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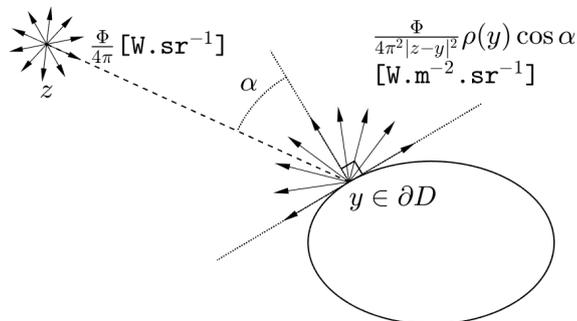
# AN EXACT RADON FORMULA FOR LAMBERTIAN TOMOGRAPHY

JEAN-BAPTISTE BELLET

**ABSTRACT.** This letter, based on an extension of the Radon transform on distributions, is a mathematical contribution to the field of reflective tomography in optics. Indeed, we tackle the reconstruction of a Lambertian convex reflector using tomography. In a bi-dimensional setup, we prove that the Lambert's cosine law can be inverted by some Radon formula involving distributions. The associated reconstruction contains the geometry and the physics of the problem: it is a Radon measure supported by the reflector, and its density is the inverse of the albedo.

## 1. INTRODUCTION

Over the past decades, there has been a considerable interest in developing new imaging modalities based on scattering of light and extensions of the Radon transform, such as Compton scattering tomography [17, 20], Bragg scattering tomography [21], or reflective tomography [2, 3, 8, 11, 12, 19]. This paper deals with reflective tomography in visible to near-infrared optics. In this field, one reconstructs a scene from optical images using a Radon inversion. The geometry of such a reconstruction is meaningful [1], because the Radon transform is a Fourier Integral Operator [13]. Nevertheless, to the author's knowledge, reflective tomography is not justified by an exact inversion formula. In this letter, we tackle this problem in the case of uniform diffuse reflection off a strictly convex surface, as in Figure 1. We derive a Radon formula which rigorously inverts a Lambert's cosine law.



**FIGURE 1.** Diffuse reflection by the Lambert's cosine law on a strictly convex surface. Here, an isotropic point source  $z$  emits light with a power per unit solid angle  $\Phi/(4\pi)$ . On the surface  $\partial D$ , an illuminated point  $y$  reflects light uniformly; for an angle of incidence  $\alpha$ , the point  $y$  reflects a radiance  $\Phi/(4\pi^2|z-y|^2) \rho(y) \cos \alpha$  in any direction above the tangent plane. The dimensionless coefficient  $\rho(y) \in [0, 1]$  is the albedo, defined as the percentage of incident irradiance which is reflected.

To begin with, let us describe uniform diffusion of light from an ideal matte opaque surface, called a *Lambertian reflector*. This modeling is based on radiometric concepts [22], and is an usual model for reflection off surfaces in optics; see for example [14, Sec. 3.A], [7]. As in Figure 1, we assume that the surface of a Lambertian reflector, denoted by  $\partial D$ , is illuminated by an isotropic point source located at  $z$ . We denote by  $\Phi$  the power of this source (energy by time unit, in [W]). The associated power per unit solid angle is  $\Phi/(4\pi)$  (in  $[\text{W} \cdot \text{sr}^{-1}]$ ). Therefore, an illuminated point  $y \in \partial D$ , with angle of incidence  $\alpha$ , receives an *irradiance*  $\Phi \cos \alpha / (4\pi |z-y|^2)$  (power per unit area,

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in  $[\text{W}\cdot\text{m}^{-2}]$ ). By assumption, this incident irradiance is uniformly reflected off the surface; it is assumed that the point  $y$  radiates the same *radiance* (power per unit projected area, and per unit solid angle, in  $[\text{W}\cdot\text{m}^{-2}\cdot\text{sr}^{-1}]$ ), in any direction above the tangent plane. This uniform radiance is given by the *Lambert's cosine law*

$$\frac{\Phi}{4\pi^2|z-y|^2}\rho(y)\cos\alpha;$$

here, the (dimensionless) *albedo*  $\rho(y) \in [0, 1]$  represents the percentage of the incident irradiance which is reflected in any direction. This model of radiance is of particular interest, since a light sensor in a camera measures a quantity which is essentially proportional to the radiance of the visible point  $y$  [10]. Finally, assuming that  $z$  is in far field with  $|z-y| \approx R$  a large constant, we consider that the records are given by  $\rho(y)\cos\alpha$ , up to a constant factor.

