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Partition function for quantum gravity in 4 dimensions as a $1/\mathcal{N}$ expansion

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Quantum gravity is the solution ascribed to rendering the geometric description of classical gravity, in any dimensions, completely consistent with principles of quantum theory. The serendipitous theoretical discovery that black holes are thermodynamical objects that must participate in the second law has led to these purely gravitational objects to be dubbed, ‘the hydrogen atom for quantum gravity’, analogous to the atomic spectrum of hydrogen, effectively used by Neils Bohr and his contemporaries to successfully formulate quantum mechanics in the early 20th century. Here, we employ the temperature and entropy formulae describing Schwarzschild black holes to consider the emergence of Einstein Field Equations from a complex-Hermitian structure, [Ricci tensor $\pm\sqrt{-1}$ Yang-Mills field strength], where the gravitational degrees of freedom are the $SU(N)$ colors with $N \in \mathbb{Z} \geq 0$ and a condensate comprising color pairs, $k = N/2$, appropriately coupled to the Yang-Mills gauge field. The foundations of our approach reveal the complex-Hermitian structure is analogous to Cayley-Dickson algebras, which aids in formulating the appropriate action principle for our formalism. The $SU(N)$ gauge group is broken into an effective $SU(4) \rightarrow SO(4) \leftrightarrow SO(1, 3)$ field theory on the tangent space with two terms: the Einstein-Hilbert action and a Gauss-Bonnet topological term. Moreover, since topologically classifying all $n = 4$ (dimensional) Riemannian manifolds is not a clear-cut endeavor, we only consider the Euclidean path integral as the sum over manifolds with distinct topologies, $h \in \mathbb{Z} \geq 0$ homeomorphic to connected sums of an arbitrary number of $n = 4$ -spheres and h number of $n = 4$ -tori. Consequently, the partition function adopts a reminiscent form of the sum of the vacuum Feynman diagrams for a large $\mathcal{N} = \exp(\beta M/2)$ theory, provided $\mathcal{S} = \beta M/2 = \pi N$ is the Schwarzschild black hole entropy, $\beta = 8\pi GM$ is the inverse temperature, G is gravitational constant, M is the black hole mass and horizon area, $\mathcal{A} = 2G\beta M = 4\pi GN$ is pixelated in units of $4\pi G$. This leads us to conclude that the partition function for quantum gravity is equivalent to the vacuum Feynman diagrams of a yet unidentified large \mathcal{N} theory in $n = 4$ dimensions. Our approach also sheds new light on the asymptotic behavior of dark matter-dominated galaxy rotation curves (the empirical baryonic Tully-Fisher relation) and emergent gravity in condensed matter systems with defects such as layered materials with cationic vacancies as topological defects.

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I. INTRODUCTION

The gravitational field in general relativity has a geometric description that, on one hand avails a myriad of visual cues which aid to build physicists' intuitions on the nature of space-time(Thorne *et al.*, 2000), but on the other hand, notoriously does not lend itself to renormalization.(Doboszewski and Linnemann, 2018; Feynman *et al.*, 2018; Hamber, 2008; Shomer, 2007; Thorne *et al.*, 2000; Zee, 2010) This has led to theoretic descriptions of gravity as an emergent phenomenon garnering some traction(Jacobson, 1995; Kleiner, 1987; Konopka *et al.*, 2008; Lee *et al.*, 2013; Markopoulou and Smolin, 2004; Oh *et al.*, 2018; Padmanabhan, 2010; Swingle and Van Raamsdonk, 2014; Van Raamsdonk, 2010; Verlinde, 2011) over quantum field theoretic approaches since emergence appears to by-pass the need for renormalization.(Burgess, 2004; Linnemann and Visser, 2017) Arguments for a successful quantum theory of gravity range from the need to understand the nature of cosmic inflation(Hartle and Hawking, 1983) and black holes(Hawking, 1976a,b, 2005), to aesthetic considerations in favor of the unification of gravity with gauge symmetries in the Standard Model of particle physics.(Ross, 2003)

Employing a semi-classical approach, Bekenstein and his contemporaries showed that black holes are thermodynamical objects that must participate in the second law.(Bardeen *et al.*, 1973; Bekenstein, 1973; Hawking, 1974, 1975) As a consequence, some proponents of emergent gravity have argued that gravity in general relativity may not be as fundamental as the other gauge theories after all, since Boltzmann's investigations demonstrated that the second law is statistical, as it emerges from underlying microscopic physics.(Jacobson, 1995; Padmanabhan, 2010; Verlinde, 2011) Nonetheless, these considerations have led to black holes being dubbed, 'the hydrogen atom for quantum gravity'(Corda and Feleppa, 2019; 't Hooft, 2016), setting up black hole thermodynamics as an indispensable tool, analogous to the atomic spectrum of hydrogen, effectively used by Neils Bohr and his contemporaries to successfully formulate quantum mechanics in the early 20th century. However, unlike the atomic spectrum of hydrogen that was well-grounded in observations prior to Bohr's insights, most effects of black hole thermodynamics are not only presently beyond experimental reach(Hossenfelder, 2010), but also

are not entirely guaranteed to be more than an analogy.(Dougherty and Callender, 2016)

Nonetheless, consistency with well-tested thermodynamical behavior(Isi *et al.*, 2021), coupled with experiments using analogue gravitational systems(Jacquet *et al.*, 2020; Steinhauer, 2014) offer a robust test for the correctness of the results, *albeit* with major unresolved problems such as the black hole information paradox(Almheiri *et al.*, 2021; Hawking, 1976b) related to the apparent violation of information conservation in such systems(Hawking, 2005), and the nature of the black hole microstates.(Bekenstein, 2008) In particular, according to the no-hair theorem(Thorne *et al.*, 2000), a charged-rotating (Kerr-Newman) black hole is completely characterized by its mass (M), charge (Q) and angular momentum (L), implying that black holes are the 'simplest' objects in the universe, with no other internal degrees of freedom. Moreover, Birkhoff's theorem(Israel, 1967; Jebsen, 2005) guarantees that the simplest such black holes are static, uncharged and non-rotating (Schwarzschild) black holes. While these theorems are consistent with classical general relativity, semi-classical approaches to black hole thermodynamics pioneered by Hawking(Hawking, 1974, 1976a, 1975) introduce a quantum field theory in a fixed space-time background, leading to the serendipitous conclusion that black holes not only are black bodies with a temperature, $T = \kappa/2\pi k_B$ proportional to their surface gravity, κ , and entropy, $\mathcal{S} = \mathcal{A}/4G$ proportional to their surface area, \mathcal{A} , where G is the gravitational constant and k_B is Boltzmann's constant, but also pack the maximum amount of information possible within a given space, often referred to as the Bekenstein bound.(Bekenstein, 2005; Bousso, 1999) The Bekenstein bound, coupled with the no-hair theorem, already implies an apparent contradiction, namely that the 'simplest' objects in the universe simultaneously harbor the most amount of information.

However, the black hole information paradox is a far more insidious problem than the aforementioned apparent contradiction, since black hole entropy and temperature require the black hole to eventually evaporate by Unruh-Hawking radiation.(Hawking, 1974; Unruh, 1977) In particular, the radiation is essentially a black body spectrum in a mixed quantum state. Whence, according to the semi-classical approach, quantum objects in a pure quantum state which fall into a black hole will eventually escape as Unruh-Hawking radiation in a mixed state via particle-pair production at the event horizon.(Giddings, 2012; Mathur, 2009) This appears to violate conservation of information since unitarity in quantum theory precludes pure states evolving into mixed states and *vice versa*. In essence, the black hole evaporation process in the semi-classical approach is not time-reversible(Hawking, 2005), since black hole entropy ap-

pears not to follow, *e.g.* the Page curve.(Almheiri *et al.*, 2019, 2021, 2020; Hashimoto *et al.*, 2020; Page, 1980, 1993) Attempts to conclusively address the black hole information paradox range from proposing black hole remnants(Chamseddine *et al.*, 2019), entangled black holes via an Einstein-Rosen bridge(Maldacena and Susskind, 2013) to introducing mechanisms for encoding black hole information in Unruh-Hawking radiation by circumventing the challenges posed by monogamy of quantum entanglement(Grudka *et al.*, 2018), with no clear consensus.

A related problem is the nature of black hole microstates.(Bekenstein, 2008) In statistical mechanics, entropy is a measure of the number of microstates, \mathcal{N} in a thermodynamical system, namely $\mathcal{S} = k_B \ln \mathcal{N}$. A myriad of approaches conjecture that black hole microstates ought to correspond to the degrees of freedom associated with quantum gravity.(Becker *et al.*, 2006; Bousso, 2002; Einstein *et al.*, 1935; Einstein and Rosen, 1935; Hawking *et al.*, 1999; Maldacena, 1999; Maldacena and Susskind, 2013; Polchinski, 1998a,b; Susskind, 1995; Swingle and Van Raamsdonk, 2014) For instance, it has been conjectured that black hole information is entanglement entropy(Swingle and Van Raamsdonk, 2014), with ideas such as the Einstein-Rosen (ER) bridge = Einstein-Podolski-Rosen (EPR)(Einstein *et al.*, 1935; Einstein and Rosen, 1935; Maldacena and Susskind, 2013) playing a major role in such approaches, with the most successful proposals centered around the holographic principle(Bousso, 2002; Susskind, 1995) and the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence(Hawking *et al.*, 1999; Hubeny *et al.*, 2007; Maldacena, 1999; Nishioka *et al.*, 2009; Ryu and Takayanagi, 2006a,b) and the reliance of extra dimensions in the context of string theory.(Becker *et al.*, 2006; Polchinski, 1998a,b)

Fairly recent developments have highlighted random tensor models/group field theories of rank n as prime candidates for quantum gravity.(Freidel, 2005; Gielen *et al.*, 2013; Gurau, 2011, 2012; Gurau and Rivasseau, 2011; Maldacena, 1999; Oriti, 2006; Thorn, 1994) This follows from the observation that their partition functions can be Taylor expanded using $1/N$ as a small parameter, where N is the size of the group and $\chi_n(\mathcal{M})$ is the order of the expansion, which also corresponds to the Euler characteristic of an emergent *compact* manifold, \mathcal{M} of dimensions n .(Aharony *et al.*, 2000; 't Hooft, 1993) For instance, for random matrix models/group field theories (tensors of rank $n = 2$), the emergent manifold is ($n = 2$ dimensional, with $\chi_2(\mathcal{M}) = 2 - 2h$, where $h \in \mathbb{Z} \geq 0$ is a positive integer counting the sum of holes and handles (genus) in the manifold. Such matrix theories have been argued to be dual to 2 dimensional quantum gravity.(Gross, 1992) Thus, it is conjectured that

the partition function of $n = 4$ dimensional quantum gravity, corresponding to a quantum theory of Einstein's general relativity, is a $1/\mathcal{N}$ expansion for a rank $n = 4$ random tensor model/group field theory.(Gurau, 2011, 2012; Gurau and Rivasseau, 2011)

Motivated by questions pertaining to the nature of the black hole microstates(Bekenstein, 2008), as well as an overall pursuit for a suitable partition function for quantum gravity in $n = 4$ dimensions, in this work, we employ the temperature and entropy formulae describing Schwarzschild black holes in order to consider the emergence of Einstein Field Equations (EFE) from a complex-Hermitian structure, [Ricci tensor $\pm\sqrt{-1}$ Yang-Mills field strength], where the gravitational degrees of freedom are the $SU(N)$ colors with $N \in \mathbb{Z} \geq 0$ and a condensate comprising color pairs, $k = N/2$, appropriately coupled to the Yang-Mills gauge field. The foundations of our approach reveal the complex-Hermitian structure is analogous to Cayley-Dickson algebras(Schafer, 1954), which aids in formulating the appropriate action principle for our formalism. A similar construction, [metric tensor $\pm\sqrt{-1}$ U(1) field strength] was considered by Einstein as an attempt to geometrize Maxwell's theory using a complex-Hermitian metric tensor, *albeit* with limited success.(Einstein, 1945, 1948)

Within the formalism, the complex-Hermitian structure avails the constraints, $\mathbf{D}_\mu \mathbb{K}_\nu^\mu = \Psi(\mathbb{D}_\nu \Psi)^\dagger$ and $\mathbf{D}_\nu \mathbf{D}_\mu \mathbb{K}^{\mu\nu} = 0$, where Ψ plays the role of wave function and $\mathbb{K}_{\mu\nu} = R_{\mu\nu} - i\mathbf{F}_{\mu\nu}$ is a complex-Hermitian tensor constructed using the Ricci tensor, $R_{\mu\nu}$ and a Yang-Mills/ $SU(N)$ field strength, $\mathbf{F}_{\mu\nu}$. The $SU(N)$ gauge group is broken into an effective $SU(4) \rightarrow SO(4) \leftrightarrow SO(1,3)$ field theory on the tangent space resulting in two terms in the effective cation: the Einstein-Hilbert action and a Gauss-Bonnet topological term.(Lovelock, 1971) The Euclidean path integral is considered as the sum over manifolds with distinct topologies, $h \in \mathbb{Z} \geq 0$ topologically equivalent (homeomorphic) to connected sums of any number of $n = 4$ -spheres and h number of $n = 4$ -tori. Consequently, the partition function for quantum gravity in $n = 4$ dimensions takes the form,

$$Z_{\text{QG}}^{\text{E}} = \sum_{\mathcal{M}^{\text{E}} \in h} \exp(\mp I_{\mathcal{M}^{\text{E}}}^{\text{E}}(\lambda)) \mathcal{N}^{\pm\chi_4(\mathcal{M}^{\text{E}})}, \quad (1a)$$

$$I_{\mathcal{M}^{\text{E}}}^{\text{E}}(\lambda) = \frac{\lambda}{\pi} \int_{\mathcal{M}^{\text{E}}} d^4 x^{\text{E}} \sqrt{\det(g_{\mu\nu}^{\text{E}})} R, \quad (1b)$$

where $\mathcal{N} = \exp(\pi N)$ is the number of microstates, $I_{\mathcal{M}^{\text{E}}}^{\text{E}}(\lambda)$ is the Einstein-Hilbert action, with $g_{\mu\nu}^{\text{E}}$ the metric tensor in Euclidean signature, $\lambda = 1/16G = m_{\text{P}}^2/16$, $m_{\text{P}} = 1/\sqrt{G}$ the Planck mass and $\chi_4(\mathcal{M}^{\text{E}})$ the Euler characteristic of a 4 dimensional Riemannian manifold, \mathcal{M}^{E} , related to the 1,3 pseudo-Riemannian space-time manifold in Einstein's general relativity by a Wick rotation, $t = \pm it^{\text{E}}$. This result requires that the horizon area, $\mathcal{A} = 2G\beta M = 4\pi GN$ is pixelated in units of

$4\pi G$.(Bekenstein and Mukhanov, 1998, 1995; Mukhanov, 1986; Vaz and Witten, 1999)

Moreover, since it is not straightforward to topologically classify all $n = 4$ dimensional Riemannian manifolds,(Freedman, 1982) the Euclidean path integral is considered as the sum over manifolds with distinct topologies, $h \in \mathbb{Z} \geq 0$ homeomorphic to connected sums of an arbitrary number of $n = 4$ -spheres and h number of $n = 4$ -tori, thus obtaining the Euler characteristic, $\chi_4 = 2 - 2h$. Consequently, eq. (1a) coincidentally corresponds to the generic form of the $1/\mathcal{N}$ expansion of the free energy of a large \mathcal{N} theory.(Aharony *et al.*, 2000; Gurau, 2012; 't Hooft, 1993) This leads to the conclusion that the quantum gravity partition function in $n = 4$ dimensions is equivalent to the sum of vacuum Feynman diagrams of a yet unidentified large \mathcal{N} theory taking the form,

$$\mathcal{Z} = \exp(Z_{\text{QG}}^{\text{E}}), \quad (2a)$$

$$\mathcal{Z} = \int \mathcal{D}[\varphi] \exp\left(-\frac{\mathcal{N}^\pm}{\lambda^\pm} \text{Tr}(\gamma(\varphi))\right), \quad (2b)$$

where $S(\varphi) = \left(\frac{\mathcal{N}}{\lambda}\right)^\pm \text{Tr}(\gamma(\varphi))$ is the action for unidentified field tensors, φ of rank $n = 4$, transforming appropriately under some unidentified group, and γ is the unidentified function of φ defining the large \mathcal{N} theory. Thus, the vacuum Feynman diagrams pave $n = 4$ (dimensional) manifolds(Gurau, 2012), contrary to random matrix large N theories where φ would be a tensor of rank 2 (matrix), with Feynman diagrams defining $n = 2$ dimensional Riemannian manifolds, \mathcal{A}^{E} with a boundary, with Euler characteristic given by $\chi_2(\mathcal{A}^{\text{E}} \in h) = 2 - 2h$, where $h = g + b/2$ with $g \in \mathbb{Z} \geq 0$ the genus, \mathbb{Z} a positive integer and b the boundary contribution.(Aharony *et al.*, 2000; Gurau, 2012; 't Hooft, 1993)

Finally, we consider the possibility of defining $\chi_4(\mathcal{M}^{\text{E}})$ in terms of $\chi_2(\mathcal{A}^{\text{E}})$, where the matrix large \mathcal{N} gauge theories taking the form of eq. (2), can be identified with quantum gravity in $n = 4$ dimensions. Our approach also sheds new light on the asymptotic behavior of dark matter-dominated galaxy rotation curves (the empirical baryonic Tully-Fisher relation)(Eisenstein and Loeb, 1997; Famaey and McGaugh, 2012; Kanyolo and Masese, 2021; Keeton, 2001; Martel and Shapiro, 2003; McGaugh, 2012; McGaugh *et al.*, 2000; Persic *et al.*, 1996) and emergent gravity in condensed matter systems with defects(Holz, 1988; Kleinert, 1987; Kleinert and Zaanen, 2004; Verçin, 1990; Zaanen *et al.*, 2004) such as layered materials(Kanyolo *et al.*, 2021; Masese *et al.*, 2021b, 2018) with cationic vacancies as topological defects.(Kanyolo and Masese, 2020)

II. NOTATION AND NOMENCLATURE

Throughout this paper, we use Einstein's summation convention(Thorne *et al.*, 2000) for all repeated Greek and Roman indices, and natural units where Planck's constant, speed of light in vacuum and Boltzmann constant, respectively are set to unity ($\hbar = c = k_{\text{B}} = 1$). We also assume a torsion-free $1 + 3$ dimensional space-time manifold(Hehl *et al.*, 1976), where ∇_μ is the metric compatible covariant derivative, $\nabla_\sigma g_{\mu\nu} = 0$ and $g_{\mu\nu} = g_{\nu\mu}$ is the metric tensor in Lorenzian signature. We shall indicate reduced quantities with an overhead bar. For instance $\bar{\beta} = \beta/2\pi$ is the reduced inverse temperature and $\bar{m}_{\text{P}} = m_{\text{P}}/\sqrt{8\pi}$ is the reduced Planck mass.

Moreover, in the tetrad formalism, $(\omega_\mu)_{\bar{b}}^{\bar{a}} = -(\omega_\mu)_{\bar{b}}^{\bar{a}} = e_{\bar{a}}^{\alpha} \nabla_\mu e_{\bar{b}}^{\alpha} = e_{\bar{a}}^{\alpha} \partial_\mu e_{\bar{b}}^{\alpha} + e_{\bar{a}}^{\alpha} \Gamma_{\mu\beta}^{\alpha} e_{\bar{b}}^{\beta}$ is the spin connection, $\Gamma_{\mu\nu}^{\alpha} = \Gamma_{\nu\mu}^{\alpha} = \frac{1}{2} g^{\alpha\beta} (\partial g_{\mu\beta} / \partial x^\nu + \partial g_{\beta\nu} / \partial x^\mu - \partial g_{\mu\nu} / \partial x^\beta)$ are the torsion-free Christoffel symbols/affine connection, $e_{\bar{a}}^{\alpha}$ are the tetrad fields, γ_μ are the gamma matrices in curved space-time satisfying $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$, $\bar{\psi} = (\psi)^{*T} (\gamma^0)^{-1} = \psi^\dagger (\gamma^0)^{-1}$ is the Dirac adjoint spinor, $\gamma^\mu = e_{\bar{a}}^{\mu} \gamma^{\bar{a}}$ with $\gamma^{\bar{a}}$ the Dirac matrices in the tangent Minkowski space-time manifold satisfying $\gamma_{\bar{a}} \gamma_{\bar{b}} + \gamma_{\bar{b}} \gamma_{\bar{a}} = 2\eta_{\bar{a}\bar{b}}$, and $e_{\bar{a}}^{\alpha}, e_{\bar{a}}^{\mu}$ are tetrad fields satisfying $e_{\bar{a}}^{\alpha} e_{\bar{a}\nu}^{\alpha} = g_{\mu\nu}$ and $e_{\bar{a}}^{\mu} e_{\bar{b}\mu}^{\nu} = \eta_{\bar{a}\bar{b}}$ with $\eta_{\bar{a}\bar{b}}$ the Minkowski metric tensor *i.e.* $\text{diag}(\eta_{\bar{a}\bar{b}}) = (1, -1, -1, -1)$.(De Felice and Clarke, 1992)

The overhead bar in $\bar{a}, \bar{b}, \bar{c}$ is used to distinguish the Minkowski tangent manifold indices from the curved space-time indices and group field theory indices, *e.g.* in the tetrad fields, $e_{\bar{a}}^0 \neq e_{\bar{a}}^0$ and $(\omega_\mu)_{\bar{b}}^{\bar{a}} = \omega_{\mu\alpha} (\lambda_a)_{\bar{b}}^{\bar{a}}$, where $(\lambda_a)_{\bar{b}}^{\bar{a}} = -(\lambda_a)_{\bar{b}}^{\bar{a}}$ are $n(n-1)/2 = 6$ (with $n = 4$) anti-symmetric matrices corresponding to the generators of rotation(Başkal *et al.*, 2021) in the tangent Euclidean space-time manifold and related to generators of $\text{SO}(1, 3)$ after Wick rotation.

A. Group scalars and vectors (tensors)

Since we will be interested in Yang-Mills theory, where $t_a \equiv \vec{\mathbf{t}}$ are the $\text{SU}(N)$ Hermitian matrices, we shall adopt the following notation for objects which transform as scalars and vectors respectively under $\text{SU}(N)$,

$$T_{0\alpha\cdots\beta}^{\gamma\cdots\delta} \equiv T_{\alpha\cdots\beta}^{\gamma\cdots\delta} I_N, \quad (3a)$$

$$T_{\alpha\cdots\beta}^{\gamma\cdots\delta} \equiv \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta}, \quad (3b)$$

$$f_{abc} A_{a\mu} T_{b\alpha\cdots\beta}^{\gamma\cdots\delta} \equiv \vec{\mathbf{A}}_\mu \times \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta}, \quad (3c)$$

where I_N is the $N \times N$ identity matrix and,

$$T_{k\alpha\cdots\beta}^{\gamma\cdots\delta} \equiv \mathbb{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} = T_{\alpha\cdots\beta}^{\gamma\cdots\delta} I_N + i \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta}, \quad (3d)$$

is the most general object of any space-time tensor that transforms as a scalar ($k = 0$) or a vector ($k = a$) under $\text{SU}(N)$ with the boldface indicating the vectors corre-

sponding to,

$$\mathbf{A}_\mu = \vec{\mathbf{A}}_\mu \cdot \vec{\mathbf{t}} \equiv A_{a\mu} t_a, \quad (3e)$$

$$\mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} \equiv \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta} \cdot \vec{\mathbf{t}}, \quad (3f)$$

in order for I_N and $i\vec{\mathbf{t}}$ in eq. (3d) to be interpreted as $N \times N$ basis matrices. This also implies that $T_{\alpha\cdots\beta}^{\gamma\cdots\delta}$ and $\mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta}$ must have the same rank in space-time indices.

Since the identity matrix is already implied for the scalars, we shall refrain from explicitly writing it unless for clarity. In this notation, the Yang-Mills field strength can be written as,

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - i[\mathbf{A}_\mu, \mathbf{A}_\nu] \\ &= (\partial_\mu \vec{\mathbf{A}}_\nu - \partial_\nu \vec{\mathbf{A}}_\mu + \vec{\mathbf{A}}_\mu \times \vec{\mathbf{A}}_\nu) \cdot \vec{\mathbf{t}}, \end{aligned} \quad (4)$$

where we choose the normalization of the $SU(N)$ matrices, t_a as,

$$\text{Tr}(t_a t_b) = \frac{N}{4} \delta_{ab}, \quad (5a)$$

$$[t_a, t_b] = i f_{abc} t_c, \quad (5b)$$

with Tr the trace, f_{abc} the structure constants and δ_{ab} the Kronecker delta. This normalization is chosen for later convenience, and it corresponds to a re-scaling, $t_a = \sqrt{N/2} T_a$ and $f_{abc} = \sqrt{N/2} F_{abc}$ of the customary $SU(N)$ matrices, T_a and the structure constants, F_{abc} of particle physics. (Zee, 2010) The **bold-face** and the overhead arrow ($\vec{}$) unambiguously serve as the disambiguation of group vectors from scalars.

