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MULTIPLE NORMALIZED SOLUTIONS FOR A SOBOLEV CRITICAL SCHRÖDINGER-POISSON-SLATER EQUATION

LOUIS JEANJEAN AND THANH TRUNG LE

ABSTRACT. We look for solutions to the Schrödinger-Poisson-Slater equation

$$(0.1) \quad -\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3,$$

which satisfy

$$\|u\|_{L^2(\mathbb{R}^3)}^2 = c$$

for some prescribed $c > 0$. Here $u \in H^1(\mathbb{R}^3)$, $\gamma \in \mathbb{R}$, $a \in \mathbb{R}$ and $p \in (\frac{10}{3}, 6]$. When $\gamma > 0$ and $a > 0$, both in the Sobolev subcritical case $p \in (\frac{10}{3}, 6)$ and in the Sobolev critical case $p = 6$, we show that there exists a $c_1 > 0$ such that, for any $c \in (0, c_1)$, (0.1) admits two solutions u_c^+ and u_c^- which can be characterized respectively as a local minima and as a mountain pass critical point of the associated *Energy* functional restricted to the norm constraint. In the case $\gamma > 0$ and $a < 0$, we show that, for any $p \in (\frac{10}{3}, 6]$ and any $c > 0$, (0.1) admits a solution which is a global minimizer. Finally, in the case $\gamma < 0$, $a > 0$ and $p = 6$ we show that (0.1) does not admit positive solutions.

1. INTRODUCTION

We consider the following Schrödinger-Poisson-Slater equation:

$$(1.1) \quad i\partial_t v + \Delta v + \gamma(|x|^{-1} * |v|^2)v + a|v|^{p-2}v = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^3,$$

where $v : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$, $\gamma \in \mathbb{R}$, $a \in \mathbb{R}$ and $p \in (\frac{10}{3}, 6]$. We look for standing wave solutions to (1.1), namely to solutions of the form $v(t, x) = e^{i\lambda t} u(x)$, $\lambda \in \mathbb{R}$. Then the function $u(x)$ satisfies the equation

$$(1.2) \quad -\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3.$$

Motivated by the fact that the L^2 -norm is a preserved quantity of the evolution we focus on the search of solutions to (1.2) with prescribed L^2 -norm. It is standard that for some prescribed $c > 0$, a solution of (1.2) with $\|u\|_{L^2(\mathbb{R}^3)}^2 = c$ can be obtained as a critical point of the *Energy* functional

$$F(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\gamma}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy - \frac{a}{p} \int_{\mathbb{R}^3} |u|^p dx$$

restricted to

$$S(c) := \{u \in H^1(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3)}^2 = c\}.$$

Then the parameter $\lambda \in \mathbb{R}$ in (1.2) appears as a Lagrange multiplier, it is an unknown of the problem.

Let us define

$$(1.3) \quad m(c) = \inf_{u \in S(c)} F(u).$$

Depending on the range of parameters we shall consider $m(c)$ will be finite or not. If, following the introduction of the Compactness by Concentration Principle of P. L. Lions [36, 37], the search of normalized solutions corresponding to a global minimizer of a functional restricted to an L^2 norm constraint is now a classical topic, the search of critical points when the functional is unbounded from below on the constraint remained for a long time much less studied. In the frame of this paper, namely for a functional corresponding to an autonomous equation lying on all the space \mathbb{R}^N , [27] was for a long time the sole contribution. This direction of research was likely brought to the attention of the community by the papers [3, 9] both published in 2013. Since then numerous contributions flourished within this topic and we just mention, among many possible choices, the works, [4, 6–8, 14, 21, 25, 30]. We also refer to [5] for non-autonomous problems set on \mathbb{R}^N and to [41–43] for contributions when the underlying equation is set on a bounded domain of \mathbb{R}^N .

In the above-mentioned papers, the involved nonlinearities were Sobolev subcritical. It was only in 2020 that was first treated in [48] a problem involving a Sobolev critical nonlinearity. Since then several works have explored further this direction [1, 28, 29, 38, 52].

The case where $\gamma < 0$ and $a > 0$ in (1.2) has been the most studied so far. When $p \in (2, \frac{10}{3})$ it can be shown that $m(c) \in (-\infty, 0]$ for any $c > 0$ and it is also the case when $p = \frac{10}{3}$ and $c > 0$ is small. It is shown in [11] that minimizer exists if $p \in (2, 3)$ and $c > 0$ is small enough, see also [45] for the special case $p = \frac{8}{3}$. The case $p \in (3, \frac{10}{3})$ was considered in [10, 31], see also [33] for a closely related problem. In [31] the existence of a threshold value $c_0 > 0$ such that $m(c)$ has a minimizer if and only if $c \in [c_0, \infty)$ was established. It was also proved in [31] that a minimizer does not exist for any $c > 0$ if $p = 3$ or $p = \frac{10}{3}$. We also refer to [18] for related results. When $p \in (\frac{10}{3}, 6]$ a scaling argument reveals that $m(c) = -\infty$ but nevertheless it was proved in [9] that, when $p \in (\frac{10}{3}, 6)$ there exists, for $c > 0$ small enough a critical point of F constrained to $S(c)$ at a strictly positive level. In this work we complement the result of [9] by showing that when $p = 6$ and for any $c > 0$ there does not exist positive solutions, see Theorem 1.9.

Even if some open problems remain when $\gamma < 0$ and $a > 0$, we shall mainly concentrate here on the others cases: $(\gamma < 0, a < 0)$, $(\gamma > 0, a > 0)$ and $(\gamma > 0, a < 0)$. We define, for short, the following quantities

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy, \quad C(u) := \int_{\mathbb{R}^3} |u|^p dx.$$

For $u \in S(c)$, we set $u^t(x) := t^{\frac{3}{2}} u(tx)$, $t > 0$, then

$$u^t \in S(c), \quad A(u^t) = t^2 A(u), \quad B(u^t) = tB(u), \quad C(u^t) = t^\sigma C(u),$$

where

$$(1.4) \quad 2 < \sigma := \frac{3(p-2)}{2} \leq 6,$$

due to $p \in (\frac{10}{3}, 6]$. For $u \in S(c)$, we define the fiber map

$$t \in (0, \infty) \mapsto g_u(t) := F(u^t) = \frac{1}{2} t^2 A(u) - \frac{\gamma}{4} t B(u) - \frac{a}{p} t^\sigma C(u).$$

Hence, we have

$$g'_u(t) = tA(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} t^{\sigma-1} C(u) = \frac{1}{t} Q(u^t),$$

where

$$Q(u) = A(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} C(u).$$

Actually the condition $Q(u) = 0$ corresponds to a Pohozaev identity and the set

$$\Lambda(c) := \{u \in S(c) : Q(u) = 0\} = \{u \in S(c) : g'_u(1) = 0\}$$

appears as a natural constraint. Indeed, if $u \in S(c)$, then $t > 0$ is a critical point for g_u if and only if $u^t \in \Lambda(c)$. In particular, $u \in \Lambda(c)$ if and only if 1 is a critical point of g_u .

First we briefly consider the case $\gamma < 0, a < 0$. For any $u \in S(c)$, we have that $g'_u(t) > 0$ for all $t > 0$, hence the fiber map $g_u(t)$ is strictly increasing and so we can state the following non-existence result:

Theorem 1.1. *Assume that $\gamma < 0, a < 0$. Then $F(u)$ has no critical point on $S(c)$.*

Next, we consider the case $\gamma > 0, a > 0$. In this case, let

$$(1.5) \quad c_1 := \left(\frac{4}{\gamma K_H} \frac{\sigma-2}{\sigma-1} \right)^{\frac{3p-10}{4(p-3)}} \left(\frac{p}{a\sigma(\sigma-1)K_{GN}} \right)^{\frac{1}{2(p-3)}} > 0,$$

where σ is defined by (1.4) and K_H, K_{GN} are defined in Lemma 2.1. We also introduce the decomposition of $\Lambda(c)$ into the disjoint union $\Lambda(c) = \Lambda^+(c) \cup \Lambda^0(c) \cup \Lambda^-(c)$, where

$$\begin{aligned} \Lambda^+(c) &:= \{u \in \Lambda(c) : g''_u(1) > 0\} = \{u \in S(c) : g'_u(1) = 0, g''_u(1) > 0\}, \\ \Lambda^0(c) &:= \{u \in \Lambda(c) : g''_u(1) = 0\} = \{u \in S(c) : g'_u(1) = 0, g''_u(1) = 0\}, \end{aligned}$$

$$\Lambda^-(c) := \{u \in \Lambda(c) : g_u''(1) < 0\} = \{u \in S(c) : g_u'(1) = 0, g_u''(1) < 0\}.$$

By [Lemma 3.3](#) and [Lemma 3.4](#), for any $c \in (0, c_1)$ we have that $\Lambda^0(c) = \emptyset$ and $\Lambda^+(c) \neq \emptyset$, $\Lambda^-(c) \neq \emptyset$. Since F is bounded from below on $\Lambda(c)$ due to [Lemma 3.1](#), we can define

$$(1.6) \quad \gamma^+(c) := \inf_{u \in \Lambda^+(c)} F(u) \quad \text{and} \quad \gamma^-(c) := \inf_{u \in \Lambda^-(c)} F(u).$$

Our first main result is

Theorem 1.2. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma > 0$, $a > 0$ and let $c_1 > 0$ be defined by (1.5). For any $c \in (0, c_1)$, there exist $u_c^+ \in \Lambda^+(c)$ such that $F(u_c^+) = \gamma^+(c)$ and $u_c^- \in \Lambda^-(c)$ such that $F(u_c^-) = \gamma^-(c)$. The functions u_c^+, u_c^- are bounded continuous positive Schwarz symmetric functions. In addition there exist $\lambda_c^+ > 0$ and $\lambda_c^- > 0$ such that (u_c^+, λ_c^+) and (u_c^-, λ_c^-) are solutions to (1.2).*

Remark 1.3. *In [Theorem 1.2](#), borrowing an approach first introduced in [21], an effort is made to optimize the limit value $c_1 > 0$. As a consequence, compared to the works [28, 29, 47, 48] we do not benefit from the property that $\gamma^-(c) \geq 0 = \sup_{u \in \Lambda^+(c)} F(u)$. Such property is a help to show the convergence of the Palais-Smale sequences in these works. Also, the fact that we may have $\gamma^-(c) < 0$ makes somehow more involved to prove that the level $\gamma^-(c)$ is reached by a radially symmetric function, a Schwartz function actually, see [Lemma 3.6](#). It is not clear to us if $c_1 > 0$ is optimal. Nevertheless, we conjecture that there exists a $c_0 \geq c_1 > 0$ such that one solution exists when $c = c_0$ and that, at least positive solutions, do not exist when $c > c_0$.*

Remark 1.4. *As we shall see $\gamma^+(c) < \gamma^-(c)$ and combined with the property that any critical point lies in $\Lambda(c)$ it implies that the solution u_c^+ obtained in [Theorem 1.2](#) is a ground state. Following [8] a ground state is defined as a solution $v \in S(c)$ to (1.2) which has minimal Energy among all the solutions which belong to $S(c)$. Namely, if*

$$F(v) = \inf \left\{ F(u), u \in S(c), (F|_{S(c)})'(u) = 0 \right\}.$$

If the geometrical structure of F restricted to $S(c)$ is identical in the Sobolev subcritical case $p \in (\frac{10}{3}, 6)$ and in the Sobolev critical case $p = 6$, the proof that the levels $\gamma^+(c)$ and $\gamma^-(c)$ are indeed reached requires additional, more involved, arguments in the case $p = 6$. In particular, showing that $\gamma^-(c)$ is attained requires to check that the following inequality holds

$$(1.7) \quad \gamma^-(c) < \gamma^+(c) + \frac{1}{3\sqrt{aK_{GN}}}.$$

It is known since the pioneering work of Brezis-Nirenberg [17] that the way to derive such a strict upper bound is through the use of testing functions. In [29], considering the equation

$$(1.8) \quad -\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N,$$

with $N \geq 3$, $\mu > 0$, $2 < q < 2 + \frac{4}{N}$ and $2^* = \frac{2N}{N-2}$ we face the need to establish a similar inequality. We constructed test functions which could be viewed as the sum of a truncated extremal function of the Sobolev inequality on \mathbb{R}^N centered at the origin and of u_c^+ translated far away from the origin. This choice of testing functions was sufficient to prove our strict inequality when $N \geq 4$ but we missed it in the case $N = 3$. Note that the approach developed in [29] proved nevertheless adequate to deal with the equation

$$\sqrt{-\Delta}u = \lambda u + \mu|u|^{q-2}u + |u|^{2^*-2}u, \quad u \in H^{\frac{1}{2}}(\mathbb{R}^N),$$

with $N \geq 2$, $q \in (2, 2 + \frac{2}{N})$, $2^* = \frac{2N}{N-1}$, that was studied in [38]. Very recently, in [52] the authors introduced an alternative choice of testing functions which allowed to treat, in a unified way, the case $N = 3$ and $N \geq 4$ for (1.8). The strategy in [52], recording of the one introduced by G. Tarantello in [50], is on the contrary, to locate the extremal functions where the solution u_c^+ takes its greater values (the origin thus). The idea behind the proof is that the interaction decreases the value of the Energy with respect to the case where the supports would be disjoint. In this paper, where (1.2) is set on \mathbb{R}^3 , we believe in view of our experience on (1.8), more appropriate to follow the approach of [52] to check the inequality (1.7) for any $c \in (0, c_1)$.

The results of [Theorem 1.2](#) are complemented in several directions. First, we show that the solution $u^+(c)$ obtained in [Theorem 1.2](#) can be characterized as a local minima for F restricted to $S(c)$. We treat here the full range $p \in (\frac{10}{3}, 6]$ with a single proof. More precisely we show,

Theorem 1.5. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma > 0$, $a > 0$ and let $c \in (0, c_1)$. Then we have $\Lambda^+(c) \subset V(c)$ and*

$$\gamma^+(c) = \inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in V(c)} F(u)$$

where

$$V(c) := \{u \in S(c) | A(u) < k_1\}$$

for some $k_1 > 0$ independent of $c \in (0, c_1)$ (see (3.45) for the definition of $k_1 > 0$). In addition, any minimizing sequence for F on $V(c)$ is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$.

Remark 1.6. *The proof of Lemma 3.20 which is a key step to established Theorem 1.5, reveals some additional properties of the set $V(c)$. Indeed, we have that $V(c) \subset S(c) \setminus \Lambda^-(c)$ and thus $V(c)$ is separating the sets $\Lambda^+(c)$ and $\Lambda^-(c)$. Also, for any $0 < c, \tilde{c} < c_1$, we have that $A(u) < k_1 < A(v)$ for all $u \in \Lambda^+(c), v \in \Lambda^-(\tilde{c})$, see (3.46) and (3.49).*

Remark 1.7. *To prove that the minimizing sequences for F on $V(c)$ are, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$ we follow an approach due to [26] that has already been used several times, see, for example, [24, 28, 38]. The first step in this approach is to show that the sequences do not vanish. When $p = 6$, we rely for this, in an essential way, on the fact that $c_1 > 0$ is sufficiently small, see Lemma 3.22. This fact is also used to end the proof. Finally, note that since we allow the possibility that $\inf_{u \in \partial V(c)} F(u) < 0$ where $\partial V(c) := \{u \in S(c) | A(u) = k_1\}$ we must check that the minimizers do ly in $V(c)$.*

Let us now denote

$$\mathcal{M}_c := \{u \in V(c) : F(u) = \gamma^+(c)\}.$$

In view of Remark 1.4, \mathcal{M}_c is the set of all ground states. The property that any minimizing sequence for F restricted to $V(c)$ is, up to translation, strongly converging is known to be a key ingredient to show that the set \mathcal{M}_c is orbitally stable. If $p \in (\frac{10}{3}, 6)$ the orbital stability of \mathcal{M}_c indeed follows directly from Theorem 1.5 by the classical arguments of [20]. In the case $p = 6$ the situation is more delicate as the existence of a uniform $H^1(\mathbb{R}^3)$ bound on the solution of (1.1) during its lifespan is not sufficient to guarantee that blow-up may not occurs. We refer to [19] for more details. We do not prove anything in that direction but strongly believe that the set \mathcal{M}_c is orbitally stable. Actually, such a result has been obtained on the equation (1.8) in [28].

We also discuss the behavior of the associated Lagrange multipliers and show that if the behavior of λ_c^+ is essentially the same for the cases $p \in (\frac{10}{3}, 6)$ and $p = 6$, see Lemma 3.24, there is a distinct behavior for λ_c^- , see Lemmas 3.25 and 3.26. In particular, Lemma 3.26 suggests that there may exist two distinct positive solutions to (1.2) for any fixed $\lambda > 0$ sufficiently small. Finally, in Lemma 3.27, we establish the property that the map $c \mapsto \gamma^-(c)$ is strictly decreasing.

Next, we consider the case $\gamma > 0$, $a < 0$. Recalling the definition of $m(c)$ given in (1.3) we show in Lemma 4.1, that $-\infty < m(c) < 0$ and then we prove the following result.

