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Local boundary feedback stabilization of a fluid-structure interaction problem under Navier slip boundary conditions with time delay

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Abstract

We consider a fluid-structure interaction system composed by a three-dimensional viscous incompressible fluid and an elastic plate located on the upper part of the fluid boundary. The fluid motion is governed by the Navier-Stokes system whereas the structure displacement satisfies the damped plate equation. We consider here the Navier slip boundary conditions. The main result of this work is the feedback stabilization of the strong solutions of the corresponding system around a stationary state for any exponential decay rate by means of a time delayed control localized on the fixed fluid boundary. This work is the application of the recent general result that is obtained in [15] that relies on the Fattorini-Hautus criterion. Then, the main tool in this work is to show the unique continuation property of the associate solution to the adjoint system.

Keywords: Navier-Stokes system, damped beam equation, strong solutions, stabilization, time delay, Navier boundary conditions

2010 Mathematics Subject Classification. 35Q30, 76D05, 76D03, 74F10

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1 Introduction

We suppose that the fluid flow occupies a 3D periodic domain and an elastic plate that is disposed on the upper part of the fluid boundary. The force exerted by the fluid on the plate influences the deformation of the elastic structure and then the fluid domain depends on the plate displacement and hence on time. Consequently, the

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interaction between the deformable structure and the fluid is modeled by a strongly coupled non linear system set in a moving domain. Our aim is to stabilize the position and the velocity of the structure as well as the velocity of the fluid around a stationary state using a time delayed control that acts on a local part of the boundary of the fluid domain provided that the initial data are close enough to the stationary state in some norm. First, we give some important notations in order to write the system.

Let ω be the rectangular torus

$$\omega = (\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_2\mathbb{Z}) \quad L_1 > 0, L_2 > 0.$$

For any function $\eta : \omega \rightarrow (-1, \infty)$, we define

$$\Omega(\eta) = \{(x_1, x_2, x_3) \in \omega \times \mathbb{R} ; 0 < x_3 < 1 + \eta(x_1, x_2)\},$$

$$\Gamma(\eta) = \{(x_1, x_2, x_3) \in \omega \times \mathbb{R} ; x_3 = 1 + \eta(x_1, x_2)\},$$

$$\Gamma_0 = \omega \times \{0\}.$$

(see the figure 1). In particular, we have

$$\partial\Omega(\eta) = \Gamma(\eta) \cup \Gamma_0.$$

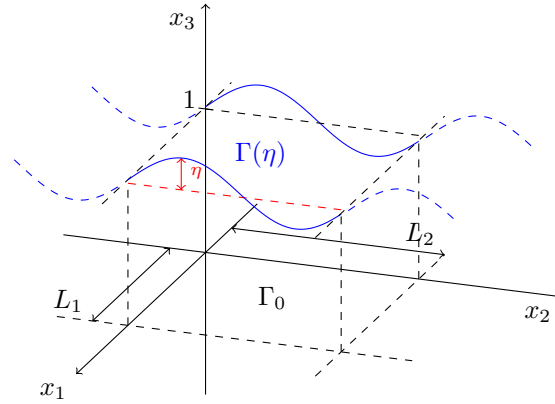


Figure 1: Configuration of the domain corresponding to the plate displacement η .

We consider the following system describing the evolution of the fluid that is governed by the Navier-Stokes equations and the displacement of the elastic plate by the damped beam equation

$$\begin{cases} \partial_t U + (U \cdot \nabla)U - \nabla \cdot \mathbb{T}(U, P) = f^S & \text{in } (0, \infty) \times \Omega(\eta), \\ \nabla \cdot U = 0 & \text{in } (0, \infty) \times \Omega(\eta), \\ \partial_{tt}\eta + \alpha\Delta^2\eta - \delta\Delta\partial_t\eta = \tilde{\mathbb{H}}_\eta(U, P) + h^S & \text{in } (0, \infty) \times \omega. \end{cases} \quad (1.1)$$

In (1.1), we have denoted by U the fluid velocity, P the fluid pressure and η the transversal displacement of the elastic structure. The functions f^S and h^S are time independent data.

The Cauchy stress tensor $\mathbb{T}(U, P)$ is given by

$$\mathbb{T}(U, P) = -PI_3 + 2\nu D(U), \quad D(U)_{i,j} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right).$$

The coefficients α , δ and ν correspond respectively to the rigidity, the damping on the structure and the fluid viscosity.

We denote by \tilde{n} the unit exterior normal vector on $\partial\Omega(\eta)$ given as follows:

$$\tilde{n} = -e_3 \text{ on } \Gamma_0,$$

and on $\Gamma(\eta)$:

$$\tilde{n}(s, 1 + \eta(s)) = \frac{\tilde{N}(s, 1 + \eta(s))}{|\tilde{N}(s, 1 + \eta(s))|}, \quad \text{where} \quad \tilde{N}(s, 1 + \eta(s)) = \begin{pmatrix} -\partial_{s_1}\eta(s) \\ -\partial_{s_2}\eta(s) \\ 1 \end{pmatrix}, \quad s \in \omega.$$

Here and in what follows, $|\cdot|$ stands for the Euclidean norm of \mathbb{R}^k , $k \geq 1$. Also, we select two tangent vectors $\tilde{\tau}^i$, $i = 1, 2$ linearly independent on $\partial\Omega(\eta)$ such that

$$\tilde{\tau}^i = e_i, \quad i = 1, 2 \text{ on } \Gamma_0, \quad (1.2)$$

and

$$\tilde{\tau}^1(s, 1 + \eta(s)) = \begin{pmatrix} 1 \\ 0 \\ \partial_{s_1}\eta(s) \end{pmatrix}, \quad \tilde{\tau}^2(s, 1 + \eta(s)) = \begin{pmatrix} 0 \\ 1 \\ \partial_{s_2}\eta(s) \end{pmatrix} \quad \text{on } \Gamma(\eta). \quad (1.3)$$

The function $\tilde{\mathbb{H}}_\eta$ is the contact force exerted by the fluid on the interface which is defined by

$$\tilde{\mathbb{H}}_\eta(U, P) = -\sqrt{1 + |\nabla\eta|^2} (\mathbb{T}(U, P)\tilde{n} \cdot e_3).$$

We complete (1.1) by the Navier slip boundary conditions. Let $a \in \mathbb{R}^3$, we denote by $a_{\tilde{n}}$ and $a_{\tilde{\tau}}$ the normal and the tangential components of a :

$$a_{\tilde{n}} = (a \cdot \tilde{n})\tilde{n}, \quad a_{\tilde{\tau}} = a - a_{\tilde{n}} = -\tilde{n} \times (\tilde{n} \times a). \quad (1.4)$$

The Navier slip boundary conditions write as follows:

$$\begin{cases} U_{\tilde{n}} = (\mathbb{M}v)_{\tilde{n}} & \text{on } (0, \infty) \times \Gamma_0, \\ [2\nu D(U)\tilde{n} + \beta_1 U]_{\tilde{\tau}} = \beta_1 (\mathbb{M}v)_{\tilde{\tau}} & \text{on } (0, \infty) \times \Gamma_0, \\ (U - \partial_t \eta e_3)_{\tilde{n}} = 0 & \text{on } (0, \infty) \times \Gamma(\eta), \\ [2\nu D(U)\tilde{n} + \beta_2 (U - \partial_t \eta e_3)]_{\tilde{\tau}} = 0 & \text{on } (0, \infty) \times \Gamma(\eta). \end{cases} \quad (1.5)$$

Here, v is the control of the system (1.1), (1.5) acting on the fixed boundary Γ_0 . In order to preserve the compatibility condition due to the incompressibility of the fluid, we use the operator \mathbb{M} defined by

$$\mathbb{M}v = mv - \left(\int_{\Gamma_0} mv \cdot \tilde{n} \, d\Gamma \right) m\tilde{n}, \quad (1.6)$$

where $m \in C^2(\Gamma_0)$ is compactly supported in Γ_0 which satisfies $\int_{\Gamma_0} m \, ds = 1$, see for example [4], [24].

Thus, the operator \mathbb{M} localizes the action of the control in a relatively compact subset of Γ_0 . We notice that $\mathbb{M} \in \mathcal{L}([L^2(\Gamma_0)]^3)$ and

$$\int_{\Gamma_0} \mathbb{M}v \cdot \tilde{n} \, d\Gamma = 0. \quad (1.7)$$

We assume that the friction coefficients β_1 and β_2 are two non negative constants

$$\beta_1 \geq 0, \quad \beta_2 \geq 0.$$

Since ω is a rectangular torus, we complement the system (1.1), (1.5) with periodic boundary conditions for the fluid and for the elastic plate on the remaining boundaries of $\Omega(\eta)$. Then, we consider data and solutions which are periodic in the both directions e_1 and e_2 , for example :

$$\begin{aligned} U(t, x_1 + L_1, x_2, x_3) &= U(t, x_1, x_2, x_3), & U(t, x_1, x_2 + L_2, x_3) &= U(t, x_1, x_2, x_3), \\ \eta(t, s_1 + L_1, s_2) &= \eta(t, s_1, s_2), & \eta(t, s_1, s_2 + L_2) &= \eta(t, s_1, s_2). \end{aligned}$$

Since U is divergence free and taking into account the equation (1.7), the velocity $\partial_t \eta$ should satisfy the condition

$$\int_{\omega} \partial_t \eta \, ds = 0. \quad (1.8)$$

Integrating the plate equation on ω and using (1.8) and the fact that η is periodic, we find

$$\int_{\omega} \tilde{\mathbb{H}}_{\eta}(U, P) + h^S \, ds = 0. \quad (1.9)$$

The above condition can be satisfied if we write

$$P = P_0 + c, \quad (1.10)$$

where P_0 satisfies

$$\int_{\Omega(\eta)} P_0 \, dx = 0,$$

and c is a constant that is chosen conveniently in such a way (1.9) is verified. Consequently, the pressure P will be uniquely defined. In order to impose the condition (1.9) without considering the normalizing constant c , we define the projection operator M on $L_0^2(\omega)$ where

$$L_0^2(\omega) = \left\{ \eta \in L^2(\omega) ; \int_{\omega} \eta \, ds = 0 \right\}. \quad (1.11)$$

Since we look for solutions such that (1.8) is verified, it is convenient to consider the restriction of the associated semigroup to the plate equation to $L_0^2(\omega)$. Then, in what follows, we assume that $\eta \in L_0^2(\omega)$. To impose the condition (1.9), we substitute the plate equation by its projection on $L_0^2(\omega)$ using the operator M

$$\partial_{tt} \eta + \alpha \Delta^2 \eta - \delta \Delta \partial_t \eta = \mathbb{H}_{\eta}(U, P) + Mh^S,$$

where

$$\mathbb{H}_{\eta}(U, P) = M\tilde{\mathbb{H}}_{\eta}(U, P).$$

We complement the system (1.1), (1.5) with the following initial conditions

$$\begin{cases} U(0, \cdot) = U^0 & \text{in } \Omega(\eta^0), \\ \eta(0, \cdot) = \eta^0 & \text{in } \omega, \\ \partial_t \eta(0, \cdot) = \eta^1 & \text{in } \omega. \end{cases} \quad (1.12)$$

Let (w^S, p^S, η^S) be a stationary state of the system (1.1), (1.5) that is a solution of the system

$$\begin{cases} (w^S \cdot \nabla) w^S - \nabla \cdot \mathbb{T}(w^S, p^S) = f^S & \text{in } \Omega(\eta^S), \\ \nabla \cdot w^S = 0 & \text{in } \Omega(\eta^S), \\ \alpha \Delta^2 \eta^S = \mathbb{H}_{\eta^S}(w^S, p^S) + Mh^S & \text{in } \omega, \\ w_n^S = 0 & \text{on } \partial\Omega(\eta^S), \\ [2\nu D(w^S)n + \beta w^S]_{\tau} = 0 & \text{on } \partial\Omega(\eta^S), \end{cases} \quad (1.13)$$

where

$$\beta(y) = \begin{cases} \beta_1 & \text{if } y \in \Gamma_0, \\ \beta_2 & \text{if } y \in \Gamma(\eta^S). \end{cases}$$

The vector n stands for the unitary exterior normal vector and τ^i , $i = 1, 2$ designate two tangent vectors on $\partial\Omega(\eta^S)$:

$$n = -e_3, \quad \tau^i = e_i, \quad i = 1, 2 \text{ on } \Gamma_0, \quad (1.14)$$

and on $\Gamma(\eta^S)$:

$$n(s, 1 + \eta^S(s)) = \frac{N(s, 1 + \eta^S(s))}{|N(s, 1 + \eta^S(s))|} \quad \text{where} \quad N(s, 1 + \eta^S(s)) = \begin{pmatrix} -\partial_{s_1} \eta^S(s) \\ -\partial_{s_2} \eta^S(s) \\ 1 \end{pmatrix} \quad s \in \omega,$$

$$\tau^1(s, 1 + \eta^S(s)) = \begin{pmatrix} 1 \\ 0 \\ \partial_{s_1} \eta^S(s) \end{pmatrix}, \quad \tau^2(s, 1 + \eta^S(s)) = \begin{pmatrix} 0 \\ 1 \\ \partial_{s_2} \eta^S(s) \end{pmatrix}. \quad (1.15)$$

Our aim is to stabilize the solution (U, P, η) of (1.1), (1.5), (1.12) around the stationary state (w^S, p^S, η) that is a solution of (1.13) by means of a feedback boundary control $v(t)$ that appears in (1.5) depending on the state (U, η) . In practice, due to a calculus time issues, we include a time delay $t_0 > 0$ in the control and the main goal is to construct this feedback control in such a way it depends at time t on the values of the state $(U(t'), \eta(t'))$ for $t' \leq t - t_0$ and the associated strong solution (U, P, η) to the system (1.1), (1.5), (1.12) goes to (w^S, p^S, η^S) exponentially. This context of stabilization has an important interest in the control theory of PDE: The analysis of the effect of the time delay on the feedback stabilization of some partial differential equations is given for instance in [13] and [12]. Recently, in [11] a feedback control is constructed for a finite dimensional system with input delay and in [23] the authors manage to obtain a stabilizing feedback boundary control to the one-dimensional reaction-diffusion system considering a constant delay, their method relies essentially on the Arstein transform and the fact that the generator operator of the system is self-adjoint. We mention also the work [18] where the author considered the stabilization of a damped Euler-Bernoulli beam equation with a time delayed control, here the system is parabolic but it is not generated by a self-adjoint operator. In [15], the authors extend the theory developed in [23] and construct a finite dimensional feedback stabilizing control with input delay to a general class of parabolic systems. The challenge in the present work is to apply the theory of [15] to deduce the stabilization of the fluid-structure problem (1.1), (1.5) since it describes a parabolic system thanks to the damping term $-\delta \Delta \partial_t \eta$ that appears in the structure equation in (1.1).

In order to state the main result, we give some notations. As it is standard in the studies of fluid-structure interaction systems, one of the main difficulties lies in the fact that the spatial domain of the fluid is variable and unknown. Since the problem consists to compare the asymptotic behavior of the solution (U, P, η) to the stationary state (w^S, p^S, η^S) , we are led to introduce a change of variable to transform the functions (U, P) into (\tilde{u}, \tilde{p}) defined in the fixed domain $\Omega(\eta^S)$. More precisely, we set

$$X_{\eta^1, \eta^2} : \Omega(\eta^1) \longrightarrow \Omega(\eta^2), \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ y_2 \\ \frac{1 + \eta^2(y_1, y_2)}{1 + \eta^1(y_1, y_2)} y_3 \end{pmatrix}. \quad (1.16)$$

We consider the change of variables

$$X(t, \cdot) = X_{\eta^S, \eta(t, \cdot)}, \quad Y(t, \cdot) = X_{\eta(t, \cdot), \eta^S}, \quad (1.17)$$

and we set

$$\Omega = \Omega(\eta^S).$$

We define the new functions

$$\tilde{u}(t, y) = (\text{Cof } \nabla X(t, y))^* U(t, X(t, y)), \quad \tilde{p}(t, y) = P(t, X(t, y)) \quad (t \geq 0, y \in \Omega). \quad (1.18)$$

We observe that we have

$$\tilde{u}^0 = (\text{Cof } \nabla X(0, \cdot))^* U^0(X(0, \cdot)).$$

In what follows, we recall that n is the unitary exterior normal vector and τ^i , $i = 1, 2$ are two tangent vectors on $\partial\Omega$:

$$n = -e_3, \quad \tau^i = e_i, \quad i = 1, 2 \text{ on } \Gamma_0, \quad (1.19)$$

and on $\Gamma(\eta^S)$:

$$n(s, 1 + \eta^S(s)) = \frac{N(s, 1 + \eta^S(s))}{|N(s, 1 + \eta^S(s))|} \quad \text{where} \quad N(s, 1 + \eta^S(s)) = \begin{pmatrix} -\partial_{s_1} \eta^S(s) \\ -\partial_{s_2} \eta^S(s) \\ 1 \end{pmatrix} \quad s \in \omega, \quad (1.20)$$

$$\tau^1(s, 1 + \eta^S(s)) = \begin{pmatrix} 1 \\ 0 \\ \partial_{s_1} \eta^S(s) \end{pmatrix}, \quad \tau^2(s, 1 + \eta^S(s)) = \begin{pmatrix} 0 \\ 1 \\ \partial_{s_2} \eta^S(s) \end{pmatrix}.$$

The operators $(A_1, \mathcal{D}(A_1))$ and $(A_2, \mathcal{D}(A_2))$ are defined by

$$A_1 \eta = \alpha \Delta^2 \eta, \quad \mathcal{D}(A_1) = H^4(\omega) \cap L_0^2(\omega), \quad (1.21)$$

$$A_2 \eta = -\delta \Delta \eta, \quad \mathcal{D}(A_2) = H^2(\omega) \cap L_0^2(\omega). \quad (1.22)$$

Let $\mathfrak{X}_1, \mathfrak{X}_2$ be two Banach spaces with the norms $\|\cdot\|_{\mathfrak{X}_1}$ and $\|\cdot\|_{\mathfrak{X}_2}$ respectively. For $s \geq 0$, we define

$$W^s(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2) = \{v \in L^2(0, \infty; \mathfrak{X}_1) ; v \in H^s(0, \infty; \mathfrak{X}_2)\},$$

with

$$\|\cdot\|_{W^s(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2)} = \|\cdot\|_{L^2(0, \infty; \mathfrak{X}_1)} + \|\cdot\|_{H^s(0, \infty; \mathfrak{X}_2)}.$$

For $s = 1$, the space $W^1(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2)$ is denoted by $W(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2)$.

