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SIMPLICIAL INTERSECTION HOMOLOGY REVISITED.

DAVID CHATAUR , MARTINTXO SARALEGI-ARANGUREN , AND DANIEL TANRÉ 

ABSTRACT. Intersection homology is defined for simplicial, singular and PL chains and it is well known that the three versions are isomorphic for a full filtered simplicial complex. In the literature, the isomorphism, between the singular and the simplicial situations of intersection homology, uses the PL case as an intermediate. Here we show directly that the canonical map between the simplicial and the singular intersection chains complexes is a quasi-isomorphism. This is similar to the classical proof for simplicial complexes, with an argument based on the concept of residual complex and not on skeletons.

This parallel between simplicial and singular approaches is also extended to the intersection blown-up cohomology that we introduced in a previous work. In the case of an orientable pseudomanifold, this cohomology owns a Poincaré isomorphism with the intersection homology, for any coefficient ring, thanks to a cap product with a fundamental class. So, the blown-up intersection cohomology of a pseudomanifold can be computed from a triangulation. Finally, we introduce a blown-up intersection cohomology for PL spaces and prove that it is isomorphic to the singular one.

The homology of a simplicial complex, K , can be computed indifferently from the simplices of the triangulation or from the singular chains complex of its realization, $|K|$. In other words, the map between chain complexes $C_*(K) \rightarrow C_*(|K|)$ is a quasi-isomorphism. Classically, the proof (see [14]) uses an induction on the skeletons $K^{(\ell)}$ of K . The crucial point is the existence of isomorphisms,

$$C_*(K^{(\ell)}, K^{(\ell-1)}) \cong \bigoplus_{\beta \in K, \dim \beta = \ell} C_*(\beta, \partial\beta) \text{ and } H_*(|K^{(\ell)}|, |K^{(\ell-1)}|) \cong \bigoplus_{\beta \in K, \dim \beta = \ell} H_*(|\beta|, |\partial\beta|).$$

Before developing the corresponding situation in intersection homology, let us make a brief historical reminder. In their first paper ([11]) on intersection homology, M. Goresky and R. MacPherson introduce it for a pseudomanifold X together with a fixed PL structure and a parameter \bar{p} , called perversity. They define the complex of \bar{p} -intersection, $\mathfrak{C}_*^{\bar{p}}(X)$, as a subcomplex of the complex of PL chains and the \bar{p} -intersection homology as the homology of $\mathfrak{C}_*^{\bar{p}}(X)$. Later, in an appendix to [16], they define a complex of \bar{p} -intersection, $C_*^{\bar{p}}(K)$, as a subcomplex of the simplicial chains of a filtered simplicial complex K . If K is a full admissible triangulation of a PL space X , they prove that the inclusion $C_*^{\bar{p}}(K) \rightarrow \mathfrak{C}_*^{\bar{p}}(X)$ induces an isomorphism in homology. They do that with a nice construction of a left inverse to this inclusion. Let us specify that, without the “full” hypothesis on K , the simplicial and PL intersection homologies may not be isomorphic, as an explicit example shows in [16, Appendix]. The barycentric subdivision of a simplicial complex being full, one can always find such triangulation of X .

A third step is to consider the topological realization, $|K|$, of K and define a complex of \bar{p} -intersection, $C_*^{\bar{p}}(|K|)$, as a subcomplex of the singular chains on $|K|$. This has been done by King ([15]) who proves the existence of an isomorphism between the singular and the PL intersection homologies, in the case of CS sets. These relationships between the three points of view are developed in a thorough way by G. Friedman in [9, Sections 3.3 and 5.4]. Let us emphasize that, unlike that of the classical case, the proof is not a direct comparison between the simplicial and the singular points of view: the PL case is used as intermediate between them and a structure of CS set is required.

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The first objective of this work is to obtain an isomorphism between singular and simplicial intersection homologies, in a direct way, from a chain map between the corresponding complexes, and without any restriction to CS sets. Let us start with a filtered simplicial complex K , that is, a simplicial complex endowed with a filtration,

$$K = K_n \supset K_{n-1} \supset K_{n-2} \supset \dots \supset K_0 \supset K_{-1} = \emptyset,$$

where each K_i is a subcomplex. Unfortunately we *do not* have the expected formula

$$C_*^{\bar{p}}(K^{(\ell)}, K^{(\ell-1)}) \cong \bigoplus_{\beta \in K, \dim \beta = \ell} C_*^{\bar{p}}(\beta, \partial\beta), \quad (1)$$

as shows the example in Subsection 2.2. To overcome this defect, we introduce for any filtered simplicial complex, K , a pair of integers, called *complexity* (see Definition 1.1) and denoted $\text{comp } K$. With the lexicographic order, the complexity provides the opportunity for reasoning by induction. As for an alternative decomposition to (1), we introduce a subcomplex of K , called the *residual complex*, denoted $\mathcal{L}(K)$ and already present in [1]. The desired decomposition comes from a particular class $\mathcal{B}(K)$ of simplices of K , called *clot*, see Definition 1.4. By noting L_β the link of a clot β , we get in Proposition 2.3, an isomorphism

$$\bigoplus_{\beta \in \mathcal{B}(K)} C_*^{\bar{p}}(\beta * L_\beta, \partial\beta * L_\beta) \xrightarrow{\cong} C_*^{\bar{p}}(K, \mathcal{L}(K)). \quad (2)$$

From it, we can adapt the proof of the classical case and prove that the canonical map between the simplicial and the singular intersection complexes, is a quasi-isomorphism.

In fact, we apply this program not only to \bar{p} -intersection homology but also to a \bar{p} -*intersection cohomology* obtained from simplicial blow up, denoted $\mathcal{H}_{\bar{p}}^*$ and called *blown-up intersection cohomology*. For stratified spaces in general, this cohomology is naturally equipped with cup and cap products for any ring coefficients and with cohomology operations ([2, 8]). Regarding duality and pseudomanifolds, since the work ([12]) of Goresky and Siegel, it is known that there is no Poincaré duality on intersection homology with coefficients in any commutative ring. However, in [5], we prove that the cap product with the fundamental class of an oriented, compact, n -dimensional pseudomanifold induces an isomorphism between $\mathcal{H}_{\bar{p}}^*(X)$ and the intersection homology $H_{n-*}^{\bar{p}}(X)$. (Similar versions exist for a paracompact not necessarily compact pseudomanifold, with compact supports in cohomology or Borel-Moore homology, [7] and [19]). This blown-up cohomology coincides with the cohomology obtained from the linear dual of the chain complexes, when \mathbb{R} is a field, or more generally with an hypothesis of \mathbb{R} -local torsion free in a part of the intersection homology of links, already present in [12]. Let us also mention that its sheafification is the Deligne sheaf ([7]). The existence of the isomorphism between simplicial and singular blown-up intersection cohomology is a new result. In short, we prove in Theorems 4.2 and 5.13:

Main Theorem. *Let K be a full filtered complex and \bar{p} be any perversity, then we have isomorphisms:*

$$\begin{array}{ccc} H_*^{\bar{p}}(K) & \cong & H_*^{\bar{p}}(|K|) & \text{and} & \mathcal{H}_{\bar{p}}^*(K) & \cong & \mathcal{H}_{\bar{p}}^*(|K|). \\ \text{simplicial} & & \text{singular} & & \text{simplicial} & & \text{singular} \end{array}$$

We also define a blown-up cohomology for PL spaces and relate it to the simplicial and the singular blown-up cohomologies in Theorem 6.7. This theorem contains a second part on intersection homology, with a proof of the existence of an isomorphism between singular and PL intersection homology, without requiring the existence of a CS set structure. This answers a question asked by G. Friedman in [9, Page 234].

In the sequel, all the complexes are considered with coefficients in an abelian group denoted \mathbb{R} , which is not explicitly mentioned. If X is a topological space, we denote $\mathfrak{c}X = X \times [0, 1]/X \times \{0\}$ the cone on X and $\mathring{\mathfrak{c}}X = X \times [0, 1[/ X \times \{0\}$ the open cone on X . The apex of a cone is \mathfrak{v} .