Theorem 1.8. *Let $p \in (\frac{10}{3}, 6]$, $\gamma > 0$ and $a < 0$. For any $c > 0$, the infimum $m(c)$ is achieved and any minimizing sequence for (1.3) is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$ to a solution of (1.2). In addition, the associated Lagrange multiplier is positive.*

Even if the proof of Theorem 1.8 follows the lines of the proof of Theorem 1.5, the change of sign in front of the power term requires some adaptations, see Lemma 4.2 and Lemma 4.4. Here again the orbital stability of the set of minimizers should follow directly from the classical arguments of [20] if $p \in (\frac{10}{3}, 6)$ and it should also be the case when $p = 6$ by adapting the arguments of [28]. Note that we also study the behavior of the associated Lagrange multipliers in Lemma 4.5.

In the last part of the paper we consider the case $\gamma < 0$, $a > 0$ and $p = 6$.

Theorem 1.9. *Let $p = 6$, $\gamma < 0$ and $a > 0$. For any $c > 0$, we have that*

(i) *If $u \in H^1(\mathbb{R}^3)$ is a non-trivial solution to (1.2) then the associated Lagrange multiplier λ is negative and*

$$F(u) > \frac{1}{3\sqrt{a}K_{GN}}.$$

(ii) *Equation (1.2) has no positive solution in $H^1(\mathbb{R}^3)$.*

Remark 1.10. Under the assumptions of [Theorem 1.9](#), it is possible to prove that

$$\inf_{u \in \Lambda(c)} F(u) = \frac{1}{3\sqrt{a}K_{GN}}.$$

Remark 1.11. In [[48](#), [Theorem 1.2](#)], considering the equation

$$(1.9) \quad -\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N,$$

with $N \geq 3$, $2 < q < 2^*$ and $\mu < 0$, it was proved that (1.9) has no positive solution $u \in H^1(\mathbb{R}^N)$ if $N = 3, 4$ or if $N \geq 5$ under the additional assumption $u \in L^p(\mathbb{R}^N)$ for some $p \in (0, \frac{N}{N-2}]$. In [Remark 5.2](#), partly using arguments used in the proof of [Theorem 1.9](#), we improve [[48](#), [Theorem 1.2](#)] showing that (1.9) has no positive solution in $H^1(\mathbb{R}^N)$ for all $N \geq 3$ and no non-trivial radial solution for $N \geq 3$ and $q > 2 + \frac{2}{N-1}$.

Remark 1.12. We propose as an open problem to investigate if there are radial solutions under the assumptions of [Theorem 1.9](#). See [Remark 5.3](#) in that direction.

The paper is organized as follows. In [Section 2](#) we recall some classical inequalities and present some preliminary results. [Section 3](#) is devoted to the treatment of the case $\gamma > 0$, $a > 0$ and $p \in (\frac{10}{3}, 6]$. In [Subsection 3.1](#) we make explicit the geometrical structure of F on $S(c)$ and show the existence of a bounded Palais-Smale sequence $(u_n^+) \subset \Lambda^+(c)$ at the level $\gamma^+(c)$ and of a bounded Palais-Smale sequence $(u_n^-) \subset \Lambda^-(c)$ at the level $\gamma^-(c)$. In [Subsection 3.2](#) we give the proof of [Theorem 1.2](#) in the Sobolev subcritical case. [Subsection 3.3](#) is devoted to the proof of [Theorem 1.2](#) in the critical case. In [Subsection 3.4](#) we prove the convergence of all minimizing sequences associated to $\gamma^+(c)$, namely [Theorem 1.5](#). The behavior of the Lagrange multipliers and the property of the map $c \mapsto \gamma^-(c)$ are studied in [Subsection 3.5](#) and [Subsection 3.6](#), respectively. In [Section 4](#) we treat the case $\gamma > 0$, $a < 0$ and $p \in (\frac{10}{3}, 6]$ and we prove [Theorem 1.8](#). Finally, in [Section 5](#), we consider the case $\gamma < 0$, $a > 0$ and $p = 6$, and prove [Theorem 1.9](#).

Notation: For $p \geq 1$, the L^p -norm of $u \in H^1(\mathbb{R}^3)$ is denoted by $\|u\|_{L^p(\mathbb{R}^3)}$. We denote by $H_r^1(\mathbb{R}^3)$ the subspace of functions in $H^1(\mathbb{R}^3)$ which are radially symmetric with respect to 0. The notation $a \sim b$ means that $Cb \leq a \leq C'b$ for some $C, C' > 0$. The open ball in \mathbb{R}^3 with center at 0 and radius $R > 0$ is denoted by B_R .

Addendum : After the completion of this paper, we were informed of the work [[53](#)] in which the authors consider a general class of problems which, when $p \in (\frac{10}{3}, 6)$, covers (1.2) as a special case. There are thus some partial overlap, in the Sobolev subcritical case, between [[53](#), [Theorem 1.3](#) (a) (ii)] and [Theorem 1.2](#) and between [[53](#), [Theorem 1.6](#) (a) (iv)] and [Theorem 1.8](#). However the scope of the two works is widely distinct.

2. PRELIMINARY RESULTS

In this section we present various preliminary results. When it is not specified they are assumed to hold for $\gamma \in \mathbb{R}$, $a \in \mathbb{R}$, $p \in (\frac{10}{3}, 6]$ and any $c > 0$. Firstly, we present the definitions of $\Lambda(c)$, $\Lambda^-(c)$, $\Lambda^0(c)$, $\Lambda^+(c)$ via $A(u)$, $B(u)$ and $C(u)$:

$$\begin{aligned} \Lambda(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u) \right\}, \\ \Lambda^+(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), A(u) > \frac{a\sigma(\sigma-1)}{p}C(u) \right\}, \\ \Lambda^0(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), A(u) = \frac{a\sigma(\sigma-1)}{p}C(u) \right\}, \\ \Lambda^-(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), A(u) < \frac{a\sigma(\sigma-1)}{p}C(u) \right\}. \end{aligned}$$

Lemma 2.1. Let $u \in S(c)$, there exists

- (i) a constant $K_H > 0$ such that $B(u) \leq K_H \sqrt{A(u)} c^{\frac{3}{2}}$.
- (ii) a constant $K_{GN} > 0$ such that $C(u) \leq K_{GN} [A(u)]^{\frac{\sigma}{2}} c^{\frac{6-p}{4}}$.

Proof. We first recall the Hardy-Littlewood-Sobolev inequality (see [35, Chapter 4]):

$$(2.1) \quad \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \right| \leq C(N, \lambda, p, q) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)},$$

where $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, $p, q > 1$, $0 < \lambda < N$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{N} = 2.$$

Let us also recall the Gagliardo-Nirenberg inequality (see [40]) and the Sobolev inequality (see [15, Theorem 9.9]) in the unified form: if $N \geq 3$ and $p \in [2, \frac{2N}{N-2}]$ then

$$\|f\|_{L^p(\mathbb{R}^N)} \leq C(N, p) \|\nabla f\|_{L^2(\mathbb{R}^N)}^\beta \|f\|_{L^2(\mathbb{R}^N)}^{(1-\beta)}, \quad \text{with } \beta = N \left(\frac{1}{2} - \frac{1}{p} \right).$$

Applying the Hardy-Littlewood-Sobolev inequality we obtain

$$(2.2) \quad B(u) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy \leq K_1 \|u\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4$$

and thus using the Gagliardo-Nirenberg inequality, we get

$$B(u) \leq K_1 \|u\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \leq K_1 K_2 \|\nabla u\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}^3 = K_H \sqrt{A(u)} c^{\frac{3}{2}}.$$

Finally, applying the Sobolev, Gagliardo-Nirenberg inequality, we have

$$C(u) = \|u\|_{L^p(\mathbb{R}^3)}^p \leq K_{GN} \|\nabla u\|_{L^2(\mathbb{R}^3)}^\sigma \|u\|_{L^2(\mathbb{R}^3)}^{\frac{6-p}{2}} = K_{GN} [A(u)]^{\frac{\sigma}{2}} c^{\frac{6-p}{4}}.$$

□

Lemma 2.2. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma \in \mathbb{R}$ and $a \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^3)$ is a weak solution to*

$$(2.3) \quad -\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0,$$

then $Q(u) = 0$. Moreover, if $u \neq 0$ then we have

- (i) $\lambda > 0$ if $\gamma > 0$ and $p \in (\frac{10}{3}, 6]$,
- (ii) $\lambda < 0$ if $\gamma < 0$ and $p = 6$.

Proof. Our proof is inspired by [9, Lemma 4.2]. The following Pohozaev type identity holds for $u \in H^1(\mathbb{R}^3)$ weak solution of (2.3) ([22], also see [44, Theorem 2.2]),

$$(2.4) \quad \frac{1}{2}A(u) + \frac{3\lambda}{2}D(u) - \frac{5\gamma}{4}B(u) - \frac{3a}{p}C(u) = 0, \quad \text{where } D(u) = \|u\|_{L^2(\mathbb{R}^3)}^2.$$

By multiplying (2.3) by u and integrating, we derive a second identity

$$(2.5) \quad A(u) + \lambda_c D(u) - \gamma B(u) - aC(u) = 0.$$

Combining (2.4) and (2.5), we get

$$A(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}C(u) = 0.$$

This means that $Q(u) = 0$. Using (2.4) and (2.5) again, we obtain

$$(2.6) \quad 2(6-p)A(u) + (5p-12)\gamma B(u) = 2(3p-6)\lambda D(u).$$

If $\gamma > 0$ and $p \in (\frac{10}{3}, 6]$, we have

$$2(6-p) \geq 0, \quad (5p-12)\gamma > 0, \quad 2(3p-6) > 0.$$

Hence, $\lambda > 0$. If $\gamma < 0$ and $p = 6$, we have

$$2(6-p) = 0, \quad (5p-12)\gamma = 18\gamma < 0, \quad 2(3p-6) = 24 > 0.$$

This implies that $\lambda < 0$.

□

Lemma 2.3. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma \in \mathbb{R}$ and $a \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^3)$ is a weak solution to*

$$(2.7) \quad -\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0,$$

then $u \in L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3)$. Moreover, in case $\gamma > 0$, $a > 0$ we have that if $u \not\equiv 0$ and $u \geq 0$ then $u > 0$.

Proof. Applying [34, Theorem 2.1], we get that $u \in W_{loc}^{2,r}(\mathbb{R}^3)$ for every $r > 1$ and hence $u \in C(\mathbb{R}^3)$. Since $u \in H^1(\mathbb{R}^3)$, the Sobolev embedding (see [15, Corollary 9.10]) implies that $|u|^2 \in L^q(\mathbb{R}^3)$ for every $q \in [1, 3]$. Now, setting $K := |x|^{-1}$, we write $K := K_1 + K_2$ where $K_1 := K$ on $B(0, 1)$, $K_1 := 0$ on $\mathbb{R}^3 \setminus B(0, 1)$ and $K_2 := K - K_1$. Clearly $K_1 \in L^2(\mathbb{R}^3)$ and $K_2 \in L^4(\mathbb{R}^3)$. Applying [35, Lemma 2.20] with $K_1 \in L^2(\mathbb{R}^3)$, $|u|^2 \in L^2(\mathbb{R}^3)$ and with $K_2 \in L^4(\mathbb{R}^3)$, $|u|^2 \in L^{\frac{4}{3}}(\mathbb{R}^3)$, we obtain that $K_1 * |u|^2$ and $K_2 * |u|^2$ are continuous. Also

$$\lim_{|x| \rightarrow \infty} (K_1 * |u|^2)(x) = 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} (K_2 * |u|^2)(x) = 0.$$

Hence, we get that $K * |u|^2$ is continuous and

$$(2.8) \quad \lim_{|x| \rightarrow \infty} (K * |u|^2)(x) = 0.$$

Therefore, $K * |u|^2$ is bounded. At this point, we deduce from [48, Proposition B.1] that $u \in L^\infty(\mathbb{R}^3)$.

Now, if we assume that $\gamma > 0$, $a > 0$, $u \not\equiv 0$, $u \geq 0$, setting $v := -u \leq 0$ we get

$$-\Delta v + \lambda v = \gamma(|x|^{-1} * |v|^2)v + a|v|^{p-2}v \leq 0.$$

By Lemma 2.2, we have that $\lambda > 0$. We assume that there exists $x_0 \in \mathbb{R}^3$ such that $v(x_0) = 0$. For all $R > |x_0|$, we have that $v \in W^{2,r}(B_R)$ for every $r > 1$, $Lv := -\Delta v + \lambda v \leq 0$ in B_R with $\lambda > 0$ and $M := \max_{x \in B_R} v = 0$. At this point, applying [51, Theorem 3.27], in the particular case where $\Gamma = \emptyset$, we obtain that $v \equiv 0$ in B_R , and hence $u \equiv 0$ in B_R . The value $R > 0$ being arbitrarily large, this contradicts our assumption that $u \not\equiv 0$ and we conclude that $u > 0$. \square

Following [13], we recall that, for any $c > 0$, $S(c)$ is a submanifold codimension 1 of $H^1(\mathbb{R}^3)$ and the tangent space at a point $u \in S(c)$ is defined as

$$T_u S(c) = \{\varphi \in H^1(\mathbb{R}^3) : \langle u, \varphi \rangle_{L^2(\mathbb{R}^3)} = 0\}.$$

The restriction $F|_{S(c)} : S(c) \rightarrow \mathbb{R}$ is a C^1 functional on $S(c)$ and for any $u \in S(c)$ and any $v \in T_u S(c)$, we have

$$\langle F'|_{S(c)}, \varphi \rangle = \langle F', \varphi \rangle.$$

We use the notation $\|dF|_{S(c)}\|_*$ to indicate the norm in the cotangent space $T_u S(c)'$, i.e the dual norm induced by the norm of $T_u S(c)$, i.e

$$(2.9) \quad \|dF|_{S(c)}(u)\|_* := \sup_{\|\varphi\| \leq 1, \varphi \in T_u S(c)} |dF(u)[\varphi]|.$$

We recall the following result, see [29, Lemma 3.1],

Lemma 2.4. *For $u \in S(c)$ and $t > 0$, the map*

$$T_u S(c) \rightarrow T_{u^t} S(c), \quad \psi \mapsto \psi^t$$

is a linear isomorphism with inverse

$$T_{u^t} S(c) \rightarrow T_u S(c), \quad \phi \mapsto \phi^{\frac{1}{t}}.$$

Next, we recall a result concerning the convergence of the term B , see [44, Lemma 2.1],

Lemma 2.5. *Let (u_n) be a sequence satisfying $u_n \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^3)$. Then we have $B(u_n) \rightarrow B(u)$.*

3. THE CASE $\gamma > 0$, $a > 0$ AND $p \in (\frac{10}{3}, 6]$.

3.1. **The geometrical structure and the existence of bounded Palais-Smale sequences for $p \in (\frac{10}{3}, 6]$.** In this subsection, we follow the approach first introduced in [21]. We shall always assume that $\gamma > 0$, $a > 0$ and $p \in (\frac{10}{3}, 6]$.