Let $f(t)$ be a vector valued function. For $\gamma > 0$, we define f_γ by

$$f_\gamma : t \mapsto e^{\gamma t} f(t).$$

Then, for $\gamma > 0$, we consider the following spaces

$$L_\gamma^p(0, \infty; \mathfrak{X}_1) = \{f \in L^p(0, \infty; \mathfrak{X}_1) ; f_\gamma \in L^p(0, \infty; \mathfrak{X}_1)\}, \quad p \in [1, +\infty], \quad (1.23)$$

and

$$W_\gamma^s(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2) = \{f \in W^s(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2) ; f_\gamma \in W^s(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2)\}, \quad (1.24)$$

with the norms

$$\|f\|_{L_\gamma^p(0, \infty; \mathfrak{X}_1)} = \|f_\gamma\|_{L^p(0, \infty; \mathfrak{X}_1)},$$

$$\|f\|_{W_\gamma^s(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2)} = \|f_\gamma\|_{W^s(0, \infty; \mathfrak{X}_1, \mathfrak{X}_2)}.$$

We set

$$\mathcal{X}_{\infty, \gamma} = W_\gamma(0, \infty; [H^2(\Omega)]^3, [L^2(\Omega)]^3) \times L_\gamma^2(0, \infty; H^1(\Omega)/\mathbb{R}) \times W_\gamma^2(0, \infty; \mathcal{D}(A_1), L_0^2(\omega)), \quad (1.25)$$

such that

$$\|(u, p, \eta)\|_{\mathcal{X}_{\infty, \gamma}} = \|u\|_{W_\gamma(0, \infty; [H^2(\Omega)]^3, [L^2(\Omega)]^3)} + \|u\|_{L_\gamma^\infty(0, \infty; [H^1(\Omega)]^3)} + \|\nabla p\|_{L_\gamma^2(0, \infty; [L^2(\Omega)]^3)} \\ + \|\eta\|_{W_\gamma^2(0, \infty; \mathcal{D}(A_1), L_0^2(\omega))} + \|\eta\|_{L_\gamma^\infty(0, \infty; H^3(\omega))} + \|\partial_t \eta\|_{L_\gamma^\infty(0, \infty; H^1(\omega))}. \quad (1.26)$$

We define the space

$$\mathcal{H} = [L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega),$$

furnished with the inner product

$$\left\langle \begin{pmatrix} \phi \\ \zeta_1 \\ \zeta_2 \end{pmatrix}, \begin{pmatrix} \psi \\ \xi_1 \\ \xi_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_{\Omega} \phi \cdot \psi \, dy + \int_{\omega} A_1^{1/2} \zeta_1 \cdot A_1^{1/2} \xi_1 \, ds + \int_{\omega} \zeta_2 \cdot \xi_2 \, ds.$$

We set

$$\widetilde{W} = \begin{pmatrix} \widetilde{u} - w^S \\ \eta - \eta^S \\ \partial_t \eta \end{pmatrix}, \quad (1.27)$$

where the function \widetilde{u} is defined by (1.18).

Finally, we suppose that the initial conditions verify

$$(\widetilde{u}^0, \eta^0, \eta^1) \in [H^1(\Omega)]^3 \times H^3(\omega) \times H^1(\omega) \quad (1.28)$$

$$\nabla \cdot \widetilde{u}^0 = 0 \text{ in } \Omega, \quad (\widetilde{u}^0 - \eta^1 e_3)_n = 0 \text{ on } \Gamma(\eta^S), \quad \widetilde{u}_n^0 = 0 \text{ on } \Gamma_0. \quad (1.29)$$

Now, we state the main result of this paper.

Theorem 1.1. *Let $t_0 > 0$, $\gamma > 0$,*

$$f^S \in [W^{2,\infty}(\Omega)]^3, \quad h^S \in L^2(\omega), \quad (1.30)$$

$$(w^S, p^S, \eta^S) \in [W^{2,\infty}(\Omega)]^3 \times W^{1,\infty}(\Omega)/\mathbb{R} \times C^4(\omega), \quad (1.31)$$

with

$$1 + \eta^S > 0, \quad (1.32)$$

such that the system (1.13) is verified and $(\widetilde{u}^0, \eta^0, \eta^1)$ satisfying (1.28), (1.29). Then, there exist

$$N_\gamma \in \mathbb{N}^*, \quad K \in L_{\text{loc}}^\infty(\mathbb{R}^2; \mathcal{L}(\mathcal{H})),$$

$$(\phi_k, \zeta_1^k, \zeta_2^k) \in [H^2(\Omega)]^3 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}), \quad v_k \in [H^{3/2}(\Gamma_0)]^3, \quad k = 1, \dots, N_\gamma,$$

and $R > 0$, such that if

$$\|\widetilde{u}^0 - w^S\|_{[H^1(\Omega)]^3} + \|\eta^0 - \eta^S\|_{H^3(\omega)} + \|\eta^1\|_{H^1(\omega)} \leq R,$$

there exists a unique strong solution (U, P, η) of (1.1), (1.5) and (1.12) associated to

$$v(t) = 1_{[t_0, +\infty)}(t) \sum_{k=1}^{N_\gamma} \left(\widetilde{W}(t - t_0) + \int_0^{t-t_0} K(t - t_0, s) \widetilde{W}(s) ds, \begin{pmatrix} \phi_k \\ \zeta_1^k \\ \zeta_2^k \end{pmatrix} \right)_{\mathcal{H}} v_k, \quad (1.33)$$

where \widetilde{W} is given by (1.27).

Moreover, $(\widetilde{u} - w^S, \widetilde{p} - p^S, \eta - \eta^S) \in \mathcal{X}_{\infty, \gamma}$ where the functions \widetilde{u} , \widetilde{p} are defined by (1.18) and we have the following estimate

$$\|(\widetilde{u} - w^S, \widetilde{p} - p^S, \eta - \eta^S)\|_{\mathcal{X}_{\infty, \gamma}} \leq C \left(\|\widetilde{u}^0 - w^S\|_{[H^1(\Omega)]^3} + \|\eta^0 - \eta^S\|_{H^3(\omega)} + \|\eta^1\|_{H^1(\omega)} \right). \quad (1.34)$$

Remark 1.2. *We can obtain analogous results if we consider the classical Dirichlet boundary conditions*

$$\begin{cases} U = \mathbb{M}v & \text{on } (0, \infty) \times \Gamma_0, \\ (U - \partial_t \eta e_3) = 0 & \text{on } (0, \infty) \times \Gamma(\eta). \end{cases} \quad (1.35)$$

We underline that if $\beta_1 = 0$, the theorem 1.1 implies that the system (1.1), (1.5) and (1.12) is exponentially stabilizable by considering only a scalar control in the impermeability boundary condition .

Remark 1.3. *Here, the regularity of η^S ensures to have a sufficient regular domain to apply elliptic estimates of the Stokes problem with Navier-slip boundary conditions proved in [8].*

with the boundary conditions

$$\begin{cases} [\tilde{u} - \mathcal{T}\partial_t\eta]_n = 1_{\Gamma_0}(\mathbb{M}v)_n & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(\tilde{u})n + \beta(\tilde{u} - \mathcal{T}\partial_t\eta)]_\tau = 1_{\Gamma_0}\beta(\mathbb{M}v)_\tau + G(\tilde{u}, \eta) & \text{on } (0, \infty) \times \partial\Omega, \end{cases} \quad (2.3)$$

and with the initial conditions

$$\begin{cases} \tilde{u}(0, \cdot) = (\text{Cof } \nabla X(0, \cdot))^* U^0(X(0, \cdot)) = \tilde{u}^0 & \text{in } \Omega, \\ \eta(0, \cdot) = \eta^0 & \text{in } \omega, \\ \partial_t\eta(0, \cdot) = \eta^1 & \text{in } \omega. \end{cases} \quad (2.4)$$

We precise here that 1_{Γ_0} is the characteristic function of the set Γ_0 . To simplify the notations, we set

$$a = (\text{Cof}(\nabla Y))^*, \quad b = (\text{Cof}(\nabla X)),$$

$$\mathbf{K}_\eta \tilde{u} = (\nabla X) \tilde{u},$$

$$\mathbf{G}_\eta \tilde{p} = b \nabla \tilde{p}.$$

We have the following formulas

$$\begin{aligned} [-\nu(\Delta - \mathbf{L}_\eta)\tilde{u}]_i &= \nu \sum_{j,k,l,m} \left(\det(\nabla X) a_{ik}(X) \frac{\partial Y_m}{\partial x_j}(X) \frac{\partial Y_l}{\partial x_j}(X) - \delta_{ik} \delta_{mj} \delta_{jl} \right) \frac{\partial^2 \tilde{u}_k}{\partial y_l \partial y_m} \\ &+ \nu \sum_{j,k,l,m} \left(\det(\nabla X) a_{jk}(X) \frac{\partial Y_m}{\partial x_i}(X) \frac{\partial Y_l}{\partial x_j}(X) - \delta_{jk} \delta_{mi} \delta_{jl} \right) \frac{\partial^2 \tilde{u}_k}{\partial y_l \partial y_m} \\ &+ \det(\nabla X) \left[\nu \sum_{j,k,l} \left(\frac{\partial a_{ik}}{\partial x_j}(X) \frac{\partial Y_l}{\partial x_j}(X) + \frac{\partial a_{jk}}{\partial x_i}(X) \frac{\partial Y_l}{\partial x_j}(X) \right) \frac{\partial \tilde{u}_k}{\partial y_l} \right. \\ &+ \nu \sum_{j,k,l} \left(\frac{\partial a_{ik}}{\partial x_j}(X) \frac{\partial Y_l}{\partial x_j}(X) + a_{ik}(X) \frac{\partial^2 Y_l}{\partial x_j^2}(X) + \frac{\partial a_{jk}}{\partial x_j}(X) \frac{\partial Y_l}{\partial x_i}(X) \right. \\ &\left. \left. + a_{jk}(X) \frac{\partial^2 Y_l}{\partial x_j \partial x_i}(X) \right) \frac{\partial \tilde{u}_k}{\partial y_l} + \nu \sum_k \left(\frac{\partial^2 a_{ik}}{\partial x_j^2}(X) + \frac{\partial^2 a_{jk}}{\partial x_j \partial x_i}(X) \right) \tilde{u}_k \right], \quad (2.5) \end{aligned}$$

$$[(\nabla - \mathbf{G}_\eta)\tilde{p}]_i = \sum_k (\delta_{ik} - b_{ik}) \frac{\partial \tilde{p}}{\partial y_k} = \sum_k (\delta_{ik} - \det(\nabla X) \frac{\partial Y_k}{\partial x_i}(X)) \frac{\partial \tilde{p}}{\partial y_k}, \quad (2.6)$$

$$[\mathbf{N}_\eta \tilde{u}]_i = - \sum_{k,l,j} \det(\nabla X) a_{kl}(X) \frac{\partial a_{ij}}{\partial x_k}(X) \tilde{u}_l \tilde{u}_j - \sum_{k,l,j,m} \det(\nabla X) a_{kl}(X) a_{ij}(X) \frac{\partial Y_m}{\partial x_k}(X) \tilde{u}_l \frac{\partial \tilde{u}_j}{\partial y_m}, \quad (2.7)$$

$$[\mathbf{M}_\eta \tilde{u}]_i = - \sum_{l,k} \det(\nabla X) a_{ik}(X) \frac{\partial \tilde{u}_k}{\partial y_l} \partial_t Y_l(X) - \sum_k \det(\nabla X) \partial_t a_{ik}(X) \tilde{u}_k, \quad i = 1, 2, 3. \quad (2.8)$$

The non linear term H appearing in the plate equation writes

$$\begin{aligned} H(\tilde{u}, \eta) &= \nu M \left[- \sum_{j,k} \left(\frac{\partial a_{3k}}{\partial x_j}(X) + \frac{\partial a_{jk}}{\partial x_3}(X) \right) \tilde{N}_j \tilde{u}_k \right. \\ &+ \sum_{j,k,l} \left(\delta_{3k} \delta_{jl}(N)_j - a_{3k}(X) \frac{\partial Y_l}{\partial x_j}(X) \tilde{N}_j \right) \frac{\partial \tilde{u}_k}{\partial y_l} \\ &\left. + \left(\delta_{3l} \delta_{jk}(N)_j - a_{jk}(X) \frac{\partial Y_l}{\partial x_3}(X) \tilde{N}_j \right) \frac{\partial \tilde{u}_k}{\partial y_l} \right]. \quad (2.9) \end{aligned}$$

Now let us deal with the boundary conditions. Let \mathcal{W} be the operator defined by

$$[\mathcal{W}(\tilde{u}, \eta)]_k = \nu \sum_{j,m} \tilde{n}_j(X) \left(\frac{\partial a_{km}}{\partial x_j}(X) \tilde{u}_m + \frac{\partial a_{jm}}{\partial x_k}(X) \tilde{u}_m \right) + \beta \left(\sum_j a_{kj}(X) \tilde{u}_j - \mathcal{T} \partial_t \eta \cdot e_k \right) \\ + \nu \sum_{j,m,q} \tilde{n}_j(X) \left(a_{km}(X) \frac{\partial \tilde{u}_m}{\partial y_q} \frac{\partial Y_q}{\partial x_j}(X) + a_{jm}(X) \frac{\partial \tilde{u}_m}{\partial y_q} \frac{\partial Y_q}{\partial x_k}(X) \right), \quad k = 1, 2, 3. \quad (2.10)$$

The boundary conditions (1.5) become after the change of variables (1.17) and (1.18)

$$\begin{cases} (\tilde{u} - \mathcal{T} \partial_t \eta)_n = 1_{\Gamma_0}(\mathbb{M}v)_n & \text{on } (0, \infty) \times \partial\Omega, \\ \mathcal{W}(\tilde{u}, \eta) \cdot (\tilde{\tau}^i(X)) = 1_{\Gamma_0} \beta(\mathbb{M}v) \cdot (\tilde{\tau}^i(X)), \quad i = 1, 2 & \text{on } (0, \infty) \times \partial\Omega, \end{cases} \quad (2.11)$$

where $\tilde{\tau}^i$ are defined by (1.2) and (1.3). The second boundary condition in (2.11) can be written as

$$(2\nu D(\tilde{u})n + \beta(\tilde{u} - \mathcal{T} \partial_t \eta)) \cdot \tau^i = \mathcal{V}^i(\tilde{u}, \eta) + 1_{\Gamma_0} \beta(\mathbb{M}v) \cdot \tau^i, \quad i = 1, 2, \quad \text{on } \partial\Omega,$$

where τ^i are the two tangent vectors on $\partial\Omega$ defined in (1.19), (1.20) and \mathcal{V}^i is given by

$$\begin{aligned} \mathcal{V}^i(\tilde{u}, \eta) &= (2\nu D(\tilde{u})n + \beta(\tilde{u} - \mathcal{T} \partial_t \eta)) \cdot (\tau^i - \tilde{\tau}^i(X)) \\ &\quad + (2\nu D(\tilde{u})n + \beta(\tilde{u} - \mathcal{T} \partial_t \eta) - \mathcal{W}(\tilde{u}, \eta)) \cdot \tilde{\tau}^i(X), \quad i = 1, 2, \quad \text{on } \partial\Omega. \end{aligned} \quad (2.12)$$

We construct an operator G such that $G \cdot n = 0$ and $G \cdot \tau^i = \mathcal{V}^i$ on $\partial\Omega$. Consequently, G is defined on $\Gamma(\eta^S)$ by

$$\begin{aligned} G_1 &= \frac{\mathcal{V}^1((\partial_{s_2} \eta^S)^2 + 1) - \mathcal{V}^2(\partial_{s_1} \eta^S \partial_{s_2} \eta^S)}{|N|^2}, \\ G_2 &= \frac{\mathcal{V}^2((\partial_{s_1} \eta^S)^2 + 1) - \mathcal{V}^1(\partial_{s_1} \eta^S \partial_{s_2} \eta^S)}{|N|^2}, \\ G_3 &= \frac{\partial_{s_1} \eta^S \mathcal{V}^1 + \partial_{s_2} \eta^S \mathcal{V}^2}{|N|^2}, \end{aligned} \quad (2.13)$$

and G is defined on Γ_0 by $G = \sum_{i=1}^2 \mathcal{V}^i \tau^i$.

We set

$$u = \tilde{u} - w^S, \quad p = \tilde{p} - p^S, \quad \xi = \eta - \eta^S.$$

Then, (u, p, ξ) satisfies the system

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{T}(u, p) + (w^S \cdot \nabla)u + (u \cdot \nabla)w^S = F(u, p, \xi) & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0, \infty) \times \Omega, \\ \partial_{tt} \xi + A_1 \xi + A_2 \partial_t \xi = \mathbb{H}_{\eta^S}(u, p) + H(u + w^S, \xi + \eta^S) & \text{in } (0, \infty) \times \omega, \end{cases} \quad (2.14)$$

where

$$\begin{aligned} F(u, p, \xi) &= (\text{Id} - \mathbf{K}_{\xi + \eta^S}) \partial_t u - \nu (\Delta - \mathbf{L}_{\xi + \eta^S})(u + w^S) + (\nabla - \mathbf{G}_{\xi + \eta^S})(p + p^S) \\ &\quad + \mathbf{M}_{\xi + \eta^S}(u + w^S) + \mathbf{N}_{\xi + \eta^S}(u + w^S) + (u \cdot \nabla)w^S + (w^S \cdot \nabla)u + (w^S \cdot \nabla)w^S \\ &\quad + \det(\nabla X) f^S(X) - f^S, \end{aligned} \quad (2.15)$$

with the boundary conditions

$$\begin{cases} [u - \mathcal{T} \partial_t \xi]_n = 1_{\Gamma_0}(\mathbb{M}v)_n & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(u)n + \beta(u - \mathcal{T} \partial_t \xi)]_\tau = 1_{\Gamma_0} \beta(\mathbb{M}v)_\tau + G(u + w^S, \xi + \eta^S) & \text{on } (0, \infty) \times \partial\Omega, \end{cases} \quad (2.16)$$

and the initial conditions

$$\begin{cases} u(0, \cdot) = \tilde{u}^0 - w^S & \text{in } \Omega, \\ \xi(0, \cdot) = \eta^0 - \eta^S & \text{in } \omega, \\ \partial_t \xi(0, \cdot) = \eta^1 & \text{in } \omega. \end{cases} \quad (2.17)$$

Remark 2.1. For the linearization of the system (2.14), (2.16), (2.17), we need to find all the linear terms coming from the change of variables. For example, we observe the term

$$\det(\nabla X)f^S(X) - f^S = \det(\nabla X)(f^S(X) - f^S) + (\det(\nabla X) - 1)f^S.$$

We recall that $\det(\nabla X) = \frac{1 + \eta}{1 + \eta^S}$. We notice that

$$X(y) = y + \frac{y_3 \xi(y_1, y_2)}{1 + \eta^S(y_1, y_2)} e_3 = y + \theta \xi(y_1, y_2) e_3.$$

We apply Taylor's formula since $f^S \in [W^{2,\infty}(\Omega)]^3$, we get

$$\begin{aligned} f^S(X(y)) - f^S(y) &= \theta \xi(y_1, y_2) \nabla f^S(y) e_3 \\ &\quad + \int_0^1 (1-s) \nabla^2 f^S(y + s\theta \xi(y_1, y_2) e_3) \theta \xi(y_1, y_2) e_3 \cdot \theta \xi(y_1, y_2) e_3 ds. \end{aligned} \quad (2.18)$$

Then

$$\begin{aligned} \det(\nabla X(y))f^S(X(y)) - f^S(y) &= \frac{\xi(y_1, y_2)}{1 + \eta^S(y_1, y_2)} f^S(y) + \theta \xi(y_1, y_2) \nabla f^S(y) e_3 + \frac{(\xi(y_1, y_2))^2}{1 + \eta^S} \theta \nabla f^S(y) e_3 \\ &\quad + \left(1 + \frac{\xi(y_1, y_2)}{1 + \eta^S(y_1, y_2)}\right) \int_0^1 (1-s) \nabla^2 f^S(y + s\theta \xi(y_1, y_2) e_3) \theta \xi(y_1, y_2) e_3 \cdot \theta \xi(y_1, y_2) e_3 ds. \end{aligned} \quad (2.19)$$

Now, let assume that $\|\xi\|_{L^2_\gamma(0,\infty;L^2(\omega))} \leq CR$, then, we have for example

$$\left\| \frac{\xi}{1 + \eta^S} f^S \right\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^3)} \leq CR.$$

Thus, the terms of this kind should be considered in the linear part of the system or else it will constitute a problem in the fixed point procedure.