Generically, since scalars commute with the $SU(N)$ gauge field, \mathbf{A}_μ but vectors do not, the $SU(N)$ gauge covariant derivative, \mathbf{D}_μ will act differently on scalars and vectors to respectively yield scalars and vectors, as follows,

$$\mathbf{D}_\mu T_{\alpha\cdots\beta}^{\gamma\cdots\delta} = \nabla_\mu T_{\alpha\cdots\beta}^{\gamma\cdots\delta}, \quad (6a)$$

$$\mathbf{D}_\mu \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta} = \nabla_\mu \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta} + \frac{1}{2} \vec{\mathbf{A}}_\mu \times \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta}. \quad (6b)$$

Here, we shall always have the factor of 1/2 in the cross-product, which guarantees the Yang-Mills field strength can be defined as,

$$\mathbf{F}_{\mu\nu} \equiv \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu, \quad (6c)$$

in order for \mathbf{D}_μ to act as a proper covariant derivative, by observing that the $SU(N) \Leftrightarrow U(1)$ exchange corresponds to $\mathbf{F}_{\mu\nu} \Leftrightarrow F_{\mu\nu}$ and $\mathbf{A}_\mu \Leftrightarrow A_\mu$, where A_μ and $F_{\mu\nu}$ are scalars under $SU(N)$. This is consistent with $\mathbf{D}_\mu \Leftrightarrow \nabla_\mu$. Thus, the gauge covariant derivative transforms scalars to scalars and vectors to vectors. Consequently, a dot-product of a vector with the $SU(N)$ matrix vector, $\vec{\mathbf{t}}$ is

not a scalar, but must transform as a vector,

$$\begin{aligned} \mathbf{D}_\mu (\mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta}) &= \mathbf{D}_\mu (\vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta} \cdot \vec{\mathbf{t}}) \\ &= (\nabla_\mu \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta} + \frac{1}{2} \vec{\mathbf{A}}_\mu \times \vec{\mathbf{T}}_{\alpha\cdots\beta}^{\gamma\cdots\delta}) \cdot \vec{\mathbf{t}} \\ &= \nabla_\mu \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} - \frac{1}{2} i [\mathbf{A}_\mu, \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta}], \end{aligned} \quad (6d)$$

where we ought to have $\mathbf{D}_\mu \vec{\mathbf{t}} = 0$. Moreover, the product of a scalar and a vector,

$$T_{\mu\cdots\nu}^{\sigma\cdots\rho} \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} = \mathbf{J}_{\alpha\cdots\nu}^{\gamma\cdots\rho}, \quad (6e)$$

must be vector. This can be verified *e.g.* by noting that differentiating by parts behaves as desired,

$$\begin{aligned} \mathbf{D}_\eta \mathbf{J}_{\alpha\cdots\nu}^{\gamma\cdots\rho} &= \mathbf{D}_\eta (T_{\mu\cdots\nu}^{\sigma\cdots\rho} \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta}) \\ &= \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} \mathbf{D}_\eta T_{\mu\cdots\nu}^{\sigma\cdots\rho} + T_{\mu\cdots\nu}^{\sigma\cdots\rho} \mathbf{D}_\eta \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} \\ &= \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} \nabla_\eta T_{\mu\cdots\nu}^{\sigma\cdots\rho} + T_{\mu\cdots\nu}^{\sigma\cdots\rho} \nabla_\eta \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta} \\ &\quad - T_{\mu\cdots\nu}^{\sigma\cdots\rho} \frac{1}{2} i [\mathbf{A}_\eta, \mathbf{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta}] \\ &= \nabla_\eta \mathbf{J}_{\alpha\cdots\nu}^{\gamma\cdots\rho} - \frac{1}{2} i [\mathbf{A}_\eta, \mathbf{J}_{\alpha\cdots\nu}^{\gamma\cdots\rho}]. \end{aligned} \quad (6f)$$

Since there exist complex objects like $\mathbb{T}_{\alpha\cdots\beta}^{\gamma\cdots\delta}$ that are neither scalars nor vectors but a combination of both, we note that the derivative,

$$\mathbb{D}_\mu = \nabla_\mu - i \mathbf{A}_\mu, \quad (7a)$$

$$\mathbb{D}_\mu^\dagger = \nabla_\mu + i \mathbf{A}_\mu, \quad (7b)$$

acting on a scalar forms such a complex object, where necessarily $\mathbf{D}_\mu \neq \mathbb{D}_\mu$. Proceeding, we can construct a complex-valued curvature tensor, $\mathbb{K}_{\mu\nu}$ using commutation with the derivative in eq. (7) and a space-time vector, \mathbb{V}^μ , which is such an object,

$$\begin{aligned} \mathbb{K}_{\mu\nu} \mathbb{V}^\mu &\equiv [\mathbb{D}_\mu, \mathbb{D}_\nu] \mathbb{V}^\mu \\ &= \mathbb{D}_\mu (\nabla_\nu - i \mathbf{A}_\nu) \mathbb{V}^\mu - \mathbb{D}_\nu (\nabla_\mu - i \mathbf{A}_\mu) \mathbb{V}^\mu \\ &= \nabla_\mu (\nabla_\nu - i \mathbf{A}_\nu) \mathbb{V}^\mu - i \mathbf{A}_\mu (\nabla_\nu - i \mathbf{A}_\nu) \mathbb{V}^\mu \\ &\quad - \nabla_\nu (\nabla_\mu - i \mathbf{A}_\mu) \mathbb{V}^\mu + i \mathbf{A}_\nu (\nabla_\mu - i \mathbf{A}_\mu) \mathbb{V}^\mu \\ &= [\nabla_\mu, \nabla_\nu] \mathbb{V}^\mu - i \mathbb{V}^\mu \nabla_\mu \mathbf{A}_\nu + i \mathbb{V}^\mu \nabla_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu] \mathbb{V}^\mu \\ &= R_{\mu\rho\nu}{}^\rho \mathbb{V}^\mu - i (\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - i [\mathbf{A}_\mu, \mathbf{A}_\nu]) \mathbb{V}^\mu \\ &= (R_{\mu\nu} - i (\mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu)) \mathbb{V}^\mu \\ &= (R_{\mu\nu} - i \mathbf{F}_{\mu\nu}) \mathbb{V}^\mu, \end{aligned} \quad (8a)$$

where $\mathbf{F}_{\mu\nu}$ is the Yang-Mills field strength defined in eq. (6c) and,

$$[\nabla_\mu, \nabla_\nu] \mathbb{V}^\sigma = R_{\rho\sigma\mu\nu} \mathbb{V}^\rho, \quad (8b)$$

with $R_{\rho\sigma\mu\nu}$ and $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$ the Riemann and Ricci tensors. Thus, we find,

$$\mathbb{K}_{\mu\nu} = R_{\mu\nu} - i \mathbf{F}_{\mu\nu}, \quad (8c)$$

is the object in question. In our notation, since $\mathbb{K}_{\mu\nu} \equiv K_{k\mu\nu}$, we discover that, $K_{0\mu\nu} = R_{\mu\nu}$ and $K_{a\mu\nu} = \vec{\mathbf{F}}_{\nu\mu}$. Consequently, $\mathbb{K}_{\mu\nu}$ has the property that $\mathbb{K}_{\mu\nu} = (\mathbb{K}_{\mu\nu})^{\text{T}*} = (\mathbb{K}_{\mu\nu})^{*\text{T}} = (\mathbb{K}_{\nu\mu})^* = (\mathbb{K}_{\mu\nu})^\dagger$, *i.e.* it is Hermitian with respect to both the space-time and group indices.

An intriguing observation is that, when the gauge group is SU(2), these objects are space-time dependent quaternion tensors. In particular, a typical quaternion, \mathbb{Q} corresponds to a space-time scalar represented as,

$$\begin{aligned} \mathbb{Q} &= Q + i\mathbf{Q} = Q + i\vec{\mathbf{Q}} \cdot \vec{\mathbf{t}} = Q + iQ_a t_a \\ &= Q + \mathbf{e}_1 Q_1/2 + \mathbf{e}_2 Q_2/2 + \mathbf{e}_3 Q_3/2, \end{aligned} \quad (9a)$$

where,

$$\mathbf{e}_1 \mathbf{e}_2 = -\mathbf{e}_2 \mathbf{e}_1 = \mathbf{e}_3 = i\sigma_3, \quad (9b)$$

$$\mathbf{e}_2 \mathbf{e}_3 = -\mathbf{e}_3 \mathbf{e}_2 = \mathbf{e}_1 = i\sigma_1, \quad (9c)$$

$$\mathbf{e}_3 \mathbf{e}_1 = -\mathbf{e}_1 \mathbf{e}_3 = \mathbf{e}_2 = i\sigma_2, \quad (9d)$$

defines the quaternion algebra, Q, Q_a are real numbers and $\mathbf{Q} = \vec{\mathbf{Q}} \cdot \vec{\mathbf{t}} = Q_a t_a$ with $t_a = \sigma_a/2$ and σ_a the Pauli matrices. (Meglicki, 2008) Thus, a dot-product of a vector with the SU(2) matrices can be treated as the imaginary part of a quaternion, whose real part transforms as a scalar under SU(2). Evidently, our notation in eq. (3) merely exploits the fact that in SU(2), the expressions with f_{abc} reduce to actual cross-products since the structure constants, $f_{abc} = \varepsilon_{abc}$ are proportional to the Levi-Civita symbol, ε_{abc} . In SU(3), the objects are analogous to octonions/Cayley numbers. (Dixon, 2013)

In particular, SU(2) objects can be related to the U(1) objects by a Cayley-Dickson construction (Schafer, 1954), where the number of generators (including the identity matrix) is doubled in the construction. We shall refer to all such objects, $\mathbb{T}_{\alpha\dots\beta}^{\gamma\dots\delta}$ with their analogous Cayley-Dickson algebra names, namely complex, quaternion, octonion, sedenion *etc.*, corresponding to complex-valued tensor objects with their imaginary parts transforming as vectors under SU(N), *i.e.* U(1), SU(2), SU(3), SU(4) *etc.* respectively. For instance, the Cayley-Dickson construction allows one to factorize a quaternion curvature tensor (eq. (8c)) into two complex tensors,

$$\begin{aligned} \mathbb{K}_{\mu\nu} &= R_{\mu\nu} - i\mathbf{F}_{\mu\nu} = R_{\mu\nu} - i\mathbf{F}_{\mu\nu} \cdot \mathbf{t} \\ &= R_{\mu\nu} + \mathbf{e}_1 F_{\mu\nu 1}/2 + \mathbf{e}_2 F_{\mu\nu 2}/2 + \mathbf{e}_3 F_{\mu\nu 3}/2, \\ &= R_{\mu\nu} + \mathbf{e}_1 F_{\mu\nu 1}/2 + \mathbf{e}_2 (F_{\mu\nu 2} - \mathbf{e}_1 F_{\mu\nu 3})/2, \end{aligned} \quad (10a)$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are given in eq. (9b) and the factor of 1/2 arises from the normalization choice of the SU(N) matrices given in eq. (5). However, the Cayley-Dickson construction is obfuscated as the number of colors, N becomes large. For instance, SU(3) objects contain $3^2 = 9$ real-valued tensors, whereas octonion tensors would only have $2 \times 4 = 8$ real-valued tensors. Nonetheless, this

discrepancy arises from the fact that t_8 and t_3 SU(3) matrices are both proportional to the single octonion element, \mathbf{e}_8 hence reducing the octonion components by one. (Shrestha *et al.*, 2012)

On the other hand, the SU(4) object has $4^2 = 16$ independent tensors (including the identity) which exactly corresponds to $2 \times 8 = 16$, the number of components of a sedenion. Since we can write, $16 = (2 \times \{2 \times [2 \times 2]\})$, we are always able to construct the SU(4) object from four quaternion sub-algebras,

$$\begin{aligned} \mathbb{K}_{\mu\nu} &= R_{\mu\nu} \mathbf{e}_0 + \mathbf{e}_1 f_{\mu\nu 1} + \mathbf{e}_2 [f_{\mu\nu 2} - \mathbf{e}_1 f_{\mu\nu 3}] \\ &\quad + \mathbf{e}_4 \{f_{\mu\nu 4} - \mathbf{e}_1 f_{\mu\nu 5} - \mathbf{e}_2 [f_{\mu\nu 6} - \mathbf{e}_1 f_{\mu\nu 7}]\} \\ &\quad + \sqrt{2} \mathbf{e}_8 (f_{\mu\nu 8} - \mathbf{e}_1 f_{\mu\nu 9} - \mathbf{e}_2 [f_{\mu\nu 10} - \mathbf{e}_1 f_{\mu\nu 11}] \\ &\quad - \mathbf{e}_4 \{f_{\mu\nu 12} - \mathbf{e}_1 f_{\mu\nu 13} - \mathbf{e}_2 [f_{\mu\nu 14} - \mathbf{e}_1 f_{\mu\nu 15}]\}) \\ &= R_{\mu\nu} + \mathbf{e}_a f_{\mu\nu a} = R_{\mu\nu} - i\mathbf{F}_{\mu\nu}, \end{aligned} \quad (10b)$$

as long as $\mathbf{e}_2 \mathbf{e}_1 = -\mathbf{e}_3$, $\mathbf{e}_4 \mathbf{e}_1 = -\mathbf{e}_5$, $\mathbf{e}_4 \mathbf{e}_2 = -\mathbf{e}_6$, $\mathbf{e}_4 \mathbf{e}_3 = -\mathbf{e}_7$, $\mathbf{e}_2 \mathbf{e}_1 = -\mathbf{e}_3$, $\mathbf{e}_8 \mathbf{e}_1 = -\mathbf{e}_9$, $\mathbf{e}_8 \mathbf{e}_2 = -\mathbf{e}_{10}$, $\mathbf{e}_8 \mathbf{e}_3 = -\mathbf{e}_{11}$, $\mathbf{e}_8 \mathbf{e}_4 = -\mathbf{e}_{12}$, $\mathbf{e}_8 \mathbf{e}_5 = -\mathbf{e}_{13}$, $\mathbf{e}_8 \mathbf{e}_6 = -\mathbf{e}_{14}$ and $\mathbf{e}_8 \mathbf{e}_7 = -\mathbf{e}_{15}$ is satisfied. In the last line, $\mathbf{e}_a f_{\mu\nu a} = \mathbf{F}_{\mu\nu}$ follows from setting $\mathbf{e}_0 = I_4$ and $\mathbb{K}_{\mu\nu}$ assumed Hermitian, since any 4×4 Hermitian matrix can always be decomposed into components with SU(4) basis matrices.

Finally, introducing $\Psi = |\Psi|U$, we note that $U = \exp(i\mathbf{S})$ and its inverse $U^{-1} = \exp(-i\mathbf{S})$ are $N \times N$ matrices, while $|\Psi|$ is a scalar under SU(N), where,

$$UU^{-1} = U^{-1}U = I_N, \quad (11a)$$

is the $N \times N$ identity matrix which satisfies $\mathbf{D}_\mu(U^{-1}U) = U^{-1}\mathbf{D}_\mu U + U\mathbf{D}_\mu U^{-1} = 0$. Consequently, it is straightforward to show, *e.g.* in SU(2), that Ψ, U and U^{-1} are quaternions by recognizing the product property given in eq. (6e) and calculating,

$$\begin{aligned} \exp(i\mathbf{S}) &= \exp(iS_1\sigma_1/2) \exp(iS_2\sigma_2/2) \exp(iS_3\sigma_3/2) \\ &= \sum_{j,k,l} \frac{1}{j!k!l!} (iS_1\sigma_1/2)^j (iS_2\sigma_2/2)^k (iS_3\sigma_3/2)^l \\ &= f(S_1, S_2, S_3)I_2 + i\mathbf{f}(S_1, S_2, S_3), \end{aligned} \quad (11b)$$

where we have used, $\sigma_a \sigma_b = \delta_{ab} + i\varepsilon_{abc} \sigma_c$ and the Taylor expansion, $\exp(x) = \sum_k x^k/k!$. Thus, eq. (11a) implies that $f^2 + f_1^2 + f_2^2 + f_3^2 = 1$, where $\vec{\mathbf{f}} = (f_1, f_2, f_3)$. This property can be generalized for the other SU(N) algebras, with $\vec{\mathbf{f}} \equiv f_a$ and $\mathbf{S} \equiv \theta$ a continuous infinitesimal variable, where $\theta^2 = 0$ vanishes in order for,

$$U = \exp(i\theta) = I_N + \vec{\theta} \cdot \vec{\mathbf{t}}, \quad (11c)$$

$$U^{-1} = \exp(i\theta) = I_N - \vec{\theta} \cdot \vec{\mathbf{t}}. \quad (11d)$$

Thus, using $\det(U) = \det(U^{-1}) = 1$ and $U^{-1}U = UU^{-1} = I_N$, it follows that U, U^{-1} are unitary.

In this formalism, a gauge transformation corresponds to,

$$\mathbf{A}'_\mu = \mathbb{U}^{-1} \mathbf{A}_\mu \mathbb{U} + \mathbb{U}^{-1} \mathbf{D}_\mu \mathbb{U}. \quad (12a)$$

This requires that the field strength must transform in a gauge invariant manner as,

$$\mathbf{F}'_{\mu\nu} = \mathbb{U}^{-1} \mathbf{F}_{\mu\nu} \mathbb{U}. \quad (12b)$$

B. Introduction to p forms

A notation usually preferred for its compactness is differential forms. (Demailly, 1997; Tu, 2017) We introduce several straightforward guidelines on differential forms particularly helpful to follow the discussion in the manuscript:

- (i) Any p form can be locally written as the contracted form of a space-time tensor of rank p with an equal number of space-time basis vectors, dx^{μ_p} such that,

$$\mathbb{A} = \frac{1}{p!} \mathbb{A}_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \quad (13a)$$

where, we can device a useful short-hand to write,

$$dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = \varepsilon^{\mu_1 \dots \mu_p} dx^1 \dots dx^p, \quad (13b)$$

with $\varepsilon^{\mu_1 \dots \mu_p}$ the totally anti-symmetric Levi-Civita symbol normalized as $\varepsilon^{1,2,3,\dots,p} = 1$ and the factorial, $p!$ introduced for later convenience. This makes the p form the complete anti-symmetrization of $\mathbb{A}_{\mu_1 \dots \mu_p}$.

- (ii) The wedge product, \wedge is associative (*i.e.* $(\mathbb{A} \wedge \mathbb{B}) \wedge \mathbb{C} = \mathbb{A} \wedge (\mathbb{B} \wedge \mathbb{C})$), and corresponds to the *multiplication* operation for p forms, The wedge product of a p and a q form yields a $p + q$ form,

$$\mathbb{A} \wedge \mathbb{B} = \frac{1}{p!q!} \mathbb{A}_{[\mu_1 \dots \mu_p] \mu_{p+1} \dots \mu_{p+q}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+q}}, \quad (14)$$

where $[\dots]$ indicates anti-symmetrization of the indices. Thus, using this, one can show that,

$$\mathbb{A} \wedge \mathbb{B} = (-1)^{pq} \mathbb{B} \wedge \mathbb{A}. \quad (15)$$

This means that $\mathbb{A} \wedge \mathbb{B} = 0$ when $\mathbb{A} = A = \mathbb{B} = B$ are group scalars and $p = q$ is odd. However, when $\mathbb{A} = \mathbf{A} = \mathbb{B} = \mathbf{B}$ are group vectors (*i.e.* can be decomposed into components multiplied by group basis matrices which do not commute), eq. (15) need not vanish, even for odd $p = q$. Finally, we can determine the wedge product, $\mathbb{A}_1 \wedge \mathbb{A}_2 \dots \mathbb{A}_n$ for n number of p forms in a similar manner.

- (iii) Since space-time indices are always hidden in the notation, only p forms with equal rank can be added or subtracted.

- (iv) The Levi-Civita symbol does not transform appropriately as a tensor. To see this, on a manifold, \mathcal{M} of n dimensions with a metric tensor, $g_{\mu\nu}$, we can attempt to lower its indices by,

$$\varepsilon^{\mu\nu\dots\sigma\rho} g_{\mu\alpha} g_{\nu\beta} \dots g_{\sigma\gamma} g_{\rho\delta} = -\det(g_{\mu\nu}) \varepsilon_{\alpha\beta\dots\gamma\delta}, \quad (16a)$$

where the right-hand side follows from the definition of determinant of an $n \times n$ matrix and the minus (-) sign originates from the Lorenzian signature of the space-time manifold (For a Euclidean signature metric, $g_{\mu\nu}^E$, we would have the determinant multiplied by a plus (+) sign instead). Thus, one divides both sides by the measure $\sqrt{-\det(g_{\mu\nu})}$ to yield the tensors,

$$v_{\alpha\beta\dots\sigma\rho} = \sqrt{-\det(g_{\mu\nu})} \varepsilon_{\alpha\beta\dots\sigma\rho}, \quad (16b)$$

and,

$$v^{\alpha\beta\dots\sigma\rho} = \frac{1}{\sqrt{-\det(g_{\mu\nu})}} \varepsilon^{\alpha\beta\dots\sigma\rho}, \quad (16c)$$

whose raising and lowering operations by the metric tensor are appropriately given by eq. (16a). We can explicitly check each transforms as a tensor using,

$$\begin{aligned} \det(g_{\sigma\rho}(x)) &= \det\left(g'_{\mu\nu}(x') \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\rho}\right) \\ &= \det(g'_{\mu\nu}(x')) \left| \frac{\partial x'}{\partial x} \right|^2, \end{aligned} \quad (16d)$$

and,

$$\begin{aligned} dx^{\sigma_1} \wedge \dots \wedge dx^{\sigma_p} &= \frac{\partial x^{\sigma_1}}{\partial x'^{\mu_1}} dx'^{\mu_1} \wedge \dots \wedge \frac{\partial x^{\sigma_p}}{\partial x'^{\mu_p}} dx'^{\mu_p} \\ &= \frac{\partial x^{\sigma_1}}{\partial x'^{\mu_1}} \dots \frac{\partial x^{\sigma_p}}{\partial x'^{\mu_p}} \varepsilon^{\mu_1 \dots \mu_p} dx'^1 \dots dx'^p \\ &= \left| \frac{\partial x}{\partial x'} \right| \varepsilon^{\mu_1 \dots \mu_p} dx'^1 \dots dx'^p \\ &= \left| \frac{\partial x}{\partial x'} \right| dx'^{\sigma_1} \wedge \dots \wedge dx'^{\sigma_p}, \end{aligned} \quad (16e)$$

where we recognize,

$$\left| \frac{\partial x}{\partial x'} \right| = \left| \frac{\partial x'}{\partial x} \right|^{-1}. \quad (16f)$$

is the determinant of the Jacobian matrix.

- (v) For an n manifold, \mathcal{M} the volume element is a $p = n$ form built from eq. (16b) as,

$$dV = \frac{1}{(p=n)!} v_{\alpha_1 \dots \alpha_p=n} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p=n}, \quad (17a)$$

where dV is the volume element. Using eq. (13), and the property of the Levi-Civita symbol,

$$\varepsilon_{\alpha_1 \dots \alpha_n} \varepsilon^{\alpha_1 \dots \alpha_n} = n!, \quad (17b)$$

the volume of the $n = p$ manifold, \mathcal{M} becomes,

$$\begin{aligned} \int_{\mathcal{M}} dV &= \int_{\mathcal{M}} dx^1 \cdots dx^{p=n} \sqrt{-\det(g_{\mu\nu})} \\ &= \int_{\mathcal{M}} d^n x \sqrt{-\det(g_{\mu\nu})}. \end{aligned} \quad (17c)$$

We now recognize the need for the factorial, $p!$ introduced in (i). For our purposes, all manifolds shall be Riemannian or pseudo-Riemannian, and hence orientable (meaning $\det(g_{\mu\nu})$ does not change sign anywhere on the manifold, to prevent the square-root from turning imaginary).

- (vi) The condition $p = n$, is referred to, in literature, as the condition that the p form is a *top* form. We note that the volume element requires this condition. This condition is vital, since the Lagrangian density of any field theory must be a *top* form. Also, due to eq. (13), the relevant Levi-Civita symbol strictly must have $p = n$ indices in order for eq. (17a) to be a *top form*.
- (vii) Any $p \leq n$ form, \mathbb{A} , has a dual $(n - p)$ form, $\star\mathbb{A}$ known as the Hodge dual such that,

$$\star\mathbb{A} = \frac{1}{p!(n-p)!} v_{\alpha_1 \cdots \alpha_n} \mathbb{A}^{\alpha_1 \cdots \alpha_p} dx^{\alpha_{p+1}} \wedge \cdots \wedge dx^{\alpha_n}. \quad (18a)$$

The appearance of $v_{\alpha_1 \cdots \alpha_n}$, defined in eq. (16b), guarantees the component of the Hodge dual,

$$(\star\mathbb{A})_{\alpha_{p+1} \cdots \alpha_n} = \frac{1}{p!} v_{\alpha_1 \cdots \alpha_n} \mathbb{A}^{\alpha_1 \cdots \alpha_p}, \quad (18b)$$

and its raised counterpart,

$$(\star\mathbb{A})^{\alpha_{p+1} \cdots \alpha_n} = \frac{1}{p!} v^{\alpha_1 \cdots \alpha_n} \mathbb{A}_{\alpha_1 \cdots \alpha_p}, \quad (18c)$$

appropriately transform as tensors. Consequently, the Hodge operation done twice yields back the p form,

$$\star\star\mathbb{A} = (-1)^{p(n-p)} \mathbb{A}_{\alpha_1 \cdots \alpha_p} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}, \quad (18d)$$

where we have used,

$$v^{\mu_1 \cdots \mu_p \alpha_{p+1} \cdots \alpha_n} v_{\alpha_1 \cdots \alpha_p \alpha_{p+1} \cdots \alpha_n} = (-1)^{n-p} (n-p)! \delta_{\alpha_1 \cdots \alpha_p}^{\mu_1 \cdots \mu_p},$$

with $\delta_{\alpha_1 \cdots \alpha_p}^{\mu_1 \cdots \mu_p}$ a generalized Kronecker delta with the property, $\delta_{\alpha_1 \cdots \alpha_p}^{\mu_1 \cdots \mu_p} \mathbb{A}^{\alpha_1 \cdots \alpha_p} = (-1)^p p! \mathbb{A}^{\mu_1 \cdots \mu_p}$.

- (viii) The exterior derivative operation, d on any p form is given by,

$$d\mathbb{A} = \frac{1}{(p+1)!} \partial_{\mu_{p+1}} \mathbb{A}_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}. \quad (19a)$$

Applying the exterior derivative operation twice annihilates the p form,

$$d^2\mathbb{A} = \frac{1}{(p+2)!} \partial_{\mu_{p+2}} \partial_{\mu_{p+1}} \mathbb{A}_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+2}} = 0,$$

where we have used the fact that partial derivatives commute, $[\partial_{\mu_{p+2}}, \partial_{\mu_{p+1}}] = 0$.

The expression translates into the Bianchi identities in gauge and gravity theories, and can be familiarized as the differential form version of the identities, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ and $\vec{\nabla} \times \vec{\nabla} \phi = 0$ for spatial vector and scalar fields in $n = 3$ dimensions. This is instructive, since the proof of these identities in vector calculus also utilizes the fact that partial derivatives commute.