In this letter, we restrict our attention to a Lambert's cosine law  $\rho(y)\cos\alpha$ , in a bi-dimensional setup, in  $\mathbb{R}^2$ . We aim at finding an explicit Radon formula which relates the geometry  $\partial D \subset \mathbb{R}^2$ , the albedo coefficient  $\rho : \partial D \rightarrow \mathbb{R}$ , and values of the "radiance"  $\rho(y)\cos\alpha$ . In comparison, in usual (X-ray) tomography, a record is an attenuated intensity along a line in a semi-transparent medium. Up to a factor, the *Beer-Lambert's law* models the record of a line  $L$  by

$$\exp\left[-\int_L a(x)d\ell\right],$$

where the function  $a(x)$  represents an attenuation coefficient of the medium [16]; in this case, the link with the Radon transform of  $a$ , defined by  $\mathcal{R}[a](L) = \int_L a(x)d\ell$ , is very clear. For a Lambertian reflector, extracting a Radon transform from the cosine law  $\rho(y)\cos\alpha$  is not so clear. Our main result, Theorem 1, answers to this question for a strictly convex reflector; it is based on an extension of the Radon transform on distributions, as described in [9, Chap. 1], [15, Chap. 2].

## 2. BACKGROUND AND NOTATION

Unless otherwise stipulated, we assume throughout the letter:

**(A1):**  $D \subset \mathbb{R}^2$  is a bounded open set with  $\mathcal{C}^1$  boundary  $\partial D$ , such that the closure  $\bar{D}$  is strictly convex, *i.e.*  $\forall x, y \in \bar{D}, \forall t \in (0, 1), tx + (1-t)y \in D$ ;  $\nu(y) \in \mathbb{S}^1$  denotes the exterior unit normal vector to  $D$  at  $y \in \partial D$ , and  $\mu$  denotes the length measure on  $\partial D$ .

**(A2):**  $\rho \in L^\infty(\partial D)$  is a positive function, bounded and bounded away from zero, *i.e.*  $\rho(\partial D) \subset [c, C]$  where  $c, C$  are two positive constants.

As displayed in Figure 2, for any angle  $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$ , and for any  $s \in \mathbb{R}$ , the set  $\{x \in \mathbb{R}^2 : x \cdot \theta = s\}$  represents a line at signed distance  $s$  from the origin, orthogonal to  $\theta$ , and oriented by  $\theta^\perp := (\theta_2, -\theta_1)$ . For our purpose, intersecting lines with  $\partial D$  is of particular interest; we refer to Lemma 1 in Appendix A for intermediate results related to this problem of convex analysis. In