To overcome this difficulty, we follow the same approach described in [7]. More precisely, we write for example

$$\det(\nabla X(y))f^S(X(y)) - f^S(y) = \ell^{(12)}(\xi) + \epsilon^{(12)}(\xi),$$

with

$$\left\| \epsilon^{(12)}(\xi) \right\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^3)} \leq C \|\xi\|_{L^2_\gamma(0,\infty;L^2(\omega))}^2 (1 + \|\xi\|_{L^2_\gamma(0,\infty;L^2(\omega))}^m), \quad m \geq 1. \quad (2.20)$$

where $\ell^{(12)}$ is the linear operator

$$\ell^{(12)}(\xi) = \frac{\xi}{1 + \eta^S} f^S,$$

and $\epsilon^{(12)}$ is the remaining non linear terms appearing in (2.19). Then, the strategy consists to integrate the linear operator $\ell^{(12)}$ in the left hand side of the fluid equation in (2.14) and to inject $\epsilon^{(12)}$ in the right hand side that will be handled in the fixed point section.

Due to remark 2.1, we need to decompose the functions F , H and G into a linear part in ξ and a nonlinear part.

The notation $\partial_s \xi$ (resp. $\partial_{ss}^2 \xi$, $\partial_{sss}^3 \xi$), designates in what follows the first derivative $\partial_{s_j} \xi$ (resp. the second derivative $\partial_{s_i s_j}^2 \xi$, the third derivative $\partial_{s_i s_j s_k}^3 \xi$).

where $\ell^{(13)}$ is a linear operator of the form (2.21) and $\epsilon^{(13)}$ is the nonlinear part. To write the linearized system in an appropriate form, we define for all \tilde{u}

$$\begin{aligned} (E_\eta(\tilde{u}))_{im} &= \nu \sum_{j,k,l} \left(\det(\nabla X) a_{ik}(X) \frac{\partial Y_m}{\partial x_j}(X) \frac{\partial Y_l}{\partial x_j}(X) - \delta_{ik} \delta_{mj} \delta_{jl} \right) \frac{\partial \tilde{u}_k}{\partial y_l} \\ &\quad + \nu \sum_{j,k,l} \left(\det(\nabla X) a_{jk}(X) \frac{\partial Y_m}{\partial x_i}(X) \frac{\partial Y_l}{\partial x_j}(X) - \delta_{jk} \delta_{mi} \delta_{jl} \right) \frac{\partial \tilde{u}_k}{\partial y_l} \\ &\quad + \det(\nabla X) \left(\nu \sum_{j,k} \left(\frac{\partial a_{ik}}{\partial x_j}(X) \frac{\partial Y_m}{\partial x_j}(X) + \frac{\partial a_{jk}}{\partial x_i}(X) \frac{\partial Y_m}{\partial x_j}(X) \right) \tilde{u}_k \right). \end{aligned} \quad (2.27)$$

Observe that

$$(\nabla \cdot E_\eta)_i = -\nu[(\Delta - \mathbf{L}_\eta)]_i + (F_\eta^1)_i, \quad (2.28)$$

with

$$\begin{aligned} [(F_\eta^1)(\tilde{u})]_i &= \nu \frac{\partial \det(\nabla X)}{\partial y_m} \sum_{j,k,l,m} \left[\left(a_{ik}(X) \frac{\partial Y_m}{\partial x_j}(X) \frac{\partial Y_l}{\partial x_j}(X) + a_{jk}(X) \frac{\partial Y_m}{\partial x_i}(X) \frac{\partial Y_l}{\partial x_j}(X) \right) \frac{\partial \tilde{u}_k}{\partial y_l} \right. \\ &\quad \left. + \sum_{j,k} \left(\frac{\partial a_{ik}}{\partial x_j}(X) \frac{\partial Y_m}{\partial x_j}(X) + \frac{\partial a_{jk}}{\partial x_i}(X) \frac{\partial Y_m}{\partial x_j}(X) \right) \tilde{u}_k \right] \\ &\quad + \nu \det(\nabla X) \left[\sum_{j,k,l,m,q} \frac{\partial^2 Y_m}{\partial x_j \partial x_q}(X) \frac{\partial X_q}{\partial y_m} \left(\left(a_{ik}(X) \frac{\partial Y_l}{\partial x_j}(X) + a_{jk}(X) \frac{\partial Y_l}{\partial x_j}(X) \right) \frac{\partial \tilde{u}_k}{\partial y_l} \right. \right. \\ &\quad \left. \left. + \left(\frac{\partial a_{ik}}{\partial x_j}(X) + \frac{\partial a_{jk}}{\partial x_i}(X) \right) \tilde{u}_k \right) \right]. \end{aligned} \quad (2.29)$$

From the lemma 4.2, we have

$$(E_{\xi+\eta^S}(w^S)) = \mathcal{L}^1(\xi) + \mathcal{N}_E(\xi), \quad (2.30)$$

where \mathcal{L}^1 is a linear operator that writes

$$\mathcal{L}^1(\xi) = Q^1 \xi + Q^2 \partial_s \xi + Q^3 \partial_{ss}^2 \xi,$$

where Q^i , $i = 1, \dots, 3$ are operators that depend only on the stationary state and are of regularity $W^{1,\infty}(\Omega)$. The quantity $\mathcal{N}_E(\xi)$ is the remaining nonlinear part that contains the terms of the form

$$c^1(w^S, \eta^S) \frac{\xi^{m_1} (\partial_s \xi)^{m_2}}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad c^2(w^S, \eta^S) \frac{\xi^{n_1} (\partial_s \xi)^{n_2} \partial_{ss}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_2}}, \quad m_1 + m_2 \geq 2, \quad n_1 + n_2 \geq 1, \quad \alpha_i \geq 1, \quad (2.31)$$

where $c^1(w^S, \eta^S)$ and $c^2(w^S, \eta^S)$ are two quantities of regularity $W^{1,\infty}(\Omega)$. In the other hand, from lemma 4.2, we have also

$$(F_{\xi+\eta^S}^1)(w^S) = -\ell^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) - \epsilon^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi), \quad (2.32)$$

where the expression of the nonlinear part $\epsilon^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi)$ contains terms of the form (2.31). Then, from (2.26), (2.28), (2.30) and (2.32), we deduce

$$\ell^{(13)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi) = \nabla \cdot (\mathcal{L}^1(\xi)) + \ell^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi), \quad (2.33)$$

$$\epsilon^{(13)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi) = \epsilon^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + \nabla \cdot (\mathcal{N}_E(\xi)). \quad (2.34)$$

Using (2.25), (2.26), (2.33) and (2.34), the term $-\nu(\Delta - \mathbf{L}_{\xi+\eta^S})(u + w^S)$ writes

$$-\nu(\Delta - \mathbf{L}_{\xi+\eta^S})(u + w^S) = \nabla \cdot (\mathcal{L}^1(\xi)) + \ell^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + \epsilon^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + \nabla \cdot (\mathcal{N}_E(\xi)) - \nu(\Delta - \mathbf{L}_{\xi+\eta^S})(u). \quad (2.35)$$

Furthermore, using again lemma 4.2, we obtain

$$[(\nabla - \mathbf{G}_{\xi+\eta^S})(p^S)]_i = \sum_k (\delta_{ki} - \det(\nabla X) \frac{\partial Y_k}{\partial x_i}(X)) \frac{\partial p^S}{\partial y_k} = \ell_i^{(5)}(\xi, \partial_s \xi) + \epsilon_i^{(5)}(\xi, \partial_s \xi), \quad (2.36)$$

where $\epsilon^{(5)}(\xi, \partial_s \xi)$ contains the terms

$$c^3(p^S, \eta^S) \frac{\xi^{m_2} (\partial_s \xi)^{m_2}}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad m_1 + m_2 \geq 2, \quad \alpha_1 \geq 1, \quad (2.37)$$

where c^3 is a function that depends on the stationary state and bounded in Ω . Then, we get

$$[(\nabla - \mathbf{G}_{\xi+\eta^S})(p + p^S)]_i = \ell_i^{(5)}(\xi, \partial_s \xi) + \epsilon_i^{(5)}(\xi, \partial_s \xi) + [(\nabla - \mathbf{G}_{\xi+\eta^S})(p)]_i. \quad (2.38)$$

Moreover, we have

$$\sum_{l,k} \det(\nabla X) a_{ik}(X) \frac{\partial w_k^S}{\partial y_l} \partial_t Y_l(X) = (\nabla X \nabla w^S \partial_t Y(X))_i,$$

where

$$(\nabla X \nabla w^S \partial_t Y(X))_i = \partial_t Y_3 \frac{\partial w_i^S}{\partial y_3}, \quad i = 1, 2,$$

$$(\nabla X \nabla w^S \partial_t Y(X))_3 = \partial_t Y_3 \left(\frac{\partial X_3}{\partial y_1} \frac{\partial w_1^S}{\partial y_3} + \frac{\partial X_3}{\partial y_2} \frac{\partial w_2^S}{\partial y_3} + \frac{1 + \eta}{1 + \eta^S} \frac{\partial w_3^S}{\partial y_3} \right).$$

We notice that

$$\partial_t Y_3 = -y_3 \frac{\partial_t \xi}{1 + \eta}, \quad \frac{\partial X_3}{\partial y_i} = y_3 \frac{\partial_{s_i} \xi (1 + \eta^S) - \partial_{s_i} \eta^S \xi}{(1 + \eta^S)^2}, \quad i = 1, 2.$$

Using (4.14), we find

$$-(\nabla X \nabla w^S \partial_t Y(X))_i = \ell_i^{(4)}(\partial_t \xi) + \epsilon_i^{(4)}(\xi, \partial_s \xi, \partial_t \xi),$$

where $\epsilon^{(4)}(\xi, \partial_s \xi, \partial_t \xi)$ has the terms

$$c^4(w^S, \eta^S) \frac{\xi^{m_1} \partial_t \xi \partial_s \xi}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad c^5(w^S, \eta^S) \frac{\xi^{m_2} \partial_t \xi}{(1 + \xi + \eta^S)^{\alpha_2}}, \quad m_1 \geq 0, \quad m_2 \geq 1, \quad \alpha_i \geq 1, \quad (2.39)$$

with c^4 and c^5 are $W^{1,\infty}(\Omega)$ functions. Moreover, using lemma 4.2, we get

$$-\sum_k \det(\nabla X) \partial_t a_{ik}(X) w_k^S = \ell_i^{(6)}(\partial_t \xi, \partial_{ts}^2 \xi) + \epsilon_i^{(6)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi), \quad (2.40)$$

where $\epsilon^{(6)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi)$ contains terms of the form

$$c^6(w^S, \eta^S) \frac{\xi^{m_1} \partial_{ts}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad c^7(w^S, \eta^S) \frac{\xi^{m_2} \partial_t \xi}{(1 + \xi + \eta^S)^{\alpha_2}}, \quad c^8(w^S, \eta^S) \frac{\xi^{m_3} \partial_t \xi \partial_s \xi}{(1 + \xi + \eta^S)^{\alpha_3}}, \quad m_1 \geq 1, \quad m_2 \geq 1, \quad m_3 \geq 0, \quad \alpha_i \geq 1, \quad (2.41)$$

where c^i , $i = 6, \dots, 8$ are $W^{2,\infty}(\Omega)$ functions. Then

$$\mathbf{M}_{\xi+\eta^S}(u+w^S) = \ell^{(4)}(\partial_t \xi) + \ell^{(6)}(\partial_t \xi, \partial_{ts}^2 \xi) + \epsilon^{(6)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi) + \epsilon^{(4)}(\xi, \partial_s \xi, \partial_t \xi) + \mathbf{M}_{\xi+\eta^S}(u). \quad (2.42)$$

We have,

$$\begin{aligned} [\mathbf{N}_{\xi+\eta^S}(u+w^S) + u \cdot \nabla w^S + w^S \cdot \nabla u + w^S \cdot \nabla w^S]_i &= - \sum_{k,l,j} \det(\nabla X) a_{kl}(X) \frac{\partial a_{ij}(X)}{\partial x_k} w_l^S w_j^S \\ &+ \sum_{k,l,j,m} \left(\delta_{ij} \delta_{kl} \delta_{km} - \det(\nabla X) a_{kl}(X) a_{ij}(X) \frac{\partial Y_m}{\partial x_k}(X) \right) w_l^S \frac{\partial w_j^S}{\partial y_m} + F_i^2(u, \xi + \eta^S) \\ &= F_i^2(u, \xi + \eta^S) + F_i^3(w^S, \xi + \eta^S), \quad i = 1, 2, 3. \end{aligned} \quad (2.43)$$

Using lemma 4.2, we obtain

$$F_i^3(w^S, \xi + \eta^S) = \ell_i^{(8)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + \epsilon_i^{(8)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi), \quad (2.44)$$

where $\epsilon^{(8)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi)$ admits terms of the form (2.31). Then, we get

$$\begin{aligned} [\mathbf{N}_{\xi+\eta^S}(u+w^S) + u \cdot \nabla w^S + w^S \cdot \nabla u + w^S \cdot \nabla w^S]_i &= \ell_i^{(8)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) \\ &+ \epsilon_i^{(8)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + F_i^2(u, \xi + \eta^S), \quad i = 1, 2, 3. \end{aligned} \quad (2.45)$$

From (2.15), (2.35), (2.38), (2.42) and (2.45), we get

$$F(u, p, \xi) = \nabla \cdot \mathcal{L}^1(\xi) + \mathcal{L}^2(\xi, \partial_t \xi) + \nabla \cdot \mathcal{N}_E(\xi) + \mathcal{N}_F(\xi) + \mathcal{F}(u, p, \xi). \quad (2.46)$$

where

$$\mathcal{L}^2(\xi, \partial_t \xi) = \ell^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + \ell^{(5)}(\xi, \partial_s \xi) + \ell^{(4)}(\partial_t \xi) + \ell^{(6)}(\partial_t \xi, \partial_{ts}^2 \xi) + \ell^{(8)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi). \quad (2.47)$$

More precisely,

$$\mathcal{L}^2(\xi, \partial_t \xi) = Q^4 \xi + Q^5 \partial_s \xi + Q^6 \partial_{ss}^2 \xi + Q^7 \partial_t \xi + Q^8 \partial_{st}^2 \xi,$$

where Q_i , $i = 4, \dots, 8$ are operators of regularity $W^{1,\infty}(\Omega)$ and we also have

$$\mathcal{N}_F(\xi) = \epsilon^{(9)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + \epsilon^{(5)}(\xi, \partial_s \xi) + \epsilon^{(4)}(\xi, \partial_s \xi, \partial_t \xi) + \epsilon^{(6)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi) + \epsilon^{(8)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi), \quad (2.48)$$

$$\mathcal{F}(u, p, \xi) = (\text{Id} - \mathbf{K}_{\xi+\eta^S}) \partial_t u - \nu(\Delta - \mathbf{L}_{\xi+\eta^S})(u) + (\nabla - \mathbf{G}_{\xi+\eta^S})(p) + \mathbf{M}_{\xi+\eta^S}(u) + F^2(u, \xi + \eta^S). \quad (2.49)$$

Decomposition $H(u, \xi + \eta^S)$:

We have

$$H(u + w^S, \xi + \eta^S) = H(u, \xi + \eta^S) + H(w^S, \xi + \eta^S).$$

We notice that

$$H(w^S, \xi + \eta^S) = -\mathcal{T}^*((E_{\xi+\eta^S}(w^S))n), \quad (2.50)$$

where \mathcal{T}^* is defined by

$$\mathcal{T}^*(\zeta)(s) = M \left(\sqrt{1 + |\nabla \eta^S|^2} \zeta(s, 1 + \eta^S(s)) \cdot e_3 \right), \quad \forall \zeta \in [L^2(\partial\Omega)]^3. \quad (2.51)$$

Using (2.30), we obtain

$$H(u + w^S, \xi + \eta^S) = -\mathcal{T}^*(\mathcal{L}^1(\xi)n) + H(u, \xi + \eta^S) - \mathcal{T}^*(\mathcal{N}_E(\xi)n). \quad (2.52)$$

Decomposition of $G(u, \xi + \eta^S)$:

Using the expression of \mathcal{W} in (2.10) and of \mathcal{V}^i in (2.12), we find

$$\begin{aligned} \mathcal{V}^i(w^S, \xi + \eta^S) &= (2\nu D(w^S)n + \beta w^S) \cdot (\tau^i - \tilde{\tau}^i(X)) \\ &\quad + \left[\nu \sum_{j,m,q} \left((n)_j \delta_{km} \delta_{qj} - (\tilde{n}(X))_j a_{km}(X) \frac{\partial Y_q}{\partial x_j}(X) \right) \frac{\partial w_m^S}{\partial y_q} \right. \\ &\quad \left. - \nu \sum_{j,m,q} \left((\tilde{n}(X))_j a_{jm}(X) \frac{\partial Y_q}{\partial x_k}(X) - (n)_j \delta_{jm} \delta_{qk} \right) \frac{\partial w_m^S}{\partial y_q} \right. \\ &\quad \left. - \nu \sum_{j,m} (\tilde{n}(X))_j \left(\frac{\partial a_{km}}{\partial x_j}(X) w_m^S + \frac{\partial a_{jm}}{\partial x_k}(X) w_m^S \right) + \beta \sum_j (\delta_{kj} - a_{kj}(X)) w_j^S \right] \tilde{\tau}_k^i(X), \quad k = 1, 2, 3. \end{aligned}$$

