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# THE ISOTHERMAL LIMIT FOR THE COMPRESSIBLE EULER EQUATIONS WITH DAMPING

QUENTIN CHAULEUR

ABSTRACT. We consider the isothermal Euler system with damping. We rigorously show the convergence of Barenblatt solutions towards a limit Gaussian profile in the isothermal limit  $\gamma \rightarrow 1$ , and we explicitly compute the propagation and the behavior of Gaussian initial data. We then show the weak  $L^1$  convergence of the density as well as the asymptotic behavior of its first and second moments.

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## 1. INTRODUCTION

We consider the isentropic compressible Euler equations with frictional damping:

$$(1.1a) \quad \begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left( \frac{m^2}{\rho} \right) + \partial_x \rho^\gamma + m = 0, \end{cases}$$

with  $\rho(0, x) = \rho_0(x) \geq 0$  and  $m(0, x) = m_0(x)$  for  $x \in \mathbb{R}$ ,  $t \geq 0$ , and the adiabatic gas exponent  $\gamma > 1$ . Such a system appears in the mathematical modeling of compressible flow through a porous medium, and its study has drawn a lot of attention over the last decades [19]. The global existence of  $L^\infty$  weak entropy solutions to the Cauchy problem of (1.1) is now well established (see for instance [20] and [14]), and it is known that their long-time asymptotic behavior [12] is governed by the limit diffusive profile:

$$(1.2a) \quad \begin{cases} \partial_t \bar{\rho} = \partial_x^2 \bar{\rho}^\gamma, \\ \bar{m} = -\partial_x \bar{\rho}^\gamma, \end{cases}$$

where equation (1.2a) is the porous media equation, whose fundamental solutions are called Barenblatt solutions [5], and equation (1.2b) is the famous Darcy law. In [18], the author constructs a

class of particular solutions for (1.1) which tend to the Barenblatt solutions  $\bar{\rho}$  asymptotically in time, with an explicit rate  $\log(t)/t$ . In [13], the authors find the explicit rates

$$\|\rho - \bar{\rho}\|_{L^2(\mathbb{R})}^2 \leq C(1+t)^{-k_1+\varepsilon} \quad \text{if } 1 < \gamma \leq 2,$$

and

$$\|\rho - \bar{\rho}\|_{L^\gamma(\mathbb{R})}^\gamma \leq C(1+t)^{-k_2+\varepsilon} \quad \text{if } \gamma > 2,$$

for any  $\varepsilon > 0$ , with  $k_1 = \min\left(\frac{\gamma^2}{(\gamma+1)^2}, \frac{\gamma-1}{\gamma}\right)$  and  $k_2 = \min\left(\frac{\gamma^2}{(\gamma+1)^2}, \frac{1}{\gamma}\right)$ , for every  $L^\infty$  weak entropy solution  $(\rho, m)$  of (1.1). Unfortunately, the decay rates are not in  $L^1$ -norm, which is the natural norm as (1.1a) and (1.2a) both satisfies the conservation of norm

$$\|\rho(t, \cdot)\|_{L^1(\mathbb{R})} = \|\rho_0\|_{L^1(\mathbb{R})} \quad \text{and} \quad \|\bar{\rho}(t, \cdot)\|_{L^1(\mathbb{R})} = \|\bar{\rho}_0\|_{L^1(\mathbb{R})} \quad \text{for all } t \geq 0.$$

Decay rates in  $L^1$ -norm were achieved for a particular range of  $\gamma$  in [15], where the authors show that

$$\|\rho - \bar{\rho}\|_{L^1(\mathbb{R})} \leq C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon} \quad \text{if } 1 < \gamma < 3,$$

which was recently improved and extended in [11] with the estimate

$$\|\rho - \bar{\rho}\|_{L^1(\mathbb{R})} \leq C(1+t)^{-\frac{1}{(\gamma+1)^2}+\varepsilon} \quad \text{for all } \gamma > 1.$$

Throughout the years, a lot of effort has been put to extend the range of  $\gamma$  to finally achieve the full range  $\gamma > 1$  of barotropic pressure laws  $P(\rho) = \rho^\gamma$ . The next natural step is the study of the case  $\gamma = 1$ , which leads to the isothermal pressure law  $P(\rho) = \rho$ . However, in the compressible fluid literature, much fewer results are known for the isothermal Euler system with damping

$$(1.3a) \quad \begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho}\right) + \partial_x \rho + m = 0, \end{cases}$$

which stands as the limit  $\gamma \rightarrow 1$  of the isentropic system (1.1). In [14], the authors show the existence of  $L^\infty$  entropy weak solutions to the Cauchy problem of (1.3). They also prove, up to a scaling in space  $z = x/\sqrt{1+t}$ , the  $L^p_{\text{loc}}$  convergence of the density  $\rho$  towards a diffusive profile  $\bar{\rho}$  which satisfies the heat equation

$$(1.4) \quad \partial_t \bar{\rho} = \partial_x^2 \bar{\rho},$$

in the case  $\max(\bar{\rho}_+, \bar{\rho}_-) > 0$ , where  $\bar{\rho}(\pm\infty) = \bar{\rho}_\pm$ . However, time dependent Gaussian functions, which stand as particular solutions of the heat equation on the whole space, do not satisfy this condition. In a bounded domain  $\Omega \subset \mathbb{R}^d$ , the author shows in [21] the exponential convergence of any global solution  $(\rho, m)$  with small initial data  $(\rho_0, m_0)$  towards the limit profile  $(\|\rho_0\|_{L^1}/|\Omega|, 0)$ . In fact, at least formally, equation (1.3) satisfies the energy inequality

$$(1.5) \quad \frac{1}{2} \int_{\mathbb{R}} \frac{m^2}{\rho} + \int_{\mathbb{R}} \rho \log \rho + \int_0^t \int_{\mathbb{R}} \frac{m^2}{\rho} \leq E_0 \quad \text{for all } t \geq 0,$$

but the left-hand side has no definite sign due to the logarithmic contribution in the potential energy, a property that causes many technical difficulties such as a lack of compactness on the whole space  $\mathbb{R}$  when the density  $t \mapsto \rho(t, \cdot) \in L^1(\mathbb{R})$  vanishes at infinity.

In [6], in the case of the compressible Euler equations without frictional damping ( $\mu = 0$ ), the authors show that Gaussian functions stand as particular solutions of their system. They also prove that every global weak solution disperses and converges to a universal asymptotic Gaussian profile, up to a rescaling by their dispersion rate, in the weak  $L^1$  topology, under some integrability

assumptions on the moment of order 2 of the initial data  $\rho_0$ . We will show in this paper that this kind of property still holds in the case with damping (1.3), and we will study the asymptotic behavior of Gaussian solutions and moments of order 1 and 2 of the density  $\rho$  as they stand as a key ingredient for the proof of this kind of feature.

The formal limit  $\gamma \rightarrow 1$  is singular regarding many features of the isentropic case, especially for the energy inequality of system (1.1):

$$(1.6) \quad \frac{1}{2} \int_{\mathbb{R}} \frac{m^2}{\rho} + \frac{1}{\gamma-1} \int_{\mathbb{R}} \rho^\gamma + \int_0^t \int_{\mathbb{R}} \frac{m^2}{\rho} \leq E_0.$$

In this paper we propose to give sense to the formal isothermal limit  $\gamma \rightarrow 1$ , in particular we will rigorously show and illustrate the convergence of the Barenblatt solutions of (1.2a) to the Gaussian solutions of (1.4), a feature which does not seem to be so much known in the community to the best of the author knowledge.

