mathbb{N} \quad \int_{|x|>R} p_n^2(T^*, x)dx < \varepsilon \tag{21}$$

An analogous lemma has already been proved on p. 1111 of [29], which is the proof of formula (14) (and can also be found in [33]). The same proof works here. We need to use Proposition 6 for T_n large enough and then make a time variation to obtain the result in T^*. We can first find R that works for $P^2(t_0)$ instead of $p_n^2(T^*)$ for a fixed $t_0 > T^*$ large enough. From Proposition 6, we see that if we take t_0 large enough, we obtain the desired lemma for $p_n^2(t_0)$ instead of $p_n^2(T^*)$. To finish, with the help of a cut-off function, we control time variations of $\int_{|x|>R} p_n^2(t)dx$, where R is taken larger if needed. That is why we obtain the result at $t = T^*$.

As a consequence of the Proposition 6 above, $(\|p_n(T^*)\|_{H^2})$ is a bounded sequence. Thus, there exists $p^* \in H^2(\mathbb{R})$ such that, up to a subsequence,

$$p_n(T^*) \rightharpoonup p^* \quad \text{in } H^2. \tag{22}$$

Thus, from Lemma 8, there holds the strong convergence

$$p_n(T^*) \to p^* \quad \text{in } L^2. \tag{23}$$

Therefore, we obtain by interpolation:

$$p_n(T^*) \to p^* \quad \text{in } H^1. \tag{24}$$

Now, let us consider the global H^1 (even H^2) solution p of (mKdV) such that $p(T^*) = p^*$. As shown in [29], the Cauchy problem for (mKdV) has a continuous dependence in H^1 on compact sets of time. Let $t \geqslant T^*$: by continuous dependence, we deduce that $p_n(t) \to p(t)$ in H^1. $(p_n(t) - P(t))$ is a bounded sequence in H^2, which admits a unique weak limit and so

$$p_n(t) - P(t) \rightharpoonup p(t) - P(t) \quad \text{in } H^2. \tag{25}$$

By weak convergence and from Proposition 6, we obtain

$$\|p(t) - P(t)\|_{H^2} \leqslant \liminf_{n \to +\infty} \|p_n(t) - P(t)\|_{H^2} \leqslant Ae^{-\theta t}. \tag{26}$$

As this is true for any $t \geqslant T^*$, this ends the proof of the Proposition 7. □

It remains to prove Proposition 6, for which we rest on a bootstrap argument. More precisely, we will reduce the proof to the following proposition:

Proposition 9. *There exists $T^* > 0$, $A > 0$, $\theta > 0$, such that for all $n \in \mathbb{N}$ such that $T_n \geqslant T^*$, for all $t^* \in [0, T_n]$, if*

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leqslant Ae^{-\theta t}, \tag{27}$$

then

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leqslant \frac{A}{2} e^{-\theta t}. \tag{28}$$

The proof of Proposotion 6 then follows from a simple continuity argument.

Proof of Proposition 6 assuming Proposition 9. We define t_n^* in the following way:

$$t_n^* := \inf\{t^* \in [T^*, T_n), \quad \forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leqslant Ae^{-\theta t}\}. \tag{29}$$

The map $t \mapsto \|p_n(t) - P(t)\|_{H^2}$ is a continuous function and $\|p_n(T_n) - P(T_n)\|_{H^2} = 0$. This means that there exists $T^* \leqslant t^* < T_n$ such that

$$\forall t \in [t^*, T_n], \quad \|p_n(t) - P(t)\|_{H^2} \leqslant Ae^{-\theta t}. \tag{30}$$

Therefore, we have

$$T^* \leqslant t_n^* < T_n. \tag{31}$$

We would like to prove that $t_n^* = T^*$. Let us argue by contradiction and assume that $t_n^* > T^*$. The Proposition 9 allows us to deduce that

$$\forall t \in [t_n^*, T_n], \ \ \|p_n(t) - P(t)\|_{H^2} \leqslant \frac{A}{2} e^{-\theta t}. \tag{32}$$

This means that

$$\|p_n(t_n^*) - P(t_n^*)\|_{H^2} \leqslant \frac{A}{2} e^{-\theta t_n^*}, \tag{33}$$

which means that t_n^* could be chosen smaller, by continuity. This is a contradiction. □

Hence, we are left to prove Proposition 9, which will be done in Section 2.

1.3.2. *The H^2 multi-breather is smooth.* We now turn to the second part of Theorem 2, which is strongly adapted from [29]. The heart of this part is to prove uniform estimates in H^s for $p_n - P$, for any $s \geqslant 0$:

Proposition 10. *There exists $T^* > 0$, $\theta > 0$, such that for any $s \geqslant 0$, there exists $A_s \geqslant 1$, such that for any $n \in \mathbb{N}$ such that $T_n \geqslant T^*$*

$$\forall t \in [T^*, T_n], \quad \|p_n(t) - P(t)\|_{H^s} \leqslant A_s e^{-\theta t}. \tag{34}$$

With this improved version of Proposition 6, one can prove by the same reasonning as in the proof of the Proposition 7, that for any $s \geqslant 0$, p actually belongs to $L^\infty([T^*, +\infty), H^s(\mathbb{R}))$ and that the convergence of $p(t) - P(t)$ occurs in H^s with an exponential decay rate. More precisely,

Theorem 11. *For all $s \geqslant 2$, $p \in C([T^*, +\infty), H^s(\mathbb{R}))$, and furthermore,*

$$\forall t \geqslant T^*, \quad \|p(t) - P(t)\|_{H^s} \leqslant A_s e^{-\theta t}. \tag{35}$$

It remains to prove Proposition 10, which will be done in Section 3.

1.3.3. *The uniqueness result.* We denote p the multi-breather constructed in the previous sections, the existence of which is established. Let u be a solution of (mKdV) such that

$$\|u - P\|_{H^2} \xrightarrow[t \to +\infty]{} 0. \tag{36}$$

Equivalently, there holds

$$\|u - p\|_{H^2} \xrightarrow[t \to +\infty]{} 0. \tag{37}$$

We denote

$$z := u - p. \tag{38}$$

The goal is to prove that $z = 0$. We prove it in two configurations: when all the velocities are positive (Theorem 4), and without any assumption on velocities (Proposition 5), but in this last case we need to assume a stronger convergence than given in (36).

The proof of Theorem 4 will be carried out in two steps.

We start with Proposition 5, which is adapted from [14]. For this, we do not study $u - P$ anymore, we deal only with $z = u - p$. z is the difference of two solutions of (mKdV), which is much more precise than $u - P$. Thus, we do not modulate parameters of the solitons, as it is needed in other parts of the proof in order to deal with the soliton part of the linear part of the Lyapunov functional, and we avoid some difficulty. In order to prove our inequalities, we need again to use coercivity of the same type of quadratic forms. In order to do this, we replace z by $\widetilde{z} = z + \sum_{j=1}^{J} c_j K_j$, where K_j, $j = 1, ..., J$ is a well chosen basis of the kernel of the quadratic form in order to have $\widetilde{z}$ orthogonal to any K_j. A important idea is to use slow variations of localized functional with adapted cut-off function of the form $\varphi\left(\frac{x - vt}{\delta t}\right)$, which provides an extra $O(1/t)$ decay when derivatives fall on the cut-off, and ultimately explain why algebraic decay comes into play.

In the context of Theorem 4, we actually prove that $u - P$ converges exponentially fast to 0: this is the purpose of Proposition 43, which uses some ideas of [29]. Due to Proposition 5, we deduce immediately from there that an exponential convergence is trivial, that is $z = 0$.

To prove Proposition 43, we use monotonicity properties combined with coercivity of an energy type functional very similar to that used for the existence result. This is why we also need to modulate, and the choice of the orthogonality condition is essential: it allows to bound linear terms in w that appear in the computations. An issue in the context of mixed breather/solitons context is that one cannot build a functional adapted to all the nonlinear objects at once, as it is done in [29]. Instead, we carry out an induction and we argue successively around each object, soliton or breather, separately.

1.3.4. *Organisation of the paper.* Sections 2 and 3 are devoted to the proof of the existence of a multi-breather: Proposition 9 is proved in Section 2, Proposition 10 is proved in Section 3. Section 4 gathers the proofs of the uniqueness results: Section 4.1 is devoted to the proof of Proposition 5, and Section 4.2 and 4.3 are devoted to the proof of Theorem 4.

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2. Construction of a multi-breather in $H^2(\mathbb{R})$

We set

$$\beta := \min\{\beta_k, 1 \leqslant k \leqslant K\} \cup \{\sqrt{c_l}, 1 \leqslant l \leqslant L\}, \qquad \tau := \min\{v_{j+1} - v_j, 1 \leqslant j \leqslant J-1\}. \tag{39}$$

Our goal in this section is to prove Proposition 9.

2.1. **Elementary results.** Let us first collect a few basic facts that will be used throughout the article. One may check an exponential decay result for any of our objects:

Proposition 12. *Let $j = 1, ..., J$, $n, m \in \mathbb{N}$. Then, there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \partial_t^m P_j(t,x)| \leqslant Ce^{-\beta|x - v_j t|}. \tag{40}$$

Corollary 13. *Let $r > 0$. For t, x such that $v_j t + r < x < v_{j+1} t - r$, we have*

$$|P(t,x)| \leqslant Ce^{-\beta r}. \tag{41}$$

The same is true for any space or time derivative of P.

We will also use the following cross-product result:

Proposition 14. *Let $i \neq j \in \{1, ..., J\}$ and $m, n \in \mathbb{N}$. There exists a constant C that depends only on P, such that for any $t \in \mathbb{R}$,*

$$\left| \int \partial_x^m P_i \partial_x^n P_j \right| \leqslant Ce^{-\beta\tau t/2}. \tag{42}$$

There is also an orthogonality result for breathers that will be useful:

Lemma 15. *Let $B := B_{\alpha,\beta}$ be a breather. We denote $B_1 := \partial_{x_1} B$ and $B_2 := \partial_{x_2} B$. Then,*

$$\int BB_1 = \int BB_2 = 0. \tag{43}$$

Proof. Note that $Span(B_1, B_2) = Span(B_x, B_t)$. Therefore, it is enough to prove that

$$\int BB_x = \int BB_t = 0. \tag{44}$$

Firstly,

$$\int BB_x = \frac{1}{2} \int \left(B^2\right)_x = 0. \tag{45}$$

Secondly,

$$\int BB_t = \frac{1}{2} \int \left(B^2\right)_t = \frac{1}{2} \frac{d}{dt} \int B^2 = 0, \tag{46}$$

by mass conservation and because a breather is a solution of (mKdV). □

2.2. **Almost-conservation of localized conservation laws.** From now on, we will fix $n \in \mathbb{N}$. This is why, for the simplicity of notations, we can write T for T_n, and p for p_n. The goal will be to find constants $T^*, A > 1, \theta$ that do not depend on n, nor on the translation parameters of the given objects, to be chosen later (T^* will depend on A and θ), such that Proposition 9 is verified. We will take a $t^* \in [T^*, T]$, and we will make the following bootstrap assumption for the remaining of the article:

$$\forall t \in [t^*, T],\ \|p(t) - P(t)\|_{H^2} \leqslant Ae^{-\theta t}, \tag{47}$$

where $p(T) = P(T)$.

Remark 16. We have the following property for solutions of (mKdV): there exists $C_0 > 0$ such that, for any solution w of (mKdV), w is global and

$$\forall t \in \mathbb{R} \quad \|w(t)\|_{H^2} \leqslant C_0 \|w(T)\|_{H^2}. \tag{48}$$

Therefore,

$$\forall t \in \mathbb{R} \quad \|p(t)\|_{H^2} \leqslant C_0 \|P(T)\|_{H^2} \leqslant C_0 \sum_{j=1}^{J} \|P_j(T)\|_{H^2} \leqslant C_0 C, \tag{49}$$

where C is a constant that depends only on problem data (because the H^s-norm of solitons or breathers can be easily bounded).

Let $\theta := \frac{\beta\tau}{32}$. Let $\min(1, \frac{\tau}{4}) > \delta > 0$ be a constant to be chosen later.

This part of the proof is adapted from [33].

Let $\psi(x)$ be a C^3 function such that

$$0 \leqslant \psi \leqslant 1 \quad \text{on } \mathbb{R}, \qquad \psi(x) = 0 \quad \text{for } x \leqslant -1, \qquad \psi(x) = 1 \quad \text{for } x \geqslant 1, \qquad \psi' \geqslant 0 \quad \text{on } \mathbb{R}, \tag{50}$$

and satisfying, for a constant $C > 0$,

$$(\psi'(x))^{4/3} \leqslant C\psi(x), \qquad (\psi'(x))^{4/3} \leqslant C(1-\psi(x)), \qquad |\psi''(x)|^{3/2} \leqslant C\psi'(x) \qquad \text{for all } x \in \mathbb{R}. \tag{51}$$

Note that it is enough to take ψ that is equal to $(1+x)^4$ on a neighbourhood of -1 and that is equal to $1-(-1+x)^4$ on a neighbourhood of 1.

These conditions on ψ will be needed for the proof of Proposition 30.

For all $j = 2, ..., J$, let

$$\sigma_j := \frac{1}{2}(v_{j-1} + v_j). \tag{52}$$

For any $j = 2, ..., J-1$, let

$$\varphi_j(t,x) := \psi\left(\frac{x-\sigma_j t}{\delta t}\right) - \psi\left(\frac{x-\sigma_{j+1} t}{\delta t}\right), \tag{53}$$

$$\varphi_1(t,x) := 1 - \psi\left(\frac{x-\sigma_2 t}{\delta t}\right), \qquad \varphi_J(t,x) := \psi\left(\frac{x-\sigma_J t}{\delta t}\right), \tag{54}$$

so that the function φ_j corresponds obviously to the object P_j. We will also use notations φ_l^s and φ_k^b, which represent the same functions, and where φ_l^s corresponds to the soliton R_l and φ_k^b corresponds to the breather B_k.

We will also denote, for $j = 2, ..., J-1$,

$$\varphi_{1,j}(t,x) := \psi'\left(\frac{x-\sigma_j t}{\delta t}\right) - \psi'\left(\frac{x-\sigma_{j+1} t}{\delta t}\right), \tag{55}$$

$$\varphi_{1,1}(t,x) := -\psi'\left(\frac{x-\sigma_2 t}{\delta t}\right), \qquad \varphi_{1,J}(t,x) := \psi'\left(\frac{x-\sigma_J t}{\delta t}\right). \tag{56}$$

Of course, notations $\varphi_{1,k}^b$, $\varphi_{1,l}^s$ or $\varphi_{2,j}$ will be used, with similar obvious definitions.

We have that, for $j = 1, ..., J$,

$$|\varphi_{1,j}| \leqslant C\varphi_j^{3/4}. \tag{57}$$

Remark 17. If $\delta \leqslant \frac{\tau}{4}$,

$$\begin{aligned}
\int_{-\infty}^{\sigma_j t + \delta t} e^{-2\beta|x-v_j t|} &= e^{-2\beta v_j t} \int_{-\infty}^{\sigma_j t + \delta t} e^{2\beta x} dx \\
&= \frac{1}{2\beta} e^{-2\beta v_j t} e^{\beta(v_j + v_{j-1})t} e^{2\beta\delta t} \\
&\leqslant C e^{-\beta\tau t} e^{2\beta\delta t} \\
&\leqslant C e^{-\beta\tau t/2},
\end{aligned} \tag{58}$$

$$\int_{\sigma_{j+1}t-\delta t}^{+\infty} e^{-2\beta|x-v_jt|} \leqslant Ce^{-\beta\tau t/2}, \tag{59}$$

for the same reason, and if $i \neq j$, e.g. $j > i$,

$$\begin{aligned}\int_{\sigma_j t-\delta t}^{\sigma_{j+1}t+\delta t} e^{-2\beta|x-v_it|} &= e^{2\beta v_i t}\int_{\sigma_j t-\delta t}^{\sigma_{j+1}t+\delta t} e^{-2\beta x}dx\\ &\leqslant \frac{1}{2\beta}e^{2\beta v_i t}e^{-\beta(v_j+v_{j-1})t}e^{2\beta\delta t}\\ &\leqslant Ce^{-\beta\tau t}e^{2\beta\delta t}\\ &\leqslant Ce^{-\beta\tau t/2}. \qquad (60)\end{aligned}$$

And finally, we set for all $j = 1, ..., J$:

(61)

$$M_j(t) := \int \frac{1}{2}p^2(t,x)\varphi_j(t,x)dx =: M_j[p](t), \qquad E_j(t) := \int \left(\frac{1}{2}p_x^2(t,x) - \frac{1}{4}p^4(t,x)\right)\varphi_j(t,x)dx =: E_j[p](t).$$

Notations $M_l^s, M_k^b, E_l^s, E_k^b$ will also be used.

These are local versions of the mass and the energy of the solution p considered (localized around each breather or soliton). We will prove the following result for the localized mass and energy:

Lemma 18. *There exists $C > 0$ and $T_1^* := T_1^*(A)$ such that, if $T^* \geqslant T_1^*$, then for all $j = 1, ..., J$, for all $t \in [t^*, T]$,*

$$|M_j(T) - M_j(t)| + |E_j(T) - E_j(t)| \leqslant \frac{C}{\delta^2 t}A^2e^{-2\theta t}. \tag{62}$$

Proof. We will use the results of the computations made on the bottom of page 1115 and on the bottom of page 1116 of [29] to claim the following facts:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int p^2 f &= \int\left(-\frac{3}{2}p_x^2 + \frac{3}{4}p^4\right)f' - \int p_x p f''\\ \frac{d}{dt}\int\left[\frac{1}{2}p_x^2 - \frac{1}{4}p^4\right]f &= \int\left[-\frac{1}{2}(p_{xx}+p^3)^2 - p_{xx}^2 + 3p_x^2p^2\right]f' - \int p_{xx}p_x f'' \qquad (63)\end{aligned}$$

where f is a C^2 function that does not depend on time.

$M_j(t)$ is a sum of quantities of the form $\frac{1}{2}\int p^2\psi(\frac{x-\sigma_j t}{\delta t})$.

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int p^2\psi\left(\frac{x-\sigma_j t}{\delta t}\right) &= \frac{1}{\delta t}\int\left(-\frac{3}{2}p_x^2 + \frac{3}{4}p^4\right)\psi'\left(\frac{x-\sigma_j t}{\delta t}\right) - \frac{1}{(\delta t)^2}\int p_x p\psi''\left(\frac{x-\sigma_j t}{\delta t}\right)\\ &\quad - \frac{1}{2}\int p^2\frac{x}{\delta t^2}\psi'\left(\frac{x-\sigma_j t}{\delta t}\right). \qquad (64)\end{aligned}$$

$\psi'(\frac{x-\sigma_j t}{\delta t})$ is zero outside of $\Omega_j(t) := (-\delta t + \sigma_j t, \delta t + \sigma_j t)$. Thus, for $x \in \Omega_j(t)$, $|\frac{x}{t}| \leqslant |\sigma_j| + |\delta| \leqslant |\sigma_j| + 1$, this means that $|\frac{x}{t}|$ is bounded by a constant (that depends only on the given parameters). We can deduce that

$$\left|\frac{d}{dt}\frac{1}{2}\int p^2\psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right| \leqslant \frac{C}{\delta^2 t}\left(\int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p^4 + \int_{\Omega_j(t)} p^2\right). \tag{65}$$

We bound $\int_{\Omega_j(t)} p^4$ by the following way:

$$\begin{aligned}\int_{\Omega_j(t)} p^4 &\leqslant \|p\|_{L^\infty}^2\int_{\Omega_j(t)} p^2\\ &\leqslant C\|p\|_{H^1}^2\int_{\Omega_j(t)} p^2 \qquad \text{by Sobolev embedding}\end{aligned}$$

$$\leqslant C \int_{\Omega_j(t)} p^2 \qquad \text{by Remark 16.} \tag{66}$$

Thus, we have for any $t \in [t^*, T]$,

$$\left| \frac{d}{dt} \frac{1}{2} \int p^2 \psi \left(\frac{x - \sigma_j t}{\delta t} \right) \right| \leqslant \frac{C}{\delta^2 t} \left(\int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p^2 \right). \tag{67}$$

$E_j(t)$ is a sum of quantities of the form $\int [\frac{1}{2} p_x^2 - \frac{1}{4} p^4] \psi(\frac{x-\sigma_j t}{\delta t})$.

$$\begin{aligned} \frac{d}{dt} \int \left[\frac{1}{2} p_x^2 - \frac{1}{4} p^4 \right] \psi \left(\frac{x - \sigma_j t}{\delta t} \right) &= \frac{1}{\delta t} \int \left[-\frac{1}{2} (p_{xx} + p^3)^2 - p_{xx}^2 + 3 p_x^2 p \right] \psi' \left(\frac{x - \sigma_j t}{\delta t} \right) \\ &\quad - \frac{1}{(\delta t)^2} \int p_{xx} p_x \psi'' \left(\frac{x - \sigma_j t}{\delta t} \right) - \int \left[\frac{1}{2} p_x^2 - \frac{1}{4} p^4 \right] \frac{x}{\delta t^2} \psi' \left(\frac{x - \sigma_j t}{\delta t} \right) \end{aligned} \tag{68}$$

We deduce from this, by using similar arguments as for the mass, that for any $t \in [t^*, T]$,

$$\left| \frac{d}{dt} \int \left[\frac{1}{2} p_x^2 - \frac{1}{4} p^4 \right] \psi \left(\frac{x - \sigma_j t}{\delta t} \right) \right| \leqslant \frac{C}{\delta^2 t} \left(\int_{\Omega_j(t)} p^2 + \int_{\Omega_j(t)} p_x^2 + \int_{\Omega_j(t)} p_{xx}^2 \right). \tag{69}$$

Now, we write $p(t) = P(t) + (p(t) - P(t))$, and we use the triangular inequality

$$\int_{\Omega_j(t)} (p^2 + p_x^2 + p_{xx}^2) \leqslant 2 \int_{\Omega_j(t)} (P^2 + P_x^2 + P_{xx}^2) + 2 \| p - P \|_{H^2}^2 \tag{70}$$

We have assumed that $\| p - P \|_{H^2}^2 \leqslant A^2 e^{-2\theta t}$, we need to study P on $\Omega_j(t)$. The following computations work also for the derivatives of P.

$$\begin{aligned} \int_{\Omega_j(t)} P^2 = \int_{\Omega_j(t)} \left(\sum_{m=1}^{J} P_m(t,x) \right)^2 &= \sum_{1 \leqslant m,l \leqslant J} \int_{\Omega_j(t)} P_m(t,x) P_l(t,x) \\ &\leqslant C \sum_{1 \leqslant m,l \leqslant J} \int_{\Omega_j(t)} e^{-\beta |x - v_m t|} e^{-\beta |x - v_l t|} dx, \end{aligned} \tag{71}$$

where we use the Proposition 12.

We assume that $m \geqslant j$ (we argue similarly if $m \leqslant j-1$). Then,

$$\begin{aligned} x \in \Omega_j(t) &\Leftrightarrow -\delta t + \sigma_j t \leqslant x \leqslant \delta t + \sigma_j t \\ &\Leftrightarrow -\delta t + (\sigma_j - v_m) t \leqslant x - v_m t \leqslant \delta t + (\sigma_j - v_m) t. \end{aligned} \tag{72}$$

We note that $\sigma_j - v_m \leqslant -\frac{1}{2}\tau < 0$, we can thus deduce from the condition on δ that $\sigma_j - v_m + \delta \leqslant -\frac{1}{4}\tau < 0$. We deduce that $x - v_m t$ is negative for $x \in \Omega_j(t)$.

Similarly, if $m \leqslant j-1$, $x - v_m t$ is positive for $x \in \Omega_j(t)$.

We will now make calculations for different cases.

If $m, l \leqslant j-1$,

$$\begin{aligned} \int_{\Omega_j(t)} e^{-\beta |x - v_m t|} e^{-\beta |x - v_l t|} dx &\leqslant \int_{\Omega_j(t)} e^{-\beta (x - v_m t)} e^{-\beta (x - v_l t)} dx \\ &= \frac{1}{2\beta} e^{\beta t (-v_j - v_{j-1} + v_m + v_l)} (e^{2\beta\delta t} - e^{-2\beta\delta t}) \\ &\leqslant C e^{\beta t (-v_j - v_{j-1} + v_m + v_l + 2\delta)} \\ &\leqslant C e^{-\beta\tau t/2}. \end{aligned} \tag{73}$$

Similarly, if $m, l \geqslant j$,

$$\int_{\Omega_j(t)} e^{-\beta |x - v_m t|} e^{-\beta |x - v_l t|} dx \leqslant C e^{-\beta\tau t/2}. \tag{74}$$

And, if $m \leqslant j-1, l \geqslant j$,

$$\begin{aligned}\int_{\Omega_j(t)} e^{-\beta|x-v_m t|}e^{-\beta|x-v_l t|}dx &\leqslant \int_{\Omega_j(t)} e^{-\beta(x-v_m t)}e^{\beta(x-v_l t)}dx\\ &\leqslant 2\delta t e^{\beta t(v_m-v_l)}\\ &\leqslant Ce^{-\frac{\beta\tau t}{2}}. \qquad (75)\end{aligned}$$

Thus,

$$\int_{\Omega_j(t)} P^2 \leqslant Ce^{-\frac{\beta\tau t}{2}}, \qquad (76)$$

and the same is valid for the derivatives of P.

Thus, for $t \in [t^*, T]$,

$$\begin{aligned}&\sum_{j=1}^{J}\left|\frac{d}{dt}\frac{1}{2}\int p^2\psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right| + \left|\frac{d}{dt}\int\left[\frac{1}{2}p_x^2-\frac{1}{4}p^4\right]\psi\left(\frac{x-\sigma_j t}{\delta t}\right)\right|\\ &\leqslant \frac{C}{\delta^2 t}A^2e^{-2\theta t}+\frac{C}{\delta^2 t}e^{-\frac{\beta\tau t}{2}}\\ &\leqslant \frac{C}{\delta^2 t}(A^2+e^{-2\theta t})e^{-2\theta t}\\ &\leqslant \frac{C}{\delta^2 t}A^2e^{-2\theta t}. \qquad (77)\end{aligned}$$

Thus, for $j = 1, ..., J$, $t \in [t^*, T]$,

$$\begin{aligned}&|M_j(T)-M_j(t)|+|E_j(T)-E_j(t)|\\ &\leqslant \int_t^T \frac{C}{\delta^2 s}A^2e^{-2\theta s}ds\\ &\leqslant \frac{C}{\delta^2 t}A^2\int_t^T e^{-2\theta s}ds\\ &= \frac{C}{\delta^2 t}A^2\frac{1}{2\theta}(e^{-2\theta t}-e^{-2\theta T})\\ &\leqslant \frac{C}{\delta^2 t}A^2e^{-2\theta t}. \qquad (78)\end{aligned}$$

□

2.3. **Modulation.**

Lemma 19. *There exists $C > 0$, $T_2^* = T_2^*(A)$ such that, if $T^* > T_2^*$, then there exist unique C^1 functions $x_{1,k} : [t^*, T] \to \mathbb{R}$, $x_{2,k} : [t^*, T] \to \mathbb{R}$ for $1 \leqslant k \leqslant K$ and $x_{0,l} : [t^*, T] \to \mathbb{R}$, $c_{0,l} : [t^*, T] \to \mathbb{R}$, such that if we set*

$$\varepsilon(t,x) = p(t,x) - \widetilde{B}(t,x) - \widetilde{R}(t,x) = p(t,x) - \widetilde{P}(t,x) \qquad (79)$$

where

$$\widetilde{B}(t,x) = \sum_{k=1}^{K}\widetilde{B_k}(t), \quad \widetilde{B_k}(t,x) = B_{\alpha_k,\beta_k}(t,x; x_{1,k}^0 + x_{1,k}(t), x_{2,k}^0 + x_{2,k}(t)), \qquad (80)$$

$$\widetilde{R}(t,x) := \sum_{l=1}^{L}\widetilde{R}_l(t), \quad \widetilde{R}_l(t,x) := R_{c_l+c_{0,l}(t),\kappa_l}(t,x; x_{0,l}^0 + x_{0,l}(t)) \quad \text{for } 1 \leqslant l \leqslant L, \qquad (81)$$

$$\widetilde{P}(t) := \widetilde{R}(t) + \widetilde{B}(t), \qquad (82)$$

and

$$\widetilde{P}(t) := \sum_{j=1}^{J}\widetilde{P}_j(t) \qquad (83)$$

where there is the usual correspondence between $\widetilde{P}_j$ *and* $\widetilde{B_k}$ *or* $\widetilde{R}_l$,

then, $\varepsilon(t)$ *satisfies, for all* $k = 1, ..., K$, *for all* $l = 1, ..., L$ *and for all* $t \in [t^*, T]$,

(84)
$$\int \widetilde{R}_l(t)\varepsilon(t)\sqrt{\varphi_l^s(t)} = \int \partial_x \widetilde{R}_l(t)\varepsilon(t)\sqrt{\varphi_l^s(t)} = \int \partial_{x_1}\widetilde{B_k}(t)\varepsilon(t)\sqrt{\varphi_k^b(t)} = \int \partial_{x_2}\widetilde{B_k}(t)\varepsilon(t)\sqrt{\varphi_k^b(t)} = 0.$$

Moreover, for all $t \in [t^*, T]$,

$$\|\varepsilon(t)\|_{H^2} + \sum_{k=1}^{K}(|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^{L}(|x_{0,l}(t)| + |x_{0,l}(t) + c_{0,l}(t)t| + |c_{0,l}(t)|) \leqslant CAe^{-\theta t}, \tag{85}$$

and

$$\sum_{k=1}^{K}(|x'_{1,k}(t)| + |x'_{2,k}(t)|) + \sum_{l=1}^{L}(|x'_{0,l}(t) + c'_{0,l}(t)t| + |c'_{0,l}(t)|) \leqslant C\|\varepsilon(t)\|_{L^2} + Ce^{-\theta t}. \tag{86}$$

Finally, $p(T) = P(T) = \widetilde{P}(T)$ *and* $\varepsilon(T) = x_{0,l}(T) = x_{1,k}(T) = x_{2,k}(T) = c_{0,l}(T) = 0$.