Lemma 3.1. *For any $c \in (0, \infty)$, F restricted to $\Lambda(c)$ is coercive on $H^1(\mathbb{R}^3)$, namely when $(u_n) \subset H^1(\mathbb{R}^3)$ satisfies $\|u_n\| \rightarrow +\infty$ then $F(u_n) \rightarrow +\infty$. In particular F restricted to $\Lambda(c)$ is bounded from below.*

Proof. Let $u \in \Lambda(c)$. Taking into account that

$$\frac{a}{p}C(u) = \frac{1}{\sigma}A(u) - \frac{\gamma}{4\sigma}B(u),$$

and using Lemma 2.1(i), we obtain

$$\begin{aligned} (3.1) \quad F(u) &= \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{1}{\sigma}A(u) + \frac{\gamma}{4\sigma}B(u) \\ &= \frac{\sigma-2}{2\sigma}A(u) - \frac{\gamma(\sigma-1)}{4}B(u) \geq \frac{\sigma-2}{2\sigma}A(u) - \frac{\gamma(\sigma-1)}{4}K_H\sqrt{A(u)}c^{\frac{3}{2}}. \end{aligned}$$

This concludes the proof. □

For any $u \in S(c)$, we recall that

$$\begin{aligned} g_u(t) &= F(u^t) = \frac{1}{2}t^2A(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}t^\sigma C(u), \\ g'_u(t) &= tA(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}t^{\sigma-1}C(u) = \frac{1}{t}Q(u^t), \\ g''_u(t) &= A(u) - \frac{a\sigma(\sigma-1)}{p}t^{\sigma-2}C(u). \end{aligned}$$

For any $u \in S(c)$, we set

$$t_u^\star := \left(\frac{pA(u)}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}}.$$

This implies that t_u^\star is the unique solution of equation $g''_u(t) = 0$. So, we have

$$(3.2) \quad g''_u(t_u^\star) = 0, \quad g''_u(t) > 0 \text{ if } 0 < t < t_u^\star, \quad g''_u(t) < 0 \text{ if } t > t_u^\star.$$

Lemma 3.2. *For any $c \in (0, c_1)$ and any $u \in S(c)$, we have $g'_u(t_u^\star) > 0$.*

Proof. Let $u \in S(c)$ be arbitrary. By the definition of t_u^\star and by $g''_u(t_u^\star) = 0$, we have

$$\begin{aligned} g'_u(t_u^\star) &= t_u^\star A(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}(t_u^\star)^{\sigma-1}C(u) = t_u^\star A(u) - \frac{\gamma}{4}B(u) - \frac{1}{\sigma-1}t_u^\star A(u) \\ &= \frac{\sigma-2}{\sigma-1}t_u^\star A(u) - \frac{\gamma}{4}B(u) = \frac{\sigma-2}{\sigma-1} \left(\frac{pA(u)}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}} A(u) - \frac{\gamma}{4}B(u) \\ &= \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{pA(u)}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}} \sqrt{A(u)} - \frac{\gamma}{4} \frac{B(u)}{\sqrt{A(u)}} \right] \\ &= \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{p[A(u)]^{\frac{\sigma}{2}}}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4} \frac{B(u)}{\sqrt{A(u)}} \right]. \end{aligned}$$

Applying Lemma 2.1, we obtain

$$g'_u(t_u^\star) \geq \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{p[A(u)]^{\frac{\sigma}{2}}}{a\sigma(\sigma-1)K_{GN}[A(u)]^{\frac{\sigma}{2}}c^{\frac{6-p}{4}}} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4} \frac{K_H\sqrt{A(u)}c^{\frac{3}{2}}}{\sqrt{A(u)}} \right]$$

$$= \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{p}{a\sigma(\sigma-1)K_{GN}c^{\frac{6-p}{4}}} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4}K_Hc^{\frac{3}{2}} \right].$$

By direct computations, we now have

$$\frac{\sigma-2}{\sigma-1} \left(\frac{p}{a\sigma(\sigma-1)K_{GN}c^{\frac{6-p}{4}}} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4}K_Hc^{\frac{3}{2}} > 0 \iff c < c_1.$$

Thus, we obtain that if $0 < c < c_1$ then $g'_u(t_u^*) > 0$. \square

Lemma 3.3. *For any $c \in (0, c_1)$, it holds that $\Lambda^0(c) = \emptyset$.*

Proof. We assume that there exists $u \in \Lambda^0(c)$. Since $g''_u(1) = 0$ and t_u^* is the unique solution of equation $g''_u(t) = 0$, we have $t_u^* = 1$. So, we have $g'_u(t_u^*) = g'_u(1) = 0$. This contradicts with $g'_u(t_u^*) > 0$ in Lemma 3.2. Thus, we obtain $\Lambda^0(c) = \emptyset$. \square

Lemma 3.4. *For any $c \in (0, c_1)$ and any $u \in S(c)$, there exists*

- (i) a unique $s_u^+ \in (0, t_u^*)$ such that s_u^+ is a unique local minimum point for g_u and $u^{s_u^+} \in \Lambda^+(c)$.
- (ii) a unique $s_u^- \in (t_u^*, \infty)$ such that s_u^- is a unique local maximum point for g_u and $u^{s_u^-} \in \Lambda^-(c)$.

Moreover, the maps $u \in S(c) \mapsto s_u^+ \in \mathbb{R}$ and $u \in S(c) \mapsto s_u^- \in \mathbb{R}$ are of class \mathbb{C}^1 .

Proof. Taking into account that

$$g'_u(t) = tA(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}t^{\sigma-1}C(u),$$

we have $g'_u(t) \rightarrow -\frac{\gamma}{4}B(u) < 0$ as $t \rightarrow 0$ and $g'_u(t) \rightarrow -\infty$ as $t \rightarrow +\infty$ due to $\sigma - 1 > 1$. By Lemma 3.2, we have $g'_u(t_u^*) > 0$. Therefore, the equation $g'_u(t) = 0$ has at least two solutions s_u^+ and s_u^- with $0 < s_u^+ < t_u^* < s_u^-$. By (3.2), we have $g''_u(t) > 0$ for all $0 < t < t_u^*$. Hence, $g'_u(t)$ is strictly increasing function on $(0, t_u^*)$ and consequently $s_u^+ \in (0, t_u^*)$ is the unique local minimum point for g_u and $u^{s_u^+} \in \Lambda^+(c)$ due to $g''_{u^{s_u^+}}(1) = g''_u(s_u^+) > 0$. By the same argument, we obtain that $s_u^- \in (t_u^*, \infty)$ is a unique local maximum point for g_u and $u^{s_u^-} \in \Lambda^-(c)$.

In order to prove that $u \mapsto s_u^-$ are of class \mathbb{C}^1 , we follow the argument in [47, Lemma 5.3]. It is a direct application of the Implicit Function Theorem on \mathbb{C}^1 -function $\varphi(t, u) = g'_u(t)$. Taking into account that $\varphi(s_u^-, u) = g'_u(s_u^-) = 0$, $\partial_t \varphi(s_u^-, u) = g''_u(s_u^-) < 0$ and $\Lambda^0(c) = \emptyset$, we obtain $u \mapsto s_u^-$ is of class \mathbb{C}^1 . The same argument proves that $u \mapsto s_u^+$ is of class \mathbb{C}^1 . \square

Lemma 3.5. *For any $c \in (0, c_1)$, it holds that*

- (i) $F(u) < 0$ for all $u \in \Lambda^+(c)$,
- (ii) there exists $\alpha := \alpha(c) > 0$ such that $A(u) \geq \alpha$ for all $u \in \Lambda^-(c)$.

Proof. Let $u \in \Lambda^+(c)$, taking into account that

$$A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), \quad A(u) > \frac{a\sigma(\sigma-1)}{p}C(u),$$

we obtain

$$\begin{aligned} F(u) &= \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}C(u) + \frac{a(\sigma-1)}{p}C(u) \\ &< \frac{1}{2}A(u) - A(u) + \frac{1}{\sigma}A(u) = \frac{2-\sigma}{2\sigma}A(u). \end{aligned}$$

Since $\sigma > 2$, we have $F(u) < 0$. The point (i) is proved.

Let $u \in \Lambda^-(c)$, taking into account that

$$A(u) < \frac{a\sigma(\sigma-1)}{p}C(u),$$

and using Lemma 2.1, we obtain that

$$A(u) < \frac{a\sigma(\sigma-1)}{p}K_{GN}c^{\frac{6-p}{4}}[A(u)]^{\frac{\sigma}{2}}.$$

Since $\sigma > 2$, the point (ii) follows. □

We define

$$S_r(c) := S(c) \cap H_r^1(\mathbb{R}^3), \quad \Lambda_r(c) := \Lambda(c) \cap H_r^1(\mathbb{R}^3), \quad \Lambda_r^\pm(c) := \Lambda^\pm(c) \cap H_r^1(\mathbb{R}^3).$$

Here $\Lambda^\pm(c)$ denotes either $\Lambda^+(c)$ or $\Lambda^-(c)$.

Lemma 3.6. *For any $c \in (0, c_1)$ it holds that*

$$\inf_{u \in \Lambda_r^\pm(c)} F(u) = \inf_{u \in \Lambda^\pm(c)} F(u).$$

Also, if $\inf_{u \in \Lambda^\pm(c)} F(u)$ is reached, it is reached by a Schwarz symmetric function.

Proof. Since $\Lambda_r^\pm(c) \subset \Lambda^\pm(c)$, we directly have

$$(3.3) \quad \inf_{u \in \Lambda_r^\pm(c)} F(u) \geq \inf_{u \in \Lambda^\pm(c)} F(u).$$

Therefore, it suffices to prove that

$$(3.4) \quad \inf_{u \in \Lambda_r^\pm(c)} F(u) \leq \inf_{u \in \Lambda^\pm(c)} F(u).$$

In this aim we start to note that

$$(3.5) \quad \inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in S(c)} \min_{0 < t \leq s_u^+} F(u^t) \quad \text{and} \quad \inf_{u \in \Lambda^-(c)} F(u) = \inf_{u \in S(c)} \max_{s_u^+ < t \leq s_u^-} F(u^t).$$

Now let $u \in S(c)$ and $v \in S_r(c)$ be the Schwarz rearrangement of $|u|$. Taking into account that $A(v) \leq A(u)$, $C(v) = C(u)$, and by the Riesz's rearrangement inequality (see [35, Section 3.7]), $B(v) \geq B(u)$, we have for all $t > 0$,

$$(3.6) \quad F(v^t) = \frac{1}{2}t^2 A(v) - \frac{\gamma}{4}tB(v) - \frac{a}{p}t^\sigma C(v) \leq \frac{1}{2}t^2 A(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}t^\sigma C(u) = F(u^t).$$

Observe that, for any $w \in S(c)$,

$$g_w'(t) = tA(w) - \frac{\gamma}{4}B(w) - \frac{a\sigma}{p}t^{\sigma-1}C(w) \quad \text{and} \quad g_w''(t) = A(w) - \frac{a\sigma(\sigma-1)}{p}t^{\sigma-2}C(w).$$

Thus we have

$$g_v'(0) \leq g_u'(0) < 0 \quad \text{and} \quad g_v''(t) \leq g_u''(t), \quad \forall t > 0.$$

This implies that $0 < s_u^+ \leq s_v^+ < s_v^- \leq s_u^-$. Hence, we deduce from (3.6) that

$$\min_{0 < t \leq s_v^+} F(v^t) \leq \min_{0 < t \leq s_u^+} F(u^t) \quad \text{and} \quad \max_{s_v^+ < t \leq s_v^-} F(v^t) \leq \max_{s_u^+ < t \leq s_u^-} F(u^t).$$

In view of (3.5), the inequality (3.4) holds. Now if $u_0 \in \Lambda^+(c)$ is such that $F(u_0) = \inf_{u \in \Lambda^+(c)} F(u)$ we see that v , the Schwarz rearrangement of $|u_0|$, belongs to $\Lambda_r^+(c)$. Indeed, if either $A(v) < A(u_0)$ or $B(v) > B(u_0)$ then $F(v^t) < F(u_0^t)$. Hence, in view of the above arguments, we get

$$\inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in S(c)} \min_{0 < t \leq s_u^+} F(u^t) \leq \min_{0 < t \leq s_v^+} F(v^t) < \min_{0 < t \leq s_u^+} F(u_0^t) = \inf_{u \in \Lambda^+(c)} F(u)$$

a contradiction. Thus $A(v) = A(u_0)$, $B(v) = B(u_0)$ and $C(v) = C(u_0)$ from which we deduce that $v \in \Lambda_r^+(c)$ and $F(v) = F(u_0)$. The case of $u_0 \in \Lambda^-(c)$ such that $F(u_0) = \inf_{u \in \Lambda^-(c)} F(u)$ is treated similarly. This ends the proof of the lemma. □

Recalling that $\gamma^+(c)$ and $\gamma^-(c)$ are defined in (1.6) we have

Lemma 3.7. *For any $c \in (0, c_1)$, there exists a bounded Palais-Smale sequence $(u_n^+) \subset \Lambda_r^+(c)$ for F restricted to $S(c)$ at level $\gamma^+(c)$ and a bounded Palais-Smale sequence $(u_n^-) \subset \Lambda_r^-(c)$ for F restricted to $S(c)$ at level $\gamma^-(c)$.*

In order to prove [Lemma 3.7](#) we define the functions

$$\begin{aligned} I^+ : S(c) &\rightarrow \mathbb{R}, & I^+(u) &= F(u^{s_u^+}), \\ I^- : S(c) &\rightarrow \mathbb{R}, & I^-(u) &= F(u^{s_u^-}). \end{aligned}$$

Note that since the maps $u \mapsto s_u^+$ and $u \mapsto s_u^-$ are of class \mathbb{C}^1 , see [Lemma 3.4](#), the functionals I^+ and I^- are of class \mathbb{C}^1 .

Lemma 3.8. *For any $c \in (0, c_1)$, we have that $dI^+(u)[\psi] = dF(u^{s_u^+})[\psi^{s_u^+}]$ and $dI^-(u)[\psi] = dF(u^{s_u^-})[\psi^{s_u^-}]$ for any $u \in S(c)$, $\psi \in T_u S(c)$.*

Proof. We first give the proof for I^+ . Let $\psi \in T_u S(c)$, then $\psi = h'(0)$ where $h : (-\epsilon, \epsilon) \mapsto S(c)$ is a \mathbb{C}^1 -curve with $h(0) = u$. We consider the incremental quotient

$$(3.7) \quad \frac{I^+(h(t)) - I^+(h(0))}{t} = \frac{F(h(t)^{s_t}) - F(h(0)^{s_0})}{t},$$

where $s_t := s_{h(t)}^+$, and hence $s_0 = s_u^+$. Recalling from [Lemma 3.4](#) that s_0 is a strict local minimum of $s \mapsto F(u^s)$ and $u \mapsto s_0$ is continuous, we get

$$\begin{aligned} &F(h(t)^{s_t}) - F(h(0)^{s_0}) \geq F(h(t)^{s_t}) - F(h(0)^{s_t}) \\ &= \frac{s_t^2}{2} \left[A(h(t)) - A(h(0)) \right] - \frac{\gamma s_t}{4} \left[B(h(t)) - B(h(0)) \right] - \frac{as_t^\sigma}{p} \left[C(h(t)) - C(h(0)) \right] \\ &= s_t^2 \int_{\mathbb{R}^3} \nabla h(\tau_1 t) \cdot \nabla h'(\tau_1 t) t dx - \gamma s_t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|h(\tau_2 t)(x)|^2 h(\tau_2 t)(y) h'(\tau_2 t)(y)}{|x-y|} dx dy - as_t^\sigma \int_{\mathbb{R}^3} |h(\tau_3 t)|^{p-2} h(\tau_3 t) h'(\tau_3 t) dx, \end{aligned}$$

for some $\tau_1, \tau_2, \tau_3 \in (0, 1)$. Analogously

$$\begin{aligned} &F(h(t)^{s_t}) - F(h(0)^{s_0}) \leq F(h(t)^{s_0}) - F(h(0)^{s_0}) \\ &= s_0^2 \int_{\mathbb{R}^3} \nabla h(\tau_4 t) \cdot \nabla h'(\tau_4 t) t dx - \gamma s_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|h(\tau_5 t)(x)|^2 h(\tau_5 t)(y) h'(\tau_5 t)(y)}{|x-y|} dx dy - as_0^\sigma \int_{\mathbb{R}^3} |h(\tau_6 t)|^{p-2} h(\tau_6 t) h'(\tau_6 t) dx, \end{aligned}$$

for some $\tau_4, \tau_5, \tau_6 \in (0, 1)$. Now, from (3.7) we deduce that

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{I^+(h(t)) - I^+(h(0))}{t} &= (s_u^+)^2 \int_{\mathbb{R}^3} \nabla u \nabla \psi dx - \gamma (s_u^+) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 u(y) \psi(y)}{|x-y|} dx dy - a (s_u^+)^\sigma \int_{\mathbb{R}^3} |u|^{p-2} u \psi dx \\ &= \int_{\mathbb{R}^3} \nabla u^{s_u^+} \nabla \psi^{s_u^+} dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u^{s_u^+}(x)|^2 u^{s_u^+}(y) \psi^{s_u^+}(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u^{s_u^+}|^{p-2} u^{s_u^+} \psi^{s_u^+} dx \\ &= dF(u^{s_u^+})[\psi^{s_u^+}], \end{aligned}$$

for any $u \in S(c)$, $\psi \in T_u S(c)$. The proof for I^- is similar. \square

Let \mathcal{G} be the set of all singletons belonging to $S_r(c)$. It is clearly a homotopy stable family of compact subsets of $S_r(c)$ with closed boundary (an empty boundary actually) in the sense of [[23](#), Definition 3.1]. In view of [Lemma 3.6](#) we have that

$$\begin{aligned} e_{\mathcal{G}}^+ &:= \inf_{A \in \mathcal{G}} \max_{u \in A} I^+(u) = \inf_{u \in S_r(c)} I^+(u) = \inf_{u \in \Lambda_r^+(c)} F(u) = \inf_{u \in \Lambda^+(c)} F(u) = \gamma^+(c). \\ e_{\mathcal{G}}^- &:= \inf_{A \in \mathcal{G}} \max_{u \in A} I^-(u) = \inf_{u \in S_r(c)} I^-(u) = \inf_{u \in \Lambda_r^-(c)} F(u) = \inf_{u \in \Lambda^-(c)} F(u) = \gamma^-(c). \end{aligned}$$

Lemma 3.9. *For any $c \in (0, c_1)$, there exists a Palais-Smale sequence $(u_n^+) \subset \Lambda^+(c)$ for F restricted to $S_r(c)$ at level $e_{\mathcal{G}}^+$ and a Palais-Smale sequence $(u_n^-) \subset \Lambda^-(c)$ for F restricted to $S(c)$ at level $e_{\mathcal{G}}^-$.*