Our program is carried out in Sections 1-6 below whose headings are self-explanatory.

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1. FILTERED SIMPLICIAL SETTING

We introduce the definitions and properties necessary for the study of the intersection homology of a filtered simplicial complex, K . In particular, by allowing proofs by induction, the notions of complexity and residual complex of K will play an important role in the sequel.

A *simplicial complex* K is a set of simplices in \mathbb{R}^p , $p \leq \infty$, such that

1. if $\sigma, \tau \in K$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a face of both σ and τ ,
2. if $\sigma \in K$ and τ is a face of σ , written $\tau \triangleleft \sigma$, then $\tau \in K$,
3. (local finiteness) every vertex of a simplex of K belongs to a finite number of simplices of K .

We say that $d \in \mathbb{N}$ is the dimension of K , denoted $\dim(K) = d$, if every simplex of K has dimension lower than or equal to d and K has at least one simplex of dimension d . If this number does not exist, we say that $\dim(K) = \infty$. (By convention, if $K = \emptyset$, we write $\dim K = -\infty$.) A subcollection $L \subset K$ is a *simplicial subcomplex* of K if it verifies properties 1 and 2. The union of the simplices of K whose dimension is smaller than or equal to a given $\ell \in \mathbb{N}$ is called the ℓ -*skeleton* of K and denoted $K^{(\ell)}$. A *simplicial map* from K to K' is a function from the set of vertices of K to the set of vertices of K' such that the images of the vertices of a simplex of K is a simplex of K' .

Let $|K|$ be the topological subspace of \mathbb{R}^p formed by the union of the simplices of the simplicial complex K . If Δ is the standard simplex, we identify Δ with $|\Delta|$.

A *filtered simplicial complex* is a simplicial complex K endowed with a filtration made up of simplicial subcomplexes,

$$K = K_n \supset K_{n-1} \supset K_{n-2} \supset \dots \supset K_0 \supset K_{-1} = \emptyset.$$

The integer k is the (*virtual*) *dimension* of the filtered simplicial complex K_k , denoted $\dim_v K_k = k$.

A *stratum* of K is a non-empty connected component, S , of a $|K_i| \setminus |K_{i-1}|$. Its dimension is $\dim_v S = i$ and its codimension is $\text{codim}_v S = n - i$. The family of strata of K is denoted \mathcal{S}_K or \mathcal{S} if there is no ambiguity. The strata included in $|K| \setminus |K_{n-1}|$ are called *regular*, the other ones being called *singular*.

We introduce a pair of integers mixing geometrical and virtual dimensions.

Definition 1.1. The *complexity* of a filtered simplicial complex $K \neq \emptyset$ is the pair $\text{comp}(K) = (a, b)$ where:

- $a = \max\{k \in \{0, \dots, n\} \mid K_{n-k} \neq \emptyset\}$, and
- $b \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the geometrical dimension of the simplicial complex K_{n-a} .

By convention, we set $\text{comp} \emptyset = (-\infty, -\infty)$.

The associated *lexicographic order* will prove to be the key in the forthcoming proofs by induction.

Recall that a subcomplex L of a simplicial complex K is *full* if any simplex of K having all its vertices in L is a simplex of L . As we will see, the following reinforcement of the notion of filtered simplicial complex is a necessary hypothesis of the main theorem.

Definition 1.2. A filtered simplicial complex K is said *full* if any K_ℓ of its filtration is full.

The standard simplex $K = \Delta$ can be endowed with different structures of filtered simplicial complex.

- The first filtration is given by the skeleta: we set $K_\ell = \Delta^{(\ell)}$. It is not full.
- A series of full filtrations is defined by induction, starting with the choice of one vertex, $K_0 = \{v\}$. For the simplicial subcomplex K_ℓ with $\ell \geq 1$, we choose an ℓ -dimensional face of Δ containing $K_{\ell-1}$.

The fullness property can be recovered with a barycentric subdivision. This is a general fact as the following result shows (see [16, Remark 2]).

Lemma 1.3. *The barycentric subdivision of a filtered simplicial complex is full.*

Proof. Consider a filtered simplicial complex

$$K = K_n \supset K_{n-1} \supset K_{n-2} \supset \cdots \supset K_0 \supset \emptyset,$$

inducing the filtered simplicial complex $K' = K'_n \supset K'_{n-1} \supset K'_{n-2} \supset \cdots \supset K'_0 \supset \emptyset$, where the upperscript $'$ indicates the barycentric subdivision. Let σ be a simplex of K' with $\sigma \cap |K'_i| \neq \emptyset$. We need to prove that $\sigma \cap |K'_i|$ is a face of σ . Notice that $\sigma \cap |K'_i| = \sigma \cap |K_i|$.

Let us denote $\widehat{\tau}$ the barycenter of a face $\tau \in K$. By definition of the barycentric subdivision, any simplex of K' is obtained from barycenters of successive faces. More precisely, the barycenters of a family of simplices of K , $\tau_0 \triangleleft \cdots \triangleleft \tau_m$, determine a simplex of K' , $\sigma = [\widehat{\tau}_0, \dots, \widehat{\tau}_m]$. Let us suppose that, for any $\alpha \in K$, we have

$$\sigma \cap \alpha = \emptyset \text{ or } \sigma \cap \alpha = [\widehat{\tau}_0, \dots, \widehat{\tau}_\ell] \text{ for some } \ell \in \{0, \dots, m\}. \quad (3)$$

As K_i is a simplicial complex, the intersection $\tau_m \cap |K_i|$ is a union of faces of τ_m and we conclude from $\sigma \subset \tau_m$ and (3):

$$\sigma \cap |K_i| = \sigma \cap \tau_m \cap |K_i| = [\widehat{\tau}_0, \dots, \widehat{\tau}_\ell]$$

for some $\ell \in \{0, \dots, m\}$. This implies the announced conclusion $\sigma \cap |K_i| \triangleleft \sigma$ and it remains to prove (3).

So, let $\alpha \in K$ and $\sigma = [\widehat{\tau}_0, \dots, \widehat{\tau}_m] \in K'$ with $\sigma \cap \alpha \neq \emptyset$. Any point $P \in \sigma$ has the canonical decomposition $P = \sum_{j \in J_P} t_j \widehat{\tau}_j$ where $t_j \in]0, 1]$ for each $j \in J_P$, $\sum_{j \in J_P} t_j = 1$ and $\emptyset \neq J_P \subset \{0, \dots, m\}$. We define $\ell = \max\{j \in J_P \mid P \in \sigma \cap \alpha\}$, which belongs to $\{0, \dots, m\}$ since $\sigma \cap \alpha \neq \emptyset$. By construction we have the inclusions $\sigma \cap \alpha \subset [\widehat{\tau}_0, \dots, \widehat{\tau}_\ell]$ and $[\widehat{\tau}_0, \dots, \widehat{\tau}_\ell] \subset \sigma$. By definition, there exists a vertex Q of $\sigma \cap \alpha$ with $\ell \in J_Q$. The condition $\sum_{j \in J_Q} t_j \widehat{\tau}_j \in \alpha$ implies $\widehat{\tau}_\ell \in \alpha$. This gives the second inclusion $[\widehat{\tau}_0, \dots, \widehat{\tau}_\ell] \subset \alpha$. \clubsuit

1.1. Canonical decomposition of $\sigma \in K$. Let us suppose that the filtered simplicial complex K is full. This implies that any simplex $\sigma \in K$ has the *canonical decomposition*

$$\sigma = \sigma_0 * \sigma_1 * \cdots * \sigma_n,$$

where $\sigma \cap |K_\ell| = \sigma_0 * \cdots * \sigma_\ell$, for each $\ell \in \{0, \dots, n\}$. (We use the convention $\emptyset * E = E$ for any simplicial complex E .) For each $\ell \in \{0, \dots, n\}$, we have:

$$\sigma_\ell \neq \emptyset \iff \sigma \cap |K_\ell| \setminus |K_{\ell-1}| \neq \emptyset \iff \exists S \in \mathcal{S}_K \text{ with } \dim_v S = \ell \text{ and } \sigma \cap S \neq \emptyset.$$

By connectedness, the stratum S is unique and we denote it $S = S_\ell$. (Notice that $\ell = n$ is equivalent to the regularity of S_ℓ .) We set

$$I_\sigma^K = \{\ell \in \{0, \dots, n\} \mid \sigma_\ell \neq \emptyset\} \quad \text{and} \quad \mathcal{S}_\sigma^K = \{S \in \mathcal{S}_K \mid S \cap \sigma \neq \emptyset\}.$$

In other words, we have $I_\sigma^K = \{\dim_v S \mid S \in \mathcal{S}_\sigma^K\}$ and $\mathcal{S}_\sigma^K = \{S_\ell \mid \ell \in I_\sigma^K\}$. We also denote $I_\sigma^K = I_\sigma$ and $\mathcal{S}_\sigma^K = \mathcal{S}_\sigma$ if there is no ambiguity.