From now on,  $(\rho_\gamma, m_\gamma)$  with  $\gamma > 1$  will denote a solution of the isentropic Euler system (1.1), whereas  $(\rho, m)$  will denote a solution of the isothermal Euler system. The overline bar will be use for solutions of the different limit diffusive profiles. The notation  $C$  will denote a generic constant  $C > 0$ .

This paper is organized as follows. In Section 2, we provide energy estimates, assumptions about existence and regularity of solutions of equations (1.3), and we state the main results of this paper. In Section 3, we show that Barenblatt solutions converge to a Gaussian profile as  $\gamma \rightarrow 1$ . In Section 4, we explicitly compute the behavior of Gaussian solutions of (1.3). In Section 5, we study the evolution of the first and second moments of every weak solution of (1.3). In Section 6, we prove the weak  $L^1$  convergence of every solution towards a universal Gaussian profile. Finally, in Section 7, we give some perspectives about the open question of explicit convergence rate in the isothermal case.

## 2. ASSUMPTIONS AND MAIN RESULTS

We first give a notion of global weak solution for the 1-dimensional isothermal Euler equations with damping, and we will assume the global existence of this kind of solutions through the rest of the paper:

**Definition 2.1.** We say that  $(\rho, m)$  is a **weak solution** of system (1.3) in  $[0, T[$  with initial data  $(\rho_0, m_0) \in L^1(\mathbb{R}) \times L^1(\mathbb{R})$ , if there exists locally integrable functions  $\sqrt{\rho}$ ,  $\Lambda$  such that, by defining  $\rho := \sqrt{\rho}^2$  and  $m = \sqrt{\rho}\Lambda$ , the following holds:

(i) The global regularity:

$$\sqrt{\rho} \in L^\infty([0, T[; L^2(\mathbb{R})), \quad \Lambda \in L^2([0, T[; L^2(\mathbb{R})),$$

with the compatibility condition

$$\sqrt{\rho} \geq 0 \text{ a.e. on } (0, \infty) \times \mathbb{R}, \quad \Lambda = 0 \text{ a.e. on } \{\rho = 0\}.$$

(ii) For any test function  $\eta \in C_0^\infty([0, T[ \times \mathbb{R})$ ,

$$\int_0^T \int_{\mathbb{R}} (\rho \partial_t \eta + m \partial_x \eta) dx dt + \int_{\mathbb{R}} \rho_0 \eta(0) dx = 0,$$

and for any test function  $\zeta \in \mathcal{C}_0^\infty([0, T[ \times \mathbb{R}; \mathbb{R})$ ,

$$\int_0^T \int_{\mathbb{R}} (m \partial_t \zeta + \Lambda^2 \partial_x \zeta + \rho \partial_x \zeta - m \zeta) dx dt + \int_{\mathbb{R}} m_0 \zeta(0) dx = 0.$$

**Assumption 2.2.** Let  $(\rho_0, m_0) \in L^1(\mathbb{R}) \times L^1(\mathbb{R})$ . We assume that there exists a weak solution  $(\rho, m)$  of system (1.3) with initial data  $(\rho_0, m_0)$  in the sense of Definition 2.1 which satisfies, for all  $t \geq 0$ , the energy estimate

$$\frac{1}{2} \int_{\mathbb{R}} \frac{m^2(t, x)}{\rho(t, x)} dx + \int_{\mathbb{R}} \rho(t, x) \log(\rho(t, x)) dx + \int_0^t \int_{\mathbb{R}} \frac{m^2(s, x)}{\rho(s, x)} dx ds \leq C,$$

with the additional regularity

$$(t, x) \mapsto x^2 \rho(t, x) \in L^\infty([0, T[; L^1(\mathbb{R})).$$

*Remark 2.3.* The  $L^\infty$  entropy weak solutions of system (1.3) described in [14] are in fact weak solutions of (1.3) when the initial data  $(\rho_0, m_0) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$  satisfies the estimates

$$0 \leq \rho_0 \leq C \quad \text{and} \quad |m_0| \leq C \rho_0 |\log \rho_0|$$

(see [14, Definition 1] for a precise definition of  $L^\infty$  entropy weak solutions of system (1.3)). Note that the existence of  $L^\infty$  entropy weak solutions of the isothermal Euler system without damping is proved in [16] under the same initial data conditions.

We now have to introduce several quantities in order to state our results. Let  $\tau$  be the unique  $\mathcal{C}^\infty([0, \infty))$  solution of the differential equation

$$\ddot{\tau} = \frac{2}{\tau} - \dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0.$$

From [8], we know that this function satisfies, as  $t \rightarrow +\infty$ ,

$$\tau(t) \sim 2\sqrt{t} \quad \text{and} \quad \dot{\tau}(t) \sim \frac{1}{\sqrt{t}}.$$

We also introduce the Gaussian function

$$\Gamma := e^{-y^2}.$$

We make the change of variable  $y = x/\tau(t)$ ,

$$\rho(t, x) = \frac{1}{\tau(t)} R\left(t, \frac{x}{\tau(t)}\right) \quad \text{and} \quad m(t, x) = \frac{1}{\tau(t)^2} M\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot{\tau}(t)}{\tau(t)} y R\left(t, \frac{x}{\tau(t)}\right).$$

We first remark that this change of variable preserves the  $L^1$ -norm, so for all  $t \geq 0$ , at least formally

$$\int_{\mathbb{R}} R(t, y) dy = \int_{\mathbb{R}} \rho(t, x) dx = \int_{\mathbb{R}} \rho_0(x) dx.$$

System (1.3) becomes, in the terms of the new unknown  $(R, M)$ ,

$$(2.1a) \quad \begin{cases} \partial_t R + \frac{1}{\tau^2} \partial_y M = 0, \end{cases}$$

$$(2.1b) \quad \begin{cases} \partial_t M + \frac{1}{\tau^2} \partial_y \left( \frac{M^2}{R} \right) + \partial_y R + 2yR + M = 0, \end{cases}$$

which have the following energy inequality:

$$(2.2) \quad \mathcal{E}(t) + \int_0^t \frac{\dot{\tau}(t)}{\tau^3(t)} \int_{\mathbb{R}} \frac{M^2}{R} dy ds \leq C,$$

where

$$(2.3) \quad \mathcal{E}(t) = \frac{1}{2\tau(t)^2} \int_{\mathbb{R}} \frac{M^2}{R} dy + \int_{\mathbb{R}} R \log \left( \frac{R}{\Gamma} \right) dy.$$

In fact, differentiating with respect to time the left part of (2.2), using equations (2.1a) and (2.1b) and by integration by parts, we get that

$$\frac{d}{dt} \left( \frac{1}{2\tau(t)^2} \int_{\mathbb{R}} \frac{M^2}{R} + \int_{\mathbb{R}} R \log R + \int_{\mathbb{R}} y^2 R + \int_0^t \frac{\dot{\tau}(s)}{\tau(t)^3} \int_{\mathbb{R}} \frac{M^2}{R} ds \right) = -\frac{1}{\tau^2} \int_{\mathbb{R}} \frac{M^2}{R} \leq 0.$$

Note that from the Csiszár-Kullback inequality (see e.g. [2])

$$\int_{\mathbb{R}} R \log \left( \frac{R}{\Gamma} \right) \geq \frac{1}{2\|R_0\|_{L^1}} \|R - \Gamma\|_{L^1}^2 \geq 0,$$

we directly get that for all  $t \geq 0$ ,  $\mathcal{E}(t) \geq 0$ .

