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SPECTRUM OF SEMISIMPLE LOCALLY SYMMETRIC SPACES AND ADMISSIBILITY OF SPHERICAL REPRESENTATIONS

S. MEHDI AND M. OLBRICH

ABSTRACT. We consider compact locally symmetric spaces $\Gamma \backslash G/H$ where G/H is a non-compact semisimple symmetric space and Γ is a discrete subgroup of G . We discuss some features of the joint spectrum of the (commutative) algebra $D(G/H)$ of invariant differential operators acting, as unbounded operators, on the Hilbert space $L^2(\Gamma \backslash G/H)$ of square integrable complex functions on $\Gamma \backslash G/H$. In the case of the Lorentzian symmetric space $SO_0(2, 2n)/SO_0(1, 2n)$, the representation theoretic spectrum is described explicitly. The strategy is to consider connected reductive Lie groups L acting transitively and co-compactly on G/H , a cocompact lattice $\Gamma \subset L$, and study the spectrum of the algebra $D(L/L \cap H)$ on $L^2(\Gamma \backslash L/L \cap H)$. Though the group G does not act on $L^2(\Gamma \backslash G/H)$, we explain how (not necessarily unitary) G -representations enter into the spectral decomposition of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ and why one should expect a continuous contribution to the spectrum in some cases. As a byproduct, we obtain a result on the L -admissibility of G -representations. These notes contain the statements of the main results, the proofs and the details will appear elsewhere.

1. INTRODUCTION.

Let G be a connected non-compact semisimple real Lie group and H a connected closed subgroup of G with complexified Lie algebras \mathfrak{g} and \mathfrak{h} respectively. Suppose that G/H is a semisimple symmetric space of the non-compact type with respect to an involution σ and consider the associated decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{where } \mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$$

Fix an invariant measure on G/H and consider the Hilbert space $L^2(G/H)$ of square integrable complex functions on G/H . The action by left translations of G on $L^2(G/H)$ defines a unitary representation whose decomposition is known as the Plancherel formula for symmetric spaces [2][3][7][20]:

$$L^2(G/H) \simeq \int_{\widehat{G}_H}^{\oplus} (W_{\tilde{\rho}, -\infty}^H)_0 \otimes W_{\rho} d\mu(\rho)$$

$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ D(G/H) & \leftarrow & \mathcal{Z}(\mathfrak{g}) \end{array}$$

where W_{ρ} is a unitary irreducible representation of G , $W_{\tilde{\rho}}$ the dual representation, $W_{\tilde{\rho}, -\infty}$ the space of distribution vectors of $W_{\tilde{\rho}}$, $W_{\tilde{\rho}, -\infty}^H$ the space of H -invariants distribution vectors, $(W_{\tilde{\rho}, -\infty}^H)_0$ is a finite dimensional Hilbert space embedded in $W_{\tilde{\rho}, -\infty}^H$ whose dimension is the multiplicity of W_{ρ} in $L^2(G/H)$, \widehat{G}_H is the H -spherical dual $\{\rho \in \widehat{G} \mid W_{\tilde{\rho}, -\infty}^H \neq \{0\}\}$ and $d\mu$ the Plancherel measure. The (commutative) algebra $D(G/H)$ of G -invariant differential operators on G/H acts on $(W_{\tilde{\rho}, -\infty}^H)_0$ and the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} acts via infinitesimal character on W_{ρ} . Except for exceptional cases where G

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is E_6 , E_7 and E_8 which we will not consider here, there is a surjective map $\mathcal{Z}(\mathfrak{g}) \twoheadrightarrow D(G/H)$ [11]. In other words, harmonic analysis on G/H is closely related to the spectrum of $D(G/H)$ on $L^2(G/H)$:

$$\text{harmonic analysis on } G/H \longleftrightarrow \text{Spec}_{L^2(G/H)}(D(G/H))$$

A compact locally symmetric space is a smooth compact manifold of the form $\Gamma \backslash G/H$ where Γ is a discrete subgroup of G , in particular the natural map $G \rightarrow \Gamma \backslash G/H$ is smooth for the quotient topology. As above one may ask about the analysis on $\Gamma \backslash G/H$. Let us consider the following group case example.