Proof. We first treat the case of $e_{\mathcal{G}}^+$. Let $(D_n) \subset \mathcal{G}$ be such that

$$\max_{u \in D_n} I^+(u) < e_{\mathcal{G}}^+ + \frac{1}{n},$$

and consider the homotopy

$$\eta : [0, 1] \times S(c) \mapsto S(c), \quad \eta(t, u) = u^{1-t+ts_u^+}.$$

From the definition of \mathcal{G} , we have

$$E_n := \eta(\{1\} \times D_n) = \{u^{s_u^+} : u \in D_n\} \in \mathcal{G}.$$

Lemma 3.4 implies that $E_n \subset \Lambda^+(c)$ for all $n \in \mathbb{N}$. Let $v \in E_n$, i.e. $v = u^{s_u^+}$ for some $u \in D_n$, and hence $I^+(v) = I^+(u)$. So, we have

$$\max_{v \in E_n} I^+(v) = \max_{u \in D_n} I^+(u).$$

Therefore, E_n is another minimizing sequence for $e_{\mathcal{G}}^+$. Applying [23, Theorem 3.2], in the particular case where the boundary $B = \emptyset$, there exists a Palais-Smale sequence (y_n) for I^+ on $S(c)$ at level $e_{\mathcal{G}}^+$ such that

$$(3.8) \quad \text{dist}_{H^1(\mathbb{R}^3)}(y_n, E_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now writing $s_n := s_{y_n}^+$ we set $u_n^+ := y_n^{s_n} \in \Lambda^+(c)$. We claim that there exists a constant $C > 0$ such that

$$(3.9) \quad \frac{1}{C} \leq s_n^2 \leq C$$

for $n \in \mathbb{N}$ large enough. Indeed, notice first that

$$s_n^2 = \frac{A(u_n^+)}{A(y_n)}.$$

By $F(u_n^+) = I^+(y_n) \rightarrow e_{\mathcal{G}}^+ = \gamma^+(c) < 0$ we deduce from (3.1) that there exists $M > 0$ such that

$$(3.10) \quad \frac{1}{M} \leq A(u_n^+) \leq M.$$

On the other hand, since $E_n \in \Lambda^+(c)$ is a minimizing sequence for $e_{\mathcal{G}}^+$ and F is $H^1(\mathbb{R}^3)$ coercive on $\Lambda^+(c)$, we obtain that E_n is uniformly bounded in $H^1(\mathbb{R}^3)$ and thus from (3.8), it implies that $\sup_n A(y_n) < \infty$. Also, since E_n is compact for every $n \in \mathbb{N}$, there exist a $v_n \in E_n$ such that $\|v_n - y_n\|_{H^1(\mathbb{R}^3)} \rightarrow 0$ as $n \rightarrow \infty$ due to (3.8). Using **Lemma 3.1** again, we have, for a $\delta > 0$,

$$A(y_n) \geq A(v_n) - A(v_n - y_n) \geq \frac{\delta}{2}.$$

This proves the claim (3.9). From (2.9), and by **Lemma 2.4**, **Lemma 3.8**, we have

$$\|dF|_{S(c)}(u_n^+)\|_* = \sup_{\|\psi\| \leq 1, \psi \in T_u S(c)} |dF(u_n^+)[\psi]| = \sup_{\|\psi\| \leq 1, \psi \in T_u S(c)} \left| dF(u_n^+) \left[\left(\psi^{\frac{1}{s_n}} \right)^{s_n} \right] \right| = \sup_{\|\psi\| \leq 1, \psi \in T_u S(c)} \left| dI^+(y_n) \left[\psi^{\frac{1}{s_n}} \right] \right|.$$

This implies that $(u_n^+) \subset \Lambda^+(c)$ is a Palais-Smale sequence for F restricted to $S(c)$ at level $e_{\mathcal{G}}^+$ since (y_n) is a Palais-Smale sequence for I^+ at level $e_{\mathcal{G}}^+$ and $\left\| \psi^{\frac{1}{s_n}} \right\| \leq C_1 \|\psi\| \leq C_1$ due to (3.9). For the case of $e_{\mathcal{G}}^-$ the proof is identical except that we use **Lemma 3.5(ii)** along with (3.1) to conclude that there exists a $M > 0$ such that (3.10) holds for $A(u_n^-)$ replacing $A(u_n^+)$. \square

Proof of Lemma 3.7. Applying **Lemma 3.9**, we deduce that there exists a Palais-Smale sequence $(u_n^+) \subset \Lambda_r^+(c)$ for F restricted to $S(c)$ at level $e_{\mathcal{G}}^+ = \gamma^+(c)$ and a Palais-Smale sequence $(u_n^-) \subset \Lambda_r^-(c)$ for F restricted to $S(c)$ at level $e_{\mathcal{G}}^- = \gamma^-(c)$. In both cases the boundedness of these sequences follows from **Lemma 3.1**. \square

3.2. The compactness of our Palais-Smale sequences in the Sobolev subcritical case $p \in (\frac{10}{3}, 6)$.

Lemma 3.10. *Let $p \in (\frac{10}{3}, 6)$. For any $c \in (0, c_1)$, if either $(u_n) \subset \Lambda^+(c)$ is a minimizing sequence for $\gamma^+(c)$ or $(u_n) \subset \Lambda^-(c)$ is a minimizing sequence for $\gamma^-(c)$, it weakly converges, up to translation, to a non-trivial limit.*

Proof. Since F restricted to $\Lambda(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see [Lemma 3.1](#)), (u_n) is bounded. Hence, up to translation, $u_n \rightharpoonup u_c$ weakly in $H^1(\mathbb{R}^3)$. Let us argue by contradiction assuming that $u_c = 0$, this means that (u_n) is vanishing. By [[37](#), Lemma I.1], we have, for $2 < q < 6$,

$$\|u_n\|_{L^q(\mathbb{R}^3)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This implies that

$$C(u_n) \rightarrow 0, \quad \text{and} \quad B(u_n) \leq K_1 \|u_n\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \rightarrow 0,$$

due to (2.2). Since $(u_n) \subset \Lambda(c)$, we have $Q(u_n) = 0$, and hence

$$(3.11) \quad A(u_n) = \frac{\gamma}{4} B(u_n) + \frac{a\sigma}{p} C(u_n) \rightarrow 0.$$

If we assume that $(u_n) \subset \Lambda^-(c)$ we recall that by [Lemma 3.5](#), there exists $\alpha > 0$ such that

$$A(u_n) \geq \alpha > 0, \quad \forall n \in \mathbb{N},$$

contradicting (3.11). If we assume that $(u_n) \subset \Lambda^+(c)$ then since

$$F(u_n) = \frac{1}{2} A(u_n) - \frac{\gamma}{4} B(u_n) - \frac{a}{p} C(u_n) \rightarrow 0$$

we reach a contradiction with the fact that

$$F(u_n) \rightarrow \gamma^+(c) = \inf_{u \in \Lambda^+(c)} F(u) < 0.$$

The lemma is proved. \square

Lemma 3.11. *Let $p \in (\frac{10}{3}, 6)$. Assume that a bounded Palais-Smale sequence $(u_n) \subset \Lambda_r(c)$ for F restricted to $S(c)$ is weakly convergent, up to translation, to the nonzero function u_c . Then, up to translation, $u_n \rightarrow u_c \in \Lambda_r(c)$ strongly in $H_r^1(\mathbb{R}^3)$. In particular u_c is a radial solution to (1.2) for some $\lambda_c > 0$ and $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.*

Proof. Since the embedding $H_r^1(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$ is compact for $q \in (2, 6)$, see [[49](#)] and, up to translation, $u_n \rightharpoonup u_c$ weakly in $H_r^1(\mathbb{R}^3)$, we have, up to translation, $u_n \rightarrow u_c$ strongly in $L^q(\mathbb{R}^3)$ for $q \in (2, 6)$ and a.e. in \mathbb{R}^3 .

Since $(u_n) \subset H^1(\mathbb{R}^3)$ is bounded, following [[13](#), Lemma 3], we know that

$$F'_{|S(c)}(u_n) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3) \iff F'(u_n) - \frac{1}{c} \langle F'(u_n), u_n \rangle u_n \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3).$$

Thus, for any $w \in H^1(\mathbb{R}^3)$, we have

$$(3.12) \quad \begin{aligned} o_n(1) &= \left\langle F'(u_n) - \frac{1}{c} \langle F'(u_n), u_n \rangle u_n, w \right\rangle \\ &= \int_{\mathbb{R}^3} \nabla u_n \nabla w dx + \lambda_n \int_{\mathbb{R}^3} u_n w dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 u_n(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_n|^{p-2} u_n w dx, \end{aligned}$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$ and

$$\lambda_n = \frac{-1}{c} [A(u_n) - \gamma B(u_n) - a C(u_n)] = \frac{1}{c} \left[\frac{3\gamma}{4} B(u_n) + a \left(1 - \frac{\sigma}{p} \right) C(u_n) \right],$$

due to $Q(u_n) = 0$. Since $u_n \in H_r^1(\mathbb{R}^3)$, we have $C(u_n) \rightarrow C(u_c)$ and $B(u_n) \rightarrow B(u_c)$ (see [Lemma 2.5](#)). Hence, we obtain that

$$\lambda_n \rightarrow \lambda_c = \frac{1}{c} \left[\frac{3\gamma}{4} B(u_c) + a \left(1 - \frac{\sigma}{p} \right) C(u_c) \right].$$

Now, using [[54](#), Lemma 2.2], the equation (3.12) leads to

$$(3.13) \quad \int_{\mathbb{R}^3} \nabla u_c \nabla w dx + \lambda_c \int_{\mathbb{R}^3} u_c w dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c(x)|^2 u_c(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_c|^{p-2} u_c w dx = 0$$

due to the weak convergence in $H_r^1(\mathbb{R}^3)$ and $\lambda_n \rightarrow \lambda_c \in \mathbb{R}$. This implies that (u_c, λ_c) satisfies

$$-\Delta u_c + \lambda_c u_c - \gamma(|x|^{-1} * |u_c|^2) u_c - a|u_c|^{p-2} u_c = 0 \quad \text{in } H^{-1}(\mathbb{R}^3).$$

By the assumption $u_c \neq 0$ and by [Lemma 2.2](#), we obtain that $Q(u_c) = 0$ and $\lambda_c > 0$.

Now choosing $w = u_n$ in [\(3.12\)](#) and choosing $w = u_c$ in [\(3.13\)](#), we obtain that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \lambda_n \int_{\mathbb{R}^3} |u_n|^2 dx - \gamma B(u_n) - aC(u_n) \rightarrow \int_{\mathbb{R}^3} |\nabla u_c|^2 dx + \lambda_c \int_{\mathbb{R}^3} |u_c|^2 dx - \gamma B(u_c) - aC(u_c).$$

We can deduce from $B(u_n) \rightarrow B(u_c)$, $C(u_n) \rightarrow C(u_c)$ and $\lambda_n \rightarrow \lambda_c$ that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \lambda_c \int_{\mathbb{R}^3} |u_n|^2 dx \rightarrow \int_{\mathbb{R}^3} |\nabla u_c|^2 dx + \lambda_c \int_{\mathbb{R}^3} |u_c|^2 dx.$$

Since $\lambda_c > 0$, we conclude that $u_n \rightarrow u_c$ strongly in $H_r^1(\mathbb{R}^3)$. The lemma is proved. \square

Proof of [Theorem 1.2](#) in the subcritical case $p \in (\frac{10}{3}, 6)$. We give the proof for $\gamma^+(c)$, the treatment for $\gamma^-(c)$ is identically. For any $c \in (0, c_1)$, by [Lemma 3.7](#), there exists a bounded Palais-Smale sequence $(u_n^+) \subset \Lambda_r^+(c)$ for F restricted to $S(c)$ at level $\gamma^+(c)$. From [Lemma 3.10](#) and [Lemma 3.11](#), we deduce that $u_n^+ \rightarrow u_c^+ \in \Lambda_r(c)$ strongly in $H_r^1(\mathbb{R}^3)$ and that there exists $\lambda_c^+ > 0$ such that (u_c^+, λ_c^+) is a solution to [\(1.2\)](#). Since $\Lambda^0(c) = \emptyset$ (see [Lemma 3.3](#)), we conclude that $u_c^+ \in \Lambda_r^+(c)$. From [Lemma 3.6](#) we can thus assume that u_c^+ is a Schwarz symmetric function. Hence, u_c^+ is non-negative. At this point, we can deduce from [Lemma 2.3](#) that u_c^+ is a bounded continuous positive function. \square

3.3. The compactness of our Palais-Smale sequences in the Sobolev critical case $p = 6$.

Our next lemma is directly inspired from [\[48, Proposition 3.1\]](#).

Lemma 3.12. *Let $c \in (0, c_1)$ and $(u_n) \subset \Lambda_r^+(c)$ or $(u_n) \subset \Lambda_r^-(c)$ be a Palais-Smale sequence for F restricted to $S(c)$ at level $m \in \mathbb{R}$ which is weakly convergent, up to subsequence, to the function u_c . If $(u_n) \subset \Lambda_r^+(c)$ we assume that $m \neq 0$ and if $(u_n) \subset \Lambda_r^-(c)$ we assume that*

$$(3.14) \quad m < \frac{1}{3\sqrt{aK_{GN}}}.$$

Then $u_c \neq 0$ and we have the following alternative:

(i) either

$$(3.15) \quad F(u_c) \leq m - \frac{1}{3\sqrt{aK_{GN}}},$$

(ii) or

$$(3.16) \quad u_n \rightarrow u_c \quad \text{strongly in } H_r^1(\mathbb{R}^3).$$

Proof. Since $u_n \rightharpoonup u_c$ weakly in $H_r^1(\mathbb{R}^3)$, we have, up to subsequence, $u_n \rightarrow u_c$ strongly in $L^q(\mathbb{R}^3)$ for $q \in (2, 6)$ and a.e in \mathbb{R}^3 .

Let us first show that $u_c \neq 0$. We argue by contradiction assuming that $u_c = 0$, this means that (u_n) is vanishing. By [\[37, Lemma I.1\]](#), we have, for $2 < q < 6$,

$$\|u_n\|_{L^q(\mathbb{R}^3)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This implies from [\(2.2\)](#) that

$$B(u_n) \leq K_1 \|u_n\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \rightarrow 0.$$

Since $(u_n) \subset \Lambda(c)$, we have

$$A(u_n) = aC(u_n) + o_n(1).$$

Passing to the limit as $n \rightarrow \infty$, up to subsequence we infer that

$$\lim_{n \rightarrow \infty} A(u_n) = \lim_{n \rightarrow \infty} aC(u_n) := \ell \geq 0.$$

Using [Lemma 2.1\(ii\)](#), we have

$$\ell = \lim_{n \rightarrow \infty} aC(u_n) \leq \lim_{n \rightarrow \infty} aK_{GN} [A(u_n)]^3 = aK_{GN} \ell^3.$$

Therefore, either $\ell = 0$ or $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. If $(u_n) \subset \Lambda^+(c)$, we have $A(u_n) > 5aC(u_n)$, and then $\ell = 0$. This implies that $F(u_n) \rightarrow 0$ and this contradicts the assumption that $m \neq 0$. Also, if $(u_n) \subset \Lambda^-(c)$, [Lemma 3.5\(ii\)](#) ensure that $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. Hence, we have

$$m + o_n(1) = F(u_n) = \frac{\sigma - 2}{2\sigma}A(u_n) - \frac{\gamma(\sigma - 1)}{4\sigma}B(u_n) = \frac{1}{3}A(u_n) + o_n(1) = \frac{1}{3}\ell + o_n(1) \geq \frac{1}{3\sqrt{aK_{GN}}} + o_n(1),$$

which contradicts our assumption [\(3.14\)](#). Thus, we have that $u_c \neq 0$.

Now, since $(u_n) \subset H^1(\mathbb{R}^3)$ is bounded, following [\[13, Lemma 3\]](#), we know that

$$F'_{|S(c)}(u_n) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3) \iff F'(u_n) - \frac{1}{c}\langle F'(u_n), u_n \rangle u_n \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3).$$

Thus, for any $w \in H^1(\mathbb{R}^3)$, we have

$$(3.17) \quad \begin{aligned} o_n(1) &= \left\langle F'(u_n) - \frac{1}{c}\langle F'(u_n), u_n \rangle u_n, w \right\rangle \\ &= \int_{\mathbb{R}^3} \nabla u_n \nabla w dx + \lambda_n \int_{\mathbb{R}^3} u_n w dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 u_n(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_n|^{p-2} u_n w dx, \end{aligned}$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$ and

$$\lambda_n = \frac{-1}{c} [A(u_n) - \gamma B(u_n) - aC(u_n)] = \frac{3\gamma}{4c} B(u_n),$$

due to $Q(u_n) = 0$. By $B(u_n) \rightarrow B(u_c)$ (see [Lemma 2.5](#)), we obtain that

$$(3.18) \quad \lambda_n \rightarrow \lambda_c = \frac{3\gamma}{4c} B(u_c).$$

Now, using [\[54, Lemma 2.2\]](#), the equation [\(3.17\)](#) leads to

$$(3.19) \quad \int_{\mathbb{R}^3} \nabla u_c \nabla w dx + \lambda_c \int_{\mathbb{R}^3} u_c w dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c(x)|^2 u_c(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_c|^{p-2} u_c w dx = 0$$

due to the weak convergence in $H_r^1(\mathbb{R}^3)$ and $\lambda_n \rightarrow \lambda_c \in \mathbb{R}$. This implies that (u_c, λ_c) satisfies

$$-\Delta u_c + \lambda_c u_c - \gamma(|x|^{-1} * |u_c|^2) u_c - a|u_c|^{p-2} u_c = 0 \quad \text{in } H^{-1}(\mathbb{R}^3).$$

By [Lemma 2.2](#), we obtain that $Q(u_c) = 0$ and $\lambda_c > 0$.