Denoting by  $\mathcal{E}_{\text{kin}}$  the kinetic energy

$$\mathcal{E}_{\text{kin}}(t) := \frac{1}{\tau^2(t)} \int_{\mathbb{R}} \frac{M^2(t, y)}{R(t, y)} dy$$

for all  $t \geq 0$ , the asymptotics of  $\tau$  and  $\dot{\tau}$  and the following Lemma 5.2 give that

$$\mathcal{E}_{\text{kin}} \in L^\infty(\mathbb{R}_+) \quad \text{and} \quad \int_0^\infty \frac{\mathcal{E}_{\text{kin}}(t)}{1+t} dt < \infty,$$

so we know there exists a sequence  $(t_n)_n$ , such that  $t_n \rightarrow \infty$  and

$$\mathcal{E}_{\text{kin}}(t_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Unfortunately, we have no proof that this property is actually satisfied uniformly in time, even if it seems to be a reasonable assumption that we state in the following:

**Assumption 2.4.** We assume that

$$\mathcal{E}_{\text{kin}}(t) = \frac{1}{\tau^2(t)} \int_{\mathbb{R}} \frac{M^2(t, y)}{R(t, y)} dy \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This assumption is for instance satisfied for Gaussians solutions of system (1.3), with an explicit convergence rate in  $t^{-1/2}$  (see Remark 5.4 below). We now state the two main results of this paper:

**Proposition 2.5.** *Under Assumption 2.2 and Assumption 2.4, we have*

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} R(t, y) dy \longrightarrow \int_{\mathbb{R}} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} \Gamma(y) dy \quad \text{as } t \rightarrow \infty.$$

**Proposition 2.6.** *Under Assumption 2.2, we have*

$$R(t) \underset{t \rightarrow \infty}{\rightharpoonup} \Gamma \text{ weakly in } L^1(\mathbb{R}).$$

*Remark 2.7.* In Proposition 2.5, Assumption 2.4 is only required to show the convergence of the moment of order 2. Without this assumption, we are only able to show that the moment of order 2 is uniformly bounded (see Lemma 5.2).

*Remark 2.8.* Note that Propositions 2.5 and 2.6 are also satisfied in higher space dimensions  $\mathbb{R}^d$  with  $d \geq 1$ , in the sense that

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} R(t, y) dy \longrightarrow \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \Gamma(y) dy \quad \text{as } t \rightarrow \infty$$

with  $\Gamma = e^{-|y|^2}$ , and

$$R(t) \xrightarrow[t \rightarrow \infty]{} \Gamma \text{ weakly in } L^1(\mathbb{R}^d).$$

Of course, our notion of global weak solution has to be adapted to the  $d$ -dimensional case, as well as system (1.3), and we refer to [6] and [8] for some  $d$ -dimensional analogue of Definition 2.1.

In the same vein, the following discussion about the particular Gaussian solutions of (1.3) can easily be generalized to any dimension  $d$  by a tensorization property. We also refer the reader to [6] and [8] for some  $d$ -dimensional analogue of system (4.1)-(4.2)-(4.3).

### 3. THE LIMIT $\gamma \rightarrow 1$ OF BARENBLATT'S SOLUTIONS

We consider the unique fundamental solution of the porous media equation (with the Dirac delta function as initial data)

$$(3.1) \quad \begin{cases} \partial_t \bar{\rho}_\gamma = \partial_x^2 [(\bar{\rho}_\gamma)^\gamma], \\ \bar{\rho}_\gamma(-1, x) = \lambda \delta(x), \quad \lambda > 0, \end{cases}$$

which is called the Barenblatt solution [4]. Note that we take the initial data at  $t = -1$  to avoid the singularity at  $t = 0$ . We recall that this solution can be written

$$(3.2) \quad \bar{\rho}_\gamma(t, x) = (1+t)^{-\frac{1}{\gamma+1}} [A - B\xi^2]_+^{\frac{1}{\gamma-1}}$$

with  $\xi = x(1+t)^{-\frac{1}{\gamma+1}}$ ,  $[f]_+ = \max\{f, 0\}$ ,  $B = \frac{\gamma-1}{2\gamma(\gamma+1)}$  and  $A$  determined by

$$(3.3) \quad 2A^{\frac{\gamma+1}{2(\gamma-1)}} B^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\cos \theta)^{\frac{\gamma+1}{\gamma-1}} d\theta = \lambda.$$

Moreover,  $\bar{\rho}_\gamma$  is continuous on  $\mathbb{R}$ , it satisfies the conservation of mass

$$\int_{\mathbb{R}} \bar{\rho}_\gamma(t, x) dx = \lambda$$

for all  $t \geq -1$ , and has compact support for any finite time  $T > 0$ , namely

$$\bar{\rho}_\gamma = 0 \quad \text{if } |\xi| \geq \sqrt{A/B}.$$

The power  $\frac{1}{\gamma-1}$  in (3.2) could make the limit  $\gamma \rightarrow 1$  unclear, however we can show:

**Proposition 3.1.** *We make the change of variable  $\xi = x(1+t)^{-\frac{1}{\gamma+1}}$  and*

$$\bar{\rho}_\gamma(t, x) = (1+t)^{-\frac{1}{\gamma+1}} \mathcal{B}_\gamma \left( x(1+t)^{-\frac{1}{\gamma+1}} \right)$$

which preserves the  $L^1$ -norm. Then,

$$\mathcal{B}_\gamma \rightarrow \frac{\lambda}{2\sqrt{\pi}} e^{-\frac{\xi^2}{4}} \quad \text{in } L^\infty(\mathbb{R}) \text{ as } \gamma \rightarrow 1.$$

In particular, the  $L^\infty$  estimate

$$|\bar{\rho}_\gamma(t, x)| \leq A^{\frac{1}{\gamma-1}} (1+t)^{-\frac{1}{\gamma+1}}$$

has the following continuous limit when  $\gamma \rightarrow 1$ :

$$|\bar{\rho}(t, x)| \leq \frac{\lambda}{2\sqrt{\pi}} (1+t)^{-\frac{1}{2}},$$

where  $\bar{\rho}$  denotes the Gaussian limit of the Barenblatt profile  $\bar{\rho}_\gamma$  as  $\gamma \rightarrow 1$  (namely  $\bar{\rho} := \bar{\rho}_1$ , we omit the index  $\gamma$  when  $\gamma = 1$ ).

*Proof.* We have, for  $|\xi| < \sqrt{A/B}$ ,

$$\mathcal{B}_\gamma(\xi) = [A - B\xi^2]_+^{\frac{1}{\gamma-1}} = \exp\left(\frac{1}{\gamma-1} \log A + \frac{1}{\gamma-1} \log\left(1 - \frac{B}{A}\xi^2\right)\right),$$

as  $A > 0$  from expression (3.3) and  $\frac{B}{A}\xi^2 < 1$ . We have now two terms to handle when  $\gamma \rightarrow 1$ . Taking the logarithm in equation (3.3), and recalling that  $B = \frac{\gamma-1}{2\gamma(\gamma+1)}$ , we have

$$\frac{1}{\gamma-1} \log A = \frac{2}{\gamma+1} \left( \log\left(\frac{\lambda}{2\sqrt{2\gamma(\gamma+1)}}\right) - \log\left(\frac{1}{\sqrt{\gamma-1}} \int_0^{\frac{\pi}{2}} (\cos \theta)^{\frac{\gamma+1}{\gamma-1}} d\theta\right) \right).$$