We notice that

$$\begin{aligned} &\nu \sum_{j,m,q} \left(n_j \delta_{km} \delta_{qj} - \tilde{n}_j(X) a_{km}(X) \frac{\partial Y_q}{\partial x_j}(X) \right) \frac{\partial w_m^S}{\partial y_q} \\ &\quad + \nu \sum_{j,m,q} \left(n_j \delta_{jm} \delta_{qk} - \tilde{n}_j(X) a_{jm}(X) \frac{\partial Y_q}{\partial x_k}(X) \right) \frac{\partial w_m^S}{\partial y_q} \\ &\quad - \nu \sum_{j,m} \tilde{n}_j(X) \left(\frac{\partial a_{km}}{\partial x_j}(X) w_m^S + \frac{\partial a_{jm}}{\partial x_k}(X) w_m^S \right) = -\frac{|N|}{|\tilde{N}|} [(E_{\xi+\eta^S}(w^S))n]_k \\ &\quad + \nu \sum_j \left((n)_j - \frac{N_j}{|\tilde{N}|} \right) \frac{\partial w_k^S}{\partial y_j} + \nu \sum_j \left((n)_j - \frac{N_j}{|\tilde{N}|} \right) \frac{\partial w_j^S}{\partial y_k}. \quad (2.53) \end{aligned}$$

In the other hand, using Taylor's formula

$$\frac{1}{|\tilde{N}|} = \frac{1}{|N|} - \frac{\nabla \eta^S \cdot \nabla \xi}{|N|^3} - \frac{|\nabla \xi|^2}{2|N|^3} + \frac{3\theta^2}{4|N|} \int_0^1 \frac{(1-s)}{(1+s\theta)^{5/2}} ds,$$

where $\theta = \frac{2\nabla \eta^S \cdot \nabla \xi + |\nabla \xi|^2}{|N|^2}$. Then, we deduce

$$\frac{1}{|\tilde{N}|} = \frac{1}{|N|} - \frac{\nabla \eta^S \cdot \nabla \xi}{|N|^3} + \epsilon^{(10)}(\partial_s \xi), \quad (2.54)$$

with

$$\epsilon^{(10)}(\partial_s \xi) = -\frac{|\nabla \xi|^2}{2|N|^3} + \frac{3\theta^2}{4|N|} \int_0^1 \frac{(1-s)}{(1+s\theta)^{5/2}} ds.$$

Then, using (2.30) and (2.54), we obtain

$$\begin{aligned} &\nu \sum_{j,m,q} \left(n_j \delta_{km} \delta_{qj} - \tilde{n}_j(X) a_{km}(X) \frac{\partial Y_q}{\partial x_j}(X) \right) \frac{\partial w_m^S}{\partial y_q} \\ &\quad + \nu \sum_{j,m,q} \left(n_j \delta_{jm} \delta_{qk} - \tilde{n}_j(X) a_{jm}(X) \frac{\partial Y_q}{\partial x_k}(X) \right) \frac{\partial w_m^S}{\partial y_q} \\ &\quad - \nu \sum_{j,m} \tilde{n}_j(X) \left(\frac{\partial a_{km}}{\partial x_j}(X) w_m^S + \frac{\partial a_{jm}}{\partial x_k}(X) w_m^S \right) = [-\mathcal{L}^1(\xi)n + \ell^{(14)}(\partial_s \xi) - \mathcal{N}_E(\xi)n \\ &\quad \quad \quad + \epsilon^{(14)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi)]_k, \quad (2.55) \end{aligned}$$

where

$$\ell_k^{(14)}(\partial_s \xi) = \nu \frac{\nabla \eta^S \cdot \nabla \xi}{|N|^2} \sum_j \left(\frac{\partial w_k^S}{\partial y_j} + \frac{\partial w_j^S}{\partial y_k} \right),$$

and $\epsilon^{(14)}$ admits the terms (2.31). Moreover, From lemma 4.2, we get

$$\beta \sum_j (\delta_{kj} - a_{kj}(X)) w_j^S = \ell_k^{(15)}(\xi, \partial_s \xi) + \epsilon_k^{(15)}(\xi, \partial_s \xi),$$

where $\epsilon_k^{(15)}$ had the terms

$$c^9(w^S, \eta^S) \frac{\xi^{m_1}}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad c^{(10)}(w^S, \eta^S) \frac{\xi^{m_2} \partial_s \xi}{(1 + \xi + \eta^S)^{\alpha_2}}, \quad m_1 \geq 2, \quad m_2 \geq 1, \quad \alpha_i \geq 1, \quad i = 1, 2,$$

where c^i , $i = 9, 10$ are of regularity $W^{3/2, \infty}(\partial\Omega)$.

$$\mathcal{V}^i(w^S, \eta) = -[\mathcal{L}^1(\xi)n + \mathcal{L}^3(\xi)] \cdot \tau^i - [\mathcal{N}_E(\xi)n] \cdot \tau^i + \epsilon_i^{(11)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi), \quad (2.56)$$

where \mathcal{L}^3 is an operator defined by

$$\mathcal{L}^3(\xi) \cdot \tau^i = - (2\nu D(w^S)n + \beta w^S) \cdot (\tau^i - \tilde{\tau}^i(X)) - \ell^{(14)}(\partial_s \xi) \cdot \tau^i, \quad \mathcal{L}^3(\xi) \cdot n = 0,$$

then \mathcal{L}^3 can be expressed as the following

$$\mathcal{L}^3(\xi) = Q^9 \xi + Q^{10} \partial_s \xi, \quad (2.57)$$

where Q^9 and Q^{10} are of regularity $W^{1/2, \infty}(\partial\Omega)$ and

$$\epsilon^{(11)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) = -\ell^{(14)}(\partial_s \xi) \cdot (\tilde{\tau}^i(X) - \tau^i) + \epsilon^{(14)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) \cdot \tilde{\tau}^i(X) - [\mathcal{N}_E(\xi)n] \cdot (\tilde{\tau}^i(X) - \tau^i).$$

Here, $\epsilon^{(11)}$ contains terms of the type (2.31). Finally, we define \mathcal{N}_G such that

$$\mathcal{N}_G(\xi) \cdot \tau^i = \epsilon_i^{(11)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi), \quad \mathcal{N}_G(\xi) \cdot n = 0.$$

Then, the system (2.14), (2.16), (2.17) becomes

$$\left\{ \begin{array}{ll} \partial_t u - \nabla \cdot \mathbb{T}(u, p) + (w^S \cdot \nabla)u + (u \cdot \nabla)w^S - \nabla \cdot \mathcal{L}^1(\xi) + \mathcal{L}^2(\xi, \partial_t \xi) & \text{in } (0, \infty) \times \Omega, \\ = \nabla \cdot \mathcal{N}_E(\xi) + \mathcal{N}_F(\xi) + \mathcal{F}(u, p, \xi) & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0, \infty) \times \Omega, \\ \partial_{tt} \xi + A_1 \xi + A_2 \partial_t \xi + \mathcal{T}^*(\mathcal{L}^1(\xi)n) = \mathbb{H}_{\eta^S}(u, p) + H(u, \xi + \eta^S) & \text{in } (0, \infty) \times \omega, \\ -\mathcal{T}^*(\mathcal{N}_E(\xi)n) & \text{in } (0, \infty) \times \omega, \end{array} \right.$$

and the boundary conditions

$$\left\{ \begin{array}{ll} [u - \mathcal{T} \partial_t \xi]_n = 1_{\Gamma_0}(\mathbb{M}v)_n & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(u)n + \mathcal{L}^1(\xi)n + \mathcal{L}^3(\xi) + \beta(u - \mathcal{T} \partial_t \xi)]_\tau = 1_{\Gamma_0} \beta(\mathbb{M}v)_\tau & \text{on } (0, \infty) \times \partial\Omega, \\ -[\mathcal{N}_E(\xi)n]_\tau + [\mathcal{N}_G(\xi)]_\tau + G(u, \xi + \eta^S) & \text{on } (0, \infty) \times \partial\Omega, \end{array} \right.$$

and the initial conditions

$$\left\{ \begin{array}{ll} u(0, \cdot) = \tilde{u}^0 - w^S & \text{in } \Omega, \\ \xi(0, \cdot) = \eta^0 - \eta^S & \text{in } \omega, \\ \partial_t \xi(0, \cdot) = \eta^1 & \text{in } \omega. \end{array} \right.$$

□

3 Feedback stabilization of the linear system

We consider the linear system associated to (2.14) and (2.16)

$$\begin{cases} \partial_t \tilde{w} + (w^S \cdot \nabla) \tilde{w} + (\tilde{w} \cdot \nabla) w^S - \nabla \cdot \mathbb{T}(\tilde{w}, \tilde{q}) - \nabla \cdot \mathcal{L}^1(\xi) + \mathcal{L}^2(\xi, \partial_t \xi) = \tilde{f} & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot \tilde{w} = 0 & \text{in } (0, \infty) \times \Omega, \\ \partial_{tt} \xi + A_1 \xi + A_2 \partial_t \xi + \mathcal{T}^*(\mathcal{L}^1(\xi)n) = -\mathcal{T}^*(\mathbb{T}(\tilde{w}, \tilde{q})n) + \tilde{h} & \text{in } (0, \infty) \times \omega, \end{cases} \quad (3.1)$$

with the boundary conditions

$$\begin{cases} [\tilde{w} - \mathcal{T} \partial_t \xi]_n = 1_{\Gamma_0}(\mathbb{M}v)_n & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(\tilde{w})n + \beta(\tilde{w} - \mathcal{T} \partial_t \xi) + \mathcal{L}^1(\xi)n + \mathcal{L}^3(\xi)]_\tau = 1_{\Gamma_0} \beta(\mathbb{M}v)_\tau + \tilde{g} & \text{on } (0, \infty) \times \partial\Omega, \end{cases} \quad (3.2)$$

and the initial conditions

$$\begin{cases} \tilde{w}(0, \cdot) = w^0 = \tilde{u}^0 - w^S & \text{in } \Omega, \\ \xi(0, \cdot) = \xi^0 = \eta^0 - \eta^S & \text{in } \omega, \\ \partial_t \xi(0, \cdot) = \xi^1 = \eta^1 & \text{in } \omega, \end{cases} \quad (3.3)$$

where

$$\begin{aligned} \mathcal{L}^1(\xi) &= Q^1 \xi + Q^2 \partial_s \xi + Q^3 \partial_{ss}^2 \xi, \\ \mathcal{L}^2(\xi, \partial_t \xi) &= Q^4 \xi + Q^5 \partial_s \xi + Q^6 \partial_{ss}^2 \xi + Q^7 \partial_t \xi + Q^8 \partial_{st}^2 \xi, \\ \mathcal{L}^3(\xi) &= Q^9 \xi + Q^{10} \partial_s \xi. \end{aligned} \quad (3.4)$$

We set

$$\mathcal{L}^{2,1}(\xi) = Q^4 \xi + Q^5 \partial_s \xi + Q^6 \partial_{ss}^2 \xi, \quad \mathcal{L}^{2,2}(\partial_t \xi) = Q^7 \partial_t \xi + Q^8 \partial_{st}^2 \xi. \quad (3.5)$$

Here, we recall that Q^i , $i = 1, \dots, 8$ are some operators that depend on the stationary state, of regularity $W^{1,\infty}(\Omega)$ and Q^9, Q^{10} are of regularity $W^{1/2,\infty}(\partial\Omega)$.

The next step consists to reformulate the system (3.1), (3.2) to get an evolution problem. To do so, we need to lift \tilde{g} in the boundary conditions. Then, we set the following lemma that concerns the instationary Stokes problem with Navier boundary conditions with a non negative friction.

Lemma 3.1. *Let $\beta \geq 0$ and $\gamma_0 > 0$. Let $(\tilde{v}, \tilde{\pi})$ that verifies the system*

$$\begin{cases} \partial_t \tilde{v} + \gamma_0 \tilde{v} - \nabla \cdot \mathbb{T}(\tilde{v}, \tilde{\pi}) = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot \tilde{v} = 0 & \text{in } (0, \infty) \times \Omega, \\ \tilde{v}_n = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(\tilde{v})n + \beta \tilde{v}]_\tau = \tilde{g} & \text{on } (0, \infty) \times \partial\Omega, \\ \tilde{v}(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (3.6)$$

If

$$\tilde{g} \in W_\gamma^{1/4}(0, \infty; [H^{1/2}(\partial\Omega)]^3, [L^2(\partial\Omega)]^3), \quad \tilde{g}_{n_0} = 0, \quad (3.7)$$

for $\gamma \in [0, \gamma_0[$, then the problem (3.6) admits a unique solution that satisfies the estimate

$$\|\tilde{v}\|_{W_\gamma(0, \infty; [H^2(\Omega)]^3, [L^2(\Omega)]^3)}^2 + \|\nabla \tilde{\pi}\|_{L_\gamma^2(0, \infty; [L^2(\Omega)]^3)}^2 \leq C \|\tilde{g}\|_{W_\gamma^{1/4}(0, \infty; [H^{1/2}(\partial\Omega)]^3, [L^2(\partial\Omega)]^3)}^2, \quad (3.8)$$

where C is a positive constant.

Remark 3.2. *Since $\gamma_0 > 0$, then the system (3.6) admits a unique solution for any $\beta \geq 0$ that is exponentially stable. If $\gamma_0 = \beta = 0$ the system is no more stable hence the importance of taking $\gamma_0 > 0$.*

Proof. In the proof, we use the fact that the condition (3.7) is equivalent to

$$\tilde{g} \in W_\gamma^{1/2}(0, \infty; [H^1(\Omega)]^3, [L^2(\Omega)]^3), \quad \tilde{g}_{n_0} = 0.$$

This fact is obtained in [20, p.21, Theorem 2.3]. This type of demonstration can be found in [29], for the sake of completeness, we recall the main steps of the proof here and see how we can adapt some points of the proof in our case.

We consider the following problem

$$\begin{cases} \gamma_0 z - \nabla \cdot \mathbb{T}(z, \chi) = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot z = 0 & \text{in } (0, \infty) \times \Omega, \\ z_n = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(z)n + \beta z]_\tau = \tilde{g} & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \quad (3.9)$$

By a density argument, we assume that $\tilde{g} \in C_0^\infty(\mathbb{R}_+, [H^1(\Omega)]^3)$ and $\tilde{g}(t) = 0$ if $t < 0$. Then, the system (3.9), admits a unique solution $(z, \chi) \in C_0^\infty(\mathbb{R}_+, [H^2(\Omega)]^3) \times C_0^\infty(\mathbb{R}_+, H^1(\Omega)/\mathbb{R})$ (see [8]). We set then, $\tilde{z} = \tilde{v} - z$, $\tilde{\chi} = \tilde{\pi} - \chi$. Then, $(\tilde{z}, \tilde{\chi})$ satisfies the system

$$\begin{cases} \partial_t \tilde{z} + \gamma_0 \tilde{z} - \nabla \cdot \mathbb{T}(\tilde{z}, \tilde{\chi}) = -\partial_t z & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot \tilde{z} = 0 & \text{in } (0, \infty) \times \Omega, \\ \tilde{z}_n = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(\tilde{z})n + \beta \tilde{z}]_\tau = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ \tilde{z}(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (3.10)$$

The system above admits a solution using the fact that the operator

$$\mathbb{A} = \mathcal{P}\Delta - \gamma_0 \text{Id}, \quad \mathcal{D}(\mathbb{A}) = \{z \in [H^2(\Omega)]^3; z \cdot n = 0, [2\nu D(z)n + \beta z]_\tau = 0\},$$

generates an analytic semigroup exponentially stable where \mathcal{P} designates the Leray projection. This is a direct consequence of [10, Theorem 2.12, p.115]. We deduce then the existence of a unique solution $(\tilde{v}, \tilde{\pi}) \in C_0^\infty(\mathbb{R}_+, [H^2(\Omega)]^3) \times C_0^\infty(\mathbb{R}_+, H^1(\Omega)/\mathbb{R})$ of the system (3.6).

