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SPECTRUM OF SEMISIMPLE LOCALLY SYMMETRIC SPACES AND ADMISSIBILITY OF SPHERICAL REPRESENTATIONS

S. MEHDI AND M. OLBRICH

Abstract. *We consider compact locally symmetric spaces $\Gamma\backslash G/H$ where G/H is a non-compact semisimple symmetric space and Γ is a discrete subgroup of G. We discuss some features of the joint spectrum of the (commutative) algebra $D(G/H)$ of invariant differential operators acting, as unbounded operators, on the Hilbert space $L^2(\Gamma\backslash G/H)$ of square integrable complex functions on $\Gamma\backslash G/H$. In the case of the Lorentzian symmetric space $SO_0(2,2n)/SO_0(1,2n)$, the representation theoretic spectrum is described explicitly. The strategy is to consider connected reductive Lie groups L acting transitively and co-compactly on G/H, a cocompact lattice $\Gamma \subset L$, and study the spectrum of the algebra $D(L/L\cap H)$ on $L^2(\Gamma\backslash L/L\cap H)$. Though the group G does not act on $L^2(\Gamma\backslash G/H)$, we explain how (not necessarily unitary) G-representations enter into the spectral decomposition of $D(G/H)$ on $L^2(\Gamma\backslash G/H)$ and why one should expect a continuous contribution to the spectrum in some cases. As a byproduct, we obtain a result on the L-admissibility of G-representations. These notes contain the statements of the main results, the proofs and the details will appear elsewhere.*

1. Introduction.

Let G be a connected non-compact semisimple real Lie group and H a connected closed subgroup of G with complexified Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose that G/H is a semisimple symmetric space of the non-compact type with respect to an involution σ and consider the associated decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \ \text{ where } \mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$$

Fix an invariant measure on G/H and consider the Hilbert space $L^2(G/H)$ of square integrable complex functions on G/H. The action by left translations of G on $L^2(G/H)$ defines a unitary representation whose decomposition is known as the Plancherel formula for symmetric spaces [2][3][7][20]:

$$\begin{array}{c} L^2(G/H) \simeq \displaystyle\int_{\widehat{G}_H}^{\oplus} \left(W^H_{\widetilde{\rho},-\infty}\right)_0 \otimes W_\rho d\mu(\rho) \\ \circlearrowleft \qquad\qquad \circlearrowleft \\ D(G/H) \twoheadleftarrow \quad \mathcal{Z}(\mathfrak{g}) \end{array}$$

where W_ρ is a unitary irreducible representation of G, $W_{\widetilde{\rho}}$ the dual representation, $W_{\widetilde{\rho},-\infty}$ the space of distribution vectors of $W_{\widetilde{\rho}}$, $W^H_{\widetilde{\rho},-\infty}$ the space of H-invariants distribution vectors, $\left(W^H_{\widetilde{\rho},-\infty}\right)_0$ is a finite dimensional Hilbert space embedded in $W^H_{\widetilde{\rho},-\infty}$ whose dimension is the multiplicity of W_ρ in $L^2(G/H)$, $\widehat{G}_H$ is the H-spherical dual $\{\rho \in \widehat{G} \mid W^H_{\widetilde{\rho},-\infty} \neq \{0\}\}$ and $d\mu$ the Plancherel measure. The (commutative) algebra $D(G/H)$ of G-invariant differential operators on G/H acts on $\left(W^H_{\widetilde{\rho},-\infty}\right)_0$ and the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ acts via infinitesimal character on W_ρ. Except for exceptional cases where G

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is E_6, E_7 and E_8 which we will not consider here, there is a surjective map $\mathcal{Z}(\mathfrak{g}) \twoheadrightarrow D(G/H)$ [11]. In other words, harmonic analysis on G/H is closely related to the spectrum of $D(G/H)$ on $L^2(G/H)$:

$$\text{harmonic analysis on } G/H \longleftrightarrow \mathrm{Spec}_{L^2(G/H)}(D(G/H))$$

A compact locally symmetric space is a smooth compact manifold of the form $\Gamma\backslash G/H$ where Γ is a discrete subgroup of G, in particular the natural map $G \to \Gamma\backslash G/H$ is smooth for the quotient topology. As above one may ask about the analysis on $\Gamma\backslash G/H$. Let us consider the following group case example.