We denote  $\varepsilon = \gamma - 1$ , so we look at the limit  $\varepsilon \rightarrow 0$  of the following Wallis integral

$$W_{1+\frac{2}{\varepsilon}} = \int_0^{\frac{\pi}{2}} (\cos \theta)^{1+\frac{2}{\varepsilon}} d\theta = \int_0^{\frac{\pi}{2}} (\cos \theta)^{\frac{\gamma+1}{\gamma-1}} d\theta.$$

We recall the well-known equivalent of Wallis integral when  $\varepsilon \rightarrow 0$ ,

$$W_{1+\frac{2}{\varepsilon}} = \sqrt{\frac{\pi}{2(1+\frac{2}{\varepsilon})}} + O\left(\frac{1}{1+\frac{2}{\varepsilon}}\right) = \sqrt{\frac{\pi\varepsilon}{4+2\varepsilon}} + O(\varepsilon),$$

so it is easy to check that

$$\frac{1}{\sqrt{\gamma-1}} \int_0^{\frac{\pi}{2}} (\cos \theta)^{\frac{\gamma+1}{\gamma-1}} d\theta = \sqrt{\frac{\pi}{4}} (1 + O(\varepsilon))$$

as  $\gamma \rightarrow 1$ , and then

$$\exp\left(\frac{1}{\gamma-1} \log A\right) = \frac{\lambda}{2\sqrt{\pi}} \frac{1}{1+O(\varepsilon)}.$$

For the second term, as we have

$$\frac{B}{A} = B \left(\frac{2W_{1+\frac{2}{\varepsilon}}}{\lambda B^{\frac{1}{2}}}\right)^{\frac{2\varepsilon}{2+\varepsilon}} = \frac{\varepsilon}{4} + O(\varepsilon^2)$$

from the previous equivalent, we write

$$\frac{1}{\varepsilon} \log\left(1 - \frac{B}{A}\xi^2\right) = -\frac{1}{\varepsilon} \sum_{n \geq 1} \frac{1}{n} \left(\frac{B}{A}\xi^2\right)^n \sim -\frac{1}{4}\xi^2 + \varepsilon \sum_{n \geq 2} \frac{1}{n} \left(\frac{\xi^2}{4}\right)^n \varepsilon^{n-2},$$



so finally

$$\exp\left(\frac{1}{\gamma-1}\log\left(1-\frac{B}{A}\xi^2\right)\right)\rightarrow e^{-\frac{\xi^2}{4}},$$

and

$$\mathcal{B}_\gamma(\xi) - \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}} = \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}}\left(\frac{1}{1+O(\varepsilon)}e^{O(\varepsilon)} - 1\right) = O(\varepsilon).$$

On  $|\xi| \geq \sqrt{A/B}$ , we know that  $\mathcal{B}_\gamma(\xi) = 0$ , and as  $\xi \rightarrow e^{-\xi^2/4}$  is a Gaussian,

$$\sup_{|\xi| \geq \sqrt{A/B}} \left| \mathcal{B}_\gamma(\xi) - \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}} \right| = \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{A}{B}} \leq Ce^{-\frac{1}{\varepsilon}},$$

so finally

$$\sup_{\xi \in \mathbb{R}} \left| \mathcal{B}_\gamma(\xi) - \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}} \right| \leq Ce^{-\frac{1}{\gamma-1}} + O(\gamma-1) \rightarrow 0 \text{ as } \gamma \rightarrow 1,$$

which ends the proof.  $\square$

In order to illustrate this result, we make the following numerical simulation of the function  $\mathcal{B}_\gamma$  for several values of  $\gamma$  (namely  $\gamma = 2, 1.5$  and  $1.1$ ), and the initial condition constant  $\lambda = 1$ . We also plot the limit Gaussian profile  $e^{-\xi^2/4}/2\sqrt{\pi}$ , which corresponds to the case  $\gamma = 1$ .

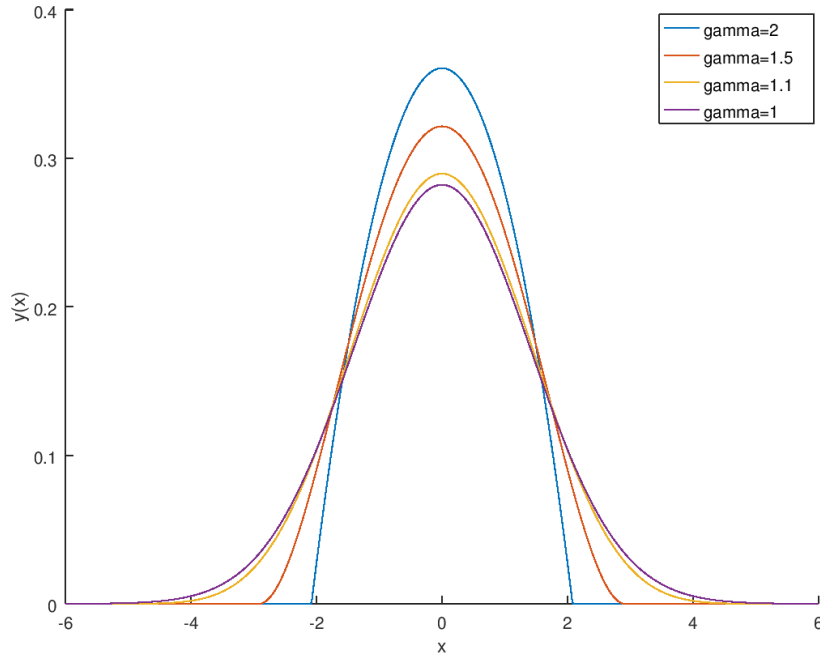


FIGURE 1. Convergence of  $\mathcal{B}_\gamma$  towards its limit Gaussian profile.

## 4. GAUSSIAN SOLUTIONS

In this section, following [6], we seek for particular Gaussian solutions of (1.3) of the form

$$\bar{\rho}(t, x) = b(t)e^{-\alpha(t)(x-\bar{x}(t))^2}$$

and

$$\bar{m}(t, x) = (\beta(t)x + c(t))\bar{\rho}(t, x),$$

with the initial conditions  $b(0) = b_0 > 0$ ,  $\alpha(0) = \alpha_0 > 0$ ,  $\beta(0) = \beta_0 \in \mathbb{R}$ ,  $c(0) = c_0 \in \mathbb{R}$ . As system (1.3) is invariant by translation, we assume that  $\bar{x}(0) = 0$ . We also denote  $\rho_0 = \bar{\rho}(0, \cdot)$  and  $m_0 = \bar{m}(0, \cdot)$ . Plugging these expressions into (1.3), we obtain the following set of differential equations:

$$(4.1) \quad \dot{\alpha} + 2\alpha b = 0, \quad \dot{\beta} + \beta^2 + \beta = 2\alpha,$$

$$(4.2) \quad \dot{\bar{x}} = \beta\bar{x} + c, \quad \dot{b} = b(\dot{\alpha}\bar{x}^2 + 2\alpha c\bar{x}(\dot{\bar{x}} - c) - \beta),$$

$$(4.3) \quad \dot{c} + \beta c + c = -2\alpha\bar{x}.$$

In order to solve this system, mimicking [17], we can check that the two equations of (4.1) are satisfied if and only if  $\alpha$  and  $\beta$  are of the form

$$\alpha(t) = \frac{\alpha_0}{\tau(t)^2}, \quad \beta(t) = \frac{\dot{\tau}(t)}{\tau(t)},$$

where  $\tau$  is the global solution of the differential equation

$$(4.4) \quad \ddot{\tau} = \frac{2\alpha_0}{\tau} - \dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = \beta_0.$$

We recall (see [8]) that there exists a unique global solution  $\tau \in \mathcal{C}^\infty([0, \infty))$  to this nonlinear ODE, and that this solution remains uniformly bounded from below by a strictly positive constant.