Proof: see for example [13] *for reference.* Let, for $t \in [t^*, T]$,

$$F_t : L^2(\mathbb{R}) \times \mathbb{R}^{2K} \times \mathbb{R}^{2L} \to \mathbb{R}^{2K+2L} \tag{87}$$

$$\begin{aligned}
&(w, x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l})\\
&\quad\longmapsto \left(\int \sqrt{\varphi_k^b(t,x)}\partial_{x_1}B_{\alpha_k,\beta_k}(t,x;x_{1,k}^0 + x_{1,k}, x_{2,k}^0 + x_{2,k})\cdot\right.\\
&\quad\cdot\left(w - \sum_{m=1}^{K} B_{\alpha_m,\beta_m}(t,x;x_{1,m}^0 + x_{1,m}, x_{2,m}^0 + x_{2,m}) - \sum_{p=1}^{L} R_{c_p+c_{0,p},\kappa_p}(t,x;x_{0,p}^0 + x_{0,p})\right),\\
&\quad\int \sqrt{\varphi_k^b(t,x)}\partial_{x_2}B_{\alpha_k,\beta_k}(t,x;x_{1,k}^0 + x_{1,k}, x_{2,k}^0 + x_{2,k})\cdot\\
&\quad\cdot\left(w - \sum_{m=1}^{K} B_{\alpha_m,\beta_m}(t,x;x_{1,m}^0 + x_{1,m}, x_{2,m}^0 + x_{2,m}) - \sum_{p=1}^{L} R_{c_p+c_{0,p},\kappa_p}(t,x;x_{0,p}^0 + x_{0,p})\right),\\
&\quad\int \sqrt{\varphi_l^s(t,x)}\partial_x R_{c_l+c_{0,l},\kappa_l}(t,x;x_{0,l}^0 + x_{0,l})\cdot\\
&\quad\cdot\left(w - \sum_{m=1}^{K} B_{\alpha_m,\beta_m}(t,x;x_{1,m}^0 + x_{1,m}, x_{2,m}^0 + x_{2,m}) - \sum_{p=1}^{L} R_{c_p+c_{0,p},\kappa_p}(t,x;x_{0,p}^0 + x_{0,p})\right),\\
&\quad\int \sqrt{\varphi_l^s(t,x)}R_{c_l+c_{0,l},\kappa_l}(t,x;x_{0,l}^0 + x_{0,l})\cdot\\
&\quad\cdot\left.\left(w - \sum_{m=1}^{K} B_{\alpha_m,\beta_m}(t,x;x_{1,m}^0 + x_{1,m}, x_{2,m}^0 + x_{2,m}) - \sum_{p=1}^{L} R_{c_p+c_{0,p},\kappa_p}(t,x;x_{0,p}^0 + x_{0,p})\right)\right)
\end{aligned} \tag{88}$$

We observe that F_t is a C^1 function and $F_t(P(t), 0, 0, 0, 0) = 0$. Now, let us consider the matrix which gives the differential of F_t (with respect to $x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}$) in $(P(t), 0, 0, 0, 0)$ (we consider diagonal and extra-diagonal terms for each bloc):

$$DF_t = \begin{pmatrix}
B_{k,k}^1 & B_{k,k}^3 & \times & \times & \times & \times & \times & \times\\
B_{k,k}^3 & B_{k,k}^2 & \times & \times & \times & \times & \times & \times\\
\times & \times & B_{k',k'}^1 & B_{k',k'}^3 & \times & \times & \times & \times\\
\times & \times & B_{k',k'}^3 & B_{k',k'}^2 & \times & \times & \times & \times\\
\times & \times & \times & \times & R_{l,l}^1 & R_{l,l}^4 & \times & \times\\
\times & \times & \times & \times & R_{l,l}^3 & R_{l,l}^2 & \times & \times\\
\times & \times & \times & \times & \times & \times & R_{l',l'}^1 & R_{l',l'}^4\\
\times & \times & \times & \times & \times & \times & R_{l',l'}^3 & R_{l',l'}^2
\end{pmatrix}, \tag{89}$$

(90)
$$B^1_{k,k} := -\int \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 \sqrt{\varphi^b_k}, \quad B^2_{k,k} := -\int \left(\partial_{x_2} B_{\alpha_k,\beta_k}\right)^2 \sqrt{\varphi^b_k}, \quad B^3_{k,k} := -\int \partial_{x_1} B_{\alpha_k,\beta_k} \partial_{x_2} B_{\alpha_k,\beta_k} \sqrt{\varphi^b_k},$$

(91)
$$R^1_{l,l} := -\int (\partial_x R_{c_l,\kappa_l})^2 \sqrt{\varphi^s_l}, \quad R^2_{l,l} := -\int R_{c_l,\kappa_l} \partial_c R_{c_l,\kappa_l} \sqrt{\varphi^s_l},$$

(92)
$$R^3_{l,l} := -\int R_{c_l,\kappa_l} \partial_x R_{c_l\kappa_l} \sqrt{\varphi^s_l}, \quad R^4_{l,l} := -\int \partial_x R_{c_l,\kappa_l} \partial_c R_{c_l\kappa_l} \sqrt{\varphi^s_l},$$

where crosses stand for exponentially decaying terms when $t \to +\infty$, and we consider variables in the following order: $x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, ..., x_{1,K}, x_{2,K}, x_{0,1}, c_{0,1}, ..., x_{0,L}, c_{0,L}$ and we order the coefficients of the function in the similar way. This is a matrix with dominant diagonal blocs.

Note that $B^1_{k,k}$ is exponentially close to $-\int \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2$, because if $P_j = B_k$ is a breather,

$$\begin{aligned}\int \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 \left(1 - \sqrt{\varphi^b_k}\right) &\leqslant \int_{-\infty}^{\sigma_j t+\delta t} \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 + \int_{\sigma_{j+1} t+\delta t}^{+\infty} \left(\partial_{x_1} B_{\alpha_k,\beta_k}\right)^2 \\ &\leqslant C \int_{-\infty}^{\sigma_j t+\delta t} \exp\left(-2\beta \left|x - v_j t\right|\right) + \int_{\sigma_{j+1} t+\delta t}^{+\infty} \exp\left(-2\beta \left|x - v_j t\right|\right) \\ &\leqslant C e^{-\frac{\beta\tau}{2} t}, \end{aligned} \tag{93}$$

and the same is true for the other dominant diagonal terms of the matrix (we can get rid of φs).

Therefore, the determinant of the matrix is exponentially close to:

$$\begin{aligned}\det(DF_t) = \prod_{k=1}^{K} &\left(\int (\partial_{x_1} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k}))^2 \int (\partial_{x_2} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k}))^2 - \left(\int \partial_{x_1} B_{\alpha_k,\beta_k} \partial_{x_2} B_{\alpha_k,\beta_k} \right)^2 \right) \\ \cdot \prod_{l=1}^{L} &\left(\int (\partial_x R_{c_l,\kappa_l}(t,x;x^0_{0,l}))^2 \int R_{c_l,\kappa_l} \partial_c R_{c_l,\kappa_l} - \int \partial_x R_{c_l,\kappa_l} \partial_c R_{c_l,\kappa_l} \int R_{c_l,\kappa_l} \partial_x R_{c_l,\kappa_l} \right). \end{aligned} \tag{94}$$

By Cauchy-Schwarz inequality and the fact that $\partial_{x_1} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k})$ and $\partial_{x_2} B_{\alpha_k,\beta_k}(t,x;x^0_{1,k},x^0_{2,k})$ are linearly independent as functions of the x variable, for all time t fixed, we see that the first product is positive. Since, each member of the product is periodic in time, then the first product is bounded below by a positive constant independent on time and translation parameters.

For the second product, we observe that $\int R_{c_l,\kappa_l} \partial_x R_{c_l,\kappa_l} = 0$. That means that to minorate it, it is enough to minorate $\int (\partial_x R_{c_l,\kappa_l}(t,x;x^0_{0,l}))^2 \int R_{c_l,\kappa_l} \partial_c R_{c_l,\kappa_l}$. We derive, by calculation with the formula of R_{c_l,κ_l}, that we have

$$\partial_c R_{c_l,\kappa_l} = \frac{1}{2c_l} \left(R_{c_l,\kappa_l} + \left(x - x^0_{0,l} - c_l t \right) \partial_x R_{c_l,\kappa_l} \right) - t \partial_x R_{c_l,\kappa_l}. \tag{95}$$

This is why, by integration by parts, if q denotes the soliton with $c = 1$, i.e. $q = Q_1$,

$$\begin{aligned}\int R_{c_l,\kappa_l} \partial_c R_{c_l,\kappa_l} &= \frac{1}{2c_l} \int R^2_{c_l,\kappa_l} + \frac{1}{2c_l} \int x R_{c_l,\kappa_l} \partial_x R_{c_l,\kappa_l} \\ &= \frac{1}{2c_l} \int R^2_{c_l,\kappa_l} + \frac{1}{4c_l} \int x \partial_x \left(R^2_{c_l,\kappa_l} \right) \\ &= \frac{1}{2c_l} \int R^2_{c_l,\kappa_l} - \frac{1}{4c_l} \int R^2_{c_l,\kappa_l} \\ &= \frac{1}{4c_l} \int R^2_{c_l,\kappa_l} = \frac{1}{4\sqrt{c_l}} \int q^2, \end{aligned} \tag{96}$$

by scaling. Again, by scaling,

$$\int \left(\partial_x R_{c_l,\kappa_l}\right)^2 = c_l^{3/2} \int \left(q'\right)^2. \tag{97}$$

Therefore,

$$\int (\partial_x R_{c_l,\kappa_l}(t,x;x_{0,l}^0))^2 \int R_{c_l,\kappa_l}\partial_c R_{c_l,\kappa_l} = \frac{1}{4}c_l \int q^2 \int (q_x)^2 \geqslant \frac{1}{4}\min\{c_p, 1\leqslant p\leqslant L\}\int q^2 \int q_x^2. \tag{98}$$

This means that the second product is bounded below by a positive constant independent on time and translation parameters.

This means that if T_2^* is large enough, the matrix considered is invertible.

Now, we may use the implicit function theorem. If w is close enough to $P(t)$, then there exists $(x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l})$ such that $F_t(w, x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) = 0$, where $(x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l})$ depend in a regular C^1 way on w. It is possible to show that the "close enough" in the previous sentence does not depend on t; for this, it is required to use a uniform implicit function theorem. This means that for T_2^* large enough (depending on A), $Ae^{-\theta t}$ is small enough for $t\in[t^*,T]$, thus for $t\in[t^*,T]$, $p(t)$ is close enough to $P(t)$ in order to apply the implicit function theorem. Therefore, we have for $t\in[t^*,T]$, the existence of $x_{1,k}(t)$, $x_{2,k}(t)$, $x_{0,l}(t)$ and $c_{0,l}(t)$. It is possible to show that these functions are C^1 in time. Basically, this comes from the fact that they are C^1 in $p(t)$ and that $p(t)$ has a similar regularity in time; (see [13] for more details).

Now, we will prove the inequalities (85) and (86). We can take the differential with respect to $p(t)$ of the implicit functions, for $t\in[t^*,T]$. For this, we differentiate with respect to $p(t)$ the equation

$$F_t(p(t), x_{1,k}(p(t)), x_{2,k}(p(t)), x_{0,l}(p(t)), c_{0,l}(p(t))) = 0. \tag{99}$$

We know that the matrix that gives the differential of F_t (with respect to $x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}$) in

$$(p(t), x_{1,k}(p(t)), x_{2,k}(p(t)), x_{0,l}(p(t), c_{0,l}(p(t))) \tag{100}$$

is invertible and its inverse is bounded with respect to the time. The differential of F_t with respect to the first variable is also bounded. Thus, by the mean-value theorem:

$$|x_{1,k}| \leqslant C\|p-P\| \leqslant CAe^{-\theta t}, \tag{101}$$

the same is true for the other implicit functions, and using the fact that $x_{0,l}(t)+c_{0,l}(t)t$ corresponds to the position of each soliton, it works also for this parameter.

By applying the mean-value theorem (inequality) for B_{α_k,β_k} with respect to $x_{1,k}$ and $x_{2,k}$, we deduce that

$$\|B_k(t)-\widetilde{B}_k(t)\|_{H^2} \leqslant C(|x_{1,k}(t)|+|x_{2,k}(t)|). \tag{102}$$

We recall that

$$R_l(t) = R_{c_l,\kappa_l}(t,\cdot;x_{0,l}^0), \qquad \widetilde{R}_l(t) = R_{c_l+c_{0,l}(t),\kappa_l}(t,\cdot;x_{0,l}^0+x_{0,l}(t)). \tag{103}$$

We need to be a bit more careful with the solitons, because unlike breathers their velocity is modified. Let us apply mean-value theorems step-by-step. Firstly, we compare solitons with different c parameter, but to do this we center them at the same point,

$$\|R_{c_l+c_{0,l}(t),\kappa_l}(t,\cdot;x_{0,l}^0+x_{0,l}(t)) - R_{c_l,\kappa_l}(t,\cdot;x_{0,l}^0+x_{0,l}(t)+c_{0,l}(t)t)\|_{H^2} \leqslant C|c_{0,l}(t)|. \tag{104}$$

Secondly, we compare solitons with the same c parameter, but with different translation parameters,

$$\|R_{c_l,\kappa_l}(t,\cdot;x_{0,l}^0+x_{0,l}(t)+c_{0,l}(t)t) - R_{c_l,\kappa_l}(t,\cdot;x_{0,l}^0)\|_{H^2} \leqslant C|x_{0,l}(t)+c_{0,l}(t)t|. \tag{105}$$

Thus, we see that,

$$\|R_l(t)-\widetilde{R}_l(t)\|_{H^2} \leqslant C\left(\left|c_{0,l}(t)\right|+\left|x_{0,l}(t)+c_{0,l}(t)t\right|\right). \tag{106}$$

Finally, by triangular inequality,

$$\begin{aligned}\|\varepsilon(t)\|_{H^2} &\leqslant \|v(t)\|_{H^2} + \|P(t)-\widetilde{P}(t)\|_{H^2}\\ &\leqslant \|v(t)\|_{H^2} + C\left(\sum_{k=1}^{K}(|x_{1,k}(t)|+|x_{2,k}(t)|) + \sum_{l=1}^{L}(|x_{0,l}(t)+c_{0,l}(t)t|+|c_{0,l}(t)|)\right)\\ &\leqslant C\|v(t)\|_{H^2} \leqslant CAe^{-\theta t}.\end{aligned} \tag{107}$$

This completes the proof of (85).

For (86), we will take time derivatives of the equations $\int \partial_{x_1}\widetilde{B_k}(t)\varepsilon(t)\sqrt{\varphi_k^b(t)}=0$, $\int \partial_{x_2}\widetilde{B_k}(t)\varepsilon(t)\sqrt{\varphi_k^b(t)}=0$, $\int \partial_x\widetilde{R_l}(t)\varepsilon(t)\sqrt{\varphi_l^s(t)}=0$ and $\int \widetilde{R_l}(t)\varepsilon(t)\sqrt{\varphi_l^s(t)}=0$. From now on, we write $\widetilde{B_{k1}}$ for $\partial_{x_1}\widetilde{B_k}$ and $\widetilde{B_{k2}}$ for $\partial_{x_2}\widetilde{B_k}$. Firstly, we write the PDE verified by ε (knowing that $p, B_1, ..., B_K, R_1, ..., R_L$ are solutions of (mKdV):

$$\begin{aligned}\partial_t\varepsilon = &-\varepsilon_{xxx}-\left[\varepsilon\left(\varepsilon^2+3\varepsilon\sum_{j=1}^{J}\widetilde{P_j}+3\sum_{i,j=1}^{J}\widetilde{P_i}\widetilde{P_j}\right)\right]_x-\sum_{h\neq i \text{ or } i\neq j}\left(\widetilde{P_h}\widetilde{P_i}\widetilde{P_j}\right)_x\\ &-\sum_{k=1}^{K}x'_{1,k}(t)\widetilde{B_{k1}}-\sum_{k=1}^{K}x'_{2,k}(t)\widetilde{B_{k2}}-\sum_{l=1}^{L}x'_{0,l}(t)\widetilde{R_l}_{x_0}-\sum_{l=1}^{L}c'_{0,l}(t)\widetilde{R_l}_c.\end{aligned}\tag{108}$$

Now, we will take the time derivative of the equation $\int \widetilde{B_{k1}}\varepsilon\sqrt{\varphi_k^b}=0$ (and perform an integration by parts):

$$\begin{aligned}&-\int(\widetilde{B_k}^3)_{1x}\varepsilon\sqrt{\varphi_k^b}+x'_{1,k}(t)\int\widetilde{B_{k11}}\varepsilon\sqrt{\varphi_k^b}+x'_{2,k}(t)\int\widetilde{B_{k12}}\varepsilon\sqrt{\varphi_k^b}\\
&+\frac{1}{2\delta t}\int\widetilde{B_{k1}}\varepsilon\left(\varepsilon^2+3\varepsilon\sum_{i=1}^{J}\widetilde{P_i}+3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}+\int\widetilde{B_{k1x}}\varepsilon\left(\varepsilon^2+3\varepsilon\sum_{i=1}^{J}\widetilde{P_i}+3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\sqrt{\varphi_k^b}\\
&+\frac{1}{2\delta t}\int\widetilde{B_{k1}}\varepsilon_{xx}\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}-\frac{1}{2\delta t}\int\widetilde{B_{k1x}}\varepsilon_x\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}+\frac{1}{2\delta t}\int\widetilde{B_{k1xx}}\varepsilon\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}\\
&-\int\widetilde{B_{k1}}\sum_{h\neq i \text{ or } g\neq h}\left(\widetilde{P_h}\widetilde{P_i}\widetilde{P_g}\right)_x\sqrt{\varphi_k^b}-\frac{1}{2\delta t^2}\int\widetilde{B_{k1}}\varepsilon x\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}\\
&=\sum_{m=1}^{K}x'_{1,m}(t)\int\widetilde{B_{k1}}\widetilde{B_{m1}}\sqrt{\varphi_k^b}\\
&+\sum_{m=1}^{K}x'_{2,m}(t)\int\widetilde{B_{k1}}\widetilde{B_{m2}}\sqrt{\varphi_k^b}-\sum_{p=1}^{L}\left(x'_{0,p}(t)+c'_{0,p}(t)t\right)\int\widetilde{B_{k1}}\widetilde{R_p}_x\sqrt{\varphi_k^b}\\
&+\sum_{p=1}^{L}\frac{c'_{0,p}(t)}{2\left(c_p+c_{0,p}(t)\right)}\int\widetilde{B_{k1}}\left(\widetilde{R_p}+\left(x-x_{0,p}^0-x_{0,p}(t)-\left(c_p+c_{0,p}(t)\right)t\right)\widetilde{R_p}_x\right)\sqrt{\varphi_k^b}\end{aligned}\tag{109}$$

Similarly, taking the time derivative of $\int \widetilde{B_{k2}}\varepsilon\sqrt{\varphi_k^b}=0$:

$$\begin{aligned}&-\int(\widetilde{B_k}^3)_{2x}\varepsilon\sqrt{\varphi_k^b}+x'_{1,k}(t)\int\widetilde{B_{k12}}\varepsilon\sqrt{\varphi_k^b}+x'_{2,k}(t)\int\widetilde{B_{k22}}\varepsilon\sqrt{\varphi_k^b}\\
&+\frac{1}{2\delta t}\int\widetilde{B_{k2}}\varepsilon\left(\varepsilon^2+3\varepsilon\sum_{i=1}^{J}\widetilde{P_i}+3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}+\int\widetilde{B_{k2x}}\varepsilon\left(\varepsilon^2+3\varepsilon\sum_{i=1}^{J}\widetilde{P_i}+3\sum_{h,i=1}^{J}\widetilde{P_h}\widetilde{P_i}\right)\sqrt{\varphi_k^b}\\
&+\frac{1}{2\delta t}\int\widetilde{B_{k2}}\varepsilon_{xx}\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}-\frac{1}{2\delta t}\int\widetilde{B_{k2x}}\varepsilon_x\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}+\frac{1}{2\delta t}\int\widetilde{B_{k2xx}}\varepsilon\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}\\
&-\int\widetilde{B_{k2}}\sum_{h\neq i \text{ or } g\neq h}\left(\widetilde{P_h}\widetilde{P_i}\widetilde{P_g}\right)_x\sqrt{\varphi_k^b}-\frac{1}{2\delta t^2}\int\widetilde{B_{k2}}\varepsilon x\frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{K} x'_{1,m}(t) \int \widetilde{B_{k}}_2 \widetilde{B_{m}}_1 \sqrt{\varphi_k^b} \\
&+ \sum_{m=1}^{K} x'_{2,m}(t) \int \widetilde{B_{k}}_2 \widetilde{B_{m}}_2 \sqrt{\varphi_k^b} - \sum_{p=1}^{L} \left(x'_{0,p}(t) + c'_{0,p}(t) t \right) \int \widetilde{B_{k}}_2 \widetilde{R_{p}}_x \sqrt{\varphi_k^b} \\
&+ \sum_{p=1}^{L} \frac{c'_{0,p}(t)}{2\left(c_p + c_{0,p}(t)\right)} \int \widetilde{B_{k}}_2 \left(\widetilde{R_p} + \left(x - x_{0,p}^0 - x_{0,p}(t) - \left(c_p + c_{0,p}(t)\right) t \right) \widetilde{R_{p}}_x \right) \sqrt{\varphi_k^b}
\end{aligned} \tag{110}
$$

Similarly, taking the time derivative of $\int \widetilde{R_l}_x(t)\varepsilon(t)\sqrt{\varphi_l^s} = 0$:

$$
\begin{aligned}
&- \int \left(\widetilde{R_l}^3 \right)_{xx} \varepsilon \sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2\left(c_l + c_{0,l}(t)\right)} \int \left(\widetilde{R_l}_x + \left(x - x_{0,l}^0 - x_{0,l}(t) - \left(c_l + c_{0,l}(t)\right) t \right) \widetilde{R_l}_{xx} \right) \varepsilon \sqrt{\varphi_l^s} \\
&- \left(x'_{0,l}(t) + c'_{0,l}(t) t \right) \int \widetilde{R_l}_{xx} \varepsilon \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R_l}_x \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{h,i=1}^{J} \widetilde{P_h} \widetilde{P_i} \right) \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \int \widetilde{R_l}_{xx} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{h,i=1}^{J} \widetilde{P_h} \widetilde{P_i} \right) \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R_l}_x \varepsilon_{xx} \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t} \int \widetilde{R_l}_{xx} \varepsilon_x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \frac{1}{2\delta t} \int \widetilde{R_l}_{xxx} \varepsilon \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \int \widetilde{R_l}_x \sum_{h \neq i \text{ or } g \neq h} \left(\widetilde{P_h} \widetilde{P_i} \widetilde{P_g} \right)_x \sqrt{\varphi_l^s} - \frac{1}{2\delta t^2} \int \widetilde{R_l}_x \varepsilon x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&= - \sum_{p=1}^{L} \left(x'_{0,p}(t) + c'_{0,p}(t) t \right) \int \widetilde{R_l}_x \widetilde{R_{p}}_x \sqrt{\varphi_l^s} \\
&+ \sum_{p=1}^{L} \frac{c'_{0,p}(t)}{2\left(c_p + c_{0,p}(t)\right)} \int \widetilde{R_l}_x \left(\widetilde{R_p} + \left(x - x_{0,p}^0 - x_{0,p}(t) - \left(c_p + c_{0,p}(t)\right) t \right) \widetilde{R_{p}}_x \right) \sqrt{\varphi_l^s} \\
&+ \sum_{m=1}^{K} x'_{1,m}(t) \int \widetilde{R_l}_x \widetilde{B_{m}}_1 \sqrt{\varphi_l^s} + \sum_{m=1}^{K} x'_{2,m}(t) \int \widetilde{R_l}_x \widetilde{B_{m}}_2 \sqrt{\varphi_l^s}
\end{aligned} \tag{111}
$$

Finally, taking the time derivative of $\int \widetilde{R_l}\varepsilon\sqrt{\varphi_l^s} = 0$:

$$
\begin{aligned}
&- \int \left(\widetilde{R_l}^3 \right)_{x} \varepsilon \sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2\left(c_l + c_{0,l}(t)\right)} \int \left(\widetilde{R_l} + \left(x - x_{0,l}^0 - x_{0,l}(t) - \left(c_l + c_{0,l}(t)\right) t \right) \widetilde{R_l}_{x} \right) \varepsilon \sqrt{\varphi_l^s} \\
&- \left(x'_{0,l}(t) + c'_{0,l}(t) t \right) \int \widetilde{R_l}_x \varepsilon \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R_l} \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{h,i=1}^{J} \widetilde{P_h} \widetilde{P_i} \right) \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \int \widetilde{R_l}_x \varepsilon \left(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P_i} + 3 \sum_{h,i=1}^{J} \widetilde{P_h} \widetilde{P_i} \right) \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R_l} \varepsilon_{xx} \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t} \int \widetilde{R_l}_x \varepsilon_x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&+ \frac{1}{2\delta t} \int \widetilde{R_l}_{xx} \varepsilon \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \int \widetilde{R_l} \sum_{h \neq i \text{ or } g \neq h} \left(\widetilde{P_h} \widetilde{P_i} \widetilde{P_g} \right)_x \sqrt{\varphi_l^s} - \frac{1}{2\delta t^2} \int \widetilde{R_l} \varepsilon x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} \\
&= - \sum_{p=1}^{L} \left(x'_{0,p}(t) + c'_{0,p}(t) t \right) \int \widetilde{R_l} \widetilde{R_{p}}_x \sqrt{\varphi_l^s} \\
&+ \sum_{p=1}^{L} \frac{c'_{0,p}(t)}{2\left(c_p + c_{0,p}(t)\right)} \int \widetilde{R_l} \left(\widetilde{R_p} + \left(x - x_{0,p}^0 - x_{0,p}(t) - \left(c_p + c_{0,p}(t)\right) t \right) \widetilde{R_{p}}_x \right) \sqrt{\varphi_l^s} \\
&+ \sum_{m=1}^{K} x'_{1,m}(t) \int \widetilde{R_l} \widetilde{B_{m}}_1 \sqrt{\varphi_l^s} + \sum_{m=1}^{K} x'_{2,m}(t) \int \widetilde{R_l} \widetilde{B_{m}}_2 \sqrt{\varphi_l^s}
\end{aligned} \tag{112}
$$

By the Proposition 21 below (that follows from the first part of the lemma we prove) and it's corollary, several terms of the equalities (109), (110), (111) and (112) are bounded by $Ce^{-\theta t}$; other terms are $O(\|\varepsilon\|_{L^2})$. We remind that $O(\|\varepsilon\|_{L^2}) \leqslant CAe^{-\theta t}$. From the basic properties of φ_j (see Section 2.2), $\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}$ is bounded. Because of the compact support of φ_j, $\frac{x}{t}\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}$ is bounded independently on x and t. Using these bounds, and after several linear combinations, we obtain the desired inequalities. □

Remark 20. As a consequence of Lemma 19, there exists a constant $C > 0$ such that,

$$\forall t \in [t^*, T] \quad \sum_{k=1}^{K}(|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^{L}(|x_{0,l}(t)| + |c_{0,l}(t)|) \leqslant CAe^{-\theta T^*}. \tag{113}$$

This means, that if we take T_2^* eventually larger (which we will assume in the following of the article), we may extend Proposition 12 to $\widetilde{P_j}$ in the following way, by integration of the bounds given by modulation (the constant C is a bit larger in a controled way, we write $\frac{\beta}{2}$ because the shape of the solitons is a bit modified in a controled way, and we write $e^{\frac{\beta\tau}{32}t}$ because the velocity of the modulated solitons is a bit modified in a controled way):

Proposition 21. *Let $j = 1, ..., J$, $n \in \mathbb{N}$. If $T^* > T_2^*$, then, there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \widetilde{P_j}(t,x)| \leqslant Ce^{-\frac{\beta}{2}|x-v_j t|}e^{\frac{\beta\tau}{32}t}. \tag{114}$$

We will also use that any $\|\partial_x^n \widetilde{P_j}\|_{H^2}$ is bounded by C.