Definition 1.4. Let K be a filtered simplicial complex of complexity (a, b) . A simplex $\sigma \in K$ is a *clot* if $\sigma = \sigma_{n-a}$ and $\dim \sigma = b$, where $\text{comp}(K) = (a, b)$. The family of clots of K is denoted $\mathcal{B}(K)$.

So, a clot is a simplex $\sigma \in K$ living in K_{n-a} with maximal geometrical dimension; i.e., $\dim \sigma = \dim K_{n-a} = b$.

1.2. Induced filtered simplicial complexes. Let K be a filtered simplicial complex. Given a simplicial subcomplex $L \subset K$ the induced filtration $L_i = L \cap K_i$ defines a filtered simplicial complex structure on L . Notice that $\text{comp}(L) \leq \text{comp}(K)$.

Since $L_i \subset K_i$ for any $i \in \{0, \dots, n\}$, for any stratum $T \in \mathcal{S}_L$ there exists a unique stratum $S \in \mathcal{S}_K$ with $T \subset S$. We say that S is the *source* of T . Notice that $\dim_v T = \dim_v S$.

Given $\sigma \in L$, we have a priori two canonical decompositions of σ : as a simplex of L or as a simplex of K . Since $\sigma \cap L_k = \sigma \cap L \cap K_k = \sigma \cap K_k$, for each $k \in \{0, \dots, n\}$, the two decompositions coincide. In particular $I_\sigma^K = I_\sigma^L$, denoted I_σ . Moreover, the family \mathcal{S}_σ^K is the family made up of the sources of the strata of \mathcal{S}_σ^L . So the source of T_ℓ is S_ℓ for each $\ell \in I_\sigma$.

1.3. Links and joins. We introduce two geometrical constructions associated to a filtered simplicial complex K , of complexity $\text{comp}(K) = (a, b)$.

The *link* of a simplex β of K is the simplicial complex

$$L_\beta = \{\sigma \in K \mid \beta * \sigma \in K\}.$$

Recall that L_β inherits from K a filtered simplicial complex structure. If the simplex β is a clot and the filtered simplicial complex K is full, then $(L_\beta)_{n-k} = \emptyset$ for $k \in \{0, \dots, a\}$. In particular, $\text{comp}(L_\beta) < \text{comp}(K)$. In this case, the *join* $\beta * L_\beta$ of a clot β inherits from K the following structure of filtered simplicial complex:

$$(\beta * L_\beta)_i = \begin{cases} \beta * (L_\beta)_i & \text{if } n - a < i \leq n, \\ \beta & \text{if } i = n - a, \\ \emptyset & \text{if } 0 \leq i < n - a. \end{cases} \quad (4)$$

The family $\mathcal{S}_{\beta * L_\beta}$ of strata of the join $\beta * L_\beta$ comes from the filtration (4).

1.4. Residual complex. The *residual complex* $\mathcal{L}(K)$ of a filtered simplicial complex $K \neq \emptyset$ with $\text{comp} K = (a, b)$ is the simplicial subcomplex defined by

$$\mathcal{L}(K) = \{\sigma \in K \mid \dim(\sigma \cap |K_{n-a}|) < b\}.$$

Given a simplicial subcomplex, $L \subset K$, we necessarily have

$$\mathcal{L}(L) \subset \mathcal{L}(K).$$

The equality $\mathcal{L}(K) = K$ occurs if $b = \infty$. When b is finite, we always have $\text{comp} \mathcal{L}(K) < \text{comp} K$. For $a = 0$ we get $\mathcal{L}(K) = K^{(b-1)}$.

If K is full, any simplex $\sigma \in K$ has a canonical decomposition, $\sigma = \sigma_{n-a} * \sigma_{n-a+1} * \dots * \sigma_n$, from which we deduce,

$$\sigma \in \mathcal{L}(K) \iff \dim \sigma_{n-a} < b \iff \sigma_{n-a} \text{ is not a clot.}$$

This gives the following decomposition

$$K = \mathcal{L}(K) \cup \bigcup_{\beta \in \mathcal{B}(K)} \beta * L_\beta, \quad (5)$$

and

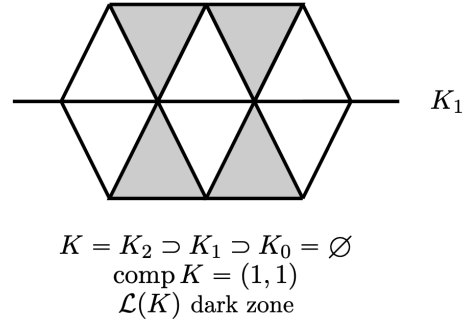
$$\mathcal{L}(K) \cap \bigcup_{\beta \in \mathcal{B}(K)} \beta * L_\beta = \bigcup_{\beta \in \mathcal{B}(K)} \partial \beta * L_\beta = \bigcup_{\beta \in \mathcal{B}(K)} \mathcal{L}(\beta * L_\beta). \quad (6)$$

The induced filtration on the join $\beta * L_\beta$ is that described in (4).

1.5. Perversities. A *perversity* on a filtered simplicial complex K is a map $\bar{p}: \mathcal{S}_K \rightarrow \bar{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$ taking the value 0 on the regular strata. The couple (K, \bar{p}) is a *perverse filtered simplicial complex*. The *dual perversity* $D\bar{p}$ is defined by $D\bar{p}(S) = \text{codim}_v S - 2 - \bar{p}(S)$ for any singular stratum $S \in \mathcal{S}_K$. For each $k \in \bar{\mathbb{Z}}$, we denote \bar{k} the perversity taking the value k on each singular stratum.

Given a simplicial subcomplex $L \subset K$ and a perversity \bar{p} on K , we also denote \bar{p} the perversity defined on the induced filtered simplicial complex L by

$$\bar{p}(T) = \bar{p}(S), \quad (7)$$



where $T \in \mathcal{S}_L$ and $S \in \mathcal{S}_K$ is the source of T . In general, in the rest of the text, the perversities considered on L are perversities induced from a perversity on K .

2. SIMPLICIAL INTERSECTION HOMOLOGY

We present the simplicial version of the intersection homology associated to a filtered simplicial complex, as it appears in the Appendix of [16], or with a more detailed wording in [9, Section 3.2]. We also develop some generic examples, such as the intersection homology of the pair $(K, \mathcal{L}(K))$ (cf. Proposition 2.3) or the intersection homology of a join (cf. Proposition 2.5).

In this work, the simplicial complexes are not supposed to be oriented, but we use oriented simplices. An *oriented simplex* of a simplicial complex K is a simplex of K with an equivalence class of orderings of its vertices, where two orderings are equivalent if they differ by an even permutation. The *simplicial chain module* $C_*(K)$ is the module generated by the oriented simplices of K . As a simplex is determined by its vertices we denote it by $[v_0, \dots, v_\ell]$. So, two writings corresponding to an even permutation of the vertices are identified. For an odd permutation, ν , we set $[v_0, \dots, v_\ell] = -[v_{\nu(0)}, \dots, v_{\nu(\ell)}]$, which makes sense at the level of chains.