Example 1.1. *$G = G_1 \times G_1$, $H = \Delta(G_1 \times G_1)$ and $\Gamma = \Gamma_1 \times \{e\}$, where G_1 is a connected non-compact semisimple real Lie group, σ is the permutation $G_1 \times G_1 \to G_1 \times G_1$, $(a,b) \mapsto (b,a)$ and Γ_1 a co-compact torsion free discrete subgroup of G_1. Then $L^2(G/H) \simeq L^2(G_1)$ is described by the Harish-Chandra Plancherel formula and*

$$L^2(\Gamma\backslash G/H) \simeq L^2(\Gamma_1\backslash G_1) \simeq \bigoplus_{\pi\in\widehat{G}_1} V^{\Gamma_1}_{\widetilde{\pi},-\infty}\widehat{\otimes}V_\pi$$

where $\dim(V^{\Gamma_1}_{\widetilde{\pi},-\infty}) < \infty$, $\dim(V_\pi) = \infty$ (except for the trivial representation π) and the set $\{\pi \in \widehat{G}_1 \mid V^{\Gamma_1}_{\widetilde{\pi},-\infty} \neq \{0\}\}$ is discrete. Moreover $D(G/H) \simeq \mathcal{Z}(\mathfrak{g})$ acts scalarly on π by infinitesimal characters. In particular, the spectrum of $D(G/H)$ on $L^2(\Gamma\backslash G/H)$ is discrete, consists of eigencharacters only, and (almost) all eigenspaces are infinite dimensional.

Already this example shows that analysis on $\Gamma\backslash G/H$ is often related to automorphic forms:

$$\text{analysis on } \Gamma\backslash G/H \longleftrightarrow \text{automorphic forms}$$

Problem 1.2. *Describe the spectrum $Spec_{L^2(\Gamma\backslash G/H)}(D(G/H))$ of $D(G/H)$ on $L^2(\Gamma\backslash G/H)$ in general.*

Issues:

(1) When H is not compact then a discrete subgroup $\Gamma \subset G$ such that $\Gamma\backslash G/H$ is a smooth compact manifold need not exist as it is illustrated by the Calabi-Markus phenomenon for $SO_0(1,n+1)/SO_0(1,n)$ [6]. Given such a Γ, the space $\Gamma\backslash G/H$ will be called a compact *Clifford-Klein form* of G/H. One can ask for deformations of Γ such that the new quotient is still a Clifford-Klein form (see [16]).

(2) The group G does not act on $\Gamma\backslash G/H$), and it is not clear how representations of G are involved in $\mathrm{Spec}_{L^2(\Gamma\backslash G/H)}(D(G/H))$.

(3) When H is not compact, the coset $\Gamma\backslash G$ is not compact and it is not clear if $L^2(\Gamma\backslash G)$ and $L^2(\Gamma\backslash G/H)$ are related.

(4) $D(G/H)$ acts on $C^\infty(\Gamma\backslash G/H)$ and $C^\infty(\Gamma\backslash G/H)$ is dense in $L^2(\Gamma\backslash G/H)$, so $D(G/H)$ acts via unbounded operators on $L^2(\Gamma\backslash G/H)$. Some care is required with the domains of the operators.

Problem 1.2 is related with a program developed by Kobayashi to study hidden symmetries (see [14][15] and references therein). In the following we shall describe and state some of the main results we prove in [21].

2. Existence of compact Clifford-Klein forms and spherical triples.

When H is compact, the existence of a discrete subgroup Γ of G such that $\Gamma\backslash G/H$ is a smooth compact manifold is equivalent to the existence of torsion free uniform lattices in G. It turns out that torsion free uniform lattices always exist in G by a result of Borel and Harish-Chandra [5].