For the wedge product of a p and a q form, \mathbb{A} and \mathbb{B} respectively, the Leibniz rule is given by,

$$d(\mathbb{A} \wedge \mathbb{B}) = d\mathbb{A} \wedge \mathbb{B} + (-1)^p \mathbb{A} \wedge d\mathbb{B}. \quad (19b)$$

For any $p = n - 1$ form, \mathbb{A} , on a non-compact manifold (*i.e.* a manifold with a boundary, $\partial\mathcal{M}$), we can elegantly write Stokes' theorem as,

$$\int_{\partial\mathcal{M}} \mathbb{A} = \int_{\mathcal{M}} d\mathbb{A}. \quad (19c)$$

- (ix) Since the wedge product of an $(n - p)$ form with a p form is a *top* form, one can exploit this property to define an inner product for two p forms, \mathbb{A} and \mathbb{B} , as the integral of their wedge product over the manifold, \mathcal{M} ,

$$\langle \mathbb{A}, \mathbb{B} \rangle = \int_{\mathcal{M}} \mathbb{A} \wedge \star\mathbb{B}. \quad (20a)$$

There is a sense in which the exterior derivative is a raising operator acting on forms since it takes a p form to a $p + 1$ form. One can define a lowering operator, d^\dagger by introducing a $p - 1$ form \mathbb{C} related to \mathbb{A} in eq. (20a) by $d\mathbb{C} = \mathbb{A}$, and differentiate by parts and drop the boundary term (eq. (19c)), $\int_{\mathcal{M}} d(\mathbb{C} \wedge \star\mathbb{B}) = \int_{\partial\mathcal{M}} \mathbb{C} \wedge \star\mathbb{B} = 0$, to yield,

$$\begin{aligned} \langle d\mathbb{C}, \mathbb{B} \rangle &= \int_{\mathcal{M}} d\mathbb{C} \wedge \star\mathbb{B} = - \int_{\mathcal{M}} \mathbb{C} \wedge d\star\mathbb{B} \\ &= - \int_{\mathcal{M}} \mathbb{C} \wedge \star(-1)^{p(n-p)} \star d\star\mathbb{B} \\ &= \int_{\mathcal{M}} \mathbb{C} \wedge \star d^\dagger \mathbb{B} = \langle \mathbb{C}, d^\dagger \mathbb{B} \rangle, \end{aligned} \quad (20b)$$

where the lowering operator is given by $d^\dagger = (-1)^{p(n-p)+1} \star d\star$.

- (x) For any $p + 1 = n$ form, \mathbb{B} on the manifold, \mathbb{B} is said to be *closed* if $d\mathbb{B} = 0$. Employing eq. (19), one

could suspect that we can always write, $\mathbb{B} = d\mathbb{A}$, with \mathbb{A} defined the same way everywhere on $\partial\mathcal{M}$. It turns out that the statement is false for topologically relevant manifolds. This stems from de Rham cohomology (Demailly, 1997), a tool exposing the extent to which the fundamental theorem of calculus fails on topological manifolds.

In particular, if $d\mathbb{A} = \mathbb{B}$, where \mathbb{A} is defined the same way everywhere on $\partial\mathcal{M}$, then \mathbb{B} is said to be *exact*. Thus, every *exact* form is *closed*, but the converse fails due to the non-trivial topology of \mathcal{M} . Thus, the topology of the manifold can be exploited to avail an avenue to apply Stokes' theorem on *compact* manifolds (*i.e.* manifolds without a boundary). In particular, the boundary operation ∂ applied to a *compact* manifold vanishes, $\partial\mathcal{M} = 0$. However, suppose \mathcal{M} can be divided into two patches, \mathcal{M}_+ and \mathcal{M}_- due to topological reasons (which will become apparent), then we can write the condition that \mathcal{M} is *compact* as,

$$\partial\mathcal{M}_\pm = \pm\mathcal{B}, \quad (21a)$$

$$\mathcal{M} = \mathcal{M}_+ + \mathcal{M}_-, \quad (21b)$$

where \mathcal{B} is the boundary shared by \mathcal{M}_+ and \mathcal{M}_- , albeit with opposite orientation (\pm), and must be a *compact* manifold, $\partial\mathcal{B} = 0$. Since \mathcal{B} is $n - 1$ dimensional when \mathcal{M} is n dimensional, and both manifolds are *compact*, this procedure can be carried out successively $n - 1$ times, introducing non-trivial topologies at each stage.

For instance, in eq. (19c), when $\mathbb{A} = A$ is the U(1) gauge field ($p = 1$ form) and $\mathcal{M} = S^2$ is the $n = 2$ -sphere, a magnetic monopole at the center of the S^2 sphere implies that A cannot be defined the same way on the northern ($\partial\mathcal{M}_+ = S_+^2$) and southern ($\partial\mathcal{M}_- = S_-^2$) hemispheres, each bounded at the equator by the $n = 2$ -sphere, $\mathcal{B} = S^1$ with opposite orientation. Nonetheless, the respective gauge fields are related by a gauge transformation, $A^+ = A^- + d\theta_1$, where θ_1 is a $p = 0$ form. Since θ_1 is the azimuthal angle, defined modulo $2\pi\nu$ on S^1 , where $\nu \in \mathbb{Z}$ is an integer, Stokes' theorem requires that,

$$\begin{aligned} \frac{1}{2\pi} \int_{S^2} dA &= \frac{1}{2\pi} \int_{S_+^2} dA + \frac{1}{2\pi} \int_{S_-^2} dA \\ &= \frac{1}{2\pi} \int_{\partial S_+^2} A^+ + \frac{1}{2\pi} \int_{\partial S_-^2} A^- \\ &= \frac{1}{2\pi} \int_{S^1} A^+ - \frac{1}{2\pi} \int_{S^1} A^- \\ &= \frac{1}{2\pi} \int_{S^1} d\theta_1 = \nu, \quad (22) \end{aligned}$$

where ν is the monopole number in Maxwell's theory. (Dirac, 1931; Konishi, 2007; Zee, 2010) In

mathematics, the monopole number is referred to as the first Chern number. (Chern, 1946) Generally, Stokes' theorem (eq. (19c)) directly relates such numbers to a class of non-*exact* but otherwise closed p forms known as characteristic classes. (Milnor and Stasheff, 2016)

Particularly useful for this paper, is the Euler characteristic of an $n = 4$ dimensional *compact* Riemannian manifold, \mathcal{M}^E (with Euclidean signature, denoted by the superscript, E) given by the integral,

$$\chi_4 = \frac{1}{8\pi^2} \int_{\mathcal{M}^E} \text{Tr}(\mathbf{R} \wedge \mathbf{R}) = \frac{1}{\pi N} \sum_{j=\pm} \int_{\partial\mathcal{M}_j^E} \text{CS}^j, \quad (23a)$$

$$\text{CS}^j = \frac{N}{8\pi} \text{Tr} \left(\omega^j \wedge d\omega^j + \frac{2}{3} \omega^j \wedge \omega^j \wedge \omega^j \right), \quad (23b)$$

where $\mathcal{M}^E = \sum_{j=\pm} \mathcal{M}_j^E$, $\mathbf{R} = d\omega + \omega \wedge \omega$ is the curvature $p = 2$ form which transforms as $\mathbf{R}' = \mathbf{\Lambda} \mathbf{R} \mathbf{\Lambda}^{-1}$ under SO(4) rotations, $\omega' = \mathbf{\Lambda}^{-1} \omega \mathbf{\Lambda} + \mathbf{\Lambda}^{-1} d\mathbf{\Lambda}$, ω is the spin connection and CS^j is the Chern-Simons $p = 3$ form (Chern and Simons, 1974) with,

$$k = \frac{N}{2} \in \mathbb{Z} \geq 0, \quad (23c)$$

a positive integer known as the Chern-Simons level. (Jackiw and Pi, 2003) This further restricts the number of colors, N to even positive integers.

The integrand in eq. (23a) is the second Euler class and is equivalent to (Lovelock, 1971),

$$\begin{aligned} &\frac{1}{\sqrt{\det(g_{\mu\nu}^E)}} \text{Tr}(\mathbf{R} \wedge \mathbf{R}) \\ &= (R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} - 4R^{\mu\nu} R_{\mu\nu} + R^2), \quad (24a) \end{aligned}$$

where $R_{\mu\nu\sigma\rho}$ is the Riemann tensor, $R_{\mu\nu}$ and R are the Ricci tensor and scalar respectively and we have used,

$$\begin{aligned} \mathbf{R} &= d\omega + \omega \wedge \omega \\ &= \mathbf{R}^{\bar{a}}_{\bar{b}} = d\omega^{\bar{a}}_{\bar{b}} + \omega^{\bar{a}}_{\bar{c}} \wedge \omega^{\bar{c}}_{\bar{b}} \\ &= \frac{1}{2} \left(\partial_\mu \omega_\nu^{\bar{a}}_{\bar{b}} - \partial_\nu \omega_\mu^{\bar{a}}_{\bar{b}} + \omega_\mu^{\bar{a}}_{\bar{c}} \wedge \omega_\nu^{\bar{c}}_{\bar{b}} \right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} R_{\mu\nu}^{\bar{a}}_{\bar{b}} dx^\mu \wedge dx^\nu, \quad (24b) \end{aligned}$$

with $R^\sigma_{\rho\mu\nu} = R_{\mu\nu}^{\bar{a}}_{\bar{b}} e^{\bar{a}\sigma} e^{\bar{b}\rho}$ the Riemann tensor.

A Wick rotation, $t^E \rightarrow \pm it$, where t^E is the Euclidean time, corresponds to the transformation, $\text{SO}(4) \rightarrow \text{SO}(1,3)$, where the rotations, $\mathbf{\Lambda}$ now correspond to local Lorentz transformations (three boosts and three rotations) on the tangent manifold.

Thus, the Chern-Simons $p = 3$ form transforms such that the wave function,

$$\begin{aligned}\psi_{\text{CS}}^+ &= \exp\left(i \int_{\mathcal{B}} \text{CS}^+\right) \\ &= \exp\left(i \int_{\mathcal{B}} \text{CS}^-\right) \exp(i\pi N \chi_4) \\ &= \psi_{\text{CS}}^- \exp(i\pi N \chi_4),\end{aligned}\quad (25)$$

is single-valued under $\text{SO}(1, 3)$, where we have used $\partial\mathcal{M}_j = j\mathcal{B}$ with $j = \pm$ from eq. (21). In fact, since the **bold-face** indicates that $\boldsymbol{\omega}$ is a group vector, we can borrow a leaf out of eq. (22), and take the two Chern-Simons $p = 3$ forms to be related to each other by local Lorentz transformations/Euclidean rotations,

$$\boldsymbol{\omega}^\pm = \boldsymbol{\Lambda}^{-1} \boldsymbol{\omega}^\mp \boldsymbol{\Lambda} + \boldsymbol{\Lambda}^{-1} d\boldsymbol{\Lambda},\quad (26)$$

which corresponds to,

$$\text{CS}^\pm = \text{CS}^\mp + N d\theta,\quad (27a)$$

$$d\theta = \frac{1}{24\pi} \text{Tr}(\boldsymbol{\Lambda}^{-1} d\boldsymbol{\Lambda})^3,\quad (27b)$$

where θ is a $p = 2$ form, defined modulo $2\pi\nu$ on \mathcal{B}^{E} corresponding to the second Chern (winding) number and,

$$\chi_4 = \frac{1}{24\pi^2} \int_{\mathcal{B}^{\text{E}}} \text{Tr}(\boldsymbol{\Lambda}^{-1} d\boldsymbol{\Lambda})^3 = 2 \int_{\mathcal{B}^{\text{E}}} d\theta = \pm 2\nu.\quad (28)$$

However, eq. (28) is an atypical result since it somewhat purports the equivalence between the second Euler and Chern classes. This is only valid if one can treat $\boldsymbol{\omega}$ as a gauge field, \mathbf{A} , which shall have great utility in the work herein.

In particular, eq. (28) exploits the fact that $\text{SU}(4)$ will be broken into $\text{SO}(4)$ ($\mathbf{F} \rightarrow \mathbf{R}$), which would permit the second Chern class integrated over S^4 to be identified as the second Euler class on the four-dimensional *compact* Riemannian manifold, \mathcal{M}^{E} , provided (S^4, S^3) are homeomorphic to $(\mathcal{M}^{\text{E}}, \mathcal{B}^{\text{E}})$ under a symmetry breaking $\mathbb{U} \rightarrow \mathbf{A}$, where the object \mathbb{U} under $\text{SU}(4)$ transforms as a vector under $\text{SO}(4)$, reflecting the apparent replacement of the complex $\text{SU}(4)$ tangent manifold with the real $\text{SO}(4)$ tangent manifold. This is explored in Section III.A.

C. Group scalars and vectors (p forms)

Any $N \times N$ Hermitian matrix, $\mathbb{A} = \mathbb{A}^\dagger$ can be expanded into basis $\text{SU}(N)$ matrices, \mathbf{t} including the identity matrix, I_N as,

$$\mathbb{A} = A - \mathbf{A},\quad (29a)$$

where $A = AI_N$, $\mathbf{A} = \vec{\mathbf{A}} \cdot \vec{\mathbf{t}}$, $\vec{\mathbf{A}} = (A_1, \dots, A_{N^2-1})$, $N^2 - 1$ is the number of $\text{SU}(N)$ generators and A, A_1, \dots, A_{N^2-1} are real-valued space-time functions. Defining anti-Hermitian basis matrices, $\vec{\boldsymbol{\lambda}}$ from the Hermitian ones, $\vec{\mathbf{t}}$ and their commutation relations as follows,

$$\vec{\boldsymbol{\lambda}} = -i\vec{\mathbf{t}},\quad (29b)$$

$$[\lambda_a, \lambda_b] = f_{abc} \lambda_c,\quad (29c)$$

we recognize that eq. (29a) takes the desired object form, *albeit* with anti-Hermitian basis vectors, $\vec{\boldsymbol{\lambda}}$ in place of the Hermitian vectors, $\vec{\mathbf{t}}$,

$$\mathbb{A} = A + i\mathbf{A}.\quad (29d)$$

In particular, taking \mathbb{A} to be a $p = 1$ form, the object form of the Yang-Mills $p = 2$ form can be defined compactly as,

$$\mathbf{D}\mathbb{A} = \mathbb{F} = d\mathbb{A} + i\mathbf{A} \wedge \mathbf{A} = F + i\mathbf{F},\quad (30a)$$

where d is the exterior derivative, $F = dA$ is the $\text{U}(1)$ field strength, while $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$ is the Yang-Mills field strength, which in component form can be written as, $\mathbb{F}_{\mu\nu} = F_{\mu\nu} + i\mathbf{F}_{\mu\nu}$.

For instance, the field equations correspond to,

$$\frac{1}{2\pi} \mathbf{D}^\dagger \mathbb{F} = \mathbb{J} = J + i\mathbf{J},\quad (30b)$$

which are equivalent to,

$$\frac{1}{2\pi} \mathbf{D}_\mu \mathbb{F}^\mu_\nu = \mathbb{J}_\nu = J_\nu + i\mathbf{J}_\nu,\quad (30c)$$

where $\mathbf{D}^\dagger F = d^\dagger F$ and $\mathbf{D}^\dagger \mathbf{F} = d^\dagger \mathbf{F} + \star(\mathbf{A} \wedge \mathbf{F})$, in conformity with eq. (6a), and a factor of 2π introduced for later convenience. Moreover, since $d^2 = d^{\dagger 2} = 0$ vanishes, $\mathbf{D}^2 = \mathbf{D}^{\dagger 2} = 0$ is also an identity of the covariant derivatives and likewise vanishes. Thus, we can retain our notation for Yang-Mills objects, namely the complex objects (\mathbb{T}), the scalars (T) and the vectors (\mathbf{T}). The **bold-face** thus indicates that we are dealing with $N \times N$ matrices, whose basis matrices are the anti-Hermitian generators of the $\text{SU}(N)$ gauge group. Consequently, gauge transformations correspond to,

$$A' = A + d\theta,\quad (31a)$$

$$\mathbf{A}' = \mathbb{U}^{-1} \mathbf{A} \mathbb{U} + \mathbb{U}^{-1} d\mathbb{U},\quad (31b)$$

which implies the respective components of the field strength object transform as a scalar,

$$F' = dA' = F = d(A + d\theta) = dA,\quad (31c)$$

and a vector,

$$\begin{aligned}\mathbf{F}' &= d(\mathbb{U}^{-1} \mathbf{A} \mathbb{U} + \mathbb{U}^{-1} d\mathbb{U}) \\ &+ (\mathbb{U}^{-1} \mathbf{A} \mathbb{U} + \mathbb{U}^{-1} d\mathbb{U}) \wedge (\mathbb{U}^{-1} \mathbf{A} \mathbb{U} + \mathbb{U}^{-1} d\mathbb{U}) \\ &= \mathbb{U}^{-1} (d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}) \mathbb{U} = \mathbb{U}^{-1} \mathbf{F} \mathbb{U},\end{aligned}\quad (31d)$$

under $SU(N)$ gauge group, where we have used $d^2\theta = d^2\mathbb{U} = d^2\mathbb{U}^{-1} = 0$ and $\mathbb{U}^{-1}d\mathbb{U} = -\mathbb{U}d\mathbb{U}^{-1}$ in eq. (31d).

Finally, it is interesting to check that Maxwell's theory in special relativity takes the form,

$$\partial_j \mathbb{F}^j = 2\pi \mathbb{J}^0, \quad (32a)$$

$$i\varepsilon^{jkl} \partial_k \mathbb{F}_l = -2\pi \mathbb{J}^j - \frac{\partial}{\partial t} \mathbb{F}^j, \quad (32b)$$

where E^j and B^j are the spatial components of the electric and magnetic fields respectively, ε^{jkl} is the $n = 3$ dimensional totally anti-symmetric Levi-Civita symbol normalized as $\varepsilon^{123} = 1$, $\mathbb{F}^j = E^j + iB^j$ is a complex vector constructed from the two $U(1)$ group scalars, E^j, B^j and $\mathbb{J}^\mu = J_e^\mu + iJ_m^\mu$ is the complex current density constructed from two $U(1)$ scalars, J_e^μ and J_m^μ , corresponding to the electric and magnetic charge densities. Thus, since Maxwell's theory with monopoles satisfies S-duality, *i.e.* Maxwell's equations are invariant under the exchange of electric quantities with their magnetic counterparts (Zee, 2010), eq. (32) can be written in a metric independent manner using the language introduced above as eq. (30), where the gauge group is $U(1)$ instead of $SU(N)$ ($\mathbf{A} \rightarrow A'$). Consequently,

$$\mathbb{A} = A + iA', \quad (32c)$$

$$\nu = \frac{1}{2\pi} \int_{S^2} dA, \quad \nu' = \frac{1}{2\pi} \int_{S^2} dA', \quad (32d)$$

where ν and ν' respectively are the monopole and electron numbers and S^2 is the $n = 2$ -sphere.

III. THEORY

A. Motivation

To gain insights into the scalar and vector structure we have considered above, we can refrain from suppressing the matrix components, and explicitly write the Yang-Mills vector field strength and gauge transformations as,

$$\mathbf{F}_{\bar{b}}^{\bar{a}} = d\mathbf{A}_{\bar{b}}^{\bar{a}} + \mathbf{A}_{\bar{c}}^{\bar{a}} \wedge \mathbf{A}_{\bar{b}}^{\bar{c}}, \quad (33a)$$

$$\mathbf{F}'_{\bar{b}}{}^{\bar{a}} = (\mathbb{U}^{-1})_{\bar{c}}^{\bar{a}} \mathbf{F}_{\bar{d}}^{\bar{c}} (\mathbb{U})_{\bar{b}}^{\bar{d}}, \quad (33b)$$

$$\mathbf{A}'_{\bar{b}}{}^{\bar{a}} = (\mathbb{U}^{-1})_{\bar{c}}^{\bar{a}} \mathbf{A}_{\bar{d}}^{\bar{c}} (\mathbb{U})_{\bar{b}}^{\bar{d}} + (\mathbb{U}^{-1})_{\bar{c}}^{\bar{a}} d(\mathbb{U})_{\bar{b}}^{\bar{c}}, \quad (33c)$$

where $\mathbf{A}_{\bar{b}}^{\bar{a}} = \mathbf{A}_{\bar{b}}^*{}^{\bar{a}}$ is a space-time dependent $N \times N$ Hermitian matrix. It is evident that, in the tetrad formalism, the curvature $p = 2$ form takes a similar form,

$$\mathbf{R}_{\bar{b}}^{\bar{a}} = d\omega_{\bar{b}}^{\bar{a}} + \omega_{\bar{c}}^{\bar{a}} \wedge \omega_{\bar{b}}^{\bar{c}}, \quad (34a)$$

$$\mathbf{R}'_{\bar{b}}{}^{\bar{a}} = (\Lambda^{-1})_{\bar{c}}^{\bar{a}} \mathbf{R}_{\bar{d}}^{\bar{c}} (\Lambda)_{\bar{b}}^{\bar{d}}, \quad (34b)$$

$$\omega'_{\bar{b}}{}^{\bar{a}} = (\Lambda^{-1})_{\bar{c}}^{\bar{a}} \omega_{\bar{d}}^{\bar{c}} (\Lambda)_{\bar{b}}^{\bar{d}} + (\Lambda^{-1})_{\bar{c}}^{\bar{a}} d(\Lambda)_{\bar{b}}^{\bar{c}}, \quad (34c)$$

where the spin connection, $\omega_{\bar{b}}^{\bar{a}} = -\omega_{\bar{b}}^{\bar{a}}$ is a 4×4 anti-symmetric matrix, and the components \bar{a} running from $\bar{0}$ to $\bar{3}$, correspond to the four coordinates of the tangent manifold. Thus, one can expand ω into its basis matrices in order to find,

$$\omega = \omega_a \lambda_a = \omega_2 \lambda_2 + \omega_5 \lambda_5 + \omega_{10} \lambda_{10} + \omega_{14} \lambda_{14} + \omega_{12} \lambda_{12} + \omega_7 \lambda_7, \quad (35)$$

where λ_a are the anti-Hermitian generators of $SU(4)$ (Sbaih *et al.*, 2013) and we require all the $SU(4)$ generators which do not appear in eq. (35) to be set to nil. Consequently, this strictly breaks $SU(4)$ gauge symmetry in favor of $SO(4)$, thus capturing the local rotation group of the tangent space in $n = 4$ dimensions.

To further exploit the formalism, we can always calculate with the full $SU(4)$ symmetry and subsequently break it to $SO(4)$ afterwards, by setting the appropriate values of \mathbf{A} to nil, followed by a Wick rotation, $SO(4) \rightarrow SO(1,3)$ in order to yield the local Lorentz group on the tangent space replacing the Euclidean rotation group. This corresponds to making the identifications,

$$\mathbf{R} \Leftrightarrow \mathbf{F}, \quad \omega \Leftrightarrow \mathbf{A}, \quad (36a)$$

$$\mathbf{\Lambda} \Leftrightarrow \mathbb{U}, \quad \eta_{\bar{a}\bar{b}} \Leftrightarrow \delta_{\bar{a}\bar{b}}, \quad (36b)$$

where $\mathbb{U} = \exp(i\theta) = \exp(i\vec{\theta} \cdot \vec{\mathbf{t}}) = \exp(\vec{\theta} \cdot \vec{\mathbf{\lambda}}) = \exp(\theta^E) = \mathbf{\Lambda}$ are local Lorentz transformations on the tangent manifold, and

$$\vec{\mathbf{\lambda}} = (K_2^E, K_5^E, K_{10}^E, L_{14}, L_{12}, L_7) = (\vec{K}^E, \vec{L}), \quad (37a)$$

corresponds to the local generators of Euclidean boost, $\vec{K}^E = (K_1^E, K_2^E, K_3^E)$ and angular momentum $\vec{L} = (L_1, L_2, L_3)$ on the tangent space. In other words, we shall have,

$$\mathbf{A}^j = \exp \left(\int \lambda_{\bar{b}}^{\bar{a}} \omega_{\mu \bar{a}}^{\bar{b}} dx^\mu \right) = \exp \left(\int \text{Tr}(\lambda \omega^j) \right), \quad (37b)$$

where $j = \pm$ and $\lambda_{\bar{b}}^{\bar{a}} = \lambda'_a \otimes \lambda_a$ is expanded as a matrix tensor product into $SU(4)$ according to eq. (35). Requiring the trace to act on the λ'_a basis matrices, but not their matrix coefficients, λ_a , the Lorentz transformation

of a topologically relevant manifold is defined as,

$$\begin{aligned}
\Lambda &= \Lambda^+(\Lambda^-)^{-1} \\
&= \exp\left(\int \text{Tr}(\lambda\omega^+) - \int \text{Tr}(\lambda\omega^-)\right) \\
&= \exp\left(\int \lambda_a\omega_b^+ \text{Tr}(\lambda'_a\lambda'_b) - \int \lambda'_a\omega_b^- \text{Tr}(\lambda'_a\lambda'_b)\right) \\
&= \exp\left(\int \lambda_a\omega_b^+ \delta_{ab} - \int \lambda_a\omega_b^- \delta_{ab}\right) \\
&= \exp\left(\int \omega^+ - \int \omega^-\right) = \exp\left(\mp \int d\theta^E\right) \\
&= \exp(\mp\theta^E) = \mathbb{U}^{\mp 1}, \quad (37c)
\end{aligned}$$

where $\mathbb{U}^{\pm 1} \equiv \mathbb{U}$ is the unitary object under $SU(4)$, and we have used eq. (5) with $N = 4$, eq. (26), eq. (29b) and set $d\theta^E = \Lambda^{-1}d\Lambda$.

Consequently, 6 of the 10 generators of Poincaré symmetry can be implemented by gauge/local Lorentz transformations via the vector component of the object, \mathbb{F} in eq. (30a). However, to capture the remaining Poincaré symmetry generators (Zee, 2010) (translations), we can simply take a linear combination of the Killing vectors and set it equal to the group scalar gauge field,

$$A^\mu = \sum_{\bar{a}=0}^3 c_{\bar{a}} \xi_{\bar{a}}^\mu, \quad (38a)$$

where $c_{\bar{a}}$ are constants with dimensions of mass and $\xi_{\bar{a}}^\mu$ are the 4 generators of space-time translations on the tangent space. This suggests that the $SU(N)$ scalar current in eq. (30) can be written as,

$$J_\mu = 2R_{\mu\nu}A^\nu. \quad (38b)$$

This result is motivated by $[\nabla_\mu, \nabla_\nu]\xi_{\sigma\bar{a}} = R^\rho_{\sigma\mu\nu}\xi_{\rho\bar{a}}$ and $\nabla_\nu\xi_{\bar{a}\mu} = -\nabla_\mu\xi_{\bar{a}\nu}$, which guarantees that the scalar current is conserved.