Now, we show (3.8). Let $\phi \in C_0^\infty(\mathbb{R}_+, [L^2(\Omega)]^3)$ such that $\nabla \cdot \phi = 0$, $\phi \cdot n = 0$, and we suppose that $\phi = 0$ on $(T, \infty) \times \Omega$ where $T > 0$. We consider the system

$$\begin{cases} \partial_t \psi - (\gamma_0 - \gamma)\psi + \nabla \cdot \mathbb{T}(\psi, \theta) = \phi & \text{in } \mathbb{R} \times \Omega, \\ \nabla \cdot \psi = 0 & \text{in } \mathbb{R} \times \Omega, \\ \psi_n = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ [2\nu D(\psi)n + \beta \psi]_\tau = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ \psi(T) = 0 & \text{in } \Omega. \end{cases} \quad (3.11)$$

We observe that $\psi = 0$, $\theta = 0$ on $(T, \infty) \times \Omega$. The system (3.11) admits a unique solution

$$(\psi, \theta) \in W(\mathbb{R}; [H^2(\Omega)]^3, [L^2(\Omega)]^3) \times L^2(\mathbb{R}; H^1(\Omega)/\mathbb{R}),$$

exponentially stable satisfying

$$\|\psi\|_{W(\mathbb{R}; [H^2(\Omega)]^3, [L^2(\Omega)]^3)} + \|\theta\|_{L^2(\mathbb{R}; H^1(\Omega)/\mathbb{R})} \leq C \|\phi\|_{L^2(\mathbb{R}; [L^2(\Omega)]^3)}. \quad (3.12)$$

We have

$$\begin{aligned} \int_{\mathbb{R}} \int_{\Omega} e^{\gamma t} \tilde{v} \cdot \phi \, dy dt &= \int_{\mathbb{R}} \int_{\Omega} e^{\gamma t} \tilde{v} \cdot (\partial_t \psi - (\gamma_0 - \gamma)\psi + \nabla \cdot \mathbb{T}(\psi, \theta)) \, dy dt \\ &= - \int_{\mathbb{R}} \int_{\Omega} e^{\gamma t} (\partial_t \tilde{v} + \gamma_0 \tilde{v} - \nabla \cdot \mathbb{T}(\tilde{v}, \tilde{\pi})) \cdot \psi \, dy dt - \int_{\mathbb{R}} \int_{\partial\Omega} e^{\gamma t} \tilde{g}_\tau \cdot \psi_\tau \, d\Gamma dt \\ &= - \int_{\mathbb{R}} \int_{\partial\Omega} e^{\gamma t} \tilde{g}_\tau \cdot \psi_\tau \, d\Gamma dt. \end{aligned}$$

Using (3.12), we get

$$\|e^{\gamma t} \tilde{v}\|_{L^2(\mathbb{R}; [L^2(\Omega)]^3)} \leq C \|e^{\gamma t} \tilde{g}\|_{L^2(\mathbb{R}; [H^1(\Omega)]^3)}. \quad (3.13)$$

In the other hand, we extend n on Ω and we obtain

$$\int_{\partial\Omega} e^{\gamma t} \partial_t \tilde{g} \cdot \psi \, d\Gamma = \sum_i \int_{\Omega} e^{\gamma t} \partial_i((n)_i \partial_t \tilde{g}) \cdot \psi \, dy + \sum_i \int_{\Omega} e^{\gamma t} ((n)_i \partial_t \tilde{g}) \cdot \partial_i \psi \, dy.$$

Then,

$$\int_{\mathbb{R}} \int_{\Omega} e^{\gamma t} \partial_t \tilde{v} \cdot \phi \, dy dt = - \sum_i \int_{\mathbb{R}} \int_{\Omega} (e^{\gamma t} \partial_i((n)_i \partial_t \tilde{g}) \cdot \psi + e^{\gamma t} ((n)_i \partial_t \tilde{g}) \cdot \partial_i \psi) \, dy dt. \quad (3.14)$$

To estimate $t \rightarrow e^{\gamma t} \partial_t \tilde{v}(t)$ in $L^2(\mathbb{R}, [L^2(\Omega)]^3)$, we notice that

$$\int_{\mathbb{R}} \int_{\Omega} e^{\gamma t} \partial_i((n)_i \partial_t \tilde{g}) \cdot \psi \, dy dt = -\gamma \int_{\mathbb{R}} \int_{\Omega} e^{\gamma t} \partial_i((n)_i \tilde{g}) \cdot \psi \, dy dt - \int_{\mathbb{R}} \int_{\Omega} e^{\gamma t} \partial_i((n)_i \tilde{g}) \cdot \partial_t \psi \, dy dt. \quad (3.15)$$

Then,

$$\begin{aligned} \left| \langle e^{\gamma t} \partial_i((n)_i \partial_t \tilde{g}), \psi \rangle_{L^2(\mathbb{R}; [L^2(\Omega)]^3)} \right| &\leq C \|e^{\gamma t} \tilde{g}\|_{L^2(\mathbb{R}; [H^1(\Omega)]^3)} \left(\|\partial_t \psi\|_{L^2(\mathbb{R}, [L^2(\Omega)]^3)} \right. \\ &\quad \left. + \gamma \|\psi\|_{L^2(\mathbb{R}, [L^2(\Omega)]^3)} \right). \end{aligned} \quad (3.16)$$

Moreover,

$$\langle e^{\gamma t} ((n)_i \partial_t \tilde{g}), \partial_t \psi \rangle_{L^2(\mathbb{R}; [L^2(\Omega)]^3)} = \langle \partial_t (e^{\gamma t} \tilde{g}), (n)_i \partial_i \psi \rangle_{L^2(\mathbb{R}; [L^2(\Omega)]^3)} - \gamma \langle e^{\gamma t} \tilde{g}, (n)_i \partial_i \psi \rangle_{L^2(\mathbb{R}; [L^2(\Omega)]^3)}.$$

Then, using [28, Proposition 2.3, Proposition 2.8], we obtain

$$\begin{aligned} \left| \langle e^{\gamma t} ((n)_i \partial_t \tilde{g}), \partial_t \psi \rangle_{L^2(\mathbb{R}; [L^2(\Omega)]^3)} \right| &\leq C \|\tilde{g}\|_{W^{1/2}(\mathbb{R}; [H^1(\Omega)]^3, [L^2(\Omega)]^3)} \|\nabla \psi\|_{W^{1/2}(\mathbb{R}; [H^1(\Omega)]^9, [L^2(\Omega)]^9)} \\ &\leq C \|\tilde{g}\|_{W^{1/2}(\mathbb{R}; [H^1(\Omega)]^3, [L^2(\Omega)]^3)} \|\psi\|_{W(\mathbb{R}; [H^2(\Omega)]^3, [L^2(\Omega)]^3)}. \end{aligned} \quad (3.17)$$

Combining (3.17), (3.16), (3.14) and (3.12) we have

$$\|e^{\gamma t} \partial_t \tilde{v}\|_{L^2(\mathbb{R}; [L^2(\Omega)]^3)} \leq C \|e^{\gamma t} \tilde{g}\|_{W^{1/2}(\mathbb{R}; [H^1(\Omega)]^3, [L^2(\Omega)]^3)}. \quad (3.18)$$

Finally, from the classical elliptic estimates of the Stokes problem, we have

$$\|\tilde{v}(t)\|_{[H^2(\Omega)]^3} + \|\nabla \tilde{\pi}(t)\|_{[L^2(\Omega)]^3} \leq C \left(\|\partial_t \tilde{v}(t)\|_{[L^2(\Omega)]^3} + \|\tilde{g}(t)\|_{[H^1(\Omega)]^3} \right). \quad (3.19)$$

Using (3.19) and (3.18), we obtain (3.8). \square

We set

$$w = \tilde{w} - \tilde{v}, \quad q = \tilde{q} - \tilde{\pi}.$$

Then, from lemma 3.1, we only need to consider the linear system

$$\begin{cases} \partial_t w + (w^S \cdot \nabla) w + (w \cdot \nabla) w^S - \nabla \cdot \mathbb{T}(w, q) - \nabla \cdot \mathcal{L}^1(\xi) + \mathcal{L}^2(\xi, \partial_t \xi) = f & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ \partial_{tt} \xi + A_1 \xi + A_2 \partial_t \xi + \mathcal{T}^*(\mathcal{L}^1(\xi) n) = -\mathcal{T}^*(\mathbb{T}(w, q) n) + h & \text{in } (0, \infty) \times \omega, \end{cases} \quad (3.20)$$

with the boundary conditions

$$\begin{cases} [w - \mathcal{T} \partial_t \xi]_n = 1_{\Gamma_0} (\mathbb{M} v)_n & \text{on } (0, \infty) \times \partial\Omega, \\ [2\nu D(w) n + \beta (w - \mathcal{T} \partial_t \xi) + \mathcal{L}^1(\xi) n + \mathcal{L}^3(\xi)]_{\tau} = 1_{\Gamma_0} \beta (\mathbb{M} v)_{\tau} & \text{on } (0, \infty) \times \partial\Omega, \end{cases} \quad (3.21)$$

and the initial conditions

$$\begin{cases} w(0, \cdot) = w^0 & \text{in } \Omega, \\ \xi(0, \cdot) = \xi^0 & \text{in } \omega, \\ \partial_t \xi(0, \cdot) = \xi^1 & \text{in } \omega, \end{cases} \quad (3.22)$$

where

$$f = \tilde{f} - (w^S \cdot \nabla) \tilde{v} - (\tilde{v} \cdot \nabla) w^S + \gamma_0 \tilde{v}, \quad h = \tilde{h} - \mathcal{T}^*(\mathbb{T}(\tilde{v}, \tilde{\pi})n). \quad (3.23)$$

In this section, we prove that the system (3.20), (3.21) and (3.22) is exponentially stabilisable.

We define

$$\mathbb{H} = \{(w, \eta_1, \eta_2) \in [L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega) ; \nabla \cdot w = 0 \text{ in } \Omega, [w - \mathcal{T}\eta_2]_n = 0 \text{ on } \partial\Omega\},$$

with the inner product

$$\left\langle \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} v \\ \xi_1 \\ \xi_2 \end{pmatrix} \right\rangle_{\mathbb{H}} = \langle w, v \rangle_{[L^2(\Omega)]^3} + \langle A_1^{1/2} \eta_1, A_1^{1/2} \xi_1 \rangle_{L^2(\omega)} + \langle \eta_2, \xi_2 \rangle_{L^2(\omega)}.$$

We define also

$$\mathbb{V} = \left([H^1(\Omega)]^3 \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}) \right) \cap \mathbb{H}, \quad (3.24)$$

and the orthogonal projection \mathbb{P}

$$\mathbb{P} : [L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega) \longrightarrow \mathbb{H}.$$

Finally, we set the operator \mathcal{A}_S

$$\mathcal{A}_S \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \nu \Delta w - (w^S \cdot \nabla) w - (w \cdot \nabla) w^S + \nabla \cdot \mathcal{L}^1(\eta_1) - \mathcal{L}^2(\eta_1, \eta_2) \\ -A_1 \eta_1 - A_2 \eta_2 - \mathcal{T}^*(2\nu D(w)n + \mathcal{L}^1(\eta_1)n) \end{pmatrix}, \quad (3.25)$$

$$\begin{aligned} \mathcal{D}(\mathcal{A}_S) = \{ & (w, \eta_1, \eta_2) \in \left([H^2(\Omega)]^3 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}) \right) \cap \mathbb{V}, \\ & [2\nu D(w)n + \beta(w - \mathcal{T}\eta_2) + \mathcal{L}^1(\eta_1)n + \mathcal{L}^3(\eta_1)]_\tau = 0 \text{ on } \partial\Omega\}, \end{aligned} \quad (3.26)$$

and the operator A_S defined as follows

$$A_S = \mathbb{P}\mathcal{A}_S, \quad \mathcal{D}(A_S) = \mathcal{D}(\mathcal{A}_S). \quad (3.27)$$

Proposition 3.3. *There exists $\lambda_0 > 0$ such that the operator $\lambda_0 \text{Id} - A_S$ is the infinitesimal generator of a strongly continuous semigroup of contractions on \mathbb{H} .*

Proof. Let $W = (w, \xi_1, \xi_2) \in \mathcal{D}(A_S)$. We show that $\lambda_0 \text{Id} - A_S$ is dissipative.

$$\begin{aligned} \langle (\lambda_0 \text{Id} - A_S)W, W \rangle &= \lambda_0 \int_{\Omega} |w|^2 dy + \int_{\Omega} (\nabla w^S)^* w \cdot w dy + 2\nu \int_{\Omega} |Dw|^2 dy \\ &+ \int_{\Omega} \mathcal{L}^1(\xi_1) : \nabla w dy + \int_{\Omega} \mathcal{L}^2(\xi_1, \xi_2) \cdot w dy + \int_{\partial\Omega} \beta |(w - \mathcal{T}\xi_2)_\tau|^2 d\Gamma \\ &+ \int_{\partial\Omega} [\mathcal{L}^3(\xi_1)]_\tau \cdot (w - \mathcal{T}\xi_2)_\tau d\Gamma + \int_{\omega} |A_2^{1/2} \xi_2|^2 ds + \lambda_0 \int_{\omega} |A_1^{1/2} \xi_1|^2 ds \\ &+ \lambda_0 \int_{\omega} |\xi_2|^2 ds. \end{aligned} \quad (3.28)$$

From (3.4) and the fact that Q^i , $i = 1, \dots, 8$ are of regularity $W^{1,\infty}(\Omega)$, then

$$\|\mathcal{L}^1(\xi_1)\|_{[L^2(\Omega)]^9}^2 \leq C \left\| A_1^{1/2} \xi_1 \right\|_{L_0^2(\omega)}^2, \quad (3.29)$$

$$\|\mathcal{L}^2(\xi_1, \xi_2)\|_{[L^2(\Omega)]^3}^2 \leq C \left(\left\| A_1^{1/2} \xi_1 \right\|_{L_0^2(\omega)}^2 + \left\| A_2^{1/2} \xi_2 \right\|_{L_0^2(\omega)}^2 \right). \quad (3.30)$$

Since ξ_1 is a periodic function, we get

$$\|\xi_1\|_{L^6(\omega)} + \|\partial_s \xi_1\|_{L^6(\omega)} \leq C \|\xi_1\|_{H^2(\omega)} \leq C \left\| A_1^{1/2} \xi_1 \right\|_{L_0^2(\omega)}.$$

Using that Q^9 and Q^{10} are of regularity $W^{1/2,\infty}(\partial\Omega) \hookrightarrow L^3(\partial\Omega)$, then we obtain

$$\|\mathcal{L}^3(\xi_1)\|_{[L^2(\Omega)]^3}^2 \leq C \left\| A_1^{1/2} \xi_1 \right\|_{L_0^2(\omega)}^2. \quad (3.31)$$

Using (3.29), (3.30) and (3.31), we find

$$\begin{aligned} & \int_{\Omega} (\nabla w^S)^* w \cdot w \, dy + \int_{\Omega} \mathcal{L}^1(\xi_1) : \nabla w \, dy + \int_{\Omega} \mathcal{L}^2(\xi_1, \xi_2) \cdot w \, dy \\ & \quad + \int_{\partial\Omega} [\mathcal{L}^3(\xi_1)]_{\tau} \cdot (w - \mathcal{T}\xi_2)_{\tau} \, d\Gamma \geq -\varepsilon \left(\|w\|_{[H^1(\Omega)]^3}^2 + \left\| A_2^{1/2} \xi_2 \right\|_{L^2(\Omega)}^2 \right) \\ & \quad - C \left(\|w\|_{[L^2(\Omega)]^3}^2 + \|\xi_2\|_{L^2(\omega)}^2 + \left\| A_1^{1/2} \xi_1 \right\|_{L^2(\omega)}^2 \right). \end{aligned} \quad (3.32)$$

Thanks to the classical Korn's inequality, we obtain for λ_0 large enough

$$\langle (\lambda_0 \text{Id} - A_S)W, W \rangle \geq 0.$$

It remains to show that the operator $\lambda_0 \text{Id} - A_S$ is m-dissipative: we prove that $\lambda_0 \text{Id} - A_S$ is onto.

Let $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathbb{H}$ and we consider the equation

$$(\lambda_0 \text{Id} - A_S) \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} = F, \quad (3.33)$$

$$\lambda_0 w - \nabla \cdot \mathbb{T}(w, q) + (w^S \cdot \nabla)w + (w \cdot \nabla)w^S - \nabla \cdot \mathcal{L}^1(\eta_1) + \mathcal{L}^2(\eta_1, \eta_2) = f \quad \text{in } \Omega, \quad (3.34a)$$

$$\nabla \cdot w = 0 \quad \text{in } \Omega, \quad (3.34b)$$

$$\lambda_0 \eta_1 - \eta_2 = g \quad \text{in } \omega, \quad (3.34c)$$

$$\lambda_0 \eta_2 + A_1 \eta_1 + A_2 \eta_2 = -\mathcal{T}^*(\mathbb{T}(w, q)n + \mathcal{L}^1(\eta_1)n) + h \quad \text{in } \omega, \quad (3.34d)$$

$$[w - \mathcal{T}\eta_2]_n = 0 \quad \text{on } \partial\Omega, \quad (3.34e)$$

$$[2\nu D(w)n + \beta(w - \mathcal{T}\eta_2) + \mathcal{L}^1(\eta_1)n + \mathcal{L}^3(\eta_1)]_{\tau} = 0 \quad \text{on } \partial\Omega. \quad (3.34f)$$

To solve the system above, we introduce the space

$$\mathcal{V} = \left\{ (\phi, \xi) \in [H^1(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \mid \nabla \cdot \phi = 0 \quad \text{in } \Omega, \quad [\phi - \mathcal{T}\xi]_n = 0 \quad \text{on } \partial\Omega \right\}.$$

Then, solving the equation (3.33) is reduced to solve the following variational problem: one need to find $(w, \eta_2) \in \mathcal{V}$ such that

$$a \left(\begin{pmatrix} w \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \xi \end{pmatrix} \right) = L \left(\begin{pmatrix} \phi \\ \xi \end{pmatrix} \right) \quad \left(\begin{pmatrix} \phi \\ \xi \end{pmatrix} \in \mathcal{V} \right), \quad (3.35)$$

with $a : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ the bilinear form given by

$$\begin{aligned} a \left(\begin{pmatrix} w \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \xi \end{pmatrix} \right) &= \lambda_0 \int_{\Omega} w \cdot \phi \, dy + \int_{\Omega} ((w^S \cdot \nabla)w + (w \cdot \nabla)w^S) \cdot \phi \, dy + 2\nu \int_{\Omega} D(w) : D(\phi) \, dy \\ &+ \frac{1}{\lambda_0} \int_{\Omega} \mathcal{L}^1(\eta_2) : \nabla \phi \, dy + \frac{1}{\lambda_0} \int_{\Omega} \mathcal{L}^{2,1}(\eta_2) \cdot \phi \, dy + \int_{\Omega} \mathcal{L}^{2,2}(\eta_2) \cdot \phi \, dy + \lambda_0 \int_{\omega} \eta_2 \cdot \xi \, ds \\ &+ \int_{\omega} (A_2 \eta_2) \cdot \xi \, ds + \frac{1}{\lambda_0} \int_{\omega} (A_1^{1/2} \eta_2) \cdot (A_1^{1/2} \xi) \, ds + \int_{\partial\Omega} \beta [w - \mathcal{T}(\eta_2)]_{\tau} \cdot [\phi - \mathcal{T}(\xi)]_{\tau} \, d\Gamma \\ &+ \frac{1}{\lambda_0} \int_{\partial\Omega} [\mathcal{L}^3(\eta_2)]_{\tau} \cdot [\phi - \mathcal{T}(\xi)]_{\tau} \, d\Gamma, \end{aligned}$$

and $L : \mathcal{V} \rightarrow \mathbb{R}$ the linear form defined by

$$\begin{aligned} L \left(\begin{pmatrix} \phi \\ \xi \end{pmatrix} \right) &= \int_{\Omega} f \cdot \phi \, dy + \int_{\omega} h \cdot \xi \, ds - \frac{1}{\lambda_0} \int_{\omega} (A_1^{1/2} g) \cdot (A_1^{1/2} \xi) \, ds \\ &- \frac{1}{\lambda_0} \int_{\Omega} \mathcal{L}^1(g) : \nabla \phi \, dy - \frac{1}{\lambda_0} \int_{\Omega} \mathcal{L}^{2,1}(g) \cdot \phi \, dy - \frac{1}{\lambda_0} \int_{\partial\Omega} [\mathcal{L}^3(g)]_{\tau} \cdot [\phi - \mathcal{T}(\xi)]_{\tau} \, d\Gamma. \end{aligned}$$