Example 1.1. $G = G_1 \times G_1$, $H = \Delta(G_1 \times G_1)$ and $\Gamma = \Gamma_1 \times \{e\}$, where G_1 is a connected non-compact semisimple real Lie group, σ is the permutation $G_1 \times G_1 \rightarrow G_1 \times G_1$, $(a, b) \mapsto (b, a)$ and Γ_1 a co-compact torsion free discrete subgroup of G_1 . Then $L^2(G/H) \simeq L^2(G_1)$ is described by the Harish-Chandra Plancherel formula and

$$L^2(\Gamma \backslash G/H) \simeq L^2(\Gamma_1 \backslash G_1) \simeq \bigoplus_{\pi \in \widehat{G}_1} V_{\pi, -\infty}^{\Gamma_1} \widehat{\otimes} V_{\pi}$$

where $\dim(V_{\pi, -\infty}^{\Gamma_1}) < \infty$, $\dim(V_{\pi}) = \infty$ (except for the trivial representation π) and the set $\{\pi \in \widehat{G}_1 \mid V_{\pi, -\infty}^{\Gamma_1} \neq \{0\}\}$ is discrete. Moreover $D(G/H) \simeq \mathcal{Z}(\mathfrak{g})$ acts scalarly on π by infinitesimal characters. In particular, the spectrum of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ is discrete, consists of eigencharacters only, and (almost) all eigenspaces are infinite dimensional.

Already this example shows that analysis on $\Gamma \backslash G/H$ is often related to automorphic forms:

$$\text{analysis on } \Gamma \backslash G/H \longleftrightarrow \text{automorphic forms}$$

Problem 1.2. Describe the spectrum $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$ of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ in general.

Issues:

- (1) When H is not compact then a discrete subgroup $\Gamma \subset G$ such that $\Gamma \backslash G/H$ is a smooth compact manifold need not exist as it is illustrated by the Calabi-Markus phenomenon for $SO_0(1, n+1)/SO_0(1, n)$ [6]. Given such a Γ , the space $\Gamma \backslash G/H$ will be called a compact *Clifford-Klein form* of G/H . One can ask for deformations of Γ such that the new quotient is still a Clifford-Klein form (see [16]).
- (2) The group G does not act on $\Gamma \backslash G/H$, and it is not clear how representations of G are involved in $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$.
- (3) When H is not compact, the coset $\Gamma \backslash G$ is not compact and it is not clear if $L^2(\Gamma \backslash G)$ and $L^2(\Gamma \backslash G/H)$ are related.
- (4) $D(G/H)$ acts on $C^\infty(\Gamma \backslash G/H)$ and $C^\infty(\Gamma \backslash G/H)$ is dense in $L^2(\Gamma \backslash G/H)$, so $D(G/H)$ acts via unbounded operators on $L^2(\Gamma \backslash G/H)$. Some care is required with the domains of the operators.

Problem 1.2 is related with a program developed by Kobayashi to study hidden symmetries (see [14][15] and references therein). In the following we shall describe and state some of the main results we prove in [21].

2. EXISTENCE OF COMPACT CLIFFORD-KLEIN FORMS AND SPHERICAL TRIPLES.

When H is compact, the existence of a discrete subgroup Γ of G such that $\Gamma \backslash G/H$ is a smooth compact manifold is equivalent to the existence of torsion free uniform lattices in G . It turns out that torsion free uniform lattices always exist in G by a result of Borel and Harish-Chandra [5].

For H noncompact, this gives a sufficient condition for the existence of Γ , employed extensively by Kobayashi starting with [18]:

$$\left. \begin{array}{l} \text{existence of a closed connected Lie group } L \subset G \text{ such that} \\ (i) \ L \text{ is reductively embedded in } G \\ (ii) \ L \text{ acts transitively on } G/H \\ (iii) \ L \cap H \text{ is compact} \end{array} \right\} \implies \text{existence of compact Clifford-Klein forms for } G/H$$

Indeed, if Γ is a co-compact lattice in L then $\Gamma \backslash G/H$ is a compact locally symmetric space. Triples (G, H, L) satisfying (i) and (ii) above were classified, when G is simple, by Onishik in the late 1960's [23] (see also [19]). We will refer to triples (G, H, L) satisfying (i)(ii)(iii) as *transitive triples*.