Finally, it will be convenient initially to work with $SU(N)$ instead of $SU(4)$, which provides a generalized framework for our approach. Comparing $\mathbb{K}_{\mu\nu}$ from eq. (8c) with $\mathbb{F}_{\mu\nu}$ from eq. (30a), it is evident that, even though $\mathbb{F}_{\mu\nu}$ is also Hermitian under the $SU(N)$ matrix indices, unlike $\mathbb{K}_{\mu\nu}$, it is not invariant under complex conjugation followed by transpose of the space-time indices. We shall refer to $\mathbb{K}_{\mu\nu}$ as complex-Hermitian since it exhibits both levels of hermiticity, while $\mathbb{F}_{\mu\nu}$ does not.

B. Equations of motion

1. Constraint 1

In the preceding work, we exploit the complex-Hermiticity of $\mathbb{K}_{\mu\nu}$ in order to introduce two constraints analogous to eq. (30). These constraints are essential to

obtaining an effective action for general relativity, with desirable topological features for quantum gravity.

We shall use our notation to introduce the gauge co-variant constraint,

$$\mathbf{D}_\mu \mathbb{K}^\mu_\nu = \Psi(\mathbb{D}_\nu \Psi)^\dagger, \quad (39a)$$

$$\Psi = \sqrt{R} \exp(i\mathbf{S}), \quad \Psi^\dagger = \sqrt{R} \exp(-i\mathbf{S}), \quad (39b)$$

where $R = g_{\mu\nu}R^{\mu\nu}$ is the Ricci scalar, $\mathbb{K}_{\mu\nu}$ and \mathbb{D}_μ are given in eq. (8) and eq. (8c) respectively, and,

$$\mathbf{S} = - \int \mathbf{p}_\mu dx^\mu + \int \mathbf{A}_\mu dx^\mu, \quad (39c)$$

is an action that transforms as a vector under $SU(N)$, where,

$$\mathbf{p}^\mu = \kappa \mathbf{u}^\mu, \quad (40a)$$

$$\mathbf{u}^\mu = \frac{dx^\mu(x^\nu)}{d\tau}, \quad (40b)$$

with \mathbf{p}^μ the $n = 4$ -momenta of N particles, tracked by a central space-time coordinate $x^\nu(\tau)$, τ is the proper time and κ is a mass parameter. We shall also choose the following normalization condition on the $n = 4$ -momenta and $n = 4$ -velocities,

$$\text{Tr}(\mathbf{p}^\mu \mathbf{p}_\mu) = \frac{N}{4} \vec{\mathbf{p}}_\mu \cdot \vec{\mathbf{p}}^\mu \equiv \lambda, \quad (41)$$

where λ is a parameter with dimensions of (mass)² to be determined. The central coordinate defines the center of mass $n = 4$ -velocity,

$$w^\mu = \frac{dx^\mu(\tau)}{d\tau}, \quad (42a)$$

$$u^\mu u_\mu = -1 \quad (42b)$$

which transforms as a scalar under $SU(N)$. Using eq. (7) and separating the real and imaginary parts of eq. (39) yields, respectively, the Bianchi identity,

$$\nabla_\mu R^\mu_\nu = \frac{1}{2} \nabla_\nu R, \quad (43)$$

and the equations of motion for the gauge field,

$$\mathbf{D}_\mu \mathbf{F}^\mu_\nu = 2\pi \mathbf{J}_\nu, \quad (44a)$$

where,

$$\mathbf{J}_\nu = \frac{1}{4\pi} i(\Psi^* \mathbb{D}_\nu \Psi - \Psi \mathbb{D}_\nu \Psi^*) = -\frac{R}{2\pi} \mathbf{p}_\nu, \quad (44b)$$

is the $SU(N)$ current density. Thus, we arrive at familiar expressions in general relativity and Yang-Mills theory.

For completeness, we can couple the Yang-Mills field strength, $\mathbf{F}_{\mu\nu}$ to a Dirac spinor, ψ with N components using the semi-classical EFE,

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi G T_{\mu\nu}, \quad (45a)$$

$$T_{\mu\nu} = \left(\langle T_{\text{Dirac}}^{\mu\nu} \rangle + T_{\text{SU}(N)}^{\mu\nu} \right), \quad (45b)$$

where,

$$T_{\text{YM}}^{\mu\nu} = \frac{2}{N\pi} \text{Tr} \left(\mathbf{F}^{\mu\alpha} \mathbf{F}^\nu_{\alpha} - \frac{1}{4} \mathbf{F}^{\alpha\beta} \mathbf{F}_{\alpha\beta} g^{\mu\nu} \right), \quad (46a)$$

is the Yang-Mills energy-momentum tensor and,

$$T_{\text{Dirac}}^{\mu\nu} = \frac{1}{4i} (\bar{\psi} \gamma^\mu \mathbb{D}^\nu \psi + \bar{\psi} \gamma^\nu \mathbb{D}^\mu \psi) - \frac{1}{4i} ((\mathbb{D}^{\dagger\mu} \bar{\psi}) \gamma^\nu \psi + (\mathbb{D}^{\dagger\nu} \bar{\psi}) \gamma^\mu \psi), \quad (46b)$$

is the energy-momentum tensor of the Dirac equations,

$$i\gamma^\mu (\mathbb{D}_\mu \psi) = M\psi, \quad (47a)$$

$$i(\mathbb{D}_\mu^\dagger \bar{\psi}) \gamma^\mu = -M\bar{\psi}, \quad (47b)$$

where M is the Dirac mass, $\mathbb{D}_\mu \psi = (\nabla_\mu - i\mathbf{A}_\mu) \psi$, $\mathbb{D}_\mu^\dagger \bar{\psi} = \nabla_\mu \bar{\psi} + i\bar{\psi} \mathbf{A}_\mu$, $\nabla_\mu \psi = (\partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b) \psi$ and $\nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \bar{\psi} (\frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b)$. The quantum average, $\langle \dots \rangle$ is taken only over the Dirac terms in a semi-classical sense, since fermionic field equations do not have a valid classical description due to the Pauli-exclusion principle. By contrast, bosonic field equations represent the equations of motion for a large number of bosons occupying the same quantum state, which avails a valid classical description.

Consequently, we can define the velocities, \mathbf{u}_ν for the Dirac degrees of freedom in the following manner,

$$\mathbf{u}_\nu = \frac{\langle \bar{\psi} \gamma_\nu \mathbf{t} \psi \rangle}{\langle \bar{\psi} \psi \rangle}. \quad (48)$$

and proceed to calculate the trace of eq. (45) using eq. (47) to yield,

$$R = -8\pi G M \langle \bar{\psi} \psi \rangle. \quad (49)$$

Consequently, using eq. (43) to ensure that the right-hand side of eq. (45) is divergence-free, we reproduce eq. (44) with,

$$\mathbf{J}_\nu = \langle \bar{\psi} \gamma_\nu \mathbf{t} \psi \rangle. \quad (50a)$$

Plugging in the results in eq. (49) into eq. (44), we find,

$$\mathbf{J}_\nu = 4GM\kappa \langle \bar{\psi} \psi \rangle \mathbf{u}_\nu. \quad (50b)$$

Notably, for eq. (50a) to correspond to eq. (50b), we use eq. (48), which requires that,

$$\kappa = \frac{1}{4GM}, \quad (51)$$

and thus motivates identifying κ as the surface gravity (Wald, 1984), provided the Dirac mass, M doubles as the central mass, M in the Schwarzschild metric (for the the non-rotating black-hole solution of eq. (45) with $T_{\mu\nu} \rightarrow 0$).

2. Defining the central mass

However, general relativity does not admit a universal definition of mass in arbitrary space-times. (Thorne *et al.*, 2000) Typically, one can exploit time translation symmetric (stationary) asymptotically flat space-times such as the Schwarzschild metric to define the mass, M at spatial infinity, where the curvature vanishes, since such a space-time admits a time-like Killing vector, ξ_0^μ as a generator of time-translations, $\xi_0^\mu \partial / \partial x^\mu = \partial / \partial t$. This introduces the notion of conserved energy, that can appropriately be compared to expected notions in special relativity. Moreover, in addition to ξ_0^μ , stationary axisymmetric asymptotic flat space-times such as the Kerr metric admit an additional unique rotational Killing vector, ξ_4^μ as a generator of rotations $\xi_4^\mu \partial / \partial x^\mu = \partial / \partial \theta$, whose orbits comprise closed curves along the azimuthal angle, $0 \leq \theta \leq 2\pi$ on the 2-surface of an oblate spheroid.

By definition, the Killing vectors obey the relations (Bardeen *et al.*, 1973),

$$\nabla_\mu \xi_{\nu\bar{0}} + \nabla_\nu \xi_{\mu\bar{0}} = 0, \quad (52a)$$

$$\nabla_\mu \xi_{\nu\bar{4}} + \nabla_\nu \xi_{\mu\bar{4}} = 0, \quad (52b)$$

$$\nabla_\mu \nabla_\nu \xi_0^\mu = R_{\mu\nu} \xi_0^\nu, \quad (52c)$$

$$\nabla_\mu \nabla_\nu \xi_4^\mu = R_{\mu\nu} \xi_4^\nu, \quad (52d)$$

where,

$$\xi_0^\mu \nabla_\mu \xi_4^\nu = \xi_4^\mu \nabla_\mu \xi_0^\nu. \quad (53)$$

Thus, a linear combination of the Killing vectors, $\xi^\mu = \xi_0^\mu + \Omega_h \xi_4^\mu$, corresponding to the null vector tangent to the generators at the horizon defined by $\xi^\mu \xi_\mu = 0$, also satisfies eq. (52),

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \quad (54a)$$

$$\nabla_\mu \nabla_\nu \xi^\mu = R_{\mu\nu} \xi^\mu, \quad (54b)$$

and the parameter, Ω_h is the angular frequency constant on the horizon and satisfies,

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\mu, \quad (55)$$

where κ is the surface gravity we seek to define in terms of the mass M , in accordance to eq. (51). Since $\nabla_\mu \xi_{\nu\bar{0}} = -\nabla_\nu \xi_{\mu\bar{0}}$ is anti-symmetric, we can use Stokes' theorem to transfer the integral of the asymptotically flat space-like hypersurface, $S = \mathcal{V}$ (tangent to the rotation Killing vector, ξ_4^μ and intersecting the horizon at a 2-surface $\partial B = \mathcal{A}$), to the boundary, ∂S of S (consisting of ∂B and a 2-surface, ∂S_∞ at spatial infinity),

$$\begin{aligned} & \int_{\partial S_\infty} d\Sigma_{\mu\nu} \nabla^\mu \xi_0^\nu + \int_{\partial B} d\Sigma_{\mu\nu} \nabla^\mu \xi_0^\nu \\ &= \int_{\partial S} d\Sigma_{\mu\nu} \nabla^\mu \xi_0^\nu = - \int_S d\Sigma_\nu \xi_{\mu\bar{0}} R^{\mu\nu}, \end{aligned} \quad (56)$$

where $d\Sigma_{\mu\nu} = \frac{1}{2}(n_\mu\xi_\nu - n_\nu\xi_\mu)d\mathcal{A}$ and $d\Sigma_\mu$ are the surface elements of ∂S and S respectively, while $d\mathcal{A}$ is the area element of the horizon, $d\mathcal{V} = d^3x\sqrt{-\det(g_{\mu\nu})}$ is the volume element of S and n_ν is the other null vector orthogonal to ∂B , normalized as $n^\mu\xi_\mu = -1$. Thus, we define the total mass as measured from infinity (corresponding to the central mass) as,

$$M = \frac{1}{4\pi G} \int_{\partial S_\infty} d\Sigma_{\mu\nu} \nabla^\mu \xi^\nu. \quad (57a)$$

In addition, the integral over ∂B yields,

$$\begin{aligned} \int_{\partial B} d\Sigma_{\mu\nu} \nabla^\mu \xi^\nu &= - \int_{\partial B} \xi_\mu \nabla^\mu \xi^\nu n_\nu d\mathcal{A} \\ &= - \int_{\partial B} \xi^\nu n_\nu \kappa d\mathcal{A} = - \int_{\partial B=\mathcal{A}} \kappa d\mathcal{A} = -\kappa\mathcal{A}, \end{aligned} \quad (57b)$$

where we have used eq. (55) and the fact that κ is constant over the horizon. (Bardeen *et al.*, 1973) Likewise, the total angular momentum as measured from infinity and the angular momentum at the horizon defined in a similar manner as the mass,

$$L = \frac{1}{8\pi G} \int_{\partial S_\infty} d\Sigma_{\mu\nu} \nabla^\mu \xi_4^\nu, \quad (58a)$$

$$L_h = \frac{1}{8\pi G} \int_{\partial B} d\Sigma_{\mu\nu} \nabla^\mu \xi_4^\nu, \quad (58b)$$

where Stokes' theorem, using eq. (54), yields,

$$\begin{aligned} \int_{\partial S_\infty} d\Sigma_{\mu\nu} \nabla^\mu \xi_4^\nu + \int_{\partial B} d\Sigma_{\mu\nu} \nabla^\mu \xi_4^\nu \\ = \int_{\partial S} d\Sigma_{\mu\nu} \nabla^\mu \xi_4^\nu = - \int_S d\Sigma_\nu \xi_{\mu\bar{4}} R^{\mu\nu}. \end{aligned} \quad (59)$$

Thus, since the EFE in eq. (45) can be transformed into in $n = 4$ dimensions,

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\alpha\beta} T^{\alpha\beta} g_{\mu\nu} \right), \quad (60)$$

we can plug in the results from eq. (57) and eq. (58) into eq. (56) to yield,

$$M = \frac{\kappa\mathcal{A}}{4\pi G} + 2\Omega_h L_h + 2E, \quad (61a)$$

where the energy, E is given by,

$$E = \int_{S=\mathcal{V}} d\Sigma_\mu \xi_{\nu\bar{0}} (T^{\mu\nu} - \frac{1}{2} g_{\alpha\beta} T^{\alpha\beta} g^{\mu\nu}), \quad (61b)$$

where \mathcal{V} is an $n = 3$ dimensional space-like hyper-surface.

For instance, it follows that, for the Kerr-Newmann solution ($T^{\mu\nu} = F^{\mu\alpha} F_\alpha^\nu - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the U(1) field strength with an appropriate choice of gauge for A_μ), the energy becomes

$$E = \sum_{i \in e, m} q_i \Phi_i, \quad (62a)$$

where Φ_e and Φ_m are the electric and magnetic potentials associated with the electric q_e and q_m monopole charges respectively. Consequently, the exact solutions for the quantities in eq. (61) are given by (Hawking, 1976a),

$$\kappa = \frac{4\pi}{\mathcal{A}} (r_+ - r_S/2), \quad \Phi_i = \frac{4\pi}{\mathcal{A}} q_i r_+, \quad \Omega_h = \frac{4\pi}{\mathcal{A}} a, \quad (62b)$$

where $r_\pm = r_S/2 \pm \sqrt{(r_S/2)^2 - a^2 - G \sum_i q_i^2}$ is the radius of the inner(-) or outer(+) horizon, $r_S = 2GM$ is the Schwarzschild radius, $\mathcal{A} = 4\pi r_+ r_S$ and $a = L_h/M$. Evidently, the surface gravity, κ is an elaborate function of M and L_h as well as q_i , and hence does not correspond to eq. (51) unless the solution is Schwarzschild ($L \rightarrow 0$ and $Q \rightarrow 0$). This implies that our approach will be a good approximation to real systems as the limits, $r_+ \rightarrow r_S$ and $q_i \rightarrow 0$ are satisfied, which correspond to the limits $\xi^\mu \rightarrow \xi_0^\mu$ and $T^{\mu\nu} \rightarrow 0$, where the energy-momentum tensor is given by eq. (45).

3. Constraint 2

We shall consider a second constraint,

$$\mathbf{D}_\nu \mathbf{D}_\mu \mathbb{K}^{\mu\nu} = 0, \quad (63)$$

whose imaginary and real parts respectively correspond to,

$$\mathbf{D}_\mu \mathbf{J}^\mu = 0, \quad (64a)$$

$$\nabla_\mu \nabla^\mu R = 0. \quad (64b)$$

The first expression in eq. (64a) is guaranteed by the SU(N) symmetry. However, the second expression is novel, since complex-Hermitian objects such as $\mathbb{K}_{\mu\nu}$ do not traditionally appear in the formulation of Einstein's general relativity.

It is prudent to define a conserved current density,

$$J_\nu = \frac{1}{2\pi} \nabla_\nu R, \quad (65)$$

which transforms as a scalar under SU(N) gauge group. For the purposes herein, we shall obtain the scalar current density using,

$$R = R_c \exp(-\Phi), \quad (66a)$$

$$\Phi = \int p_\mu dx^\mu, \quad (66b)$$

$$p_\mu = \bar{\beta}^{-1} u_\mu, \quad (66c)$$

where R_c is a constant and $\bar{\beta} = \beta/2\pi$ is a reduced inverse temperature to be defined. Using the Killing vector ξ^μ , we can exploit the Killing relations in eq. (54) by taking the trace to find, $\nabla_\nu \xi^\nu = 0$ and setting,

$$J_\nu = J_c \xi_\nu = -\frac{R}{2\pi} p_\nu, \quad (67)$$

in order to guarantee that the current is conserved, $\nabla_\nu J^\nu = 0$. When there is more than one space-time isometry, one should take a linear combination of the Killing vectors (such as in eq. (38a)), in order to construct a suitable ξ^μ . This ensures that the constraint in eq. (63) represents a combined space-time and $SU(N)$ gauge symmetry, where the conserved charge is a complex object given by,

$$\mathbb{Q} = \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \text{Tr}(\mathbb{J}^0), \quad (68)$$

related to this combined symmetry. Here, \mathcal{V} is the $n = 3$ (dimensional) space-like hyper-surface corresponding to spatial volume. Thus, $\mathbb{J}_\mu = \frac{1}{2\pi} \Psi(\mathbb{D}_\mu \Psi)^\dagger = J_\mu + i\mathbf{J}_\mu$ is the conserved current. Nonetheless, the trace, Tr in \mathbb{Q} ensures that $\mathbf{Q} = 0$ vanishes by virtue of the $SU(N)$ matrices being trace-less. This reduces the number of finite charges to unity, where $\mathbb{Q} = Q + i0$ is purely a scalar under $SU(N)$. Since Q is the only finite charge, it is reasonable to identify $Q = N$ with the $SU(N)$ colors, which corresponds to the scalar charge associated with the Killing vectors, ξ^μ . Moreover, when $N \rightarrow 1$, we obtain a trivial $U(1)$ gauge symmetry with the identity replacing the trace-less matrices, and hence \mathbf{Q} remains finite satisfying $\mathbf{Q} \rightarrow 2Q$. For $U(1)$, this suggests eq. (39) and eq. (63) respectively reduce to (Kanyolo and Masese, 2021),

$$\nabla_\mu K^\mu_\nu = \Psi(D_\nu \Psi)^\dagger, \quad (69a)$$

$$\Psi = \sqrt{R} \exp(iS), \quad \Psi^\dagger = \sqrt{R} \exp(-iS), \quad (69b)$$

$$\nabla_\mu \nabla_\nu K^{\mu\nu} = 0, \quad (69c)$$

where,

$$D_\mu = \nabla_\mu - iA_\mu, \quad (70a)$$

$$K_{\mu\nu} = R_{\mu\nu} - iF_{\mu\nu}, \quad (70b)$$

$$S = - \int p_\mu dx^\mu + \int A_\mu dx^\mu. \quad (70c)$$

Proceeding, we employ the definitions in eq. (65) and eq. (67) to discover that the $n = 4$ -velocity is related to the Killing vector by,

$$u^\nu = - \exp(\Phi) \xi^\nu, \quad (71)$$

where $J_c = R_c/\bar{\beta}$ defines relates the constants. Applying eq. (71), we obtain equations of motion for the center of mass coordinate,

$$\begin{aligned} u^\mu \nabla_\mu u_\nu &= \exp(\Phi) \xi^\mu \nabla_\mu (\exp(\Phi) \xi_\nu) \\ &= \exp(2\Phi) \xi^\mu \nabla_\mu \xi_\nu + \exp(\Phi) \xi_\nu \xi^\mu \nabla_\mu \exp(\Phi) \\ &= -\frac{1}{2} \exp(2\Phi) \nabla_\nu (\xi^\mu \xi_\mu) \\ &= \frac{1}{2} \exp(2\Phi) \nabla_\nu \exp(-2\Phi) = -\nabla_\nu \Phi, \end{aligned}$$

where we have used eq. (71), eq. (42) and,

$$\begin{aligned} R^{\mu\nu} \nabla_\mu \xi_\nu &= 0, \\ \nabla_\mu \xi_\nu &= -\nabla_\nu \xi_\mu, \\ \frac{1}{2} \xi^\mu \nabla_\mu R &= \nabla^\nu (R_{\mu\nu} \xi^\mu) = \nabla^\nu \nabla_\mu \nabla_\nu \xi^\mu = 0, \end{aligned}$$

from eq. (54), which guarantee that $\xi^\mu \nabla_\mu \Phi = 0$. Thus, we obtain the equation of motion,

$$u^\mu \nabla_\mu u_\nu = -\bar{\beta}^{-1} u_\nu + \eta_\nu, \quad (72a)$$

which takes the form of a Langevin equation (Lemons and Gythiel, 1997), where β plays the role of the mean free path between collisions and η_ν the random acceleration (which is nil in eq. (72a)). Moreover, since,

$$\begin{aligned} \bar{\beta}^{-1} \exp(-\Phi) \xi_\mu &= -\bar{\beta}^{-1} \exp(-2\Phi) u_\mu \\ &= -\exp(-2\Phi) \nabla_\mu \Phi = \frac{1}{2} \nabla_\mu (\exp(-2\Phi)) \\ &= -\frac{1}{2} \nabla_\mu (\xi^\nu \xi_\nu) = -\xi^\nu \nabla_\mu \xi_\nu = \xi^\nu \nabla_\nu \xi_\mu, \end{aligned} \quad (72b)$$

we obtain eq. (55), where $\kappa = \bar{\beta}^{-1} \exp(-\Phi)$ indeed is the surface gravity. When $\xi^\mu \rightarrow \xi_0^\mu = (-1, \vec{0})$, the Killing vector is time-like. Assuming a diagonalized metric tensor, and using the fact that Φ is real-valued, we obtain $u^0 = \exp(\Phi) \geq 0$ and $-\exp(-2\Phi) = \exp(-2\Phi) u^\mu u_\mu = \xi^\mu \xi_\mu = \xi^\mu \xi^\nu g_{\mu\nu} = g_{00} \leq 0$, which means that,

$$\beta^{-1} = \frac{\kappa}{2\pi \sqrt{|g_{00}|}} \geq 0, \quad (73a)$$

is the Tolman relation (Tolman, 1930; Tolman and Ehrenfest, 1930), where κ is given in eq. (51), defined at the Killing horizon, $\xi^\mu \xi_\mu = 0$. In order to treat Φ as the Newtonian potential, we shall assume it is small, $\Phi \ll 1$ such that, $g_{00} \simeq 1 - 2\Phi$, implying $u^\mu \rightarrow \xi^\mu$, which also lead to,

$$\beta^{-1} \simeq \frac{\kappa}{2\pi} = \frac{1}{8\pi GM}, \quad (73b)$$

as expected. (Bekenstein, 2008; Hawking, 1976a; Zee, 2010) Consequently, we can introduce the complex object, $\mathbb{S} = S/2 + i\mathbf{S}$, where $S = \kappa \int d\tau$ in order for $\Psi = \sqrt{\rho_c} \exp(\mathbb{S})$ and ρ_c is a constant.

Finally, using eq. (73b), eq. (65) and the trace of EFE given in eq. (49), we find,

$$J^\mu = \langle \bar{\psi} \psi \rangle u^\mu I_N. \quad (74)$$

In the same spirit as eq. (48), we can define the $n = 4$ -velocity as,

$$u_\nu = \frac{\langle \bar{\psi} \gamma_\nu \psi \rangle}{\langle \bar{\psi} \psi \rangle} I_N, \quad (75)$$

where, u_ν is a scalar under $SU(N)$ and hence an $N \times N$ matrix. Thus, we proceed to choose the normalization,

$$\begin{aligned} & \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} J^0(\vec{x}) \\ &= \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \langle \bar{\psi}(\vec{x}) \psi(\vec{x}) \rangle u^0(\vec{x}) I_N \\ &= \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \langle \bar{\psi}(\vec{x}) \gamma^0(\vec{x}) \psi(\vec{x}) \rangle I_N \\ &= \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \langle \psi^\dagger(\vec{x}) \psi(\vec{x}) \rangle I_N = I_N, \end{aligned} \quad (76a)$$

where we have used $(\gamma^0)^{-1} \gamma^0 = 1$, in order for $J^0(\vec{x}) = \langle \psi^\dagger \psi \rangle$ to have the proper normalization for a probability density. Thus, the scalar charge, Q from eq. (68) becomes,

$$Q = \text{Tr}(I_N) = N, \quad (76b)$$

as expected. Likewise, we can employ eq. (76a) to obtain the vector charges, \mathbf{Q} using,

$$\begin{aligned} & \int_{\mathcal{M}} d^4x \sqrt{-\det(g_{\mu\nu})} \mathbf{J}^\mu \\ &= \int d\tau \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \mathbf{J}^\mu \frac{dt}{d\tau} \\ &= \int d\tau \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \mathbf{J}^\mu u^0(\vec{x}) \\ &= \int d\tau \mathbf{u}^\mu(\tau) \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} J^0(\vec{x}) \\ &= \int d\tau \mathbf{u}^\mu(\tau) = \int d\tau \frac{d\mathbf{x}^\mu(\tau)}{d\tau} = \Delta \mathbf{x}^\mu(\tau), \end{aligned} \quad (77a)$$

where, $\mathcal{M} = (t, \mathcal{V})$ is the 1,3 dimensional pseudo-Riemannian manifold. Here, we have applied eq. (48) and eq. (50b), and parametrized the vector current as, $\mathbf{J}^\mu(\tau, \vec{x}) = \langle \bar{\psi}(\vec{x}) \psi(\vec{x}) \rangle \mathbf{u}^\mu(\tau)$. Note that, $\Delta \mathbf{x}^\mu(\tau) = \mathbf{x}^\mu(\tau) - \mathbf{x}^\mu(0)$ is the displacement with $\mathbf{x}^\mu(0)$ the integration constant. Thus, eq. (77a) implies that,

$$\text{Tr} \left(\frac{d\mathbf{x}^0}{dt} \right) = \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \text{Tr}(\mathbf{J}^0) = 0, \quad (77b)$$

and thus we shall have,

$$\mathbf{Q} = 0, \quad (77c)$$

by virtue of the $SU(N)$ matrices being trace-less, $\text{Tr}(\mathbf{t}) = 0$. Consequently, when $N \rightarrow 1$ *i.e.* ($SU(N) \rightarrow U(1)$) in eq. (76b), the trace-less matrices can be replaced by twice the identity matrix, $2I_N$ implying that $\mathbf{x}^0 \rightarrow 2t$, in order for $\mathbf{Q} \rightarrow 2Q$ as required. An approach, considering only this limit, has been employed in a preceding paper (Kanyolo and Masese, 2021), which provides the necessary framework to reproduce asymptotic behavior of galaxy rotation curves, within the context of dark matter (Section V).