The bilinear form a is continuous and coercive on \mathcal{V} thanks to the classical Korn's inequality and L is continuous on \mathcal{V} . Using the Lax-Milgram theorem, there exists a unique $(w, \eta_2) \in \mathcal{V}$ that is solution to (3.35). Now, taking $\xi = 0$ and $\phi \in \mathcal{D}_{\sigma}(\Omega)$, the equation (3.35) becomes

$$\begin{aligned} \lambda_0 \int_{\Omega} w \cdot \phi \, dy + \int_{\Omega} ((w^S \cdot \nabla)w + (w \cdot \nabla)w^S) \cdot \phi \, dy + 2\nu \int_{\Omega} D(w) : D(\phi) \, dy \\ + \int_{\Omega} \mathcal{L}^1(\eta_1) : \nabla \phi \, dy + \int_{\Omega} \mathcal{L}^2(\eta_1, \eta_2) \cdot \phi \, dy = \int_{\Omega} f \cdot \phi \, dy, \end{aligned}$$

that is equivalent to

$$\langle \lambda_0 w + (w^S \cdot \nabla)w + (w \cdot \nabla)w^S - \nu \Delta w - \nabla \cdot \mathcal{L}^1(\eta_1) + \mathcal{L}^2(\eta_1, \eta_2) - f, \phi \rangle = 0, \quad \forall \phi \in \mathcal{D}_{\sigma}(\Omega).$$

Using the De Rham theorem [30, Proposition 1.2, p.14], we deduce the existence of a unique element $q \in L^2(\Omega)/\mathbb{R}$ such that (3.34a) is satisfied. In particular, we have $\nabla \cdot \mathbb{T}(w, q) \in [L^2(\Omega)]^3$ and $\mathbb{T}(w, q) \in [L^2(\Omega)]^9$. Consequently, we get $\mathbb{T}(w, q)n \in [H^{-1/2}(\partial\Omega)]^3$ and

$$\begin{aligned} \int_{\Omega} \mathbb{T}(w, q) : D(\phi) \, dy - \langle \mathbb{T}(w, q)n, \phi \rangle_{H^{-1/2}, H^{1/2}} \\ = \int_{\Omega} (f - \lambda_0 w - (w^S \cdot \nabla)w - (w \cdot \nabla)w^S - \mathcal{L}^2(\eta_1, \eta_2)) \cdot \phi \, dy - \int_{\Omega} \mathcal{L}^1(\eta_1) : \nabla \phi \, dy \\ + \int_{\partial\Omega} [\mathcal{L}^1(\eta_1)n]_{\tau} \cdot \phi_{\tau} \, d\Gamma, \quad (3.36) \end{aligned}$$

for all $\phi \in [H^1(\Omega)]^3$, $\phi_n = 0$. Taking $\xi = 0$ in (3.35), we obtain

$$\begin{aligned} \lambda_0 \int_{\Omega} w \cdot \phi \, dy + 2\nu \int_{\Omega} D(w) : D(\phi) \, dy + \int_{\Omega} \mathcal{L}^1(\eta_1) : \nabla \phi \, dy + \int_{\Omega} \mathcal{L}^2(\eta_1, \eta_2) \cdot \phi \, dy \\ + \int_{\Omega} ((w^S \cdot \nabla)w + (w \cdot \nabla)w^S) \cdot \phi \, dy + \langle [\beta(w - \mathcal{T}(\eta_2)) + \mathcal{L}^3(\eta_1)]_{\tau}, \phi \rangle_{H^{-1/2}, H^{1/2}} \\ = \int_{\Omega} f \cdot \phi \, dy, \quad (3.37) \end{aligned}$$

for any $\phi \in [H^1(\Omega)]^3$, $\nabla \cdot \phi = 0$, $\phi_n = 0$. Comparing (3.36) and (3.37) and taking into account the fact

$$\int_{\Omega} \mathbb{T}(w, q) : D(\phi) \, dy = 2\nu \int_{\Omega} D(w) : D(\phi) \, dy, \quad \forall \phi \in [H^1(\Omega)]^3, \nabla \cdot \phi = 0, \phi_n = 0,$$

we obtain for all $\phi \in [H^1(\Omega)]^3$ satisfying $\nabla \cdot \phi = 0$ and $\phi_n = 0$

$$- \langle \mathbb{T}(w, q)n, \phi \rangle_{H^{-1/2}, H^{1/2}} = \langle [\beta(w - \mathcal{T}\eta_2) + \mathcal{L}^1(\eta_1)n + \mathcal{L}^3(\eta_1)]_{\tau}, \phi \rangle_{H^{-1/2}, H^{1/2}}. \quad (3.38)$$

Then, (3.38) is also satisfied for all $\phi \in [H^1(\Omega)]^3$, $\phi_n = 0$ since we can always construct a divergence free function on Ω that coincides with ϕ at the boundary. Plugging (3.38) in (3.36), we obtain

$$\begin{aligned} \int_{\Omega} ((w^S \cdot \nabla)w + (w \cdot \nabla)w^S) \cdot \phi \, dy + 2\nu \int_{\Omega} D(w) : D(\phi) \, dy - \int_{\Omega} q \nabla \cdot \phi \, dy \\ + \langle [\beta(w - \mathcal{T}\eta_2) + \mathcal{L}^3(\eta_1)]_{\tau}, \phi_{\tau} \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega} (f - \lambda_0 w - \mathcal{L}^2(\eta_1, \eta_2)) \cdot \phi \, dy \\ - \int_{\Omega} \mathcal{L}^1(\eta_1) : \nabla \phi \, dy, \end{aligned} \quad (3.39)$$

for any $\phi \in [H^1(\Omega)]^3$, $\phi_n = 0$.

Then, we deduce that (w, q) is a weak solution of (3.34a), (3.34b), (3.34e) and (3.34f) in the sens of the [9, Definition, p.10]. Since $\eta_2 \in H^2(\omega)$ then $\mathcal{T}\eta_2 \in [H^2(\partial\Omega)]^3$, we can apply [9, Théorème 1.2] to get that $(w, q) \in [H^2(\Omega)]^3 \times H^1(\Omega)/\mathbb{R}$.

Then, going back to (3.35), we get

$$\begin{aligned} \int_{\omega} (A_1^{1/2} \eta_1) \cdot (A_1^{1/2} \xi) \, ds = -\lambda \int_{\omega} \eta_2 \cdot \xi \, ds - \int_{\omega} (A_2 \eta_2) \cdot \xi \, ds \\ - \int_{\omega} \mathcal{T}^*(\mathbb{T}(u, q)n + \mathcal{L}^1(\eta_1)n) \cdot \xi \, ds + \int_{\omega} h \cdot \xi \, ds, \end{aligned} \quad (3.40)$$

for all $\xi \in \mathcal{D}(A_1^{1/2})$ where $\eta_1 = \frac{1}{\lambda}(g + \eta_2)$. We notice that we have $\eta_1 \in \mathcal{D}(A_1^{1/2})$, then $\mathcal{T}^*(\mathcal{L}^1(\eta_1)n) \in L_0^2(\omega)$. In the other hand, we have $\mathbb{T}(w, q)n \in [H^{1/2}(\partial\Omega)]^3$, then it implies that $\mathcal{T}^*(\mathbb{T}(w, q)n) \in L_0^2(\omega)$. Moreover, since $\eta_2 \in H^2(\omega)$, we deduce that $\eta_2 \in \mathcal{D}(A_2)$. Thus, using the fact that $\mathcal{D}(A_1^{1/2})$ is dense in $L_0^2(\omega)$, we obtain $A_1 \eta_1 \in L_0^2(\omega)$.

Applying the Lumer-Phillips theorem, we deduce that $\lambda_0 \text{Id} - A_S$ generates a strongly continuous semigroup of contractions on \mathbb{H} . \square

Proposition 3.4. *The adjoint operator of A_S is given by*

$$A_S^* \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathbb{P} \begin{pmatrix} \nu \Delta w + (w^S \cdot \nabla)w - (\nabla w^S)^* w \\ -\eta_2 - A_1^{-1} ((\mathcal{L}^1)^*(\nabla w) + (\mathcal{L}^{2,1})^* w + (\mathcal{L}^3)^* w - ((\mathcal{L}^3)^*(\mathcal{T}\eta_2)) \\ A_1 \eta_1 - A_2 \eta_2 - \mathcal{T}^*(2\nu D(w)n) - (\mathcal{L}^{2,2})^* w \end{pmatrix}, \quad (3.41)$$

and

$$\begin{aligned} \mathcal{D}(A_S^*) = \{(w, \eta_1, \eta_2) \in ([H^2(\Omega)]^3 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})) \cap \mathbb{V}, \\ [2\nu D(w)n + \beta(w - \mathcal{T}\eta_2)]_{\tau} = 0 \text{ on } \partial\Omega\}. \end{aligned} \quad (3.42)$$

Proof. Let $(w, \eta_1, \eta_2) \in \mathcal{D}(A_S)$ and $(\varphi, \zeta_1, \zeta_2) \in \mathcal{D}(A_S^*)$. We have

$$\begin{aligned} \langle A_S(w, \eta_1, \eta_2), (\varphi, \zeta_1, \zeta_2) \rangle &= -2\nu \int_{\Omega} D(w) : D(\varphi) \, dy + \int_{\Omega} (w^S \cdot \nabla) \varphi \cdot w \\ &\quad - \int_{\Omega} (\nabla w^S)^* \varphi \cdot w \, dy - \int_{\Omega} \mathcal{L}^1(\eta_1) : \nabla w \, dy - \int_{\Omega} \mathcal{L}^2(\eta_1, \eta_2) \cdot w \, dy \\ &\quad - \int_{\partial\Omega} \beta(w - \mathcal{T}\eta_2)_{\tau} \cdot (\varphi - \mathcal{T}\zeta_2)_{\tau} \, d\Gamma - \int_{\partial\Omega} [\mathcal{L}^3(\eta_1)]_{\tau} \cdot (\varphi - \mathcal{T}\zeta_2)_{\tau} \, d\Gamma \\ &\quad - \int_{\omega} (A_2\zeta_2) \cdot \eta_2 \, ds. \end{aligned} \quad (3.43)$$

Then

$$\begin{aligned} \langle A_S(w, \eta_1, \eta_2), (\varphi, \zeta_1, \zeta_2) \rangle &= \nu \int_{\Omega} \Delta \varphi \cdot w \, dy + \int_{\Omega} (w^S \cdot \nabla) \varphi \cdot w \\ &\quad - \int_{\Omega} (\nabla w^S)^* \varphi \cdot w \, dy - \int_{\Omega} \mathcal{L}^1(\eta_1) : \nabla \varphi \, dy - \int_{\Omega} \mathcal{L}^2(\eta_1, \eta_2) \cdot \varphi \, dy \\ &\quad - \int_{\partial\Omega} [\mathcal{L}^3(\eta_1)]_{\tau} \cdot (\varphi - \mathcal{T}\zeta_2)_{\tau} \, d\Gamma - \int_{\omega} (A_2\zeta_2) \cdot \eta_2 \, ds - \int_{\omega} \mathcal{T}^*(2\nu D(\varphi)n) \eta_2 \, ds. \end{aligned} \quad (3.44)$$

Using (3.4) and (3.5), we obtain (3.41). \square

Proposition 3.5. For $\theta \in [0, 1]$, we have

$$\mathcal{D}((\lambda_0 \text{Id} - A_S)^{\theta}) = [\mathcal{D}(A), \mathbb{H}]_{1-\theta}, \quad \mathcal{D}((\lambda_0 \text{Id} - A_S^*)^{\theta}) = [\mathcal{D}(A^*), \mathbb{H}]_{1-\theta}, \quad (3.45)$$

for $\lambda_0 > 0$ large enough.

Proof. The proof is a direct consequence of proposition 3.3 and of [10, Proposition 6.1, p.171]. \square

Proposition 3.6. The operator A_S defined by (3.25) and (3.27) is the infinitesimal generator of an analytical semigroup on \mathbb{H} .

Proof. We notice that $A_S^* = -A^* + O_S$ where A is the operator defined by

$$\begin{aligned} \mathcal{D}(A) &= \{(w, \eta_1, \eta_2) \in ([H^2(\Omega)]^3 \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})) \cap \mathbb{V}, \\ &\quad [2\nu D(w)n + \beta(w - \mathcal{T}\eta_2)]_{\tau} = 0 \text{ on } \partial\Omega\}, \end{aligned} \quad (3.46)$$

$$\mathcal{A} \begin{pmatrix} w \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\nu \Delta w \\ -\eta_2 \\ A_1 \eta_1 + A_2 \eta_2 + \mathcal{T}^*(2\nu D(w)n) \end{pmatrix}, \quad (3.47)$$

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}), \quad A = \mathbb{P}\mathcal{A}, \quad (3.48)$$

and

$$O_S \begin{pmatrix} w \\ \xi_1 \\ \xi_2 \end{pmatrix} = \mathbb{P} \begin{pmatrix} (w^S \cdot \nabla)w - (\nabla w^S)^* w \\ -A_1^{-1} ((\mathcal{L}^1)^*(\nabla w) + (\mathcal{L}^{2,1})^* w + (\mathcal{L}^3)^* w - ((\mathcal{L}^3)^*(\mathcal{T}\xi_2)) \\ -(\mathcal{L}^{2,2})^* w \end{pmatrix}.$$

We precise here

$$\mathcal{D}((A^*)^{1/2}) = \mathbb{V}.$$

From [14], A is the infinitesimal generator of an analytical semigroup on \mathbb{H} . Then, from [22, Corollaire 2.4, p.81], we only need to show that $\mathbb{V} \subset \mathcal{D}(O_S)$. In fact, let $(w, \xi_1, \xi_2) \in \mathbb{V}$ and $(\phi, \eta_1, \eta_2) \in \mathbb{H}$. First, we use (3.5) and we observe that for $\eta_2 \in H^1(\omega) \cap L_0^2(\omega)$

$$\begin{aligned} - \int_{\omega} (\mathcal{L}^{2,2})^* w \cdot \eta_2 \, ds &= - \int_{\Omega} w \cdot \mathcal{L}^{2,2}(\eta_2) \, dy = - \int_{\Omega} w \cdot (Q^7 \eta_2 + Q^8 \nabla \eta_2) \, dy \\ &= - \int_{\Omega} w \cdot Q^7 \eta_2 \, dy - \int_{\Omega} w \cdot Q^8 \nabla \eta_2 \, dy. \end{aligned} \quad (3.49)$$

By integration by parts, we find that

$$\left| \int_{\Omega} w \cdot Q^7 \eta_2 \, dy + \int_{\Omega} w \cdot Q^8 \nabla \eta_2 \, dy \right| \leq C \|w\|_{[H^1(\Omega)]^3} \|\eta_2\|_{L^2(\omega)}.$$

Then, by density, we get

$$\begin{aligned} \left\langle O_S \begin{pmatrix} w \\ \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \varphi \\ \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle_{\mathbb{H}} &\leq C \left(\|w\|_{[H^1(\Omega)]^3} \left(\|\varphi\|_{[L^2(\Omega)]^3} + \|A^{1/2} \eta_1\|_{L^2(\omega)} \right) \right. \\ &\quad \left. + \|A^{1/2} \eta_1\|_{L^2(\omega)} \|\xi_2\|_{L^2(\omega)} + \|w\|_{[H^1(\Omega)]^3} \|\eta_2\|_{L^2(\omega)} \right) \\ &\leq C \|(w, \xi_1, \xi_2)\|_{\mathbb{V}} \|(\varphi, \eta_1, \eta_2)\|_{\mathbb{H}}. \end{aligned} \quad (3.50)$$

We deduce that A_S is the infinitesimal generator of an analytical semigroup on \mathbb{H} . Moreover, A_S admits a compact resolvent. \square

Let

$$\mathcal{V}_n^s(\Gamma_0) = \{v \in H^s(\Gamma_0) ; \int_{\Gamma_0} v \cdot n \, d\Gamma = 0\}, \quad s \geq 0.$$

From [27], the system (3.20), (3.21), (3.22) is equivalent to

$$\begin{aligned} \mathbb{P}W' &= A_S \mathbb{P}W + \mathbb{P}F + Bv, & \mathbb{P}W(0) &= \mathbb{P}W^0, \\ (\text{Id} - \mathbb{P})W' &= (\text{Id} - \mathbb{P})Dv, \end{aligned} \quad (3.51)$$

with

$$\xi_1 = \xi, \quad \xi_2 = \partial_t \xi, \quad W = \begin{pmatrix} w \\ \xi_1 \\ \xi_2 \end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0 \\ h \end{pmatrix}, \quad W^0 = \begin{pmatrix} w^0 \\ \xi^0 \\ \xi^1 \end{pmatrix},$$

where $D \in \mathcal{L}(\mathcal{V}_n^0(\Gamma_0), [L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega))$ such that $Dv = (\bar{w}, \eta_1, \eta_2)$ verifies the system

$$\left\{ \begin{array}{ll} \lambda_0 \bar{w} - \nabla \cdot (\mathbb{T}(\bar{w}, \pi)) + (w^S \cdot \nabla) \bar{w} + (\bar{w} \cdot \nabla) w^S - \nabla \cdot \mathcal{L}^1(\eta_1) + \mathcal{L}^2(\eta_1, \eta_2) = 0 & \text{in } \Omega, \\ \nabla \cdot \bar{w} = 0 & \text{in } \Omega, \\ \lambda_0 \eta_1 - \eta_2 = 0 & \text{in } \omega, \\ \lambda_0 \eta_2 + A_1 \eta_1 + A_2 \eta_2 + \mathcal{T}^*(\mathbb{T}(\bar{w}, \pi)n + \mathcal{L}^1(\eta_1)n) = 0 & \text{in } \omega, \\ [\bar{w} - \mathcal{T} \eta_2]_n = 1_{\Gamma_0} v_n & \text{on } \partial\Omega, \\ [2\nu D(\bar{w})n + \beta(\bar{w} - \mathcal{T} \eta_2) + \mathcal{L}^1(\eta_1)n + \mathcal{L}^3(\eta_1)]_{\tau} = 1_{\Gamma_0} (\beta v_{\tau}) & \text{on } \partial\Omega. \end{array} \right.$$

with $\lambda_0 \in \rho(A_S)$, and the operator B is defined by

$$B : v \in \mathbb{U} = [H^{3/2}(\Gamma_0)]^3 \longrightarrow (\lambda_0 \text{Id} - A_S) \mathbb{P}D(\mathbb{M}v) \in (\mathcal{D}(A_S^*))'.$$

We notice that

$$(\lambda_0 \text{Id} - A_S)^{-1+\varepsilon} B \in \mathcal{L}([H^{3/2}(\Gamma_0)]^3, \mathbb{H}), \quad \forall \varepsilon \in (0, 1/4).$$