Let (K, \bar{p}) be a perverse filtered simplicial complex. A simplex σ of K is \bar{p} -allowable if, for each stratum $S \in \mathcal{S}_K$, we have

$$\|\sigma\|_S = \dim(\sigma \cap S) \leq \dim \sigma - \text{codim}_v S + \bar{p}(S). \quad (8)$$

This condition is always satisfied when the stratum S is regular. Thus, the \bar{p} -allowability condition is equivalent to the inequality,

$$\|\sigma\|_S = \dim(\sigma \cap S) \leq \dim \sigma - 2 - D\bar{p}(S), \quad (9)$$

for each *singular* stratum $S \in \mathcal{S}_K$. Notice that $\sigma \cap S$ is a union of open faces of σ , so the dimension of $\sigma \cap S$ makes sense. A chain $c \in C_*(K)$ is a \bar{p} -allowable chain if any simplex with a non-zero coefficient in c is \bar{p} -allowable. It is a \bar{p} -intersection chain if c and ∂c are \bar{p} -allowable chains. The complex of \bar{p} -intersection chains is denoted $C_*^{\bar{p}}(K)$. The associated homology is the *simplicial \bar{p} -intersection homology* $H_*^{\bar{p}}(K)$, or simply the *simplicial intersection homology* if there is no ambiguity.

If $K_{n-1} = \emptyset$, then $C_*^{\bar{p}}(K)$ is the usual simplicial chain complex $C_*(K)$.

If K is full, for each $\ell \in I_\sigma$ the number $\|\sigma\|_{S_\ell}$ is equal to $\|\sigma\|_\ell = \dim(\sigma_0 * \dots * \sigma_\ell)$ and the allowability condition (8) becomes, for each $\ell \in I_\sigma$ (i.e., $\sigma_\ell \neq \emptyset$),

$$\|\sigma\|_\ell \leq \dim \sigma - (n - \ell) + \bar{p}(S_\ell). \quad (10)$$

There is an important difference between the complex of simplicial chains and that of \bar{p} -intersection chains. The first one is a free module over the family of simplices. This is not the case in the second context since there are \bar{p} -allowable simplices which are not \bar{p} -intersection chains.

2.1. Relative intersection homology. Let (K, \bar{p}) be a perverse filtered simplicial complex and $L \subset K$ be a simplicial subcomplex, endowed with the induced filtration and perversity. The allowability condition of a simplex $\sigma \in L$ can be understood in L itself or in K . Both points of view are equivalent. Let us see that. The simplex σ is \bar{p} -allowable in K if, and only if,

$$\|\sigma\|_\ell \leq \dim \sigma - (n - \ell) + \bar{p}(S_\ell),$$

for each $\ell \in I_\sigma^K$. The simplex σ is \bar{p} -allowable in L if, and only if,

$$\|\sigma\|_\ell \leq \dim \sigma - (n - \ell) + \bar{p}(T_\ell),$$

for each $\ell \in I_\sigma^L$. These conditions are equivalent since the two decompositions of σ are the same, $I_\sigma^K = I_\sigma^L$, and $\bar{p}(S_\ell) = \bar{p}(T_\ell)$. The natural inclusion $L \hookrightarrow K$ gives the following exact sequence

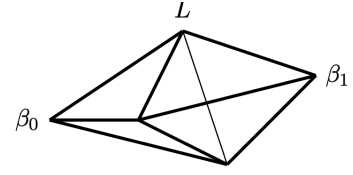
$$0 \longrightarrow C_*^{\bar{p}}(L) \hookrightarrow C_*^{\bar{p}}(K) \longrightarrow \frac{C_*^{\bar{p}}(K)}{C_*^{\bar{p}}(L)} = C_*^{\bar{p}}(K, L) \longrightarrow 0, \quad (11)$$

defining the *relative simplicial intersection complex* $C_*^{\bar{p}}(K, L)$. Its homology is the *relative simplicial intersection homology* denoted $H_*^{\bar{p}}(K, L)$.

2.2. Example. The determination of the relative complex $C_*^{\bar{p}}(K^{(\ell)}, K^{(\ell-1)})$ does not go through the family of ℓ -simplices of K as in the classical case. Let us see an example.

As simplicial complex K , we consider the suspension of a triangle L . We denote β_0, β_1 the apexes of the suspension. The 2-skeleton of K is K and its 1-skeleton is the union of the edges. This simplicial complex K is endowed with the filtration

$$\emptyset = K_{-1} \subset \{\beta_0, \beta_1\} = K_0 = K_1 \subset K_2 = K.$$



A straightforward calculation gives, for the $\bar{0}$ -perversity, $H_j^{\bar{0}}(K^{(2)}, K^{(1)}) = 0$, for $j \neq 2$ and $H_2^{\bar{0}}(K^{(2)}, K^{(1)}) = \mathbb{R} \oplus \mathbb{R}$. On the other hand, we have $\bigoplus_{\sigma \in K, \dim \sigma = 2} H_2^{\bar{0}}(\sigma, \partial\sigma) = 0$. Therefore the formula (1) cannot be true.

For having a decomposition of the relative intersection homology, we must replace the skeleton by the residual complex. Let us notice that $\mathcal{L}(K) = L$, which is also the link of β_0, β_1 . Here, we do have the following decomposition, for any j ,

$$H_j^{\bar{0}}(K, \mathcal{L}(K)) = H_j^{\bar{0}}(K, L) = \bigoplus_{i=0,1} H_j^{\bar{0}}(\beta_i * L, L) = \bigoplus_{i=0,1} H_j^{\bar{0}}(\beta_i * L, \mathcal{L}(\beta_i * L)).$$

This decomposition exists in the general case of a full simplicial complex, as we prove in Proposition 2.3. The proof proceeds by induction on the complexity. Firstly, we need the following Lemma.

Lemma 2.1. *Let (K, \bar{p}) be a perverse full filtered simplicial complex. Consider a face α of a clot $\beta \in \mathcal{B}(K)$ and a simplex ε of the link L_β . If the stratum $Q \in \mathcal{S}_K$ containing β is singular, we have the following equivalence:*

$$\alpha * \varepsilon \text{ is a } \bar{p}\text{-allowable simplex} \iff \begin{cases} \dim \varepsilon \geq D\bar{p}(Q) + 1 \text{ and} \\ \varepsilon \text{ is a } \bar{p}\text{-allowable simplex.} \end{cases} \quad (12)$$

Proof. We write $q = \dim_v Q$. We have

$$\begin{aligned} I_{\alpha * \varepsilon} &= \{\ell \in \{0, \dots, n\} \mid (\alpha * \varepsilon) \cap (|(\alpha * L_\beta)_\ell| \setminus |(\alpha * L_\beta)_{\ell-1}|) \neq \emptyset\} \\ &= \{q\} \cup \{\ell \in \{a+1, \dots, n\} \mid \varepsilon \cap (|(L_\beta)_\ell| \setminus |(L_\beta)_{\ell-1}|) \neq \emptyset\} = \{q\} \cup I_\varepsilon. \end{aligned}$$

So,

$$\begin{aligned} \alpha * \varepsilon \text{ is a } \bar{p}\text{-allowable simplex} &\iff \begin{cases} \|\alpha * \varepsilon\|_q \leq \dim(\alpha * \varepsilon) - (n - q) + \bar{p}(Q) \text{ and} \\ \|\alpha * \varepsilon\|_\ell \leq \dim(\alpha * \varepsilon) - (n - \ell) + \bar{p}(S_\ell), \forall \ell \in I_\varepsilon \setminus \{n\}, \end{cases} \\ &\iff \begin{cases} \dim \alpha \leq \dim \alpha + \dim \varepsilon + 1 - (n - q) + \bar{p}(Q) \text{ and} \\ \dim \alpha + \|\varepsilon\|_\ell + 1 \leq \dim \alpha + 1 + \dim \varepsilon - (n - \ell) + \bar{p}(S_\ell), \forall \ell \in I_\varepsilon \setminus \{n\}, \end{cases} \\ &\iff \begin{cases} 0 \leq \dim \varepsilon + 1 - (n - q) + \bar{p}(Q) \text{ and} \\ \|\varepsilon\|_\ell \leq \dim \varepsilon - (n - \ell) + \bar{p}(S_\ell), \forall \ell \in I_\varepsilon \setminus \{n\}, \end{cases} \iff \begin{cases} \dim \varepsilon \geq D\bar{p}(Q) + 1 \text{ and} \\ \varepsilon \text{ is a } \bar{p}\text{-allowable simplex.} \end{cases} \end{aligned}$$