Plugging these expressions into the second equation of (4.2), we also get that

$$b(t) = \frac{b_0}{\tau(t)}.$$

We also get an expression of  $c$  in terms of the center  $\bar{x}$  of our Gaussian:

$$c(t) = c_0 - \left(1 + \frac{\dot{\tau}}{\tau}\right)\bar{x}.$$

Let first show that the center of the Gaussian has an explicit expression, which does not depend on the function  $\tau$ :

**Proposition 4.1.** *We have*

$$\bar{x}(t) = \frac{1}{\|\rho_0\|_{L^1(\mathbb{R})}} \left( \int_{\mathbb{R}} x \rho_0 - (1 - e^{-t}) \int_{\mathbb{R}} m_0 \right).$$

*In particular, there exists  $\bar{x}_\infty \in \mathbb{R}$  such that*

$$\bar{x}(t) \rightarrow \bar{x}_\infty \quad \text{as } t \rightarrow \infty.$$

*Proof.* We know that  $\bar{\rho}$  is a Gaussian function centered in  $\bar{x}$ , so we get that for all  $t \geq 0$ ,

$$\int_{\mathbb{R}} (x - \bar{x}(t)) \bar{\rho}(t, x) dx = 0,$$

hence we only have to study the first moment of  $\bar{\rho}$  keeping in mind that

$$\bar{x}(t) = \frac{1}{\|\rho_0\|_{L^1(\mathbb{R})}} \int_{\mathbb{R}} x \bar{\rho}(t, x) dx.$$

Integrating equation (1.3b) over  $\mathbb{R}$  we get that

$$\frac{d}{dt} \left( \int_{\mathbb{R}} \bar{m}(t, x) dx \right) = \int_{\mathbb{R}} \partial_t \bar{m}(t, x) dx = - \int_{\mathbb{R}} \bar{m}(t, x) dx,$$

so we get by integration by parts that

$$\frac{d}{dt} \left( \int_{\mathbb{R}} x \bar{\rho}(t, x) dx \right) = \int_{\mathbb{R}} x \partial_t \bar{\rho}(t, x) dx = - \int_{\mathbb{R}} \bar{m}(t, x) dx = -e^{-t} \int_{\mathbb{R}} m_0(x) dx,$$

so finally integrating this expression over  $[0, t]$  we get the result.  $\square$

In order to know the behavior of our Gaussian functions when  $t \rightarrow \infty$ , we only need to get some equivalents of the real function  $\tau$ , which is the goal of the following proposition. We refer to [8] for a complete study of the differential equation (4.4).

**Proposition 4.2.** *Let  $\tau$  be the unique  $C^\infty([0, \infty))$  solution of the differential equation (4.4), then we have*

$$\tau(t) \sim 2\sqrt{\alpha_0 t} \quad \text{and} \quad \dot{\tau}(t) \sim \sqrt{\frac{\alpha_0}{t}}.$$

*In particular,*

$$\alpha(t) \sim \frac{1}{4t}, \quad \beta(t) \sim \frac{1}{2t}, \quad b(t) \sim \frac{b_0}{2\sqrt{\alpha_0 t}} \quad \text{and} \quad c(t) \rightarrow c_0 - \bar{x}_\infty.$$

## 5. EVOLUTION OF CERTAIN QUANTITIES

In this section, we are going to give two propositions that describe the behavior of respectively the first and second moment of the renormalized density  $R$ .

**Proposition 5.1.** *We denote*

$$I_1 = \int_{\mathbb{R}} M dy \quad \text{and} \quad I_2 = \int_{\mathbb{R}} y R dy.$$

*Then, there exists  $C \geq 0$  such that*

$$I_1(t) = e^{-t} I_1(0) \quad \text{and} \quad I_2(t) \sim \frac{C}{\sqrt{t}} \quad \text{as } t \rightarrow \infty.$$

*Proof.* First off, mimicking the calculus of the proof of Proposition 4.2, we have by integrating equation (1.3b) over  $\mathbb{R}$  and integration by parts that

$$\frac{d}{dt} \left( \int_{\mathbb{R}} M dy \right) = - \int_{\mathbb{R}} M dy,$$

so we easily get that

$$I_1(t) = e^{-t} I_1(0).$$

Now, by integration by parts, and using (2.1), we get the system of coupled differential equations:

$$\dot{I}_1 = \int_{\mathbb{R}} \partial_t M = - \int_{\mathbb{R}} \left[ \frac{1}{\tau^2} \partial_x \left( \frac{M^2}{R} \right) + \partial_y R + 2yR + M \right] = -I_1 - 2I_2,$$

and

$$\dot{I}_2 = \int_{\mathbb{R}} y \partial_t R = - \int_{\mathbb{R}} y \frac{\partial_x M}{\tau^2} = \frac{1}{\tau^2} I_1.$$

We denote  $\tilde{I}_2 = \tau I_2$ , so that

$$\dot{\tilde{I}}_2 = \dot{\tau} I_2 + \tau \dot{I}_2 = \dot{\tau} I_2 + \frac{1}{\tau} I_1,$$

and

$$\ddot{\tilde{I}}_2 = \ddot{\tau} I_2 + \dot{\tau} \dot{I}_2 - \frac{\dot{\tau}}{\tau} I_1 + \frac{1}{\tau} \dot{I}_1 = -(\dot{\tau} I_2 + \frac{1}{\tau} I_1) = -\dot{\tilde{I}}_2,$$

hence

$$\dot{\tilde{I}}_2(t) = \dot{\tilde{I}}_2(0) e^{-t}$$

and

$$\tilde{I}_2(t) = \tilde{I}_2(0) + \dot{\tilde{I}}_2(0)(e^{-t} - 1).$$

We easily compute the initial conditions

$$\dot{\tilde{I}}_2(0) = I_1(0) \quad \text{and} \quad \tilde{I}_2(0) = I_2(0),$$

so finally

$$I_2(t) = \frac{1}{\tau(t)} [I_2(0) - I_1(0)(1 - e^{-t})] \sim \frac{1}{\tau(t)} [I_2(0) - I_1(0)]$$

when the initial condition are not well prepared in the sense that  $I_2(0) \neq I_1(0)$ . Note that if  $I_2(0) = I_1(0)$ , we have

$$I_2(t) = \frac{e^{-t}}{\tau(t)} I_1(0).$$

□

We now give an useful lemma, which induces the boundedness of the second moment of the density  $R$  uniformly with respect to time:

**Lemma 5.2.** *There holds*

$$(5.1) \quad \sup_{t \geq 0} \int_{\mathbb{R}} R(t, y) (1 + y^2 + |\log R(t, y)|) dy < \infty$$

and

$$(5.2) \quad \int_0^\infty \frac{\dot{\tau}(t)}{\tau^3(t)} \int_{\mathbb{R}} \frac{M^2}{R} dy dt < \infty.$$

*Proof.* Since  $\mathcal{E}(t) \geq 0$ , (5.2) follows from (2.2). We define

$$\mathcal{E}_+ := \frac{1}{2\tau^2} \int_{\mathbb{R}} \frac{M^2}{R} dy + \int_{R>1} R \log R + \int_{\mathbb{R}} y^2 R,$$

such that Equation (2.2) gives

$$\mathcal{E}_+(t) + \int_0^t \frac{\dot{\tau}(s)}{\tau^3(s)} \int_{\mathbb{R}} \frac{M^2}{R} dy ds \leq C + \int_{R<1} R \log \left( \frac{1}{R} \right).$$