Corollary 22. *Let $i \neq j \in \{1, ..., J\}$ and $m, n \in \mathbb{N}$. If $T^* > T_2^*$, then there exists a constant C that depends only on P, such that for any $t \in \mathbb{R}$,*

$$\left|\int \partial_x^m \widetilde{P_i}\partial_x^n \widetilde{P_j}\right| \leqslant Ce^{-\beta\tau t/8}. \tag{115}$$

2.4. **Study of coercivity.** In [5], the Lyapunov functional that was introduced to study the orbital stability of a breather was the following conserved-in-time functional:

$$F[p](t) + 2(\beta^2 - \alpha^2)E[p](t) + (\alpha^2 + \beta^2)^2 M[p](t) \tag{116}$$

The functional that we will consider here is adapted from the latter. For $t \in [t^*, T]$:

(117)

$$\mathcal{H}[p](t) := F[p](t) + \sum_{k=1}^{K}\left(2(\beta_k^2 - \alpha_k^2)E_k^b[p](t) + (\alpha_k^2 + \beta_k^2)^2 M_k^b[p](t)\right) + \sum_{l=1}^{L}\left(2c_l E_l^s[p](t) + c_l^2 M_l^s[p](t)\right).$$

For the simplicity of notations, for $j \in \{1, ..., J\}$, a_j will design α_k if P_j is the breather B_k or 0 if P_j is a soliton, and b_j will design β_k if P_j is the breather B_k or $c_l^{1/2}$ if P_j is the soliton R_l. With these notations, we may write:

$$\mathcal{H}[p](t) = F[p](t) + \sum_{j=1}^{J}\left(2(b_j^2 - a_j^2)E_j[p](t) + (a_j^2 + b_j^2)^2 M_j[p](t)\right). \tag{118}$$

We would like to study locally this functional around the sum considered of breathers and solitons. The aim of this section will be to prove two following propositions:

Proposition 23 (Expansion of H^2 conserved quantity)**.** *There exists $T_4^* > 0$ such that if $T^* \geqslant T_4^*$, for all $t \in [t^*, T]$, we have*

$$\begin{aligned}\mathcal{H}[p](t) = &\sum_{j=1}^{J}\left(F[\widetilde{P_j}](t) + 2(b_j^2 - a_j^2)E[\widetilde{P_j}](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P_j}](t)\right)\\ &+ H_2[\varepsilon](t) + O(\|\varepsilon(t)\|_{H^2}^3) + O(e^{-2\theta t}\|\varepsilon(t)\|_{H^2}) + O(e^{-2\theta t}),\end{aligned} \tag{119}$$

where

$$H_2[\varepsilon](t) := \frac{1}{2}\int \varepsilon_{xx}^2 - \frac{5}{2}\int \widetilde{P}^2\varepsilon_x^2 + \frac{5}{2}\int \widetilde{P}_x^2\varepsilon^2 + 5\int \widetilde{P}\widetilde{P}_{xx}\varepsilon^2 + \frac{15}{4}\int \widetilde{P}^4\varepsilon^2$$
$$+ \sum_{j=1}^{J}(b_j^2 - a_j^2)\Big(\int \varepsilon_x^2\varphi_j - 3\int \widetilde{P}^2\varepsilon^2\varphi_j\Big) + \sum_{j=1}^{J}(a_j^2+b_j^2)^2\frac{1}{2}\int \varepsilon^2\varphi_j. \tag{120}$$

Proposition 24 (Coercivity of H_2)**.** *There exists $\mu > 0$, $T_3^* = T_3^*(A)$ such that, if $T^* \geqslant T_3^*$, for all $t \in [t^*, T]$,*

$$H_2[\varepsilon](t) \geqslant \mu\|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu}\sum_{k=1}^{K}\left(\int \varepsilon\widetilde{B_k}\sqrt{\varphi_k^b}\right)^2. \tag{121}$$

The Propositions 23 and 24 will be used in the next concluding subsection to prove the Proposition 9.

Firstly, let us prove the Proposition 23.

Proof of Proposition 23. We would like to compare $\mathcal{H}[\widetilde{P}+\varepsilon](t)$ and $\mathcal{H}[\widetilde{P}](t)$ (recall that $p = \widetilde{P}+\varepsilon$) by studying the difference asymptotically when ε is small. Firstly, let us see how we could simplify the expression of $\mathcal{H}[\widetilde{P}](t)$.

Step 1:

Claim 25. If T^* is large enough, for all $t \in [t^*, T]$, we have

$$\mathcal{H}[\widetilde{P}](t) = \sum_{j=1}^{J}\Big(F[\widetilde{P_j}](t) + 2(b_j^2-a_j^2)E[\widetilde{P_j}](t) + (a_j^2+b_j^2)^2M[\widetilde{P_j}](t)\Big) + O(e^{-2\theta t}). \tag{122}$$

Proof. We prove that for $t \in [t^*, T]$, we have

$$\left|\mathcal{H}[\widetilde{P}] - \sum_{j=1}^{J}\Big(F[\widetilde{P_j}] + 2(b_j^2-a_j^2)E[\widetilde{P_j}] + (a_j^2+b_j^2)^2M[\widetilde{P_j}]\Big)\right| \leqslant Ce^{-2\theta t}. \tag{123}$$

Let us compare $F_j[\widetilde{P}]$ and $F[\widetilde{P_j}]$.

$$F_j[\widetilde{P}] = \int\left(\frac{1}{2}\widetilde{P}_{xx}^2 - \frac{5}{2}\widetilde{P}^2\widetilde{P}_x^2 + \frac{1}{4}\widetilde{P}^6\right)\varphi_j(t,x)dx, \tag{124}$$

$$F[\widetilde{P_j}] = \int\left(\frac{1}{2}\widetilde{P_j}_{xx}^2 - \frac{5}{2}\widetilde{P_j}^2\widetilde{P_j}_x^2 + \frac{1}{4}\widetilde{P_j}^6\right)dx. \tag{125}$$

We compare the corresponding terms of these equalities. Let us start by the first one:

$$\begin{aligned}
\left|\int\Big(\widetilde{P}_{xx}^2\varphi_j(t,x) - \widetilde{P_j}_{xx}^2\Big)\right| &\leqslant \int \widetilde{P_j}_{xx}^2\big|1-\varphi_j(t,x)\big| + \sum_{(r,s)\neq(j,j)}\int\Big|\widetilde{P}_{rxx}\widetilde{P}_{sxx}\Big|\varphi_j(t,x)\\
&\leqslant C\int e^{-\frac{\beta}{2}|x-v_jt|}e^{\frac{\beta\tau}{32}t}\big|1-\varphi_j(t,x)\big| + C\sum_{i\neq j}\int e^{-\frac{\beta}{2}|x-v_it|}e^{\frac{\beta\tau}{32}t}\varphi_j(t,x)\\
&\leqslant Ce^{\frac{\beta\tau}{32}t}\left[\int_{-\infty}^{\sigma_jt+\delta t}e^{-\frac{\beta}{2}|x-v_jt|}dx + \int_{\sigma_{j+1}t-\delta t}^{+\infty}e^{-\frac{\beta}{2}|x-v_jt|}dx + \sum_{i\neq j}\int_{\sigma_jt-\delta t}^{\sigma_{j+1}t+\delta t}e^{-\frac{\beta}{2}|x-v_it|}dx\right]\\
&\leqslant Ce^{-\beta\tau t/16},
\end{aligned} \tag{126}$$

by Proposition 21 and Remark 17.

For the other terms of the difference to be bounded, we reason in a similar way. This completes the proof of the claim. □

Step 2:

Therefore, when we manage to compare $\mathcal{H}[p](t)$ and $\mathcal{H}[\widetilde{P}](t)$, we are also able to compare $\mathcal{H}[p](t)$ and $\sum_{j=1}^{J}(F[\widetilde{P}_j](t)+2(b_j^2-a_j^2)E[\widetilde{P}_j](t)+(a_j^2+b_j^2)^2M[\widetilde{P}_j](t))$.

$$\begin{aligned}\mathcal{H}[\widetilde{P}+\varepsilon] &= \frac{1}{2}\int(\widetilde{P}+\varepsilon)_{xx}^2-\frac{5}{2}\int(\widetilde{P}+\varepsilon)^2(\widetilde{P}+\varepsilon)_x^2+\frac{1}{4}\int(\widetilde{P}+\varepsilon)^6\\ &\quad+\sum_{j=1}^{J}\left[(b_j^2-a_j^2)\left(\int(\widetilde{P}+\varepsilon)_x^2\varphi_j-\frac{1}{2}\int(\widetilde{P}+\varepsilon)^4\varphi_j\right)\right]+\sum_{j=1}^{J}\left[(a_j^2+b_j^2)^2\frac{1}{2}\int(\widetilde{P}+\varepsilon)^2\varphi_j\right]\\ &= \frac{1}{2}\int\widetilde{P}_{xx}^2-\frac{5}{2}\int\widetilde{P}^2\widetilde{P}_x^2+\frac{1}{4}\int\widetilde{P}^6+\int\widetilde{P}_{(4x)}\varepsilon+5\int\widetilde{P}\widetilde{P}_x^2\varepsilon+5\int\widetilde{P}^2\widetilde{P}_{xx}\varepsilon+\frac{3}{2}\int\widetilde{P}^5\varepsilon\\ &\quad+\frac{1}{2}\int\varepsilon_{xx}^2-\frac{5}{2}\int\widetilde{P}^2\varepsilon_x^2+\frac{5}{2}\int\widetilde{P}_x^2\varepsilon^2+5\int\widetilde{P}\widetilde{P}_{xx}\varepsilon^2+\frac{15}{4}\int\widetilde{P}^4\varepsilon^2+O(\|\varepsilon(t)\|_{H^2}^3)\\ &\quad+\sum_{j=1}^{J}(b_j^2-a_j^2)\left(\int\widetilde{P}_x^2\varphi_j-\frac{1}{2}\int\widetilde{P}^4\varphi_j-2\int\widetilde{P}_{xx}\varepsilon\varphi_j\right.\\ &\quad\left.-2\int\widetilde{P}_x\varepsilon\varphi_{j,x}-2\int\widetilde{P}^3\varepsilon\varphi_j+\int\varepsilon_x^2\varphi_j-3\int\widetilde{P}^2\varepsilon^2\varphi_j\right)\\ &\quad+\sum_{j=1}^{J}(a_j^2+b_j^2)^2\frac{1}{2}\left(\int\widetilde{P}^2\varphi_j+2\int\widetilde{P}\varepsilon\varphi_j+\int\varepsilon^2\varphi_j\right).\end{aligned}\tag{127}$$

We can observe that the sum (127) is composed of 0-order terms in ε, of 1^{st}-order terms in ε, of 2^{nd}-order terms in ε. 3^{rd} and larger-order terms in ε are contained in $O(\|\varepsilon(t)\|_{H^2}^3)$. The sum of the 0-order terms is actually $\mathcal{H}[\widetilde{P}]$. The sum of 2^{nd}-order terms in ε is $H_2[\varepsilon](t)$.

Let us study for closely the 1-order terms:

$$\begin{aligned}H_1 &= \int\widetilde{P}_{(4x)}\varepsilon+5\int\widetilde{P}\widetilde{P}_x^2\varepsilon+5\int\widetilde{P}^2\widetilde{P}_{xx}\varepsilon+\frac{3}{2}\int\widetilde{P}^5\varepsilon\\ &\quad+\sum_{j=1}^{J}(b_j^2-a_j^2)\left(2\int\widetilde{P}_x\varepsilon_x\varphi_j-2\int\widetilde{P}^3\varepsilon\varphi_j\right)+\sum_{j=1}^{J}(a_j^2+b_j^2)^2\int\widetilde{P}\varepsilon\varphi_j.\end{aligned}\tag{128}$$

From [5], we know that a breather $A=A_{\alpha,\beta}$ satisfies for any fixed $t\in\mathbb{R}$, the following nonlinear equation:

$$A_{(4x)}-2(\beta^2-\alpha^2)(A_{xx}+A^3)+(\alpha^2+\beta^2)^2A+5AA_x^2+5A^2A_{xx}+\frac{3}{2}A^5=0.\tag{129}$$

This equation is also satisfied for $A=\widetilde{B}_k$ with $\alpha=\alpha_k$ and $\beta=\beta_k$ for any $k=1,...,K$ (the key parameters of a breather are not changed by modulation).

For a soliton $Q=R_{c,\kappa}$, we know from $Q_{xx}=cQ-Q^3$ that Q satisfies for any fixed $t\in\mathbb{R}$, the following nonlinear equation (see Section 5.1 (Appendix)):

$$Q_{(4x)}-2c(Q_{xx}+Q^3)+c^2Q+5QQ_x^2+5Q^2Q_{xx}+\frac{3}{2}Q^5=0.\tag{130}$$

This equation is not exactly satisfied for $Q=\widetilde{R}_l$ for any $l=1,...,L$ (the key parameters of a soliton are changed by modulation). The exact equation satisfied by $Q=\widetilde{R}_l$ is:

$$Q_{(4x)}-2c_l\left(Q_{xx}+Q^3\right)+c_l^2Q+5QQ_x^2+5Q^2Q_{xx}+\frac{3}{2}Q^5=2c_{0,l}(t)\left(Q_{xx}+Q^3\right)-2c_lc_{0,l}(t)Q-c_{0,l}(t)^2Q.\tag{131}$$

We will compare H_1 and

$$H_1':=\int\widetilde{P}_{(4x)}\varepsilon+5\sum_{j=1}^{J}\int\widetilde{P}_j\widetilde{P}_{j_x}^2\varepsilon+5\sum_{j=1}^{J}\int\widetilde{P}_{j_x}^2\widetilde{P}_{j_{xx}}\varepsilon+\frac{3}{2}\sum_{j=1}^{J}\int\widetilde{P}_j^5\varepsilon$$

$$-2\sum_{j=1}^{J}(b_j^2-a_j^2)\left(\int \widetilde{P}_{j_{xx}}\varepsilon+\int \widetilde{P}_j^{\,3}\varepsilon\right)+\sum_{j=1}^{J}(a_j^2+b_j^2)^2\int \widetilde{P}_j\varepsilon. \tag{132}$$

Firstly, let us compare $\int \widetilde{P}\widetilde{P}_x^2\varepsilon$ and $\sum_{j=1}^{J}\int \widetilde{P}_j\widetilde{P}_{j_x}^{\,2}\varepsilon$.

$$\begin{aligned}\int \widetilde{P}\widetilde{P}_x^2\varepsilon &= \int\left(\sum_{j=1}^{J}\widetilde{P}_j\right)\left(\sum_{j=1}^{J}\widetilde{P}_{j_x}\right)^2\varepsilon\\ &= \sum_{j=1}^{J}\int \widetilde{P}_j\widetilde{P}_{j_x}^{\,2}\varepsilon+\sum_{h\neq i \text{ or } i\neq j}\int \widetilde{P_h}\widetilde{P}_{i_x}\widetilde{P}_{j_x}\varepsilon. \end{aligned}\tag{133}$$

Therefore, we need to find a bound on a term of the type $\int \widetilde{P_h}\widetilde{P}_{i_x}\widetilde{P}_{j_x}\varepsilon$ where $h\neq i$ or $i\neq j$. We can perform the following upper bounding (where without loss of generality, we suppose that $i\neq j$):

$$\begin{aligned}\left|\int \widetilde{P_h}\widetilde{P}_{i_x}\widetilde{P}_{j_x}\varepsilon\right| &\leqslant Ce^{\frac{\beta\tau}{16}t}\int e^{-\frac{\beta}{2}|x-v_it|}e^{-\frac{\beta}{2}|x-v_jt|}|\varepsilon|\\ &\leqslant C\|\varepsilon\|_{L^\infty}e^{\frac{\beta\tau}{16}t}\int e^{-\frac{\beta}{2}|x-v_it|}e^{-\frac{\beta}{2}|x-v_jt|}\\ &\leqslant C\|\varepsilon\|_{H^2}e^{-\beta\tau t/8}, \end{aligned}\tag{134}$$

by Sobolev embedding and Proposition 14.

The bounding is quite similar for $\int \widetilde{P}^2\widetilde{P}_{xx}\varepsilon$ and $\int \widetilde{P}^5\varepsilon$. We observe that $-\int \widetilde{P}_{j_{xx}}\varepsilon=\int \widetilde{P}_{j_x}\varepsilon_x$. To compare $\int \widetilde{P}_x\varepsilon_x\varphi_j$ and $\int_{\mathbb{R}}\widetilde{P}_{j_x}\varepsilon_x$, and for similar terms, we can use computations that we have already performed at the beginning of this proof. Therefore,

$$\left|\int \widetilde{P}_x\varepsilon_x\varphi_j-\int_{\mathbb{R}}\widetilde{P}_{j_x}\varepsilon_x\right|\leqslant C\|\varepsilon\|_{H^2}e^{-\frac{\beta\tau t}{16}}. \tag{135}$$

That enables us to bound the difference between H_1 and H_1':

$$\left|H_1-H_1'\right|\leqslant C\|\varepsilon(t)\|_{H^2}e^{-\frac{\beta\tau t}{16}}. \tag{136}$$

Now, because our objects are not only breathers, H_1' is not equal to 0. Actually, we have

$$H_1'=2\sum_{l=1}^{L}c_{0,l}(t)\left(\int \widetilde{R}_{l_{xx}}\varepsilon+\int \widetilde{R}_l^{\,3}\varepsilon\right)-2\sum_{l=1}^{L}c_lc_{0,l}(t)\int \widetilde{R}_l\varepsilon-\sum_{l=1}^{L}c_{0,l}(t)^2\int \widetilde{R}_l\varepsilon. \tag{137}$$

Now, we introduce

$$H_1''=2\sum_{l=1}^{L}c_{0,l}(t)\left(\int \widetilde{R}_{l_{xx}}\varepsilon\sqrt{\varphi_l^s}+\int \widetilde{R}_l^{\,3}\varepsilon\sqrt{\varphi_l^s}\right)-2\sum_{l=1}^{L}c_lc_{0,l}(t)\int \widetilde{R}_l\varepsilon\sqrt{\varphi_l^s}-\sum_{l=1}^{L}c_{0,l}(t)^2\int \widetilde{R}_l\varepsilon\sqrt{\varphi_l^s}. \tag{138}$$

By reasonning the same way as for H_1 and H_1', we see that

$$\left|H_1'-H_1''\right|\leqslant C\|\varepsilon(t)\|_{H^2}e^{-2\theta t}. \tag{139}$$

Because of (84) and because of the elliptic equation satisfied by a soliton, we have that

$$H_1''=0. \tag{140}$$

Thus,

$$|H_1|=|H_1-H_1'|+|H_1'-H_1''|+|H_1''|\leqslant C\|\varepsilon(t)\|_{H^2}e^{-2\theta t}. \tag{141}$$

The proof of Proposition 23 is now completed. □

Now, we would like to study the quadratic terms in ε of the development of $\mathcal{H}[\widetilde{P}+\varepsilon]$. They are contained in $H_2[\varepsilon](t)$.

Let $A = B_{\alpha,\beta}$ be a breather (we note $A_1 := \partial_{x_1} A$ and $A_2 := \partial_{x_2} A$). We define a quadratic form associated to this breather:

$$\begin{aligned} \mathcal{Q}^b_{\alpha,\beta}[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int A^2\epsilon_x^2 + \frac{5}{2}\int A_x^2\epsilon^2 + 5\int AA_{xx}\epsilon^2 + \frac{15}{4}\int A^4\epsilon^2 \\ + (\beta^2-\alpha^2)\left(\int \epsilon_x^2 - 3\int A^2\epsilon^2\right) + (\alpha^2+\beta^2)^2\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{\alpha,\beta}[\epsilon]. \end{aligned} \tag{142}$$

From [5], we know that the kernel of this quadratic form is of dimension 2 and is generated by $\partial_{x_1} B_{\alpha,\beta}$ and $\partial_{x_2} B_{\alpha,\beta}$, and that this quadratic form has only one negative eigenvalue that is of multiplicity 1:

Proposition 26 (Proposition 4.11, [33])**.** *There exists $\mu^b_{\alpha,\beta} > 0$ that depends only on α, β (and does not depend on time), such that if $\epsilon \in H^2(\mathbb{R})$ is such that*

$$\int A_1\epsilon = \int A_2\epsilon = 0, \tag{143}$$

then

$$\mathcal{Q}^b_{\alpha,\beta}[\epsilon] \geqslant \mu^b_{\alpha,\beta}\|\epsilon\|^2_{H^2} - \frac{1}{\mu^b_{\alpha,\beta}}\left(\int \epsilon A\right)^2. \tag{144}$$

Remark 27. $\mu^b_{\alpha,\beta}$ is continuous in α,β. Note that translation parameters are implicit in $\mathcal{Q}^b_{\alpha,\beta}$.

Let $Q = R_{c,\kappa}$ be a soliton. We define a quadratic form associated to this soliton:

$$\begin{aligned} \mathcal{Q}^s_c[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ + c\left(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\right) + c^2\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{0,\sqrt{c}}[\epsilon]. \end{aligned} \tag{145}$$

By the same techniques, such as those presented in [5], adapted to the quadratic form of a soliton, we may establish that the kernel of this quadratic form is of dimension 2, and is generated by $\partial_x Q$ and $\partial_c Q$, and that this quadratic form does not have any negative eigenvalue (see Section 5.2 (Appendix)). After that, from Seciton 5.3 (Appendix), we deduce that the coercivity still works when we are orthogonal to Q and $\partial_x Q$. More precisely,

Proposition 28. *There exists $\mu^s_c > 0$ that depends only on c (and does not depend on time), such that if $\epsilon \in H^2(\mathbb{R})$ is such that*

$$\int Q\epsilon = \int Q_x\epsilon = 0, \tag{146}$$

then

$$\mathcal{Q}^s_c[\epsilon] \geqslant \mu^s_c\|\epsilon\|^2_{H^2}. \tag{147}$$

Remark 29. μ^s_c is continuous in c. Note that translation and sign parameters are implicit in the notation $\mathcal{Q}^s_c$.

We would like to find a similar minoration for H_2 (which is a generalization of $\mathcal{Q}$).

For $j = 1, ..., J$, let us define for $\epsilon \in H^2$,

$$\begin{aligned} \mathcal{Q}'_j[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2\varphi_j - \frac{5}{2}\int \widetilde{P}^2\epsilon_x^2\varphi_j + \frac{5}{2}\int \widetilde{P}_x^2\epsilon^2\varphi_j + 5\int \widetilde{P}\widetilde{P}_{xx}\epsilon^2\varphi_j + \frac{15}{4}\int \widetilde{P}^4\epsilon^2\varphi_j \\ + (b_j^2-a_j^2)\left(\int \epsilon_x^2\varphi_j - 3\int \widetilde{P}^2\epsilon^2\varphi_j\right) + (a_j^2+b_j^2)^2\frac{1}{2}\int \epsilon^2\varphi_j, \end{aligned} \tag{148}$$

such that

$$H_2[\varepsilon(t)] = \sum_{j=1}^{J} \mathcal{Q}'_j[\varepsilon(t)]. \tag{149}$$

Let us define

$$\begin{aligned} Q_j[\epsilon] := &\frac{1}{2}\int \epsilon_{xx}^2\varphi_j - \frac{5}{2}\int \widetilde{P_j}^2\epsilon_x^2\varphi_j + \frac{5}{2}\int \widetilde{P_j}_x^2\epsilon^2\varphi_j + 5\int \widetilde{P_j}\widetilde{P_j}_{xx}\epsilon^2\varphi_j + \frac{15}{4}\int \widetilde{P_j}^4\epsilon^2\varphi_j \\ &+ (b_j^2 - a_j^2)\left(\int \epsilon_x^2\varphi_j - 3\int \widetilde{P_j}^2\epsilon^2\varphi_j\right) + (a_j^2+b_j^2)^2\frac{1}{2}\int \epsilon^2\varphi_j. \end{aligned} \tag{150}$$

Notations Q_k^b, $\left(Q_k^b\right)'$, Q_l^s and $\left(Q_l^s\right)'$ will also be used.

We note that the support of φ_j increases with time, so that Q_j is near a $Q^b_{\alpha_k,\beta_k}$ or a $Q^s_{c_l}$ when time is large (note that $Q^b_{\alpha_k,\beta_k}$ is the canonical quadratic form associated to the breather $\widetilde{B_k}$, but the canonical quadratic form associated to the soliton $\widetilde{R_c}$ is $Q^s_{c_l+c_{0,l}(t)}$). However, firstly, let us study the difference between Q_j and Q_j'. Using the computations carried out at the beginning of this part (those done for the linear part) and Sobolev inequalities, we obtain:

$$|Q_j[\epsilon] - Q_j'[\epsilon]| \leqslant Ce^{-2\theta t}\|\epsilon\|^2_{H^2(\mathbb{R})}. \tag{151}$$

Lemma 30. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that, if $T^* \geqslant T_3^*$, for all $\epsilon \in H^2(\mathbb{R})$, for all $t \in [t^*, T]$,*

if

$$\int \widetilde{B_k}_1(t)\epsilon\sqrt{\varphi_k^b(t)} = \int \widetilde{B_k}_2(t)\epsilon\sqrt{\varphi_k^b(t)} = 0, \tag{152}$$

then

$$Q_k^b[\epsilon] \geqslant \mu\int(\epsilon^2+\epsilon_x^2+\epsilon_{xx}^2)\varphi_k^b(t) - \frac{1}{\mu}\left(\int \epsilon\widetilde{B_k}(t)\sqrt{\varphi_k^b(t)}\right)^2 - \rho\|\epsilon\|^2_{H^2}. \tag{153}$$

Proof of Lemma 30. The idea is to write $Q_k^b[\epsilon]$ as a $Q_{\alpha_k,\beta_k}[\epsilon\sqrt{\varphi_k^b}]$ plus several error terms. Let j such that $\widetilde{P_j} = \widetilde{B_k}$. We will note $\varphi_{1,j} := \psi'(\frac{x-\sigma_j t}{\delta t}) - \psi'(\frac{x-\sigma_{j+1}t}{\delta t})$ and $\varphi_{2,j} := \psi''(\frac{x-\sigma_j t}{\delta t}) - \psi''(\frac{x-\sigma_{j+1}t}{\delta t})$, which will be useful to write the derivatives of φ_j. We recall that they have the same support and bounding properties as φ_j. We have

$$\begin{aligned} \int(\epsilon\sqrt{\varphi_j})_{xx}^2 = &\int \epsilon_{xx}^2\varphi_j + \int\frac{\epsilon_x^2}{(\delta t)^2}\frac{\varphi_{1,j}^2}{\varphi_j} + \frac{1}{4}\int\frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{2,j}^2}{\varphi_j} + \frac{1}{16}\int\frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{1,j}^4}{\varphi_j^3} - \frac{1}{4}\int\frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{2,j}\varphi_{1,j}^2}{\varphi_j^2} \\ &+ 2\int\frac{\epsilon_{xx}\epsilon_x}{\delta t}\varphi_{1,j} + \int\frac{\epsilon_{xx}\epsilon}{(\delta t)^2}\varphi_{2,j} - \frac{1}{2}\int\frac{\epsilon_{xx}\epsilon}{(\delta t)^2}\frac{\varphi_{1,j}^2}{\varphi_j} + \int\frac{\epsilon_x\epsilon}{(\delta t)^3}\frac{\varphi_{1,j}\varphi_{2,j}}{\varphi_j} - \frac{1}{2}\int\frac{\epsilon_x\epsilon}{(\delta t)^3}\frac{\varphi_{1,j}^3}{\varphi_j^2}. \end{aligned} \tag{154}$$

We observe that, for T_3^* large enough, and by using the inequalities that define ψ, the error terms can be bounded by $\frac{C}{\delta t}\|\epsilon\|^2_{H^2} \leqslant \frac{\rho}{100}\|\epsilon\|^2_{H^2}$. The computation for the other terms is similar and the same bound can be used for the error terms.

Now, $\epsilon\sqrt{\varphi_k^b}$ satisfies the orthogonality conditions.

This means that we can apply Proposition 26, and we have

$$Q_{\alpha_k,\beta_k}[\epsilon\sqrt{\varphi_k^b}] \geqslant \mu_k^b\|\epsilon\sqrt{\varphi_k^b}\|^2_{H^2} - \frac{1}{\mu_k^b}\left(\int\epsilon\sqrt{\varphi_k^b}\widetilde{B_k}\right)^2. \tag{155}$$

To finish, $\|\epsilon\sqrt{\varphi_k^b}\|^2_{H^2}$ is $\int(\epsilon^2+\epsilon_x^2+\epsilon_{xx}^2)\varphi_k^b(t)$ plus several error terms as in (154). □

Lemma 31. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that, if $T^* \geqslant T_3^*$, for all $\epsilon \in H^2(\mathbb{R})$, for all $t \in [t^*, T]$,*

if

$$\int \widetilde{R_l}(t)\epsilon\sqrt{\varphi_l^s(t)} = \int \widetilde{R_l}_x(t)\epsilon\sqrt{\varphi_l^s(t)} = 0, \tag{156}$$

then

$$Q_l^s[\epsilon] \geqslant \mu \int \left(\epsilon^2 + \epsilon_x^2 + \epsilon_{xx}^2\right) \varphi_l^s(t) - \rho\|\epsilon\|_{H^2}^2. \tag{157}$$

Proof. As in the previous proof, we write $Q_l^s[\epsilon]$ as $Q_{c_l}^s\left[\epsilon\sqrt{\varphi_l^s}\right]$ (with $Q = \widetilde{R}_l$) plus several error terms, that are all bounded by $\rho\|\epsilon\|_{H^2}^2$ if T_3^* is chosen large enough. However, $Q_{c_l}^s\left[\epsilon\sqrt{\varphi_l^s}\right]$ is not appropriate in order to have coercivity, it is with $Q_{c_l+c_{0,l}(t)}^s\left[\epsilon\sqrt{\varphi_l^s}\right]$ that we have coercivity. That is why, we need to bound the difference between $Q_{c_l}^s\left[\epsilon\sqrt{\varphi_l^s}\right]$ and $Q_{c_l+c_{0,l}(t)}^s\left[\epsilon\sqrt{\varphi_l^s}\right]$. This difference is

$$c_{0,l}(t)\left(\int \epsilon_x^2\varphi_j - 3\int \widetilde{R}_l^{\,2}\epsilon^2\varphi_j\right) + c_l c_{0,l}(t)\int \epsilon^2\varphi_j + c_{0,l}(t)^2\frac{1}{2}\int \epsilon^2\varphi_j, \tag{158}$$

which can, because of the bound of $c_{0,l}(t)$, for T_3^* large enough (depending on A), be bounded by $\rho\|\epsilon\|_{H^2}^2$.