For H noncompact, this gives a sufficient condition for the existence of Γ, employed extensively by Kobayashi starting with [18]:

$$\left.\begin{array}{l}\text{existence of a closed connected Lie group } L\subset G \text{ such that}\\ (i)\ L \text{ is reductively embedded in } G\\ (ii)\ L \text{ acts transitively on } G/H\\ (iii)\ L\cap H \text{ is compact}\end{array}\right\}\Longrightarrow \begin{array}{l}\text{existence of compact Clifford-Klein}\\ \text{forms for } G/H\end{array}$$

Indeed, if Γ is a co-compact lattice in L then $\Gamma\backslash G/H$ is a compact locally symmetric space. Triples (G,H,L) satisfying (i) and (ii) above were classified, when G is simple, by Onishik in the late 1960's [23] (see also [19]). We will refer to triples (G,H,L) satisfying (i)(ii)(iii) as *transitive triples.*

In fact, the above implication remains true, if we weaken *transitively* in (ii) to *cocompactly.* However, we are able to show that this does not change anything: *cocompactly* implies *transitively* [21].

Next fix $P=MAN$ the Langlands decomposition of a minimal parabolic subgroup in G and consider a principal series representation $\mathrm{Ind}_P^G\tau\otimes e^\nu\otimes 1$ of G, where $\tau\in\widehat{M}$ is a finite dimensional irreducible representation of M and ν is a linear form on the complexified Lie algebra of A. Then van den Ban proved in [1] that the space of H-invariants distributions vectors in $\mathrm{Ind}_P^G\tau\otimes e^\nu\otimes 1$ is finite dimensional:

$$\dim\left(\mathrm{Ind}_P^G\tau\otimes e^\nu\otimes 1\right)^H_{-\infty}<\infty$$

Now pick a transitive triple (G,H,L) and let $P_L=M_LA_LN_L$ be the Langlands decomposition of a minimal parabolic in L. Let $\tau_L\in\widehat{M_L}$ be a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L. Observe that L is not σ-stable, $L/L\cap H$ need not be a symmetric space and the space of $L\cap H$-invariants distributions $\left(\mathrm{Ind}_{P_L}^L\tau_L\otimes e^{\nu_L}\otimes 1\right)^{L\cap H}_{-\infty}$ could be infinite dimensional as it is the case for the triple $(G_1\times G_1,\Delta(G_1\times G_1),G_1\times\{e\})$. In fact we obtain the following criterion for the space of $L\cap H$-invariants to be finite dimensional.

Proposition 2.1. *For any $\tau_L\in\widehat{M_L}$ a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L, one has*

$$\dim\left(Ind_{P_L}^L\tau_L\otimes e^{\nu_L}\otimes 1\right)^{L\cap H}_{-\infty}<\infty\Leftrightarrow P_L \text{ acts transitively on } L/L\cap H.$$

Transitive triples (G,H,L) such that P_L acts transitively on $L/L\cap H$ will be called *spherical triples.* The use of the qualifier 'spherical' is motivated by the following facts:

$$L/L\cap H \text{ is a spherical homogeneous space} \Longleftrightarrow P_L \text{ acts with an open orbit in } L/L\cap H$$

$$\xLeftrightarrow[\text{is compact}]{\text{if } L\cap H} P_L \text{ acts transitively on } L/L\cap H$$

Example 2.2.

(1) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$ *is not spherical.*

(2) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1)$ *is spherical (where K_1 is maximal compact in G_1).*

(3) $(SO_0(2,2n), SO_0(1,2n), U(1,n))$ *is spherical ($n \geq 2$).*

(4) $(SO_0(4,3), SO_0(4,1) \times SO(2), G_{2(2)})$ *is not spherical.*

3. L-admissibility.

Suppose $G' \subset G$ is a connected closed reductive subgroup and π is an irreducible unitary representation of G. From a general result of Mautner and Teleman, the restriction of π to G' can be decomposed as the direct integral sum of irreducible unitary representations:

$$\pi_{|_{G'}} \simeq \int_{\widehat{G'}}^{\oplus} M_\pi(\tau) \widehat{\otimes} V_\tau \, d\mu(\tau)$$

where $M_\pi(\tau)$ is the multiplicity (Hilbert) space.

Definition 3.1. π is G'-admissible if $\pi_{|_{G'}}$ decomposes discretely and $\dim M_\pi(\tau) < \infty$

Example 3.2.