The last term is controlled by

$$\int_{R < 1} R \log \left( \frac{1}{R} \right) \leq C_\varepsilon \int_{\mathbb{R}} R^{1-\varepsilon}$$

for all  $\varepsilon > 0$  arbitrary small. By an interpolation formula, we get that

$$\int_{\mathbb{R}} R^{1-\varepsilon} \lesssim \|R\|_{L^1}^{1-\frac{3\varepsilon}{2}} \|y^2 R\|_{L^1}^{\varepsilon/2}$$

for all  $0 < \varepsilon < \frac{2}{3}$ . This implies that for all  $t \geq 0$ ,

$$\mathcal{E}_+(t) + \int_0^t \frac{\dot{\tau}(s)}{\tau^3(s)} \int_{\mathbb{R}} \frac{M^2}{R} dy ds \lesssim 1 + \mathcal{E}_+^{\varepsilon/2}(t),$$

thus  $\mathcal{E}_+(t) \in L^\infty(\mathbb{R}_+)$ , and equation (5.1) follows.  $\square$

With Assumption 2.4, we can then show a much better result on the second moment of the density  $R$  than its boundedness, which is the convergence of this second moment towards the second moment of the Gaussian  $\Gamma$ :

**Proposition 5.3.** *We denote*

$$J_1 = \int_{\mathbb{R}} y^2 (R - \Gamma) dy.$$

*Then, under Assumption 2.4,*

$$|J_1(t)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

*Proof.* Denoting

$$J_2 = \int_{\mathbb{R}} y M dy,$$

we first write the system of differential equations satisfied by  $J_1$  and  $J_2$ . By integration by parts and using equation (2.1a), we compute

$$\dot{J}_1 = \frac{d}{dt} \int_{\mathbb{R}} y^2 \partial_t (R - \Gamma) = - \int_{\mathbb{R}} y^2 \frac{\partial_y M}{\tau^2} = \frac{2}{\tau^2} \int_{\mathbb{R}} y M = \frac{2}{\tau^2} J_2.$$

Still by integration by parts, and using equation (2.1b), we get that

$$\dot{J}_2 = \int_{\mathbb{R}} y \partial_t M = - \int_{\mathbb{R}} y \left[ \frac{1}{\tau^2} \partial_y \left( \frac{M^2}{R} \right) + \partial_y R + 2yR + M \right] = \frac{1}{\tau^2} \int_{\mathbb{R}} \frac{M^2}{R} + \int_{\mathbb{R}} R - \int_{\mathbb{R}} 2y^2 R - \int_{\mathbb{R}} y M.$$

Here we introduce the function  $\Gamma$ , writing

$$\int_{\mathbb{R}} R - 2 \int_{\mathbb{R}} y^2 R = \int_{\mathbb{R}} \Gamma - 2 \int_{\mathbb{R}} y^2 \Gamma - \int_{\mathbb{R}} 2y^2 (R - \Gamma),$$

and we can remark by an easy calculation that

$$\int_{\mathbb{R}} y^2 \Gamma = \frac{1}{2} \int_{\mathbb{R}} \Gamma,$$

so that finally

$$\dot{J}_2 + J_2 + 2J_1 = \frac{1}{\tau^2} \int_{\mathbb{R}} \frac{M^2}{R}.$$

We denote  $\tilde{J}_1 = \tau J_1$ , so that

$$\dot{\tilde{J}}_1 = \dot{\tau} J_1 + \tau \dot{J}_1 = \dot{\tau} J_1 + \frac{2}{\tau} J_2,$$

and

$$\ddot{J}_1 = \ddot{\tau}J_1 + \dot{\tau}\dot{J}_1 - \frac{2\dot{\tau}}{\tau}J_2 + \frac{2}{\tau}\dot{J}_2 = -\dot{J}_1 - \frac{2}{\tau}J_1 + \frac{2}{\tau^3} \int_{\mathbb{R}} \frac{M^2}{R},$$

so we write

$$\ddot{J}_1 + \dot{J}_1 + \frac{1}{\tau^2}\tilde{J}_1 = \frac{2}{\tau^3} \int_{\mathbb{R}} \frac{M^2}{R}.$$

Rather than trying to solve directly this non-autonomous differential equation of order 2, we will work on the following approximate equation, for  $t \geq 1$ ,

$$(5.3) \quad \ddot{f} + \dot{f} + \frac{1}{4t}f = \frac{2}{\tau}\mathcal{E}_{\text{kin}}.$$

In fact, denoting  $w = \tilde{J}_1 - f$ , we see that  $w$  satisfies the differential equation

$$\ddot{w} + \dot{w} + \frac{1}{4t}w + \left(\frac{1}{\tau^2} - \frac{1}{4t}\right)\tilde{J}_1 = 0,$$

with

$$\left(\frac{1}{\tau^2} - \frac{1}{4t}\right) = \frac{(\tau - 2\sqrt{t})(\tau + 2\sqrt{t})}{4t\tau^2} = O\left(\frac{1}{t^{\frac{3}{2}}}\right)$$

as we actually know from [8] that  $\tau(t) - 2\sqrt{t}$  is uniformly bounded. The homogeneous part of equation (5.3) can be written under a Kummer's type equation

$$t\ddot{f} + t\dot{f} + \frac{1}{4}f = 0$$

which has two fundamental independent solutions (see [1])

$$f_1(t) = t\mathcal{M}\left(\frac{5}{4}, 2, -t\right) = te^{-t}\mathcal{M}\left(\frac{3}{4}, 2, t\right)$$

and

$$f_2(t) = e^{-t}\mathcal{U}\left(-\frac{1}{4}, 0, t\right) = te^{-t}\mathcal{U}\left(\frac{3}{4}, 2, t\right),$$

where  $\mathcal{M}(a, b, z)$  and  $\mathcal{U}(a, b, z)$  respectively denotes the Kummer's and the Tricomi's function, which both stand as confluent hypergeometric functions and are independent solutions of the Kummer's equation

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0.$$

In particular, from the asymptotic properties [1]

$$\mathcal{M}(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b} (1 + O(|z|^{-1}))$$

and

$$\mathcal{U}(a, b, z) = z^{-a} (1 + O(|z|^{-1})),$$

we get the following asymptotic for our fundamental solutions

$$(5.4) \quad f_1(t) \sim \frac{C}{t^{\frac{1}{4}}} \left(1 + O\left(\frac{1}{t}\right)\right)$$

and

$$(5.5) \quad f_2(t) \sim e^{-t}t^{\frac{1}{4}} \left(1 + O\left(\frac{1}{t}\right)\right).$$