♣

In the case of a face α of the clot β , we have

$$\alpha \text{ is a } \bar{p}\text{-allowable simplex} \iff \begin{cases} Q \text{ is singular and } D\bar{p}(Q) + 2 \leq 0, \text{ or} \\ Q \text{ is regular.} \end{cases} \quad (13)$$

Lemma 2.2. *Let (K, \bar{p}) be a perverse full filtered simplicial complex. Then, we have*

$$C_*^{\bar{p}}(K) = C_*^{\bar{p}}(\mathcal{L}(K)) + \bigoplus_{\beta \in \mathcal{B}(K)} C_*^{\bar{p}}(\beta * L_\beta).$$

Proof. It suffices to prove the inclusion \subseteq . From (5), we know that any chain $c \in C_*^{\bar{p}}(K)$ can be written $c = f + \sum_{\beta \in \mathcal{B}(K)} c_\beta$ with $f \in C_*(\mathcal{L}(K))$ and $c_\beta \in C_*(\beta * L_\beta)$. As (see (6)) $\mathcal{L}(K) \cap \bigcup_{\beta \in \mathcal{B}(K)} \beta * L_\beta = \bigcup_{\beta \in \mathcal{B}(K)} \partial\beta * L_\beta$, we obtain a unique writing of c as

$$c = f + \sum_{\beta \in \mathcal{B}(K)} c_\beta = f + \sum_{\beta \in \mathcal{B}(K)} n_\beta \beta * e_\beta + m_\beta \beta,$$

with $n_\beta, m_\beta \in \mathbb{R}$ and $e_\beta \in L_\beta$. Since the elements of this decomposition are independent, the \bar{p} -allowability of c gives the \bar{p} -allowability of the chains f, c_β, e_β (see Lemma 2.1). Let Q_β be the stratum containing β . From Lemma 2.1 and (13), we also deduce

$$\begin{cases} \dim e_\beta \geq D\bar{p}(Q_\beta) + 1 & \text{if } n_\beta \neq 0, \text{ and} \\ D\bar{p}(Q_\beta) + 2 \leq 0 & \text{if } m_\beta \neq 0 \text{ and } Q_\beta \text{ singular.} \end{cases}$$

The boundary ∂c of c is given by

$$\partial c = \sum_{\beta \in \mathcal{B}(K)} n_\beta \partial\beta * e_\beta + \sum_{\beta \in \mathcal{B}(K)} (-1)^{\dim \beta + 1} n_\beta \beta * \partial e_\beta + \sum_{\beta \in \mathcal{B}(K)} m_\beta \partial\beta + \partial f. \quad (14)$$

(In the case $\dim \beta = 0$, the join $\partial\beta * e_\beta$ means e_β , and similarly if $\dim e_\beta = 0$.) Following a new time Lemma 2.1 and (13), we get that each element of the first sum in (14) is a \bar{p} -allowable chain. Therefore, the chain

$$\sum_{\beta \in \mathcal{B}(K)} (-1)^{\dim \beta + 1} n_\beta \beta * \partial e_\beta + \sum_{\beta \in \mathcal{B}(K)} m_\beta \partial\beta + \partial f$$

is \bar{p} -allowable and so is every element of this sum. In short, we get $f \in C_*^{\bar{p}}(\mathcal{L}(K))$ and $c_\beta \in C_*^{\bar{p}}(\beta * L_\beta)$, for each $\beta \in \mathcal{B}(K)$, which gives the claim. \clubsuit

Proposition 2.3. *Let (K, \bar{p}) be a perverse full filtered simplicial complex. The inclusion maps induce an isomorphism of chain complexes,*

$$\bigoplus_{\beta \in \mathcal{B}(K)} C_*^{\bar{p}}(\beta * L_\beta, \partial\beta * L_\beta) \xrightarrow{\cong} C_*^{\bar{p}}(K, \mathcal{L}(K)). \quad (15)$$

Proof. Notice that, for each clot $\beta \in \mathcal{B}(K)$, we have $\partial\beta * L_\beta = \mathcal{L}(\beta * L_\beta) \subset \mathcal{L}(K)$. (The case $\dim \beta = 0$ corresponds to $L_\beta = \mathcal{L}(\beta * L_\beta) \subset \mathcal{L}(K)$.) Thus the inclusion maps are well defined. The result comes from Lemma 2.2 and (6). \clubsuit

Remark 2.4. Let us consider a filtered simplicial complex K with $\dim K = b < \infty$ and the trivial filtration $K = K_n \supset \emptyset$. So, we have $\text{comp } K = (0, b)$, $K = K^{(b)}$, $\beta(K) = \{\beta \in K \mid \dim \beta = b\}$ and $\mathcal{L}(K) = K^{(b-1)}$. The previous formula (15) becomes

$$C_*(K, \mathcal{L}(K)) \cong \bigoplus_{\substack{\beta \in K \\ \dim \beta = b}} C_*(\beta, \partial\beta) \cong C_*\left(K^{(b)}, K^{(b-1)}\right).$$

Thus our approach contains the classical formula (1).

2.3. Intersection homology of the join. To make the writing easier, we employ the dual perversity $D\bar{p}$ of \bar{p} . Notice that the intersection homology $H_0^{\bar{p}}(K)$ of a filtered simplicial complex can be 0 when there are no regular strata. This is why the following statement contains more cases to consider than that of [4, Proposition 1.49] for instance. The same phenomenon appears for the calculation of the intersection homology of a cone between King's statement [15, Proposition 5] and Friedman's [9, Theorem 4.2.1].

Proposition 2.5. *Consider a perverse full filtered simplicial complex (K, \bar{p}) . Let $\beta \in \mathcal{B}(K)$ be a clot such that the stratum $Q \in \mathcal{S}_K$ containing β is a singular stratum. We have,*

$$H_i^{\bar{p}}(\beta * L_\beta) = \begin{cases} H_i^{\bar{p}}(L_\beta) & \text{if } i \leq D\bar{p}(Q), \\ 0 & \text{if } i > D\bar{p}(Q), i \neq 0, \\ \mathbb{R} & \text{if } i = 0 \text{ and } D\bar{p}(Q) < -1, \\ \mathbb{R} & \text{if } i = 0, D\bar{p}(Q) = -1 \text{ and } H_0^{\bar{p}}(L_\beta) \neq 0, \\ 0 & \text{if } i = 0, D\bar{p}(Q) = -1 \text{ and } H_0^{\bar{p}}(L_\beta) = 0. \end{cases}$$

The first isomorphism is given by the inclusion $L_\beta \hookrightarrow \beta * L_\beta$. On the third and fourth lines, a generator of \mathbb{R} is a point in β or any \bar{p} -allowable point in L_β , respectively.