In fact, the above implication remains true, if we weaken *transitively* in (ii) to *cocompactly*. However, we are able to show that this does not change anything: *cocompactly* implies *transitively* [21].

Next fix $P = MAN$ the Langlands decomposition of a minimal parabolic subgroup in G and consider a principal series representation $\text{Ind}_P^G \tau \otimes e^\nu \otimes 1$ of G , where $\tau \in \widehat{M}$ is a finite dimensional irreducible representation of M and ν is a linear form on the complexified Lie algebra of A . Then van den Ban proved in [1] that the space of H -invariants distributions vectors in $\text{Ind}_P^G \tau \otimes e^\nu \otimes 1$ is finite dimensional:

$$\dim (\text{Ind}_P^G \tau \otimes e^\nu \otimes 1)_{-\infty}^H < \infty$$

Now pick a transitive triple (G, H, L) and let $P_L = M_L A_L N_L$ be the Langlands decomposition of a minimal parabolic in L . Let $\tau_L \in \widehat{M}_L$ be a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L . Observe that L is not σ -stable, $L/L \cap H$ need not be a symmetric space and the space of $L \cap H$ -invariants distributions $(\text{Ind}_{P_L}^L \tau_L \otimes e^{\nu_L} \otimes 1)_{-\infty}^{L \cap H}$ could be infinite dimensional as it is the case for the triple $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$. In fact we obtain the following criterion for the space of $L \cap H$ -invariants to be finite dimensional.

Proposition 2.1. *For any $\tau_L \in \widehat{M}_L$ a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L , one has*

$$\dim (\text{Ind}_{P_L}^L \tau_L \otimes e^{\nu_L} \otimes 1)_{-\infty}^{L \cap H} < \infty \Leftrightarrow P_L \text{ acts transitively on } L/L \cap H.$$

Transitive triples (G, H, L) such that P_L acts transitively on $L/L \cap H$ will be called *spherical triples*. The use of the qualifier 'spherical' is motivated by the following facts:

$$\begin{array}{ccc} L/L \cap H \text{ is a spherical homogeneous space} & \iff & P_L \text{ acts with an open orbit in } L/L \cap H \\ & \iff & \begin{array}{l} \text{if } L \cap H \\ \text{is compact} \end{array} P_L \text{ acts transitively on } L/L \cap H \end{array}$$

Example 2.2.

- (1) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$ is not spherical.
- (2) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1)$ is spherical (where K_1 is maximal compact in G_1).
- (3) $(SO_0(2, 2n), SO_0(1, 2n), U(1, n))$ is spherical ($n \geq 2$).
- (4) $(SO_0(4, 3), SO_0(4, 1) \times SO(2), G_{2(2)})$ is not spherical.

3. L -ADMISSIBILITY.

Suppose $G' \subset G$ is a connected closed reductive subgroup and π is an irreducible unitary representation of G . From a general result of Mautner and Teleman, the restriction of π to G' can be decomposed as the direct integral sum of irreducible unitary representations:

$$\pi|_{G'} \simeq \int_{G'}^{\oplus} M_{\pi}(\tau) \widehat{\otimes} V_{\tau} d\mu(\tau)$$

where $M_{\pi}(\tau)$ is the multiplicity (Hilbert) space.

Definition 3.1. π is G' -admissible if $\pi|_{G'}$ decomposes discretely and $\dim M_{\pi}(\tau) < \infty$

Example 3.2.

- (1) $\widehat{G} \ni \pi$ is K -admissible (Harish-Chandra admissibility theorem [8]).
- (2) $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ and $H = \Delta(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))$. Let $\widehat{G}_H \ni \pi = \pi_1 \otimes \pi_1$ where π_1 is a holomorphic discrete series representation of $SL(2, \mathbb{R})$. From [24], $\pi|_H$ has continuous spectrum (but multiplicities are finite). Therefore π is not H -admissible.