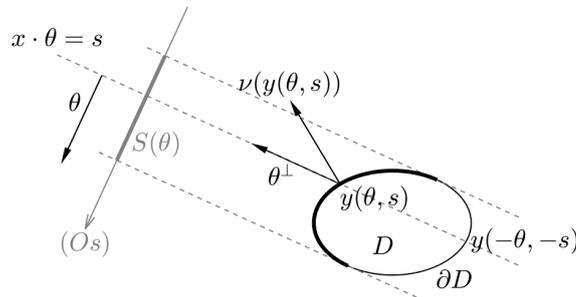


FIGURE 2. Geometrical setup of Lemma 1 and Definition 1. For any angle  $\theta \in \mathbb{S}^1$ , the set  $S(\theta)$  contains the values  $s \in \mathbb{R}$  such that the line  $x \cdot \theta = s$  intersects  $D$ . For any  $s \in S(\theta)$ ,  $\partial D \cap \{x \cdot \theta = s\} = \{y(\theta, s), y(-\theta, -s)\}$  with  $\nu(y(\theta, s)) \cdot \theta^\perp > 0$ . For  $s \in \partial S(\theta)$ , the line  $x \cdot \theta = s$  is tangent to  $\partial D$  at a unique  $y(\theta, s)$ , and  $\nu(y(\theta, s)) \cdot \theta^\perp = 0$ . For any  $s \notin \bar{S}(\theta)$ ,  $\bar{D} \cap \{x \cdot \theta = s\} = \emptyset$ . The bold curve represents a set of visible points,  $\{y(\theta, s), s \in \bar{S}(\theta)\}$ .

particular, Lemma 1.(i) proves that if a line  $x \cdot \theta = s$  intersects  $\bar{D}$ , either  $\bar{D} \cap \{x \cdot \theta = s\} = \{y(\theta, s)\}$  with  $\nu(y(\theta, s)) \cdot \theta^\perp = 0$ , either  $\partial D \cap \{x \cdot \theta = s\} = \{y(\theta, s), y(-\theta, -s)\}$  with  $\nu(y(\theta, s)) \cdot \theta^\perp > 0$ . The set of values  $s$  associated to the latter case is the set  $S(\theta)$  given by (3). Lemma 1.(i) justifies the following definition.

**Definition 1.** Assume (A1-A2). The *lambertian projection* of  $(D, \rho)$ , denoted by  $\mathcal{L}[D, \rho]$ , is a function defined on  $\mathbb{S}^1 \times \mathbb{R}$  as follows. For every  $(\theta, s) \in \mathbb{S}^1 \times \mathbb{R}$ ,

- if the line  $x \cdot \theta = s$  does not intersect  $\bar{D}$ , then  $\mathcal{L}[D, \rho](\theta, s) := 0$ ;
- if the line  $x \cdot \theta = s$  intersects  $\bar{D}$ , the *visible point*  $y(\theta, s)$  is defined as the unique point  $y \in \partial D \cap \{x \cdot \theta = s\}$  such that  $\nu(y) \cdot \theta^\perp \geq 0$ , and

$$\mathcal{L}[D, \rho](\theta, s) := \rho(y(\theta, s)) \nu(y(\theta, s)) \cdot \theta^\perp.$$

### 3. MAIN RESULT

**Theorem 1.** Assume (A1-A2). Let  $\mathcal{R} : \mathcal{E}'(\mathbb{R}^2) \rightarrow \mathcal{E}'(\mathbb{S}^1 \times \mathbb{R})$  denote the Radon transform extended on distributions by duality. Then,

$$\mathcal{R} \left[ \frac{1}{\rho} d\mu \right] = \left( \frac{1}{\mathcal{L}[D, \rho](\theta, s)} + \frac{1}{\mathcal{L}[D, \rho](-\theta, -s)} \right) \mathbb{1}_{\mathcal{L}[D, \rho](\theta, s) > 0}, \quad (1)$$

where, by convention, the right member takes the value 0 if (and only if)  $\mathcal{L}[D, \rho](\theta, s) = 0$ .

*Proof.* The proof is a combination of the definitions of a Radon measure [4, Chap. 4], a Radon transform on distributions [9, Chap. 1], and the formula (5) of Appendix A for integration on  $\partial D$ . Indeed, the function  $\frac{1}{\rho}$  is bounded on  $\partial D$ ; therefore  $\frac{1}{\rho} \in L^1(\partial D, d\mu)$ , and  $\frac{1}{\rho} d\mu \in \mathcal{E}'(\mathbb{R}^2)$  is defined as a compactly supported distribution by

$$\left\langle \frac{1}{\rho} d\mu, \psi \right\rangle = \int_{\partial D} \frac{\psi(y)}{\rho(y)} d\mu(y), \quad \psi \in \mathcal{E}(\mathbb{R}^2).$$