From the classical theory of linear differential equations, we know that every solution of equation (5.3) can be written under the form

$$(5.6) \quad f(t) = c_1 f_1(t) + c_2 f_2(t) + f_0(t),$$

where  $c_1$  and  $c_2$  denote two real numbers, and  $f_0$  is a particular solution of (5.3). We easily compute the Wronskian function

$$W(t) = f_1(t)\dot{f}_2(t) - \dot{f}_1(t)f_2(t)$$

by solving the differential equation

$$\dot{W} = \dot{f}_1\dot{f}_2 + f_1\ddot{f}_2 - \ddot{f}_1f_2 - \dot{f}_1\dot{f}_2 = f_1(-\dot{f}_2 - \frac{1}{\tau^2}f_2) + (\dot{f}_1 + \frac{1}{\tau^2}f_1)f_2 = -W,$$

which leads to

$$W(t) = W_0 e^{-t}.$$

Then, by a variation of constant formula, we can find a particular solution of (5.3) under the form

$$f_0(t) = f_2(t) \int_0^t \frac{f_1(s)}{W(s)} \frac{\mathcal{E}_{\text{kin}}(s)}{\tau(s)} ds - f_1(t) \int_0^t \frac{f_2(s)}{W(s)} \frac{\mathcal{E}_{\text{kin}}(s)}{\tau(s)} ds,$$

so that using Assumption 2.4 and the equivalents (5.4) and (5.5), we get that every solution of (5.3) has the asymptotic:

$$|f(t)| = o(\sqrt{t}).$$

In fact, in the expression of (5.6),

$$\left| f_1(t) \int_0^t e^s f_2(s) \frac{\mathcal{E}_{\text{kin}}(s)}{\tau(s)} ds \right| \lesssim \frac{1}{t^{\frac{3}{4}}} \int_1^t s^{-\frac{1}{4}} \mathcal{E}_{\text{kin}}(s) ds = o(\sqrt{t}),$$

and every other terms is bounded as  $t \rightarrow \infty$ . The same result applies for the function  $w = \tilde{J}_1 - f$ , so we can write that for all  $t \geq t_0 \geq 0$ ,

$$(5.7) \quad \begin{aligned} \tilde{J}_1(t) = f(t) + c_1 f_1(t) + c_2 f_2(t) + f_2(t) \int_{t_0}^t \frac{f_1(s)}{W(s)} \left( \frac{1}{4s} - \frac{1}{\tau(s)^2} \right) \tilde{J}_1(s) ds \\ - f_1(t) \int_{t_0}^t \frac{f_2(s)}{W(s)} \left( \frac{1}{4s} - \frac{1}{\tau(s)^2} \right) \tilde{J}_1(s) ds. \end{aligned}$$

We already know from Lemma 5.2 that  $\tilde{J}_1$  is  $O(\sqrt{t})$ , and as we have just shown that  $f$  is  $o(\sqrt{t})$ , for  $\varepsilon > 0$  fixed, there exists  $t_0 \geq 0$  such that, for all  $t \geq t_0$ ,

$$|\tilde{J}_1(t)| \leq C\sqrt{t}, \quad \left| \frac{1}{4s} - \frac{1}{\tau(s)^2} \right| \leq Ct^{-\frac{3}{2}} \quad \text{and} \quad \frac{1}{\sqrt{t}} |f(t) + c_1 f_1(t) + c_2 f_2(t)| \leq \frac{\varepsilon}{2}.$$

Injecting these inequalities in the right-hand side of (5.7), we have

$$\frac{1}{\sqrt{t}} |\tilde{J}_1(t)| \leq \frac{\varepsilon}{2} + \frac{C}{t^{\frac{1}{4}}} + \frac{C}{t^{\frac{3}{4}}} \leq \varepsilon$$

for  $t$  large enough, so we finally get by a bootstrap argument that

$$|\tilde{J}_1(t)| = o(\sqrt{t}).$$

Finally, recalling that  $\tilde{J}_1 = \tau J_1$  and that  $\tau \sim 2\sqrt{t}$ , we can conclude that

$$|J_1| = \left| \int_{\mathbb{R}} y^2 (R - \Gamma) dy \right| = o(1),$$

which ends the proof.  $\square$

*Remark 5.4.* Note that in the Gaussian case, we can explicitly compute the kinetic energy

$$\mathcal{E}_{\text{kin}}(t) = \frac{1}{\tau^2(t)} \int_{\mathbb{R}} \frac{M^2(t, y)}{R(t, y)} dy = \frac{c(t)}{\tau(t)} \|\rho_0\|_{L^1(\mathbb{R})} = O\left(\frac{1}{\sqrt{t}}\right),$$

unless the initial data are well prepared in the sense that  $c_0 = \bar{x}_\infty$  (which would actually improve this rate). This feature gives an explicit rate of convergence of  $\mathcal{E}_{\text{kin}}$  towards 0 that can be propagated to the convergence of the second momentum of  $R$  by adapting the proof of Proposition 5.3, namely

$$|J_1| = \left| \int_{\mathbb{R}} y^2 (R - \Gamma) dy \right| = O\left(\frac{1}{\sqrt{t}}\right).$$

## 6. CONVERGENCE

*Proof of Proposition 2.6.* We are first going to try to eliminate the momentum  $M$  of our target equation. Differentiating equation (2.1b) with respect to  $y$ , and using equation (2.1a) in order to express

$$\partial_t(\partial_y(RM)) = -\partial(\tau^2 \partial_t R),$$

we get that

$$-\partial_t(\tau^2 \partial_t R) - \tau^2 \partial_t R + LR = -\frac{1}{\tau^2} \partial_y \left( \frac{M^2}{R} \right).$$

where we have defined the Fokker-Planck operator

$$L := \partial_y^2 + 2\partial_y(y \cdot).$$

Following [8], we introduce another scaling in time  $s$  defined by

$$(6.1) \quad \partial_s = \tau^2 \partial_t,$$

and the notation

$$\tilde{R}(s(t), y) := R(t, y).$$

We calculate the quantities

$$\partial_t(\tau^2 \partial_t R) = \frac{1}{\tau^2} \partial_s^2 \tilde{R} \text{ and } \tau^2 \partial_t R = \partial_s \tilde{R},$$

hence we obtain the following equation:

$$(6.2) \quad -\frac{1}{\tau^2} \partial_s^2 \tilde{R} - \partial_s \tilde{R} + L\tilde{R} = -\frac{1}{\tau^2} \partial_y \left( \frac{M^2}{\tilde{R}} \right).$$

Now we remark that equation (5.1) induces

$$(6.3) \quad \sup_{s \geq 0} \int_{\mathbb{R}} \tilde{R}(s, y) (1 + |y|^2 + |\log \tilde{R}(s, y)|) dy < \infty,$$

that equation (5.2) gives

$$(6.4) \quad \int_0^\infty \dot{\tau}(t) \int_{\mathbb{R}} \frac{\tilde{M}^2}{\tilde{R}} dy dt < \infty,$$

and that

$$\tau(s) \sim 2e^{2s} \text{ and } \dot{\tau}(s) \sim e^{-2s},$$

so we can conclude like in [7]. Let a sequence  $s_n \rightarrow \infty$ , take  $s \in [-1, 2]$ , and denote

$$\tilde{R}_n(s, y) := \tilde{R}(s + s_n, y).$$



From (6.3) along with the de la Vallée-Poussin [9] and Dunford-Pettis theorems [10], we get the following weak convergence (up to a subsequence, not relabeled for reader's convenience), for all  $p \in [1, \infty)$ ,

$$\tilde{R}_n \rightharpoonup_{t \rightarrow \infty} \tilde{R}_\infty \quad \text{in } L^p(-1, 2; L^1(\mathbb{R})).$$

We also get the weak convergence of the initial datum, up to another subsequence:

$$\tilde{R}_n(0) \rightharpoonup_{t \rightarrow \infty} \tilde{R}_{0,\infty} \quad \text{in } L^1(\mathbb{R}).$$

Thanks to (6.3), we also get that the family  $(\tilde{R}(s_n, \cdot))_n$  is tight, so

$$\int_{\mathbb{R}} \tilde{R}_{0,\infty}(y) dy = \int_{\mathbb{R}} \Gamma(y) dy$$

and

$$\int_{\mathbb{R}} \tilde{R}_{0,\infty}(y)(1 + |y|^2 + |\log \tilde{R}_{0,\infty}(y)|) dy < \infty.$$