Let $v_n := u_n - u_c \rightarrow 0$ in $H_r^1(\mathbb{R}^3)$. By Brezis-Lieb lemma (see [\[16\]](#)), we obtain that

$$(3.20) \quad A(u_n) = A(u_c) + A(v_n) + o_n(1), \quad C(u_n) = C(u_c) + C(v_n) + o_n(1).$$

By $B(u_n) \rightarrow B(u_c)$ (see [Lemma 2.5](#)) and by $Q(u_n) = 0$, we have

$$A(u_c) + A(v_n) - \frac{\gamma}{4} B(u_c) - a[C(u_c) - C(v_n)] = o_n(1).$$

Taking into account that $Q(u_c) = 0$, we get $A(v_n) = aC(v_n) + o_n(1)$. Passing to the limit as $n \rightarrow \infty$, up to subsequence we infer that

$$\lim_{n \rightarrow \infty} A(v_n) = \lim_{n \rightarrow \infty} aC(v_n) := k \geq 0.$$

Using [Lemma 2.1\(ii\)](#), we have

$$k = \lim_{n \rightarrow \infty} aC(v_n) \leq \lim_{n \rightarrow \infty} aK_{GN}[A(v_n)]^3 = aK_{GN}k^3.$$

Therefore, either $k = 0$ or $k \geq (aK_{GN})^{-\frac{1}{2}}$.

If $k \geq (aK_{GN})^{-\frac{1}{2}}$, then by [\(3.20\)](#) and by $B(u_n) \rightarrow B(u_c)$, we have

$$\begin{aligned} m &= \lim_{n \rightarrow \infty} F(u_n) = \lim_{n \rightarrow \infty} \left[\frac{1}{2}A(u_n) - \frac{\gamma}{4}B(u_n) - \frac{a}{6}C(u_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2}A(u_c) + \frac{1}{2}A(v_n) - \frac{\gamma}{4}B(u_c) - \frac{a}{6}C(u_c) - \frac{a}{6}C(v_n) \right] \\ &= F(u_c) + \frac{1}{3}k \geq F(u_c) + \frac{1}{3}(aK_{GN})^{-\frac{1}{2}}. \end{aligned}$$

This implies that alternative (i) holds.

If instead $k = 0$, then by (3.20), we have $A(u_n) \rightarrow A(u_c)$ and $C(u_n) \rightarrow C(u_c)$. Choosing $w = u_n$ in (3.17) and $w = u_c$ in (3.19), we obtain that

$$A(u_n) + \lambda_n \|u_n\|_{L^2(\mathbb{R}^3)}^2 - \gamma B(u_n) - aC(u_n) \rightarrow A(u_c) + \lambda_c \|u_c\|_{L^2(\mathbb{R}^3)}^2 - \gamma B(u_c) - aC(u_c).$$

This implies that $\|u_n\|_{L^2(\mathbb{R}^3)} \rightarrow \|u_c\|_{L^2(\mathbb{R}^3)}$. Thus, we conclude that $u_n \rightarrow u_c$ strongly in $H_r^1(\mathbb{R}^3)$. \square

Proof of Theorem 1.2 in the critical case $p = 6$ for $\gamma^+(c)$. Since $\gamma^+(c) < 0$, the fact that it is reached is a direct consequence of Lemma 3.7, Lemma 3.12 and of the property, which is established in Lemma 3.21(iii) to come, that the map $c \mapsto \gamma^+(c)$ is non-increasing. The rest of the proof is identical to the one in the case $p \in (\frac{10}{3}, 6)$. \square

In the rest of this subsection, we shall prove Theorem 1.2 in the critical case $p = 6$ for $\gamma^-(c)$.

Lemma 3.13. *Let $c \in (0, c_1)$. If*

$$(3.21) \quad \gamma^-(c) < \gamma^+(c) + \frac{1}{3\sqrt{a}K_{GN}}$$

then there exists a $u_c \in \Lambda_r^-(c)$ with $F(u_c) = \gamma^-(c)$ which is a radial solution to (1.2) for some $\lambda_c > 0$ with $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.

Proof. By Lemma 3.7 there exists a Palais-Smale sequence $(u_n) \subset \Lambda^-(c)$ for F restricted to $S(c)$ at the level $\gamma^-(c)$. If (3.21) holds then necessarily (3.14), with $m = \gamma^-(c)$ holds, and (3.15) cannot hold. We deduce from Lemma 3.12 that $u_n \rightarrow u_c$ strongly in $H^1(\mathbb{R}^N)$ and the conclusions follow. \square

Now we shall show that

Lemma 3.14. *For any $c \in (0, c_1)$, we have that*

$$\gamma^-(c) < \gamma^+(c) + \frac{1}{3\sqrt{a}K_{GN}}.$$

As already indicated our proof is inspired by [52, Lemma 3.1]. Let u_ε be an extremal function for the Sobolev inequality in \mathbb{R}^3 defined by

$$(3.22) \quad u_\varepsilon(x) := \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{N-2}{2}}}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^3.$$

Let $\xi \in C_0^\infty(\mathbb{R}^N)$ be a radial non-increasing cut-off function with $\xi \equiv 1$ in B_1 , $\xi \equiv 0$ in $\mathbb{R}^N \setminus B_2$. Setting $U_\varepsilon(x) = \xi(x)u_\varepsilon(x)$ we recall the following result, see [29, Lemma 7.1].

Lemma 3.15. *Denoting ω the area of the unit sphere in \mathbb{R}^3 , we have*

(i)

$$\|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 = \sqrt{\frac{1}{K_{GN}}} + O(\varepsilon) \quad \text{and} \quad \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^3)}^{2^*} = \sqrt{\frac{1}{K_{GN}}} + O(\varepsilon^3).$$

(ii) *For some positive constant $K > 0$,*

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^3)}^q = \begin{cases} K\varepsilon^{3-\frac{q}{2}} + o(\varepsilon^{3-\frac{q}{2}}) & \text{if } q \in (3, 6), \\ \omega\varepsilon^{\frac{3}{2}}|\log \varepsilon| + O(\varepsilon^{\frac{3}{2}}) & \text{if } q = 3, \\ \omega\left(\int_0^2 \frac{\xi^q(r)}{r^{q-2}} dr\right)\varepsilon^{\frac{q}{2}} + o(\varepsilon^{\frac{q}{2}}) & \text{if } q \in [1, 3). \end{cases}$$

In the rest of the subsection we assume that $c \in (0, c_1)$ is arbitrary but fixed. Let u_c^+ be as provided by Theorem 1.2. We recall that $u_c^+ \in \Lambda^+(c)$ satisfies $F(u_c^+) = \gamma^+(c)$ and is a bounded continuous positive Schwarz symmetric function.

Lemma 3.16. *For any $1 \leq p, q < \infty$, it holds that*

$$\int_{\mathbb{R}^3} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx \sim \int_{\mathbb{R}^3} |U_\varepsilon(x)|^q dx.$$

Proof. On one hand, since u_c^+ is bounded, we have that

$$\int_{\mathbb{R}^3} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx \leq \|u_c^+\|_{L^\infty(\mathbb{R}^3)}^p \int_{\mathbb{R}^3} |U_\varepsilon(x)|^q dx.$$

On the other hand, since $u_c^+ > 0$ on \mathbb{R}^3 is continuous and the function U_ε is compactly supported in B_2 , we have that

$$\int_{\mathbb{R}^3} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx = \int_{B_2} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx \geq \min_{x \in B_2} |u_c^+(x)|^p \int_{B_2} |U_\varepsilon(x)|^q dx = \min_{x \in B_2} |u_c^+(x)|^p \int_{\mathbb{R}^3} |U_\varepsilon(x)|^q dx.$$

The lemma is proved. \square

For any $\varepsilon > 0$ and any $t > 0$, we have

$$(3.23) \quad A(u_c^+ + tU_\varepsilon) = \|\nabla(u_c^+ + tU_\varepsilon)\|_{L^2(\mathbb{R}^3)}^2 = A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + A(tU_\varepsilon)$$

and

$$(3.24) \quad \|u_c^+ + tU_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 = c + 2 \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + \|tU_\varepsilon\|_{L^2(\mathbb{R}^3)}^2.$$

Using that, for all $a, b \geq 0$, $(a+b)^6 \geq a^6 + 6a^5b + 6ab^5 + b^6$, and that both $u_c^+ \in H^1(\mathbb{R}^N)$ and U_ε are non negative, we readily derive that

$$(3.25) \quad C(u_c^+ + tU_\varepsilon) = \|u_c^+ + tU_\varepsilon\|_{L^6(\mathbb{R}^3)}^6 \geq C(u_c^+) + C(tU_\varepsilon) + 6 \int_{\mathbb{R}^3} (u_c^+(x))^5 (tU_\varepsilon(x)) dx + 6 \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))^5 dx.$$

Also, still using that $u_c^+ \in H^1(\mathbb{R}^N)$ and U_ε are non negative, we get by direct calculations that

$$(3.26) \quad \begin{aligned} B(u_c^+ + tU_\varepsilon) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x) + tU_\varepsilon(x)|^2 |u_c^+(y) + tU_\varepsilon(y)|^2}{|x-y|} dx dy \\ &\geq B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy. \end{aligned}$$

Finally, since u_c^+ is solution of the following equation

$$-\Delta u + \lambda_c^+ u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3$$

for a $\lambda_c^+ > 0$, we have that

$$(3.27) \quad \begin{aligned} -\lambda_c^+ \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx &= \int_{\mathbb{R}^3} \nabla u_c^+(x) \nabla(tU_\varepsilon(x)) dx \\ &\quad - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy - a \int_{\mathbb{R}^3} (u_c^+(x))^5 (tU_\varepsilon(x)) dx. \end{aligned}$$

Now, we define for $t > 0$, $w_{\varepsilon,t} = u_c^+ + tU_\varepsilon$ and $\bar{w}_{\varepsilon,t}(x) = \sqrt{\theta} w_{\varepsilon,t}(\theta x)$ with $\theta^2 = \frac{1}{c} \|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2$. The proof of [Lemma 3.14](#) will follow directly from the three lemmas below.

Lemma 3.17. *It holds that*

$$\gamma^-(c) \leq \sup_{t \geq 0} F(\bar{w}_{\varepsilon,t})$$

for $\varepsilon > 0$ sufficiently small.

Lemma 3.18. *There exist a $\varepsilon_0 > 0$ and $0 < t_0 < t_1 < \infty$ such that*

$$F(\bar{w}_{\varepsilon,t}) < \gamma^+(c) + \frac{1}{6\sqrt{a}K_{GN}}$$

for $t \in [t_0, t_1]$ and any $\varepsilon \in (0, \varepsilon_0]$.

Lemma 3.19. *It holds that*

$$\max_{t \in [t_0, t_1]} F(\bar{w}_{\varepsilon,t}) < \gamma^+(c) + \frac{1}{3\sqrt{a}K_{GN}},$$

for any $\varepsilon \in (0, \varepsilon_0]$ where ε_0 and t_0, t_1 are provided by [Lemma 3.18](#).

Proof of Lemma 3.17. By direct calculation we get

$$(3.28) \quad A(\bar{w}_{\varepsilon,t}) = A(w_{\varepsilon,t}), \quad C(\bar{w}_{\varepsilon,t}) = C(w_{\varepsilon,t}),$$

and

$$(3.29) \quad \|\bar{w}_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2 = \theta^{-2} \|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2, \quad B(\bar{w}_{\varepsilon,t}) = \theta^{-3} B(w_{\varepsilon,t}).$$

Since $\theta^2 = \frac{1}{c} \|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2$, we have that $\bar{w}_{\varepsilon,t} \in S(c)$. By Lemma 3.4 there exists $s_{\varepsilon,t}^- > 0$ such that $(\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-} \in \Lambda^-(c)$. We claim that $s_{\varepsilon,t}^- \rightarrow 0$ as $t \rightarrow +\infty$ uniformly for $\varepsilon > 0$ sufficiently small. Indeed, we have

$$A((\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-}) = \frac{\gamma}{4} B((\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-}) + aC((\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-})$$

or equivalently

$$(s_{\varepsilon,t}^-)A(\bar{w}_{\varepsilon,t}) = \frac{\gamma}{4} B(\bar{w}_{\varepsilon,t}) + a(s_{\varepsilon,t}^-)^5 C(\bar{w}_{\varepsilon,t}).$$

This implies that

$$(3.30) \quad A(\bar{w}_{\varepsilon,t}) \geq a(s_{\varepsilon,t}^-)^4 C(\bar{w}_{\varepsilon,t}).$$

In view of (3.23), (3.28), Lemma 3.15(i) and using Hölder's inequality, we have

$$(3.31) \quad \begin{aligned} A(\bar{w}_{\varepsilon,t}) &= A(w_{\varepsilon,t}) = A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + A(tU_\varepsilon) \\ &\leq A(u_c^+) + 2t \|\nabla u_c^+\|_{L^2(\mathbb{R}^3)} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^3)} + t^2 A(U_\varepsilon) \rightarrow A(u_c^+) + 2J\sqrt{A(u_c^+)}t + Jt^2 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

In view of (3.25), (3.28) and Lemma 3.15(i), we also have

$$(3.32) \quad C(\bar{w}_{\varepsilon,t}) = C(w_{\varepsilon,t}) \geq C(tU_\varepsilon) = t^6 C(U_\varepsilon) \rightarrow Lt^6 \text{ as } \varepsilon \rightarrow 0.$$

Combining (3.30)-(3.32), we obtain that, for $\varepsilon > 0$ sufficiently small

$$A(u_c^+) + J\sqrt{A(u_c^+)}t + Jt^2 \geq a(s_{\varepsilon,t}^-)^4 Lt^6,$$

which implies the claim. Since $\bar{w}_{\varepsilon,0} = w_{\varepsilon,0} = u_c^+$ and $u_c^+ \in \Lambda^+(c)$ we obtain, see Lemma 3.4, that $s_{\varepsilon,0}^- > 1$. Still by Lemma 3.4, the map $t \mapsto s_{\varepsilon,t}^-$ is continuous which implies that there exists $t_\varepsilon > 0$ such that $s_{\varepsilon,t_\varepsilon}^- = 1$. It follows that $\bar{w}_{\varepsilon,t_\varepsilon} \in \Lambda^-(c)$ and thus

$$\sup_{t \geq 0} F(\bar{w}_{\varepsilon,t}) \geq F(\bar{w}_{\varepsilon,t_\varepsilon}) \geq \gamma^-(c).$$

The lemma is proved. □

Proof of Lemma 3.18. In view of (3.28) and (3.29), we have that

$$F(\bar{w}_{\varepsilon,t}) = \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4}\theta^{-3}B(w_{\varepsilon,t}) - \frac{a}{6}C(w_{\varepsilon,t}).$$

Hence, by (3.23), (3.25) and (3.26), we get that

$$\begin{aligned} F(\bar{w}_{\varepsilon,t}) &\leq \frac{1}{2} \left[A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + A(tU_\varepsilon) \right] - \frac{\gamma}{4}\theta^{-3}B(u_c^+) - \frac{a}{6} \left[C(u_c^+) + C(tU_\varepsilon) \right] \\ &= F(u_c^+) + \frac{\gamma}{4}(1 - \theta^{-3})B(u_c^+) + \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + \frac{1}{2}A(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \\ &\leq F(u_c^+) + \frac{\gamma}{4}(1 - \theta^{-3})B(u_c^+) + t \|\nabla u_c^+\|_{L^2(\mathbb{R}^3)} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^3)} + \frac{1}{2}t^2 A(U_\varepsilon) - \frac{a}{6}t^6 C(U_\varepsilon) := I(t). \end{aligned}$$

By Lemma 3.15(i), we see that, uniformly for $\varepsilon > 0$ small, $I(t) \rightarrow -\infty$ as $t \rightarrow +\infty$ and $I(t) \rightarrow F(u_c^+)$ as $t \rightarrow 0$ due to $\theta \rightarrow 1$. Hence, there exists $\varepsilon_0 > 0$ and $0 < t_0 < t_1 < \infty$ such that

$$F(\bar{w}_{\varepsilon,t}) < \gamma^+(c) + \frac{1}{6\sqrt{aK_{GN}}}$$

for $t \notin [t_0, t_1]$ and $\varepsilon \in (0, \varepsilon_0]$. The lemma is proved. □

Proof of Lemma 3.19. We assume throughout the proof that $t \in [t_0, t_1]$. By using (3.26), we can write,

$$\begin{aligned}
 (3.33) \quad F(\bar{w}_{\varepsilon,t}) &= \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4}\theta^{-3}B(w_{\varepsilon,t}) - \frac{a}{6}C(w_{\varepsilon,t}) \\
 &\leq \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4}\theta^{-3} \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \right] - \frac{a}{6}C(w_{\varepsilon,t}) \\
 &= I_1 + I_2,
 \end{aligned}$$

where

$$I_1 := \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4} \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \right] - \frac{a}{6}C(w_{\varepsilon,t}),$$

and

$$(3.34) \quad I_2 := \frac{\gamma}{4}(1 - \theta^{-3}) \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \right].$$