Now, $\epsilon\sqrt{\varphi_l^s}$ satisfies the right orthogonality conditions, and as in the previous proof we may apply coercivity. □

Proof of Proposition 24. We will now use the Lemma 30 and it's version for solitons for $\epsilon = \varepsilon(t)$. From this, we deduce that for $\rho > 0$ small enough,

$$\sum_{j=1}^{J} Q_j[\varepsilon(t)] \geqslant \mu\|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu}\sum_{k=1}^{K}\left(\int \varepsilon(t)\widetilde{B_k}\sqrt{\varphi_k^b}\right)^2, \tag{159}$$

for a suitable constant $\mu > 0$. This means that for T_3^* large enough, by taking, if needed, a smaller constant μ,

$$H_2[\varepsilon(t)] \geqslant \mu\|\varepsilon\|_{H^2}^2 - \frac{1}{\mu}\sum_{k=1}^{K}\left(\int \varepsilon\widetilde{B_k}\sqrt{\varphi_k^b}\right)^2. \tag{160}$$

The proof of Proposition 24 is now completed. □

2.5. **Proof of Proposition 9 (Bootstrap).** We recall that p_n from Proposition 9 is denoted by p and T_n is denoted by T in what follows, in order to simplify the notations. We do the proof that follows under the assumption (47), so that the Propositions proved above are true for $t \in [t^*, T]$.

The aim of this subsection is to complete the proof of Proposition 9 by using the Propositions 23 and 24.

We note that by Lemma 18, the conservation of $F[p](t)$ and the definition of $\mathcal{H}[p]$, we have for all $t \in [t^*, T]$,

$$|\mathcal{H}[p](T) - \mathcal{H}[p](t)| \leqslant \frac{CA^2}{\delta^2 t}e^{-2\theta t}. \tag{161}$$

Thus, for all $t \in [t^*, T]$,

$$\mathcal{H}[p](t) \leqslant \mathcal{H}[p](T) + \frac{CA^2}{\delta^2 t}e^{-2\theta t}. \tag{162}$$

From Proposition 23:

$$\begin{aligned}\left|\mathcal{H}[\widetilde{P}+\varepsilon](t) - \sum_{j=1}^{J}(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2)E[\widetilde{P}_j](t) + (a_j^2+b_j^2)^2 M[\widetilde{P}_j](t)) - H_2[\varepsilon](t)\right| \\ \leqslant Ce^{-2\theta t} + C\|\varepsilon\|_{H^2}e^{-2\theta t} + C\|\varepsilon\|_{H^2}^3 \\ \leqslant Ce^{-2\theta t} + \frac{\mu}{100}\|\varepsilon\|_{H^2}^2. \end{aligned} \tag{163}$$

To obtain the last line, we use $\|\varepsilon(t)\|_{H^2} \leqslant CAe^{-\theta t}$, and by taking $T^* \geqslant T_5^*$ for T_5^* large enough (depending on A), $\|\varepsilon\|_{H^2} \leqslant C$ and $C\|\varepsilon(t)\|_{H^2} \leqslant \frac{\mu}{100}$, and thus $C\|\varepsilon(t)\|_{H^2}^3 \leqslant \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2$.

We remark that if $P_j = B_k$ is a breather, then $F[\widetilde{P_j}]$, $E[\widetilde{P_j}]$ and $M[\widetilde{P_j}]$ are all constants in time. If $P_j = R_l$ is a soliton and we denote q the basic ground state (i.e. the ground state for $c = 1$), we have the following:

$$M[\widetilde{R_l}](t) = \left(c_l + c_{0,l}(t)\right)^{1/2} M[q], \tag{164}$$

$$E[\widetilde{R_l}](t) = \left(c_l + c_{0,l}(t)\right)^{3/2} E[q], \tag{165}$$

$$F[\widetilde{R_l}](t) = \left(c_l + c_{0,l}(t)\right)^{5/2} F[q]. \tag{166}$$

Using that, we can simplify $\mathcal{R}_l(t) := F[\widetilde{R_l}](t) + 2c_l E[\widetilde{R_l}](t) + c_l^2 M[\widetilde{R_l}](t)$ as follows:

$$\begin{aligned}\mathcal{R}_l(t) &= \left(c_l + c_{0,l}(t)\right)^{5/2} F[q] + 2c_l \left(c_l + c_{0,l}(t)\right)^{3/2} E[q] + c_l^2 \left(c_l + c_{0,l}(t)\right)^{1/2} M[q] \\ &= c_l^{5/2}\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{5/2} F[q] + 2c_l^{5/2}\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{3/2} E[q] + c_l^{5/2}\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{1/2} M[q].\end{aligned} \tag{167}$$

Note that from Lemma 19, $|c_{0,l}(t)|^3 \leqslant CA^3 e^{-\theta t} e^{-2\theta t}$. That is why, if we take T_5^* eventually larger, $|c_{0,l}(t)|^3 \leqslant Ce^{-2\theta t}$. For this reason, we will do Taylor expansions of order 2 of (167):

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{5/2} = 1 + \frac{5}{2}\frac{c_{0,l}(t)}{c_l} + \frac{15}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O\left(e^{-2\theta t}\right), \tag{168}$$

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{3/2} = 1 + \frac{3}{2}\frac{c_{0,l}(t)}{c_l} + \frac{3}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O\left(e^{-2\theta t}\right), \tag{169}$$

$$\left(1 + \frac{c_{0,l}(t)}{c_l}\right)^{1/2} = 1 + \frac{1}{2}\frac{c_{0,l}(t)}{c_l} - \frac{1}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O\left(e^{-2\theta t}\right). \tag{170}$$

That allows us to write

$$\begin{aligned}\mathcal{R}_l(t) &= c_l^{5/2}\left(F[q] + 2E[q] + M[q]\right) + c_l^{3/2} c_{0,l}(t)\left(\frac{5}{2}F[q] + 3E[q] + \frac{1}{2}M[q]\right) \\ &\quad + c_l^{1/2} c_{0,l}(t)^2\left(\frac{15}{8}F[q] + \frac{3}{4}E[q] - \frac{1}{8}M[q]\right) + O\left(e^{-2\theta t}\right).\end{aligned} \tag{171}$$

Now, $c_l^{5/2}\left(F[q] + 2E[q] + M[q]\right)$ is constant in time. For both other terms, we use that $M[q] = 2$, $E[q] = -\frac{2}{3}$ and $F[q] = \frac{2}{5}$, and we see that $\frac{5}{2}F[q] + 3E[q] + \frac{1}{2}M[q] = 0$ and $\frac{15}{8}F[q] + \frac{3}{4}E[q] - \frac{1}{8}M[q] = 0$. That allows us to write

$$\mathcal{R}_l(t) = \frac{16}{15}c_l^{5/2} + O\left(e^{-2\theta t}\right). \tag{172}$$

From this, we deduce that

$$\mathcal{R}_l(t) - \mathcal{R}_l(T) = O\left(e^{-2\theta t}\right). \tag{173}$$

By using that $\mathcal{H}[p](T) = \mathcal{H}[P](T) = \mathcal{H}[\widetilde{P}](T)$, the equations (163) and (162), Claim 25, and the fact that for $t \geqslant T_4^*$, $O(\|\varepsilon(t)\|_{H^2}^3) \leqslant \frac{\mu}{100}\|\varepsilon\|_{H^2}^2$, we have

$$\begin{aligned}H_2[\varepsilon](t) &\leqslant \mathcal{H}[p](t) - \sum_{j=1}^{J}(F[\widetilde{P_j}](t) + 2(b_j^2 - a_j^2)E[\widetilde{P_j}](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P_j}](t)) + Ce^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2 \\ &\leqslant \mathcal{H}[\widetilde{P}](T) - \sum_{j=1}^{J}(F[\widetilde{P_j}](t) + 2(b_j^2 - a_j^2)E[\widetilde{P_j}](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P_j}](t)) \\ &\quad + C\left(\frac{A^2}{\delta^2 t} + 1\right)e^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2\end{aligned}$$

$$\leqslant \mathcal{H}[\widetilde{P}](T) - \sum_{j=1}^{J}(F[\widetilde{P}_j](T) + 2(b_j^2 - a_j^2)E[\widetilde{P}_j](T) + (a_j^2+b_j^2)^2 M[\widetilde{P}_j](T))$$

$$+ C\left(\frac{A^2}{\delta^2 t}+1\right)e^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2 + \sum_{l=1}^{L}(\mathcal{R}_l(T) - \mathcal{R}_l(t))$$

$$\leqslant C\left(\frac{A^2}{\delta^2 t}+1\right)e^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2. \tag{174}$$

From Proposition 24, we deduce (by taking a smaller constant μ):

$$\mu\|\varepsilon\|_{H^2}^2 \leqslant C\left(\frac{A^2}{\delta^2 t}+1\right)e^{-2\theta t} + \frac{1}{\mu}\sum_{k=1}^{K}\left(\int \varepsilon\widetilde{B_k}\sqrt{\varphi_k^b}\right)^2. \tag{175}$$

We will now need to establish a result close to Lemma 18.
We set for all $j = 1, ..., J$:

$$m_j(t) := \int \frac{1}{2}p^2(t,x)\sqrt{\varphi_j(t,x)}dx := m_j[p](t). \tag{176}$$

Lemma 32. *There exists $C > 0$, $T_6^* = T_6^*(A)$ such that, if $T^* \geqslant T_6^*$, for all $j = 1, ..., J$, for all $t \in [t^*, T]$,*

$$|m_j(T) - m_j(t)| \leqslant \frac{C}{\delta^2 t}A^2 e^{-2\theta t}. \tag{177}$$

Proof.

$$\frac{d}{dt}\int \frac{1}{2}p^2(t,x)\sqrt{\varphi_j(t,x)}dx = \frac{1}{2\delta t}\int\left(-\frac{3}{2}p_x^2 + \frac{3}{4}p^4\right)\frac{\varphi_{1,j}}{\sqrt{\varphi_j}} - \frac{1}{2(\delta t)^2}\int p_x p\frac{\varphi_{2,j}}{\sqrt{\varphi_j}}$$

$$+ \frac{1}{4(\delta t)^2}\int p_x p\frac{\varphi_{1,j}^2}{\varphi_j^{3/2}} - \frac{1}{4}\int p^2\frac{x}{\delta t^2}\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}. \tag{178}$$

From the inequalities that define ψ, we find

$$\left|\frac{d}{dt}\int \frac{1}{2}p^2(t,x)\sqrt{\varphi_j(t,x)}dx\right| \leqslant \frac{C}{\delta^2 t}\int_{\Omega_j(t)\cup\Omega_{j+1}(t)}(p_x^2 + p^2 + p^4). \tag{179}$$

From now on, we can follow the proof of Lemma 18. □

Now, we observe the following:

$$\int(\widetilde{P}+\varepsilon)^2\sqrt{\varphi_k^b} = \int\widetilde{B_k}^2 + 2\int\widetilde{B_k}\varepsilon\sqrt{\varphi_k^b} + \int\varepsilon^2\sqrt{\varphi_k^b} + Err, \tag{180}$$

where Err designates the other terms of the sum, which we consider as error terms, and for which we will show now that they are bounded by $Ce^{-\theta t}$.

For $i \neq j$ and any h (if $P_j = B_k$ is a breather),

$$\left|\int \widetilde{P_i}\widetilde{P_h}\sqrt{\varphi_j}\right| \leqslant C\int_{-\delta t+\sigma_j t}^{\delta t+\sigma_{j+1}t} e^{-\frac{\beta}{2}|x-v_i t|}dx$$

$$\leqslant Ce^{-\theta t}, \tag{181}$$

and

$$\left|\int \widetilde{P_i}\varepsilon\sqrt{\varphi_j}\right| \leqslant \sqrt{\left(\int \widetilde{P_i}^2\varphi_j\right)\left(\int\varepsilon^2\right)}$$

$$\leqslant Ce^{-\frac{\theta}{2}t}\|\varepsilon\|_{H^2}$$

$$\leqslant CAe^{-\theta t}e^{-\frac{\theta}{2}t} \leqslant Ce^{-\theta t}, \tag{182}$$

where $T^* \geqslant T_7^*$ with T_7^* being large enough depending on A.

If we use the calculations we made in the proof of Claim 25,

$$\left| \int \widetilde{P}_j^{\,2} - \int \widetilde{P}_j^{\,2} \sqrt{\varphi_j} \right| \leqslant C e^{-\theta t}. \tag{183}$$

That proves the bound for the error terms. Now, we study the variations of (180).

We know that $\int \widetilde{P}_j^{\,2} = \int \widetilde{B_k}^2$ has no variations.

We can apply Lemma 32 for $\int (\widetilde{P} + \varepsilon)^2 \sqrt{\varphi_j}$.

By writing the difference of the equation (180) between t and T, and recall that $\varepsilon(T) = 0$, we deduce, for $T^* \geqslant \max(T_6^*, T_7^*)$,

$$\begin{aligned} \left| \int \widetilde{P}_j \varepsilon \sqrt{\varphi_j}(t) \right| &\leqslant C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-\theta t} + \|\varepsilon\|_{H^2}^2 \\ &\leqslant C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}. \end{aligned} \tag{184}$$

Thus,

$$\begin{aligned} \mu \|\varepsilon\|_{H^2}^2 &\leqslant C \left(\frac{A^2}{\delta^2 t} + 1 \right) e^{-2\theta t} + \frac{1}{\mu} \sum_{j=1}^{J} \left(\int \varepsilon \widetilde{P}_j \sqrt{\varphi_j} \right)^2 \\ &\leqslant C \left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t} + \frac{\mu}{100} \|\varepsilon(t)\|_{H^2}^2. \end{aligned} \tag{185}$$

Therefore,

$$\|\varepsilon(t)\|_{H^2}^2 \leqslant C \left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t}. \tag{186}$$

Before finishing the proof, we need to find a better bound for $|c_{0,l}(t)|$ than $CAe^{-2\theta t}$ given by the modulation. For this, we study the localized mass around R_l:

$$\begin{aligned} M_l^s(t) &= \frac{1}{2} \int p^2(t,x) \varphi_l^s(t,x) dx \\ &= \frac{1}{2} \int \left(\widetilde{P}(t) + \varepsilon(t) \right)^2 \varphi_l^s(t) \\ &= \frac{1}{2} \int \widetilde{P}(t)^2 \varphi_l^s(t) + \int \widetilde{P}(t) \varepsilon(t) \varphi_l^s(t) + \frac{1}{2} \int \varepsilon(t)^2 \varphi_l^s(t) \\ &= \frac{1}{2} \int \widetilde{R}_l(t)^2 + \int \widetilde{R}_l(t) \varepsilon(t) \varphi_l^s(t) + O\left(e^{-2\theta t}\right) + O\left(\|\varepsilon(t)\|_{H^2} e^{-2\theta t} \right) + O\left(\|\varepsilon(t)\|_{H^2}^2 \right) \\ &= \left(c_l + c_{0,l}(t) \right)^{1/2} M[q] + O\left(\left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t} \right), \end{aligned} \tag{187}$$

by modulation assumptions and (186).

That is why,

$$\begin{aligned} M_l^s(T) - M_l^s(t) &= \left[\left(c_l + c_{0,l}(T) \right)^{1/2} - \left(c_l + c_{0,l}(t) \right)^{1/2} \right] M[q] + O\left(\left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t} \right) \\ &= \left[c_l^{1/2} - \left(c_l + c_{0,l}(t) \right)^{1/2} \right] M[q] + O\left(\left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t} \right) \\ &= \left[-\frac{1}{2} c_l^{-1/2} c_{0,l}(t) + O\left(c_{0,l}(t)^2 \right) \right] M[q] + O\left(\left(\frac{A^4}{\delta^4 t} + 1 \right) e^{-2\theta t} \right). \end{aligned} \tag{188}$$

On the other hand, from Lemma 18,

$$M_l^s(T) - M_l^s(t) = O\left(\frac{A^2}{\delta^2 t} e^{-2\theta t} \right). \tag{189}$$

We note that

$$\left|-\frac{1}{2}c_l^{-1/2}c_{0,l}(t)+O\left(c_{0,l}(t)^2\right)\right| = \left|c_{0,l}(t)\right|\left|-\frac{1}{2}c_l^{-1/2}+O\left(c_{0,l}(t)\right)\right|$$
$$\geqslant \frac{1}{4}\left|c_{0,l}(t)\right|\frac{1}{|c_l|^{1/2}}, \tag{190}$$

if T_7^* is eventually larger, with respect to A. Therefore,

$$\left|c_{0,l}(t)\right| \leqslant C\left(\frac{A^4}{\delta^4 t}+1\right)e^{-2\theta t}. \tag{191}$$

By using (186), the mean-value theorem and Lemma 19, for $t \in [t^*, T]$,

$$\begin{aligned}
\|p(t)-P(t)\|_{H^2} &\leqslant \|\varepsilon(t)\|_{H^2} + \|\widetilde{P}(t)-P(t)\|_{H^2}\\
&\leqslant C\left(\sqrt{\frac{A^4}{\delta^4 t}+1}\right)e^{-\theta t} + C\left(\sum_{k=1}^{K}(|x_{1,k}(t)|+|x_{2,k}(t)|)+\sum_{l=1}^{L}(|x_{0,l}(t)+c_{0,l}(t)t|+|c_{0,l}(t)|)\right)\\
&\leqslant C\left(\sqrt{\frac{A^4}{\delta^4 t}+1}\right)e^{-\theta t} + C\sum_{k=1}^{K}\left(\left|\int_t^T x'_{1,k}(s)ds\right|+\left|\int_t^T x'_{2,k}(s)ds\right|\right)\\
&+ C\sum_{l=1}^{L}\left(\left|\int_t^T\left(x'_{0,l}(s)+c'_{0,l}(s)s\right)ds\right|+\left|\int_t^T c_{0,l}(s)ds\right|+\left|\int_t^T c'_{0,l}(s)ds\right|\right)\\
&\leqslant C\left(\frac{A^4}{\delta^4 t}+1\right)e^{-\theta t} + C\left(\int_t^T\|\varepsilon(s)\|_{H^2}ds+\int_t^T e^{-\theta s}ds\right)\\
&\leqslant C\left(\frac{A^4}{\delta^4 t}+1\right)e^{-\theta t}. \qquad (192)
\end{aligned}$$

We take $A = 4C$ (where C is a constant that can be used anywhere in this proof) and

$$T^* := \max(T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, T_6^*, T_7^*, T_8^*)$$

(depending on A), where $T_8^* := T_8^*(A)$ is such that for $t \geqslant T_8^*$, $\frac{A^4}{\delta^4 t} \leqslant 1$. And thus, for all $t \in [t^*, T]$,

$$C\left(\frac{A^4}{\delta^4 t}+1\right) \leqslant 2C = \frac{A}{2}, \tag{193}$$

which is exactly what we wanted to prove.

3. p is a smooth multi-breather

Our goal here is to prove Proposition 10.

3.1. **Estimates in higher order Sobolev norms.** Firstly, we notice that the proposition is already established for $s = 2$. We note also that if this proposition is proved for an $s \geqslant 2$ with a corresponding constant A_s, then this proposition is also valid for any $s' \leqslant s$ with the same constant A_s. This means that A_s can possibly increase with s and that this proposition is already established for $0 \leqslant s \leqslant 2$. From now on, we can denote (as before) p_n by p, T_n by T and $p_n - P$ by v, and make sure that the constant A_s that we will obtain in the proof does not depend on n (although it will depend on s). For the constant θ, we will take the usual value: $\theta := \frac{\beta\tau}{32}$. For the constant T^*, we will also take the constant T^* that works for Proposition 6.

We will prove the proposition by induction on s (it is sufficient to prove the proposition for any integer s). We assume that for any $s \geqslant 3$, the proposition is true for all $0 \leqslant s' \leqslant s-1$. And we prove the proposition for s.

Let us deduce from the (mKdV) equation the equation satisfied by v:

$$v_t = p_t - \sum_{j=1}^{J} P_{jt}$$

$$
\begin{aligned}
&= -\left(p_{xx} + p^3 - \sum_{j=1}^{J} P_{jxx} - \sum_{j=1}^{J} P_j^3 \right)_x \\
&= -\left(v_{xx} + (v+P)^3 - \sum_{j=1}^{J} P_j^3 \right)_x \\
&= -\left(v_{xx} + v^3 + 3v^2P + 3vP^2 + P^3 - \sum_{j=1}^{J} P_j^3 \right)_x . \qquad (194)
\end{aligned}
$$

Firstly, we compute $\frac{d}{dt}\int(\partial_x^s v)^2$ by integration by parts:

$$
\begin{aligned}
\frac{d}{dt}\int(\partial_x^s v)^2 &= 2\int(\partial_x^s v_t)(\partial_x^s v) \\
&= -2\int \partial_x^{s+1}\left(v_{xx} + v^3 + 3v^2P + 3vP^2 + P^3 - \sum_{j=1}^{J} P_j^3 \right)(\partial_x^s v) \\
&= 2(-1)^{s+1}\int \partial_x^{2s+1}\left(P^3 - \sum_{j=1}^{J} P_j^3 \right) v - 2\int \partial_x^{s+1}(v^3)(\partial_x^s v) \\
&\quad - 6\int \partial_x^{s+1}(v^2P)(\partial_x^s v) - 6\int \partial_x^{s+1}(vP^2)(\partial_x^s v), \qquad (195)
\end{aligned}
$$

because $\int(\partial_x^{s+3}v)(\partial_x^s v) = -\int(\partial_x^{s+2}v)(\partial_x^{s+1}v) = 0$.

We will now majorate each of the terms of the obtained sum.

$$
\begin{aligned}
\left| \int \partial_x^{2s+1}\left(P^3 - \sum_{j=1}^{J} P_j^3 \right) v \right| &\leqslant \|v\|_{L^\infty} \int \left| \partial_x^{2s+1}\left(P^3 - \sum_{j=1}^{J} P_j^3 \right) \right| \\
&\leqslant C\|v\|_{H^1} e^{-\beta\tau t/2} \quad \text{Sobolev embedding and cross-product result} \\
&\leqslant CAe^{-\theta t}e^{-\beta\tau t/2} \quad \text{by Proposition 6} \\
&\leqslant CAe^{-2\theta t} \leqslant CA_{s-1}^2 e^{-2\theta t}, \qquad (196)
\end{aligned}
$$

where C is a constant that depends only on s.

We observe that

$$
\begin{aligned}
\partial_x^{s+1}(v^3) &= 3(\partial_x^{s+1}v)v^2 + 6(s+1)(\partial_x^s v)v_x v + Z_1(v, v_x, ..., \partial_x^{s-1}v), \\
\partial_x^{s+1}(v^2P) &= 2(\partial_x^{s+1}v)vP + 2(s+1)(\partial_x^s v)(vP)_x + Z_2(v, v_x, ..., \partial_x^{s-1}v, P, P_x, ..., \partial_x^{s+1}P), \qquad (197)
\end{aligned}
$$

where Z_1 and Z_2 are homogeneous polynomials of degree 3 with constant coefficients.

Now, let us look for a bound for $\int \partial_x^{s+1}(v^3)(\partial_x^s v)$. Firstly, by integration by parts,

$$
\begin{aligned}
\int \partial_x^{s+1}(v^3)(\partial_x^s v) &= \frac{3}{2}\int \left((\partial_x^s v)^2\right)_x v^2 + 3(s+1)\int(\partial_x^s v)^2(v^2)_x + \int(\partial_x^s v)Z_1 \\
&= \frac{6(s+1)-3}{2}\int(\partial_x^s v)^2(v^2)_x + \int(\partial_x^s v)Z_1. \qquad (198)
\end{aligned}
$$

Then, we majorate each of the terms of the obtained sum:

$$
\begin{aligned}
\left| \int(\partial_x^s v)^2(v^2)_x \right| &\leqslant C\|v\|_{L^\infty}\|v_x\|_{L^\infty}\int(\partial_x^s v)^2 \\
&\leqslant C\|v\|_{H^2}^2\int(\partial_x^s v)^2 \\
&\leqslant C(\|p\|_{H^2} + \|P\|_{H^2})Ae^{-\theta t}\int(\partial_x^s v)^2
\end{aligned}
$$

$$\leqslant CC_0Ae^{-\theta t}\int(\partial_x^s v)^2 \leqslant CA_{s-1}e^{-\theta t}\int(\partial_x^s v)^2. \tag{199}$$

We have actually shown in the computation above that $\|v\|_{H^2}^2$ can be bounded above as $\|v\|_{H^2}$, and therefore the degree of $\|v\|_{H^2}$ can be lowered without harm in the upper bound. We will use this fact again for the rest of the proof. In fact, all what it means is that, for several terms, what we have is more than what we need.

By the Cauchy-Schwarz and Gagliardo-Nirenberg-Sobolev inequalities,

$$\begin{aligned}\left|\int(\partial_x^s v)Z_1\right| &\leqslant C\int|\partial_x^s v|\left(\sum_{s'=0}^{s-1}|\partial_x^{s'}v|^3\right)\\ &\leqslant C\left(\int|\partial_x^s v|^2\right)^{1/2}\sum_{s'=0}^{s-1}\left(\int|\partial_x^{s'}v|^6\right)^{1/2}\\ &\leqslant C\left(\int|\partial_x^s v|^2\right)^{1/2}\sum_{s'=0}^{s-1}\left(\int|\partial_x^{s'}v|^2\right)\left(\int|\partial_x^{s'+1}v|^2\right)^{1/2}\\ &\leqslant C\sum_{s'=0}^{s-1}\left(\int|\partial_x^{s'}v|^2\right)\left(\int|\partial_x^s v|^2+\int|\partial_x^{s'+1}v|^2\right)\\ &\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}e^{-\theta t}\int|\partial_x^s v|^2. \end{aligned}\tag{200}$$

Similarly, we bound $\int\partial_x^{s+1}(v^2P)(\partial_x^s v)$. By integration by parts,

$$\begin{aligned}\int\partial_x^{s+1}(v^2P)(\partial_x^s v) &= \int\left((\partial_x^s v)^2\right)_x vP+2(s+1)\int(\partial_x^s v)^2(vP)_x+\int(\partial_x^s v)Z_2\\ &= (2s+1)\int(\partial_x^s v)^2(vP)_x+\int(\partial_x^s v)Z_2. \end{aligned}\tag{201}$$

We majorate each of the terms of the obtained sum.

$$\begin{aligned}\left|\int(\partial_x^s v)^2(vP)_x\right| &\leqslant C(\|v\|_{L^\infty}+\|v_x\|_{L^\infty})\int(\partial_x^s v)^2\\ &\leqslant CAe^{-\theta t}\int(\partial_x^s v)^2. \end{aligned}\tag{202}$$

The upper bound of $\left|\int(\partial_x^s v)Z_2\right|$ is similar to (200) above:

$$\left|\int(\partial_x^s v)Z_2\right| \leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}e^{-\theta t}\int|\partial_x^s v|^2. \tag{203}$$

$\int\partial_x^{s+1}(vP^2)(\partial_x^s v)$ remains to be bounded.

By integration by parts,

$$\begin{aligned}\int\partial_x^{s+1}(vP^2)(\partial_x^s v) &= -\int\partial_x^{s+2}(vP^2)(\partial_x^{s-1}v)\\ &= -\int\left(\partial_x^{s+2}v\right)(\partial_x^{s-1}v)P^2-(s+2)\int\left(\partial_x^{s+1}v\right)(\partial_x^{s-1}v)(P^2)_x\\ &-\frac{(s+2)(s+1)}{2}\int(\partial_x^s v)\left(\partial_x^{s-1}v\right)\left(P^2\right)_{xx}+\int\left(\partial_x^{s-1}v\right)Z_3^0(v,v_x,...,\partial_x^{s-1}v)\\ &= \frac{1}{2}\int\left((\partial_x^s v)^2\right)_x P^2+(s+1)\int(\partial_x^s v)^2(P^2)_x\\ &-\frac{s(s+1)}{4}\int\left((\partial_x^{s-1}v)^2\right)_x(P^2)_{xx}+\int\left(\partial_x^{s-1}v\right)Z_3^0(v,v_x,...,\partial_x^{s-1}v)\\ &= \frac{2s+1}{2}\int(\partial_x^s v)^2(P^2)_x+\int\left(\partial_x^{s-1}v\right)Z_3(v,v_x,...,\partial_x^{s-1}v), \end{aligned}\tag{204}$$

where Z_3^0 and Z_3 are homogeneous polynomials of degree 1 whose coefficients are polynomials in P and its space derivatives. We have: $|Z_3| \leqslant C\left(\sum_{s'=0}^{s-1} |\partial_x^{s'} v|\right)$.