C. Lagrangian density

To ground the equations of motion on a firmer footing, we provide a suitable action principle for our formalism. We consider the following action on the space-time manifold, \mathcal{M} given by,

$$\mathcal{I}_{\mathcal{M}} = \int_{\mathcal{M}} d^4x \sqrt{-\det(g_{\mu\nu})} \sum_{j=1}^3 \text{Tr}(\mathcal{L}_j), \quad (78a)$$

$$\mathcal{L}_1 = i\bar{\psi} \gamma^\mu \mathbb{D}_\mu \psi - M \bar{\psi} \psi, \quad (78b)$$

$$\mathcal{L}_2 = \frac{1}{2\pi} \mathbb{K}_{\mu\nu} (\mathbb{K}^{\mu\nu})^\dagger, \quad (78c)$$

$$\mathcal{L}_3 = \frac{1}{2\pi} (\mathbb{D}_\mu \Psi) (\mathbb{D}^\mu \Psi)^\dagger + \frac{\lambda}{\pi N} |\Psi|^2 - \frac{1}{8\pi} |\Psi|^4, \quad (78d)$$

where λ satisfies eq. (41), and $\mathbb{K}_{\mu\nu}$ and Ψ are given in eq. (8) and eq. (39) respectively. Note that the Lagrangian densities are $N \times N$ identity matrices, which necessitates the trace, Tr to be appear in the Lagrangian.

Proceeding to set $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$, we derive the constraints in eq. (39) and eq. (63) respectively by,

$$\mathbf{D}_\nu \frac{\delta \mathcal{L}}{\delta \mathbb{K}_{\mu\nu}} = \frac{\delta \mathcal{L}}{\delta \mathbb{D}_\mu}, \quad \mathbf{D}_\nu \frac{\delta \mathcal{L}}{\delta (\mathbb{K}_{\mu\nu})^\dagger} = \frac{\delta \mathcal{L}}{\delta (\mathbb{D}_\mu)^\dagger}, \quad (79a)$$

$$\mathbf{D}_\mu \mathbf{D}_\nu \frac{\delta \mathcal{L}}{\delta \mathbb{K}_{\mu\nu}} = 0, \quad \mathbf{D}_\mu \mathbf{D}_\nu \frac{\delta \mathcal{L}}{\delta (\mathbb{K}_{\mu\nu})^\dagger} = 0, \quad (79b)$$

where $\mathbb{K}_{\mu\nu}$, $(\mathbb{K}_{\mu\nu})^\dagger$ and $\mathbb{D}_\mu = \nabla_\mu - i\mathbf{A}_\mu$, $\mathbb{D}_\mu^\dagger = \nabla_\mu + i\mathbf{A}_\mu$ are treated as independent fields. Moreover, recall that the trace, Tr in the action, $\mathcal{I}_{\mathcal{M}}$ will introduce a multiplicative factor of N to all $SU(N)$ scalars in the action. Thus, plugging in eq. (8) into \mathcal{L}_2 and using $\mathbf{F}_{\mu\nu} R^{\mu\nu} = 0$, we can take the trace, $\text{Tr}(\mathcal{L}_2)$ to find,

$$\frac{N}{2\pi} R_{\mu\nu} R^{\mu\nu} - \frac{N}{8\pi} \vec{\mathbf{F}}_{\mu\nu} \cdot \vec{\mathbf{F}}^{\mu\nu}. \quad (80a)$$

Consequently, this corresponds to the Lagrangian density of the gravity and gauge fields, $g_{\mu\nu}$ and \mathbf{A}_μ respectively. However, currently, eq. (78) does not yet resemble the Einstein-Hilbert action we should obtain in order to be consistent with Einstein's general relativity.

Moreover, the third Lagrangian density, $\text{Tr}(\mathcal{L}_3)$ takes the Ginzburg-Landau form for a Bose-Einstein condensate of N colors of mass $\sqrt{2\lambda/\pi N}$, whose momenta are given in eq. (40a). Substituting $\Psi = \sqrt{R} \exp(i\mathbf{S})$ from eq. (39) into eq. (78) and taking the trace, $\text{Tr}(\mathcal{L}_3)$ we obtain,

$$\frac{1}{2\pi} \text{Tr}(|\mathbb{D}_\mu \Psi|^2) + \frac{\lambda}{\pi} R - \frac{N}{8\pi} R^2, \quad (80b)$$

where R is the Ricci scalar. Thus, for eq. (78) to reproduce the Einstein-Hilbert action as expected (neglecting the possible leading order terms), we ought to set the coupling constant, λ to,

$$\lambda \equiv \frac{1}{16G} = \left(\frac{m_P}{4} \right)^2, \quad (81)$$

where $m_{\text{P}} = 1/\sqrt{G}$ is the Planck mass. This suggests that eq. (40a) describes the energy-momentum relation of Planckian particles. Moreover, we can perform the symmetry breaking, $\text{SU}(N) \rightarrow \text{SU}(4) \rightarrow \text{SO}(4) \rightarrow \text{SO}(1,3)$, in accordance with eq. (35) and eq. (36), by setting $(\mathbf{R}_{\mu\nu})_{\bar{b}}^{\bar{a}} = (\mathbf{F}_{\mu\nu})_{\bar{b}}^{\bar{a}}$, thus obtaining $\vec{\mathbf{F}}_{\mu\nu} \cdot \vec{\mathbf{F}}^{\mu\nu} = (\mathbf{F}_{\mu\nu})_{\bar{b}}^{\bar{a}}(\mathbf{F}_{\mu\nu})_{\bar{a}}^{\bar{b}} = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$, where $R_{\mu\nu\sigma\rho}$ is the Riemann tensor. In particular, since the tangent space must have the same dimensions ($n = 4$) as the manifold, \mathcal{M} , the symmetry breaking (subsequently employed in eq. (80a) to arrive at eq. (83)), can be viewed as a Cayley-Dickson (de-)construction of $\mathbb{K}_{\mu\nu}$ from $\text{SU}(N)$ to $\text{SU}(4)$, in a similar fashion to eq. (10b), setting the irrelevant terms to nil and a subsequent Wick rotation to introduce the Lorenzian signature.

Consequently, it is now evident that the coefficients in eq. (78) and the normalization of the $\text{SU}(N)$ matrices in eq. (5) were *a priori* chosen in order to obtain the factors, $N/8\pi$, $N/2\pi$ and $N/8\pi$ as the coefficients of the terms $-\vec{\mathbf{F}}_{\mu\nu} \cdot \vec{\mathbf{F}}^{\mu\nu}$, $R^{\mu\nu}R_{\mu\nu}$ and $-R^2$ respectively, where the former two arise from the $\mathbb{K}_{\mu\nu}(\mathbb{K}^{\mu\nu})^\dagger$ term and the latter the $|\Psi|^4$ term in the Lagrangian density. In turn, this ensures that the resultant corrections to the Einstein-Hilbert action (arising from the $|\Psi|^2$ term) correspond to the Gauss-Bonnet term (Lovelock, 1971),

$$\mathcal{L}_{\text{GB}} = -\frac{1}{8\pi}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}), \quad (82a)$$

which is topological in nature (a total derivative). From eq. (23a),

$$d\text{CS} = -d^4x \sqrt{-\det(g_{\mu\nu})} \text{Tr}(\mathcal{L}_{\text{GB}}). \quad (82b)$$

Consequently, we have,

$$\begin{aligned} \chi_4(\mathcal{M}) &= \frac{1}{8\pi^2} \int_{\mathcal{M}} \text{Tr}(\mathbf{R} \wedge \mathbf{R}) = \frac{1}{\pi N} \int_{\mathcal{M}} d\text{CS} \\ &= \frac{1}{\pi N} \int_{\partial\mathcal{M}_+ = \mathcal{B}} \text{CS}^+ + \frac{1}{\pi N} \int_{\partial\mathcal{M}_- = -\mathcal{B}} \text{CS}^- \\ &\quad - \frac{1}{\pi N} \int_{\mathcal{B}} \text{CS}^+ - \frac{1}{\pi N} \int_{\mathcal{B}} \text{CS}^- \\ &= \frac{1}{24\pi^2} \int_{\mathcal{B}} d\text{Tr}(\mathbf{\Lambda}^{-1}d\mathbf{\Lambda}) = \pm 2\nu, \end{aligned} \quad (82c)$$

where, we have used eq. (21) and eq. (27). Thus, χ_4 is the Euler characteristic in $n = 4$ dimensions, and hence does not contribute to the equations of motion. Finally, collecting all the terms, we have,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \text{Tr}(|\mathbb{D}_\mu \Psi|^2) + i \langle \bar{\psi} \gamma^\mu \mathbb{D}_\mu \psi \rangle \\ &\quad - M \langle \bar{\psi} \psi \rangle + \frac{\lambda}{\pi} R + \text{Tr}(\mathcal{L}_{\text{GB}}), \end{aligned} \quad (83)$$

where we have used the definition, $\text{Tr}(\mathcal{L}_1) \equiv \langle \mathcal{L}_1 \rangle$ for the Dirac field Lagrangian density. It is now clear that

varying this action with respect to the metric, $g_{\mu\nu}$ will reproduce eq. (45) where $T_{\text{SU}(N)}^{\mu\nu} \rightarrow 0$, provided we can find the conditions that allow the kinetic term of Ψ to identically vanish.

In $n = 4$ dimensions, the constraint in eq. (39) is invariant under $T_{\text{SU}(N)}^{\mu\nu} \rightarrow 0$. This arises from the fact that mass-less $\text{SU}(N)$ gauge field theories in $n = 4$ dimensions are conformal field theories, requiring that their energy-momentum tensors are mass-less and hence traceless. On the contrary, a finite mass, M_{gauge} introduces a preferred length scale comparable to the Compton wavelength, $\ell \sim 1/M_{\text{gauge}}$. (Francesco *et al.*, 2012) Moreover, in Ginzburg-Landau theories (Huebener, 2001), one would typically assume Ψ varies slowly, resulting in the sombrero potential dominating the free energy contribution over the kinetic term. Within this paradigm, we can introduce the fluctuations, $|\Psi|^2 - \langle |\Psi|^2 \rangle = \delta|\Psi|^2$ and $\mathbf{S} - \langle \mathbf{S} \rangle = \delta\mathbf{S}$, and define the Ricci scalar as the fluctuation, $R = \delta|\Psi|^2$ instead of $|\Psi|^2$, where the mean values of $|\Psi|^2$ and \mathbf{S} are kept independent of space-time coordinates ($\langle |\Psi|^2 \rangle / 2 = -\Lambda$ and $\langle \mathbf{S} \rangle = \mathbf{S}_c$). This does not alter the real and imaginary parts of eq. (39), since the Bianchi identity and Yang-Mills equations, given in eq. (43) and eq. (44) respectively, are invariant under the shift $R \rightarrow R - 2\Lambda$ and $\mathbf{S} \rightarrow \mathbf{S} - \mathbf{S}_c$. Thus, we can identify Λ as the cosmological constant. Consequently, the kinetic term in eq. (83) leads to gauge symmetry breaking by generating the cosmological constant and hence a mass term for the gauge field assuming $\Lambda \gg R$, and $\mathbf{S}_c \gg \delta\mathbf{S}$. However, since a mass term (spontaneous symmetry breaking) breaks the aforementioned conformal symmetry by generating an additional energy-momentum tensor proportional to $\frac{1}{2}M_{\text{gauge}}^2 \text{Tr}(\mathbf{A}^\mu \mathbf{A}^\nu)$ which is not traceless, this introduces a length scale $\ell \sim M_{\text{gauge}} = 1/\sqrt{\Lambda}$ in the gauge field theory. (Zee, 2010) On the other hand, in order to maintain conformal symmetry, we shall explore another approach of identically getting rid of the kinetic term of Ψ .

In particular, we exploit the second constraint given in eq. (63), which serendipitously avails an avenue for the kinetic term to identically vanish. To see this, we plug in $\Psi = \sqrt{R} \exp(i\mathbf{S})$, employ eq. (7), eq. (39c), eq. (67) and eq. (73b), and the normalization conditions in eq. (41) and eq. (42) to yield,

$$\frac{1}{2} \text{Tr}(|\mathbb{D}_\mu \Psi|^2) = \frac{R}{2} \left(\text{Tr}(\mathbf{p}^\mu \mathbf{p}_\mu) + \frac{\kappa^2 N}{4} u^\mu u_\mu \right), \quad (84a)$$

which identically vanishes when,

$$\pi N = \frac{\lambda}{\pi \bar{\kappa}^2} = 4\pi G M^2 = \frac{\beta M}{2} = \frac{4\pi r_{\text{S}}^2}{4l_{\text{P}}^2} = \mathcal{S}, \quad (84b)$$

which implies (Kanyolo and Masese, 2021),

$$\beta M = 2\pi N, \quad (85a)$$

where we have used eq. (51), eq. (81), the Schwarzschild radius, $r_S = 2GM$ and Planck length, $l_P = \sqrt{G} = 1/m_P$ to identify the entropy, \mathcal{S} . In fact, eq. (85a) is the already proposed black hole area quantization condition by several authors (Bekenstein and Mukhanov, 1998, 1995; Mukhanov, 1986; Vaz and Witten, 1999) since $N \in \mathbb{Z} \geq 0$ is the number of colors and hence must be discrete. Thus, to maintain conformal invariance, Killing horizon areas must be pixelated in units of $4\pi G$. Another curious observation is, eq. (85a) can be rearranged in order for the mass to take the form of bosonic, $N = m$ or fermionic, $N = (2m + 1)$ Matsubara frequencies (Abrikosov, 1975),

$$\omega_b = M = 2\pi N/\beta = 2\omega_f, \quad (85b)$$

where $m \in \mathbb{Z} \geq 0$ is a positive integer. This could have some deep significance for thermal Green functions not explored further in the present work.

Finally, plugging in eq. (84b) into eq. (83), the sum of the traces in eq. (78) can be transformed into an equivalent sum,

$$\sum_{j=1}^3 \text{Tr}(\mathcal{L}_j) = \mathcal{L}(\lambda) + \langle \mathcal{L}_1 \rangle + \text{Tr}(\mathcal{L}_{\text{GB}}), \quad (86a)$$

where, $\mathcal{L}(\lambda) = \lambda R/\pi$ corresponds to the Einstein-Hilbert Lagrangian density and,

$$\int_{\mathcal{M}} d^4x \sqrt{-\det(g_{\mu\nu})} \text{Tr}(\mathcal{L}_{\text{GB}}) = -\mathcal{S}\chi_4(\mathcal{M}), \quad (86b)$$

. is the Gauss-Bonnet term. (Lovelock, 1971) Thus, we have demonstrated that eq. (78) is a candidate action for the formalism earlier introduced, since it reproduces the Einstein-Hilbert action, *albeit* with a finite topological term. The significance of this topological term will subsequently be explored within the context of quantum gravity.

IV. RESULTS

A. Generating all mass terms

The path integral approach requires that the quantum gravity wave function (Gibbons and Hawking, 1977; Hamber, 2008; Hawking, 1978) be defined by,

$$\Psi_{\text{QG}} = \int \mathcal{D}[g_{\mu\nu}, \mathbf{A}_\mu, \bar{\psi}, \psi] \exp(i\mathcal{I}_{\mathcal{M}}), \quad (87)$$

where $\mathcal{I}_{\mathcal{M}}$ is given in eq. (86a). We shall first treat the gauge field, $(\mathbf{A}_\mu)_{\bar{b}}^a$ and the spin connection, $(\omega_\mu)_{\bar{b}}^a$ as different fields and first consider the path integral over \mathbf{A}_μ . This appears straight-forward to perform since only $\langle \mathcal{L}_1 \rangle$ in eq. (86a) depends on \mathbf{A} . It will be clear that we can effectively generate the $|\Psi|^4/8\pi$ term in eq. (78) by a

path integral with respect to \mathbf{A}^μ , where the Lagrangian initially has a term,

$$\frac{1}{2} \int_{\mathcal{M}} d^4y \sqrt{-\det(g_{\mu\nu})} \mathbf{A}_\mu(x) (G^{-1})^{\mu\nu} (x-y) \mathbf{A}_\nu(y),$$

with,

$$(G^{-1})^{\mu\nu} = \frac{\text{Tr}(\mathbf{p}^\alpha \mathbf{p}_\alpha)}{\pi N \sqrt{-\det(g_{\mu\nu})}} g^{\mu\nu} \delta^4(x-y), \quad (88a)$$

the inverse propagator, $\delta^4(x-y)$ the Dirac delta function normalized as $\int d^4x \delta^4(x-y) = 1$, $\text{Tr}(\mathbf{p}^\alpha \mathbf{p}_\alpha) = \lambda$ given in eq. (41) and gauge field mass corresponding to the reduced Planck mass, $\sqrt{2\lambda/\pi} = \bar{m}_P = 1/\sqrt{8\pi G}$ (eq. (81)). Consequently, the propagator, $G_{\mu\nu}(x-y)$ is given by,

$$G_{\mu\nu} = \frac{\pi N \sqrt{-\det(g_{\mu\nu})}}{\text{Tr}(\mathbf{p}^\alpha \mathbf{p}_\alpha)} g_{\mu\nu} \delta^4(x-y), \quad (88b)$$

which satisfies,

$$\int d^4z (G^{-1})^\mu_\alpha(x-z) G^\alpha_\nu(z-y) = \delta^4(x-y) \delta^\mu_\nu, \quad (88c)$$

where $\delta^\mu_\nu = g^{\mu\alpha} g_{\alpha\nu}$ is the Kronecker delta.

Consequently, performing the path integral, $\int \mathcal{D}[\mathbf{A}_\mu]$ obtains the action,

$$-\frac{1}{2} \int_{\mathcal{M}} d^4x \int_{\mathcal{M}} d^4y \text{Tr}(\mathbf{J}^\mu(x) \mathbf{J}^\nu(y)) G_{\mu\nu}(x-y). \quad (88d)$$

where the Dirac current, $\mathbf{J}^\mu = \langle \bar{\psi} \gamma^\mu \mathbf{t} \psi \rangle$ is the Dirac current in the term, $\langle \mathcal{L}_1 \rangle$. Moreover, plugging in the equivalent form of \mathbf{J}_μ from eq. (44b) into eq. (88d) obtains,

$$\begin{aligned} & -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{-\det(g_{\mu\nu})} \pi N \left(\frac{R}{2\pi} \right)^2 \\ & = -\frac{1}{8\pi} \int_{\mathcal{M}} d^4x \sqrt{-\det(g_{\mu\nu})} \text{Tr}(|\Psi|^4), \end{aligned} \quad (89)$$

as earlier remarked. Moreover, the masses of \mathbf{A}_μ and Ψ can effectively be generated by spontaneous symmetry breaking in the following manner,

$$\mathcal{L}_4 = \frac{1}{\pi N} |\phi|^2 |\Psi|^2 - \frac{1}{2\pi} (\mathbb{D}_\mu \phi)^\dagger (\mathbb{D}^\mu \phi) - V(\phi), \quad (90a)$$

$$V(\phi) = -\frac{\lambda}{2} |\phi|^2 + \frac{1}{4} |\phi|^4, \quad (90b)$$

where the sombrero potential $V(\phi)$ has its minima at $|\phi_c|^2 = 0, \lambda$, with λ the mass of the ϕ field as expected from eq. (41). Notably, the first term in eq. (90a) yields the Einstein-Hilbert action, while the kinetic term of ϕ yields the mass term for the gauge field, $|\phi_c|^2 \mathbf{A}_\mu \mathbf{A}^\mu / 2\pi$, which is integrated out by $\mathcal{D}[\mathbf{A}_\mu]$ as discussed in order to yield eq. (89). Conversely, the presence of eq. (89) in

the Lagrangian density gauge invariance is already spontaneously broken by the condensate, ϕ and not Ψ . Likewise, recall that the kinetic term of Ψ governs whether the Killing horizon is quantized as in eq. (85a) or cosmological ($\Lambda \neq 0$) in nature, corresponding to either preserving or breaking the conformal symmetry of the underlying gauge theory, \mathbf{A}_μ . Consequently, this can be viewed as exploring conformal and non-conformal regimes of the theory, corresponding respectively to the length scales, $\ell \sim r_S = 2GM$ comparable to the black hole radius and $\ell \sim 1/\sqrt{\Lambda}$ comparable to the size of the de Sitter universe.

Finally, since ψ no longer couples to \mathbf{A}^μ after integrating it out, we recognize that the essence of the above maneuvers in the case of $\ell \sim r_S$ is already captured in eq. (87) by simply setting $\mathbf{A}_\mu = 0$.

B. Quantum gravity

We proceed by considering the path integral, $\int \mathcal{D}[\bar{\psi}, \psi]$. We shall perform this integral by the stationary phase/steepest descent approach (Zee, 2010), where we simply plug in the classical solutions of motion of $\psi, \bar{\psi}$ given in eq. (47), which require, $\langle \mathcal{L}_1 \rangle = 0$. Consequently, the path integral simply acquires an unimportant phase, $\ln(\int \mathcal{D}[\bar{\psi}, \psi])$. Proceeding, we are interested in the path integral involving the metric tensor, $g_{\mu\nu}$ as the dynamical field. However, we face a daunting challenge to appropriately define this path integral since there lacks a consensus in literature on how the measure, $\mathcal{D}[g_{\mu\nu}]$ ought to be defined in order to yield consistent results. (Hamber, 2008)

Nonetheless, we make progress by defining the path integral with the measure as a discrete sum over an ensemble of varied space-time manifolds, *i.e.* $\int \mathcal{D}[g_{\mu\nu}] \rightarrow \sum_{\mathcal{M}}$. Moreover, a neat observation from eq. (25) and eq. (28) implies,

$$\mathcal{S}\chi_4(\mathcal{M}) = \pi N\chi_4(\mathcal{M}) = \pm 2\pi N\nu, \quad (91)$$

where $\nu \in \mathbb{Z}$ is the second Chern (winding) number. Consequently, this implies that the topological action can identically be excluded from the exponent. Moreover, we can rewrite eq. (87) as,

$$\begin{aligned} \Psi_{\text{QG}} &= \sum_{\mathcal{M}} \exp(iI_{\mathcal{M}}(\lambda)) \psi_{\text{CS}}^{*+}(\mathcal{B}) \psi_{\text{CS}}^{-}(\mathcal{B}) \\ &= \sum_{\mathcal{M}} \exp(iI_{\mathcal{M}}(\lambda)), \end{aligned} \quad (92a)$$

where,

$$I_{\mathcal{M}}(\lambda) = \frac{\lambda}{\pi} \int_{\mathcal{M}} d^4x \sqrt{-\det(g_{\mu\nu})} R, \quad (92b)$$

is the Einstein-Hilbert action (in Lorenzian signature).

Nonetheless, since a Wick rotation, defined by $t = \pm it^{\text{E}}$ and $\Psi_{\text{QG}} \rightarrow Z_{\text{QG}}^{\text{E}}$ where Z_{QG}^{E} is the partition function, transforms the exponent to a real value, the topological term now contributes in the Euclidean path integral,

$$Z_{\text{QG}}^{\text{E}} = \sum_{\mathcal{M}^{\text{E}}} \exp(\mp I_{\mathcal{M}^{\text{E}}}^{\text{E}}(\lambda)) \exp(\pm \mathcal{S}\chi_4(\mathcal{M}^{\text{E}})), \quad (93a)$$

where,

$$I_{\mathcal{M}^{\text{E}}}^{\text{E}}(\lambda) = \frac{\lambda}{\pi} \int_{\mathcal{M}^{\text{E}}} d^4x^{\text{E}} \sqrt{\det(g_{\mu\nu}^{\text{E}})} R, \quad (93b)$$

is the Einstein-Hilbert action in Euclidean signature and $\mathcal{M}^{\text{E}} \in (\beta, \mathcal{V})$ is the $n = 4$ (dimensional) Riemannian manifold.

C. $1/\mathcal{N}$ expansion

Since \mathcal{S} is the thermodynamic entropy, it is expected to be statistical, arising from quantum gravity degrees of freedom of some quantum theory. Such a quantum theory ought to reproduce the partition function given in eq. (93), with the constraint, $\mathcal{S} = \ln \mathcal{N}$, where \mathcal{N} is the black hole/quantum gravity number of microstates. As earlier remarked, since the path integral measure, $\mathcal{D}[g_{\mu\nu}]$ is not well-understood, the aspiration is for such a successful quantum theory reproducing eq. (93) to have a well-defined path integral measure, *e.g.* $\mathcal{D}[\varphi]$, where φ are dynamical fields, in order for quantum gravity to emerge from the path integral formulation, hence circumventing this issue.

To make some progress in identifying such a theory, we plug into eq. (93) the Boltzmann entropy formula,

$$\mathcal{S} = \ln \mathcal{N}, \quad (94)$$

thus obtaining the homestretch,

$$\begin{aligned} Z_{\text{QG}}^{\text{E}} &= \sum_{\mathcal{M}^{\text{E}} \in h} \exp(\mp I_{\mathcal{M}^{\text{E}}}^{\text{E}}(\lambda)) \mathcal{N}^{\pm \chi_4(\mathcal{M}^{\text{E}})} \\ &= \sum_{\mathcal{M}^{\text{E}} \in h} \exp\left(-\int d\beta \mathcal{H}_{\mathcal{M}^{\text{E}}}(\beta)\right) \\ &= \text{Tr}\left(\exp\left(-\int d\beta \mathcal{H}_{\mathcal{M}^{\text{E}}}(\beta)\right)\right), \end{aligned} \quad (95)$$

corresponding to eq. (1a), where $\exp(-\int d\beta \mathcal{H}_{\mathcal{M}^{\text{E}}}(\beta))$ takes the form of a Boltzmann factor with the trace, Tr going over the ensemble of manifolds, $\mathcal{M}^{\text{E}} \in h$ with distinct topologies characterized by the index, h to be later defined. Note that the classification of all $n = 4$ dimensional Riemannian manifolds into distinct topologies is not a straightforward exercise (Freedman, 1982), requiring additional assumptions, tackled in subsequent sections.