Proof. We write $\beta = \langle v_0, \dots, v_\ell \rangle$ and $q = \dim_v Q$. (Since Q is singular, we have $D\bar{p}(Q) = n - q - 2 - \bar{p}(Q)$.) Any chain $c \in C_*(\beta * L_\beta)$ can be written as,

$$c = f + \sum_{\alpha \triangleleft \beta} c_\alpha = f + \sum_{\alpha \triangleleft \beta} n_\alpha \alpha * e_\alpha + m_\alpha \alpha, \quad (16)$$

where $f \in C_*(L_\beta)$, $n_\alpha, m_\alpha \in \mathbb{R}$ and $e_\alpha \in L_\beta$, for any $\alpha \triangleleft \beta$. With these notations, the characterizations (12) and (13) become

$$c \text{ is } \bar{p}\text{-allowable} \iff \begin{cases} \dim e_\alpha \geq D\bar{p}(Q) + 1 \text{ and } e_\alpha \text{ is } \bar{p}\text{-allowable,} & \text{if } n_\alpha \neq 0, \\ D\bar{p}(Q) + 2 \leq 0, & \text{if } m_\alpha \neq 0, \\ \text{and } f \text{ is } \bar{p}\text{-allowable.} \end{cases} \quad (17)$$

We determine $H_i^{\bar{p}}(\beta * L_\beta)$ by distinguishing the three cases boxed below.

• $i = 0$. A chain $c \in C_0^{\bar{p}}(\beta * L_\beta)$ is of the form $c = \sum_{k=0}^b n_k v_k + f$ with $f \in C_0^{\bar{p}}(L_\beta)$ or $c = f$ if $D\bar{p}(Q) + 1 \geq 0$. We distinguish 3 cases. The first one, $D\bar{p}(Q) + 1 > 0$, is postponed in the third item, for any degree i . So, we are left with:

- $D\bar{p}(Q) + 1 = 0$. Here, $c = f \in C_0^{\bar{p}}(L_\beta)$. Two \bar{p} -allowable points $p', p'' \in L_\beta$ define the same class in $H_0^{\bar{p}}(\beta * L_\beta)$, since $p' - p'' = \partial(v_0 * p' - v_0 * p'')$, where $v_0 * p'$ and $v_0 * p''$ are \bar{p} -allowable (see (17)). As a point cannot be the boundary of a 1-chain, the vanishing of $H_0^{\bar{p}}(\beta * L_\beta)$ is equivalent to the non-existence of \bar{p} -allowable points in $C_0^{\bar{p}}(L_\beta)$ which gives the two last lines of the statement.
- $D\bar{p}(Q) + 1 < 0$. Let us write $f = \sum_{i \in I} m_i p_i$, with $m_i \in \mathbb{R}$ and $p_i \in L_\beta^{(0)}$. From (17), we deduce that $\sum_{i \in I} m_i v_0 * p_i$ is a \bar{p} -intersection chain of boundary $f - \sum_{i \in I} m_i v_0$. So the map $\iota_0: H_0^{\bar{p}}(\beta) \rightarrow H_0^{\bar{p}}(\beta * L_\beta)$, induced by the canonical inclusion, is surjective. Recall that $H_0^{\bar{p}}(\beta) = H_0(\beta) = \mathbb{R}$ is generated by $[v_0]$. If $n[v_0]$, $n \in \mathbb{R}$, is sent to zero by ι_0 , we have $n v_0 = \partial\gamma$, with $\gamma \in C_1^{\bar{p}}(\beta * L_\beta)$. Thus the augmentation map ϵ gives $n = \epsilon(\partial\gamma) = 0$. We deduce $H_0^{\bar{p}}(\beta) \cong H_0^{\bar{p}}(\beta * L_\beta) \cong \mathbb{R}$ and the third line of the statement.

• $i > 0$ and $D\bar{p}(Q) \leq -1$. Let $\eta \triangleleft \beta$ be the opposite face to the vertex v_0 or $\eta = \emptyset$ when $\dim \beta = 0$. We consider a cycle $\gamma \in C_i^{\bar{p}}(\beta * L_\beta)$ and we prove that it is a boundary. We decompose $\gamma = \gamma_0 + \gamma_1$ where γ_0 is a chain of simplices having v_0 as vertex and $\gamma_1 = \sum_i m_i \gamma'_i$, with $\gamma'_i \in \eta * L_\beta$. As γ is a cycle, a direct calculation shows that $\gamma = \partial(\sum_i m_i v_0 * \gamma'_i)$, since $i > 0$. To prove the second line of the statement, we only need to establish that $v_0 * \gamma'_i$ is a \bar{p} -allowable simplex. For that we use (17). As γ'_i is \bar{p} -allowable by hypothesis, we have the three different cases:

- $\gamma'_i = \alpha$ where $\alpha \triangleleft \eta$ and $D\bar{p}(Q) + 2 \leq 0$,
- $\gamma'_i = \alpha * e_\alpha$ where $\alpha \triangleleft \eta$ and $e_\alpha \in L_\beta$ is a \bar{p} -allowable simplex,
- $\gamma'_i = e_\alpha$ where $e_\alpha \in L_\beta$ is a \bar{p} -allowable simplex,

since $D\bar{p}(Q)+1 \leq 0$. The same analysis for the simplex $v_0 * \gamma'_i$, again with (17) and $D\bar{p}(Q)+1 \leq 0$, gives the claim.

• $D\bar{p}(Q) \geq 0$. Let us introduce the following truncation of the complex $C_*^{\bar{p}}(L_\beta)$,

$$\tau C_*^{\bar{p}}(L_\beta) = \begin{cases} C_i^{\bar{p}}(L_\beta) & \text{if } i > D\bar{p}(Q) + 1, \\ C_i^{\bar{p}}(L_\beta) \cap \partial^{-1}(0) & \text{if } i = D\bar{p}(Q) + 1, \\ 0 & \text{if } i < D\bar{p}(Q) + 1. \end{cases}$$

Notice that $H_i(\tau C_*^{\bar{p}}(L_\beta)) = H_i^{\bar{p}}(L_\beta)$, if $i \geq D\bar{p}(Q) + 1$, and $H_i(\tau C_*^{\bar{p}}(L_\beta)) = 0$ otherwise. Let us consider the map,

$$\Psi: C_*(\beta) \otimes \tau C_*^{\bar{p}}(L_\beta) \rightarrow C_{*+1}(\beta * L_\beta),$$

defined at the level of simplices by $\Psi(\alpha \otimes \gamma) = (-1)^{\deg \alpha} \alpha * \gamma$. (Notice that $\deg \gamma > 0$.) By abuse of notation, we will also denote $*$ the extension of Ψ to chains. We will use it mainly in the case of the boundary of a simplex. Ce map Ψ verifies the following properties.

i) If $\deg \alpha > 0$ we have

$$\begin{aligned} \Psi(\partial(\alpha \otimes \gamma)) &= \Psi(\partial\alpha \otimes \gamma) + (-1)^{\deg \alpha} \Psi(\alpha \otimes \partial\gamma) = (-1)^{\deg \alpha - 1} (\partial\alpha * \gamma) + (\alpha * \partial\gamma) \\ &= (-1)^{\deg \alpha - 1} (\partial\alpha * \gamma + (-1)^{\deg \alpha - 1} \alpha * \partial\gamma) = (-1)^{\deg \alpha - 1} \partial(\alpha * \gamma) \\ &= -\partial\Psi(\alpha \otimes \gamma). \end{aligned}$$

ii) If $\deg \alpha = 0$, we have

$$\Psi(\partial(\alpha \otimes \gamma)) = \Psi(\alpha \otimes \partial\gamma) = \alpha * \partial\gamma = \gamma - \partial(\alpha * \gamma) = \gamma - \partial\Psi(\alpha \otimes \gamma).$$

iii) The simplex $\Psi(\alpha \otimes \gamma)$ is \bar{p} -allowable, see (17).

From these points, we deduce that the map

$$\psi: C_*(\beta) \otimes \tau C_*^{\bar{p}}(L_\beta) \rightarrow \frac{C_{*+1}^{\bar{p}}(\beta * L_\beta)}{C_{*+1}^{\bar{p}}(L_\beta)},$$

defined by $\psi(\alpha \otimes \gamma) = (-1)^{\deg \alpha} \llbracket \alpha * \gamma \rrbracket$ is a well-defined chain map of degree 1. It is clearly a monomorphism. Let us now prove that ψ is surjective.