Therefore,

$$\left| \int \left(\partial_x^{s-1} v\right) Z_3 \right| \leqslant C A_{s-1}^2 e^{-2\theta t}. \tag{205}$$

Thus, by taking the sum of all those inequalities, we obtain:

$$\left| \frac{d}{dt} \int (\partial_x^s v)^2 + 3(2s+1) \int (\partial_x^s v)^2 (P^2)_x \right| \leqslant C A_{s-1}^2 e^{-2\theta t} + C A_{s-1} e^{-\theta t} \int |\partial_x^s v|^2. \tag{206}$$

Next, we perform similar computations for $\frac{d}{dt}\int (\partial_x^{s-1} v)^2 P^2$:

$$\begin{aligned} \frac{d}{dt} \int (\partial_x^{s-1} v)^2 P^2 &= 2 \int (\partial_x^{s-1} v_t)(\partial_x^{s-1} v) P^2 + 2 \int (\partial_x^{s-1} v)^2 P_t P \\ &= -2 \int \partial_x^s \left(v_{xx} + v^3 + 3v^2 P + 3vP^2 + P^3 - \sum_{j=1}^{J} P_j^3 \right) (\partial_x^{s-1} v) P^2 \\ &\quad - 2 \int (\partial_x^{s-1} v)^2 \left(P_{xx} + \sum_{j=1}^{J} P_j^3 \right)_x P. \end{aligned} \tag{207}$$

Let us study each of the obtained terms.

Firstly,

$$\begin{aligned} -2 \int (\partial_x^{s+2} v)(\partial_x^{s-1} v) P^2 &= 2 \int (\partial_x^{s+1} v)(\partial_x^s v) P^2 + 2 \int (\partial_x^{s+1} v)(\partial_x^{s-1} v)(P^2)_x \\ &= -3 \int (\partial_x^s v)^2 (P^2)_x - 2 \int (\partial_x^s v)(\partial_x^{s-1} v)(P^2)_{xx} \\ &= -3 \int (\partial_x^s v)^2 (P^2)_x + \int (\partial_x^{s-1} v)^2 (P^2)_{xxx}. \end{aligned} \tag{208}$$

Indeed,

$$\left| \int (\partial_x^{s-1} v)^2 (P^2)_{xxx} \right| \leqslant C A^2 e^{-2\theta t}. \tag{209}$$

Secondly,

$$\left| \int \partial_x^s \left(P^3 - \sum_{j=1}^{J} P_j^3 \right) (\partial_x^{s-1} v) P^2 \right| \leqslant C A_{s-1}^2 e^{-2\theta t} \tag{210}$$

can be obtained similarly to the first part of the proof (by starting by an integration by parts to have $\partial_x^{s-2} v$ at the place of $\partial_x^{s-1} v$).

Thirdly,

$$\begin{aligned} \int \partial_x^s \left(v^3\right) (\partial_x^{s-1} v) P^2 &= 3 \int (\partial_x^s v)(\partial_x^{s-1} v) v^2 P^2 + \int Z_4(v, v_x, ..., \partial_x^{s-1} v) P^2 \\ &= -\frac{3}{2} \int (\partial_x^{s-1} v)^2 (v^2 P^2)_x + \int Z_4 P^2, \end{aligned} \tag{211}$$

where Z_4 is a homogeneous polynomial of degree 4 with constant coefficients. Both terms are easily bounded by $C A_{s-1}^2 e^{-2\theta t}$.

Fourthly, for $\int \partial_x^s \left(v^2 P\right) (\partial_x^{s-1} v) P^2$ and $\int \partial_x^s \left(v P^2\right) (\partial_x^{s-1} v) P^2$, we reason similarly.

Fifthly,

$$\left| \int (\partial_x^{s-1} v)^2 \left(P_{xx} + \sum_{j=1}^{J} P_j^3 \right)_x P \right| \leqslant C A_{s-1}^2 e^{-2\theta t} \tag{212}$$

is clear.

Therefore,

$$\left|\frac{d}{dt}\int(\partial_x^{s-1}v)^2P^2+3\int(\partial_x^s v)^2(P^2)_x\right|\leqslant CA_{s-1}^2e^{-2\theta t}. \tag{213}$$

We set

$$F(t):=\int(\partial_x^s v)^2-(2s+1)\int(\partial_x^{s-1}v)^2P^2. \tag{214}$$

By putting the both parts of the proof together:

$$\left|\frac{d}{dt}F(t)\right|\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}e^{-\theta t}\int|\partial_x^s v|^2. \tag{215}$$

Because $\left|\int(\partial_x^{s-1}v)^2P^2\right|\leqslant CA^2e^{-2\theta t}$, we can write the following upper bound:

$$\int(\partial_x^s v)^2\leqslant|F(t)|+CA_{s-1}^2e^{-2\theta t}. \tag{216}$$

Therefore, we have, for a suitable constant $C>0$ that depends only on s,

$$\left|\frac{d}{dt}F(t)\right|\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}e^{-\theta t}|F(t)|. \tag{217}$$

For $t\in[T^*,T]$, by integration between t and T (we recall that $F(T)=0$),

$$\begin{aligned}|F(t)|=|F(T)-F(t)|&=\left|\int_t^T\frac{d}{dt}F(\sigma)d\sigma\right|\\&\leqslant\int_t^T\left|\frac{d}{dt}F(\sigma)\right|d\sigma\\&\leqslant CA_{s-1}^2\int_t^Te^{-2\theta\sigma}d\sigma+CA_{s-1}\int_t^Te^{-\theta\sigma}|F(\sigma)|d\sigma\\&\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}\int_t^Te^{-\theta\sigma}|F(\sigma)|d\sigma.\end{aligned} \tag{218}$$

By Gronwall lemma, for all $t\in[T^*,T]$,

$$\begin{aligned}|F(t)|&\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}\int_t^Te^{-\theta\sigma}CA_{s-1}^2e^{-2\theta\sigma}\exp\left(\int_t^\sigma CA_{s-1}e^{-\theta u}du\right)d\sigma\\&\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}^3\exp\left(\frac{CA_{s-1}}{\theta}e^{-\theta t}\right)\int_t^Te^{-3\theta\sigma}\exp\left(-\frac{CA_{s-1}}{\theta}e^{-\theta\sigma}\right)d\sigma\\&\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}^3\exp\left(\frac{CA_{s-1}}{\theta}\right)\int_t^Te^{-3\theta\sigma}d\sigma\\&\leqslant CA_{s-1}^2e^{-2\theta t}+CA_{s-1}^3\exp\left(\frac{CA_{s-1}}{\theta}\right)e^{-3\theta t}\\&\leqslant CA_{s-1}^3\exp\left(\frac{CA_{s-1}}{\theta}\right)e^{-2\theta t}.\end{aligned} \tag{219}$$

Therefore,

$$\int(\partial_x^s v)^2\leqslant A_se^{-2\theta t}, \tag{220}$$

where $A_s:=CA_{s-1}^3\exp\left(\frac{CA_{s-1}}{\theta}\right)$ with C is a constant large enough that depends only on s.

This conclude the proof of Proposition 10, and so of Theorem 2

3.2. **Uniformity of constants.** We conclude this section with an explanation regarding Remark 3.

In the proof above, the constants that we obtain A, T^*, θ do depend on $P_j(0)$ $(1 \leqslant j \leqslant J)$. Actually, we may characterize this dependence. In fact, they do not depend on the initial positions of our objects in the case when our objects are initial ordered in the right order and sufficiently far from each other.

Theorem 33. *Given parameters (9), (10), (11) and (12) which satisfy (13), there exists $D > 0$ large enough that depends only on α_k, β_k, c_l such that if*

$$\forall j \geqslant 2, \quad x_j(0) \geqslant x_{j-1}(0) + D, \tag{221}$$

then the following holds. We set $\theta := \frac{\beta\tau}{32}$, with β and τ given by (39) and $p(t)$ the multi-breather associated to P by Proposition 7. There exists $A_s \geqslant 1$ for any $s \geqslant 2$ that depend only on α_k, β_k, c_l and D such that

$$\forall t \geqslant 0, \quad \|p(t) - P(t)\|_{H^s} \leqslant A_s e^{-\theta t}. \tag{222}$$

First, we prove that for any $D > 0$, if (221) is satisfied then the constants A_s and T^* do depend only on α_k, β_k, c_l and D. At the end, we will prove that if $D > 0$ is large enough with respect to the given parameters, then we can take $T^* = 0$.

To establish the validity of this theorem, it is enough to read again the whole article and to make sure that on any step of the proof, there is no dependence on initial positions of our objects when our objects are initially far from each other for the constants C and then A and T^* (but, these constants may depend on D). This is the case, but we should change a bit the way we write our results.

For Proposition 3, we should write

$$|\partial_x^n \partial_t^m P_j(t,x)| \leqslant C e^{-\beta|x - v_j t - x_j(0)|}. \tag{223}$$

Therefore, in Proposition 4, we have nothing to change, but the constant C do depend on D. This will also be the case in the following propositions and lemmas of this proof.

We should replace $\sigma_j t$ for the definition of φ_j by $\sigma_j t + \frac{x_{j-1}(0)+x_j(0)}{2}$ to take account of initial positions. More precisely, we will have for any $j = 2, ..., J-1$,

$$\varphi_j(t,x) := \psi\left(\frac{x - \sigma_j t - \frac{x_{j-1}(0)+x_j(0)}{2}}{\delta t}\right) - \psi\left(\frac{x - \sigma_{j+1} t - \frac{x_j(0)+x_{j+1}(0)}{2}}{\delta t}\right), \tag{224}$$

and similarly for other definitions.

After having done the modulation with C and T^* depending on D, for Proposition 15, we should write:

$$|\partial_x^n \widetilde{P}_j(t,x)| \leqslant C e^{-\frac{\beta}{2}|x - v_j t - x_j(0)|} e^{\frac{\beta\tau}{32} t}. \tag{225}$$

Therefore, with these adaptations, the same proof works to prove that A_s and T^* do depend only on α_k, β_k, c_l and D.

Now, given α_k, β_k, c_l, we choose $D_0 > 0$ in an arbitrary maner. Therefore, we get $A_s(D_0)$ and $T^*(D_0)$ associated to D_0. Let $\Lambda := v_J - v_1$ the maximal difference between two velocities. We set $D := D_0 + \Lambda \cdot T^*(D_0)$. Therefore, if we suppose (221) in $t = 0$ for D, then we have (221) in $t = -T^*(D_0)$ for D_0. Therefore, by appliying the intermediate result for D_0, we obtain the desired conclusion where $A_s = A_s(D_0)$.

4. Uniqueness

p is the multi-breather constructed in the existence part. The goal here is to prove that if a solution u converges to p when $t \to +\infty$ (in some sense), then $u = p$ (under well chosen assumptions). We prove here two propositions. For both of them, we assume that the velocities of all our objects are distinct (this was also an assumption for the existence). The first proposition does not make more assumptions on velocities of our objects, but it is a partial uniqueness result as we restrict ourselves to the class of super polynomial convergence to the multi-breather. The second proposition assumes that the velocities of all our objects are positive (that is a new assumption and it is needed because this proof uses monotonicity arguments).

4.1. **A solution converging super polynomialy to a multi-breather is this multi-breather.** The goal of this subsection is to prove Proposition 5.

Remark 34. Note that in Proposition 5, we don't make any assumption on the sign of v_1 or v_2. This uniqueness proposition has the same degree of generality as Theorem 2.

Proof of Proposition 5. Let $p(t)$ be the multi-breather associated to P by Theorem 2. Recall that for any s,

$$\left\|p(t) - P(t)\right\|_{H^s} = O\left(e^{-\theta t}\right), \tag{226}$$

for a suitable $\theta > 0$.

Let $N > 2$ to be chosen later. We take $u(t)$ an H^2 solution of (mKdV) such that there exists $C_0 > 0$ such that for t large enough,

$$\|u(t) - P(t)\|_{H^2} \leq \frac{C_0}{t^N}. \tag{227}$$

From that, we may deduce that for t large enough (namely, $t \geq 2C_0$ along with the previous condition),

$$\|u(t) - P(t)\|_{H^2} \leq \frac{1}{2}\frac{1}{t^{N-1}}. \tag{228}$$

Our goal is to find a condition on N that do not depend on u, such that the condition (228) on u for t large enough implies that $u \equiv p$.

Because of (226), the condition (228) for t large enough is equivalent to: for t large enough,

$$\left\|u(t) - p(t)\right\|_{H^2} \leq \frac{1}{t^{N-1}}. \tag{229}$$

We denote $z(t) := u(t) - p(t)$. Our goal is to find N large enough that do not depend on z, for which we will be able to prove that $z \equiv 0$, given

$$\|z(t)\|_{H^2} \leq \frac{1}{t^{N-1}}, \tag{230}$$

for t large enough.

Because z is a difference of two solutions of (mKdV), we may derive the following equation for z:

$$z_t + \left(z_{xx} + (z+p)^3 - p^3\right)_x = 0. \tag{231}$$

We divide our proof in several steps.

Step 1. Modulation on z.

For $j = 1, ..., J$, if $P_j = B_k$ is a breather, we denote

$$K_j := \begin{pmatrix} \partial_{x_1} B_k \\ \partial_{x_2} B_k \end{pmatrix}, \tag{232}$$

if $P_j = R_l$ is a soliton, we denote

$$K_j = \partial_x R_l. \tag{233}$$

We may derive the following equation for K_j:

$$\left(K_j\right)_t + \left(\left(K_j\right)_{xx} + 3P_j^2 K_j\right)_x = 0. \tag{234}$$

For $j = 1, ..., J$, if $P_j = B_k$ is a breather, let $c_j(t) \in \mathbb{R}^2$ defined for t large enough and if $P_j = R_l$ is a soliton, let $c_j(t) \in \mathbb{R}$ defined for t large enough such that for

$$\widetilde{z}(t) := z(t) + \sum_{j=1}^{J} c_j(t) K_j(t), \tag{235}$$

where $c_j K_j$ is either a product of two numbers of $\mathbb{R}$ or a scalar product of two vectors of $\mathbb{R}^2$, the following condition is satisfied. For any $j = 1, ..., J$, for t large enough,

$$\int \widetilde{z}(t) K_j(t) \sqrt{\varphi_j(t)} = 0, \tag{236}$$

where φ_j is defined in Section 2.2 (in this proof, it is OK to take $\delta = 1$).

It is possible to do so in a unique way, because the Gram matrix associated to $K_j(t)\sqrt{\varphi_j(t)}$, $1 \leq j \leq J$, is invertible; which is the case because $K_j(t)\sqrt{\varphi_j(t)}$, $1 \leq j \leq J$, is linearly independent. This is why $c_j(t)$, $1 \leq j \leq J$, are defined in a unique way. For the same reason, $c_j(t)$ is obtained linearly from $\int K_k(t)z(t)\sqrt{\varphi_k(t)}$, $1 \leq k \leq J$, with coefficients that depend only on K_k, $1 \leq k \leq J$. This is why, from Cauchy-Schwarz, we may deduce that

Lemma 35. *For any $j = 1, ..., J$, for t large enough, there exists $C > 0$ that do not depend on z, such that*

$$|c_j(t)| \leq C\|z(t)\|_{L^2}, \tag{237}$$

$$\|\widetilde{z}(t)\|_{H^2} \leq C\|z(t)\|_{H^2}. \tag{238}$$

Moreover, the Gram matrix is C^1 in time and invertible. This is why, it's inverse is C^1 in time. Because $\int K_j z\sqrt{\varphi_j}$ are C^1 in time, we deduce by multiplication that $c_j(t)$ are C^1 in time.

By differentiating in time the linear relation that defines $c_j(t)$, we see that $c_j'(t)$ is obtained linearly from $\int K_k(t)z(t)\sqrt{\varphi_k(t)}$, $1 \leq k \leq J$, with coefficients that depend on K_k, $1 \leq k \leq J$ (and their derivatives) and linearly from $\frac{d}{dt}\int K_k(t)z(t)\sqrt{\varphi_k(t)}$, $1 \leq k \leq J$, with coefficients that depend on K_k, $1 \leq k \leq J$. Because it is easy to see that $\frac{d}{dt}\int K_k(t)z(t)\sqrt{\varphi_k(t)}$ may still be bounded by $C\|z(t)\|_{L^2}$, we deduce that for any $j = 1, ..., J$, for t large enough, there exists $C > 0$ that do not depend on z, such that

$$|c_j'(t)| \leq C\|z(t)\|_{L^2}. \tag{239}$$

We may derive the following equation for $\widetilde{z}$:

$$\widetilde{z}_t + \left(\widetilde{z}_{xx} + 3\widetilde{z}p^2\right)_x = -\left(3z^2p + z^3\right)_x + \sum_{k=1}^{J} c_k'(t)K_k - 3\sum_{k=1}^{J} c_k(t)\left((P_k^2 - p^2)K_k\right)_x. \tag{240}$$

Step 2. A bound for $|c_j'(t)|$. The goal here is to improve (239).

Lemma 36. *For any $j = 1, ..., J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z, such that*

$$|c_j'(t)| \leq C\|\widetilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \tag{241}$$

Proof. We may differentiate (236):

$$\begin{aligned}
0 &= \frac{d}{dt}\int \widetilde{z}K_j\sqrt{\varphi_j} \\
&= \int \widetilde{z}_t K_j\sqrt{\varphi_j} + \int \widetilde{z}\left(K_j\right)_t\sqrt{\varphi_j} + \int \widetilde{z}K_j\left(\sqrt{\varphi_j}\right)_t \\
&= -\int \left(\widetilde{z}_{xx} + 3\widetilde{z}p^2\right)_x K_j\sqrt{\varphi_j} - \int \left(3z^2p + z^3\right)_x K_j\sqrt{\varphi_j} + \sum_{k=1}^{J}\int \left(c_k'(t)\cdot K_k\right)K_j\sqrt{\varphi_j} \\
&\quad - 3\sum_{k=1}^{J} c_k(t)\int \left(c_k(t)\cdot\left((P_k^2 - p^2)K_k\right)_x\right)K_j\sqrt{\varphi_j} - \int \widetilde{z}\left((K_j)_{xx} + 3K_jP_j^2\right)_x\sqrt{\varphi_j} + \int \widetilde{z}K_j\left(\sqrt{\varphi_j}\right)_t.
\end{aligned}$$

We know that $\left(\sqrt{\varphi_j}\right)_x$ and $\left(\sqrt{\varphi_j}\right)_t$ are bounded (from inequalities established in Section 2.2). This is why, $\int \widetilde{z}K_j\left(\sqrt{\varphi_j}\right)_t$ is bounded by $C\|\widetilde{z}(t)\|_{H^2}$. For the same reason, after eventually doing an integration by parts, $\int \left(\widetilde{z}_{xx} + 3\widetilde{z}p^2\right)_x K_j\sqrt{\varphi_j}$ and $\int \widetilde{z}\left((K_j)_{xx} + 3K_jP_j^2\right)_x\sqrt{\varphi_j}$ are bounded by $C\|\widetilde{z}(t)\|_{H^2}$. $\int \left(3z^2p + z^3\right)_x K_j\sqrt{\varphi_j}$ is clearly bounded by $C\|z(t)\|_{H^2}^2$. Finally, we see that $(P_k^2 - p^2)K_k$ is exponentially bounded in time (in Sobolev or L^∞ norm), and using Lemma 35, we deduce that

$$\int \left(c_k(t)\cdot\left((P_k^2 - p^2)K_k\right)_x\right)K_j\sqrt{\varphi_j} \tag{242}$$

is bounded by $Ce^{-\theta t}\|z(t)\|_{H^2}$, for a suitable $\theta > 0$ that do not depend on z.

This is why, we deduce that for any $j = 1, ..., J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z, such that

$$\left| \sum_{k=1}^{J} \int \left(c_k'(t) \cdot K_k \right) K_j \sqrt{\varphi_j} \right| \leq C \|\widetilde{z}(t)\|_{H^2} + Ce^{-\theta t} \|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2. \tag{243}$$

We recall that for any $(v_1, v_2) \in (\mathbb{R})^2$ *or* $\left(\mathbb{R}^2\right)^2, v_3 \in \mathbb{R}$ *or* $\mathbb{R}^2$, we have the following equality between two elements of $\mathbb{R}$ or $\mathbb{R}^2$ (where vectors are denoted as a colon)

$$(v_1 \cdot v_2)\, v_3 = \left(v_1^T \left(v_2 v_3^T \right) \right)^T, \tag{244}$$

where ^{T} denotes the transpose.

First of all, because $\int K_k K_j^T \sqrt{\varphi_j}$ converges exponentially to $\int K_k K_j^T$, for $k \neq j$ $\int K_k K_j^T$ is exponentially decreasing, and from (239), we may write that for any $j = 1, ..., J$, for t large enough, there exists $C > 0$ and $\theta > 0$ that do not depend on z, such that

$$\left| \left(c_j'(t)^T \int K_j K_j^T \right)^T \right| \leq C \|\widetilde{z}(t)\|_{H^2} + Ce^{-\theta t} \|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2, \tag{245}$$

Now, in the case when $K_j \in \mathbb{R}^2$, using that it's components are linearly independent, using Cauchy-Schwarz inequality, we deduce the desired lemma. □

Step 3. Coercivity.

We define the following functional quadratic in $\widetilde{z}$:

$$\begin{aligned} H(t) = \frac{1}{2} \int \widetilde{z}_{xx}^2 - \frac{5}{2} \int p^2 \widetilde{z}_x^2 + \frac{5}{2} \int p_x^2 \widetilde{z}^2 + 5 \int p p_{xx} \widetilde{w}^2 + \frac{15}{4} \int p^4 \widetilde{z}^2 \\ + \sum_{j=1}^{J} \left(b_j^2 - a_j^2 \right) \left(\int \widetilde{z}_x^2 \varphi_j - 3 \int p^2 \widetilde{z}^2 \varphi_j \right) + \sum_{j=1}^{J} \left(a_j^2 + b_j^2 \right)^2 \frac{1}{2} \int \widetilde{z}^2 \varphi_j. \end{aligned}$$

We will prove the following lemma:

Lemma 37. *There exists $C > 0$ that do not depend on z, such that for t large enough,*

$$\|\widetilde{z}(t)\|_{H^2}^2 \leq CH(t) + C \sum_{j=1}^{J} \left(\int \widetilde{z} P_j \right)^2. \tag{246}$$

Proof. We denote Q_j the quadratic form associated to P_j. We remind that

$$\begin{aligned} Q_j[\varepsilon] := \frac{1}{2} \int \varepsilon_{xx}^2 - \frac{5}{2} \int P_j^2 \varepsilon_x^2 + \frac{5}{2} \int \left(P_j \right)_x^2 \varepsilon^2 + 5 \int P_j \left(P_j \right)_{xx} \varepsilon^2 + \frac{15}{4} \int P_j^4 \varepsilon^2 \\ + \left(b_j^2 - a_j^2 \right) \left(\int \varepsilon_x^2 - 3 \int P_j^2 \varepsilon^2 \right) + \left(a_j^2 + b_j^2 \right)^2 \frac{1}{2} \int \varepsilon^2. \end{aligned}$$

In any case, we have that for any $j = 1, ..., J$, there exists $\mu_j > 0$, such that if $\varepsilon \in H^2$ satisfies $\int K_j \varepsilon = 0$, then we have

$$Q_j[\varepsilon] \geq \mu_j \|\varepsilon\|_{H^2}^2 - \frac{1}{\mu_j} \left(\int \varepsilon P_j \right)^2. \tag{247}$$

Here, we apply this coercivity result with $\varepsilon = \widetilde{z}\sqrt{\varphi_j}$ for which the orthogonality conditions are satisfied. Thus,

$$\|\widetilde{z}\sqrt{\varphi_j}\|_{H^2}^2 \leq CQ_j[\widetilde{z}\sqrt{\varphi_j}] + C \left(\int \widetilde{z} P_j \sqrt{\varphi_j} \right)^2. \tag{248}$$

We denote

$$\begin{aligned} Q_j'[\varepsilon] := \frac{1}{2} \int \varepsilon_{xx}^2 \varphi_j - \frac{5}{2} \int p^2 \varepsilon_x^2 \varphi_j + \frac{5}{2} \int p_x^2 \varepsilon^2 \varphi_j + 5 \int p p_{xx} \varepsilon^2 \varphi_j + \frac{15}{4} \int p^4 \varepsilon^2 \varphi_j \\ + \left(b_j^2 - a_j^2 \right) \left(\int \varepsilon_x^2 \varphi_j - 3 \int p^2 \varepsilon^2 \varphi_j \right) + \left(a_j^2 + b_j^2 \right)^2 \frac{1}{2} \int \varepsilon^2 \varphi_j, \end{aligned}$$

and we observe that

$$H(t) = \sum_{j=1}^{J} Q_j'[\widetilde{z}(t)]. \tag{249}$$

In $Q_j'[\widetilde{z}(t)]$, we may replace p by P_j with an error bounded by $Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}^2$, because of (226) mainly. After that, the expression obtained may be replaced by $Q_j[\widetilde{z}(t)\sqrt{\varphi_j(t)}]$ with an error bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$ (cf. calculations done in the proof of Lemma 30). For the same reason, $\|\widetilde{z}\sqrt{\varphi_j}\|_{H^2}^2$ may be replaced by $\int \left(\widetilde{z}^2 + \widetilde{z}_x^2 + \widetilde{z}_{xx}^2\right)\varphi_j$ with an error bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$. Therefore, because of

$$\|\widetilde{z}\|_{H^2}^2 = \sum_{j=1}^{J} \int \left(\widetilde{z}^2 + \widetilde{z}_x^2 + \widetilde{z}_{xx}^2\right)\varphi_j, \tag{250}$$

because of the fact that $P_j\sqrt{\varphi_j}$ converges exponentially to P_j, and the fact that $\frac{C}{t}$ may be as small as we want if we take t large enough, we deduce the desired lemma. □

Step 4. Modification of H for the sake of simplification.
We define

$$\widetilde{H}(t) := \int \left[\frac{1}{2}\widetilde{z}_{xx}^2 - \frac{5}{2}\left(\left(\widetilde{z}+p\right)^2\left(\widetilde{z}+p\right)_x^2 - p^2p_x^2 - 2\widetilde{z}pp_x^2 - 2\widetilde{z}_x p^2 p_x\right) + \frac{1}{4}\left(\left(\widetilde{z}+p\right)^6 - p^6 - 6\widetilde{z}p^5\right)\right]$$
$$+ 2\sum_{j=1}^{J}\left(b_j^2 - a_j^2\right)\int\left[\frac{1}{2}\widetilde{z}_x^2 - \frac{1}{4}\left(\left(\widetilde{z}+p\right)^4 - p^4 - 4\widetilde{z}p^3\right)\right]\varphi_j + \frac{1}{2}\sum_{j=1}^{J}\left(a_j^2+b_j^2\right)^2\int\widetilde{z}^2\varphi_j.$$

We observe that the difference between H and $\widetilde{H}$ is bounded by $O\left(\|\widetilde{z}(t)\|_{H^2}^3\right)$. We can thus claim:

Lemma 38. *There exists $C > 0$ that do not depend on z, such that for t large enough,*

$$\|\widetilde{z}(t)\|_{H^2}^2 \leq C\widetilde{H}(t) + C\sum_{j=1}^{J}\left(\int \widetilde{z}P_j\right)^2. \tag{251}$$

Step 5. A bound for $\frac{d\widetilde{H}}{dt}$.

Lemma 39. *There exists $C > 0$ and $\theta > 0$ that do not depend on z, such that for t large enough,*

$$\left|\frac{d\widetilde{H}}{dt}\right| \leq \frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2 + Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2. \tag{252}$$

Proof. We develop the expression of $\widetilde{H}(t)$, we differentiate each term obtained and we use (240), the fact that p is a solution of (mKdV) and the fact that $\left(\varphi_j\right)_t = -\frac{x}{t}\left(\varphi_j\right)_x$, where $\frac{x}{t}$ is bounded independently from z because of the compact support of φ_j. We obtain several sorts of terms after doing several integrations by parts and several obvious simplifications.

Several terms are clearly bounded by one the bounds of the lemma, because z or $\widetilde{z}$ appears with a degree larger than 2. As an example, we show how to deal with $\int z_{xxx}z\widetilde{z}_{xx}p$. We use that $z = \widetilde{z} - \sum_{j=1}^{J} c_jK_j$, and we obtain

$$\int z_{xxx}z\widetilde{z}_{xx}p = \int \widetilde{z}_{xxx}\widetilde{z}\widetilde{z}_{xx}p - \int \widetilde{z}_{xxx}\left(\sum_{j=1}^{J}c_jK_j\right)\widetilde{z}_{xx}p - \int\left(\sum_{j=1}^{J}c_j\left(K_j\right)_{xxx}\right)\widetilde{z}\widetilde{z}_{xx}p$$
$$+ \int\left(\sum_{j=1}^{J}c_j\left(K_j\right)_{xxx}\right)\left(\sum_{j=1}^{J}c_jK_j\right)\widetilde{z}_{xx}p.$$

It is easy to see that any of those terms is bounded as we want in the lemma (several of them are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$, the last one is bounded by $C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2$), because of Lemma 35 and of (230).