In view of (3.23), (3.25) and using crucially (3.27), we have

$$\begin{aligned}
 (3.35) \quad I_1 &\leq \frac{1}{2} \left[A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla (tU_\varepsilon(x)) dx + A(tU_\varepsilon) \right] \\
 &\quad - \frac{\gamma}{4} \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \right] \\
 &\quad - \frac{a}{6} \left[C(u_c^+) + C(tU_\varepsilon) + 6 \int_{\mathbb{R}^3} (u_c^+(x))^5 (tU_\varepsilon(x)) dx + 6 \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))^5 dx \right] \\
 &= F(u_c^+) + F(tU_\varepsilon) - \lambda_c^+ \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx - a \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))^5 dx.
 \end{aligned}$$

Now, we shall evaluate I_2 . By (3.24) and Lemma 3.15(ii), we get that

$$\begin{aligned}
 (3.36) \quad \theta^2 &= \frac{\|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2}{c} = 1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + \frac{t^2}{c} \|U_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \\
 &= 1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + \frac{t^2}{c} \left[\omega \left(\int_0^2 \xi(r) dr \right) \varepsilon + O(\varepsilon^2) \right] = 1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + O(\varepsilon).
 \end{aligned}$$

Note that, by Lemma 3.15(ii) and Lemma 3.16,

$$(3.37) \quad \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx \sim \|\nabla U_\varepsilon\|_{L^1(\mathbb{R}^3)} = O(\varepsilon^{\frac{1}{2}}).$$

Observing that the Taylor expansion of $(1+x)^{-\frac{3}{2}}$ around $x=0$ is given by $(1+x)^{-\frac{3}{2}} = 1 - \frac{3}{2}x + O(x^2)$, we get, in view of (3.36) and (3.37), that

$$\begin{aligned}
 (3.38) \quad 1 - \theta^{-3} &= 1 - (\theta^2)^{-\frac{3}{2}} = 1 - \left[1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + O(\varepsilon) \right]^{-\frac{3}{2}} \\
 &= 1 - \left[1 - \frac{3}{c} \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + O(\varepsilon) \right] = \frac{3}{c} \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + O(\varepsilon).
 \end{aligned}$$

Concerning the term $B(tU_\varepsilon)$, in view of (2.2) and Lemma 3.15(ii), we have

$$(3.39) \quad B(tU_\varepsilon) = t^4 B(U_\varepsilon) \leq t^4 K_1 \|U_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 = t^4 K_1 \left(\|U_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)} \right)^{\frac{10}{6}} = t^4 K_1 \left(K_2 \varepsilon^{\frac{6}{5}} + o(\varepsilon^{\frac{6}{5}}) \right)^{\frac{10}{6}} = O(\varepsilon^2).$$

Also, using the Hardy-Littlewood-Sobolev inequality (2.1), Lemma 3.16 and Lemma 3.15(ii) we have

$$(3.40) \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \leq K_2 \|u_c^+\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^2 \|u_c^+ U_\varepsilon\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \leq K_3 \|U_\varepsilon\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} = O(\varepsilon^{\frac{3}{5}}).$$

From (3.39) and (3.40) we deduce that

$$(3.41) \quad B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy = B(u_c^+) + O(\varepsilon^{\frac{3}{5}}).$$

Taking into account, see (3.18), that

$$c\lambda_c^+ = \frac{3\gamma}{4}B(u_c^+)$$

we obtain, combining (3.34), (3.37), (3.38) and (3.41), the following evaluation of I_2

$$(3.42) \quad I_2 \leq \frac{3\gamma}{4c}B(u_c^+) \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))dx + O(\varepsilon) = \lambda_c^+ \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))dx + O(\varepsilon).$$

At this point, in view of (3.33), (3.35) and (3.42) we deduce that

$$(3.43) \quad \begin{aligned} F(\bar{w}_{\varepsilon,t}) &\leq F(u_c^+) + F(tU_\varepsilon) - a \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))^5 dx + O(\varepsilon) \\ &\leq F(u_c^+) + F(tU_\varepsilon) - at_0^5 \int_{\mathbb{R}^3} u_c^+(x)(U_\varepsilon(x))^5 dx + O(\varepsilon). \end{aligned}$$

In view of Lemma 3.15(i), a direct calculation shows that

$$(3.44) \quad \begin{aligned} \max_{t \in [t_0, t_1]} F(tU_\varepsilon) &= \max_{t \in [t_0, t_1]} \left[\frac{1}{2}A(tU_\varepsilon) - \frac{\gamma}{4}B(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \right] \\ &\leq \max_{t \in [t_0, t_1]} \left[\frac{1}{2}A(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \right] \leq \max_{t>0} \left[\frac{1}{2}A(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \right] = \frac{1}{3\sqrt{aK_{GN}}} + O(\varepsilon). \end{aligned}$$

In view of (3.43) and (3.44), by Lemma 3.15(ii) and Lemma 3.16, we conclude by observing that

$$-at_0^5 \int_{\mathbb{R}^3} u_c^+(x)(U_\varepsilon(x))^5 dx \sim -\|U_\varepsilon(x)\|_{L^5(\mathbb{R}^3)}^5 = -K\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}}).$$

□

Proof of Theorem 1.2 in the critical case $p = 6$ for $\gamma^-(c)$. We conclude that $\gamma^-(c)$ is reached by combining Lemma 3.7, Lemma 3.13 and Lemma 3.14. The rest of the proof is identical to the one in the case $p \in (\frac{10}{3}, 6)$. □

3.4. The compactness of any minimizing sequence associated to $\gamma^+(c)$ for $p \in (\frac{10}{3}, 6]$. In this subsection we give the proof of Theorem 1.5. For short we introduce the following notations,

$$(3.45) \quad M := \frac{p}{a\sigma(\sigma-1)K_{GN}}, \quad N := \frac{4(\sigma-2)}{\gamma(\sigma-1)K_H}, \quad k_0 := N^{-2}, \quad \text{and} \quad k_1 := k_0 c_1^3.$$

Note that

$$c_1 = N^{\frac{3p-10}{4(p-3)}} M^{\frac{1}{2(p-3)}}.$$

Lemma 3.20. *Let $p \in (\frac{10}{3}, 6]$ and $c \in (0, c_1)$.*

(i) *If $u \in \Lambda^+(c)$ then we have*

$$(3.46) \quad A(u) < k_0 c^3.$$

(ii) $\Lambda^+(c) \subset V(c)$ and

$$\gamma^+(c) = \inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in V(c)} F(u).$$

(iii) *If u_c is a minimizer for the minimization problem*

$$\inf_{u \in V(c)} F(u)$$

then $u_c \in V(c)$ and $\gamma^+(c)$ is reached.

Proof. i) Since $u \in \Lambda^+(c)$,

$$A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u) \quad \text{and} \quad A(u) > \frac{a\sigma(\sigma-1)}{p}C(u).$$

Using Lemma 2.1(i), we have

$$\frac{\sigma-2}{\sigma-1}A(u) < \frac{\gamma}{4}B(u) \leq \frac{\gamma}{4}K_H\sqrt{A(u)}c^{\frac{3}{2}},$$

which implies that

$$(3.47) \quad A(u) < \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3 = N^{-2}c^3 = k_0c^3 < k_0c_1^3 = k_1.$$

Hence, the point (i) holds.

ii) By (3.47), we obtain that $\Lambda^+(c) \subset V(c)$ and hence

$$\inf_{u \in \Lambda^+(c)} F(u) \geq \inf_{u \in V(c)} F(u).$$

To prove the point (ii), it is sufficient to show that

$$(3.48) \quad \inf_{u \in \Lambda^+(c)} F(u) \leq \inf_{u \in \overline{V(c)}} F(u).$$

Firstly, we claim that $\Lambda^-(c) \cap \overline{V(c)} = \emptyset$. Indeed, let $v \in \Lambda^-(c)$. Taking into account that

$$A(v) < \frac{a\sigma(\sigma-1)}{p}C(v),$$

and using Lemma 2.1(ii), we obtain that

$$A(v) < \frac{a\sigma(\sigma-1)}{p}K_{GN}c^{\frac{6-p}{4}}[A(v)]^{\frac{\sigma}{2}} = M^{-1}c^{\frac{6-p}{4}}[A(v)]^{\frac{\sigma}{2}}.$$

This implies that

$$A(v) > M^{\frac{4}{3p-10}}c^{-\frac{6-p}{3p-10}}.$$

By direct computations, we can check that

$$N^{-2}c_1^3 = M^{\frac{4}{3p-10}}c_1^{-\frac{6-p}{3p-10}},$$

which implies that for all $0 < c < c_1$,

$$(3.49) \quad k_1 = N^{-2}c_1^3 = M^{\frac{4}{3p-10}}c_1^{-\frac{6-p}{3p-10}} < M^{\frac{4}{3p-10}}c^{-\frac{6-p}{3p-10}} < A(v).$$

Therefore, the claim holds. Next, let $u \in S(c)$. Since the mapping $t \mapsto A(u^t)$ is continuous increasing, there exists a unique $t_u^1 > 0$ such that $A(u^{t_u^1}) = k_1$. By Lemma 3.4 and (3.47), (3.49), we have

$$A(u^{s_u^+}) < A(u^{t_u^1}) < A(u^{s_u^-}),$$

which implies that

$$s_u^+ < t_u^1 < s_u^-.$$

Since $g'_u(t) > 0$ for all $t \in (s_u^+, s_u^-)$, we get that $g'_u(t) > 0$ for all $t \in (s_u^+, t_u^1]$ and hence

$$(3.50) \quad F(u^{s_u^+}) = g_u(s_u^+) < g_u(t) = F(u^t) \quad \forall t \in (s_u^+, t_u^1].$$

Since s_u^+ is the unique local minimum point for g_u on $(0, s_u^-)$, we have that $F(u^{s_u^+}) \leq F(u^t)$ for all $t \in (0, t_u^1]$. Therefore, we obtain that

$$F(u^{s_u^+}) = \min\{F(u^t) | 0 < t \leq t_u^1\} = \min\{F(u^t) | t \in \mathbb{R}, A(u^t) \leq k_1\}.$$

In particular, if $u \in \overline{V(c)}$ we have

$$F(u^{s_u^+}) = \min\{F(u^t) | t \in \mathbb{R}, A(u^t) \leq k_1\} = \min\{F(u) | u \in \overline{V(c)}\} \leq F(u).$$

This implies (3.48) and the point (ii) is proved.

(iii) If we assume that $u_c \in \partial V(c)$, namely $A(u_c) = k_1$ and

$$F(u_c) = \min\{F(u) \mid u \in \overline{V(c)}\} = \min\{F(u^t) \mid t \in \mathbb{R}, A(u^t) \leq k_1\},$$

and we have a contradiction with (3.50). Thus, we have $u_c \in V(c)$. Now, since the minimizer u_c lies in the open (with respect to $S(c)$) set $V(c)$, we deduce from Lemma 2.2 that $u_c \in \Lambda(c)$. By $\Lambda^-(c) \cap \overline{V(c)} = \emptyset$ and $\Lambda^0(c) = \emptyset$, we conclude that $u_c \in \Lambda^+(c)$ and thus $\gamma^+(c)$ is reached. \square

Lemma 3.21. *It holds that*

- (i) $\gamma^+(c) < 0$, $\forall c \in (0, c_1)$.
- (ii) $c \in (0, c_1) \mapsto \gamma^+(c)$ is a continuous mapping.
- (iii) Let $c \in (0, c_1)$, for all $\alpha \in (0, c)$, we have $\gamma^+(c) \leq \gamma^+(\alpha) + \gamma^+(c - \alpha)$ and if $\gamma^+(\alpha)$ or $\gamma^+(c - \alpha)$ is reached then the inequality is strict.

Proof. Point (i) follows from Lemma 3.5. To prove (ii), let $c \in (0, c_1)$ be arbitrary and $(c_n) \subset (0, c_1)$ be such that $c_n \rightarrow c$. From the definition of $\gamma^+(c_n)$, for any $\varepsilon > 0$, there exists $u_n \in \Lambda^+(c_n)$ such that

$$(3.51) \quad F(u_n) \leq \gamma^+(c_n) + \varepsilon.$$

By (3.46), we have $A(u_n) < k_0 c_n^3$. We set $y_n := \sqrt{\frac{c}{c_n}} \cdot u_n$. Hence, we have $y_n \in S(c)$ and

$$A(y_n) = \frac{c}{c_n} A(u_n) < \frac{c}{c_n} k_0 c_n^3 = k_0 c_n^2 c < k_0 c_1^3 = k_1.$$

This implies that $y_n \in \overline{V(c)}$. Taking into account that $\frac{c}{c_n} \rightarrow 1$, we have

$$(3.52) \quad \gamma^+(c) \leq F(y_n) = F(u_n) + o_n(1).$$

Combining (3.51) and (3.52), we get

$$\gamma^+(c) \leq \gamma^+(c_n) + \varepsilon + o_n(1).$$

Reversing the argument we obtain similarly that

$$\gamma^+(c_n) \leq \gamma^+(c) + \varepsilon + o_n(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $\gamma^+(c_n) \rightarrow \gamma^+(c)$. The point (ii) follows.

(iii) Note that, fixed $\alpha \in (0, c)$, it is sufficient to prove that the following holds

$$(3.53) \quad \forall \theta \in \left(1, \frac{c}{\alpha}\right] : \gamma^+(\theta\alpha) \leq \theta\gamma^+(\alpha)$$

and that, if $\gamma^+(\alpha)$ is reached, the inequality is strict. Indeed, if (3.53) holds then it follows directly that

$$\gamma^+(c) = \frac{c-\alpha}{c}\gamma^+(c) + \frac{\alpha}{c}\gamma^+(c) = \frac{c-\alpha}{c}\gamma^+\left(\frac{c}{c-\alpha}(c-\alpha)\right) + \frac{\alpha}{c}\gamma^+\left(\frac{c}{\alpha}\alpha\right) \leq \gamma^+(c-\alpha) + \gamma^+(\alpha)$$

with a strict inequality if $\gamma^+(\alpha)$ is reached. To prove that (3.53) holds, note that for any $\varepsilon > 0$ sufficiently small, there exist $u \in \Lambda^+(\alpha)$ such that

$$(3.54) \quad F(u) \leq \gamma^+(\alpha) + \varepsilon.$$

By (3.46), we have $A(u) < k_0 \alpha^3$. Consider now $v := \sqrt{\theta}u$, we have

$$\|v\|_{L^2(\mathbb{R}^3)} = \theta\|u\|_{L^2(\mathbb{R}^3)}, \quad A(v) = \theta A(u), \quad B(v) = \theta^2 B(u), \quad C(v) = \theta^3 C(u).$$

Therefore, we obtain that $v \in S(\theta\alpha)$ and

$$A(v) = \theta A(u) < k_0 \theta \alpha^3 < k_0 (\theta\alpha)^3 \leq k_0 c^3 < k_1.$$

Hence, $v \in \overline{V(\theta\alpha)}$ and we can write

$$\begin{aligned} \gamma^+(\theta\alpha) &\leq F(v) = \frac{1}{2}A(v) - \frac{\gamma}{4}B(v) - \frac{a}{p}C(v) = \frac{1}{2}\theta A(u) - \frac{\gamma}{4}\theta^2 B(u) - \frac{a}{p}\theta^3 C(u) \\ &< \frac{1}{2}\theta A(u) - \frac{\gamma}{4}\theta B(u) - \frac{a}{p}\theta C(u) = \theta F(u) \leq \theta(\gamma^+(\alpha) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that $\gamma^+(\theta\alpha) \leq \theta\gamma^+(\alpha)$. If $\gamma^+(\alpha)$ is reached then we can let $\varepsilon = 0$ in (3.54) and thus the strict inequality follows. \square

Lemma 3.22. *Let $(v_n) \subset H^1(\mathbb{R}^3)$ be such that $B(v_n) \rightarrow 0$ and $A(v_n) \leq k_1$. Then there exists a $b > 0$ such that*

$$(3.55) \quad F(v_n) \geq bA(v_n) + o_n(1).$$

Proof. Indeed, using $B(v_n) \rightarrow 0$ and Lemma 2.1(ii), we have

$$F(v_n) = \frac{1}{2}A(v_n) - \frac{a}{p}C(v_n) + o_n(1) \geq \frac{1}{2}A(v_n) - \frac{a}{p}K_{GN}c^{\frac{6-p}{4}}[A(v_n)]^{\frac{\sigma}{2}} + o_n(1) = bA(v_n) + o_n(1),$$

where

$$b := \frac{1}{2} - \limsup_{n \rightarrow \infty} \frac{a}{p}K_{GN}c^{\frac{6-p}{4}}[A(v_n)]^{\frac{\sigma}{2}-1} \geq \frac{1}{2} - \frac{a}{p}K_{GN}c_1^{\frac{6-p}{4}}k_1^{\frac{\sigma}{2}-1} = \frac{1}{2} - \frac{1}{\sigma(\sigma-1)}.$$

Hence, $b > 0$ due to $\sigma > 2$. The lemma is proved. \square

Lemma 3.23. *For any $c \in (0, c_1)$, any minimizing sequence (u_n) for F on $\overline{V(c)}$ is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$. In addition all minimizers lies in $V(c)$. In particular $\gamma^+(c)$ is reached.*

Proof. Since $(u_n) \subset \overline{V(c)}$, it is bounded in $H^1(\mathbb{R}^3)$. Also, from $\gamma^+(c) < 0$ we deduce from Lemma 3.22 that there exists a $\beta_0 > 0$ and a sequence $(y_n) \subset \mathbb{R}^3$ such that

$$\int_{B(y_n, R)} |u_n|^2 dx \geq \beta_0 > 0, \quad \text{for some } R > 0.$$

This implies that

$$u_n(x - y_n) \rightharpoonup u_c \neq 0 \quad \text{in } H^1(\mathbb{R}^3), \quad \text{for some } u_c \in H^1(\mathbb{R}^3).$$

Our aim is to prove that $w_n(x) := u_n(x - y_n) - u_c(x) \rightarrow 0$ in $H^1(\mathbb{R}^3)$. Clearly

$$\|u_n\|_{L^2(\mathbb{R}^3)}^2 = \|u_n(x - y_n)\|_{L^2(\mathbb{R}^3)}^2 = \|u_n(x - y_n) - u_c(x)\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) = \|w_n\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1).$$

Thus, we have

$$(3.56) \quad \|w_n\|_{L^2(\mathbb{R}^3)}^2 = \|u_n\|_{L^2(\mathbb{R}^3)}^2 - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) = c - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1).$$

By the similar argument,

$$(3.57) \quad A(w_n) = A(u_n) - A(u_c) + o_n(1).$$

More generally, taking into account that any term in F fulfills the splitting properties of Brezis-Lieb (see [16] for terms A and C ; see [54, Lemma 2.2] or [11, Proposition 3.1] for term B), we have

$$F(u_n - u_c) + F(u_c) = F(u_n) + o_n(1),$$

and by the translational invariance, we obtain

$$(3.58) \quad F(u_n) = F(u_n(x - y_n)) = F(u_n(x - y_n) - u_c(x)) + F(u_c) + o_n(1) = F(w_n) + F(u_c) + o_n(1).$$

Now, we claim that

$$(3.59) \quad \|w_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In order to prove this, let us denote $\tilde{c} := \|u_c\|_{L^2(\mathbb{R}^3)}^2 > 0$. By (3.56), if we show that $\tilde{c} = c$ then the claim follows.