Then the Radon transform of  $\frac{1}{\rho} d\mu$  is a distribution in  $\mathcal{E}'(\mathbb{S}^1 \times \mathbb{R})$ , defined by

$$\left\langle \mathcal{R} \left[ \frac{1}{\rho} d\mu \right], \phi \right\rangle = \left\langle \frac{1}{\rho} d\mu, \mathcal{R}^* \phi \right\rangle, \quad \phi \in \mathcal{E}(\mathbb{S}^1 \times \mathbb{R}),$$

where  $\mathcal{R}^* \phi \in \mathcal{E}(\mathbb{R}^2)$  is defined by

$$\mathcal{R}^* \phi(x) = \int_{\mathbb{S}^1} \phi(\theta, x \cdot \theta) d\theta, \quad x \in \mathbb{R}^2.$$

Fix  $\phi \in \mathcal{E}(\mathbb{S}^1 \times \mathbb{R})$ . By Fubini's theorem,

$$\left\langle \mathcal{R} \left[ \frac{1}{\rho} d\mu \right], \phi \right\rangle = \int_{\partial D} \frac{\mathcal{R}^* \phi(y)}{\rho(y)} d\mu(y) = \int_{\mathbb{S}^1} \int_{\partial D} \frac{\phi(\theta, y \cdot \theta)}{\rho(y)} d\mu(y) d\theta.$$

For any  $\theta \in \mathbb{S}^1$ , the relation (5), with  $f(y) = \frac{\phi(\theta, y \cdot \theta)}{\rho(y)} \in L^\infty(\partial D) \subset L^1(\partial D)$ , proves that

$$\int_{\partial D} \frac{\phi(\theta, y \cdot \theta)}{\rho(y)} d\mu = \int_{S(\theta)} \frac{\phi(\theta, s)}{\rho(y(\theta, s)) \nu(y(\theta, s)) \cdot \theta^\perp} + \frac{\phi(\theta, s)}{\rho(y(-\theta, -s)) \nu(y(-\theta, -s)) \cdot (-\theta)^\perp} ds.$$

Therefore,

$$\left\langle \mathcal{R} \left[ \frac{1}{\rho} d\mu \right], \phi \right\rangle = \int_{\mathbb{S}^1} \int_{\mathbb{R}} \phi(\theta, s) \left( \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](\theta, s)} + \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](-\theta, -s)} \right) ds d\theta.$$

This computation shows that the compactly supported function

$$\frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](\theta, s)} + \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](-\theta, -s)}, \quad \text{with } \mathbb{1}_{s \in S(\theta)} = \mathbb{1}_{\mathcal{L}[D, \rho](\theta, s) > 0},$$

is in  $L^1(\mathbb{S}^1 \times \mathbb{R})$  and coincides as a distribution with  $\mathcal{R} \left[ \frac{1}{\rho} d\mu \right] \in \mathcal{E}'(\mathbb{S}^1 \times \mathbb{R})$ . This concludes the proof.  $\square$

**Corollary 1.** *Assume (A1-A2). Then, the distribution  $\frac{1}{\rho}d\mu \in \mathcal{E}'(\mathbb{R}^2)$  is uniquely determined by the Lambertian projection  $\mathcal{L}[D, \rho]$ ; moreover, it can be reconstructed by the formula*

$$\frac{1}{\rho}d\mu = \frac{1}{2\pi} \mathcal{R}^* \mathcal{H}_s \partial_s \frac{\mathbb{1}_{\mathcal{L}[D, \rho](\theta, s) > 0}}{\mathcal{L}[D, \rho](\theta, s)}; \quad (2)$$

here  $\mathcal{R}^* : \mathcal{D}'(\mathbb{S}^1 \times \mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R}^2)$  denotes the backprojection and  $\mathcal{H} : \mathcal{E}'(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$  denotes the Hilbert transform (both extended on distributions).