(1) $\widehat{G} \ni \pi$ *is K-admissible (Harish-Chandra admissibility theorem* [8]*).*

(2) $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ *and* $H = \Delta(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$*. Let* $\widehat{G}_H \ni \pi = \pi_1 \otimes \pi_1$ *where π_1 is a holomorphic discrete series representation of $SL(2,\mathbb{R})$. From* [24]*, $\pi_{|_H}$ has continuous spectrum (but multiplicities are finite). Therefore π is not H-admissible.*

There is a criterion for H-admissibility in terms of associated varieties due to Kobayashi [17]. We prove the following admissibility result for spherical representations associated with triples (G, H, L).

Theorem 3.3. *Suppose (G, H, L) is a spherical triple. If $\pi \in \widehat{G}_H$ then π is L-admissible. Moreover, if π is not trivial then there are infinitely many summands in $\pi_{|_L}$ all having $L \cap H$-invariants.*

4. Embedding of Casimir operators.

Fix a transitive triple (G, H, L). Write $\mathcal{U}(\mathfrak{g})^H$ for the H-invariant elements in the enveloping algebra of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})\mathfrak{h}$ the left $\mathcal{U}(\mathfrak{g})$-ideal generated by $\mathfrak{h}$. The algebra $D(G/H)$ of G-invariant differential operators on G/H is isomorphic to the following quotient algebra:

$$D(G/H) \simeq \mathcal{U}(\mathfrak{g})^H / \mathcal{U}(\mathfrak{g})\mathfrak{h} \cap \mathcal{U}(\mathfrak{g})^H$$

It is known that since G/H is a symmetric space, $D(G/H)$ is commutative (see [10] and references therein).

By definition, if (G, H, L) is a transitive triple then one has a diffeomorphism of homogeneous spaces

$$G/H \simeq L/L \cap H$$

and an isomorphism of algebra

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{l} \cap \mathfrak{h})} \mathcal{U}(\mathfrak{l}).$$

In particular, we get an embedding of algebras

$$\imath : D(G/H) \hookrightarrow D(L/L \cap H) \ .$$

The algebra $D(L/L \cap H)$ need not be commutative, though it is for many transitive triples (see [12]). Of special interest in $D(G/H)$ (resp. $D(L/L \cap H)$, $D(L \cap K/L \cap H)$) is the Casimir element Ω_G (resp. Ω_L, $\Omega_{L\cap K}$). The algebras $D(L/L \cap H)$ and $D(L \cap K/L \cap H)$ are defined in an analogous way to $D(G/H)$. Observe that when G/H has rank one then $D(G/H)$ is generated by the Casimir Ω_G.

Proposition 4.1. *For each transitive triple (G, H, L) with G/H irreducible, the embedding $\imath(\Omega_G)$ is computed explicitly in terms of the decomposition of $\mathfrak{l}/\mathfrak{l} \cap \mathfrak{h}$ into $Ad(L \cap H)$-irreducibles.*

In particular, based on the restriction to $\mathfrak{l}$ and $\mathfrak{l} \cap \mathfrak{k}$ of the Killing form of $\mathfrak{g}$, we have:

Example 4.2.

(1) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$: $\imath(\Omega_G) = 2\Omega_L$.
(2) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1)$: $\imath(\Omega_G) = 2\Omega_L - \Omega_{L\cap K}$.
(3) $(SO_0(2, 2n), SO_0(1, 2n), U(1, n))$: $\imath(\Omega_G) = 2\Omega_L - \Omega_{L\cap K}$.
(4) $(SO_0(4, 3), SO_0(4, 1) \times SO(2), G_{2(2)})$: $\imath(\Omega_G) = 3\Omega_L - \frac{3}{2}\Omega_{L\cap K} + 2\Omega_{\mathfrak{l}\cap\mathfrak{s}\cap\mathfrak{q}}$

In case (3), Schlichtkrull, Trapa and Vogan obtained a similar formula for $O(2n)/U(n)$ [25].

Remark 4.3. By inspection, we observe that $\imath(\Omega_G)$ involves 'non-compact terms' terms from $\mathfrak{l} \cap \mathfrak{q} \cap \mathfrak{s}$ only when (G, H, L) is not spherical, except in the first group case (1).