We assume by contradiction that $\tilde{c} < c$. In view of (3.56) and (3.57), for n large enough, we have $\|w_n\|_{L^2(\mathbb{R}^3)}^2 \leq c$ and $A(w_n) \leq A(u_n) \leq k_1$. Hence, we obtain that $w_n \in \overline{V(\|w_n\|_{L^2(\mathbb{R}^3)}^2)}$ and $F(w_n) \geq \gamma^+(\|w_n\|_{L^2(\mathbb{R}^3)}^2)$. Recording that $F(u_n) \rightarrow \gamma^+(c)$, in view of (3.58), we have

$$\gamma^+(c) = F(w_n) + F(u_c) \geq \gamma^+(\|w_n\|_{L^2(\mathbb{R}^3)}^2) + F(u_c).$$

Since the map $c \mapsto \gamma^+(c)$ is continuous (see Lemma 3.21(ii)) and in view of (3.56), we deduce that

$$(3.60) \quad \gamma^+(c) \geq \gamma^+(c - \tilde{c}) + F(u_c).$$

We also have that $u_c \in \overline{V(\bar{c})}$ by the weak limit. This implies that $F(u_c) \geq \gamma^+(\bar{c})$. If $F(u_c) > \gamma^+(\bar{c})$, then it follows from (3.60) and Lemma 3.21(iii) that

$$\gamma^+(c) > \gamma^+(c - \bar{c}) + \gamma^+(\bar{c}) \geq \gamma^+(c - \bar{c} + \bar{c}) = \gamma^+(c),$$

which is impossible. Hence, we have $F(u_c) = \gamma^+(\bar{c})$, namely u_c is local minimizer on $\overline{V(\bar{c})}$. So, we can using Lemma 3.21(iii) with the strict inequality and we deduce from (3.60) that

$$\gamma^+(c) \geq \gamma^+(c - \bar{c}) + F(u_c) = \gamma^+(c - \bar{c}) + \gamma^+(\bar{c}) > \gamma^+(c - \bar{c} + \bar{c}) = \gamma^+(c),$$

which is impossible. Thus, the claim follows and $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.

Let us now show that $A(w_n) \rightarrow 0$. This will complete the proof of the lemma. In this aim first observe that since (w_n) is a bounded sequence in $H^1(\mathbb{R}^N)$ we have, using Lemma 2.1(i), not only that $\|w_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0$ but also that $B(w_n) \rightarrow 0$. Now we remember that

$$(3.61) \quad F(u_n) = F(u_c) + F(w_n) + o_n(1) \rightarrow \gamma^+(c).$$

Since $u_c \in \overline{V(c)}$ by weak convergence property, we have, by Lemma 3.20(ii), that $F(u_c) \geq \gamma^+(c)$. Thus from (3.61) we deduce, on one hand, that necessarily $F(w_n) \leq o(1)$. On the other hand, since $A(w_n) \leq A(u_n) \leq k_1$, Lemma 3.22 implies that $F(w_n) \geq bA(w_n) + o_n(1)$ for some $b > 0$. Hence, we conclude $A(w_n) \rightarrow 0$ and thus that $u_n \rightarrow u_c \in \overline{V(c)}$ strongly in $H^1(\mathbb{R}^3)$. Finally, by Lemma 3.20(iii), we have $u_c \in V(c)$ and $\gamma^+(c)$ is reached. The lemma is proved. \square

3.5. Asymptotic behavior of the Lagrange multipliers.

Lemma 3.24. *Let $p \in (\frac{10}{3}, 6]$. There exist two constants $K_1 > 0$ and $K_2 > 0$ such that for any $c \in (0, c_1)$, if λ_c^+ is the Lagrange parameter associated to a solution u_c^+ lying at the level $\gamma^+(c)$ then we have*

$$|\gamma^+(c)| \leq K_1 c^3 \quad \text{and} \quad \lambda_c^+ \leq K_2 c^2.$$

Proof. By Lemma 3.20(i), we have

$$A(u_c^+) < N^{-2} c^3 = \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3.$$

Hence, we can deduce from Lemma 2.1(i) that

$$B(u_c^+) \leq K_H \sqrt{A(u_c^+)} c^{\frac{3}{2}} < \frac{\gamma(\sigma-1)K_H^2}{4(\sigma-2)} c^3.$$

Therefore, we have

$$\begin{aligned} |\gamma^+(c)| &= |F(u_c^+)| = \left| \frac{\sigma-2}{2\sigma} A(u_c^+) - \frac{\gamma(\sigma-1)}{4\sigma} B(u_c^+) \right| \leq \frac{\sigma-2}{2\sigma} A(u_c^+) + \frac{\gamma(\sigma-1)}{4\sigma} B(u_c^+) \\ &< \frac{\sigma-2}{2\sigma} \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3 + \frac{\gamma(\sigma-1)}{4\sigma} \frac{\gamma(\sigma-1)K_H^2}{4(\sigma-2)} c^3 \\ &= \frac{3\gamma^2(\sigma-1)^2 K_H^2}{32\sigma(\sigma-2)} c^3 := K_1 c^3. \end{aligned}$$

We deduce from (2.6) that

$$\begin{aligned} 2(3p-6)c\lambda_c^+ &= 2(6-p)A(u_c^+) + (5p-12)\gamma B(u_c^+) \\ &< 2(6-p) \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3 + (5p-12)\gamma \frac{\gamma(\sigma-1)K_H^2}{4(\sigma-2)} c^3. \end{aligned}$$

This implies that there exists a constant $K_2 > 0$ such that $\lambda_c^+ \leq K_2 c^2$. The lemma is proved. \square

Lemma 3.25. *Let $p \in (\frac{10}{3}, 6)$. There exist two constants $K_1 > 0$ and $K_2 > 0$ such that is λ_c^- denotes the Lagrange parameter associated to a solution u_c^- lying at the level $\gamma^-(c)$,*

$$|\gamma^-(c)| > K_1 c^{-\frac{6-p}{3p-10}} \quad \text{and} \quad \lambda_c^- > K_2 c^{-\frac{2p-4}{3p-10}}.$$

Proof. By $u_c^- \in \Lambda^-(c)$, we have

$$A(u_c^-) < \frac{a\sigma(\sigma-1)}{p} C(u_c^-).$$

Using [Lemma 2.1\(ii\)](#), we obtain that

$$A(u_c^-) < \frac{a\sigma(\sigma-1)}{p} K_{GN} c^{\frac{6-p}{4}} [A(u_c^-)]^{\frac{\sigma}{2}},$$

which implies that

$$A(u_c^-) > \left[\frac{p}{a\sigma(\sigma-1)K_{GN}} \right]^{\frac{2}{\sigma-2}} c^{-\frac{6-p}{3p-10}}.$$

We have that

$$|\gamma^-(c)| = |F(u_c^-)| = \left| -\frac{1}{2}A(u_c^-) + \frac{a(\sigma-1)}{p}C(u_c^-) \right| > \frac{\sigma-2}{2\sigma}A(u_c^-) > \frac{\sigma-2}{2\sigma} \left[\frac{p}{a\sigma(\sigma-1)K_{GN}} \right]^{\frac{\sigma-2}{2}} c^{-\frac{6-p}{3p-10}} := K_1 c^{-\frac{6-p}{3p-10}}.$$

We deduce from [\(2.6\)](#) that

$$\begin{aligned} \lambda_c^- &= \frac{1}{c} \frac{1}{2(3p-6)} [2(6-p)A(u_c^-) + (5p-12)\gamma B(u_c^-)] \\ &> \frac{1}{c} \frac{6-p}{3p-6} A(u_c^-) > \frac{1}{c} \frac{6-p}{3p-6} \left[\frac{p}{a\sigma(\sigma-1)K_{GN}} \right]^{\frac{\sigma-2}{2}} c^{-\frac{6-p}{3p-10}} := K_2 c^{-\frac{2p-4}{3p-10}}. \end{aligned}$$

The lemma is proved. \square

Lemma 3.26. *Let $p = 6$. There exists a constant $K_1 > 0$ such that if λ_c^- denote the Lagrange parameter associated to a solution u_c^- lying at the level $\gamma^-(c)$ then we have*

$$\gamma^-(c) \rightarrow \frac{1}{3\sqrt{aK_{GN}}} \quad \text{as } c \rightarrow 0 \quad \text{and} \quad \lambda_c^- \leq K_1 c^{\frac{1}{2}}.$$

Proof. Since $F(u)$ restricted to $\Lambda(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see [Lemma 3.1](#)) we have that $A(u_c^-)$ remain bounded. We deduce from [\(2.6\)](#) and [Lemma 2.1\(i\)](#) that

$$\lambda_c^- = \frac{1}{c} \frac{3\gamma}{4} B(u_c^-) \leq \frac{1}{c} \frac{3\gamma}{4} K_H \sqrt{A(u_c^-)} c^{\frac{3}{2}} := K_1 c^{\frac{1}{2}}.$$

We have that $B(u_c^-) \rightarrow 0$ as $c \rightarrow 0$ due to $B(u_c^-) \leq K_H \sqrt{A(u_c^-)} c^{\frac{3}{2}}$. Since $Q(u_c^-) = 0$, we have

$$A(u_c^-) = aC(u_c^-) + o_c(1),$$

where $o_c(1) \rightarrow 0$ as $c \rightarrow 0$. Passing to the limit as $c \rightarrow 0$, up to subsequence we infer that

$$\lim_{c \rightarrow 0} A(u_c^-) = \lim_{c \rightarrow 0} aC(u_c^-) := \ell \geq 0.$$

Using [Lemma 2.1\(ii\)](#), we have

$$\ell = \lim_{c \rightarrow 0} aC(u_c^-) \leq \lim_{c \rightarrow 0} aK_{GN} [A(u_c^-)]^3 = aK_{GN} \ell^3.$$

Therefore, either $\ell = 0$ or $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. Using [Lemma 3.5\(ii\)](#), we ensure that $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. Hence, we have

$$\gamma^-(c) + o_c(1) = F(u_c^-) = \frac{\sigma-2}{2\sigma} A(u_c^-) - \frac{\gamma(\sigma-1)}{4\sigma} B(u_c^-) = \frac{1}{3} A(u_c^-) + o_c(1) = \frac{1}{3} \ell + o_c(1) \geq \frac{1}{3\sqrt{aK_{GN}}} + o_c(1),$$

which implies that

$$\gamma^-(c) \geq \frac{1}{3\sqrt{aK_{GN}}} \quad \text{as } c \rightarrow 0.$$

Recording [Lemma 3.14](#), the lemma is proved. \square

3.6. The monotonicity of the map $c \mapsto \gamma^-(c)$.

Lemma 3.27. *When $p \in (\frac{10}{3}, 6]$, the function $c \mapsto \gamma^-(c)$ is strictly decreasing on $(0, c_1)$.*

Proof. Let $0 < c_2 < c_3 < c_1$. Since $\gamma^-(c_2)$ is reached, there exists $u \in S(c_2)$ such that $F(u) = \gamma^-(c_2)$. We define $v \in S(c_3)$ by $v(x) = \sqrt{\theta}u(\theta x)$ where $\theta = \sqrt{\frac{c_2}{c_3}} < 1$. By direct calculations we have

$$(3.62) \quad A(v) = A(u), \quad B(v) = \theta^{-3}B(u) \quad \text{and} \quad C(v) = \theta^{\frac{p}{2}-3}C(u).$$

Now observe that, since $\theta < 1$, for all $t > 0$,

$$(3.63) \quad F(v^t) = \frac{1}{2}t^2A(v) - \frac{\gamma}{4}tB(v) - \frac{a}{p}t^\sigma C(v) = \frac{1}{2}t^2A(u) - \frac{\gamma}{4}t\theta^{-3}B(u) - \frac{a}{p}t^\sigma\theta^{\frac{p}{2}-3}C(u) < F(u^t).$$

By (3.46) and (3.49), we have that

$$A(u^{s_u^+}) < k_1 < A(v^{s_v^-}),$$

and thus $s_u^+ < s_v^-$ due to $A(v) = A(u)$. Hence, we can deduce from (3.63) that

$$F(v^{s_v^-}) < \max_{s_u^+ < t} F(u^t) = F(u) = \gamma^-(c_2).$$

This implies that $\gamma^-(c_3) < \gamma^-(c_2)$ and hence, the lemma is proved. \square

4. THE CASE $\gamma > 0$, $a < 0$ AND $p \in (\frac{10}{3}, 6]$.

Throughout this section, we assume that $\gamma > 0$, $a < 0$ and $p \in (\frac{10}{3}, 6]$.

Lemma 4.1. *F restricted to $S(c)$ is coercive on $H^1(\mathbb{R}^3)$ and bounded from below.*

Proof. Let $u \in S(c)$. Using Lemma 2.1(i), we obtain

$$F(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) \geq \frac{1}{2}A(u) - \frac{\gamma}{4}K_H\sqrt{A(u)}c^{\frac{3}{2}} - \frac{a}{p}C(u).$$

Since $\gamma > 0$, $a < 0$, this concludes the proof. \square

In what follows, we collect some basic properties of $m(c)$ defined in (1.3).

Lemma 4.2. *It holds that*

- (i) $m(c) < 0$, $\forall c > 0$.
- (ii) $c \mapsto m(c)$ is a continuous mapping.
- (iii) For any $c_2 > c_1 > 0$, we have $c_1 m(c_2) \leq c_2 m(c_1)$. If $m(c_1)$ is reached then the inequality is strict.
- (iv) For any $c_2, c_1 > 0$, we have $m(c_1 + c_2) \leq m(c_1) + m(c_2)$. If $m(c_1)$ or $m(c_2)$ is reached then the inequality is strict.

Proof. i) For any $u \in S(c)$, we recall that $u^t \in S(c)$ and

$$g_u(t) = F(u^t) = \frac{1}{2}t^2A(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}t^\sigma C(u) \quad \text{and also} \quad g'_u(t) = tA(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}t^{\sigma-1}C(u).$$

We observe that $g_u(t) \rightarrow 0$ and $g'_u(t) \rightarrow -\frac{\gamma}{4}B(u) < 0$ as $t \rightarrow 0$. Therefore, there exists $t_0 > 0$ such that $F(u^{t_0}) = g_u(t_0) < 0$. Thus, we have $m(c) < 0$.

ii) We assume that $c_n \rightarrow c$. From the definition of $m(c_n)$, for any $\varepsilon > 0$, there exists $u_n \in S(c_n)$ such that

$$(4.1) \quad F(u_n) \leq m(c_n) + \varepsilon.$$

We set $y_n := \sqrt{\frac{c}{c_n}} \cdot u_n$. Taking into account that $y_n \in S(c)$ and $\frac{c}{c_n} \rightarrow 1$, we have

$$(4.2) \quad m(c) \leq F(y_n) = F(u_n) + o_n(1).$$

Combining (4.1) and (4.2), we get

$$m(c) \leq m(c_n) + \varepsilon + o_n(1).$$

Reversing the argument we obtain similarly that

$$m(c_n) \leq m(c) + \varepsilon + o_n(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $m(c_n) \rightarrow m(c)$. The point (ii) follows.