*Proof.* We apply the Radon inversion formula extended on distributions [9, Theorem 5.5], with  $T = \frac{1}{\rho}d\mu \in \mathcal{E}'(\mathbb{R}^2)$ ,

$$T = \frac{1}{4\pi} \mathcal{R}^* \mathcal{H}_s \partial_s \mathcal{R} T, \quad T \in \mathcal{E}'(\mathbb{R}^2).$$

Here, by definition of  $\mathcal{R}^* : \mathcal{D}'(\mathbb{S}^1 \times \mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R}^2)$ , and  $\mathcal{H} : \mathcal{E}'(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ , the formula means

$$\left\langle \frac{1}{\rho}d\mu, \psi \right\rangle = \frac{1}{4\pi} \int_{\mathbb{S}^1 \times \mathbb{R}} \left( \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](\theta, s)} + \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](\theta, -s)} \right) \partial_s \mathcal{H}_s [\mathcal{R}\psi(\theta, s)] d\theta ds, \quad \psi \in \mathcal{D}(\mathbb{R}^2),$$

with the usual Radon transform  $\mathcal{R}$ , Hilbert transform  $\mathcal{H}_s$  (with respect to  $s$ ), defined by

$$\begin{aligned} \mathcal{R}\psi(\theta, s) &= \int_{x \cdot \theta = s} \psi(x) dl \in \mathcal{D}(\mathbb{S}^1 \times \mathbb{R}), \quad \psi \in \mathcal{D}(\mathbb{R}^2), \\ \mathcal{H}_s g(s) &= \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{g(t)}{s - t} dt \in \mathcal{E}(\mathbb{R}), \quad g \in \mathcal{D}(\mathbb{R}). \end{aligned}$$

Since  $\partial_s \mathcal{H}_s [g(-s)] = [\partial_s \mathcal{H}_s g](-s)$ , the change of variable  $(\theta, s) := (-\theta, -s)$  yields

$$\int_{\mathbb{S}^1 \times \mathbb{R}} \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](\theta, s)} \partial_s \mathcal{H}_s [\mathcal{R}\psi(\theta, s)] d\theta ds = \int_{\mathbb{S}^1 \times \mathbb{R}} \frac{\mathbb{1}_{-s \in S(-\theta)}}{\mathcal{L}[D, \rho](\theta, s)} \partial_s \mathcal{H}_s [\mathcal{R}\psi(-\theta, -s)] d\theta ds.$$

By the way, we have the symmetry  $S(\theta) = -S(\theta)$  and  $\mathcal{R}\psi(\theta, s) = \mathcal{R}\psi(-\theta, -s)$ , and we obtain

$$\begin{aligned} \left\langle \frac{1}{\rho}d\mu, \psi \right\rangle &= \frac{2}{4\pi} \int_{\mathbb{S}^1 \times \mathbb{R}} \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](\theta, s)} \partial_s \mathcal{H}_s [\mathcal{R}\psi(\theta, s)] d\theta ds \\ &= \left\langle \frac{1}{2\pi} \mathcal{R}^* \mathcal{H}_s \partial_s \frac{\mathbb{1}_{s \in S(\theta)}}{\mathcal{L}[D, \rho](\theta, s)}, \psi \right\rangle, \quad \psi \in \mathcal{D}(\mathbb{R}^2). \end{aligned}$$

□

#### 4. CONCLUSION AND DISCUSSION

Theorem 1 provides a way to extract a Radon transform from a Lambert's cosine law  $\rho(y) \cos \alpha$ , for a Lambertian convex reflector  $(D, \rho)$  in two dimensions. The formula (1) exhibits an appropriate pre-processing to obtain an element in the range of the transform: the cosine law  $\mathcal{L}[D, \rho]$  must be inverted, then symmetrized. In this case, the relevant mathematical object to represent the scene is the Radon measure  $\frac{d\mu}{\rho}$ . This object contains simultaneously the geometry and the physics of the problem. Indeed, the support of this Radon measure is exactly the boundary  $\partial D$  of the reflector, while the density is directly the inverse of the albedo  $\rho$ . As a consequence of the Radon formula (1), the representation  $\frac{d\mu}{\rho}$  of the scene can be deduced from Lambertian projections, by the filtered backprojection (2), well-defined mathematically.