Nonetheless, we recognize eq. (95) as the $1/\mathcal{N}$ expansion of a large \mathcal{N} theory with the partition function given by,

$$\mathcal{Z} = \exp(Z_{\text{QG}}^{\text{E}}) = \int \mathcal{D}[\varphi] \exp\left(-\frac{\mathcal{N}^{\pm}}{\lambda^{\pm}} \text{Tr}(\gamma(\varphi))\right), \quad (96)$$

where $S(\varphi) = \left(\frac{\mathcal{N}}{\lambda}\right)^{\pm} \text{Tr}(\gamma(\varphi))$ is the action for unidentified field tensors, φ of rank $n = 4$ transforming appropriately as vectors under an unknown group, and γ is the unidentified function of φ defining the large \mathcal{N} theory. In this case, the vacuum Feynman diagrams pave $n = 4$ manifolds (Gurau, 2012), differing from random matrix large N theories, where φ is a rank $n = 2$ tensor (matrix) with vacuum Feynman diagrams defining $n = 2$ manifolds. (Gurau, 2011, 2012; Gurau and Rivasseau, 2011)

D. Topology

1. Old quantum condition

Considering the Langevin equation given in eq. (72a), we define the geodesic curvature as,

$$k_{\text{g}}(\tau) = u^{\nu} u^{\mu} \nabla_{\mu} u_{\nu}, \quad (97a)$$

motivated by the fact that $k_{\text{g}}(\tau)$ identically vanishes as the $n = 4$ -velocity approaches the Killing vector $u^{\mu}(\tau) \rightarrow \xi^{\mu}(\tau)$ due to the anti-symmetry relation in eq. (54) or more favorably when the center of mass equations of motion are geodesics,

$$u^{\mu} \nabla_{\mu} u^{\nu} = 0. \quad (97b)$$

In particular, when the random acceleration vanishes, $\eta_{\mu} = 0$ as in eq. (72a), we can employ $u^{\mu} u_{\mu} = -1$ to discover the geodesic curvature corresponds to reduced temperature, $k(\tau) = 1/\bar{\beta}(\tau)$. Moreover, observe that,

$$\begin{aligned} I(\lambda) &= \frac{\lambda}{\pi} \int d^4x \sqrt{-\det(g_{\mu\nu})} R \\ &= -\frac{M}{2} \int dt \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \langle \bar{\psi} \psi \rangle \\ &= -\frac{M}{2N} \int d\tau \int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} \text{Tr}(J^0(\vec{x})) \\ &= -\frac{M}{2} \int d\tau = \frac{M}{2} \int u_{\mu}(\tau) dx^{\mu}(\tau), \end{aligned} \quad (98)$$

where $u_{\mu}(x^{\mu}(\tau)) = dx^{\mu}(\tau)/d\tau$, $u^{\mu}(\tau)u_{\mu}(\tau) = -1$. Note that, we have used eq. (49), eq. (74), eq. (76) and eq. (81). Thus, eq. (98) guarantees that the Einstein-Hilbert action satisfies,

$$\begin{aligned} I_{\mathcal{M}}(\lambda) &= -\frac{M}{2} \int_{\mathcal{C}(\mathcal{M})} d\tau \\ &= \frac{M}{2} \int_{\mathcal{C}(\mathcal{M})} u_{\mu}(x^{\mu}(\tau)) dx^{\mu}(\tau), \end{aligned} \quad (99)$$

where \mathcal{C} identifies a world-line in space-time corresponding to the trajectory for the center of mass. Under a wick rotation, $t \rightarrow \pm it^{\text{E}}$ and $\tau \rightarrow \pm is$, and closed world-lines become possible, where s is the arc length for the closed trajectory $\mathcal{C}^{\text{E}} \in [0, \beta]$.

Consequently, the integral over the Langevin eq. (72a) after Wick rotation yields,

$$\oint_{\mathcal{C}^{\text{E}}} k_{\text{g}}(s) ds + \frac{M}{N} \oint_{\mathcal{C}^{\text{E}}} u_{\mu}(s) dx^{\mu}(s) = \oint_{\mathcal{C}^{\text{E}}} \eta_{\mu} dx^{\mu}, \quad (100a)$$

where,

$$M \int_0^{\beta} ds = \beta M = 2\pi N = 4\pi k, \quad (100b)$$

$u^{\mu}(s)u_{\mu}(s) = 1$, $u^{\mu}(s) = dx^{\mu}/ds$ and we have used $\beta M = 2\pi N$ from eq. (85a) and eq. (23c).

2. Entropy as an adiabatic invariant

Recalling that $\hbar = 1$, $N \in \mathbb{Z} \geq 0$ and Mu^{μ} has the right form for momentum, we recognize eq. (100b) as the old quantum condition (Wilson–Sommerfeld/Ishiwara rule) (Ishiwara, 2017; Pauling and Wilson, 2012), which is equivalent to the condition of black hole area quantization, implying that black hole entropy (proportional to area) is an adiabatic invariant. (Hernard, 1993)

For instance, considering the constraint equations in eq. (69) when the gauge field is broken to $U(1)$ $U(1)$, we can appropriately re-scale the complex function using the trace of EFE, $R = -\beta \langle \bar{\psi} \psi \rangle$ with $\beta = 8\pi GM$ in order to have, $\Psi = \sqrt{\langle \bar{\psi} \psi \rangle} \exp(iS)$, $S = -\frac{M}{2} \int ds$ and,

$$\nabla^{\mu} K_{\mu\nu} = -\beta \Psi^{\dagger} \partial_{\nu}^{\text{E}} \Psi, \quad (101a)$$

where $\partial_{\mu}^{\text{E}}$ the partial derivative in Euclidean signature and $\partial_{\mu}^{\text{E}} - iA_{\mu} \rightarrow \partial_{\mu}^{\text{E}}$. Consequently, the normalization of $\Psi = \sqrt{\rho} \exp(iS)$ in Euclidean signature, equivalent to eq. (76a), corresponds to,

$$\begin{aligned} 1 &= \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^{\text{E}})} u^0(\vec{x}(t^{\text{E}})) \langle \bar{\psi}(\vec{x}(t^{\text{E}})) \psi(\vec{x}(t^{\text{E}})) \rangle \\ &= \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^{\text{E}})} u^0(\vec{x}(t^{\text{E}})) \Psi^{\dagger}(\vec{x}(t^{\text{E}})) \Psi(\vec{x}(t^{\text{E}})) \\ &= \frac{2}{M} \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^{\text{E}})} \xi^{\nu} \Psi^{\dagger}(\vec{x}(t^{\text{E}})) i \partial_{\nu}^{\text{E}} \Psi(\vec{x}(t^{\text{E}})) \\ &= \frac{2}{M} \langle \Psi(\vec{x}(t^{\text{E}})) | i \frac{\partial}{\partial t^{\text{E}}} | \Psi(\vec{x}(t^{\text{E}})) \rangle, \end{aligned} \quad (101b)$$

where $\xi^{\mu} = (-1, \vec{0})$ is the time-like Killing vector and we have used $\xi^{\mu} \nabla^{\nu} R_{\mu\nu} = \frac{1}{2} \xi^{\mu} \partial_{\mu}^{\text{E}} R = 0$ satisfied by eq. (52) and eq. (101a). Computing the integrals below using eq. (101a) and eq. (101b), we find,

$$-i \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^{\text{E}})} \xi^{\nu} \nabla^{\mu} K_{\mu\nu} = \frac{\beta M}{2} = \mathcal{S}, \quad (102a)$$

and consequently,

$$\begin{aligned} & -i\beta^{-1} \int_0^\beta dt^E \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^E)} \xi^\nu \nabla^\mu K_{\mu\nu} \\ & = i \int_0^\beta dt^E \langle \Psi(\vec{x}(t^E)) | \frac{\partial}{\partial t^E} | \Psi(\vec{x}(t^E)) \rangle = \mathcal{S}. \end{aligned} \quad (102b)$$

Moreover, if Ψ is interpreted as the wave function of a quantum gravity system, the entropy, \mathcal{S} becomes the geometric phase/Berry phase (Berry, 1984). In addition, if a Schwarzschild black hole is described by Ψ and the black hole does not undergo quantum transitions as it accretes/evaporates, the wave function should evolve adiabatically. For instance, this requires that the Unruh-Hawking radiation from the black hole or the accreted energy content to be negligible such that the space-time can be assumed fairly static, $\partial M/\partial t^E \simeq 0$. Since the system at finite temperature is periodic in β , a standard calculation of the adiabatic invariant (Berry, 1984; Cohen *et al.*, 2019) yields,

$$\begin{aligned} & i \int_0^\beta dt^E \langle \Psi(\vec{x}(t^E)) | \frac{\partial}{\partial t^E} | \Psi(\vec{x}(t^E)) \rangle \\ & = i \int_{\partial \mathcal{A}^E = \mathcal{C}^E} d\vec{x} \cdot \langle \Psi(\vec{x}(t^E)) | \vec{\nabla} | \Psi(\vec{x}(t^E)) \rangle \\ & = \int_{\mathcal{A}^E} d^2x \vec{n} \cdot \vec{\Omega} = \int_{\mathcal{A}^E} \Omega = 2\pi k, \end{aligned} \quad (102c)$$

where \mathcal{A}^E is the $n = 2$ manifold enclosed by the Euclidean path, $\vec{x}(t^E)$, $\vec{\Omega} = \vec{\nabla} \times \vec{\mathcal{P}}$ is the Berry curvature (Berry, 1984) which can be written as a $p = 2$ form,

$$\Omega = d\mathcal{P} = \frac{1}{2} \partial_j \mathcal{P}_l dx^j \wedge dx^l, \quad (103a)$$

whereas the Berry connection is given by $\vec{\mathcal{P}} = i \langle \Psi(\vec{x}) | \vec{\nabla} | \Psi(\vec{x}) \rangle$, which can be written as a $p = 1$ form,

$$\mathcal{P} = i \langle \Psi(\vec{x}) | \partial_l | \Psi(\vec{x}) \rangle dx^l, \quad (103b)$$

\vec{n} is the unit vector normal to \mathcal{A}^E and $k = N/2 \in \mathbb{Z} \geq 0$ is the first Chern number (Chern, 1946), corresponding to the Chern-Simons level (eq. (23c)). (Jackiw and Pi, 2003)

In addition, since the object, $K_{\mu\nu}$ under $U(1)$, corresponds to the complex charge, $\mathbb{Q} = Q + i2Q$, the real and imaginary parts of the adiabatic wave function, $\Psi = \sqrt{\rho_c} \exp(\mathbb{S})$ capture the same information about the quantum system. Thus, we write the adiabatic wave function as $\Psi = \sqrt{\rho_c} \exp(\mathbb{S})$ where,

$$\begin{aligned} \mathbb{S}(t^E, \vec{x}) & = -\frac{\beta M}{4} \Phi(\vec{x}) + iS(t^E, \vec{x}) \\ & = S(0, \vec{x})/2 + iS(t^E, \vec{x}), \end{aligned} \quad (104a)$$

where $S(0, \vec{x}) = -\beta M \Phi(\vec{x})/2$ is defined in order for $\partial \rho/\partial t^E = 0$ to vanish, ρ_c is a constant distribution with

$\langle \bar{\psi} \psi \rangle = \rho_c \exp(-\beta M \Phi(\vec{x})) = \rho$ the equilibrium Boltzmann density distribution function which clearly satisfies, $\partial \rho(\vec{x})/\partial t^E = 0$ and $\Phi(\vec{x})$ is the Newtonian potential. Introducing fluctuations by $\rho \rightarrow \tilde{\rho} = \rho f^2(t^E, \vec{x})$ where $\tilde{\rho}$ is the fluctuating density distribution, the manifold will no longer admit a time-like Killing vector, leading to $\partial \tilde{\rho}/\partial t^E = \rho \partial f^2(t^E, \vec{x})/\partial t^E \neq 0$. Nonetheless, since the form of the wave function is constrained by the complex charge, $\mathbb{Q} = Q + i2Q$, we must have $\Psi = \sqrt{\tilde{\rho}} \exp(iS) = \sqrt{\rho_c} \exp(\tilde{\mathbb{S}})$ and $\tilde{\rho} = \rho_c \exp(S)$ where,

$$\begin{aligned} \mathbb{S}(t^E, \vec{x}) & = \ln f(t^E, \vec{x}) - \frac{\beta M}{4} \Phi(\vec{x}) + iS(t^E, \vec{x}) \\ & = S(t^E, \vec{x})/2 + iS(t^E, \vec{x}). \end{aligned} \quad (104b)$$

It follows from $S(t^E, \vec{x}) = \ln f^2(t^E, \vec{x}) - \beta M \Phi(\vec{x})/2$ that,

$$\frac{\partial \ln f^2}{\partial t^E} = \frac{\partial S}{\partial t^E}, \quad (105)$$

where we can set $f(0, \vec{x}) = 1$.

Applying these assumptions, we proceed to calculate the Kullback-Leibler (KL) divergence, $D_{\text{KL}}(\rho||\tilde{\rho})$ for the two distributions ρ and $\tilde{\rho}$ by (Kullback and Leibler, 1951),

$$\begin{aligned} D_{\text{KL}}(\rho||\tilde{\rho}) & = \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^E)} \rho \ln \left(\frac{\rho}{\tilde{\rho}(t^E)} \right) \\ & = - \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^E)} \rho \ln f^2(t^E) \\ & = - \int_0^{t^E} dt^E \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^E)} \frac{\partial}{\partial t^E} (\rho \ln f^2(t^E)) \\ & = - \int_0^{t^E} dt^E \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^E)} \rho \frac{\partial S}{\partial t^E} \\ & = \int_0^{t^E} dt^E \int_{\mathcal{V}} d^3x \sqrt{\det(g_{\mu\nu}^E)} \Psi^\dagger i \frac{\partial}{\partial t^E} \Psi \\ & = i \int_0^{t^E} dt^E \langle \Psi | \frac{\partial}{\partial t^E} | \Psi \rangle = 2\pi k = \mathcal{S} \geq 0, \end{aligned} \quad (106)$$

where we have used eq. (105), $\partial \rho/\partial t^E = 0$ and set $t^E = \beta$ in the last line. Thus, by the positive definite property of the KL divergence, the black hole entropy is always positive, $\mathcal{S} = \pi N \geq 0$, consistent with $N \in \mathbb{Z} \geq 0$ as expected.

3. Random acceleration

Proceeding to appropriately define the random acceleration, η_ν we note that, since the quantum gravity amplitude satisfies,

$$\begin{aligned} \Psi_{\text{QG}} & = \sum_{\mathcal{M}} \exp \left(ik \int_{\mathcal{C}(\mathcal{M})} p \right) \exp(-i2\pi k \chi_4(\mathcal{M})) \\ & = \sum_{\mathcal{M}} \exp \left(ik \int_{\mathcal{C}(\mathcal{M})} p \right), \end{aligned} \quad (107)$$

it implies we set, $\psi_{\text{CS}}^{*+} \equiv \exp(ik \int_{\mathcal{C}} p_{\mu} dx^{\mu})$ from eq. (25) in order for Ψ_{QG} to transform appropriately. This also implies that,

$$\int_{\mathcal{C}^{\text{E}}(\mathcal{M}^{\text{E}})} p = -\frac{1}{k} \int_{\mathcal{B}^{\text{E}}(\mathcal{M}^{\text{E}})} \text{CS}^+, \quad (108\text{a})$$

where $\mathcal{B}^{\text{E}} = \pm \partial \mathcal{M}_{\pm}^{\text{E}}$. Moreover, to find $\mathcal{C}^{\text{E}}(\mathcal{M}^{\text{E}})$, it is instructive to define η_{μ} such that it generates the $1/\mathcal{N}$ expansion terms in eq. (96), corresponding to the path integral contributions. In other words, we ought to set,

$$\int_{\mathcal{C}^{\text{E}}(\mathcal{M}^{\text{E}})} k_{\text{g}}(s) ds = \frac{1}{k} \int_{\mathcal{B}^{\text{E}}(\mathcal{M}^{\text{E}})} \text{CS}^- = -\frac{1}{k} \mathcal{I}_{\mathcal{M}^{\text{E}}}, \quad (108\text{b})$$

and,

$$\int_{\mathcal{C}^{\text{E}}(\mathcal{M}^{\text{E}})} \eta = \pm 2 \int_{\mathcal{B}^{\text{E}}(\mathcal{M}^{\text{E}})} d\theta, \quad (108\text{c})$$

where θ is given in eq. (27). To make further progress, we consider a series of *compact* manifolds homeomorphic to the n -sphere, S^n where $n = 4, 3, 2, 1$, together with their non-*compact* hemispheres (indicated by \pm) with boundaries homeomorphic to the $n-1$ spheres, *i.e.* $\partial S_{\pm}^n = \pm S^{n-1}$. Thus, we employ the definitions,

$$\mathcal{M}^{\text{E}} = \mathcal{M}_{+}^{\text{E}} + \mathcal{M}_{-}^{\text{E}}, \quad \partial \mathcal{M}_{\pm}^{\text{E}} = \pm \mathcal{B}^{\text{E}}, \quad (109\text{a})$$

$$\mathcal{B}^{\text{E}} = \mathcal{B}_{+}^{\text{E}} + \mathcal{B}_{-}^{\text{E}}, \quad \partial \mathcal{B}_{\pm}^{\text{E}} = \pm \mathcal{D}^{\text{E}}, \quad (109\text{b})$$

$$\mathcal{D}^{\text{E}} = \mathcal{D}_{+}^{\text{E}} + \mathcal{D}_{-}^{\text{E}}, \quad \partial \mathcal{D}_{\pm}^{\text{E}} = \pm \mathcal{C}^{\text{E}}, \quad (109\text{c})$$

where $\mathcal{M}^{\text{E}}, \mathcal{B}^{\text{E}}, \mathcal{D}^{\text{E}}, \mathcal{C}^{\text{E}}$ are *compact* manifolds with dimensions $n = 4, 3, 2, 1$ respectively. Since \mathcal{B}^{E} is *compact*, we expect that if $d\theta$ is non-*exact* on $\mathcal{B}^{\text{E}}(\mathcal{M}^{\text{E}})$, we would have,

$$\int_{\mathcal{B}^{\text{E}}} d\theta = \int_{\partial \mathcal{B}_{+}^{\text{E}}} \theta^{+} + \int_{\partial \mathcal{B}_{-}^{\text{E}}} \theta^{-} = \int_{\mathcal{D}^{\text{E}}} d\alpha, \quad (110\text{a})$$

where $\theta^{+} - \theta^{-} = d\alpha$ is also assumed and with the $p = 1$ form, α defined modulo $2\pi\nu$ on \mathcal{D}^{E} . This procedure can be repeated successively under a similar treatment. Proceeding, when $d\alpha$ is non-*exact* on $\mathcal{D}^{\text{E}}(\mathcal{M}^{\text{E}})$, we have,

$$\int_{\mathcal{D}^{\text{E}}} d\alpha = \int_{\partial \mathcal{D}_{+}^{\text{E}}} \alpha^{+} + \int_{\partial \mathcal{D}_{-}^{\text{E}}} \alpha^{-} = \pm \frac{1}{2} \int_{\mathcal{C}^{\text{E}}} d\eta, \quad (110\text{b})$$

where $\alpha^{+} - \alpha^{-} = \pm \frac{1}{2} d\eta$ as before and the random acceleration, η is defined modulo $2\pi\nu$ on $\mathcal{C}^{\text{E}}(\mathcal{M}^{\text{E}})$.

4. Gauss-Bonnet theorem

We can thus introduce the Gaussian curvature (Wu, 2008), K of the non-*compact* $n = 2$ Euclidean manifold, \mathcal{A}^{E} bounded by $\partial \mathcal{A}^{\text{E}} = \mathcal{C}^{\text{E}}(\mathcal{M}^{\text{E}})$, by,

$$dp = \star K, \quad (111\text{a})$$

$$\int_{\partial \mathcal{A}^{\text{E}}} p = \int_{\mathcal{A}^{\text{E}}} \star K, \quad (111\text{b})$$

where $\star K$ is the $p = 2$ Hodge dual of $p = 0$ form, K . Consequently, the integral form of the Langevin equation summarizes to,

$$\int_{\partial \mathcal{A}^{\text{E}}} k_{\text{g}} + \int_{\mathcal{A}^{\text{E}}} \star K = 2\pi \chi_4. \quad (112)$$

Evidently, this is the Gauss-Bonnet theorem in $n = 2$ dimensions (Wu, 2008), where the Euler characteristic of the $n = 4$ *compact* manifold, \mathcal{M}^{E} has been transformed into the Euler characteristic of the $n = 2$ non-*compact* manifold, \mathcal{A}^{E} , *i.e.* $\chi_4(\mathcal{M}^{\text{E}}) = \chi_2(\mathcal{A}^{\text{E}})$. Consequently, computing $\chi_4(\mathcal{M}^{\text{E}})$ can be done on \mathcal{A}^{E} instead of \mathcal{M}^{E} , suggesting that the partition function in eq. (95) can actually arise from a random matrix large \mathcal{N} group theory where φ transforms as a vector under a rank $n = 2$ tensor group theory, instead of rank $n = 4$, as earlier remarked. Thus, these observations take us closer to defining the appropriate large \mathcal{N} theory for the field variable, φ by edging us closer to consistently defining the topology index, h in eq. (96) for $n = 4$ manifolds of interest to our study.

Finally, recall that the Gauss-Bonnet theorem in eq. (112) relates the Langevin equation (Lemons and Gythiel, 1997) (eq. (72a)) to the action in the integrand of the path integral in eq. (87). Since varying the action, $\mathcal{I}_{\mathcal{M}}$ leads to the field equations of gravity, we conclude that EFE are analogous to the Fokker-Planck equation (Risken, 1996), where the Ricci scalar R is proportional to the probability density. In diffusion models, the Fokker-Planck equation is related to the Langevin equation by Itô's lemma, under a non-standard calculus known as Itô calculus. (Øksendal, 2003)

5. Euler characteristic, $\chi_4 = \chi_2$

We shall consider some implications of our approach by setting,

$$\chi_2 = 2 - 2g - b, \quad (113)$$

where g is the genus of \mathcal{A}^{E} and $b = \frac{1}{2\pi} \int_{\mathcal{C}^{\text{E}}} k(s) ds$ is the finite contribution of the geodesic curvature integral over the boundary, \mathcal{C}^{E} . Moreover, we shall only consider manifolds homeomorphic to the connected sums of the $n = 4$ -sphere, S^4 and the $n = 4$ -torus, T^4 where \mathcal{M}^{E} is thus compact. Their respective Euler characteristics are given by,

$$\chi_4(S^4) = \sum_{p=0}^4 (-1)^p b_p(S^4) = 2, \quad (114\text{a})$$

$$\chi_4(T^4) = \sum_{p=0}^4 (-1)^p b_p(T^4) = 0, \quad (114\text{b})$$

respectively, where,

$$b_{p=0}(S^n) = \binom{n}{0} = 1, \quad b_{p=n}(S^n) = \binom{n}{n} = 1, \quad (114c)$$

$$b_{p \neq 0,4}(S^n) = 0, \quad b_p(T^n) = \binom{n}{p} = \frac{n!}{(n-p)!p!}, \quad (114d)$$

and b_p is the p -th Betti number.(Bochner and Yano, 2016) The Betti numbers have the property that the b_0 and b_n numbers are not additive (remain invariant) under connected sum operations, while the rest of the Betti numbers are additive.(Zagier, 2017) For instance, the connected sum of manifolds homeomorphic to S^4 has the Euler characteristic,

$$\begin{aligned} \chi_4(S^4 \# S^4 \# \dots \# S^4) \\ = b_0(S^4) + b_4(S^4) = \chi_4(S^4), \end{aligned} \quad (115a)$$

since the Betti numbers, $b_0 = 1$ and $b_4 = 1$ are invariant, while $b_{p \neq 0,4} = 0$ are additive but do not contribute to additional terms since they are vanishing. Meanwhile, the connected sum of manifolds homeomorphic to S^4 and T^4 has the Euler characteristic,

$$\begin{aligned} \chi_4(S^4 \# S^4 \# \dots \# S^4 \# \underbrace{T^4 \# T^4 \dots \# T^4}_h) \\ = \binom{4}{0} + \binom{4}{4} + h \sum_{p \neq 0,4} (-1)^p \binom{4}{p} = 2 - 2h, \end{aligned} \quad (115b)$$

where $h \in \mathbb{Z} \geq 0$ is the number of tori in the connected sum. Since, we can always reproduce eq. (115a) by setting $h = 0$, we have,

$$\chi_4(\mathcal{M}^E) = 2 - 2h, \quad (116)$$

for all connected sums. Consequently, we note that, $\chi_2(\mathcal{A}^E) = \chi_4(\mathcal{M}^E)$ implies,

$$h = g + b/2. \quad (117a)$$

Thus, when \mathcal{A}^E is *compact*, the contribution to the Gauss-Bonnet theorem from the geodesic curvature vanishes, $b = 0$ and h corresponds to the genus of \mathcal{A}^E .