The adjoint operator $D^* \in \mathcal{L}([L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega), [L^2(\Gamma_0)]^3)$ is defined for all $(\bar{f}, \bar{g}, \bar{h}) \in [L^2(\Omega)]^3 \times \mathcal{D}(A_1^{1/2}) \times L_0^2(\omega)$ by

$$\begin{cases} D^*(\bar{f}, \bar{g}, \bar{h}) = ((\mathbb{T}(\phi, r)n)_n + (\mathbb{T}(\phi, r)n)_\tau)|_{\Gamma_0}, & \text{if } \beta_1 > 0, \\ D^*(\bar{f}, \bar{g}, \bar{h}) = (\mathbb{T}(\phi, r)n)_n|_{\Gamma_0}, & \text{if } \beta_1 = 0, \end{cases}$$

such that $(\phi, r, \zeta_1, \zeta_2) \in [H^2(\Omega)]^3 \times H^1(\Omega)/\mathbb{R} \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})$ is solution of the system

$$\begin{cases} \lambda_0 \phi - \nabla \cdot (\mathbb{T}(\phi, r)) - (w^S \cdot \nabla) \phi + (\nabla w^S)^* \phi = \bar{f} & \text{in } \Omega, \\ \nabla \cdot \phi = 0 & \text{in } \Omega, \\ \lambda_0 \zeta_1 + \zeta_2 + A_1^{-1} (((\mathcal{L}^1)^*)(\nabla \phi) + (\mathcal{L}^{2,1})^* \phi - (\mathcal{L}^3)^* \phi - (\mathcal{L}^3)^*(\mathcal{T}\zeta_2)) = \bar{g} & \text{in } \omega, \\ \lambda_0 \zeta_2 - A_1 \zeta_1 + A_2 \zeta_2 + \mathcal{T}^*(\mathbb{T}(\phi, r)n) + (\mathcal{L}^{2,2})^* \phi = \bar{h} & \text{in } \omega, \\ (\phi - \mathcal{T}\zeta_2)_n = 0 & \text{on } \partial\Omega, \\ [2\nu D(\phi)n + \beta(\phi - \mathcal{T}\zeta_2)]_\tau = 0 & \text{on } \partial\Omega. \end{cases}$$

We precise that (ϕ, ζ_1, ζ_2) is solution of the system

$$(\lambda_0 - A_S^*)(\phi, \zeta_1, \zeta_2) = \mathbb{P}(\bar{f}, \bar{g}, \bar{h}). \quad (3.52)$$

If $\beta_1 > 0$, the operator B^* is given by

$$B^*(\phi, \zeta_1, \zeta_2) = m \left((\mathbb{T}(\phi, r)n) \cdot n - \int_{\Gamma_0} m \mathbb{T}(\phi, r)n \cdot n \, d\Gamma \right) n + m (\mathbb{T}(\phi, r)n)_\tau \Big|_{\Gamma_0},$$

and if $\beta_1 = 0$, we get

$$B^*(\phi, \zeta_1, \zeta_2) = m \left((\mathbb{T}(\phi, r)n) \cdot n - \int_{\Gamma_0} m \mathbb{T}(\phi, r)n \cdot n \, d\Gamma \right) n \Big|_{\Gamma_0},$$

where $r \in H^1(\Omega)/\mathbb{R}$, is the solution of the problem

$$\begin{pmatrix} \nabla r \\ 0 \\ -\mathcal{T}^*(rn) \end{pmatrix} = (I - \mathbb{P})\mathcal{A}_S^* \begin{pmatrix} \phi \\ \zeta_1 \\ \zeta_2 \end{pmatrix}. \quad (3.53)$$

Now, we establish a first result of stabilization of the linear system (3.20), (3.21) and (3.22). To do so, it suffices to verify that (A_S, B) satisfies the Fattorini-hautus criterion (UC_σ) and to apply [15, Theorem 1.1].

Let $\sigma > 0$ and $\lambda \in \mathbb{C}$ such that $\text{Re } \lambda \geq -\sigma$. Let $(\phi, r, \zeta_1, \zeta_2)$ solution of the system

$$\begin{cases} \lambda \phi - \nabla \cdot (\mathbb{T}(\phi, r)) - (w^S \cdot \nabla) \phi + (\nabla w^S)^* \phi = 0 & \text{in } \Omega, \\ \nabla \cdot \phi = 0 & \text{in } \Omega, \\ \lambda \zeta_1 + \zeta_2 + A_1^{-1} (((\mathcal{L}^1)^*)(\nabla \phi) + (\mathcal{L}^{2,1})^* \phi - (\mathcal{L}^3)^* \phi + (\mathcal{L}^3)^*(\mathcal{T}\zeta_2)) = 0 & \text{in } \omega, \\ \lambda \zeta_2 - A_1 \zeta_1 + A_2 \zeta_2 + \mathcal{T}^*(\mathbb{T}(\phi, r)n) + (\mathcal{L}^{2,2})^* \phi = 0 & \text{in } \omega, \\ (\phi - \mathcal{T}\zeta_2)_n = 0 & \text{on } \partial\Omega, \\ [2\nu D(\phi)n + \beta(\phi - \mathcal{T}\zeta_2)]_\tau = 0 & \text{on } \partial\Omega. \end{cases}$$

and assume that $B^*(\phi, \zeta_1, \zeta_2) = 0$, it implies that for $\beta_1 \geq 0$, we have

$$m \left((\mathbb{T}(\phi, r)n) \cdot n - \int_{\Gamma_0} m \mathbb{T}(\phi, r)n \cdot n \, d\Gamma \right) n + m (\mathbb{T}(\phi, r)n)_\tau = 0. \quad (3.54)$$

We set

$$c(\phi, r) = \int_{\Gamma_0} m \mathbb{T}(\phi, r)n \cdot n \, d\Gamma.$$

The equation (3.54) becomes

$$m(\mathbb{T}(\phi, r - c(\phi, r))n)_n + m(\mathbb{T}(\phi, r - c(\phi, r))n)_\tau = 0. \quad (3.55)$$

We deduce that

$$\mathbb{T}(\phi, r - c(\phi, r))n = 0, \quad \text{on } \Gamma_0.$$

Then, from [16], we get

$$\phi = 0, \quad \text{in } \Omega, \quad r = c(\phi, r), \quad \text{on } \Omega.$$

In particular

$$\mathbb{T}(\phi, r - c(\phi, r))n = 0, \quad \text{on } \partial\Omega. \quad (3.56)$$

Thus, we obtain

$$\mathcal{T}^*(\mathbb{T}(\phi, r)n) = -c(\phi, r)\mathcal{T}^*(n), \quad \text{on } \partial\Omega. \quad (3.57)$$

From the definition of the operator \mathcal{T}^* given in (2.51), we have that $\mathcal{T}^*(n) = 0$. Then, from (3.57), we obtain

$$\mathcal{T}^*(\mathbb{T}(\phi, r)n) = 0, \quad \text{on } \partial\Omega. \quad (3.58)$$

Moreover, we have $\mathcal{T}\zeta_2 = 0$ on $\partial\Omega$, then $\zeta_2 = 0$. The equation (3.58) implies that $A_1\zeta_1 = 0$ and taking into account the periodicity of ζ_1 , we find $\zeta_1 = 0$. Then, the condition (UC_σ) is verified. We deduce then the following theorem.

Theorem 3.7. *Let $t_0 > 0$, $\sigma > 0$ and $W^0 \in \mathbb{V}$. Then, there exists $N_+ \in \mathbb{N}^*$, $K \in L_{\text{loc}}^\infty(\mathbb{R}^2; \mathcal{L}(\mathbb{H}))$, $(\phi_k, \zeta_1^k, \zeta_2^k) \in \mathcal{D}(A_S^*)$ and $v_k \in B^*(\mathcal{D}(A_S^*))$, $k = 1, \dots, N_+$, such that*

$$v(t) = 1_{[t_0, +\infty)}(t) \sum_{k=1}^{N_+} \left(W(t - t_0) + \int_0^{t-t_0} K(t - t_0, s)W(s) ds, \begin{pmatrix} \phi_k \\ \zeta_1^k \\ \zeta_2^k \end{pmatrix} \right)_{\mathbb{H}} v_k, \quad (3.59)$$

stabilizes exponentially the system (3.51) and we have

$$\|W\|_{L^2_\sigma(0, \infty; \mathcal{D}(A_S)) \cap C^0_\sigma([0, \infty); \mathbb{V}) \cap H^1_\sigma(0, \infty; \mathbb{H})} \leq C (\|W^0\|_{\mathbb{V}} + \|F\|_{L^2_\sigma(0, \infty; \mathbb{H})}). \quad (3.60)$$

Finally, using (3.60), (3.8) and (3.23), we obtain the following corollary.

Corollary 3.8. *Let $\gamma_0 > 0$. For all $\gamma \in [0, \gamma_0[$, the solution $(\tilde{w}, \tilde{q}, \xi)$ of the system (3.1), (3.2), (3.3) verifies the estimate*

$$\begin{aligned} \|(\tilde{w}, \tilde{q}, \xi)\|_{\mathcal{X}_{\infty, \gamma}} \leq C \left(\|(w^0, \xi^0, \xi^1)\|_{\mathbb{V}} + \|\tilde{f}\|_{L^2_\gamma(0, +\infty; [L^2(\Omega)]^3)} + \|\tilde{h}\|_{L^2_\gamma(0, +\infty; L^2(\omega))} \right. \\ \left. + \|\tilde{g}\|_{W_\gamma^{1/4}(0, +\infty; [H^{1/2}(\partial\Omega)]^3, [L^2(\partial\Omega)]^3)} \right). \quad (3.61) \end{aligned}$$

4 Fixed point

Let \mathcal{Y}_∞ and $\mathcal{B}_{\infty, R}$ be given respectively by

$$\mathcal{Y}_\infty = L^2_\gamma(0, \infty; [L^2(\Omega)]^3) \times W_\gamma^{1/4}(0, \infty; [H^{1/2}(\partial\Omega)]^3, [L^2(\partial\Omega)]^3) \times L^2_\gamma(0, \infty; L^2(\omega)). \quad (4.1)$$

$$\mathcal{B}_{\infty, R} = \{(\tilde{f}, \tilde{g}, \tilde{h}) \in \mathcal{Y}_\infty \mid \|(\tilde{f}, \tilde{g}, \tilde{h})\|_{\mathcal{Y}_\infty} \leq R\}. \quad (4.2)$$

Let $(\tilde{f}, \tilde{g}, \tilde{h}) \in \mathcal{B}_{\infty, R}$ and let (u, p, ξ) be the solution of the system (3.1), (3.2), (3.3) associated to $(\tilde{f}, \tilde{g}, \tilde{h})$. From (3.61), we obtain

$$\|(u, p, \xi)\|_{\mathcal{X}_{\infty, \gamma}} \leq CR, \quad (4.3)$$

where we supposed that

$$\|(w^0, \xi^0, \xi^1)\|_{\mathbb{V}} \leq R.$$

From (4.3), we have the following estimates

$$\begin{aligned} \|\xi\|_{L_\gamma^\infty(0, \infty; L^\infty(\omega))} + \|\partial_{s_j} \xi\|_{L_\gamma^\infty(0, \infty; L^\infty(\omega))} + \left\| \partial_{s_j s_k}^2 \xi \right\|_{L_\gamma^\infty(0, \infty; L^2(\omega))} \\ + \left\| \partial_{s_j s_k s_l}^3 \xi \right\|_{L_\gamma^\infty(0, \infty; L^2(\omega))} \leq C \|\xi\|_{L_\gamma^\infty(0, \infty; H^3(\omega))} \leq CR, \end{aligned} \quad (4.4)$$

$$\|\partial_t \xi\|_{L_\gamma^4(0, \infty; L^\infty(\omega))} + \left\| \partial_{t s_j}^2 \xi \right\|_{L_\gamma^6(0, \infty; L^2(\omega))} \leq CR, \quad (4.5)$$

$$\|\xi\|_{H_\gamma^{7/8}(0, \infty; L^\infty(\omega))} + \|\partial_{s_j} \xi\|_{H_\gamma^{7/8}(0, \infty; L^\infty(\omega))} + \left\| \partial_{s_j s_k}^2 \xi \right\|_{H_\gamma^{7/8}(0, \infty; L^{8/3}(\omega))} \leq CR, \quad (4.6)$$

$$\|\xi\|_{L_\gamma^\infty(0, \infty; H^2(\omega))} + \|\partial_{s_j} \xi\|_{L_\gamma^\infty(0, \infty; H^{3/2}(\omega))} + \left\| \partial_{s_j s_k}^2 \xi \right\|_{L_\gamma^\infty(0, \infty; H^{1/2}(\omega))} \leq CR. \quad (4.7)$$

In particular, taking into account the condition (1.32), there exists $R_0 > 0$ such that, if $R \leq R_0$, then

$$\left\| \frac{1}{1 + \eta} \right\|_{L^\infty(0, \infty; L^\infty(\omega))} \leq C(R_0). \quad (4.8)$$

In fact, if $1 + \eta^S > \bar{\delta}$, then R_0 is chosen such that $0 < R_0 < \bar{\delta}$.

Here, we recall a well known result about product in Sobolev spaces.

Proposition 4.1. *Let $s \geq 0$, $s_1 \geq s$ and $s_2 \geq s$ such that $s_1 + s_2 > s + 1/2$. Let \mathfrak{X}_1 , \mathfrak{X}_2 and \mathfrak{X}_3 be three Banach spaces such that for all $f \in \mathfrak{X}_2$ and $g \in \mathfrak{X}_3$, we have*

$$\|fg\|_{\mathfrak{X}_1} \leq C \|f\|_{\mathfrak{X}_2} \|g\|_{\mathfrak{X}_3}.$$

Then

$$\forall u \in H^{s_1}(0, \infty; \mathfrak{X}_2), \forall v \in H^{s_2}(0, \infty; \mathfrak{X}_3), \quad \|uv\|_{H^s(0, \infty; \mathfrak{X}_1)} \leq C \|u\|_{H^{s_1}(0, \infty; \mathfrak{X}_2)} \|v\|_{H^{s_2}(0, \infty; \mathfrak{X}_3)}.$$

Proof. The proof is similar to the one given in [32, Theorem 2] extending data by reflexion on \mathbb{R} . \square

Lemma 4.2. *Let X and Y given in (1.17). Then, we have*

$$\nabla Y(X) - I_3 = \ell^{(1)}(\xi, \partial_s \xi) + \epsilon^{(1)}(\xi, \partial_s \xi), \quad (4.9)$$

$$\frac{\partial^2 Y_3}{\partial x_k \partial x_j}(X) = \ell^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) + \epsilon^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi), \quad (4.10)$$

$$\frac{\partial^2 a_{ik}}{\partial x_m \partial x_l}(X) = \ell^{(3)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi) + \epsilon^{(3)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi), \quad (4.11)$$

$$\partial_t Y = \ell^{(13)}(\partial_t \xi) + \epsilon^{(13)}(\xi, \partial_t \xi), \quad (4.12)$$

$$\partial_t a_{ik}(X) = \ell^{(14)}(\partial_t \xi, \partial_{ts}^2 \xi) + \epsilon^{(14)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi), \quad (4.13)$$

where the operators $\ell^{(i)}$ have the form (2.21) and $\epsilon^{(i)}$ are given formally by

$$\epsilon^{(1)}(\xi, \partial_s \xi) = O(\xi^{m_1} (\partial_s \xi)^{m_2}), \quad m_1 + m_2 \geq 2,$$

$$\epsilon^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) = O(\xi_1^{m_1} (\partial_s \xi)^{m_2} (\partial_{ss}^2 \xi)^{m_3}), \quad m_1 + m_2 + m_3 \geq 2,$$

$$\epsilon^{(3)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi) = O(\xi_1^{m_1} (\partial_s \xi)^{m_2} (\partial_{ss}^2 \xi)^{m_3} (\partial_{sss}^3 \xi)^{m_4}), \quad m_1 + m_2 + m_3 + m_4 \geq 2,$$

$$\epsilon^{(13)}(\xi, \partial_t \xi) = O(\xi^{m_1} (\partial_t \xi)^{m_2}), \quad m_1 + m_2 \geq 2,$$

$$\epsilon^{(14)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi) = O(\xi^{m_1} (\partial_t \xi)^{m_2} (\partial_{ts}^2 \xi)^{m_3}), \quad m_1 + m_2 + m_3 \geq 2.$$

Proof. We recall that $\xi = \eta - \eta^S$, then we have

$$\frac{1}{1+\eta} = \left(\frac{1}{1+\eta^S} - \frac{\xi}{(1+\eta^S)^2} \right) + \frac{\xi^2}{(1+\eta^S)^2(1+\eta)}. \quad (4.14)$$

After standard calculations, $\nabla Y(X) - I_3$ writes

$$\frac{\partial Y_3}{\partial x_3}(X) - 1 = \frac{\eta^S - \eta}{1+\eta} = -\frac{\xi}{1+\eta^S} + \frac{\xi^2}{(1+\eta^S)^2} - \frac{\xi^3}{(1+\eta^S)^2(1+\eta)}, \quad (4.15)$$

$$\frac{\partial Y_3}{\partial x_j}(X) = -y_3 \frac{\partial_{s_j} \xi}{1+\eta^S} + y_3 \partial_{s_j} \eta^S \frac{\xi}{(1+\eta)(1+\eta^S)} + y_3 \partial_{s_j} \xi \frac{\xi}{(1+\eta)(1+\eta^S)}, \quad j = 1, 2. \quad (4.16)$$

Using (4.14), we obtain (4.9) where $\epsilon^{(1)}(\xi, \partial_s \xi)$ contains the terms of the form

$$\gamma^1(\eta^S) \frac{\xi^{m_1}}{(1+\xi+\eta^S)^{\alpha_1}}, \quad \gamma^2(\eta^S) \frac{\xi^{m_2} \partial_s \xi}{(1+\xi+\eta^S)^{\alpha_2}}, \quad m_1 \geq 2, \quad m_2 \geq 1, \quad \alpha_i \geq 1, \quad i = 1, 2, \quad (4.17)$$

where γ^i , $i = 1, 2$ are functions of regularity $C^3(\omega)$. We have for all $k, j \in \{1, 2\}$,

$$\begin{aligned} \frac{\partial^2 Y_3}{\partial x_k \partial x_j}(X) &= -y_3 \frac{\partial_{s_j s_k}^2 \xi}{(1+\eta^S)} + y_3 \partial_{s_k} \eta \frac{\partial_{s_j} \xi}{(1+\eta)(1+\eta^S)} + y_3 \partial_{s_j} \eta \frac{\partial_{s_k} \xi}{(1+\eta)(1+\eta^S)} \\ &\quad + y_3 \xi \left(\frac{\partial_{s_k s_j}^2 \eta}{(1+\eta^S)(1+\eta)} - 2 \frac{\partial_{s_k} \eta \partial_{s_j} \eta}{(1+\eta^S)(1+\eta)^2} \right). \end{aligned} \quad (4.18)$$

Then, we deduce (4.10) where $\epsilon^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi)$ had the terms

$$\gamma^3(\eta^S) \frac{\xi^{m_1} (\partial_s \xi)^{m_2}}{(1+\xi+\eta^S)^{\alpha_1}}, \quad m_1 + m_2 \geq 2, \quad \gamma^4(\eta^S) \frac{\xi^{m_3} \partial_{ss}^2 \xi}{(1+\xi+\eta^S)^{\alpha_2}}, \quad m_3 \geq 1, \quad \alpha_i \geq 1, \quad i = 1, 2, \quad (4.19)$$

where γ^i , $i = 3, 4$ are $C^2(\omega)$ functions. The third derivative $\frac{\partial^3 Y_3}{\partial x_i \partial x_k \partial x_j}(X)$ admits the terms:

$$\begin{aligned} &y_3 \frac{\partial_{s_j s_k s_i}^3 \xi}{(1+\eta^S)}, \quad y_3 \frac{\partial_{s_i} \eta \partial_{s_j s_k}^2 \xi}{(1+\eta)(1+\eta^S)}, \quad y_3 \frac{\partial_{s_i s_k}^2 \eta \partial_{s_j} \xi}{(1+\eta)(1+\eta^S)}, \quad y_3 \frac{\partial_{s_k} \eta \partial_{s_i} \eta \partial_{s_j} \xi}{(1+\eta)^2(1+\eta^S)}, \\ &y_3 \frac{\partial_{s_j s_k}^2 \eta \partial_{s_i} \xi}{(1+\eta)^2(1+\eta^S)}, \quad y_3 \frac{\xi \partial_{s_j s_k s_i}^3 \eta}{(1+\eta^S)(1+\eta)}, \quad y_3 \frac{\xi \partial_{s_k s_j}^2 \eta \partial_{s_i} \eta}{(1+\eta)^2(1+\eta^S)}, \quad y_3 \frac{\xi \partial_{s_k} \eta \partial_{s_j} \eta \partial_{s_i} \eta}{(1+\eta)^4(1+\eta^S)}. \end{aligned} \quad (4.20)$$

Then, we get (4.11) where $\epsilon^{(3)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi)$ contains the terms

$$\gamma^5(\eta^S) \frac{\xi^{m_1} (\partial_s \xi)^{m_2}}{(1+\xi+\eta^S)^{\alpha_1}}, \quad \gamma^6(\eta^S) \frac{\xi^{n_1} (\partial_s \xi)^{n_2} \partial_{ss}^2 \xi}{(1+\xi+\eta^S)^{\alpha_2}}, \quad \gamma^7(\eta^S) \frac{\xi^{m_3} \partial_{sss}^3 \xi}{(1+\xi+\eta^S)^{\alpha_3}}, \quad m_1 + m_2 \geq 2, \quad n_1 + n_2 \geq 2, \quad m_3 \geq 1, \quad \alpha_i \geq 1, \quad i = 1 \dots 3, \quad (4.21)$$

with γ^i , $i = 5, \dots, 7$ are bounded functions on ω .