To the author's knowledge, it is the first time that such a mathematical proof is obtained for a problem of Lambertian tomography, with an exact formula. This result potentially opens new perspectives concerning reflective tomography in a more general framework. For instance, studying the following question is of particular interest: given a set of bi-dimensional optical images of a three-dimensional scene with occlusions, is there a Radon measure supported by surfaces, whose Radon transform (or X-ray transform) reproduces some transformation of the optical images? Also, the results of this work may open new practical applications in optics. It is worth investigating practical implementation of (2) and testing on real optical data.

## APPENDIX A. CONVEX ANALYSIS

For completeness, we prove some intermediate results of convex analysis; they appear clearly in Figure 2.

**Lemma 1.** *Assume (A1) and let  $\theta \in \overline{\mathbb{S}^1}$  be an angle.*

(i) *Let  $s \in \mathbb{R}$  be such that the line  $x \cdot \theta = s$  intersects  $\bar{D}$ , i.e.  $s \in \overline{S(\theta)}$  where  $S(\theta)$  denote the set*

$$S(\theta) := \{x \cdot \theta, x \in D\} \subset \mathbb{R}. \quad (3)$$

*Then, there exists a unique point  $y \in \partial D$  such that*

$$y \in \partial D, \quad y \cdot \theta = s, \quad \nu(y) \cdot \theta^\perp \geq 0. \quad (4)$$

(ii) *The function  $s \in S(\theta) \mapsto y(\theta, s)$ , where  $y(\theta, s)$  is defined by (4), is a  $\mathcal{C}^1$  parametrization of the set*

$$\partial D(\theta) := \{x \in \partial D : \nu(x) \cdot \theta^\perp > 0\},$$

*with*

$$|\partial_s y(\theta, s)| = (\nu(y(\theta, s)) \cdot \theta^\perp)^{-1}.$$

(iii) *Let  $f \in L^1(\partial D, d\mu)$  be integrable for the length measure  $\mu$  on  $\partial D$ . Then,*

$$\int_{\partial D} f d\mu = \int_{S(\theta)} \left( \frac{f(y(\theta, s))}{\nu(y(\theta, s)) \cdot \theta^\perp} + \frac{f(y(-\theta, -s))}{\nu(y(-\theta, -s)) \cdot (-\theta)^\perp} \right) ds. \quad (5)$$

*Proof.* As in [18, pp. 51-53], fix  $d : \mathbb{R}^2 \rightarrow \mathbb{R}$  a defining function of  $D$ , i.e.  $d \in \mathcal{C}^1(\mathbb{R}^2)$  such that

$$D = \{x \in \mathbb{R}^2 : d(x) < 0\}, \quad \partial D = \{y \in \mathbb{R}^2 : d(y) = 0\}, \quad \nabla d(y) \neq 0, \quad y \in \partial D.$$

The exterior normal vector satisfies  $\nu(y) = \frac{\nabla d(y)}{|\nabla d(y)|}$ ,  $y \in \partial D$ .

(i) The set  $S(\theta)$  represents a projection of the bounded open convex set  $D$  on the line  $x \cdot \theta^\perp = 0$ ; then  $S(\theta)$  is a bounded open interval. By construction, for every  $s \in \mathbb{R}$ , the line  $x \cdot \theta = s$  meets  $D$  if and only if  $s \in S(\theta)$ . The closure  $\overline{S(\theta)}$  is a line segment and satisfies  $\overline{S(\theta)} = \{x \cdot \theta, x \in \bar{D}\}$ ; the line  $x \cdot \theta = s$  meets  $\bar{D}$  if and only if,  $s \in \overline{S(\theta)}$ .

Fix  $s \in \partial S(\theta)$ . Then the line  $x \cdot \theta = s$  is a supporting hyperplane of the convex set  $D$  (see [5, pp. 50-51]). Hence,  $\bar{D} \cap \{x \cdot \theta = s\}$  is contained in  $\partial D$ , and it contains a unique point  $y_0 \in \partial D$ , because  $\bar{D}$  is strictly convex. The boundary  $\partial D$  is smooth, so the supporting hyperplane  $x \cdot \theta = s$  is tangent to  $\partial D$  at  $y_0$ , i.e.  $\nu(y_0) \cdot \theta^\perp = 0$ . Indeed, the  $\mathcal{C}^1$  function  $g : t \mapsto d(s\theta + t\theta^\perp)$  has a minimum at  $t_0 = y_0 \cdot \theta^\perp$ , because  $g(t_0) = 0$  and  $g(t) > 0$ ,  $t \neq t_0$ ; this implies  $g'(t_0) = \nabla d(y_0) \cdot \theta^\perp = 0$ .