Other terms contain $\widetilde{z}$ quadratically and contain $\left(\varphi_j\right)_x$. And, $\left(\varphi_j\right)_x$ is bounded by $\frac{C}{t}$. This is why that sort of terms is bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$.

Several other terms can be, by doing suitable integrations by parts transformed in two following expressions:

$$6\sum_{j=1}^{J}\int \widetilde{z}\widetilde{z}_x p\left[p_{xxxx}-2\left(b_j^2-a_j^2\right)\left(p_{xx}+p^3\right)+\left(a_j^2+b_j^2\right)^2 p+5pp_x^2+5p^2p_{xx}+\frac{3}{2}p^5\right]\varphi_j, \tag{253}$$

$$3\sum_{j=1}^{J}\int \widetilde{z}^2 p_x\left[p_{xxxx}-2\left(b_j^2-a_j^2\right)\left(p_{xx}+p^3\right)+\left(a_j^2+b_j^2\right)^2 p+5pp_x^2+5p^2p_{xx}+\frac{3}{2}p^5\right]\varphi_j. \tag{254}$$

To deal with these two expressions, we use the elliptic equation satisfied by P_j:

$$\left(P_j\right)_{xxxx}-2\left(b_j^2-a_j^2\right)\left(\left(P_j\right)_{xx}+P_j^3\right)+\left(a_j^2+b_j^2\right)^2 P_j+5P_j\left(P_j\right)_x^2+5P_j^2\left(P_j\right)_{xx}+\frac{3}{2}P_j^5=0, \tag{255}$$

and the fact that $\left[p_{xxxx}-2\left(b_j^2-a_j^2\right)\left(p_{xx}+p^3\right)+\left(a_j^2+b_j^2\right)^2 p+5pp_x^2+5p^2p_{xx}+\frac{3}{2}p^5\right]\varphi_j$ converges exponentially to $\left(P_j\right)_{xxxx}-2\left(b_j^2-a_j^2\right)\left(\left(P_j\right)_{xx}+P_j^3\right)+\left(a_j^2+b_j^2\right)^2 P_j+5P_j\left(P_j\right)_x^2+5P_j^2\left(P_j\right)_{xx}+\frac{3}{2}P_j^5$, which is a direct consequence from (226). This is why that sort of terms is bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$.

Other terms contain $\left(\left(P_j^2-p^2\right)K_j\right)$, which is bounded exponentially, with c_j bounded by $\|z\|_{H^2}$. Those terms are obviously bounded by $Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$.

Other terms contain K_k (or a derivative) and φ_j with $j\neq k$. In this case, this product gives an exponential decreasing, and such a term is bounded by $Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$, using (239).

Therefore, we are left with the following terms:

$$\sum_{j=1}^{J}c_j'(t)\int\left[\left(K_j\right)_{xx}\widetilde{z}_{xx}-10K_j\widetilde{z}_x pp_x-5K_j\widetilde{z}p_x^2-10\left(K_j\right)_x\widetilde{z}pp_x-5\left(K_j\right)_x\widetilde{z}_x p^2+\frac{15}{4}K_j\widetilde{z}p^4\right.$$
$$\left.+2\left(b_j^2-a_j^2\right)\left(K_j\right)_x\widetilde{z}_x-6\left(b_j^2-a_j^2\right)K_j\widetilde{z}p^2+\left(a_j^2+b_j^2\right)^2K_j\widetilde{z}\right]\varphi_j.$$

We may replace p by P_j in the preceeding expression with an error of $Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$, because of (239) and (226). This is acceptable, knowing the result we want to prove.

By integration by parts, we obtain several terms of the form $c_j'(t)\int\left(K_j\right)_{xx}\widetilde{z}_x\left(\varphi_j\right)_x$, which are bounded by $\frac{C}{t}|c_j'(t)|\|\widetilde{z}(t)\|_{H^2}$. Now, from Lemma 36, we deduce that they are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2+Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}+C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2$, which is exactly the bound we want. And, we are left with the following terms:

$$\sum_{j=1}^{J}c_j'(t)\int\left[\left(K_j\right)_{xxxx}+10\left(K_j\right)_x P_j\left(P_j\right)_x+5K_j\left(P_j\right)_x^2+10K_jP_j\left(P_j\right)_{xx}+5\left(K_j\right)_{xx}P_j^2+\frac{15}{2}K_jP_j^4\right.$$
$$\left.-2\left(b_j^2-a_j^2\right)\left(K_j\right)_{xx}-6\left(b_j^2-a_j^2\right)K_jP_j^2+\left(a_j^2+b_j^2\right)^2K_j\right]\widetilde{z}\varphi_j.$$

The last expression equals zero, because of the elliptic equation satisfied by K_j, which we may derive by differentiating (255). □

Step 6. A bound for $\frac{d}{dt}\int\widetilde{z}P_j$.

Lemma 40. *There exists $C>0$ and $\theta>0$ that do not depend on z, such that for t large enough, for any $j=1,...,J$,*

$$\left|\frac{d}{dt}\int\widetilde{z}P_j\right|\leq Ce^{-\theta t}\|z(t)\|_{H^2}+C\|z(t)\|_{H^2}^2. \tag{256}$$

Proof. We observe that

$$\int \widetilde{z} P_j = \int z P_j + \sum_{k=1}^{J} c_k(t) \int K_k P_j. \tag{257}$$

First, for $k = j$,

$$\int K_j P_j = 0, \tag{258}$$

for $k \neq j$,

$$\frac{d}{dt}\left[c_k(t) \int K_k P_j\right] = c_k'(t) \int K_k P_j + c_k(t) \int (K_k)_t P_j + c_k(t) \int K_k \left(P_j\right)_t, \tag{259}$$

and it is obvious, from Lemma 35 and (239), that this is bounded by $Ce^{-\theta t}\|z(t)\|_{H^2}$.

It is left to bound $\frac{d}{dt}\int z P_j$. We use (231) and we obtain

$$\frac{d}{dt}\int z P_j = -\int \left(z_{xx} + \left(z+p\right)^3 - p^3\right)_x P_j - \int z \left(\left(P_j\right)_{xx} + P_j^3\right)_x. \tag{260}$$

Several terms are immediately boundable by $C\|z(t)\|_{H^2}^2$, we kill several others by integration by parts and we are left with

$$\int z \left(p^2 - P_j^2\right)\left(P_j\right)_x, \tag{261}$$

which is obviously bounded by $Ce^{-\theta t}\|z(t)\|_{H^2}$, because of (226). □

By differentiation of a square, we obtain that

Lemma 41. *There exists $C > 0$ and $\theta > 0$ that do not depend on z, such that for t large enough, for any $j = 1, ..., J$,*

$$\left|\frac{d}{dt}\left(\int \widetilde{z} P_j\right)^2\right| \leq Ce^{-\theta t}\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2. \tag{262}$$

Step 7. A bound for $\|z(t)\|_{H^2}$ in function of $\widetilde{z}(t)$.

Because we have chosen $N > 2$, because of (230), we may claim that for t large enough, the integral

$$\int_t^{+\infty} \|z(s)\|_{H^2} ds \tag{263}$$

is finite.

Because of Lemma 35 and (230), we deduce that

$$c_j(t) \to_{t\to+\infty} 0. \tag{264}$$

Knowing this, from Lemma 36, we deduce by integration that

$$\begin{aligned} |c_j(t)| &\leq \int_t^{+\infty} |c_j'(s)| ds \\ &\leq C\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds + C\int_t^{+\infty} e^{-\theta s}\|z(s)\|_{H^2} ds + \int_t^{+\infty} \|z(s)\|_{H^2}^2 ds. \end{aligned}$$

Knowing this and using (235), we may deduce that

$$\begin{aligned} \|z(t)\|_{H^2} &\leq C\|\widetilde{z}(t)\|_{H^2} + C\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds + C\int_t^{+\infty} e^{-\theta s}\|z(s)\|_{H^2} ds + \int_t^{+\infty} \|z(s)\|_{H^2}^2 ds \\ &\leq C\|\widetilde{z}(t)\|_{H^2} + C\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds + C\sup_{s\geq t}\|z(s)\|_{H^2} e^{-\theta t} + C\sup_{s\geq t}\|z(s)\|_{H^2}\int_t^{+\infty} \|z(s)\|_{H^2} ds, \end{aligned}$$

which implies, because $\int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds$, $\sup_{s\geq t} \|z(s)\|_{H^2} e^{-\theta t}$ and $\sup_{s\geq t} \|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2} ds$ are decreasing in time, that

(265)

$$\sup_{s\geq t} \|z(s)\|_{H^2} \leq C \sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} + C \int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds + C \sup_{s\geq t} \|z(s)\|_{H^2} e^{-\theta t} + C \sup_{s\geq t} \|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2} ds,$$

and because $e^{-\theta t}$ and $\int_t^{+\infty} \|z(s)\|_{H^2} ds$ may be as small as we want for t large enough (dependent on z), we may deduce that

Lemma 42. *There exists $C > 0$ that do not depend on z, such that for t large enough,*

$$\|z(t)\|_{H^2} \leq \sup_{s\geq t} \|z(s)\|_{H^2} \leq C \sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} + C \int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds. \tag{266}$$

Step 8. Conclusion.

By integration, from Lemmas 38, 39 and 41, for t large enough (depending on z), with constants C and θ that do not depend on z,

$$\begin{aligned}
\|\widetilde{z}(t)\|_{H^2}^2 &\leq C \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2}^2 ds + C \int_t^{+\infty} e^{-\theta s} \|\widetilde{z}(s)\|_{H^2} \|z(s)\|_{H^2} ds + C \int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} \|z(s)\|_{H^2}^2 ds \\
&\leq C \sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2} ds + C \sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \int_t^{+\infty} e^{-\theta s} \|z(s)\|_{H^2} ds \\
&+ C \sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2}^2 ds.
\end{aligned}$$

Because $\sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2} ds$, $\sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \int_t^{+\infty} e^{-\theta s} \|z(s)\|_{H^2} ds$ and $\sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \int \|z(s)\|_{H^2}^2 ds$ are decreasing in time, we deduce after taking the supremum of the previous inequality and after simplification, that for t large enough,

$$\begin{aligned}
\sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} &\leq C \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2} ds + C \int_t^{+\infty} e^{-\theta s} \|z(s)\|_{H^2} ds + C \int_t^{+\infty} \|z(s)\|_{H^2}^2 ds \\
&\leq C \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2} ds + C \sup_{s\geq t} \|z(s)\|_{H^2} e^{-\theta t} + C \sup_{s\geq t} \|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2} ds,
\end{aligned}$$

and using (230), the fact that $N - 1 > 1$ and the fact that $e^{-\theta t}$ is decreasing faster than $\frac{1}{t^{N-2}}$, we deduce that for t large enough,

$$\sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \leq C \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2} ds + C \frac{1}{t^{N-2}} \sup_{s\geq t} \|z(s)\|_{H^2}, \tag{267}$$

and using Lemma 42,

$$\sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \leq C \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2} ds + C \frac{1}{t^{N-2}} \sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} + C \frac{1}{t^{N-2}} \int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds, \tag{268}$$

and because $\frac{1}{t^{N-2}}$ can be as small as we want for t large enough, we deduce that for t large enough and for a constant $C > 0$ that do not depend on z or on N,

$$\|\widetilde{z}(t)\|_{H^2} \leq \sup_{s\geq t} \|\widetilde{z}(s)\|_{H^2} \leq C \int_t^{+\infty} \frac{1}{s} \|\widetilde{z}(s)\|_{H^2} ds + C \frac{1}{t^{N-2}} \int_t^{+\infty} \|\widetilde{z}(s)\|_{H^2} ds. \tag{269}$$

Let us pick $T > 0$ large enough such that for $t \geq T$, the inequality (269) works.

From (235) and Lemma 35, we know that for $t \geq T$ (by taking T larger if needed),

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{C}{t^{N-1}}. \tag{270}$$

This is why, the following quantity is well defined:

$$A := \sup_{t\geq T} \{t^{N-1} \|\widetilde{z}(t)\|_{H^2}\}, \tag{271}$$

which means that for $t \geq T$,

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{A}{t^{N-1}}. \tag{272}$$

Now, using (270) and (272), we deduce from (269) that for $t \geq T$, with $C > 0$ that do not depend on z, on N or on A,

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{CA}{N-1}\frac{1}{t^{N-1}} + \frac{CA}{N-2}\frac{1}{t^{2N-4}} \leq \frac{CA}{N-2}\frac{1}{t^{N-1}}, \tag{273}$$

if we assume that $N > 3$.

Now, from (271), we deduce that there exists $T^* > T$ such that

$$(T^*)^{N-1}\,\|\widetilde{z}(T^*)\|_{H^2} \geq \frac{A}{2}. \tag{274}$$

This is why, by evaluating (273) in $t = T^*$, we find that

$$\frac{A}{2\,(T^*)^{N-1}} \leq \frac{CA}{N-2}\frac{1}{(T^*)^{N-1}}, \tag{275}$$

which, if we assume that $A > 0$, after simplification yields

$$N - 2 \leq 2C. \tag{276}$$

This means that if we assume that $N > 2C + 2$ and $N > 3$, the assumption $A > 0$ leads to a contradiction. Therefore, $A = 0$ under that assumption on N, which implies $\|\widetilde{z}(t)\|_{H^2} = 0$, and from Lemma 42, this implies that $z \equiv 0$. This means that the condition that we have esatblished for N:

$$N > \max(2C + 2, 3), \tag{277}$$

that do not depend on z, allows us to deduce that under (230), we may establish that $z \equiv 0$. The Proposition 5 is now proved. □

4.2. **A solution converging to a multi-breather converges exponentially to this multi-breather, if the velocities are positive.**

Proposition 43. *Let $u(t)$ be an H^2 solution of* (mKdV) *on $[T, +\infty)$, for $T \in \mathbb{R}$. We assume that*

$$\|u(t) - p(t)\|_{H^2} \to_{t\to+\infty} 0, \tag{278}$$

where p is the multi-breather constructed in Section 2. If $v_1 > 0$, then there exists $\gamma > 0$, $T_0 \geqslant T$ and $C > 0$ such that for all $t \geqslant T_0$,

$$\|u(t) - p(t)\|_{H^2} \leqslant Ce^{-\gamma t}. \tag{279}$$

Proof. We set $z(t) := u(t) - p(t)$, such that $\|z(t)\|_{H^2} \to_{t\to+\infty} 0$. From the fact that u and p are solutions of (mKdV), we can write the equation of $z(t)$:

$$z_t = -\left(z_{xx} + \left(z + p\right)^3 - p^3\right)_x. \tag{280}$$

We denote:

$$\Psi(x) := \frac{2}{\pi}\arctan\left(\exp\left(-\sqrt{\sigma}x/2\right)\right), \tag{281}$$

where $\sigma > 0$ is small enough (with precise conditions that will be mentioned throughout the proof). By direct calculations,

$$\Psi'(x) = \frac{-\sqrt{\sigma}}{2\pi\cosh\left(\sqrt{\sigma}x/2\right)}. \tag{282}$$

Thus,

$$|\Psi'(x)| \leqslant C\exp(-\sqrt{\sigma}|x|/2). \tag{283}$$

We have the following properties: $\lim_{+\infty}\Psi = 0$, $\lim_{-\infty}\Psi = 1$, for all $x \in \mathbb{R}$ $\Psi(-x) = 1 - \Psi(x)$, $\Psi'(x) < 0$, $|\Psi''(x)| \leqslant \frac{\sqrt{\sigma}}{2}|\Psi'(x)|$, $|\Psi'''(x)| \leqslant \frac{\sqrt{\sigma}}{2}|\Psi''(x)|$, $|\Psi'(x)| \leqslant \frac{\sqrt{\sigma}}{2}\Psi$ and $|\Psi'(x)| \leqslant \frac{\sqrt{\sigma}}{2}(1 - \Psi)$.

For $j = 2, ..., J$, let m_j be such that

$$m_j := \frac{v_{j-1} + v_j}{2}. \tag{284}$$

Let us denote $\tau_0 > 0$ the minimal distance between a v_j and a m_j.

From this, we define for $j = 2, ..., J$,

$$\Phi_j(t,x) := \Psi(x - m_j t). \tag{285}$$

We may extend this definition to $j = 1$ and $j = J+1$ in the following way: $\Phi_1 := 0$ and $\Phi_{J+1} := 1$.

Thus, the function that allows us to study properties around each breather (for $j = 1, ..., J$) are $\chi_j := \Phi_{j+1} - \Phi_j$.

The goal is to prove that:

$$\|z(t)\|_{H^2} \leqslant Ce^{-\gamma t}, \tag{286}$$

where $\gamma > 0$ is a constant to be deduced from the constants of the problem, and for t large enough.

Let $\gamma > 0$ be deduced from the constants of the problem with respect to the needs of the following proof.

We will prove this by induction. We will prove, for $j = 2, ..., J+1$, that $\int \left(z^2 + z_x^2 + z_{xx}^2\right) \Phi_j \leqslant Ce^{-2\gamma t}$ for t large enough, knowing that $\int \left(z^2 + z_x^2 + z_{xx}^2\right) \Phi_{j-1} \leqslant Ce^{-2\gamma t}$ for t large enough (note that this assumption is empty when $j = 2$). This implies the desired inequality.

Let us write the j-th step of our reasoning by induction (where $j \in \{2, ..., J+1\}$). Thus, j is fixed in the rest of the proof.

We assume that

$$\int \left(z^2 + z_x^2 + z_{xx}^2\right) \Phi_{j-1} \leqslant Ce^{-2\gamma t}. \tag{287}$$

We divide our proof in several steps.

Step 1. Almost-conservation of localized conservation laws. We define quantities that are similar to quantities defined in Section 2.2. We note that we localize around the first $j-1$ breathers (or solitons), not only around the $(j-1)$-th breather (or soliton). Notations defined in Section 2.2 should not be considered in the following proof and should be replaced by notations we define here:

$$M_j(t) := \frac{1}{2} \int u^2(t) \Phi_j(t), \tag{288}$$

$$E_j(t) := \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4\right] \Phi_j(t), \tag{289}$$

$$F_j(t) := \int \left[\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6\right] \Phi_j(t). \tag{290}$$

Lemma 44. *Let $\omega_2, \omega_6 > 0$, as small as desired. There exists $T_1 \geqslant T$ and $C > 0$ such that for $t \geqslant T_1$,*

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) \geqslant -Ce^{-2\gamma t}, \tag{291}$$

$$\sum_{i=1}^{j-1} (E[P_i] + \omega_2 M[P_i]) - \left(E_j(t) + \omega_2 M_j(t)\right) \geqslant -Ce^{-2\gamma t}, \tag{292}$$

$$\sum_{i=1}^{j-1} (F[P_i] + \omega_6 M[P_i]) - \left(F_j(t) + \omega_6 M_j(t)\right) \geqslant -Ce^{-2\gamma t}. \tag{293}$$

Proof. We will use the results of the computations made at the bottom of page 1115 and at the bottom of page 1116 of [29], as well as in Section 5.5 (Appendix) to claim the three following facts:

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int u^2 f &= \int \left(-\frac{3}{2} u_x^2 + \frac{3}{4} u^4\right) f' + \frac{1}{2} \int u^2 f''', \\ \frac{d}{dt} \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4\right] f &= \int \left[-\frac{1}{2} (u_{xx} + u^3)^2 - u_{xx}^2 + 3 u_x^2 u^2\right] f' + \frac{1}{2} \int u_x^2 f''', \\ \frac{d}{dt} \int \left(\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6\right) f & \end{aligned} \tag{294}$$

$$= \int \left(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2u^2 + 15u_x^2uu_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 - \frac{45}{4}u^4u_x^2\right)f'$$

$$+5\int u^2u_xu_{xx}f'' + \frac{1}{2}\int u_{xx}^2f''''. \tag{295}$$

where f is a C^3 function that does not depend on time.

For the mass:

If $j \leqslant J$,

$$2\frac{d}{dt}M_j(t) = -\int \left(3u_x^2 + m_ju^2 - \frac{3}{2}u^4\right)\Phi_{jx}(t) + \int u^2\Phi_{jxxx}(t). \tag{296}$$

We recall that

$$|\Phi_{jxx}| \leqslant \frac{\sqrt{\sigma}}{2}|\Phi_{jx}|, \quad |\Phi_{jxxx}| \leqslant \frac{\sigma}{4}|\Phi_{jx}|, \quad \Phi_{jx} \leqslant 0, \tag{297}$$

where we can choose σ as small as desired. For this proof, we would like to ask for σ:

$$0 < \sigma \leqslant v_1 \leqslant m_j. \tag{298}$$

Thus,

$$2\frac{d}{dt}M_j(t) \geqslant \int \left(3u_x^2 + \frac{3\sigma}{4}u^2 - \frac{3}{2}u^4\right)\left|\Phi_{jx}(t)\right|. \tag{299}$$

By Corollary 13, for $r > 0$, if t, x satisfy $v_{j-1}t + r < x < v_jt - r$, then

$$\begin{aligned} |u(t,x)| &\leqslant |P(t,x)| + \|z(t)\|_{L^\infty} \\ &\leqslant Ce^{-\beta r} + C\|z(t)\|_{H^2}, \end{aligned} \tag{300}$$

the same could be said for u_x.

We can thus deduce that for r large enough and for T_1 large enough, for $x \in (v_{j-1}t + r, v_jt - r)$, we can obtain that $|u|$ is bounded by any fixed constant, that can be taken as small as desired. Here, we will use the latter to bound $\frac{3}{2}u^2$ by $\frac{\sigma}{4}$.

For $t \geqslant T_1$ and $x \leqslant v_{j-1}t + r$ or $x \geqslant v_jt - r$, we have $|x - m_jt| \leqslant \tau_0t - r$, and therefore for such t, x:

$$\begin{aligned} |\Phi_{jx}(t,x)| &\leqslant C\exp\left(-\sqrt{\sigma}|x - m_jt|/2\right) \\ &\leqslant C\exp\left(-\sqrt{\sigma}\tau_0t/2\right)\exp\left(\sqrt{\sigma}r/2\right). \end{aligned} \tag{301}$$

Because $\int u^4$ is bounded by a constant for any time and $\exp\left(\sqrt{\sigma}r/2\right)$ is a fixed constant (r is already chosen), we have, for $t \geqslant T_1$,

$$\frac{d}{dt}M_j(t) \geqslant \int \left(\frac{3}{2}u_x^2 + \frac{\sigma}{4}u^2\right)|\Phi_{jx}(t)| - Ce^{-2\gamma t} \geqslant -Ce^{-2\gamma t}, \tag{302}$$

where γ is chosen as a suitable function of σ and τ_0.

By integration, we deduce that for any $t_1 \geqslant t$, with a constant $C > 0$ that does not depend on t_1, we have:

$$M_j(t_1) - M_j(t) \geqslant -Ce^{-2\gamma t}. \tag{303}$$

We note that this conclusion is immediate when $j = J + 1$, because we have exactly the conserved quantity.

We have:

$$\begin{aligned} \left|\sum_{i=1}^{j-1} M[P_i] - M_j(t_1)\right| &\leqslant \left|\sum_{i=1}^{j-1}\frac{1}{2}\int P_i^2 - \frac{1}{2}\int P^2\Phi_j(t_1)\right| + \frac{1}{2}\left|\int P^2\Phi_j(t_1) - \int u^2\Phi_j(t_1)\right| \\ &\leqslant Ce^{-\kappa(\beta,\sigma,\tau_0)t_1} + \frac{1}{2}\int \left|P^2 - u^2\right|\Phi_j(t_1) \\ &\leqslant Ce^{-\kappa(\beta,\sigma,\tau_0)t_1} + C\int \left|P^2 - u^2\right| \to_{t_1\to+\infty} 0. \end{aligned} \tag{304}$$

This means that when we take the limit of (303) when $t_1 \to +\infty$, we obtain, for $t \geqslant T_1$:

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) \geqslant -Ce^{-2\gamma t}, \tag{305}$$

which is exactly what we wished to prove.

For the energy:

If $j \leqslant J$,

$$\begin{aligned} 2\frac{d}{dt}E_j(t) &= \int \left[-\left(u_{xx}+u^3\right)^2 - 2u_{xx}^2 + 6u_x^2u^2 \right] \Phi_{jx}(t) - m_j \int \left(u_x^2 - \frac{1}{2}u^4 \right) \Phi_{jx}(t) + \frac{1}{2}\int u_x^2 \Phi_{jxxx}(t) \\ &\geqslant \int \left[\left(u_{xx}+u^3\right)^2 + 2u_{xx}^2 - 6u_x^2u^2 + \frac{3\sigma}{4}u_x^2 - \frac{m_j}{2}u^4 \right] |\Phi_{jx}(t)|. \end{aligned} \tag{306}$$

We can do the same reasoning as for the mass, to majorate $\frac{m_j}{2}u^2$ by ω_1, a constant that we can choose as small as desired, and to majorate $6u^2$ by $\frac{\sigma}{4}$. We obtain that if T_1 is large enough (dependently on the chosen constant ω_1):

$$2\frac{d}{dt}E_j(t) \geqslant \int \left[\left(u_{xx}+u^3\right)^2 + 2u_{xx}^2 + \frac{\sigma}{2}u_x^2 - \omega_1 u^2 \right] |\Phi_{jx}(t)| - Ce^{-2\gamma t}. \tag{307}$$

By using what we have performed for the mass, we have that if we take ω_1 small enough with respect to $\frac{v_1}{2}$,

$$\frac{d}{dt}\left(E_j + \omega_2 M_j\right)(t) \geqslant -Ce^{-2\gamma t}. \tag{308}$$

Then, by integration and similarly as for the mass, we obtain the desired conclusion that is true for any j.

For F:

If $j \leqslant J$,

$$\begin{aligned} 2\frac{d}{dt}F_j(t) &= \int \left(-3u_{xxx}^2 + 18u_{xx}^2u^2 + 30u_x^2uu_{xx} + \frac{9}{8}u^8 + \frac{1}{2}u_x^4 + 3u_{xx}u^5 - \frac{45}{2}u^4u_x^2 \right) \Phi_{jx}(t) \\ &\quad - m_j \int \left(u_{xx}^2 - 5u^2u_x^2 + \frac{1}{2}u^6 \right) \Phi_{jx}(t) + 10\int u^2u_{xx}u_x\Phi_{jxx}(t) + \int u_{xx}^2\Phi_{jxxx}(t) \\ &\geqslant \int \left(3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 - 18u_{xx}^2u^2 - 15u_x^2u^2 - 15u_x^2u_{xx}^2 - \frac{9}{8}u^8 - \frac{1}{2}u_x^4 - \frac{3}{2}u_{xx}^2u^4 - \frac{3}{2}u^6 \right) \left|\Phi_{jx}(t)\right| \\ &\quad + \int \left(\sigma u_{xx}^2 + \frac{\sigma}{2}u^6 - 5m_ju^2u_x^2 \right) |\Phi_{jx}(t)| - 5\int u^2u_x^2|\Phi_{jxx}(t)| - 5\int u^2u_{xx}^2|\Phi_{jxx}(t)| - \int u_{xx}^2|\Phi_{jxxx}(t)|. \end{aligned} \tag{309}$$

By the same reasoning as for the energy and the mass, if we set $\omega_3, \omega_4, \omega_5 > 0$ constants that we can take as small as desired, and if T_1 is large enough dependently on these constants, for $t \geqslant T_1$:

$$2\frac{d}{dt}F_j(t) \geqslant \int \left(3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 + \frac{3\sigma}{4}u_{xx}^2 + \frac{\sigma}{2}u^6 - \omega_3 u_{xx}^2 - \omega_4 u_x^2 - \omega_5 u^2 \right) \left|\Phi_{jx}(t)\right| - Ce^{-2\gamma t}. \tag{310}$$

By using what we have carried out for the mass, we have that if we take $\omega_3, \omega_4, \omega_5$ small enough,

$$\frac{d}{dt}\left(F_j + \omega_6 M_j\right)(t) \geqslant -Ce^{-2\gamma t}. \tag{311}$$

Then, by integration and similarly as before, we obtain the desired conclusion that is true for any j. □

Remark 45. If $j = J+1$, we have that:

$$\sum_{i=1}^{J} M[P_i] - M_{J+1}(t) = 0, \tag{312}$$

$$\sum_{i=1}^{J} E[P_i] - E_{J+1}(t) = 0, \tag{313}$$

$$\sum_{i=1}^{J} F[P_i] - F_{J+1}(t) = 0. \tag{314}$$

Step 2. Modulation. Notations that were defined in Section 2.3 should not be taken into consideration in the following proof and should be replaced by notations we define here.