There is a criterion for H -admissibility in terms of associated varieties due to Kobayashi [17]. We prove the following admissibility result for spherical representations associated with triples (G, H, L) .

Theorem 3.3. Suppose (G, H, L) is a spherical triple. If $\pi \in \widehat{G}_H$ then π is L -admissible. Moreover, if π is not trivial then there are infinitely many summands in $\pi|_L$ all having $L \cap H$ -invariants.

4. EMBEDDING OF CASIMIR OPERATORS.

Fix a transitive triple (G, H, L) . Write $\mathcal{U}(\mathfrak{g})^H$ for the H -invariant elements in the enveloping algebra of \mathfrak{g} and $\mathcal{U}(\mathfrak{g})\mathfrak{h}$ the left $\mathcal{U}(\mathfrak{g})$ -ideal generated by \mathfrak{h} . The algebra $D(G/H)$ of G -invariant differential operators on G/H is isomorphic to the following quotient algebra:

$$D(G/H) \simeq \mathcal{U}(\mathfrak{g})^H / \mathcal{U}(\mathfrak{g})\mathfrak{h} \cap \mathcal{U}(\mathfrak{g})^H$$

It is known that since G/H is a symmetric space, $D(G/H)$ is commutative (see [10] and references therein).

By definition, if (G, H, L) is a transitive triple then one has a diffeomorphism of homogeneous spaces

$$G/H \simeq L/L \cap H$$

and an isomorphism of algebra

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{l} \cap \mathfrak{h})} \mathcal{U}(\mathfrak{l}).$$

In particular, we get an embedding of algebras

$$\iota : D(G/H) \hookrightarrow D(L/L \cap H) .$$

The algebra $D(L/L \cap H)$ need not be commutative, though it is for many transitive triples (see [12]). Of special interest in $D(G/H)$ (resp. $D(L/L \cap H)$, $D(L \cap K/L \cap H)$) is the Casimir element Ω_G (resp. Ω_L , $\Omega_{L \cap K}$). The algebras $D(L/L \cap H)$ and $D(L \cap K/L \cap H)$ are defined in an analogous way to $D(G/H)$. Observe that when G/H has rank one then $D(G/H)$ is generated by the Casimir Ω_G .

Proposition 4.1. *For each transitive triple (G, H, L) with G/H irreducible, the embedding $\iota(\Omega_G)$ is computed explicitly in terms of the decomposition of $\mathfrak{l}/\mathfrak{l} \cap \mathfrak{h}$ into $Ad(L \cap H)$ -irreducibles.*

In particular, based on the restriction to \mathfrak{l} and $\mathfrak{l} \cap \mathfrak{k}$ of the Killing form of \mathfrak{g} , we have:

Example 4.2.

- (1) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$: $\iota(\Omega_G) = 2\Omega_L$.
- (2) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1)$: $\iota(\Omega_G) = 2\Omega_L - \Omega_{L \cap K}$.
- (3) $(SO_0(2, 2n), SO_0(1, 2n), U(1, n))$: $\iota(\Omega_G) = 2\Omega_L - \Omega_{L \cap K}$.
- (4) $(SO_0(4, 3), SO_0(4, 1) \times SO(2), G_{2(2)})$: $\iota(\Omega_G) = 3\Omega_L - \frac{3}{2}\Omega_{L \cap K} + 2\Omega_{\mathfrak{l} \cap \mathfrak{s} \cap \mathfrak{q}}$

In case (3), Schlichtkrull, Trapa and Vogan obtained a similar formula for $O(2n)/U(n)$ [25].

Remark 4.3. By inspection, we observe that $\iota(\Omega_G)$ involves ‘non-compact terms’ terms from $\mathfrak{l} \cap \mathfrak{q} \cap \mathfrak{s}$ only when (G, H, L) is not spherical, except in the first group case (1).