5. On the spectrum $\mathrm{Spec}_{L^2(\Gamma\backslash G/H)}(D(G/H))$.

Fix a transitive triple (G, H, L). We have the following sequence of isomorphisms:

$$\begin{array}{lll}
L^2(\Gamma\backslash G/H) \simeq L^2(\Gamma\backslash L/L\cap H) \simeq L^2(\Gamma\backslash L)^{L\cap H} \simeq \widehat{\bigoplus}_{\pi\in\widehat{L}} V^{\Gamma}_{\tilde{\pi},-\infty} \otimes V^{L\cap H}_{\pi} & & \text{(Hilbert sum)} \\
\bigcup \qquad\qquad \bigcup \qquad\qquad \bigcup & & \\
C^{\infty}(\Gamma\backslash G/H) \simeq C^{\infty}(\Gamma\backslash L/L\cap H) \simeq C^{\infty}(\Gamma\backslash L)^{L\cap H} \simeq \widehat{\bigoplus}_{\pi\in\widehat{L}} V^{\Gamma}_{\tilde{\pi},-\infty} \otimes V^{L\cap H}_{\pi,\infty} & & \text{(closure of algebraic direct sum)} \\
\circlearrowleft \qquad\qquad\qquad\qquad \circlearrowleft & & \\
D \in D(G/H) \qquad\qquad\qquad\qquad \imath(D) \in D(L/L\cap H) & &
\end{array}$$

Suppose now that (G, H, L) is spherical, i.e $\dim\left(\mathrm{Ind}^{L}_{P_L}\tau_L \otimes e^{\nu_L} \otimes 1\right)^{L\cap H}_{-\infty} < \infty$, for any $\tau_L \in \widehat{M_L}$ a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L. Then by Proposition 2.1 and Casselman embedding Theorem, we deduce that

Proposition 5.1. *For any irreducible unitary representation (π, V_π) of L, one has*

$$\dim V^{L\cap H}_{\pi,\infty} < \infty$$

This means that the spectrum of $D(G/H)$ on $C^\infty(\Gamma\backslash G/H)$ may be reduced to the diagonalization of the operator $\pi(\imath(D))$ on the (infinitely many) finite dimensional blocks $V_{\pi,\infty}^{L\cap H}$. Then we obtain the following description of the spectrum pf $D(G/H)$ on $L^2(\Gamma\backslash G/H)$.

Theorem 5.2. *Suppose (G, H, L) is a spherical triple.*

(1) *There is a direct sum decomposition of $C^\infty(\Gamma\backslash G/H)$ and $L^2(\Gamma\backslash G/H)$ into joint eigenspaces of $D(G/H)$. $Spec_{L^2(\Gamma\backslash G/H)}(D(G/H))$ consists of the corresponding eigencharacters and their possible accumulation points.*

(2) *In the Lorentzian case $(SO_0(2,2n), SO_0(1,2n), U(1,n))$, $D(G/H)$ is generated by the Casimir Ω_G, the set $\{\pi \in \widehat{L} \mid V_{\pi,\infty}^{L\cap H} \neq \{0\}\}$ is computed explicitly and the representations π are identified as well as their contributions to eigenvalues for $\imath(\Omega_G)$. More precisely, if Δ denotes the Laplace operator on $\Gamma\backslash G/H$ induced by Ω_G, one has*
   - $\mathrm{spec}(\Delta)\cap(-\infty, -n^2]$ *comes from unitary principal series and, at $-n^2$, also from limits of discrete series of L.*
   - $\mathrm{spec}(\Delta) \cap (-n^2, 0]$ *consists of contributions from complementary series, ends of complementary series and non-integrable discrete series.*
   - $\mathrm{spec}(\Delta) \cap (0, \infty) = \bigcup_{\ell=n+1}^{\infty} \{\ell^2 - n^2\}$ *is the contribution of integrable discrete series. The corresponding eigenspaces are infinite dimensional.*

The last assertion about the contribution of integrable discrete series with $L \cap H$-invariants generalizes to arbitrary spherical triples (G, H, L). These contributions combine to infinite dimensional eigenspaces of $D(G/H)$ [21]. The shortest proof of this uses Theorem 3.3 instead of the embedding

$$\imath : D(G/H) \hookrightarrow D(L/L \cap H) \ .$$

In view of Theorem 3.3 the result can be rephrased in terms of G-representations as follows. Each integrable discrete series for G/H contributes an infinite dimensional eigenspace of $D(G/H)$ in $L^2(\Gamma\backslash G/H)$. Compare related results obtained in [13].