Lemma 46. *There exists $C > 0$, $T_2 \geqslant T$, such that there exist unique C^1 functions $y_1, y_2 : [T_2, +\infty) \to \mathbb{R}$ such that if we set*

$$w(t,x) := u - \widetilde{P}, \tag{315}$$

where

$$\widetilde{P}(t,x) := \sum_{i=1}^{J} \widetilde{P_i}(t,x), \tag{316}$$

for $i \neq j-1$,

$$\widetilde{P_i}(t,x) := P_i(t,x), \tag{317}$$

and if $P_{j-1} = R_{c,\kappa}(t,x;x_0)$ is a soliton,

$$\widetilde{P_{j-1}}(t,x) := R_{c+y_1(t),\kappa}(t,x;x_0+y_2(t)), \tag{318}$$

if $P_{j-1} = B_{\alpha,\beta}(t,x;x_1,x_2)$ is a breather,

$$\widetilde{P_{j-1}}(t,x) := B_{\alpha,\beta}(t,x;x_1+y_1(t),x_2+y_2(t)), \tag{319}$$

in the case when P_{j-1} is a breather we denote:

$$\widetilde{P_{j-1}}_1(t,x) := \partial_{y_1}\widetilde{P_{j-1}}, \quad \widetilde{P_{j-1}}_2(t,x) := \partial_{y_2}\widetilde{P_{j-1}}, \tag{320}$$

then, $w(t)$ satisfies, for all $t \in [T_2, +\infty)$,

$$\int \widetilde{P_{j-1}}_1(t)w(t) = \int \widetilde{P_{j-1}}_2(t)w(t) = 0, \tag{321}$$

if P_{j-1} is a breather, and

$$\int \widetilde{P_{j-1}}(t)w(t) = \int \widetilde{P_{j-1}}_x(t)w(t) = 0, \tag{322}$$

if P_{j-1} is a soliton.

Moreover, for all $t \in [T_2, +\infty)$,

$$\|w(t)\|_{H^2} + |y_1(t)| + |y_2(t)| \leqslant C\|z(t)\|_{H^2}, \tag{323}$$

and if P_{j-1} is a breather,

$$\left|y_1'(t)\right| + \left|y_2'(t)\right| \leqslant C\left(\int w(t)^2\Phi_j\right)^{1/2} + Ce^{-\gamma t}, \tag{324}$$

if P_{j-1} is a soliton,

$$|y_2(t) + y_1(t)t| \leqslant C\|z(t)\|_{H^2}, \tag{325}$$

$$\left|y_1'(t)\right| + \left|y_2'(t) + y_1'(t)t\right| \leqslant C\left(\int w(t)^2\Phi_j\right)^{1/2} + Ce^{-\gamma t}. \tag{326}$$

Proof. The proof that has to be performed is similar to the proof of Lemma 19, which is a consequence of the implicit function theorem. The proof of (324) is also similar, as in the proof of Lemma 19; we take the time derivative of $\int \widetilde{P_{j-1}}_1(t)w(t) = \int \widetilde{P_{j-1}}_2(t)w(t) = 0$. □

Step 3. Quadratic approximations of localized conservation laws.

Lemma 47. *Let $\omega > 0$ as small as we want. There exists $C > 0, T_3 \geqslant T$ such that the following hold for $t \geqslant T_3$:*

$$\left| M_j(t) - \sum_{i=1}^{j-1} M[\widetilde{P}_i] - \sum_{i=1}^{j-1} \int \widetilde{P}_i w - \frac{1}{2}\int w^2\Phi_j \right| \leqslant Ce^{-2\gamma t}, \tag{327}$$

$$\left| E_j(t) - \sum_{i=1}^{j-1} E[\widetilde{P}_i] - \sum_{i=1}^{j-1} \int \left[\widetilde{P}_{ix}w_x - \widetilde{P}_i^{\,3} w\right] - \int \left[\frac{1}{2}w_x^2 - \frac{3}{2}\widetilde{P}^2 w^2\right]\Phi_j \right| \leqslant Ce^{-2\gamma t} + \omega \int w^2\Phi_j, \tag{328}$$

$$\left| F_j(t) - \sum_{i=1}^{j-1} F[\widetilde{P}_i] - \sum_{i=1}^{j-1} \int \left[\widetilde{P}_{ixx}w_{xx} - 5\widetilde{P}_i\widetilde{P}_{ix}^{\,2} w - 5\widetilde{P}_i^{\,2}\widetilde{P}_{ix}w_x + \frac{3}{2}\widetilde{P}_i^{\,5} w\right] \right.$$
$$\left. - \int \left[\frac{1}{2}w_{xx}^2 - \frac{5}{2}w^2\widetilde{P}_x^2 - 10\widetilde{P}w\widetilde{P}_x w_x - \frac{5}{2}\widetilde{P}^2 w_x^2 + \frac{15}{4}\widetilde{P}^4 w^2\right]\Phi_j(t) \right| \leqslant Ce^{-2\gamma t} + \omega \int \left(w^2 + w_x^2\right)\Phi_j. \tag{329}$$

Proof. For the mass:

$$\begin{aligned} M_j(t) &= \frac{1}{2}\int \left(\widetilde{P} + w\right)^2 \Phi_j \\ &= \frac{1}{2}\int \widetilde{P}^2\Phi_j + \int \widetilde{P}w\Phi_j + \frac{1}{2}\int w^2\Phi_j. \end{aligned} \tag{330}$$

As in Step 1, we can show that $\frac{1}{2}\int \widetilde{P}^2\Phi_j$ converges exponentially (we choose γ with respect to this exponential convergence) to $\sum_{i=1}^{j-1} M[\widetilde{P}_i]$. Similarly, the difference between $\int \widetilde{P}w\Phi_j$ and $\sum_{i=1}^{j-1}\widetilde{P}_i w$ converges exponentially to 0 (the velocity of a soliton is not modified a lot by modulation, this is why it works in any cases).

For E and F, we perform similar basic computations with the only difference that there will also be terms of degree 3 or more in w. We know that $\|w(t)\|_{H^2} \to_{t\to+\infty} 0$, this is the reason why for t large enough, such terms are boundable by $\omega\int w^2\Phi_j$ or $\omega\int w_x^2\Phi_j$. □

Step 4. Approximation of the Lyapunov functional. By analogy with the existence part, we introduce the following Lyapunov functional:

$$\mathcal{H}_j(t) := F_j(t) + 2\left(b_{j-1}^2 - a_{j-1}^2\right)E_j(t) + \left(a_{j-1}^2 + b_{j-1}^2\right)^2 M_j(t). \tag{331}$$

We will use the previous step to approximate $\mathcal{H}_j(t)$.

Lemma 48. *There exists $T_4 \geqslant T$ such that the following hold for $t \geqslant T_4$:*

$$\begin{aligned} \mathcal{H}_j(t) = &\sum_{i=1}^{j-1} F[\widetilde{P}_i] + 2\left(b_{j-1}^2 - a_{j-1}^2\right)\sum_{i=1}^{j-1} E[\widetilde{P}_i] + \left(a_{j-1}^2 + b_{j-1}^2\right)^2 \sum_{i=1}^{j-1} M[\widetilde{P}_i] \\ &+ H_j(t) + O(e^{-2\gamma t}) + o\left(\int \left(w^2 + w_x^2\right)\Phi_j\right), \end{aligned} \tag{332}$$

where

$$\begin{aligned} H_j(t) := &\int \left[\frac{1}{2}w_{xx}^2 - \frac{5}{2}w_x^2\widetilde{P_{j-1}}^2 + \frac{5}{2}w^2\widetilde{P_{j-1}}_x^2 + 5w^2\widetilde{P_{j-1}}\,\widetilde{P_{j-1}}_{xx} + \frac{15}{4}w^2\widetilde{P_{j-1}}^4\right]\Phi_j(t) \\ &+ \left(b_{j-1}^2 - a_{j-1}^2\right)\int \left[w_x^2 - 3w^2\widetilde{P_{j-1}}^2\right]\Phi_j(t) + \frac{1}{2}\left(a_{j-1}^2 + b_{j-1}^2\right)^2 \int w^2\Phi_j(t). \end{aligned} \tag{333}$$

Proof. This lemma is obtained from the summation of the facts established in the previous lemma. We get rid of the linear terms in the following way, by integrations by parts:

$$\sum_{i=1}^{j-1} \int \left(\widetilde{P}_{ixx}w_{xx} - 5\widetilde{P}_i\widetilde{P}_{ix}^{\,2} w - 5\widetilde{P}_i^{\,2}\widetilde{P}_{ix}w_x + \frac{3}{2}\widetilde{P}_i^{\,5} w\right)$$

$$
\begin{aligned}
&+2\left(b_{j-1}^2-a_{j-1}^2\right)\sum_{i=1}^{j-1}\int\left(\widetilde{P}_{ix}w_x-\widetilde{P}_i{}^3w\right)+\left(a_{j-1}^2+b_{j-1}^2\right)^2\sum_{i=1}^{j-1}\widetilde{P}_iw\\
&=\sum_{i=1}^{j-1}\int\left(\widetilde{P}_{ixxxx}+5\widetilde{P}_i\widetilde{P}_{ix}^2+5\int\widetilde{P}_i{}^2\widetilde{P}_{ixx}+\frac{3}{2}\widetilde{P}_i{}^5\right)w\\
&+2\left(b_{j-1}^2-a_{j-1}^2\right)\sum_{i=1}^{j-1}\int\left(-\widetilde{P}_{ixx}-\widetilde{P}_i{}^3\right)w+\left(a_{j-1}^2+b_{j-1}^2\right)^2\sum_{i=1}^{j-1}\int\widetilde{P}_iw. \qquad (334)
\end{aligned}
$$

If we consider that this sum goes from $i=1$ to $j-2$, we see that for $1\leqslant i\leqslant j-2$, this sum is exponentially bounded by induction assumption (we use that for $i\leqslant j-2$, a polynomial in $\widetilde{P}_i$ and its derivatives is bounded by $C\Phi_{j-1}$ and that $w=z+\left(P_{j-1}-\widetilde{P_{j-1}}\right)$). It is left to consider the sum of the terms with $i=j-1$.

For $i=j-1$, we have nearly the elliptic equation satisfied by $\widetilde{P_{j-1}}$. It is actually exactly this equation in the case when $\widetilde{P_{j-1}}$ is a breather. When $\widetilde{P_{j-1}}$ is a soliton, it's shape parameter is modified by modulation. That is why, in this case, the sum of the terms with $i=j-1$ is equal to

$$
2y_1(t)\int\left(-\widetilde{P_{j-1}}_{xx}-\widetilde{P_{j-1}}^3\right)w+2b_{j-1}^2y_1(t)\int\widetilde{P_{j-1}}w+y_1(t)^2\int\widetilde{P_{j-1}}w, \qquad (335)
$$

which vanishes because of the orthogonality condition from the modulation and the elliptic equation satisfied by a soliton.

H_j is obtained as the sum of the quadratic parts of the previous lemma and its form is slightly modified by integration by parts, and by the fact that for $i\geqslant j$, $\widetilde{P}_i\Phi_j(t)$ is exponentially decreasing, and the fact that for $i\leqslant j-2$, $\int\widetilde{P}_iw^2$ is exponentially decreasing by the induction assumption. Therefore, H_j corresponds to the sum of the quadratic parts of previous lemma to which we have to add $5\int w^2\widetilde{P}\widetilde{P}_x\Phi_{jx}$, which is bounded exponentially. □

Step 5. Bound from above for $H_j(t)$. Because $v_1>0$, we have that $b_{j-1}^2-a_{j-1}^2>0$. By taking ω_2 and ω_6 small enough (with respect to $\left(a_{j-1}^2+b_{j-1}^2\right)^2$), we obtain, by summation of the facts of Lemma 44, the following inequality:

$$
\mathcal{H}_j(t)-\sum_{i=1}^{j-1}F[P_i]-2\left(b_{j-1}^2-a_{j-1}^2\right)\sum_{i=1}^{j-1}E[P_i]-\left(a_{j-1}^2+b_{j-1}^2\right)^2\sum_{i=1}^{j-1}M[P_i]\leqslant Ce^{-2\gamma t}. \qquad (336)
$$

From Lemma 48, for $t\geqslant T_3$,

$$
\begin{aligned}
H_j(t)&\leqslant F[P_{j-1}]-F[\widetilde{P_{j-1}}]+2\left(b_{j-1}^2-a_{j-1}^2\right)\left(E[P_{j-1}]-E[\widetilde{P_{j-1}}]\right)\\
&+\left(a_{j-1}^2+b_{j-1}^2\right)^2\left(M[P_{j-1}]-M[\widetilde{P_{j-1}}]\right)+Ce^{-2\gamma t}+\omega\int\left(w^2+w_x^2\right)\Phi_j. \qquad (337)
\end{aligned}
$$

In the case if P_{j-1} is a breather, we have immediately

$$
H_j(t)\leqslant Ce^{-2\gamma t}+\omega\int\left(w^2+w_x^2\right)\Phi_j. \qquad (338)
$$

The case when P_{j-1} is a soliton needs more inspection. As in the existence part, we have the following relations:

$$
M[\widetilde{P_{j-1}}](t)=\left(b_{j-1}^2+y_1(t)\right)^{1/2}M[q], \qquad (339)
$$

$$
E[\widetilde{P_{j-1}}](t)=\left(b_{j-1}^2+y_1(t)\right)^{3/2}E[q], \qquad (340)
$$

$$
F[\widetilde{P_{j-1}}](t)=\left(b_{j-1}^2+y_1(t)\right)^{5/2}F[q]. \qquad (341)
$$

We set $\mathcal{R}_{j-1}(t) := F[\widetilde{P_{j-1}}](t) + 2b_{j-1}^2 E[\widetilde{P_{j-1}}](t) + b_{j-1}^4 M[\widetilde{P_{j-1}}](t)$, and we simplify it as follows:

$$\mathcal{R}_{j-1}(t) = b_{j-1}^5 \left(1 + \frac{y_1(t)}{b_{j-1}^2}\right)^{5/2} F[q] + 2b_{j-1}^5 \left(1 + \frac{y_1(t)}{b_{j-1}^2}\right)^{3/2} E[q] + b_{j-1}^5 \left(1 + \frac{y_1(t)}{b_{j-1}^2}\right)^{1/2} M[q]. \tag{342}$$

After making a Taylor expansion as in the existence,

$$\mathcal{R}_{j-1}(t) - F[P_{j-1}] - 2b_{j-1}^2 E[P_{j-1}] - b_{j-1}^4 M[P_{j-1}] = O(y_1(t)^3). \tag{343}$$

Therefore, if T_4 is large enough, $\|z(t)\|_{H^2}$ can be as small as we want, and for $t \geqslant T_4$ and P_{j-1} a soliton, we may write:

$$H_j(t) \leqslant Ce^{-2\gamma t} + \omega \int \left(w^2 + w_x^2\right) \Phi_j + \omega y_1(t)^2. \tag{344}$$

Step 6. Coercivity. H_j can be seen as the quadratic form associated to $\widetilde{P_{j-1}}$ and evaluated in $w\sqrt{\Phi_j}$, modulo several terms that can be bounded by $C\sqrt{\sigma}\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j$ (because these terms depend on derivatives of Φ_j). Let us prove that we can apply Section 5.4 (Appendix) for $w\sqrt{\Phi_j}$.

More precisely, we need to prove that for $\nu > 0$ small enough (from Section 5.4), $\left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}_1\right| + \left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}_2\right| \leq \nu\|w\sqrt{\Phi_j}\|_{H^2}$ if P_{j-1} is a breather or that $\left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}\right| + \left|\int w\sqrt{\Phi_j}\widetilde{P_{j-1}}_x\right| \leq \nu\|w\sqrt{\Phi_j}\|_{H^2}$ if P_{j-1} is a soliton. In any case, the proof is the same and let us write K at the place of $\widetilde{P_{j-1}}_1$, $\widetilde{P_{j-1}}_2$, $\widetilde{P_{j-1}}$ or $\widetilde{P_{j-1}}_x$. This means that we want to bound $\int w\sqrt{\Phi_j}K$.

From (321), (322), we see that it is enough to bound $\int w(1-\sqrt{\Phi_j})K$ by $\nu\|w\sqrt{\Phi_j}\|_{H^2}$. The reasonning that follows works for $j \leq J$, for $j = J+1$ the result is immediate because $\Phi_{J+1} = 1$. Φ_j is a translate of Ψ, and, using the Taylor expansion when $z \to 0$, $\sqrt{1+z} = 1 + O(z)$,

$$\begin{aligned} 1 - \sqrt{\Psi} &= 1 - \sqrt{1 + \Psi - 1} \\ &= 1 - \sqrt{1 - \Psi(-x)} \\ &= O(\Psi(-x)), \end{aligned} \tag{345}$$

which means that $1 - \sqrt{\Phi_j} \leq C\min(1, \exp(\sqrt{\sigma}(x - m_j t)/2))$. We may deduce now that

$$\begin{aligned} \left|\int w(1-\sqrt{\Phi_j})K\right| &= \left|\int w\sqrt{\Phi_j}\frac{1-\sqrt{\Phi_j}}{\sqrt{\Phi_j}}K\right| \\ &\leq \left\|\frac{1-\sqrt{\Phi_j}}{\sqrt{\Phi_j}}K\right\|_{L^2} \|w\sqrt{\Phi_j}\|_{L^2} \\ &\leq Ce^{\sqrt{\sigma}(m_j - v_{j-1})t}\|w\sqrt{\Phi_j}\|_{L^2}, \end{aligned} \tag{346}$$

if $\frac{\sqrt{\sigma}}{4} < \frac{\beta}{2}$. And so, if t is large enough, we get the bound we want.

Thus, there exists $\mu > 0$ such that for $t \geqslant T_5$ (where T_5 is large enough and depends on σ):

$$\begin{aligned} \mu\|w\sqrt{\Phi_j}\|_{H^2}^2 &\leqslant H_j(t) + C\sqrt{\sigma}\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j + \frac{1}{\mu}\left(\int \widetilde{P_{j-1}} w\sqrt{\Phi_j}\right)^2 \\ &\leqslant Ce^{-2\gamma t} + \omega\int \left(w^2 + w_x^2\right)\Phi_j + C\sqrt{\sigma}\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j \\ &\quad + \omega y_1(t)^2 + \frac{1}{\mu}\left(\int \widetilde{P_{j-1}} w\sqrt{\Phi_j}\right)^2, \end{aligned} \tag{347}$$

where the term $\frac{1}{\mu}\left(\int \widetilde{P_{j-1}} w\sqrt{\Phi_j}\right)^2$ is present only if $\widetilde{P_{j-1}}$ is a breather and the term $\omega y_1(t)^2$ is present only if $\widetilde{P_{j-1}}$ is a soliton.

For σ small enough and ω small enough, we deduce that:

$$\int \left(w^2 + w_x^2 + w_{xx}^2\right) \Phi_j \leqslant Ce^{-2\gamma t} + \omega y_1(t)^2 + C\left(\int \widetilde{P_{j-1}} w \sqrt{\Phi_j}\right)^2. \tag{348}$$

We set $T_0 := \max(T_1, T_2, T_3, T_4, T_5)$.

Step 7. Bound from above for $\left|\int \widetilde{P_{j-1}} w \sqrt{\Phi_j}\right|$ (to do in the case if $\widetilde{P_{j-1}}$ is a breather). We would like to prove that $\int \widetilde{P_{j-1}} w \sqrt{\Phi_j}$ is exponentially decreasing.To do so, we would like to get rid of $\sqrt{\Phi_j}$.

It is clear that $\int \widetilde{P_{j-1}} w \left(1 - \sqrt{\Phi_j}\right)$ is exponentially decreasing. Thus, it is enough to prove that $\int \widetilde{P_{j-1}} w$ is exponentially decreasing.

If $i \leqslant j-2$, we know that $\int \widetilde{P_i} w$ is exponentially decreasing by the induction assumption. Thus, it is enough to prove that $\sum_{i=1}^{j-1} \int \widetilde{P_i} w$ is exponentially decreasing.

From the mass approximation of Lemma 47 and Lemma 44, we observe that, for $t \geqslant T_0$:

$$\begin{aligned}
\sum_{i=1}^{j-1} \int \widetilde{P_i} w &= O(e^{-2\gamma t}) + M_j(t) - \sum_{i=1}^{j-1} M[P_i] - \frac{1}{2}\int w^2 \Phi_j \\
&\leqslant Ce^{-2\gamma t} - \frac{1}{2}\int w^2 \Phi_j \\
&\leqslant Ce^{-2\gamma t}.
\end{aligned} \tag{349}$$

Now, we use the fact that the sum of the linear parts of our localized conservation laws is exponentially decreasing, which we have established in the proof of Lemma 48. Therefore, the linear terms of $F_j + 2(b_{j-1}^2 - a_{j-1}^2)E_j$ are $O(e^{-2\gamma t}) - \left(a_{j-1}^2 + b_{j-1}^2\right)^2 \sum_{i=1}^{j-1} \int \widetilde{P_i} w$.

Now, from the energy and F approximation of Lemma 47 and Lemma 44, we observe that (we recall that $b_{j-1}^2 - a_{j-1}^2 > 0$), for $t \geqslant T_0$:

$$\begin{aligned}
-\left(a_{j-1}^2 + b_{j-1}^2\right)^2 \sum_{i=1}^{j-1} \int \widetilde{P_i} w &= O(e^{-2\gamma t}) + o\left(\int \left(w^2 + w_x^2\right)\Phi_j\right) + F_j(t) + 2\left(b_{j-1}^2 - a_{j-1}^2\right) E_j(t) \\
&\quad - \sum_{i=1}^{j-1} F[P_i] - 2\left(b_{j-1}^2 - a_{j-1}^2\right) \sum_{i=1}^{j-1} E[P_i] \\
&\quad - \int \left[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w^2 \widetilde{P}_x^2 - 10 \widetilde{P} w \widetilde{P}_x w_x - \frac{5}{2} \widetilde{P}^2 w_x^2 + \frac{15}{4} \widetilde{P}^4 w^2\right] \Phi_j \\
&\quad - 2\left(b_{j-1}^2 - a_{j-1}^2\right) \int \left[\frac{1}{2} w_x^2 - \frac{3}{2} \widetilde{P}^2 w^2\right] \Phi_j \\
&= O(e^{-2\gamma t}) + o\left(\int \left(w^2 + w_x^2\right)\Phi_j\right) \\
&\quad + F_j(t) + \omega_6 M_j(t) - \sum_{i=1}^{j-1} F[P_i] - \omega_6 \sum_{i=1}^{j-1} M[P_i] \\
&\quad + 2\left(b_{j-1}^2 - a_{j-1}^2\right)\left[E_j(t) + \omega_2 M_j(t) - \sum_{i=1}^{j-1} E[P_i] - \omega_2 \sum_{i=1}^{j-1} M[P_i]\right] \\
&\quad + \left(\omega_6 + 2\omega_2\left(b_{j-1}^2 - a_{j-1}^2\right)\right)\left(\sum_{i=1}^{j-1} M[P_i] - M_j(t)\right) \\
&\quad - \int \left[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w^2 \widetilde{P}_x^2 - 10 \widetilde{P} w \widetilde{P}_x w_x - \frac{5}{2} \widetilde{P}^2 w_x^2 + \frac{15}{4} \widetilde{P}^4 w^2\right] \Phi_j \\
&\quad - 2\left(b_{j-1}^2 - a_{j-1}^2\right) \int \left[\frac{1}{2} w_x^2 - \frac{3}{2} \widetilde{P}^2 w^2\right] \Phi_j
\end{aligned}$$

$$\leqslant Ce^{-2\gamma t} + C\int \left(w^2 + w_x^2\right)\Phi_j$$

$$- \left(\omega_6 + 2\omega_2\left(b_{j-1}^2 - a_{j-1}^2\right)\right)\left(\sum_{i=1}^{j-1}\int \widetilde{P_i}w + \frac{1}{2}\int w^2\Phi_j\right), \tag{350}$$

and therefore, for ω_2 and ω_6 small enough,

$$-\sum_{i=1}^{j-1}\int \widetilde{P_i}w \leqslant Ce^{-2\gamma t} + C\int \left(w^2 + w_x^2\right)\Phi_j. \tag{351}$$

We thus deduce the following bound:

$$\left|\int \widetilde{P_{j-1}}w\sqrt{\Phi_j}\right| \leqslant Ce^{-2\gamma t} + C\int \left(w^2 + w_x^2\right)\Phi_j. \tag{352}$$

Because $\|w(t)\|_{H^2} \to_{t\to+\infty} 0$, we deduce that:

$$\left(\int \widetilde{P_{j-1}}w\sqrt{\Phi_j}\right)^2 = o(e^{-2\gamma t}) + o\left(\int \left(w^2 + w_x^2\right)\Phi_j\right). \tag{353}$$

Step 8. Conclusion. From (348) and (353), we deduce for $t \geqslant T_0$,

$$\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j = O(e^{-2\gamma t}) + o\left(y_1(t)^2\right) + o\left(\int \left(w^2 + w_x^2\right)\Phi_j\right). \tag{354}$$

This means that if we take T_0 large enough, we have:

$$\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j = o\left(y_1(t)^2\right) + O(e^{-2\gamma t}), \tag{355}$$

where the term $o\left(y_1(t)^2\right)$ is present only if P_{j-1} is a soliton.

Before finishing the proof, we need to find a better bound for $y_1(t)$ than just a convergence to 0 given by the modulation (in the case when P_{j-1} is a soliton). For this, we study $M_j(t)$:

$$\begin{aligned}
M_j(t) &= \frac{1}{2}\int u^2(t)\Phi_j(t)\\
&= \frac{1}{2}\int \left(\widetilde{P}(t) + w(t)\right)^2\Phi_j(t)\\
&= \frac{1}{2}\int \widetilde{P}(t)^2\Phi_j(t) + \int \widetilde{P}(t)w(t)\Phi_j(t) + \frac{1}{2}\int w(t)^2\Phi_j(t)\\
&= \frac{1}{2}\sum_{i=1}^{j-1}\int \widetilde{P_i}(t)^2 + \sum_{i=1}^{j-1}\int \widetilde{P_i}(t)w(t) + O\left(e^{-2\gamma t}\right) + \frac{1}{2}\int w(t)^2\Phi_j(t)\\
&= \frac{1}{2}\int \widetilde{P_{j-1}}(t)^2 + \int \widetilde{P_{j-1}}(t)w(t) + O\left(e^{-2\gamma t}\right) + \frac{1}{2}\int w(t)^2\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2,
\end{aligned} \tag{356}$$

from the induction assumption, then

$$M_j(t) = \frac{1}{2}\int \widetilde{P_{j-1}}(t)^2 + O\left(e^{-2\gamma t}\right) + \frac{1}{2}\int w(t)^2\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2, \tag{357}$$

from the orthogonality condition obtained in the modulation. Therefore,

$$M_j(t) = \left(b_{j-1}^2 + y_1(t)\right)^{1/2} M[q] + O\left(e^{-2\gamma t}\right) + \frac{1}{2}\int w(t)^2\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2. \tag{358}$$

Now, if we take $t_1 \geqslant t$, we obtain from (355),

$$M_j(t_1) - M_j(t) = \left[\left(b_{j-1}^2 + y_1(t_1)\right)^{1/2} - \left(b_{j-1}^2 + y_1(t)\right)^{1/2}\right] M[q] + O\left(e^{-2\gamma t}\right) + o\left(y_1(t)^2\right) + o\left(y_1(t_1)^2\right). \tag{359}$$

By doing a Taylor expansion of order 1, as in the existence part, we obtain:

$$\left(b_{j-1}^2 + y_1(t_1)\right)^{1/2} = b_{j-1}\left(1 + \frac{1}{2}\frac{y_1(t_1)}{b_{j-1}^2} + O\left(y_1(t_1)^2\right)\right). \tag{360}$$

Therefore,

$$\left(b_{j-1}^2 + y_1(t_1)\right)^{1/2} - \left(b_{j-1}^2 + y_1(t)\right)^{1/2} = \frac{1}{2b_{j-1}}\left(y_1(t_1) - y_1(t)\right) + O\left(y_1(t_1)^2\right) + O\left(y_1(t)^2\right). \tag{361}$$

Now, we recall that when $t_1 \to +\infty$, we have $y_1(t_1) \to 0$. Therefore, by taking the limit of the previous formula when $t_1 \to +\infty$, we obtain:

$$b_{j-1} - \left(b_{j-1}^2 + y_1(t)\right)^{1/2} = -\frac{y_1(t)}{2b_{j-1}} + O\left(y_1(t)^2\right). \tag{362}$$

Therefore, from (359), with $t_1 \to +\infty$,

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) = -\frac{y_1(t)}{2b_{j-1}} M[q] + O\left(e^{-2\gamma t}\right) + O\left(y_1(t)^2\right). \tag{363}$$

The second step is to study $E_j(t)$ (and to do the same reasonning as for M_j):

$$\begin{aligned} E_j(t) &= \int \left[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\right]\Phi_j(t) \\ &= \int \left[\frac{1}{2}\widetilde{P}_x^2 - \frac{1}{4}\widetilde{P}^4\right]\Phi_j(t) + \int \left[\widetilde{P}_x w_x - \widetilde{P}^3 w\right]\Phi_j(t) + O\left(\int w^2\Phi_j(t)\right), \end{aligned} \tag{364}$$

and after simplications by Φ_j due to exponential convergences, induction assumption and orthogonality conditions,

$$\begin{aligned} E_j(t) &= E[\widetilde{P_{j-1}}(t)] + \sum_{i=1}^{j-2} E[P_i] + O\left(e^{-2\gamma t}\right) + O\left(\int w^2\Phi_j(t)\right) \\ &= \left(b_{j-1}^2 + y_1(t)\right)^{3/2} E[q] + \sum_{i=1}^{j-2} E[P_i] + O\left(e^{-2\gamma t}\right) + O\left(\int w^2\Phi_j(t)\right) \\ &= \left(b_{j-1}^2 + y_1(t)\right)^{3/2} E[q] + \sum_{i=1}^{j-2} E[P_i] + O\left(e^{-2\gamma t}\right) + o\left(y_1(t)^2\right), \end{aligned} \tag{365}$$

from (355). And then, by taking the difference for $t_1 \geqslant t$,

$$E_j(t_1) - E_j(t) = \left[\left(b_{j-1}^2 + y_1(t_1)\right)^{3/2} - \left(b_{j-1}^2 + y_1(t)\right)^{3/2}\right] E[q] + O\left(e^{-2\gamma t}\right) + o\left(y_1(t_1)^2\right) + o\left(y_1(t)^2\right). \tag{366}$$

By taking a Taylor expansion of order 1, we obtain:

$$\left(b_{j-1}^2 + y_1(t_1)\right)^{3/2} = b_{j-1}^3\left(1 + \frac{3}{2}\frac{y_1(t_1)}{b_{j-1}^2} + O\left(y_1(t_1)^2\right)\right). \tag{367}$$

Therefore, after taking $t_1 \to +\infty$, we obtain:

$$\sum_{i=1}^{j-1} E[P_i] - E_j(t) = -\frac{3}{2} b_{j-1} y_1(t) E[q] + O\left(e^{-2\gamma t}\right) + O\left(y_1(t)^2\right). \tag{368}$$

That is why, from (363), (368) and Lemma 44, we obtain

$$-\frac{y_1(t)}{2b_{j-1}} M[q] + O\left(e^{-2\gamma t}\right) + O\left(y_1(t)^2\right) \geqslant -Ce^{-2\gamma t}, \tag{369}$$

and

$$-\frac{3}{2} b_{j-1} y_1(t) E[q] + O\left(e^{-2\gamma t}\right) + O\left(y_1(t)^2\right) \geqslant -Ce^{-2\gamma t}. \tag{370}$$

Because $M[q] = 2$ and $E[q] = -\frac{2}{3}$, we rewrite both previous inequalities (369) and (370) in the following way (and we pass $O\left(e^{-2\gamma t}\right)$ on the other side of each inequality):

$$-\frac{y_1(t)}{b_{j-1}} + O\left(y_1(t)^2\right) \geqslant -Ce^{-2\gamma t}, \tag{371}$$

and

$$b_{j-1}y_1(t) + O\left(y_1(t)^2\right) \geqslant -Ce^{-2\gamma t}. \tag{372}$$

Because $y_1(t) \to +\infty$, by taking T_0 larger if needed, $O\left(y_1(t)^2\right)$ can be bounded above by any positive constant multiplied by $|y_1(t)|$, so by taking this constant small enough (by taking T_0 large enough) and combining both previous inequalities (371) and (372), we obtain:

$$|y_1(t)| \leqslant Ce^{-2\gamma t}. \tag{373}$$

Therefore, we have obtained a better bound for $y_1(t)$ in the case when P_{j-1} is a soliton. Therefore, we may conclude that in any case, for $t \geqslant T_0$ for T_0 large enough:

$$\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j(t) = O\left(e^{-2\gamma t}\right). \tag{374}$$

Then, we deduce from (324) that:

$$|y_1'(t)| + |y_2'(t)| = O(e^{-\gamma t}), \tag{375}$$

in the case when $\widetilde{P_{j-1}}$ is a breather, and

$$\left|y_1'(t)\right| + \left|y_2'(t) + y_1'(t)t\right| = O\left(e^{-\gamma t}\right), \tag{376}$$

in the case when $\widetilde{P_{j-1}}$ is a soliton. In this case, we have also $|y_1(t)| = O\left(e^{-\gamma t}\right)$.