5. ON THE SPECTRUM $\text{SPEC}_{L^2(\Gamma \backslash G/H)}(D(G/H))$.

Fix a transitive triple (G, H, L) . We have the following sequence of isomorphisms:

$$\begin{array}{ccc} L^2(\Gamma \backslash G/H) \simeq L^2(\Gamma \backslash L/L \cap H) \simeq L^2(\Gamma \backslash L)^{L \cap H} \simeq \widehat{\bigoplus}_{\pi \in \widehat{L}} V_{\pi, -\infty}^{\Gamma} \otimes V_{\pi}^{L \cap H} & & \text{(Hilbert sum)} \\ \cup & & \cup \\ C^{\infty}(\Gamma \backslash G/H) \simeq C^{\infty}(\Gamma \backslash L/L \cap H) \simeq C^{\infty}(\Gamma \backslash L)^{L \cap H} \simeq \widehat{\bigoplus}_{\pi \in \widehat{L}} V_{\pi, -\infty}^{\Gamma} \otimes V_{\pi, \infty}^{L \cap H} & & \text{(closure of algebraic direct sum)} \\ \circlearrowleft & & \circlearrowleft \\ D \in D(G/H) & & \iota(D) \in D(L/L \cap H) \end{array}$$

Suppose now that (G, H, L) is spherical, i.e $\dim(\text{Ind}_{P_L}^L \tau_L \otimes e^{\nu_L} \otimes 1)_{-\infty}^{L \cap H} < \infty$, for any $\tau_L \in \widehat{M_L}$ a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L .

Then by Proposition 2.1 and Casselman embedding Theorem, we deduce that

Proposition 5.1. *For any irreducible unitary representation (π, V_{π}) of L , one has*

$$\dim V_{\pi, \infty}^{L \cap H} < \infty$$

This means that the spectrum of $D(G/H)$ on $C^\infty(\Gamma \backslash G/H)$ may be reduced to the diagonalization of the operator $\pi(\iota(D))$ on the (infinitely many) finite dimensional blocks $V_{\pi, \infty}^{L \cap H}$. Then we obtain the following description of the spectrum of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$.

Theorem 5.2. *Suppose (G, H, L) is a spherical triple.*

- (1) *There is a direct sum decomposition of $C^\infty(\Gamma \backslash G/H)$ and $L^2(\Gamma \backslash G/H)$ into joint eigenspaces of $D(G/H)$. $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$ consists of the corresponding eigencharacters and their possible accumulation points.*
- (2) *In the Lorentzian case $(SO_0(2, 2n), SO_0(1, 2n), U(1, n))$, $D(G/H)$ is generated by the Casimir Ω_G , the set $\{\pi \in \widehat{L} \mid V_{\pi, \infty}^{L \cap H} \neq \{0\}\}$ is computed explicitly and the representations π are identified as well as their contributions to eigenvalues for $\iota(\Omega_G)$. More precisely, if Δ denotes the Laplace operator on $\Gamma \backslash G/H$ induced by Ω_G , one has*
 - $\text{spec}(\Delta) \cap (-\infty, -n^2]$ *comes from unitary principal series and, at $-n^2$, also from limits of discrete series of L .*
 - $\text{spec}(\Delta) \cap (-n^2, 0]$ *consists of contributions from complementary series, ends of complementary series and non-integrable discrete series.*
 - $\text{spec}(\Delta) \cap (0, \infty) = \bigcup_{\ell=n+1}^{\infty} \{\ell^2 - n^2\}$ *is the contribution of integrable discrete series. The corresponding eigenspaces are infinite dimensional.*

The last assertion about the contribution of integrable discrete series with $L \cap H$ -invariants generalizes to arbitrary spherical triples (G, H, L) . These contributions combine to infinite dimensional eigenspaces of $D(G/H)$ [21]. The shortest proof of this uses Theorem 3.3 instead of the embedding

$$\iota : D(G/H) \hookrightarrow D(L/L \cap H) .$$

In view of Theorem 3.3 the result can be rephrased in terms of G -representations as follows. Each integrable discrete series for G/H contributes an infinite dimensional eigenspace of $D(G/H)$ in $L^2(\Gamma \backslash G/H)$. Compare related results obtained in [13].