Let $c \in C_i^{\bar{p}}(\beta * L_\beta)$. From (17), we have

$$c = f + \sum_{\alpha \triangleleft \beta} n_\alpha \alpha * e_\alpha + \sum_{\alpha \triangleleft \beta} m_\alpha \alpha,$$

with $f \in C_*(L_\beta)$, $n_\alpha, m_\alpha \in \mathbb{R}$ and $e_\alpha \in L_\beta^{(j)}$ with $j \geq D\bar{p}(Q) + 1$. Notice that $\dim e_\alpha > 0$ for each $n_\alpha \neq 0$. The boundary of c is also \bar{p} -allowable, with

$$\partial c = \partial f + \sum_{\substack{\alpha \triangleleft \beta \\ \dim \alpha = 0}} n_\alpha e_\alpha + \sum_{\substack{\alpha \triangleleft \beta \\ \dim \alpha > 0}} n_\alpha \partial\alpha * e_\alpha + \sum_{\alpha \triangleleft \beta} (-1)^{\deg \alpha - 1} n_\alpha \alpha * \partial e_\alpha + \sum_{\alpha \triangleleft \beta} m_\alpha \partial\alpha.$$

From (17), we deduce $f \in C_*^{\bar{p}}(L_\beta)$ and $e_\alpha \in \tau C_*^{\bar{p}}(L_\beta)$, which implies $\sum_{\alpha \triangleleft \beta} (-1)^{\deg \alpha} n_\alpha \alpha \otimes e_\alpha \in C_*(\beta) \otimes \tau C_*^{\bar{p}}(L_\beta)$. This gives the claim since

$$\psi\left(\sum_{\alpha \triangleleft \beta} (-1)^{\deg \alpha} n_\alpha \alpha * e_\alpha\right) = \llbracket \sum_{\alpha \triangleleft \beta} n_\alpha \alpha * e_\alpha \rrbracket = \llbracket c \rrbracket.$$

Also, the map $\Phi: \tau C_*^{\bar{p}}(L_\beta) \rightarrow C_*(\beta) \otimes \tau C_*^{\bar{p}}(L_\beta)$, defined by $\Phi(\gamma) = -v_0 \otimes \gamma$, is a quasi-isomorphism. Let us consider the short exact sequence

$$0 \longrightarrow C_*^{\bar{p}}(L_\beta) \hookrightarrow C_*^{\bar{p}}(\beta * L_\beta) \longrightarrow \frac{C_*^{\bar{p}}(\beta * L_\beta)}{C_*^{\bar{p}}(L_\beta)} \longrightarrow 0. \quad (18)$$

By using Ψ and Φ , we may replace the homology of the quotient by the homology of the truncation, with a shift of one degree. Therefore, the long exact sequence associated to (18) can be written,

$$\dots \longrightarrow H_{i+1}^{\bar{p}}(L_\beta) \longrightarrow H_{i+1}^{\bar{p}}(\beta * L_\beta) \longrightarrow H_i(\tau C_*^{\bar{p}}(L_\beta)) \xrightarrow{\delta} H_i^{\bar{p}}(L_\beta) \longrightarrow \dots,$$

where the connecting map δ becomes the inclusion. Replacing $H_i(\tau C_*^{\bar{p}}(L_\beta))$ by its value, we get $H_i^{\bar{p}}(\beta * L_\beta) \cong H_i^{\bar{p}}(L_\beta)$, if $i \leq D\bar{p}(Q)$, and $H_i^{\bar{p}}(\beta * L_\beta) = 0$ otherwise. Moreover, the isomorphisms are induced by the inclusion $L_\beta \hookrightarrow \beta * L_\beta$. \clubsuit

When Q is a regular stratum, we have $L_\beta = \emptyset$, $H_i^{\bar{p}}(\beta * L_\beta) = H_i^{\bar{p}}(\beta) = H_i(\beta) = 0$ if $i > 0$ and $H_0^{\bar{p}}(\beta * L_\beta) = H_0^{\bar{p}}(\beta) = H_0(\beta) = \mathbb{R}$.

3. SINGULAR INTERSECTION HOMOLOGY

The singular version of the intersection homology goes back to King [15]. We focus here on the singular intersection homology of the realization of a filtered simplicial complex K . As in the simplicial case of Section 2, we develop the relative intersection homology of the pair $(|K|, |\mathcal{L}(K)|)$ (cf. Proposition 3.3) and the intersection homology of the realization of a join (cf. Proposition 3.4).

Definition 3.1. A *filtered space* is a Hausdorff topological space X endowed with a filtration by closed subspaces,

$$X = X_n \supset X_{n-1} \supset \dots \supset X_0 \supset X_{-1} = \emptyset.$$

The integer n is the dimension of X . The i -strata of X are the non-empty connected components of $X_i \setminus X_{i-1}$. The open strata are called *regular*, the other ones being called *singular*. The set of singular strata of X is denoted \mathcal{S}_X , or \mathcal{S} if there is no ambiguity. The *dimension* of a stratum $S \subset X_i \setminus X_{i-1}$ is $\dim_v S = i$. Its *codimension* is $\text{codim}_v S = n - i$.

Given a n -dimensional filtered simplicial complex K , the associated filtration

$$|K| = |K_n| \supset |K_{n-1}| \supset |K_{n-2}| \supset \dots \supset |K_0| \supset K_{-1} = \emptyset,$$

defines a n -dimensional filtered space. By definition, there is a canonical bijection $\mathcal{S}_{|K|} \cong \mathcal{S}_K$.

3.1. Induced filtered spaces. Let $X = X_n \supset X_{n-1} \supset X_{n-2} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$ be a filtered space. Given a subset $Y \subset X$, the induced filtration $Y_i = Y \cap X_i$ defines a filtered space structure on Y . Since $Y_i \subset X_i$ for any $i \in \{0, \dots, n\}$, then for any stratum $T \in \mathcal{S}_Y$ there exists a unique stratum $S \in \mathcal{S}_X$ with $T \subset S$. We say that S is the *source* of T . Notice that $\dim_v T = \dim_v S$.

3.2. Perversities. A *perversity* on a filtered space X is a map $\bar{p}: \mathcal{S}_X \rightarrow \bar{\mathbb{Z}}$ taking the value 0 on the regular strata. The couple (X, \bar{p}) is a *perverse filtered space*. Given a subset $Y \subset X$ and a perversity \bar{p} on X , we also denote \bar{p} the perversity defined on the induced filtered space Y by

$$\bar{p}(T) = \bar{p}(S),$$

where $T \in \mathcal{S}_Y$ and $S \in \mathcal{S}_X$ is the source of T .

3.3. Perverse degree. For each stratum $S \in \mathcal{S}_X$, the *perverse degree of a singular simplex* $\sigma: \Delta \rightarrow X$ along S is

$$\|\sigma\|_S = \begin{cases} -\infty & \text{if } S \cap \sigma(\Delta) = \emptyset, \\ \dim \sigma^{-1}(S) & \text{if not.} \end{cases}$$

By definition, the *dimension* of a non-empty subset $A \subset \Delta$ is the smallest $s \in \mathbb{N}$ for which the subset A is included in the s -skeleton of Δ .

3.4. The definition. We have all the ingredients needed for the definition of the intersection homology of a filtered space. (For further information, the reader should consult the historical definition in [15], or [9, Section 3.4], [6].)

Definition 3.2. Let (X, \bar{p}) be a perverse filtered space. A singular simplex $\sigma: \Delta \rightarrow X$, is \bar{p} -allowable if,

$$\|\sigma\|_S \leq \dim \Delta - \text{codim}_v S + \bar{p}(S), \quad (19)$$

for each stratum $S \in \mathcal{S}_X$. This condition is always satisfied if the stratum S is regular. Thus, the \bar{p} -allowability condition is equivalent to the inequality

$$\|\sigma\|_S \leq \dim \Delta - D\bar{p}(S) - 2, \quad (20)$$

for each *singular* stratum $S \in \mathcal{S}_X$. A singular chain c is \bar{p} -allowable if any simplex with a non-zero coefficient in c is \bar{p} -allowable. It is a \bar{p} -intersection chain if c and ∂c are \bar{p} -allowable chains. The associated homology is the *singular intersection homology* denoted $H_*^{\bar{p}}(X)$.