Because $|y_1(t)| + |y_2(t)| \to_{t\to+\infty} 0$, we obtain by integration in the case when $\widetilde{P_{j-1}}$ is a breather:

$$|y_1(t)| + |y_2(t)| = O(e^{-\gamma t}). \tag{377}$$

Because $|y_1(t)| + |y_2(t) + y_1(t)t| \to_{t\to+\infty} 0$, we obtain by integration in the case when $\widetilde{P_{j-1}}$ is a soliton:

$$|y_1(t)| + |y_2(t) + y_1(t)t| = O\left(e^{-\gamma t}\right). \tag{378}$$

And, so, by the mean-value theorem,

$$\left\|\widetilde{P_{j-1}} - P_{j-1}\right\|_{H^2} \leqslant C\left(|y_1(t)| + |y_2(t)|\right) \leqslant Ce^{-\gamma t}, \tag{379}$$

when P_{j-1} is a breather, and

$$\left\|\widetilde{P_{j-1}} - P_{j-1}\right\|_{H^2} \leqslant C\left(|y_1(t)| + |y_2(t) + y_1(t)t|\right) \leqslant Ce^{-\gamma t}, \tag{380}$$

when P_{j-1} is a soliton.

From $z = w + \widetilde{P_{j-1}} - P_{j-1}$, we deduce:

$$\begin{aligned}\int \left(z^2 + z_x^2 + z_{xx}^2\right)\Phi_j &\leqslant C\int \left(w^2 + w_x^2 + w_{xx}^2\right)\Phi_j \\ &\quad + C\int \left[\left(\widetilde{P_{j-1}} - P_{j-1}\right)^2 + \left(\widetilde{P_{j-1}} - P_{j-1}\right)_x^2 + \left(\widetilde{P_{j-1}} - P_{j-1}\right)_{xx}^2\right]\Phi_j \\ &\leqslant Ce^{-2\gamma t},\end{aligned} \tag{381}$$

and this finishes the induction. □

4.3. **Proof of Theorem 4.**

Proof of Theorem 4. We suppose all the velocities of our objects positive, i.e. we suppose that $v_1 > 0$. Let p be the associated multi-breather given by Theorem 2. Let u be a solution of (mKdV) such that

$$\|u(t) - p(t)\|_{H^2} \to_{t\to+\infty} 0. \tag{382}$$

From Proposition 43, we deduce that there exists a constant $C > 0$ and a constant $\gamma > 0$ such that for t large enough

$$\|u(t) - p(t)\|_{H^2} \leq Ce^{-\gamma t}. \tag{383}$$

This implies that u satisfies the assumptions of Proposition 5. Thus, $u = p$ and Theorem 4 is proved.

□

5. Appendix

The first two subsections of the Appendix show that a soliton has similar properties as a "limit breather" of parameter $\alpha = 0$. Firstly, the corresponding elliptic equation is satisfied by a soliton. Secondly, the corresponding quadratic form is coercive for a soliton, and we see that it's kernel is spanned by $\partial_x Q$ and $\partial_c Q$. In the third subsection, we prove that it is possible for ϵ to be orthogonal to Q and $\partial_x Q$ (instead of $\partial_x Q$ and $\partial_c Q$) in order to satisfy coercivity for the quadratic form. We will use this fact for the proof of the existence, as well as for the first part of the proof of the uniqueness. In the fourth subsection, we prove that we can have coercivity for quadratic forms when the orthogonality condition is not exactly satisfied. We will use this result for the proof of the uniqueness. The last subsection is about computations for the third conservation law. It will be useful for the monotonicity property for localized F that we need in the proof of the uniqueness.

5.1. **Elliptic equation satisfied by a soliton.**

Lemma 49. *A soliton $Q = R_{c,\kappa}$ satisfies for any time $t \in \mathbb{R}$, the following nonlinear elliptic equation:*

$$Q_{(4x)} - 2c(Q_{xx} + Q^3) + c^2 Q + 5QQ_x^2 + 5Q^2 Q_{xx} + \frac{3}{2}Q^5 = 0. \tag{384}$$

Proof. In order to derive this equation, we will use the equation that defines a soliton (and that is satisfied by Q at any time):

$$Q_{xx} = cQ - Q^3. \tag{385}$$

We will also need the following equation:

$$Q_x^2 = cQ^2 - \frac{1}{2}Q^4, \tag{386}$$

that can be derived by taking the space derivative of $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$, and by showing that this derivative is zero. From this, we deduce that $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$ is constant, and by taking its limit when $x \to \pm\infty$, we see that this constant is zero. More precisely, the derivative of $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$ is:

$$2Q_xQ_{xx} - 2cQQ_x + 2Q^3Q_x = 2Q_x\left(Q_{xx} - cQ + Q^3\right) = 0. \tag{387}$$

From now on, the derivation of (384) is straight forward. It is sufficient to take space derivatives of $Q_{xx} = cQ - Q^3$ and to inject them into the right hand side of the equation (384), which we want to prove to be equal to zero. By doing this, we make the maximal order of a derivative of Q present in the right hand side equation lower. At the end, we have only, zero and first order derivatives. To have only a polynomial in Q, we have to use $Q_x^2 = cQ^2 - \frac{1}{2}Q^4$, and the calculations show that this polynomial is zero. □

5.2. **Study of coercivity of the quadratic form associated to a soliton.** In this article, we adapt the argument for the breathers in [5] to the soliton case. We consider:

$$\begin{aligned} Q_c^s[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ + c\left(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\right) + c^2\frac{1}{2}\int \epsilon^2 =: Q_{0,\sqrt{c}}[\epsilon]. \end{aligned} \tag{388}$$

Firstly, we prove, by simple calculations, as in the previous section, that Q_x and $Q + xQ_x$ are in the kernel of this quadratic form. It is easy to see, by asymptotic study that these two functions are linearly independent.

The self-adjoint linear operator associated to this quadratic form is:

$$\begin{aligned} \mathcal{L}_c^s[\epsilon] := \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon + 5Q^2\epsilon_{xx} + 10QQ_x\epsilon_x \\ + \left(5Q_x^2 + 10QQ_{xx} + \frac{15}{2}Q^4 - 6cQ^2\right)\epsilon, \end{aligned} \tag{389}$$

so that $Q_c^s[\epsilon] = \int \epsilon\mathcal{L}_c^s[\epsilon]$.

$\mathcal{L}_c^s$ is a compact perturbation of the constant coefficients operator:

$$\mathcal{M}[\epsilon] := \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon. \tag{390}$$

A direct analysis involving ODE shows that the null space of $\mathcal{M}$ is spawned by four linearly independent functions:

$$e^{\pm\sqrt{c}x}, \quad xe^{\pm\sqrt{c}x}. \tag{391}$$

Among these four functions, there are only two L^2-integrable ones in the semi-infinite line $[0, +\infty)$. Therefore, the null space of $\mathcal{L}_c^s\,|_{H^4(\mathbb{R})}$ is spanned by at most two L^2-functions.

Therefore,

$$\ker(\mathcal{L}_c^s) = Span(\partial_x Q, Q + x\partial_x Q). \tag{392}$$

Lemma 50. *The operator $\mathcal{L}_c^s$ does not have any negative eigenvalue.*

Proof. $\mathcal{L}_c^s$ has

$$\sum_{x\in\mathbb{R}} \dim\ker W[Q_x, Q + xQ_x](t, x) \tag{393}$$

negative eigenvalues, counting multiplicity, where W is the Wronskian matrix:

$$W[Q_x, Q + xQ_x](t, x) := \begin{bmatrix} Q_x & Q + xQ_x \\ Q_{xx} & (Q + xQ_x)_x \end{bmatrix}. \tag{394}$$

For this result, see [20], where the finite interval case was considered. As shown in several articles [24, 28], the extension to the real line is direct.

Thus, it is sufficient to see that $\det W[Q_x, Q + xQ_x](t, x)$ is never zero. For this, let us simply calculate this determinant:

$$\begin{aligned} Q_x(2Q_x + xQ_{xx}) - (Q + xQ_x)Q_{xx} &= 2Q_x^2 - QQ_{xx} \\ &= 2cQ^2 - Q^4 - Q(cQ - Q^3) \\ &= cQ^2 > 0. \end{aligned} \tag{395}$$

□

5.3. **Coercivity of the quadratic form associated to a soliton.** For $Q = R_{c,\kappa}$, let

$$\begin{aligned} Q_c^s[\epsilon] : = \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ + c\left(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\right) + c^2\frac{1}{2}\int \epsilon^2. \end{aligned} \tag{396}$$

Lemma 51. *There exists $\mu_c > 0$ such that for any $\epsilon \in H^2$ such that $\int \epsilon Q = \int \epsilon Q_x = 0$, we have*

$$Q_c^s[\epsilon] \geqslant \mu_c\|\epsilon\|_{H^2}^2. \tag{397}$$

Proof. From Section 6.2, we know that if $\int \epsilon \partial_x Q = \int \epsilon \partial_c Q = 0$, then, for a constant $\nu_c > 0$, we have

$$\mathcal{Q}_c^s[\epsilon] \geqslant \nu_c \|\epsilon\|_{H^2}^2. \tag{398}$$

Let $\epsilon \in H^2$ be such that $\int \epsilon Q = \int \epsilon \partial_x Q = 0$. There exists $a \in \mathbb{R}$ and $\epsilon_\perp \in \mathrm{Span}(\partial_x Q, \partial_c Q)^\perp$ such that

$$\epsilon = a\partial_c Q + \epsilon_\perp. \tag{399}$$

From $\int \epsilon Q = 0$, we have that

$$a \int \partial_c Q \cdot Q + \int \epsilon_\perp Q = 0, \tag{400}$$

thus

$$\frac{a}{2} \int Q^2 + \int \epsilon_\perp Q = 0, \tag{401}$$

which allows us to derive

$$a = -2 \frac{\int \epsilon_\perp Q}{\int Q^2}. \tag{402}$$

Because $\partial_c Q$ is in the kernel of $\mathcal{Q}_c^s$, we have

$$\mathcal{Q}_c^s[\epsilon] = \mathcal{Q}_c^s[\epsilon_\perp] \geqslant \nu_c \|\epsilon_\perp\|_{H^2}^2. \tag{403}$$

Now, from

$$\epsilon = -2 \frac{\int \epsilon_\perp Q}{\int Q^2} \partial_c Q + \epsilon_\perp, \tag{404}$$

we have by trinagular and Cauchy-Schwarz inequality that

$$\begin{aligned} \|\epsilon\|_{H^2} &\leqslant \|\epsilon_\perp\|_{H^2} + 2 \frac{\left|\int \epsilon_\perp Q\right|}{\|Q\|_{L^2}^2} \|\partial_c Q\|_{H^2} \\ &\leqslant \|\epsilon_\perp\|_{H^2} + 2 \frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}} \|\epsilon_\perp\|_{L^2} \\ &\leqslant \left(1 + 2 \frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}}\right) \|\epsilon_\perp\|_{H^2}. \end{aligned} \tag{405}$$

Therefore, we may derive a constant μ_c (independent on ϵ) such that

$$\mathcal{Q}_c^s[\epsilon] \geqslant \mu_c \|\epsilon\|_{H^2}^2. \tag{406}$$

□

5.4. **Coercivity with almost orthogonality conditions (to be used for the uniqueness).** For $B := B_{\alpha,\beta}$ or any of its translations, we define the canonical quadratix form associated to B:

$$\begin{aligned} \mathcal{Q}_{\alpha,\beta}^b[\epsilon] := &\frac{1}{2} \int \epsilon_{xx}^2 - \frac{5}{2} \int B^2 \epsilon_x^2 + \frac{5}{2} \int B_x^2 \epsilon^2 + 5 \int B B_{xx} \epsilon^2 + \frac{15}{4} \int B^4 \epsilon^2 \\ &+ \left(\beta^2 - \alpha^2\right) \left(\int \epsilon_x^2 - 3 \int B^2 \epsilon^2 \right) + \left(\alpha^2 + \beta^2\right)^2 \frac{1}{2} \int \epsilon^2, \end{aligned} \tag{407}$$

and we know that $\partial_{x_1} B$ and $\partial_{x_2} B$ is the kernel of $\mathcal{Q}_{\alpha,\beta}^b$. More precisely, there exists $\mu_{\alpha,\beta}^b > 0$ such that if ϵ is orthogonal to $\partial_{x_1} B$ and $\partial_{x_2} B$, we have

$$\mathcal{Q}_{\alpha,\beta}^b[\epsilon] \geqslant \mu_{\alpha,\beta}^b \|\epsilon\|_{H^2}^2 - \frac{1}{\mu_{\alpha,\beta}^b} \left(\int \epsilon B \right)^2. \tag{408}$$

We would like to prove the following lemma (adapted from the Appendix A of [34]):

Lemma 52. *There exists $\nu := \nu^b_{\alpha,\beta} > 0$ such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\left|\int (\partial_{x_1} B_{\alpha,\beta})\epsilon\right| + \left|\int (\partial_{x_2} B_{\alpha,\beta})\epsilon\right| < \nu\|\epsilon\|_{H^2}, \tag{409}$$

then

$$Q^b_{\alpha,\beta}[\epsilon] \geqslant \frac{\mu^b_{\alpha,\beta}}{4}\|\epsilon\|^2_{H^2} - \frac{4}{\mu^b_{\alpha,\beta}}\left(\int \epsilon B_{\alpha,\beta}\right)^2, \tag{410}$$

where $B_{\alpha,\beta}$ denotes the breather of parameters α, β or any of its translations (in space or in time).

Proof. Take $\nu > 0$ (we will find a condition on ν later in the proof) and take ϵ satisfying the assumption of the lemma. Then (by denoting $B = B_{\alpha,\beta}$)

$$\epsilon = \epsilon_1 + aB_1 + bB_2 = \epsilon_1 + \epsilon_2, \tag{411}$$

where $\int \epsilon_1 B_1 = \int \epsilon_1 B_2 = \int \epsilon_1\epsilon_2 = 0$.

By performing a L^2-scalar product of (411) with B_1 and B_2, we obtain, by assumption,

$$\left|a\int B_1^2 + b\int B_1B_2\right| \leqslant \nu\|\epsilon\|_{H^2}, \tag{412}$$

$$\left|a\int B_1B_2 + b\int B_2^2\right| \leqslant \nu\|\epsilon\|_{H^2}. \tag{413}$$

Therefore, by making linear combinations of these two inequalities, using triangular and Cauchy-Schwarz inequalities, we obtain

$$|a| + |b| \leqslant C\nu\|\epsilon\|_{H^2}. \tag{414}$$

We can take space derivatives of (411). And thus, we obtain, for ν small enough, , that will be useful in the last section of this article

$$\frac{1}{2}\|\epsilon\|^2_{H^2} \leqslant \|\epsilon_1\|^2_{H^2} \leqslant 2\|\epsilon\|^2_{H^2}. \tag{415}$$

Because $\int BB_1 = \int BB_2 = 0$,

$$\int \epsilon B = \int \epsilon_1 B. \tag{416}$$

By bilinearity,

$$\begin{aligned} Q^b_{\alpha,\beta}[\epsilon] &= Q^b_{\alpha,\beta}[\epsilon_1] + Q^b_{\alpha,\beta}[\epsilon_2] + \int \epsilon_{1,xx}\epsilon_{2,xx} - 5\int B^2\epsilon_{1,x}\epsilon_{2,x} + 5\int B_x^2\epsilon_1\epsilon_2 + 10\int BB_{xx}\epsilon_1\epsilon_2 \\ &+ \frac{15}{2}\int B^4\epsilon_1\epsilon_2 + (\beta^2-\alpha^2)\left(2\int \epsilon_{1,x}\epsilon_{2,x} - 6\int B^2\epsilon_1\epsilon_2\right) + (\alpha^2+\beta^2)^2\int \epsilon_1\epsilon_2 \end{aligned} \tag{417}$$

We know from the coercivity of $Q^b_{\alpha,\beta}$ that

$$\begin{aligned} Q^b_{\alpha,\beta}[\epsilon_1] &\geqslant \mu^b_{\alpha,\beta}\|\epsilon_1\|^2_{H^2} - \frac{1}{\mu^b_{\alpha,\beta}}\left(\int \epsilon_1 B\right)^2 \\ &\geqslant \frac{\mu^b_{\alpha,\beta}}{2}\|\epsilon\|^2_{H^2} - \frac{2}{\mu^b_{\alpha,\beta}}\left(\int \epsilon B\right)^2. \end{aligned} \tag{418}$$

Also, if we denote by $\mathcal{L}^b_{\alpha,\beta}$ the self-adjoint operator associated to the quadratic form $Q^b_{\alpha,\beta}$,

$$\begin{aligned} Q^b_{\alpha,\beta}[\epsilon_2] &= a^2Q^b_{\alpha,\beta}[B_1] + b^2Q^b_{\alpha,\beta}[B_2] + 2ab\int \mathcal{L}^b_{\alpha,\beta}[B_1]B_2 \\ &\leqslant C\nu^2\|\epsilon\|^2_{H^2}, \end{aligned} \tag{419}$$

actually in this case $Q^b_{\alpha,\beta}[\epsilon_2] = 0$, because ϵ_2 is in the kernel of $Q^b_{\alpha,\beta}$ (but, when we adapt this proof for solitons, we can only write the bound).

Now, we recall that $\int \epsilon_1\epsilon_2 = 0$, and study the other terms by using Cauchy-Schwarz:

$$\left| \int \epsilon_{1,xx}\epsilon_{2,xx} - 5\int B^2\epsilon_{1,x}\epsilon_{2,x} + 5\int B_x^2\epsilon_1\epsilon_2 + 10\int BB_{xx}\epsilon_1\epsilon_2 \right.$$
$$\left. + \frac{15}{2}\int B^4\epsilon_1\epsilon_2 + (\beta^2-\alpha^2)\left(2\int \epsilon_{1,x}\epsilon_{2,x} - 6\int B^2\epsilon_1\epsilon_2\right)\right| \leqslant C(|a|+|b|)\|\epsilon_1\|_{H^2} \leqslant C\nu\|\epsilon\|^2_{H^2(\mathbb{R})}. \tag{420}$$

We observe that if we take ν small enough, the claim of the lemma is proved. □

We prove in the same way that we have the similar lemma for solitons:

Lemma 53. *There exists $\nu := \nu_c^s > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\left|\int (\partial_c R_{c,\kappa})\,\epsilon\right| + \left|\int (\partial_x R_{c,\kappa})\epsilon\right| \leqslant \nu\|\epsilon\|_{H^2}, \tag{421}$$

then

$$Q_c^s[\epsilon] \geqslant \frac{\mu_c^s}{4}\|\epsilon\|^2_{H^2}, \tag{422}$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

And even,

Lemma 54. *There exists $\nu := \nu_c^s > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\left|\int R_{c,\kappa}\epsilon\right| + \left|\int (\partial_x R_{c,\kappa})\,\epsilon\right| \leqslant \nu\|\epsilon\|_{H^2}, \tag{423}$$

then

$$Q_c^s[\epsilon] \geqslant \frac{\mu_c^s}{4}\|\epsilon\|^2_{H^2}, \tag{424}$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

5.5. **Computations for the third localized integral (to be used for the uniqueness).**

Lemma 55. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 function that do not depend on time and u a solution of* (mKdV). *Then,*

$$\frac{d}{dt}\int \left(\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\right) f$$
$$= \int \left(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2u^2 + 15u_x^2uu_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 - \frac{45}{4}u^4u_x^2\right) f'$$
$$+ 5\int u^2u_xu_{xx}f'' + \frac{1}{2}\int u_{xx}^2 f'''. \tag{425}$$

Proof. We perform by doing integrations by parts when needed and basic calculations.

$$\frac{d}{dt}\int \left(\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\right) f = \int u_{txx}u_{xx}f - 5\int u_tuu_x^2f - 5\int u^2u_{tx}u_xf + \frac{3}{2}\int u_tu^5f$$
$$= -\int \left(u_{xx}+u^3\right)_{xxx} u_{xx}f + 5\int \left(u_{xx}+u^3\right)_x uu_x^2f + 5\int u^2\left(u_{xx}+u^3\right)_{xx} u_xf - \frac{3}{2}\int \left(u_{xx}+u^3\right)_x u^5f$$
$$= \int \left(u_{xx}+u^3\right)_{xx} u_{xxx}f + \int \left(u_{xx}+u^3\right)_{xx} u_{xx}f' + 5\int \left(u_{xx}+u^3\right)_x uu_x^2f$$
$$+ 5\int u^2\left(u_{xx}+u^3\right)_{xx} u_xf - \frac{3}{2}\int \left(u_{xx}+u^3\right)_x u^5f$$
$$= -\frac{1}{2}\int u_{xxx}^2f' + \int \left(u^3\right)_{xx} u_{xxx}f + \int \left(u_{xx}+u^3\right)_{xx} u_{xx}f' + 5\int u_{xxx}uu_x^2f + 5\int \left(u^3\right)_x uu_x^2f$$
$$+ 5\int u^2u_{xxxx}u_xf + 5\int u^2\left(u^3\right)_{xx} u_xf - \frac{3}{2}\int u_{xxx}u^5f - \frac{3}{2}\int \left(u^3\right)_x u^5f$$
$$= -\frac{1}{2}\int u_{xxx}^2f' + \int \left(u_{xx}+u^3\right)_{xx} u_{xx}f' + \int \left(3u_{xx}u^2 + 6u_x^2u\right) u_{xxx}f + 5\int u_{xxx}uu_x^2f$$

$$
\begin{aligned}
&+15\int u_x^3u^3f+5\int u^2u_{xxxx}u_xf+5\int u^2\left(3u_{xx}u^2+6u_x^2u\right)u_xf-\frac{3}{2}\int u_{xxx}u^5f-\frac{9}{2}\int u_xu^7f\\
&=-\frac{1}{2}\int u_{xxx}^2f'+\int\left(u_{xx}+u^3\right)_{xx}u_{xx}f'+3\int u^2u_{xx}u_{xxx}f+5\int u^2u_{xxxx}u_xf\\
&+11\int uu_x^2u_{xxx}f+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u_{xxx}u^5f+\frac{9}{16}\int u^8f'\\
&=-\frac{1}{2}\int u_{xxx}^2f'+\int\left(u_{xx}+u^3\right)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-2\int u^2u_{xx}u_{xxx}f\\
&+\int uu_x^2u_{xxx}f-5\int u^2u_xu_{xxx}f'+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u^5u_{xxx}f\\
&=-\frac{1}{2}\int u_{xxx}^2f'+\int\left(u_{xx}+u^3\right)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'-\int u^2\left(u_{xx}^2\right)_xf\\
&+\int uu_x^2u_{xxx}f+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u^5u_{xxx}f\\
&=-\frac{1}{2}\int u_{xxx}^2f'+\int\left(u_{xx}+u^3\right)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'+\int u^2u_{xx}^2f'+2\int uu_xu_{xx}^2f\\
&-\int u_x^3u_{xx}f-2\int uu_xu_{xx}^2f-\int uu_x^2u_{xx}f'+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\frac{3}{2}\int u^5u_{xxx}f\\
&=-\frac{1}{2}\int u_{xxx}^2f'+\int\left(u_{xx}+u^3\right)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'+\int u^2u_{xx}^2f'\\
&-\int uu_x^2u_{xx}f'-\frac{1}{4}\int\left(u_x^4\right)_xf+45\int u^3u_x^3f+\frac{45}{4}\int u^4\left(u_x^2\right)_xf+\frac{3}{2}\int u^5u_{xx}f'\\
&=-\frac{1}{2}\int u_{xxx}^2f'+\int\left(u_{xx}+u^3\right)_{xx}u_{xx}f'+\frac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'+\int u^2u_{xx}^2f'\\
&-\int uu_x^2u_{xx}f'+\frac{1}{4}\int u_x^4f'+\frac{3}{2}\int u^5u_{xx}f'+45\int u^3u_x^3f-45\int u^3u_x^3f-\frac{45}{4}\int u^4u_x^2f'\\
&=-\frac{3}{2}\int u_{xxx}^2f'-\int u_{xxx}u_{xx}f''+4\int u_{xx}^2u^2f'+5\int u_x^2uu_{xx}f'-5\int u^2u_xu_{xxx}f'\\
&+\frac{9}{16}\int u^8f'+\frac{1}{4}\int u_x^4f'+\frac{3}{2}\int u^5u_{xx}f'-\frac{45}{4}\int u^4u_x^2f'\\
&=-\frac{3}{2}\int u_{xxx}^2f'+9\int u_{xx}^2u^2f'+15\int u_x^2uu_{xx}f'+\frac{9}{16}\int u^8f'+\frac{1}{4}\int u_x^4f'\\
&+\frac{3}{2}\int u^5u_{xx}f'-\frac{45}{4}\int u^4u_x^2f'-\int u_{xxx}u_{xx}f''+5\int u^2u_xu_{xx}f''\\
&=\int\left(-\frac{3}{2}u_{xxx}^2+9u_{xx}^2u^2+15u_x^2uu_{xx}+\frac{9}{16}u^8+\frac{1}{4}u_x^4+\frac{3}{2}u_{xx}u^5-\frac{45}{4}u^4u_x^2\right)f'
\end{aligned}
$$

(426)

$$
+5\int u^2u_xu_{xx}f''+\frac{1}{2}\int u_{xx}^2f'''.
$$

which is exactly the desired expression. □

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