6. GENERALIZED MATRIX COEFFICIENTS, UNITARITY AND CONTINUOUS SPECTRUM.

The Hilbert spaces $L^2(G/H)$ and $L^2(\Gamma \backslash G)$ are both unitary G -representations which decompose into a direct integral of irreducible ones. Though the Hilbert space $L^2(\Gamma \backslash G/H)$ is not a G -module, there is a natural way to involve representations of G .

For an admissible representation (ρ, W_ρ) of G of finite length, one may consider generalized matrix coefficients (see [9]):

$$W_{\tilde{\rho}, -\infty} \otimes W_{\rho, -\infty} \rightarrow C^{-\infty}(G), \quad \tilde{w} \otimes w \mapsto c_{\tilde{w}, w}$$

with

$$\begin{aligned} c_{\tilde{w},w}(f) &= \langle \tilde{w}, \rho(f)w \rangle \\ \langle \rho(f)w, v \rangle &= \int_G f(g) \langle \rho(g)w, v \rangle dg \quad \text{for } f \in C_c^\infty(G), v \in W_{\rho,\infty} \end{aligned}$$

Then there is a $D(G/H)$ -equivariant map (which is injective if ρ is irreducible):

$$(6.1) \quad \begin{array}{ccc} W_{\tilde{\rho},-\infty}^\Gamma \otimes W_{\rho,-\infty}^H & \longrightarrow & C^{-\infty}(\Gamma \backslash G/H) \simeq \widehat{\bigoplus_{\pi \in \hat{L}} V_{\pi,-\infty}^\Gamma} \otimes V_{\pi,-\infty}^{L \cap H} \\ \circlearrowleft & & \circlearrowleft \\ & D(G/H) & D(L/L \cap H) \end{array}$$

A comparison of (6.1) with Theorem 5.2 leads, among other things, to the following funny observation. On one hand, from Theorem 5.2, there are complementary series representations π of $U(1, n)$ contributing to finite dimensional eigenspaces. On the other hand, if ρ were unitary then $\rho|_L$ would contain, by Theorem 3.3, infinitely many unitary summands having $L \cap H$ -invariants. Moreover, a result of Bergeron and Clozel [4], Thm. 6.5.1, tell us that there are indeed lattices $\Gamma \subset U(1, n)$ such that $V_{\pi,-\infty}^\Gamma \neq \{0\}$ for complementary series π with certain integral parameters. Therefore there must exist non-unitary representations ρ of G that are involved in $L^2(\Gamma \backslash G/H)$ via (6.1).

One can also try to use (6.1) to produce families of eigendistributions on $\Gamma \backslash G/H$ depending continuously on the spectral parameter. Let $0 \neq \tilde{v} \in V_{\pi,-\infty}^\Gamma$ and assume we have a continuous nonconstant family $t \mapsto \rho_t$, $t \in (a, b) \subset \mathbb{R}$ of H -spherical principal series of G with $0 \neq w_t \in W_{\rho_t,-\infty}^H$ and, the crucial ingredient, a continuous family of non-zero intertwining operators $\Psi_t \in \text{Hom}_L(W_{\rho_t,\infty}, V_{\pi,\infty})$. Then $\Psi_t^* v \in W_{\rho_t,-\infty}^\Gamma$ and

$$t \mapsto c_{\Psi_t^* v, w_t}$$

is the desired continuous family of eigendistributions. Note that the existence of such a family for $\Gamma \backslash G/H$ excludes the existence of a discrete spectral decomposition as in Theorem 5.2. Therefore such families can only exist for non-spherical triples (G, H, L) . One expects that such families are the building blocks for the continuous part of the spectral decomposition of $L^2(\Gamma \backslash G/H)$. We are able to find such families of intertwiners for some non-spherical triples, and expect their existence for all non-spherical (G, H, L) except for the group case as in Example 2.2 (1). For instance, for the non-spherical triple $(SO(8, \mathbb{C}), SO_0(1, 7), Spin(7, \mathbb{C}))$ the existence of many families Ψ_t is ensured by the work of Jan Frahm [22].

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