E. Average energy of manifolds

The energies associated with a given manifold can be obtained from eq. (95) by, $\mathcal{H}_{\mathcal{M}^E}(\beta) = -\partial \mathcal{I}_{\mathcal{M}^E}^E(\beta)/\partial \beta$, which yields,

$$\begin{aligned} \mathcal{H}_{\mathcal{M}^E} = \mp \frac{M}{2} \chi_4(h) \\ \mp \frac{M}{2\beta^2} \int_{\mathcal{M}^E \in h} d^4 x^E \sqrt{\det(g_{\mu\nu}^E)} R, \end{aligned} \quad (118)$$

where we have used $\lambda = 1/16G$ from eq. (81), $\beta = 8\pi GM$ and $\mathcal{S} = \ln \mathcal{N} = \beta M/2 = \pi N$ from eq. (84b). Meanwhile, individual probabilities, $P_{\mathcal{M}^E}$ for a given manifold correspond to,

$$P_{\mathcal{M}^E} = (Z_{\text{QG}}^E)^{-1} \exp\left(-\int d\beta \mathcal{H}_{\mathcal{M}^E}(\beta)\right). \quad (119)$$

Thus, given the above probability distribution for each manifold configuration, the expression for average energy becomes,

$$\langle \mathcal{H} \rangle = \sum_{\mathcal{M}^E \in h} P_{\mathcal{M}^E} \mathcal{H}_{\mathcal{M}^E}(\beta) = -\frac{\partial}{\partial \beta} \ln(Z_{\text{QG}}^E). \quad (120)$$

Moreover, when there are no sources in the EFE or when the energy momentum tensor is trace-less, the Ricci scalar vanishes and the stationary phase/steepest descent approach(Zee, 2010) allows the energy of the manifold to be transformed into a more palatable expression by plugging in $R = 0$. Thus, the average energy becomes purely the sum of the topological contributions,

$$\langle \mathcal{H} \rangle = \frac{M}{2} \sum_{\mathcal{M}^E \in h} P_{\mathcal{M}^E} \chi_4(h), \quad (121)$$

$$P_{\mathcal{M}^E \in h} = \frac{\exp\left(-\beta \frac{M}{2} \chi_4(h)\right)}{\sum_{\mathcal{M}^E \in h} \exp\left(-\beta \frac{M}{2} \chi_4(h)\right)}. \quad (122)$$

V. RAMIFICATIONS

A. Fermion/boson picture

Typically, in the $1/\mathcal{N}$ expansion of a random matrix theory such as in eq. (96), terms corresponding to the topology, $\chi_4(S^4 \# S^4 \# \dots \# S^4) = \chi_4(S^4) = 2$ (planar vacuum Feynman diagrams) would be expected to dominate the expansion.(Gurau and Rivasseau, 2011) However, here, the \pm sign arising from opposite-direction Wick rotations slightly complicates the situation. To understand how, recall that we are considering $R = 0$, where R is the Ricci scalar. We can proceed by the assumption above that the S^4 topology dominates with $P_{\mathcal{M}^E} \simeq 1$, where the manifold energy $\mathcal{M}_{\text{BH}}^E \in h = 0$ is the central mass, $\mathcal{H}_{\mathcal{M}^E \in h=0} = M$ and,

$$\chi_4(\mathcal{M}_{\text{BH}}^E) = \chi_4(S^4) = 2, \quad (123)$$

with $\mathcal{M}_{\text{BH}}^E$ the manifold corresponding to a space-time with a single Schwarzschild black hole of mass, M while the sign of the energy function chosen to correspond to the Wick rotation, $t = -it_E$. Even though black hole solutions are vacuum solutions of EFE, this cannot be the ground state of our theory since we could always lower the energy of the manifold by removing the central mass, by setting, $M = 0$.

In fact, the ground state corresponding to $\mathcal{H}_{\mathcal{M}^E} = 0$ is characterized by three quantum states: $(M = 0, \chi_4(\mathcal{M}^E) \neq 0)$, $(M \neq 0, \chi_4(\mathcal{M}^E) = 0)$ and $(M = 0, \chi_4(\mathcal{M}^E) = 0)$ states; hence it is three-fold degenerate. Nonetheless, since the $1/\mathcal{N}$ expansion requires we take the 't Hooft limit (corresponding to $\mathcal{N} \rightarrow \infty$, $\lambda = 1/16G$ fixed), and eq. (84b) requires $M^2 \propto \pi N = \ln \mathcal{N} \rightarrow \infty$, this lifts the degeneracy by selecting the unique ground state to be $(M \neq 0, \chi(\mathcal{M}^E) = 0)$. Thus, for our approach to be physical, $(M = 0, \chi_4(\mathcal{M}^E))$ ought to include Minkowski space-time/Euclidean space, *albeit* alongside other Riemannian manifolds with non-trivial topologies, $\chi_4(\mathcal{M}^E) \neq 2 - 2h$. This is encouraging since it classifies relevant manifolds consistent with the assumption, $\mathcal{M}^E \in h$ in the partition function, Z_{QG}^E .

Proceeding, we already have, $\chi_4(\mathcal{M}^E \in h) = 2 - 2h$, with $h = 0, 1$. To avoid negative mass states, $M < 0$ we observe that, the manifold states with $h \geq 1$ correspond to a Wick rotation in the opposite direction, *i.e.* $t = +it^E$. In fact, the Wick rotations, $t = \pm it^E$ correspond to the following $1/\mathcal{N}$ expansions,

$$\Psi_{\text{QG}}(t, h) \Big|_{t=-it^E, h \leq 1} = \mathcal{N}^{-2} + \mathcal{N}^0, \quad (124a)$$

$$\Psi_{\text{QG}}(t, h) \Big|_{t=+it^E, h \geq 1} = \mathcal{N}^0 + \mathcal{N}^{-2} + \dots \mathcal{N}^{2-2h}, \quad (124b)$$

where both select the unique ground state, $\mathcal{N}^0 = 1$ for large \mathcal{N} . Moreover, there is a sense in which the $h = 1$ state is fermionic whereas the $h > 1$ states are bosonic in nature. This follows from their average energy expressions,

$$\begin{aligned} \langle \mathcal{H} \rangle &= \frac{M}{2} \frac{\sum_{h=1}^{h=1} \chi_4(h) \exp\left(-\frac{\beta M}{2} \chi_4(h)\right)}{\sum_{h'=0}^{h'=1} \exp\left(-\frac{\beta M}{2} \chi_4(h')\right)} \\ &= M \frac{\exp(-\beta M)}{1 + \exp(-\beta M)} \\ &= \frac{M}{\exp(\beta M) + 1} = M \langle c^\dagger c \rangle, \quad (125a) \end{aligned}$$

where c^\dagger, c are the fermionic creation and annihilation operators respectively, satisfying the anti-commutation relations, $\{c, c^\dagger\} = 1$. Thus, when the Ricci scalar in eq. (118) is non-vanishing, $R \neq 0$, the presence of a massive Dirac field restores the fermionic zero-point energy term, $-M/2$ missing in eq. (125). Consequently, a substitution of the trace of EFE into eq. (118) yields,

$$\begin{aligned} &\frac{M}{2\beta^2} \int_{\mathcal{M}^E} d^4 x^E \sqrt{\det(g_{\mu\nu}^E)} R \\ &= -\frac{M}{2\beta} \int_0^\beta ds \int_{\mathcal{V}} d^3 x \sqrt{\det(g_{\mu\nu}^E)} \langle \bar{\psi}(\vec{x}) \psi(\vec{x}) \rangle u^0(\vec{x}) \\ &= -\frac{M}{2\beta} \int_0^\beta ds = -\frac{M}{2}, \quad (125b) \end{aligned}$$

as expected. Here, $R = -\beta \langle \bar{\psi} \psi \rangle$ and $\beta = 8\pi G M$ have been used from eq. (49), and $u^0(\vec{x}) = dt/d\tau = dt^E/ds$ with $s = i\tau$ the proper distance and $t^E = -it$ Euclidean time, where the normalization condition given in eq. (76a) applies. Thus, we conclude that the Euler characteristic is proportional to the fermion number operator acting on the quantum states (vacuum state, $h = 1$ and the single fermion state, $h = 0$),

$$\hat{\chi}_4 = 2c^\dagger c = c^\dagger c - cc^\dagger + 1, \quad (126a)$$

$$|\nu\rangle \equiv |\mathcal{M}^E \in h \leq 1\rangle, \quad (126b)$$

where $h = cc^\dagger$ and $\nu = 1 - h = c^\dagger c$ is the winding number given in eq. (28) in order for,

$$\hat{\mathcal{H}} = \frac{M}{2} (\hat{\chi}_4 - 1), \quad (126c)$$

$$\hat{\mathcal{H}}|\nu\rangle = \mathcal{H}_{\mathcal{M}^E \in h}(\nu)|\nu\rangle. \quad (126d)$$

The expressions are consistent with the plus sign option of the Hamiltonian given in eq. (118).

Similarly, the manifolds with $h \geq 1$ correspond to bosonic states. In particular, performing a Wick rotation in the opposite direction, $t = +it^E$ selects the appropriate sign corresponding to the bosonic average energy,

$$\begin{aligned} \langle \mathcal{H} \rangle &= -\frac{M}{2} \frac{\sum_{h=1}^{h=\infty} \chi_4(h) \exp\left(\frac{\beta M}{2} \chi_4(h)\right)}{\sum_{h'=1}^{h'=\infty} \exp\left(\frac{\beta M}{2} \chi_4(h')\right)} \\ &= -\frac{\partial}{\partial \beta} \ln \left(\sum_{h=1}^{h=\infty} \exp\left(\frac{\beta M}{2} \chi_4(h)\right) \right) \\ &= \frac{\partial}{\partial \beta} \ln(1 - \exp(-\beta M)) \\ &= \frac{M}{\exp(\beta M) - 1} = M \langle a^\dagger a \rangle, \quad (127a) \end{aligned}$$

where a^\dagger, a are the bosonic creation and annihilation operators respectively satisfying the commutation relations, $[a, a^\dagger] = 1$. However, caution should be exercised to replace the Dirac field in the trace of EFE with $R = -\beta\rho$, arising from the trace with a dust energy-momentum tensor, $T_{\text{Dirac}}^{\mu\nu} \rightarrow T_{\text{Dust}}^{\mu\nu} = M\rho u^\mu u^\nu$, where ρu^0 is interpreted as the bosonic probability density normalized as,

$$\int_{\mathcal{V}} d^3 x \sqrt{\det(g_{\mu\nu}^E)} \rho(\vec{x}) u^0(\vec{x}) = 1, \quad (127b)$$

to unity. Thus, we obtain,

$$\begin{aligned} &-\frac{M}{2\beta^2} \int_{\mathcal{M}^E} d^4 x^E \sqrt{\det(g_{\mu\nu}^E)} R \\ &= \frac{M}{2\beta} \int_0^\beta ds \int_{\mathcal{V}} d^3 x \sqrt{\det(g_{\mu\nu}^E)} \rho(\vec{x}) u^0(\vec{x}) \\ &= \frac{M}{2\beta} \int_0^\beta ds = \frac{M}{2}, \quad (127c) \end{aligned}$$

as expected. Consequently, parallel to the fermionic case, the Euler characteristic operator is proportional to the boson number operator, and the manifold states span the states of the quantum harmonic oscillator,

$$\hat{\chi}_4 = -2a^\dagger a = -a^\dagger a - aa^\dagger + 1, \quad (128a)$$

$$|\nu\rangle \equiv |\mathcal{M}^E \in h \geq 1\rangle, \quad (128b)$$

as expected, where $h = aa^\dagger$ and $\nu = h - 1 = a^\dagger a$ is the winding number given in eq. (28), in order for,

$$\hat{\mathcal{H}} = -\frac{M}{2}(\hat{\chi}_4 - 1), \quad (128c)$$

$$\hat{\mathcal{H}}|\nu\rangle = \mathcal{H}_{\mathcal{M}^E \in h}|\nu\rangle. \quad (128d)$$

The expressions are consistent with the minus sign option of the Hamiltonian given in eq. (118).

Finally, eq. (126) and eq. (128) satisfy the free energy equation,

$$\langle \hat{\mathcal{F}} \rangle = \langle \hat{\mathcal{H}} \rangle \pm \beta^{-1} \mathcal{S}, \quad (129a)$$

where $\hat{\mathcal{F}} = \pm \frac{M}{2} \hat{\chi}_4$ is the minimized free energy operator. By inspection, we see that eq. (129) corresponds to the quantum version of the classical Gauss-Bonnet theorem given in eq. (112), where the Gaussian curvature term is responsible for the black hole entropy, \mathcal{S} whereas the random acceleration term is responsible for the $1/\mathcal{N}$ expansion of the free energy since,

$$\langle \hat{\mathcal{F}} \rangle = -\beta^{-1} \ln(\mathcal{Z}_{\text{QG}}^E), \quad (129b)$$

Consequently, geodesics ($k_g(s) = 0$) are allowed if and only if $M = 0$. However, since $M \neq 0$ is imposed by the 't Hooft limit, geodesics must be precluded even for the vacuum state ($h = 1$) unless $R = 0$. Conversely, $R \neq 0$ gives rise to a finite zero-point term, $\pm M/2$ that cannot otherwise vanish. Consequently, \mathcal{A}^E must be non-compact when $M \neq 0$ unless $R = 0$ and $h = 1$, since the geodesic curvature term would otherwise identically vanish, contradicting our previous statements.

B. Asymptotic behavior in galaxy rotation curves

A direct way of deriving the empirical baryonic Tully-Fisher relation (McGaugh, 2012; McGaugh *et al.*, 2000), relevant for explaining the asymptotic behavior of galaxy rotation curves is to employ eq. (85a) with additional minor considerations. (Kanyolo and Masese, 2021) First, we shall consider the EFE coupled to a pressure-less dust, where $R = -8\pi GM\rho$ is the trace. This corresponds to the bosonic case given in eq. (127b), where M is the mass of the baryonic matter in the galaxy. However, since EFE are analogous to the Fokker-Planck equations, we can modify the normalization condition and by treating ρ as the number density of bosonic particles instead of a

probability density. A second adjustment is to relax the expression for temperature, $\beta \neq 8\pi GM$, since this was motivated by black hole thermodynamics, and hence is not necessarily guaranteed to apply to this scenario, as we shall see. Nonetheless, the condition, eq. (85a) can be expected to be robust across varied (quantum) gravitational systems, since it simply corresponds to the old quantization. (Ishiwara, 2017; Pauling and Wilson, 2012)

To simplify the problem, we assume that the periphery of the galaxy contains a negligible amount of baryonic matter compared to the interior, where $\rho(x) \simeq \rho_D(x)$ largely corresponds to the number density contribution from dark matter. (Persic *et al.*, 1996) In addition, we impose the time-like Killing vector, ξ^μ on the space-time manifold, in order to guarantee the equilibrium conditions,

$$\xi^\mu \partial_\mu \rho_D = 0, \quad (130a)$$

$$\rho_D(x) = \rho_c \exp(-\bar{\beta} M \Phi(x)), \quad (130b)$$

where ρ_c is a critical number density to be defined. Respectively, eq. (130) follows from the Killing equations in eq. (54) while assuming the equilibrium distribution takes the form of a Boltzmann factor with $\Phi(x)$ the Newtonian potential appropriately defined. We take the dark matter particles to be equivalent to a Bose-Einstein condensate (Kanyolo and Masese, 2021) of the colors, $k = N/2$ where k is the number of bosons/Cooper-pairs (Tinkham, 2004) forming the condensate normalization condition,

$$\int_{\mathcal{V}} d^3x \sqrt{-\det(g_{\mu\nu})} |\Psi(\vec{x})|^2 u^0(\vec{x}) = k, \quad (131a)$$

with $|\Psi(\vec{x})|^2 = \rho_D(\vec{x})$ the number density, $u^0(\vec{x})$ the time component of the $n = 4$ -velocity vector and $\Psi(\vec{x})$ satisfies eq. (69). For brevity, we consider the Newtonian limit corresponding to,

$$\begin{aligned} \nabla^2 \Phi(\vec{x}) &= 4\pi GM |\Psi(\vec{x})|^2 \\ &= 4\pi GM \rho_c \exp(-2k\Phi(\vec{x})), \end{aligned} \quad (131b)$$

where $u^0(x) \simeq 1$ and $k = 2\bar{\beta}M$ from eq. (85a). Here, we have used the time-like Killing vector, $\xi^\mu = (-1, \vec{0})$ and $\xi^\mu \partial_\mu R = 8\pi GM \partial \rho_D / \partial t = 0$, from eq. (130), in order to guarantee, $\partial \Phi(x) / \partial t = 0$.

Assuming the conditions of spherical symmetry of the dark matter halo (Kramer and Randall, 2016; Persic *et al.*, 1996), we can re-write eq. (131b) in spherical coordinates as,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi(r)}{\partial r} \right) = 4\pi GM \rho_c \exp(-2k\Phi(r)), \quad (132)$$

and proceed to solve it. This yields the potential and number density corresponding to the singular isothermal

profile(Keeton, 2001),

$$\Phi(r) = \frac{1}{2k} \ln(Kr^2), \quad (133a)$$

$$\rho_{\text{D}}(r) = \frac{\rho_{\text{c}}}{Kr^2} = \frac{1}{4\pi GMkr^2}, \quad (133b)$$

$$\rho_{\text{c}} = \frac{K}{4\pi GMk}, \quad (133c)$$

where K is a constant. However, the isothermal density profile is singular at the origin. This may not be *a priori* unphysical since it is not peculiar for vortices in nature to be described by singularities at the origin. In fact, $\Phi(r) = \int p_{\mu} dx^{\mu}$ takes the form of a logarithmic spiral with $\arctan(k) = \theta_{\text{p}}$ the pitch angle, where eq. (66) and eq. (107) imply $\Phi(r)$ is the quantum phase. However, since we have only considered conditions where dark matter dominates over baryonic matter at the periphery of the spiral galaxy, it is entirely feasible that predictions with eq. (133) better approximate the asymptotic behavior of spiral galaxies where dark matter dominates, rather than in the interior where there would be a significant energy density contribution from the neglected baryonic matter. Consequently, we shall introduce the condensate wave function as,

$$\Psi = \sqrt{\rho_{\text{D}}} \exp(ik\Phi(r)) = \sqrt{\rho_{\text{c}}} \exp(k(1+2i)\Phi), \quad (134)$$

implying the approach admits the complex charge, $\mathbb{Q} = Q + i2Q$ corresponding to the subsequent breaking of $\text{SU}(N)$ to $\text{U}(1)$, $\mathbf{F}_{\mu\nu} \rightarrow \mathcal{F}_{\mu\nu}$, where in eq. (69), $\mathcal{F}_{\mu\nu}$ is the $\text{U}(1)$ field strength and $K_{\mu\nu} = R_{\mu\nu} - i\mathcal{F}_{\mu\nu}$ is the complex-valued object. In a previous paper, we argued that the ensuing $\text{U}(1)$ gauge field strength corresponds to $\mathcal{F}_{\mu\nu} \propto \nabla_{\nu}\xi_{\mu}$, where ξ_{μ} is the time-like Killing vector, and thus ensuring dark matter remains effectively non-interacting and charge-less.(Kanyolo and Masese, 2021)

Moreover, the normalization condition in eq. (131a) can be exploited to find an expression for the number of bosons, k ,

$$\begin{aligned} k &= \int_{\mathcal{V}} d^3x \rho_{\text{D}}(r) = \int_0^{1/a_0} 4\pi r^2 \rho_{\text{D}}(r) \\ &= \int_0^{1/a_0} \frac{dr}{GMk} = \frac{1}{GMka_0}, \end{aligned} \quad (135)$$

where we have introduced a cut-off scale, $1/a_0$ for the $n = 3$ manifold, \mathcal{V} since the integral would otherwise diverge. In addition, since the logarithmic spiral is scale invariant but the cut-off introduces a length scale, this suggests that conformal invariance must be broken, while preserving scale invariance.(Milgrom, 2017) Consequently, one would suspect that the cut-off scale would be comparable to the size of the dark matter halo. However, since $\rho_{\text{D}} = \Psi^{\dagger}\Psi$ in eq. (131a) is the number density of the bosons (which are quantum mechanical objects), the cut-off in the normalization of the condensate wave

function ought to define the size of the manifold, \mathcal{V} and not necessarily the halo. Thus, it is reasonable to set the cut-off scale to be $1/a_0 \simeq \sqrt{3/\Lambda}$, which is comparable to the size of the de Sitter universe, where Λ is the cosmological constant. Consequently, from eq. (135), we have,

$$k = \frac{1}{\sqrt{GMa_0}} \simeq \frac{1}{\sqrt{GM(\Lambda/3)^{1/2}}}. \quad (136)$$

However, the number of bosons, k in eq. (136) appears not to take positive integer values. Of course, this is not a problem since a finite cosmological constant requires the kinetic term of Ψ to spontaneously break $\text{SU}(N)$ gauge symmetry in eq. (83), and hence does not require eq. (84b) nor eq. (85a) to be satisfied. This corresponds to breaking conformal invariance of the gauge theory, as earlier discussed. Equivalently, whether k is a good quantum number or not depends on the number/phase regime(Kanyolo, 2020; Kanyolo and Shimada, 2020) of the condensate governed by the commutation relation,

$$[k, \Phi] = -i. \quad (137)$$

For a more rigorous treatment, the commutation relation should be replaced by the Susskind–Glogower operators.(Susskind and Glogower, 1964) Thus, we are considering here the phase regime of the condensate, where the quantum phase $\Phi(r)$ obeys classical equations of motion analogous to the Josephson relations in large tunnel junctions(Josephson, 1974), and the Cooper-pair number, k is not a good quantum number. Meanwhile, the galactic size is determined by the binding condition(Kanyolo and Masese, 2021), $\Phi(r) \leq 0$, which corresponds to $r_{\text{c}} \leq 1/\sqrt{K}$, where $\rho(r_{\text{c}}) = \rho_{\text{c}}$. The total mass, $M_{\text{tot.}}(r)$ within radius, $r \leq r_{\text{c}}$ becomes,

$$M_{\text{tot.}}(r) = M \int dr 4\pi r^2 \rho(r) = M_{\text{D}} + M, \quad (138a)$$

$$M_{\text{D}} = M \int_0^r dr' 4\pi r'^2 \rho(r') = \frac{r}{Gk}, \quad (138b)$$

where M_{D} is the mass of dark matter and is defined as the mass, M of baryonic matter, arising as the integration constant.

Consequently, a star of mass, m_{\star} in orbit with speed, v at the periphery of the galaxy will experience a gravitational attractive force,

$$F_{\text{g}}(r) = -\frac{Gm_{\star}M_{\text{tot.}}(r)}{r^2} = -\frac{m_{\star}}{kr} - \frac{GMm_{\star}}{r^2}, \quad (139a)$$

as well as the centrifugal repulsion,

$$F_{\text{c}} = \frac{m_{\star}v^2}{r}. \quad (139b)$$

The first term in eq. (139a) is the gravitational contribution from dark matter, arising from the *particular*

solution of the non-homogeneous equation (eq. (132)) given by eq. (133), while the second term is the Newtonian inverse-square law arising from the *complementary* solution of the homogeneous equation (eq. (132) with $\rho_D = 0$) given by $\Phi_N = -GM/r$. At large radius, $r \gg GM$, the $1/r$ term dominates over the inverse-square term, $F_g \simeq m_*/kr$. Consequently, the critical speed, v_c of the star in orbit can be solved by setting, $F_c + F_g = 0$, which yields the celebrated mass-asymptotic speed relation (Milgrom, 1983),

$$v_c \simeq \frac{1}{\sqrt{k}} = (GMa_0)^{1/4}, \quad (140)$$

where a_0 takes the form predicted by Modified Newtonian dynamics (MOND) for the acceleration parameter. (McGaugh, 2012; McGaugh *et al.*, 2000) However, our path to this result vastly differs from standard MOND, since we explicitly rely on a pressure-less source at the right-hand side of EFE as well as the constraints introduced herein which ultimately reproduce eq. (140), consistent with the empirical baryonic Tully-Fisher relation. (McGaugh, 2012; McGaugh *et al.*, 2000)

C. Layered materials with cationic vacancies

The derivation of eq. (140) largely makes use of the old quantization condition, $\beta M = N = 2k$ from eq. (85a) and a single (time-like) Killing vector, $\xi^\mu = (-1, \vec{0})$. For a gravitational system with two Killing vectors such as the Schwarzschild black hole (time-like and azimuthal-like), eq. (130) requires that the gravitational potential, Φ cannot depend on more than two coordinates out of the $n = 4$ coordinates of \mathcal{M} . However, due to the spherical symmetry, the radial coordinate, $r(x, y, z)$ depends on three Cartesian coordinates, x, y, z instead of two, which means the Newtonian limit of the theory is effectively three dimensional.

Here, we are interested in a two dimensional emergent quantum gravitational system whose dynamics exploit the Gauss-Bonnet theorem in $n - 2$ dimensions, particularly in the Newtonian limit, where $n = 4$ is the dimensions of the manifold, \mathcal{M} and 2 is the number of translation Killing vectors. Following our approach, we shall consider a condensed matter system with desirable properties which favor the topological aspects discussed herein to emerge. In particular, we proceed to highlight a wide class of layered materials where positively charged mobile ions (cations) are sandwiched between the layers of immobile ions forming adjacent series of slabs within a stable crystalline structure, *viz.*,

- (i) Layered transition metal oxides such as A_xMO_2 (where $A = \text{Li, Na, K, Ag, etc.}$, M is a transition metal or a combination of multiple transition metals and $0 < x < 1$), $A_yV_2O_5$ (where

$0 < y < 2$), $D_xV_2O_5$ (where $D = \text{Mg, Ca, Al, Ag}$) and $\text{Ca}_3\text{Co}_4\text{O}_9$; (Delmas *et al.*, 2021; Galy, 1992; Goodenough and Park, 2013; Masset *et al.*, 2000; Shannon *et al.*, 1971a,b,c; Shirpour *et al.*, 2014; Whittingham, 2004; Xu *et al.*, 2017)

- (ii) Layered metal (di)chalcogenides such as $A_x\text{TiS}_2$ and $A_x\text{CrS}_2$ (where $0 < x < 1$); (Chia *et al.*, 2015; Johnson and Worrell, 1982; Whittingham, 1978)
- (iii) Graphite intercalation compounds such as LiC_6 , KC_8 , RbC_8 and CsC_8 , including their intermediate compositions, for instance, KC_{12n} ($n > 1$), LiC_{6n} ($n > 1$) and LiC_{9n} ($n \geq 2$); (Dresselhaus and Dresselhaus, 2002, 1981; Guerard and Herold, 1975; Hosaka *et al.*, 2020; Jian *et al.*, 2015)
- (iv) Layered polyanion-based compounds comprising pyrophosphates such as $\text{Na}_2\text{CoP}_2\text{O}_7$ and $\text{K}_2\text{MP}_2\text{O}_7$ ($M = \text{Co, Ni, Cu}$), pyrovanadates such as $\text{K}_2\text{MnV}_2\text{O}_7$ and $\text{Rb}_2\text{MnV}_2\text{O}_7$, oxyphosphates such as NaVOPO_4 and LiVOPO_4 , layered KVOPO_4 , diphosphates such as $\text{Na}_3\text{V}(\text{PO}_4)_2$, fluorophosphates such as $\text{Na}_2\text{FePO}_4\text{F}$, hydroxysulphates such as LiFeSO_4OH and oxysilicates such as $\text{Li}_2\text{VOSiO}_4$. (Barpanda *et al.*, 2018, 2012; Jin *et al.*, 2020; Liao *et al.*, 2019; Liu *et al.*, 2018; Masquelier and Croguennec, 2013; Niu *et al.*, 2019; Prakash *et al.*, 2006; Yahia *et al.*, 2007)

In a majority of the aforementioned exemplars of layered materials, the mobility of the cations can be traced to extremely weak chemical bonds whose strength is correlated with the strength of emergent forces such as Van der Waals interactions and the inter-layer distance between the slabs. (Delmas *et al.*, 2021; Dresselhaus and Dresselhaus, 1981; Kanyolo *et al.*, 2021; Sun *et al.*, 2019; Whittingham, 2004)

Layered transition metal oxides display a wide swath of crystal structural versatility and composition tuneability. They have thus been the subject of passionate research in various realms of solid-state (electro)chemistry, materials science and condensed matter physics. (He *et al.*, 2012; Kalantar-zadeh *et al.*, 2016; Kanyolo *et al.*, 2021; Kubota, 2020; Liu *et al.*, 2019; McClelland *et al.*, 2020; Schnelle *et al.*, 2021) A specific class of layered transition metal oxide materials has recently emerged adopting, *inter alia*, chemical compositions embodied mainly by A_4MDO_6 , $A_3M_2DO_6$ or $A_2M_2DO_6$ wherein A represents an alkali-ion (Li, Na, K, etc.) or coinage metal ions suchlike Ag, whereas M is mainly a transition metal species such as Co, Ni, Cu, Zn, etc. and D depicts a pnictogen or chalcogen metal species such as Sb, Bi, Te and so forth. (Berthelot *et al.*, 2012; Brown *et al.*, 2019; Derakhshan *et al.*, 2007; Evstigneeva *et al.*, 2011; Grundish *et al.*, 2019; Kumar *et al.*, 2012;

Nagarajan *et al.*, 2002; Nalbandyan *et al.*, 2013; Politaev *et al.*, 2010; Roudebush *et al.*, 2013; Seibel *et al.*, 2013; Skakle *et al.*, 1997; Smirnova *et al.*, 2005; Stratan *et al.*, 2019; Uma and Gupta, 2016; Viciu *et al.*, 2007; Yadav *et al.*, 2019; Zvereva *et al.*, 2013, 2016, 2012) In these materials, mobile A cations are sandwiched between slabs entailing M atoms coordinated with oxygen around D atoms in a hexagonal (honeycomb) arrangement. We shall thus refer to these materials as honeycomb layered oxides.(Kanyolo *et al.*, 2021)

Of particular interest are the dynamics of the cations within the aforementioned materials,(Kanyolo and Masese, 2020) since their diffusion contributes a net current when a sufficient external electric field arises in an electrode-electrolyte setup forming a cell or battery.(Goodenough and Park, 2013) The polarity of the electric field defines the charging and discharging processes corresponding to de-intercalation (cation extraction) and intercalation (cation insertion/reinsertion) processes respectively. Theoretical computations show that the diffusion paths are largely restricted to honeycomb pathways in honeycomb layered tellurates (for instance, $\text{K}_2\text{Ni}_2\text{TeO}_6$ and $\text{Na}_2\text{Ni}_2\text{TeO}_6$)(Bera and Yusuf, 2020; Masese *et al.*, 2018), where locations of the cations are correlated with specific sites defined by the honeycomb octahedral structures within the slabs. Thus, the Van der Waals forces initially localize the cations, forming a loosely-bound two dimensional hexagonal lattice where mobility of the cations is only possible when sufficient activation energy can offset this localization leaving cationic vacancies.(Matsubara *et al.*, 2020; Wang *et al.*, 2018) Since the number of cationic vacancies should correspond to the number of mobile cations if the material had no initial vacancy defects, it is reasonable to expect that the diffusion in the material can be completely captured either by the dynamics of the cations within the lattice or by the dynamics of the cationic vacancies.