6. Generalized matrix coefficients, unitarity and continuous spectrum.

The Hilbert spaces $L^2(G/H)$ and $L^2(\Gamma\backslash G)$ are both unitary G-representations which decompose into a direct integral of irreducible ones. Though the Hilbert space $L^2(\Gamma\backslash G/H)$ is not a G-module, there is a natural way to involve representations of G.

For a an admissible representation (ρ, W_ρ) of G of finite length, one may consider generalized matrix coefficients (see [9]):

$$W_{\widetilde{\rho},-\infty} \otimes W_{\rho,-\infty} \to C^{-\infty}(G),\ \widetilde{w} \otimes w \mapsto c_{\widetilde{w},w}$$

with

$$\begin{aligned} c_{\widetilde{w},w}(f) &= <\widetilde{w},\rho(f)w> \\ <\rho(f)w,v> &= \int_G f(g)<\rho(g)w,v>dg \quad \text{for } f\in C_c^\infty(G),\ v\in W_{\rho,\infty} \end{aligned}$$

Then there is a $D(G/H)$-equivariant map (which is injective if ρ is irreducible):

$$\begin{array}{ccc} W^\Gamma_{\widetilde{\rho},-\infty}\otimes W^H_{\rho,-\infty} & \longrightarrow C^{-\infty}(\Gamma\backslash G/H)\simeq \widehat{\bigoplus}_{\pi\in\widehat{L}} V^\Gamma_{\widetilde{\pi},-\infty}\otimes V^{L\cap H}_{\pi,-\infty} \\ \circlearrowleft & \qquad\qquad\qquad\qquad \circlearrowleft \\ D(G/H) & \qquad\qquad\qquad\qquad D(L/L\cap H) \end{array} \tag{6.1}$$

A comparison of (6.1) with Theorem 5.2 leads, among other things, to the following funny observation. On one hand, from Theorem 5.2, there are complementary series representations π of $U(1,n)$ contributing to finite dimensional eigenspaces. On the other hand, if ρ were unitary then $\rho_{|_L}$ would contain, by Theorem 3.3, infinitely many unitary summands having $L\cap H$-invariants. Moreover, a result of Bergeron and Clozel [4], Thm. 6.5.1, tell us that there are indeed lattices $\Gamma\subset U(1,n)$ such that $V^\Gamma_{\widetilde{\pi},-\infty}\neq\{0\}$ for complementary series π with certain integral parameters. Therefore there must exist non-unitary representations ρ of G that are involved in $L^2(\Gamma\backslash G/H)$ via (6.1).

One can also try to use (6.1) to produce families of eigendistributions on $\Gamma\backslash G/H$ depending continuously on the spectral parameter. Let $0\neq\tilde{v}\in V^\Gamma_{\widetilde{\pi},-\infty}$ and assume we have a continuous nonconstant family $t\mapsto\rho_t$, $t\in(a,b)\subset\mathbb{R}$ of H-spherical principal series of G with $0\neq w_t\in W^H_{\rho_t,-\infty}$ and, the crucial ingredient, a continuous family of non-zero intertwining operators $\Psi_t\in\mathrm{Hom}_L(W_{\rho_t,\infty},V_{\pi,\infty})$. Then $\Psi_t^*v\in W^\Gamma_{\widetilde{\rho}_t,-\infty}$ and

$$t\mapsto c_{\Psi_t^*v,w_t}$$

is the desired continuous family of eigendistributions. Note that the existence of such a family for $\Gamma\backslash G/H$ excludes the existence of a discrete spectral decomposition as in Theorem 5.2. Therefore such families can only exist for non-spherical triples (G,H,L). One expects that such families are the building blocks for the continuous part of the spectral decomposition of $L^2(\Gamma\backslash G/H)$. We are able to find such families of intertwiners for some non-spherical triples, and expect their existence for all non-spherical (G,H,L) except for the group case as in Example 2.2 (1). For instance, for the non-spherical triple $(SO(8,\mathbb{C}),SO_0(1,7),Spin(7,\mathbb{C}))$ the existence of many families Ψ_t is ensured by the work of Jan Frahm [22].

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