Then, denoting  $\tau_n(s) = \tau(s + s_n)$ , equation (6.4) implies that

$$-\frac{1}{\tau_n^2} \partial_y \left( \frac{M_n^2}{\tilde{R}_n} \right) \rightharpoonup_{t \rightarrow \infty} 0 \quad \text{in } L^1(-1, 2; W^{-2,1}(\mathbb{R})).$$

In addition, in (6.2), all the other terms but two obviously go weakly to zero, which yields

$$(6.5) \quad \partial_s \tilde{R}_\infty = L \tilde{R}_\infty$$

in  $\mathcal{D}'((-1, 2) \times \mathbb{R})$ , with  $\tilde{R}_\infty(0, \cdot) = \tilde{R}_{0,\infty} \in L^1(\mathbb{R})$ . Thanks to the above bounds on  $\tilde{R}_{0,\infty}$ , it is known (see [3]) that the solution  $\tilde{R}_\infty$  to (6.5) is actually defined for all  $s \geq 0$  and satisfies

$$(6.6) \quad \|\tilde{R}_\infty - \Gamma\|_{L^1(\mathbb{R})} \xrightarrow{t \rightarrow \infty} 0.$$

Going back to system (2.1), we need to show that  $\tilde{R}_\infty$  is independent of  $s$ . In the  $s$  variable, equation (2.1a) becomes

$$(6.7) \quad \partial_s \tilde{R} + \partial_y \tilde{M} = 0,$$

and (6.4) implies that  $\tilde{M} \in L^2(-1, 2; L^1(\mathbb{R}))$ . With  $\tilde{M}_n(s) := \tilde{M}(s + s_n)$ , we have

$$\partial_y \tilde{M}_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } L^2(-1, 2; W^{-1,1}(\mathbb{R})),$$

so

$$\partial_s \tilde{R}_\infty = 0.$$

Combining this last equality with equation (6.6), we infer that  $\tilde{R}_\infty = \Gamma$ . The limit being unique, no extraction of a subsequence is needed, and we conclude that

$$\tilde{R}(s) \rightharpoonup_{s \rightarrow \infty} \Gamma \quad \text{weakly in } L^1(\mathbb{R}).$$

## 7. CONCLUSION

We have shown that Gaussian functions play an important role in the study of the isothermal compressible Euler equation, standing both as particular solutions of the system (1.3) and time-asymptotic limit of solutions of this system. They also make the link between the isentropic and the isothermal system, as limit of the Barenblatt solutions of (1.2a).

The next step of the analysis of the long-time behavior of this system would be to find an explicit convergence rate of any  $L^\infty$  weak solution of (1.3) towards the limit Gaussian profile, adapting the work from [13], [15] and [11]. A good approach seems to be the use of the Csiszár-Kullback inequality that gives a lower bound of the entropy by the  $L^1$ -norm of  $\rho - \bar{\rho}$ :

$$\|\rho - \bar{\rho}\|_{L^1}^2 \leq 2\|\rho_0\|_{L^1} \int_{\mathbb{R}} \rho \log\left(\frac{\rho}{\bar{\rho}}\right) = 2\|\rho_0\|_{L^1} \int_{\mathbb{R}} R \log\left(\frac{R}{\Gamma}\right).$$

Unfortunately, no decreasing rate of the entropy function is currently known. This is still an open question that appears in other fields of the analysis of PDEs, for instance the study of the logarithmic Schrödinger equation [7]. Note that in [21], the author indeed has an upper bound for the entropy, assuming  $\bar{\rho} \geq c > 0$ , which is true on the compact set  $\Omega \subset \mathbb{R}$  but obviously false on the whole space  $\mathbb{R}$ .

## REFERENCES

- [1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, No. 55. U. S. Government Printing Office, Washington, D. C., 1964. For sale by the Superintendent of Documents.
- [2] Cécile Ané, Sébastien Blachère, Djali Chafaï, Pierre Fougères, Ivan Gentil, Florent Malrieu, Cyril Roberto, and Grégory Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.
- [3] Anton Arnold, Peter Markowich, and Andreas Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Communications in Partial Differential Equations*, 26, 05 2000.
- [4] D. G. Aronson. The porous medium equation. In *Nonlinear diffusion problems (Montecatini Terme, 1985)*, volume 1224 of *Lecture Notes in Math.*, pages 1–46. Springer, Berlin, 1986.
- [5] G. I. Barenblatt. On a class of exact solutions of the plane one-dimensional problem of unsteady filtration of a gas in a porous medium. *Akad. Nauk SSSR. Prikl. Mat. Meh.*, 17:739–742, 1953.
- [6] Rémi Carles, Kleber Carrapatoso, and Matthieu Hillaire. Rigidity results in generalized isothermal fluids. *Annales Henri Lebesgue*, 1:47–85, 2018.
- [7] Rémi Carles and Isabelle Gallagher. Universal dynamics for the defocusing logarithmic Schrödinger equation. *Duke Math. J.*, 167(9):1761–1801, 2018.
- [8] Quentin Chauleur. Dynamics of the Schrödinger-Langevin equation. *Nonlinearity*, 34(4):1943–1974, 2021.
- [9] Claude Dellacherie and Paul-André Meyer. *Probabilités et potentiel*. Publications de l’Institut de Mathématique de l’Université de Strasbourg, No. XV. Hermann, Paris, 1975.
- [10] Nelson Dunford and Jacob T. Schwartz. *Linear Operators. I. General Theory*. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.
- [11] Shifeng Geng and Feimin Huang.  $L^1$ -convergence rates to the Barenblatt solution for the damped compressible Euler equations. *J. Differential Equations*, 266(12):7890–7908, 2019.
- [12] Feimin Huang. Large time behavior for compressible Euler equations with damping and vacuum. Number 1247, pages 57–66. 2002. *Mathematical analysis in fluid and gas dynamics (Japanese)* (Kyoto, 2001).
- [13] Feimin Huang, Pierangelo Marcati, and Ronghua Pan. Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum. *Arch. Ration. Mech. Anal.*, 176(1):1–24, 2005.
- [14] Feimin Huang and Ronghua Pan. Asymptotic behavior of the solutions to the damped compressible Euler equations with vacuum. *J. Differential Equations*, 220(1):207–233, 2006.

- [15] Feimin Huang, Ronghua Pan, and Zhen Wang.  $L^1$  convergence to the Barenblatt solution for compressible Euler equations with damping. *Arch. Ration. Mech. Anal.*, 200(2):665–689, 2011.
- [16] Philippe G. LeFloch and Vladimir Shelukhin. Symmetries and global solvability of the isothermal gas dynamics equations. *Arch. Ration. Mech. Anal.*, 175(3):389–430, 2005.
- [17] Tianhong Li and Dehua Wang. Blowup phenomena of solutions to the Euler equations for compressible fluid flow. *Journal of Differential Equations*, 221(1):91 – 101, 2006.
- [18] Tai-Ping Liu. Compressible flow with damping and vacuum. *Japan J. Indust. Appl. Math.*, 13(1):25–32, 1996.
- [19] Pierangelo Marcati and Ronghua Pan. Cauchy problem for compressible Euler equations with damping. In *International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999)*, pages 315–317. World Sci. Publ., River Edge, NJ, 2000.
- [20] Takaaki Nishida. *Nonlinear hyperbolic equations and related topics in fluid dynamics*. Publications Mathématiques d’Orsay, No. 78-02. Département de Mathématique, Université de Paris-Sud, Orsay, 1978.
- [21] Kun Zhao. On the isothermal compressible Euler equations with frictional damping. *Commun. Math. Anal.*, 9(2):77–97, 2010.

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