(iii) Let $t := \frac{c_2}{c_1} > 1$. For any $\varepsilon > 0$, there exist $u \in S(c_1)$ such that

$$(4.3) \quad F(u) \leq m(c_1) + \varepsilon.$$

Let $v := u(t^{-\frac{1}{3}}x)$. Then we have $\|v\|_{L^2(\mathbb{R}^3)}^2 = t\|u\|_{L^2(\mathbb{R}^3)}^2 = c_2$, hence $v \in S(c_2)$. Moreover, we have

$$A(v) = t^{\frac{1}{3}}A(u), \quad B(v) = t^{\frac{5}{3}}B(u), \quad C(v) = tC(u).$$

Therefore, we have

$$\begin{aligned} m(c_2) &\leq F(v) = \frac{1}{2}A(v) - \frac{\gamma}{4}B(v) - \frac{a}{p}C(v) = \frac{1}{2}t^{\frac{1}{3}}A(u) - \frac{\gamma}{4}t^{\frac{5}{3}}B(u) - \frac{a}{p}tC(u) \\ &< \frac{1}{2}tA(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}tC(u) = t \left(\frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) \right) \\ &= tF(u) \leq t(m(c_1) + \varepsilon) = \frac{c_2}{c_1}m(c_1) + \frac{c_2}{c_1}\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $c_1 m(c_2) \leq c_2 m(c_1)$. If $m(c_1)$ is reached then we can let $\varepsilon = 0$ in (4.3) and thus the strict inequality follows.

(iv) Assume first that $0 < c_1 \leq c_2$. Then, by (iii), we have that

$$m(c_1 + c_2) \leq \frac{c_1 + c_2}{c_2}m(c_2) = m(c_2) + \frac{c_1}{c_2}m(c_2) \leq m(c_2) + \frac{c_1}{c_2} \frac{c_2}{c_1}m(c_1) = m(c_1) + m(c_2).$$

If $m(c_1)$ or $m(c_2)$ is reached, then we can use the strict inequality in (iii) and thus the strict inequality follows. The case $0 < c_2 < c_1$ can be treated reversing the role of c_1 and c_2 . \square

Lemma 4.3. *Let $(u_n) \subset S(c)$ be any minimizing sequence for $m(c)$. Then, there exist a $\beta_0 > 0$ and a sequence $(y_n) \in \mathbb{R}^3$ such that*

$$(4.4) \quad \int_{B(y_n, R)} |u_n|^2 dx \geq \beta_0 > 0, \quad \text{for some } R > 0.$$

Proof. Since F restricted to $S(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see Lemma 4.1), the sequence (u_n) is bounded. Now, we assume that (4.4) does not hold. By [37, Lemma I.1], we have, for $q \in (2, 6)$, $\|u_n\|_{L^q(\mathbb{R}^3)} \rightarrow 0$, as $n \rightarrow \infty$. This implies that

$$B(u_n) \leq K_1 \|u_n\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \rightarrow 0,$$

due to (2.2). Hence, we obtain

$$F(u_n) = \frac{1}{2}A(u_n) - \frac{\gamma}{4}B(u_n) - \frac{a}{p}C(u_n) \rightarrow \frac{1}{2}A(u_n) - \frac{a}{p}C(u_n) \geq 0,$$

due to $a < 0$. This contradicts $F(u_n) \rightarrow m(c) < 0$, see Lemma 4.2(i). \square

Lemma 4.4. *Any minimizing sequence $(u_n) \subset S(c)$ for $m(c)$ is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$.*

Proof. Since F restricted to $S(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see Lemma 4.1), the sequence (u_n) is bounded in $H^1(\mathbb{R}^3)$. We deduce from the weak convergence in $H^1(\mathbb{R}^3)$, the local compactness in $L^2(\mathbb{R}^3)$ and Lemma 4.3 that

$$u_n(x - y_n) \rightharpoonup u_c \neq 0 \quad \text{in } H^1(\mathbb{R}^3).$$

Our aim is to prove that $w_n(x) := u_n(x - y_n) - u_c(x) \rightarrow 0$ in $H^1(\mathbb{R}^3)$. Now, taking into account that F fulfills the following splitting properties of Brezis-Lieb type (see [16] for terms A and C ; see [54, Lemma 2.2] or [11, Proposition 3.1] for term B),

$$F(u_n - u_c) + F(u_c) = F(u_n) + o_n(1),$$

and by the translational invariance, we obtain

$$(4.5) \quad F(u_n) = F(u_n(x - y_n)) = F(u_n(x - y_n) - u_c(x)) + F(u_c) + o_n(1) = F(w_n) + F(u_c) + o_n(1),$$

and

$$\|u_n\|_{L^2(\mathbb{R}^3)}^2 = \|u_n(x - y_n)\|_{L^2(\mathbb{R}^3)}^2 = \|u_n(x - y_n) - u_c(x)\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) = \|w_n\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1).$$

Thus, we have

$$(4.6) \quad \|w_n\|_{L^2(\mathbb{R}^3)}^2 = \|u_n\|_{L^2(\mathbb{R}^3)}^2 - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) = c - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1).$$

We claim that

$$(4.7) \quad \|w_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In order to prove this, let us denote $c_1 := \|u_c\|_{L^2(\mathbb{R}^3)}^2 > 0$. By (4.6), if we show that $c_1 = c$ then the claim follows.

We assume by contradiction that $c_1 < c$. Recording that $F(u_n) \rightarrow m(c)$, in view of (4.5), we have

$$m(c) = F(w_n) + F(u_c) \geq m\left(\|w_n\|_{L^2(\mathbb{R}^3)}^2\right) + F(u_c).$$

Since the map $c \mapsto m(c)$ is continuous (see Lemma 4.2(ii)) and (4.6), we deduce that

$$(4.8) \quad m(c) \geq m(c - c_1) + F(u_c).$$

If $F(u_c) > m(c_1)$, then it follows from Lemma 4.2(iv) that

$$m(c) > m(c - c_1) + m(c_1) \geq m(c - c_1 + c_1) = m(c),$$

which is impossible. Hence, we have $F(u_c) = m(c_1)$, namely u_c is global minimizer with respect to c_1 . So, we can use Lemma 4.2(iv) with the strict inequality and we deduce from (4.8) that

$$m(c) \geq m(c - c_1) + F(u_c) = m(c - c_1) + m(c_1) > m(c - c_1 + c_1) = m(c),$$

which is impossible. Thus, the claim follows and $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.

At this point, since w_n is a bounded sequence in $H^1(\mathbb{R}^3)$ and by Lemma 2.1(i), we have

$$B(w_n) \leq K_H \sqrt{A(w_n)} \|w_n\|_{L^2(\mathbb{R}^3)}^3 \rightarrow 0.$$

Thus, we obtain that

$$(4.9) \quad \liminf_{n \rightarrow \infty} F(w_n) = \liminf_{n \rightarrow \infty} \left[A(w_n) - \frac{a}{p} C(w_n) \right] \geq 0.$$

On the other hand, since $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$, we deduce from (4.5) that

$$F(u_n) = F(w_n) + F(u_c) + o_n(1) \geq F(w_n) + m(c) + o_n(1),$$

and by $F(u_n) \rightarrow m(c)$, we have that

$$(4.10) \quad \limsup_{n \rightarrow \infty} F(w_n) \leq 0.$$

Combining (4.9) and (4.10), we obtain that $F(w_n) \rightarrow 0$. Hence, by (4.9) and by $a < 0$, we have $A(w_n) \rightarrow 0$ and $C(w_n) \rightarrow 0$. Thus, we get $w_n \rightarrow 0$ in $H^1(\mathbb{R}^3)$. The lemma is completed. \square

Proof of Theorem 1.8. The proof follows directly from Lemma 4.4 for the convergence of the minimizing sequence and from Lemma 2.2 for the sign of the Lagrange parameter. \square

Lemma 4.5. *There exist three constants $K_1, K_2, K_3 > 0$ such that if λ_c denote the Lagrange parameter associated to a solution u_c lying at the level $m(c)$ then we have*

$$|m(c)| \leq K_1 c^3 + K_2 c^{2p-3} \quad \text{and} \quad \lambda_c \leq K_3 c^2.$$

Proof. By the fact that $m(c) < 0$ and by using Lemma 2.1(i), we get that

$$0 > m(c) = F(u_c) = \frac{1}{2}A(u_c) - \frac{\gamma}{4}B(u_c) - \frac{a}{p}C(u_c) \geq \frac{1}{2}A(u_c) - \frac{\gamma}{4}K_H \sqrt{A(u_c)}c^{\frac{3}{2}} - \frac{a}{p}C(u_c) \geq \frac{1}{2}A(u_c) - \frac{\gamma}{4}K_H \sqrt{A(u_c)}c^{\frac{3}{2}},$$

due to our assumption $\gamma > 0$ and $a < 0$. This implies that

$$\sqrt{A(u_c)} < \frac{\gamma K_H}{2} c^{\frac{3}{2}}.$$

Therefore, using again [Lemma 2.1](#), we obtain that

$$\begin{aligned} |m(c)| = |F(u_c)| &= \left| \frac{1}{2}A(u_c) - \frac{\gamma}{4}B(u_c) - \frac{a}{p}C(u_c) \right| \leq \frac{1}{2}A(u_c) + \frac{\gamma}{4}B(u_c) - \frac{a}{p}C(u_c) \\ &\leq \frac{1}{2}A(u_c) + \frac{\gamma K_H}{4} \sqrt{A(u_c)} c^{\frac{3}{2}} - \frac{a K_{GN}}{p} [A(u_c)]^{\frac{\sigma}{2}} c^{\frac{6-p}{4}} \\ &\leq \frac{\gamma^2 K_H^2}{8} c^3 - \frac{a K_{GN}}{p} \left[\frac{\gamma K_H}{2} \right]^{\sigma} c^{\frac{3\sigma}{2}} c^{\frac{6-p}{4}} := K_1 c^3 + K_2 c^{2p-3}. \end{aligned}$$

We deduce from [\(2.6\)](#) that

$$\begin{aligned} \lambda_c &= \frac{6-p}{3p-6} \frac{1}{c} A(u_c) + \frac{\gamma(5p-12)}{2(3p-6)} \frac{1}{c} B(u_c) \leq \frac{6-p}{3p-6} \frac{1}{c} A(u_c) + \frac{\gamma(5p-12)K_H}{2(3p-6)} \frac{1}{c} \sqrt{A(u_c)} c^{\frac{3}{2}} \\ &\leq \frac{6-p}{3p-6} \frac{1}{c} \frac{\gamma^2 K_H^2}{4} c^3 + \frac{\gamma(5p-12)K_H}{2(3p-6)} \frac{1}{c} \frac{\gamma K_H}{2} c^{\frac{3}{2}} c^{\frac{3}{2}} := K_3 c^2. \end{aligned}$$

Thus, the lemma is proved. \square

5. THE CASE $\gamma < 0$, $a > 0$ AND $p = 6$.

Throughout this section, we assume that $\gamma < 0$, $a > 0$ and $p = 6$. To prove the non-existence of the positive solution to [\(1.2\)](#), we first recall a Liouville-type result, see [\[2, Theorem 2.1\]](#),

Proposition 5.1. *Assume that $N \geq 3$ and the nonlinearity $f : (0, \infty) \mapsto (0, \infty)$ is continuous and satisfies*

$$\liminf_{s \rightarrow 0} s^{-\frac{N}{N-2}} f(s) > 0.$$

Then the differential inequality $-\Delta u \geq f(u)$ has no positive solution in any exterior domain of \mathbb{R}^N .

Proof of Theorem 1.9. Let $u \in H^1(\mathbb{R}^3)$ be a non-trivial solution to [\(1.2\)](#). By [Lemma 2.2](#), we have $\lambda < 0$ and $Q(u) = 0$. Hence,

$$aC(u) = A(u) - \frac{\gamma}{4}B(u) > A(u)$$

and using [Lemma 2.1\(ii\)](#), we obtain that

$$A(u) < aC(u) \leq aK_{GN}[A(u)]^3.$$

This implies that

$$A(u) > \sqrt{\frac{1}{aK_{GN}}}.$$

Using again $Q(u) = 0$ we have that

$$F(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{6}C(u) = \frac{5a}{6}C(u) - \frac{1}{2}A(u) > \frac{1}{3}A(u) > \frac{1}{3\sqrt{aK_{GN}}},$$

proving point [\(i\)](#). To prove point [\(ii\)](#), we assume by contradiction that there exists a positive solution $u \in H^1(\mathbb{R}^3)$ to [\(1.2\)](#). Then, by point [\(i\)](#), the associated Lagrange multiplier λ is strictly negative. In view of [\(2.8\)](#), there exists $R_0 > 0$ large enough such that

$$(|x|^{-1} * |u|^2)(x) \leq -\frac{\lambda}{2\gamma} \quad \text{for } |x| > R_0.$$

Therefore, we get that

$$-\Delta u(x) = \left(-\lambda + \gamma(|x|^{-1} * |u|^2)(x) + a|u(x)|^4 \right) u(x) \geq \left(-\lambda + \gamma(|x|^{-1} * |u|^2)(x) \right) u(x) \geq -\frac{\lambda}{2} u(x) \quad \text{for } |x| > R_0.$$

By applying [Proposition 5.1](#) with $f(s) = -\frac{\lambda}{2}s$, we obtain a contradiction, and thus point [\(ii\)](#) holds. \square

Remark 5.2. In [48, Theorem 1.2], the author considers the equation

$$(5.1) \quad -\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N,$$

with $N \geq 3$, $2 < q < 2^*$ and $\mu < 0$. If $u \in H^1(\mathbb{R}^N)$ is a non-trivial solution to (5.1) then by [48, Theorem 1.2], the associated Lagrange multiplier λ is positive and following the arguments in [48, Proof of Theorem 1.2], one obtains that

$$-\Delta u \geq \frac{\lambda}{2}u \quad \text{for } |x| > R_1,$$

with $R_1 > 0$ large enough. Hence, by applying Proposition 5.1, we see that (5.1) has no positive solution $u \in H^1(\mathbb{R}^N)$ for all $N \geq 3$, improving slightly the conclusions of [48, Theorem 1.2]. Actually, borrowing an observation from [12], the non-existence results of [48, Theorem 1.2] can be further extended by showing that (5.1) has no non-trivial radial solutions in $H^1(\mathbb{R}^N)$ when $N \geq 3$ and $q > 2 + \frac{2}{N-1}$. Indeed, if $u \in H^1(\mathbb{R}^N)$ is a radial function by [12, Radial Lemma A.II], there exist constants $C > 0$ and $R_2 > 0$ such that

$$|u(x)| \leq C|x|^{-\frac{N-1}{2}} \quad \text{for } |x| > R_2.$$

Setting $V(x) = -\mu|u(x)|^{q-2} - |u(x)|^{2^*-2}$, we obtain that any radial solution $u \in H^1(\mathbb{R}^N)$ satisfies

$$(5.2) \quad -\Delta u(x) + V(x)u(x) = \lambda u(x),$$

where, since $q > 2 + \frac{2}{N-1}$,

$$\lim_{|x| \rightarrow \infty} |x||V(x)| \leq \lim_{|x| \rightarrow \infty} \left[-\mu C|x|^{-\frac{(N-1)(q-2)}{2}+1} + C|x|^{-\frac{(N-1)(2^*-2)}{2}+1} \right] = 0.$$

Then (5.2) has no solution in view of Kato's result [32, page 404], also see [46] which states that Schrödinger operator $H = -\Delta + p(x)$ has no positive eigenvalue with an L^2 -eigenfunction if $p(x) = o(|x|^{-1})$.

Remark 5.3. One may wonder if a non-existence result for radial solutions also holds for (1.2) under the assumptions of Theorem 1.9. The difficulty one faces is that, for any $u \in H^1(\mathbb{R}^N)$, $(|x|^{-1} * |u|^2)(x) \geq C|x|^{-1}$ for $|x| > R$ for some $C, R > 0$ (see [9] or [39, Appendix A.4]). Thus, the result of Kato used in Remark 5.2 cannot be directly applied and the non-existence of radial solutions to (1.2) when $\gamma < 0$, $a > 0$ and $p = 6$ is an open problem.

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LOUIS JEANJEAN

LABORATOIRE DE MATHÉMATIQUES (CNRS UMR 6623), UNIVERSITÉ DE BOURGOGNE FRANCHE-COMTÉ, BESANÇON 25030, FRANCE
Email address: louis.jeanjean@univ-fcomte.fr

THANH TRUNG LE

LABORATOIRE DE MATHÉMATIQUES (CNRS UMR 6623), UNIVERSITÉ DE BOURGOGNE FRANCHE-COMTÉ, BESANÇON 25030, FRANCE
Email address: thanh_trung.le@univ-fcomte.fr