We also have,

$$\partial_t Y_3 = -y_3 \frac{\partial_t \xi}{1+\eta}.$$

We deduce (4.12) where $\epsilon^{(13)}(\xi, \partial_t \xi)$ had the terms

$$\frac{\xi^{m_1} \partial_t \xi}{(1+\xi+\eta^S)^{\alpha_1}}, \quad m_1 \geq 1, \quad \alpha_1 \geq 1. \quad (4.22)$$

The terms appearing in $\partial_t a_{ik}(X)$ have the form

$$y^3 \frac{\partial_t \xi \partial_{s_j} \eta}{(1+\eta)^2}, \quad y^3 \frac{\partial_t \xi \partial_{s_j} \eta^S}{(1+\eta)(1+\eta^S)}, \quad y^3 \frac{\partial_{ts_j}^2 \xi}{(1+\eta)}, \quad -\frac{(1+\eta^S) \partial_t \xi}{(1+\eta)^2}.$$

Then, we get (4.13) with $\epsilon^{(14)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi)$ contains the terms

$$\gamma^8(\eta^S) \frac{\xi^{m_1} \partial_{ts}^2 \xi}{(1+\xi+\eta^S)^{\alpha_1}}, \quad \gamma^9(\eta^S) \frac{\xi^{m_2} \partial_t \xi}{(1+\xi+\eta^S)^{\alpha_2}}, \quad \gamma^{10}(\eta^S) \frac{\xi^{m_3} \partial_t \xi \partial_s \xi}{(1+\xi+\eta^S)^{\alpha_3}},$$

$$m_1 \geq 1, \quad m_2 \geq 1, \quad m_3 \geq 0, \quad \alpha_i \geq 1, \quad i = 1, \dots, 3, \quad (4.23)$$

where γ^i , $i = 8, \dots, 10$ are $C^3(\omega)$ functions. \square

Let suppose that $R < 1$.

Lemma 4.3. *Let $\epsilon^{(i)}$ given in lemma 4.2 and let*

$$(w^S, p^S, \eta^S) \in [W^{2,\infty}(\Omega)]^3 \times W^{1,\infty}(\Omega) \times C^4(\omega). \quad (4.24)$$

Then, we have

$$\begin{aligned} & \left\| \epsilon^{(1)}(\xi, \partial_s \xi) \right\|_{L_\gamma^\infty(0,\infty;[L^\infty(\Omega)]^9)} + \left\| \epsilon^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) \right\|_{L_\gamma^\infty(0,\infty;L^2(\Omega))} \\ & \quad + \left\| \epsilon^{(3)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi) \right\|_{L_\gamma^\infty(0,\infty;L^2(\Omega))} + \left\| \epsilon^{(13)}(\xi, \partial_t \xi) \right\|_{L_\gamma^4(0,\infty;[L^\infty(\Omega)]^3)} \\ & \quad + \left\| \epsilon^{(14)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi) \right\|_{L_\gamma^6(0,\infty;L^2(\Omega))} \leq CR^2, \end{aligned} \quad (4.25)$$

$$\begin{aligned} & \left\| \epsilon^{(1)}(\xi, \partial_s \xi) \right\|_{L_\gamma^\infty(0,\infty;[H^{3/2}(\partial\Omega)]^9)} + \left\| \epsilon^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) \right\|_{L_\gamma^\infty(0,\infty;H^{1/2}(\partial\Omega))} \\ & \quad + \left\| \epsilon^{(1)}(\xi, \partial_s \xi) \right\|_{H_\gamma^{7/8}(0,\infty;[L^\infty(\partial\Omega)]^9)} + \left\| \epsilon^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) \right\|_{H_\gamma^{7/8}(0,\infty;L^{8/3}(\partial\Omega))} \leq CR^2. \end{aligned} \quad (4.26)$$

Proof. Using (4.24), (4.4), (4.8) and (4.17), we get

$$\left\| \epsilon^{(1)}(\xi, \partial_s \xi) \right\|_{L_\gamma^\infty(0,\infty;[L^\infty(\Omega)]^9)} \leq CR^2. \quad (4.27)$$

From (4.8), (4.24), (4.4) and (4.19), we find

$$\left\| \epsilon^{(2)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi) \right\|_{L_\gamma^\infty(0,\infty;L^2(\Omega))} \leq CR^2. \quad (4.28)$$

From (4.8), (4.24), (4.4) and (4.21), we obtain

$$\left\| \epsilon^{(3)}(\xi, \partial_s \xi, \partial_{ss}^2 \xi, \partial_{sss}^3 \xi) \right\|_{L_\gamma^\infty(0,\infty;L^2(\Omega))} \leq CR^2. \quad (4.29)$$

From (4.8), (4.24), (4.5), (4.4) and (4.22), we get

$$\left\| \epsilon^{(13)}(\xi, \partial_t \xi) \right\|_{L_\gamma^4(0,\infty;[L^\infty(\Omega)]^3)} \leq CR^2. \quad (4.30)$$

Using (4.8), (4.24), (4.4), (4.5) and (4.23), we get

$$\left\| \epsilon^{(14)}(\xi, \partial_t \xi, \partial_{ts}^2 \xi) \right\|_{L_\gamma^6(0,\infty;L^2(\Omega))} \leq CR^2. \quad (4.31)$$

Using (4.27), (4.28), (4.29), (4.30) and (4.31), we deduce (4.25). In the other hand, we have

$$\|\xi \partial_s \xi\|_{L^\infty(0,\infty;H^{3/2}(\omega))} \leq C \|\xi\|_{L^\infty(0,\infty;H^2(\omega))} \|\partial_s \xi\|_{L^\infty(0,\infty;H^{3/2}(\omega))} \leq CR^2. \quad (4.32)$$

Thus, using (4.8) and (4.4), we have

$$\left\| \frac{1}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{L^\infty(0,\infty;H^2(\omega))} \leq C, \quad \alpha_i \geq 1. \quad (4.33)$$

Then,

$$\left\| \frac{\xi^{m_1} \partial_s \xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{L^\infty(0,\infty;H^{3/2}(\omega))} \leq CR^2. \quad (4.34)$$

Then, using the fact that $\partial_s \xi \in L^\infty(0,\infty;H^2(\omega))$ and $\|\partial_s \xi\|_{L^\infty(0,\infty;H^2(\omega))} \leq CR$, we get

$$\|\xi^{m_1} (\partial_s \xi)^{m_2}\|_{L^\infty(0,\infty;H^{3/2}(\omega))} \leq CR^2. \quad (4.35)$$

We have also

$$\|\xi \partial_{ss}^2 \xi\|_{L^\infty(0,\infty;H^{1/2}(\omega))} \leq C \|\xi\|_{L^\infty(0,\infty;H^{3/2}(\omega))} \|\partial_{ss}^2 \xi\|_{L^\infty(0,\infty;H^{1/2}(\omega))} \leq CR^2, \quad (4.36)$$

Then, using (4.33), we obtain

$$\left\| \frac{\xi^{m_1} \partial_{ss}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{L^\infty(0,\infty;H^{1/2}(\omega))} \leq CR^2, \quad (4.37)$$

Furthermore, using (4.8), we get

$$\begin{aligned} \left\| \frac{\xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{H_\gamma^{5/4}(0,\infty;L^\infty(\omega))} &\leq C \left\| \frac{\xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{H_\gamma^2(0,\infty;L^2(\omega))}^{5/8} \left\| \frac{\xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{L_\gamma^2(0,\infty;H^4(\omega))}^{3/8} \\ &\leq CR, \quad \alpha_i \geq 1. \end{aligned} \quad (4.38)$$

hence, we obtain

$$\begin{aligned} \left\| \frac{\xi \partial_s \xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{H_\gamma^{7/8}(0,\infty;L^\infty(\omega))} &\leq C \left\| \frac{\xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{H_\gamma^{5/4}(0,\infty;L^\infty(\omega))} \|\partial_s \xi\|_{H_\gamma^{7/8}(0,\infty;L^\infty(\omega))} \\ &\leq CR^2. \end{aligned} \quad (4.39)$$

Since $7/8 > 1/2$, we use the proposition 4.1, we obtain

$$\|(\partial_s \xi)^{m_2}\|_{H^{7/8}(0,\infty;L^\infty(\omega))} \leq C \|(\partial_s \xi)\|_{H^{7/8}(0,\infty;L^\infty(\omega))}^{m_2}.$$

We deduce

$$\left\| \frac{\xi^{m_1} (\partial_s \xi)^{m_2}}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{H_\gamma^{7/8}(0,\infty;L^\infty(\omega))} \leq CR^2. \quad (4.40)$$

Moreover,

$$\left\| \frac{\xi \partial_{ss}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{H_\gamma^{7/8}(0,\infty;L^{8/3}(\omega))} \leq C \|\partial_{ss}^2 \xi\|_{H_\gamma^{7/8}(0,\infty;L^{8/3}(\omega))} \|\xi\|_{H_\gamma^{5/4}(0,\infty;L^\infty(\omega))} \leq CR^2. \quad (4.41)$$

From (4.38), we have

$$\left\| \frac{\xi^{m_1} \partial_{ss}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_i}} \right\|_{H_\gamma^{7/8}(0,\infty;L^{8/3}(\omega))} \leq CR^2, \quad (4.42)$$

Using (4.35), (4.37), (4.40) and (4.42), we obtain (4.26). \square

Lemma 4.4. *We have*

$$\|\nabla \cdot \mathcal{N}_E(\xi)\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^3)} + \|\mathcal{N}_F(\xi)\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^3)} \leq CR^2, \quad (4.43)$$

$$\|\mathcal{N}_E(\xi)\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^9)} \leq CR^2, \quad (4.44)$$

$$\|\mathcal{N}_E(\xi)n\|_{W_\gamma^{1/4}(0,\infty;[H^{1/2}(\partial\Omega)]^3;[L^2(\partial\Omega)]^3)} + \|\mathcal{N}_G(\xi)\|_{W_\gamma^{1/4}(0,\infty;[H^{1/2}(\partial\Omega)]^3;[L^2(\partial\Omega)]^3)} \leq CR^2, \quad (4.45)$$

$$\|(\mathcal{F}(u, p, \xi), G(u, \xi + \eta^S), H(u, \xi + \eta^S))\|_{\mathcal{Y}_\infty} \leq CR^2. \quad (4.46)$$

Proof. From (2.30) and lemma 4.2, $\mathcal{N}_E(\xi)$ admits the following terms

$$c^1(\eta^S, w^S) \frac{\xi^{m_1} (\partial_s \xi)^{m_2}}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad c^2(\eta^S, w^S) \frac{\xi^{n_1} (\partial_s \xi)^{n_2} \partial_{ss}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_2}}, \quad (4.47)$$

$m_1 + m_2 \geq 2, \quad n_1 + n_2 \geq 2, \quad \alpha_i \geq 1, \quad i = 1 \dots 2,$

where $c^i, i = 1, 2$ are $W^{1,\infty}(\Omega)$ functions.

From (4.8), (4.24) and (4.4), we obtain

$$\|\mathcal{N}_E(\xi)\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^3)} \leq CR^2. \quad (4.48)$$

From (2.28), (2.30), (2.32), (4.24) and lemma 4.2, the terms that appear in $\nabla \cdot \mathcal{N}_E(\xi)$ are the following

$$d^1(\eta^S, w^S) \frac{\xi^{m_1} (\partial_s \xi)^{m_2}}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad d^2(\eta^S, w^S) \frac{\xi^{n_1} (\partial_s \xi)^{n_2} \partial_{ss}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_2}}, \quad d^3(\eta^S, w^S) \frac{\xi^{m_3} \partial_{sss}^3 \xi}{(1 + \xi + \eta^S)^{\alpha_3}}, \quad (4.49)$$

$m_1 + m_2 \geq 2, \quad n_1 + n_2 \geq 2, \quad m_3 \geq 1, \quad \alpha_i \geq 1, \quad i = 1 \dots 3,$

where $d^i, i = 1, \dots, 3$ are bounded functions in Ω .

From (4.8), (4.24) and (4.4), we obtain

$$\|\nabla \cdot \mathcal{N}_E(\xi)\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^3)} \leq CR^2. \quad (4.49)$$

Moreover, from (2.48), $\mathcal{N}_F(\xi)$ has the terms

$$a^1(\eta^S, w^S) \frac{\xi^{m_1} (\partial_s \xi)^{m_2}}{(1 + \xi + \eta^S)^{\alpha_1}}, \quad a^2(\eta^S, w^S) \frac{\xi^{n_1} (\partial_s \xi)^{n_2} \partial_{ss}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_2}}, \quad a^3(\eta^S, w^S) \frac{\xi^{m_3} \partial_{ts}^2 \xi}{(1 + \xi + \eta^S)^{\alpha_3}},$$

$$a^4(\eta^S, w^S) \frac{\xi^{m_4} \partial_t \xi}{(1 + \xi + \eta^S)^{\alpha_4}}, \quad a^5(\eta^S, w^S) \frac{\xi^{m_5} \partial_t \xi \partial_s \xi}{(1 + \xi + \eta^S)^{\alpha_5}},$$

$m_1 + m_2 \geq 2, \quad n_1 + n_2 \geq 2, \quad m_3 \geq 1, \quad m_4 \geq 1, \quad m_5 \geq 0, \quad \alpha_i \geq 1, \quad i = 1 \dots 5,$

where $a^i, i = 1, \dots, 5$ are bounded functions in Ω .

Thus, using (4.8), (4.24), (4.4) and (4.5), we obtain

$$\|\mathcal{N}_F(\xi)\|_{L^2_\gamma(0,\infty;[L^2(\Omega)]^9)} \leq CR^2.$$

The functions $\mathcal{N}_E(\xi)$ and $\mathcal{N}_G(\xi)$ admit terms of the form (4.47). Then, from lemma 4.3 and (4.24), we have the estimate (4.45).

Finally, using lemma 4.3, (4.3) and the same procedure described in [14], we get (4.46). \square

4.1 Proof of theorem 1.1

We define the application $\Phi : \mathcal{B}_{\infty,R} \rightarrow \mathcal{Y}_{\infty}$ by

$$\begin{aligned} \Phi(\tilde{f}, \tilde{g}, \tilde{h}) = & (\nabla \cdot \mathcal{N}_E(\xi) + \mathcal{N}_F(\xi) + \mathcal{F}(u, p, \xi), \\ & - [\mathcal{N}_E(\xi)n]_{\tau} + [\mathcal{N}_G(\xi)]_{\tau} + G(u, \xi + \eta^S), H(u, \xi + \eta^S) - \mathcal{T}^*(\mathcal{N}_E(\xi)n)). \end{aligned} \quad (4.50)$$

In this case, we show that for R small enough $\Phi(\mathcal{B}_{\infty,R}) \subset \mathcal{B}_{\infty,R}$ and $\Phi|_{\mathcal{B}_{\infty,R}}$ is a strict contraction. Let $(\tilde{f}, \tilde{g}, \tilde{h}) \in \mathcal{B}_{\infty,R}$, from lemma 4.4, we obtain

$$\left\| \Phi(\tilde{f}, \tilde{g}, \tilde{h}) \right\|_{\mathcal{Y}_{\infty}} \leq CR^2.$$

We get similarly

$$\left\| \Phi(\tilde{f}^{(1)}, \tilde{g}^{(1)}, \tilde{h}^{(1)}) - \Phi(\tilde{f}^{(2)}, \tilde{g}^{(2)}, \tilde{h}^{(2)}) \right\|_{\mathcal{Y}_{\infty}} \leq CR \left\| (\tilde{f}^{(1)}, \tilde{g}^{(1)}, \tilde{h}^{(1)}) - (\tilde{f}^{(2)}, \tilde{g}^{(2)}, \tilde{h}^{(2)}) \right\|_{\mathcal{Y}_{\infty}}.$$

for $(\tilde{f}, \tilde{g}, \tilde{h}), (\tilde{f}^{(i)}, \tilde{g}^{(i)}, \tilde{h}^{(i)}) \in \mathcal{B}_{\infty,R}$.

Thus, we deduce the principal result of this paper.

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