Fix now  $s \in S(\theta)$ . Then the line  $x \cdot \theta = s$  meets  $D$ , and  $\partial D \cap \{x \cdot \theta = s\}$  contains only the two distinct extremities of the line segment  $\bar{D} \cap \{x \cdot \theta = s\}$ . Denote these extremities by  $y_1 = s\theta + t_1\theta^\perp$ ,  $y_2 = s\theta + t_2\theta^\perp$ , with  $t_1 \neq t_2$ . By strict convexity,  $z = y_1 + \frac{1}{2}(y_2 - y_1) \in D$ ; therefore,  $\nabla d(y_1) \cdot (y_1 - z) > 0$  (see for instance [6, Lemma 2.1]), which is equivalent to  $(t_1 - t_2)\nu(y_1) \cdot \theta^\perp > 0$ . Analogously,  $(t_2 - t_1)\nu(y_2) \cdot \theta^\perp > 0$ , which proves  $\nu(y_1) \cdot \theta^\perp \nu(y_2) \cdot \theta^\perp < 0$ .

(ii) We deduce from the proof of (i) that  $\partial D(\theta) = \{y(\theta, s), s \in S(\theta)\}$ ; indeed, for any  $s \in S(\theta)$ ,  $y(\theta, s) \in \partial D(\theta)$ , and for any  $x \in \partial D(\theta)$ ,  $x = y(\theta, x \cdot \theta)$  with  $x \cdot \theta \in S(\theta)$ . Furthermore, the set  $\partial D(\theta)$  satisfies an implicit equation

$$d(s\theta + t\theta^\perp) = 0, \quad \text{with} \quad \partial_t d(s\theta + t\theta^\perp) = |\nabla d(s\theta + t\theta^\perp)| \nu(s\theta + t\theta^\perp) \cdot \theta^\perp \neq 0.$$

Around any  $s \in S(\theta)$ , by the implicit function theorem,  $\partial D(\theta)$  can be locally represented by an equation  $t = \phi(s)$ , where  $\phi$  is a (local)  $\mathcal{C}^1$  function satisfying  $\phi'(s) = -\partial_s d(s\theta + t\theta^\perp) / \partial_t d(s\theta + t\theta^\perp)$ . Finally,  $y(\theta, s) = s\theta + \phi(s)\theta^\perp$  is  $\mathcal{C}^1$  on the whole open interval  $S(\theta)$ , and

$$\partial_s y(\theta, s) \cdot \theta^\perp = \phi'(s) = -\frac{\nu(y(\theta, s)) \cdot \theta}{\nu(y(\theta, s)) \cdot \theta^\perp}, \quad |\partial_s y(\theta, s)| = \sqrt{1 + \phi'(s)^2}.$$

(iii) The set  $\partial D$  can be decomposed as a disjoint union

$$\begin{aligned} \partial D &= \{\nu_y \cdot \theta^\perp > 0\} \cup \{\nu_y \cdot (-\theta)^\perp > 0\} \cup \{\nu_y \cdot \theta^\perp = 0\} \\ &= \partial D(\theta) \cup \partial D(-\theta) \cup \{y(\theta, s), s \in \partial S(\theta)\}, \end{aligned}$$

which proves that

$$\int_{\partial D} f d\mu = \int_{\partial D(\theta)} f(y) d\mu(y) + \int_{\partial D(-\theta)} f(y) d\mu(y).$$

We use the parametrization of (ii), once for  $\theta$ , once for  $-\theta$ . To finish with, we change the variable in the second integral;  $s := -s$ , the domain of integration becomes  $-S(-\theta) = S(\theta)$ .  $\square$

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