In particular, when the cations are bosons, a Fermi level does not exist, implying that a particle-hole picture, where the particle and the vacancy carry separate pieces of information is precluded. Thus, the vacancies cannot be treated as holes, but an equivalent description for the dynamics of the cations carrying the same (thermodynamic) information. Consequently, a Bose-Einstein condensate of the cations(Kanyolo and Masese, 2020) avails a prime avenue for an emergent geometric description of such vacancies as topological defects within a theory of diffusion in the context of emergent quantum gravity. Conversely, describing the diffusion in layered materials comprising fermionic cations such as ${}^6\text{Li}$ with this approach would pose some significant challenge. Nonetheless, since their magnetic moment is readily trace-able in Nuclear Magnetic Resonance experiments, the fermionic cations are typically introduced in

meager amounts via doping techniques in order to improve resolution.(Lee *et al.*, 2000; Pan *et al.*, 2002) Consequently, their overall effects on the diffusion properties are expected to be negligible. Nonetheless, if the vacancies are treated as holes it is expected that this particle-hole symmetry is rather befitting to cationic Majorana modes *e.g.* with twist defects(Beenakker, 2013; Bombín, 2010; Zheng *et al.*, 2015) which could be exploited to incorporate fermionic behavior in the formalism.(Kanyolo, 2019)

Considering emergent gravity to describe defects in crystals is not entirely a novel idea.(Holz, 1988; Kleinert, 1988, 1987, 2005; Verçin, 1990; Yajima and Nagahama, 2016) For instance, it has long been proposed that considering finite torsion (non-symmetric Christoffel symbols/affine connection, $\Gamma_{\mu\nu}^\rho \neq \Gamma_{\nu\mu}^\rho$) within the context of Einstein-Cartan theory ought to capture various intriguing aspects related to disclinations and dislocations within crystals.(Holz, 1988; Kleinert, 1988, 1987; Verçin, 1990; Yajima and Nagahama, 2016) Moreover, it has been further argued that Einstein gravity can still emerge in a crystal whose kinetic energy order terms are restricted to second-order in derivatives(Kleinert, 2005) in accordance with Lovelock's theorem.(Lovelock, 1971) Thus, since the connection considered herein is torsion-free ($\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$) and topological defects can still be non-vanishing even for torsion-free manifolds as long as higher order derivatives of the Gauss-Bonnet type(Lovelock, 1971) are present in the crystal, herein we shall consider a description of bosonic cationic vacancies as topological defects.

In particular, the radial distribution function (pair correlation function), $g(\vec{x}_a)$ is the conditional probability density that a cation will be found at \vec{x}_a at each inter-layer, relative to another within the same inter-layer. Equivalently, it is the average density of the cations at \vec{x}_a relative to a tagged particle.(Chandler, 1987) This means the number density, $\rho_b g(\vec{x}_a) = \rho_{2D}^a(\vec{x}_a)$ is normalized as(Tuckerman, 2010),

$$\int_{\mathcal{A}_a^E} d^2x_a \sqrt{\det(g_{ij}^a(\vec{x}_a))} \rho_{2D}^a(\vec{x}_a) = N_a - 1, \quad (141)$$

where N_a is the number of cations within the inter-layer, a , ρ_b is the bulk number density and the integration is performed over the Euclidean $n = 2$ manifolds, \mathcal{A}_a^E at each inter-layer with a metric tensor, $g_{ij}^a(\vec{x}_a)$. Note that Einstein summation convention should not be applied for the index, a . The -1 in the normalization can be thought to arise from excluding the contribution of the reference cation, as per the above definition. The center of mass coordinates describe average diffusion properties and hence must obey the uncoupled Langevin equations,

$$\frac{d^2\vec{x}_a}{ds^2} = -\vec{p}_a(s) + \vec{\eta}_a(s), \quad (142)$$

where $s = \pm it$ is Euclidean time, $\vec{\eta}_a(s)$ is the acceleration and $\vec{p}_a = \bar{m}d\vec{x}_a(s)/ds$ are the center of mass momenta with \bar{m} the average mass of the cations and $1/\bar{m}$ playing the role of a mean time between collisions, assumed to be equivalent in all slabs due to translation invariance, $z_{a+1} = z_a + \Delta z_a$ along the vector, $\vec{n} = (0, 0, 1)$ normal to the slabs (the z direction). This guarantees that in the continuum limit and when the energy of cations is conserved, the crystal must admit not only a time-like but also a z -like Killing vector.

Observe that, we can follow eq. (112) and define the Gauss-Bonnet theorem at each inter-layer by making the identification,

$$\frac{d^2\vec{x}_a}{ds^2} = k_g^a(s), \quad (143)$$

$$\mathcal{H}_a = M \int_{\mathcal{C}_a^E} k_g^a(s) ds, \quad (144)$$

where \mathcal{H}_a is the Hamiltonian at each inter-layer, a and M is the total mass of the mobile cations in the structure. Since we shall be concerned with momentum conservation, $d^2\vec{x}_a/ds^2 = 0$, we can set $k_g^a(s) = 0$, which implies the Gauss-Bonnet theorem expression is devoid boundary terms and hence simplifies to,

$$\int_{\partial\mathcal{A}_a^E} p_a = \int_{\mathcal{A}_a^E} \star K_a = \int_{\mathcal{A}_a^E} d\eta_a = 2\pi(2 - 2h_a), \quad (145)$$

where h_a is the genus of \mathcal{A}_a^E . If we interpret the cationic vacancies in \mathcal{A}^E as the genus, then eq. (145) reveals that the energy needed to create vacancies in the vacuum must always balance the energy due to motion, since $p^a \neq 0$ when $h_a \neq 1$.

On the other hand, the Fokker-Planck equation corresponds to the Newtonian limit,

$$\nabla^2\Phi(x) = \frac{1}{\bar{m}}\rho(x)\exp(2\Phi(x)), \quad (146a)$$

where we have used $u^0(x) = \exp(\Phi(x)) \neq 1$ and $1/\bar{m} \leftrightarrow 4\pi GM = \beta/2$, requiring \bar{m} to play the role of temperature. In order to guarantee $\rho(\vec{x})$ can be related to the two dimensional number density, we impose time-like, $\xi_t^\mu = (-1, \vec{0})$ and z -like, $\xi_z^\mu = (\vec{0}, 1)$ Killing vectors which guarantee that $\partial\rho(x)/\partial t = \partial\rho(x)/\partial z = 0$, where,

$$\rho(x) = \rho_b \exp(-\bar{\beta}M\Phi(x)), \quad (146b)$$

is the Boltzmann factor with $\bar{\beta} = 1/\bar{m}$. Moreover, since \bar{m} is also defined as the average mass of the cations, we have $\bar{\beta}M = N$ as required by eq. (85a).

Thus, the Newtonian potential ought to split into z -like inter-layer slices labeled by index a , each satisfying,

$$\nabla_a^2\Phi_a(\vec{x}_a) = \frac{1}{\bar{m}}\rho_a(\vec{x}_a)\exp(2\Phi_a(\vec{x}_a)), \quad (146c)$$

where $\nabla_a^2 = \partial^2/\partial x_a^2 + \partial^2/\partial y_a^2$ is the two dimensional Laplacian operator. Thus, to be consistent with eq. (141), the number density normalization is given by,

$$\begin{aligned} N - \Sigma &= \int_{\mathcal{V}^E} d^3x \exp(2\Phi(x))\rho(x) \\ &= \lim_{\Delta z_a \rightarrow 0} \sum_{a=1}^{\Sigma} \Delta z_a \int_{\mathcal{A}_a^E} d^2x_a \exp(2\Phi_a(\vec{x}_a))\rho_a(\vec{x}_a) \\ &= \sum_{a=1}^{\Sigma} (N_a - 1), \end{aligned} \quad (147a)$$

where $N = \sum_{a=1}^{\Sigma} N_a$ is the total number of cations, Σ is the total number of two dimensional manifolds corresponding to the space between the layers (inter-layers) of the material and,

$$\Delta z_a \rho_a(\vec{x}_a) = \rho_{2D}^a(\vec{x}_a) \quad (147b)$$

$$\exp(2\Phi(\vec{x}_a)) = \sqrt{\det(g_{ij}^a(\vec{x}_a))}, \quad (147c)$$

with Δz_a the inter-layer spacing.

Thus, we recognize eq. (146c) as Liouville's equation, with the Gaussian curvature given by,

$$K_a(\vec{x}_a) = -\frac{1}{\bar{m}}\rho_a(\vec{x}_a). \quad (148a)$$

This means that the $n = 2$ manifold, \mathcal{A}_a^E is described by the conformal metric,

$$d\sigma_a^2 = f^2(dx_a^2 + dy_a^2), \quad (148b)$$

where $f^2 = \exp(2\Phi(\vec{x}_a)) = \sqrt{\det(g_{ij}^a(\vec{x}_a))}$ is the conformal factor. To find the order of magnitude for Δz_a , we should use eq. (145), eq. (148a) and eq. (147b) to find,

$$h_a = N_a, \quad (149a)$$

$$\Delta z_a = 1/4\pi\bar{m}. \quad (149b)$$

A potential problem with the model is that the cations are charged while typically the gravitational field, $\Phi_a(\vec{x}_a)$ is not. However, this poses no problem since we only need to require that the $SU(N)$ symmetry in our approach to break to $\mathbf{F} \rightarrow F = dA$, where $A = A_\mu dx^\mu$ is the electromagnetic (U(1)) gauge potential and hence requiring the wave function, Ψ to be charged. To consistently introduce the electric field, we shall require the (random) accelerations and the friction terms respectively, in the Langevin equation to take the forms,

$$\vec{\eta}_a = q_m(\vec{n} \times \vec{E}_a), \quad (150a)$$

$$\vec{\nabla}_a\Phi(\vec{x}_a) = \vec{p}_a, \quad (150b)$$

where $\vec{E}_a = (E_{x_a}, E_{y_a}, 0)$ is the electric field on \mathcal{A}_a^E responsible for the (de-)intercalation process in the cell, q_m

is the magnetic charge and $\vec{n} = (0, 0, 1)$ is the unit vector normal to the slabs. The Gauss-Bonnet theorem then requires that the Dirac quantization condition (Dirac, 1931) is satisfied,

$$\begin{aligned}\Phi_a &= \int_{\mathcal{C}^E} p_a = \int_{\mathcal{C}^E} \eta_a \\ &= q_m \int_{\mathcal{C}^E} d\vec{x}_a \cdot (\vec{n} \times \vec{E}_a) = q_m \int_{\mathcal{A}^E} d^2x \vec{\nabla}_a \cdot \vec{E}_a \\ &= q_m q_e = 2\pi\nu_a = \pi\chi_2(h_a),\end{aligned}\quad (150c)$$

where $q_e = \int_{\mathcal{A}^E} d^2x \vec{\nabla}_a \cdot \vec{E}_a$ is the electric charge, $\vec{\nabla}_a \cdot \vec{E}_a = -f^2 K_a / 2q_m$, ν_a is the monopole number and $\chi_2(h_a) = 2 - 2h_a$. Notably, Φ_a is analogous to the Aharonov-Casher phase (Aharonov and Casher, 1984), where the magnetic moment corresponds to, $\vec{\mu} = q_m \vec{n}$. Thus, it is intuitive to view the cations diffusing along curves around neutral vacancies with a magnetic moment.

Consequently, the pair correlation function can be written as a Boltzmann factor,

$$g(h_a) = \exp\left(\frac{M}{\bar{m}} \pi \chi_2(h_a)\right) = \mathcal{N}^{\chi_2(h_a)},\quad (151)$$

where $\mathcal{N} = \exp(\pi N)$ and $M/\bar{m} = N$. Thus, we recognize the $1/\mathcal{N}$ expansion factor appearing in the pair correlation function. We expect the pair correlations to be calculated for varied cationic vacancies as cations are created/annihilated on the manifold. Thus, a weighted sum over the distinct topologies yields,

$$\langle g_a \rangle = \sum_{h_a} P_{h_a}(\lambda) \mathcal{N}^{\chi_2(h_a)},\quad (152)$$

where $P_{h_a}(\lambda)$ is the probability for the topology h_a to occur in a given (de-)intercalation process and $\lambda(\bar{m})$ is a parameter that ought to depend only on \bar{m} and hence the temperature, $\bar{\beta}^{-1} = \bar{m}$. Consequently, eq. (152) corresponds to the sum of vacuum Feynman diagrams of a large \mathcal{N} theory, where the 't Hooft limit ($\mathcal{N} \rightarrow \infty$, keeping λ fixed) makes physical sense, since it can be interpreted as a consequence of considering a fixed equilibrium temperature environment with a large number of particles in the material.

VI. DISCUSSION

Gravity, as formulated by Einstein, is not simply a force like electro-magnetism or the weak and strong nuclear forces but rather manifests itself as space-time curvature in a pseudo-Riemannian $n = 4$ manifold. (Thorne *et al.*, 2000) Thus, the emergence of such manifolds from an underlying theory with desirable classical or quantum properties would be sufficient to test

certain thermodynamic and topological features in general relativity and specifically quantum gravity. In the present work, we have shown that the foundations of our approach reveal a rich complex-Hermitian structure analogous to Cayley-Dickson algebras (Schafer, 1954), which can be exploited to formulate the appropriate action principle to yield general relativity in $n = 4$ dimensions as the effective theory, *albeit* with a Gauss-Bonnet (topological) term. Moreover, we treat the ill-defined Euclidean path integral measure of quantum gravity, $\mathcal{D}[g_{\mu\nu}^E]$ by taking the sum over topologically distinct manifolds, $\int \mathcal{D}[g_{\mu\nu}^E] \rightarrow \sum_{\mathcal{M}^E \in h}$, which yields a partition function reminiscent of the $1/\mathcal{N}$ expansion of an unknown large \mathcal{N} theory. (Freidel, 2005; Gielen *et al.*, 2013; Gurau, 2011, 2012; Gurau and Rivasseau, 2011; Maldacena, 1999; Oriti, 2006; Thorn, 1994)

However, the (pseudo-)Riemannian manifold in our approach is not emergent, rather it is introduced *ab initio* in the treatment. Nonetheless, large \mathcal{N} theories have the feature that the topologies of the vacuum Feynman diagrams pave an emergent manifold, with dimensions equal to the rank of the tensor group theory. Moreover, it has been conjectured that specific large \mathcal{N} theories are equivalent to some string theories. (Freidel, 2005; Gielen *et al.*, 2013; Gurau, 2011, 2012; Gurau and Rivasseau, 2011; Maldacena, 1999; Oriti, 2006; Thorn, 1994) Thus, it would be interesting to investigate whether these ideas are compatible with the work herein. Nonetheless, the expectation is that rank $n = 4$ tensor group theories are prime candidates for the large \mathcal{N} theory we seek (Gurau, 2011, 2012; Gurau and Rivasseau, 2011), where the group field, φ need not be space-time dependent. A particularly curious fact is that, unlike random matrix large N theories in $n = 4$ dimensions, \mathcal{N}/λ is not dimensionless, but has dimensions of $1/(\text{mass})^2$.

For illustration purposes, we can substitute, in eq. (96), the bosonic action, $S(\varphi) = \frac{\mathcal{N}}{\lambda} \text{Tr}(\gamma(\varphi))$ with $\gamma(\varphi) = \varphi^2/2 - \varphi^4/4g^2$, where the free propagator after appropriate re-scaling, $\varphi \rightarrow \varphi/g$ corresponds to $\langle \varphi^2 \rangle \simeq \lambda/g^2 \mathcal{N}$ and g is a coupling constant with mass dimensions. Thus, the field, φ becomes dimensionless after re-scaling, is assumed Hermitian and need not be space-time dependent. Assuming φ transforms as a vector under the group, $U(\mathcal{N})$ constrained by $\lambda/\pi = g^2 N = \ln(\mathcal{N}) = \mathcal{S}$, where the 't Hooft limit corresponds to $N, \mathcal{N} \rightarrow \infty$ at fixed λ/π .

The free energy contribution from any vacuum Feynman diagram, $\mathcal{F}_{hF}(N, \mathcal{N})$ at a given loop level, F is proportional to the product of three terms ('t Hooft, 1993),

$$\begin{aligned}\mathcal{F}_{hF}(N, \mathcal{N}) &= f_F \left(\frac{g^2 \mathcal{N}}{\lambda}\right)^V \left(\frac{g^2 \mathcal{N}}{\lambda}\right)^{-E} \mathcal{N}^F \\ &= (\pi N)^{F - \chi_2(h)} f_F \mathcal{N}^{\chi_2(h)},\end{aligned}\quad (153)$$

where F is the number of loops, E is the number of propagators, V is the number of vertices, corresponding to two-particle interactions, f_F is the proportionality constant

which depends only on F , and $F - E + V = \chi_2(h) = 2 - 2h$ is the Euler characteristic of some emergent $n = 2$ dimensional Riemannian manifold, \mathcal{A}^E , with the Feynman diagrams acting as the simplex triangulation of the manifold with F faces, E edges and V vertices. Consequently, the free energy contribution at a given topology level, $h = g + b/2$ (where g is the genus of \mathcal{A}^E and b the boundary contribution) corresponds to the sum over loop diagrams,

$$\mathcal{F}_h(N, \mathcal{N}) = \sum_{F=0}^{+\infty} \mathcal{F}_{hF}(N, \mathcal{N}) = f_h(N) \mathcal{N}^{\chi_2(h)}, \quad (154a)$$

where,

$$f_h(N) = (\pi N)^{-\chi_2(h)} \sum_{F=0}^{+\infty} (\pi N)^F f_F. \quad (154b)$$

In fact, eq. (154a) is robust and is obtained by most rank $n = 2$ large \mathcal{N} group theories, *albeit* with differing $f_h(N)$ values which completely characterize the specific theory under consideration.

In our approach, we argued that $\chi_2(\mathcal{A}^E) = \chi_4(\mathcal{M}^E)$ is possible, where \mathcal{M}^E is the $n = 4$ dimensional *compact* Riemannian manifold corresponding to the connected sum of h number of $n = 4$ -tori (T^4) and an arbitrary number of $n = 4$ -spheres (S^4). Thus, the $1/\mathcal{N}$ expansion in eq. (95) corresponds to a large \mathcal{N} theory with $f_h(N) \simeq \exp(-g^2\lambda) = \exp(-\pi N) = 1/\mathcal{N}$, where the coupling constant satisfies,

$$g^{-2} = -\frac{1}{\pi} \int_{\mathcal{M}^E} \sqrt{\det(g_{\mu\nu}^E)} R = \frac{M}{2\lambda} \int_0^\beta ds = \frac{\pi N}{\lambda}. \quad (155)$$

Here, we have used eq. (98) in Euclidean signature and eq. (100b) respectively. Consequently, due to the form of eq. (154b), it is prudent to re-scale the path integral measure as, $\sum_{\mathcal{M}^E \in h} \rightarrow \sum_{\mathcal{M}^E \in h} (\pi N)^{\chi_4(h)}$ in order for $f_F = 1/F!$ to appropriately yield eq. (95) with the quantum gravity partition function corresponding to the $1/\mathcal{N}$ expansion, $\mathcal{Z}_{\text{QG}}^E = \sum_h \mathcal{F}_h(N, \mathcal{N})$. The fermionic case should follow similar treatment and considerations with a suitable group field theory.

Finally, using the temperature and entropy of the Kerr-Newmann black hole (Bekenstein, 2008) instead of Schwarzschild's should alter the expressions considered, but not the conclusions herein. Nonetheless, we are left to puzzle how Unruh-Hawking radiation (Hawking, 1974, 1976a; Unruh, 1977), and more importantly, the black hole information paradox (Hawking, 1976b; Mathur, 2009) fits into this picture. Since a space-time with an evaporating black hole will be subject to fluctuation-dissipation effects (Banerjee and Majhi, 2020), one expects that our approach lacks key features which ought to be incorporated in future works. Another interesting observation is that breaking $\text{SU}(N)$ to $\text{SO}(N)$ by the

identifications in eq. (36) is possible since the number of anti-symmetric $\text{SU}(N)$ generators is given by $(N^2 - N)/2$, which is equivalent to the number of $\text{SO}(n)$ generators, $(n^2 - n)/2$, which allows the decomposition of ω in eq. (35) when $N = n$ is the number of dimensions. Consequently, due to eq. (106), eq. (23c) can be interpreted as a relation between the central charge of a black hole, k and the number of dimensions, n . Such a relationship has been conjectured in the context of the sphere packing optimization problem in n dimensions and quantum gravity. (Hartman *et al.*, 2019)

In summary, we have employed the thermodynamic expressions for entropy and temperature of Schwarzschild black holes to constrain an $\text{SU}(N)$ gauge theory leading to the emergence of a quantum framework for gravity, where the gravitational degrees of freedom are the $N \in \mathbb{Z} \geq 0$ colors and a condensate comprising color pairs, $k = N/2$, appropriately coupled to the Yang-Mills gauge field. The foundations of our approach reveal a complex-Hermitian structure, constructed as [Ricci tensor $\pm\sqrt{-1}$ Yang-Mills field strength], whose structure is analogous to Cayley-Dickson algebras (Schafer, 1954), which aids in formulating the appropriate action principle. The $\text{SU}(N)$ gauge group is broken into an effective $\text{SU}(4) \rightarrow \text{SO}(4) \leftrightarrow \text{SO}(1, 3)$ field theory with two terms: the Einstein-Hilbert action and a Gauss-Bonnet topological term. Moreover, the Euclidean path integral is considered as the sum over manifolds with distinct topologies, $h \in \mathbb{Z} \geq 0$ homeomorphic to connected sums of an arbitrary number of $n = 4$ -spheres and h number of $n = 4$ -tori.

Consequently, the partition function takes a reminiscent form of the sum of the vacuum Feynman diagrams for a large $\mathcal{N} = \exp(\beta M/2)$ theory, provided $\mathcal{S} = \beta M/2 = \pi N$ is the Schwarzschild black hole entropy, $\beta = 8\pi GM$ is the inverse temperature, G is gravitational constant, M is the black hole mass and horizon area, $\mathcal{A} = 2G\beta M = 4\pi GN$ is pixelated in units of $4\pi G$. This leads us to conclude that the partition function for quantum gravity is equivalent to the vacuum Feynman diagrams of a yet unidentified large \mathcal{N} theory in $n = 4$ dimensions. Our approach also sheds new light on the asymptotic behavior of dark matter-dominated galaxy rotation curves (the empirical baryonic Tully-Fisher relation) (Eisenstein and Loeb, 1997; Famaey and McGaugh, 2012; Kanyolo and Masese, 2021; Keeton, 2001; Martel and Shapiro, 2003; McGaugh, 2012; McGaugh *et al.*, 2000; Persic *et al.*, 1996) and emergent gravity in condensed matter systems with defects (Holz, 1988; Kleinert, 1987; Kleinert and Zaanen, 2004; Verçin, 1990; Zaanen *et al.*, 2004) such as layered materials (Kanyolo *et al.*, 2021; Masese *et al.*, 2021a,b,c, 2018) which admit cationic vacancies as topological defects. (Kanyolo and Masese, 2020)

While we acknowledge quantum gravity is a broad sub-

ject whose rapid progress largely remains uncovered by the present work, we consider the demonstration of the intriguing ramifications, not only in quantum gravity research but also in condensed matter and materials science, as warranting considerable merit.

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