When $X_{n-1} = \emptyset$, the complex $C_*^{\bar{p}}(X)$ is the usual singular chain complex $C_*(X)$.

3.5. Relative intersection homology. Let (X, \bar{p}) be a perverse filtered space and $Y \subset X$ be a subspace endowed with the induced filtration and perversity. The natural inclusion $Y \hookrightarrow X$ gives the following exact sequence,

$$0 \longrightarrow C_*^{\bar{p}}(Y) \hookrightarrow C_*^{\bar{p}}(X) \longrightarrow \frac{C_*^{\bar{p}}(X)}{C_*^{\bar{p}}(Y)} = C_*^{\bar{p}}(X, Y) \longrightarrow 0, \quad (21)$$

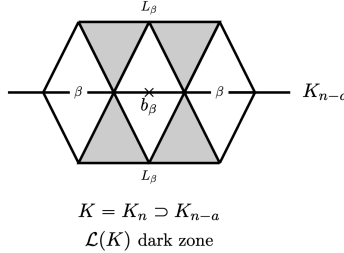
defining the relative singular intersection complex $C_*^{\bar{p}}(X, Y)$. Its homology is the *relative singular intersection homology* denoted $H_*^{\bar{p}}(X, Y)$.

Proposition 3.3. Let (K, \bar{p}) be a perverse full filtered simplicial complex. The inclusion map induces the isomorphism,

$$\bigoplus_{\beta \in \mathcal{B}(K)} H_*^{\bar{p}}(|\beta * L_\beta|, |\mathcal{L}(\beta * L_\beta)|) \xrightarrow{\cong} H_*^{\bar{p}}(|K|, |\mathcal{L}(K)|).$$

Proof. Recall $\mathcal{L}(\beta * L_\beta) = \partial\beta * L_\beta$. The proof is divided into two steps.

Step 1: Thickening. The barycenter of $\beta \in \mathcal{B}(K)$ is denoted b_β . We have the equality $|\beta * L_\beta| = |b_\beta * \partial\beta * L_\beta|$, where $\partial\beta = \emptyset$ if $\dim \beta = 0$.



So, any point of $x \in |\beta * L_\beta|$ can be written as $x = (1-t)b_\beta + ta$, where $t \in [0, 1]$ and $a \in |\partial\beta * L_\beta|$. This writing is unique when $x \neq b_\beta$ (i.e., $t \neq 0$). Notice that the assignment $(1-t)b_\beta + tc \mapsto (c, t)$ induces the homeomorphism

$$|\beta * L_\beta| \setminus \{b_\beta\} \cong |\partial\beta * L_\beta| \times]0, 1]. \quad (22)$$

Under this homeomorphism, the filtration on the subset $|\beta * L_\beta| \setminus \{b_\beta\}$ becomes the product filtration, with the trivial filtration on $]0, 1]$. Inspired by Subsection 1.4, we define the open subset

$$W = |K| \setminus \bigcup_{\beta \in \mathcal{B}(K)} \{b_\beta\} = |\mathcal{L}(K)| \cup \bigcup_{\beta \in \mathcal{B}(K)} (|\beta * L_\beta| \setminus \{b_\beta\}).$$

Associated to this open subset, we define a map, $F: W \times [0, 1] \rightarrow W$ by

$$F(x, s) = \begin{cases} x & \text{if } x \in |\mathcal{L}(K)|, \\ sx + (1-s)y & \text{if } x = (1-t)b_\beta + ty \in |\beta * L_\beta| \setminus \{b_\beta\}. \end{cases}$$

Let us specify the restriction of F to $|\beta * L_\beta| \setminus \{b_\beta\} \times [0, 1]$, with $\beta \in \mathcal{B}(K)$. By definition, we have $F((1-t)b_\beta + ty, s) = s(1-t)b_\beta + (1+st-s)y$. Composing it with the homeomorphism (22), we get a map, still denoted F , from $|\partial\beta * L_\beta| \times]0, 1] \times [0, 1]$ to $|\partial\beta * L_\beta| \times]0, 1]$, defined by $F(y, t, s) = (y, 1+st-s)$. This is the identity on the factor $|\partial\beta * L_\beta|$. Combined with the fact that F is the identity on the factor $\mathcal{L}(K)$, we conclude that F is a stratified homotopy in the sense of [9, Definition 4.1.9].

Let us notice that $F(-, 1)$ is the identity on W . The map $F(-, 0)$ is the identity on $|\mathcal{L}(K)|$ by construction, and sends $x \notin |\mathcal{L}(K)|$ on $y \in |\partial\beta * L_\beta| \subset |\mathcal{L}(K)|$. Thus, $F(-, 0)$ gives a map $\nu: W \rightarrow |\mathcal{L}(K)|$. If we denote $j: |\mathcal{L}(K)| \hookrightarrow W$ the inclusion we have $\nu \circ j = \text{id}_{|\mathcal{L}(K)|}$. On the other hand, F is a stratified homotopy between $F(-, 0) = j \circ \nu$ and $F(-, 1) = \text{id}_W$. Therefore ([9, Proposition 4.1.10]), the map $j \circ \nu$ induces the identity map in homology.

In short, the inclusion j induces an isomorphism $H_*^{\bar{p}}(W) \cong H_*^{\bar{p}}(|\mathcal{L}(K)|)$ and therefore an isomorphism

$$H_*^{\bar{p}}(|K|, W) \cong H_*^{\bar{p}}(|K|, |\mathcal{L}(K)|).$$

Step 2: Excision. By excision (see [9, Corollary 4.4.18]), we get an isomorphism,

$$H_*^{\bar{p}}(|K|, W) \cong H_*^{\bar{p}}(|K| \setminus |\mathcal{L}(K)|, W \setminus |\mathcal{L}(K)|).$$

From Subsection 1.4, we have the disjoint unions

$$|K| \setminus |\mathcal{L}(K)| = \bigsqcup_{\beta \in \mathcal{B}(K)} |\beta * L_\beta| \setminus |\partial\beta * L_\beta| \quad \text{and} \quad W \setminus |\mathcal{L}(K)| = \bigsqcup_{\beta \in \mathcal{B}(K)} |\beta * L_\beta| \setminus (\{b_\beta\} \cup |\partial\beta * L_\beta|). \quad (23)$$

This implies

$$H_*^{\bar{p}}(|K| \setminus |\mathcal{L}(K)|, W \setminus |\mathcal{L}(K)|) \cong \bigoplus_{\beta \in \mathcal{B}(K)} H_*^{\bar{p}}(|\beta * L_\beta| \setminus |\partial\beta * L_\beta|, |\beta * L_\beta| \setminus (\{b_\beta\} \cup |\partial\beta * L_\beta|)).$$

An excision relatively to the closed subset $|\partial\beta * L_\beta|$ gives

$$H_*^{\bar{p}}(|\beta * L_\beta| \setminus |\partial\beta * L_\beta|, |\beta * L_\beta| \setminus (\{b_\beta\} \cup |\partial\beta * L_\beta|)) \cong H_*^{\bar{p}}(|\beta * L_\beta|, |\beta * L_\beta| \setminus \{b_\beta\}).$$

Finally, from (22), we may replace $|\beta * L_\beta| \setminus \{b_\beta\}$ by $|\partial\beta * L_\beta| \times]0, 1]$ and obtain

$$H_*^{\bar{p}}(|\beta * L_\beta|, |\beta * L_\beta| \setminus \{b_\beta\}) \cong H_*^{\bar{p}}(|\beta * L_\beta|, |\partial\beta * L_\beta|),$$

which gives the claim. ♣

3.6. Intersection homology of the join. As in Subsection 2.3, we use the dual perversity. The following determination meets the same pattern as in Proposition 2.5.

Proposition 3.4. *Consider a perverse full filtered simplicial complex (K, \bar{p}) . Let $\beta \in \mathcal{B}(K)$ be a clot such that the stratum $Q \in \mathcal{S}_K$ containing β is a singular stratum. We have*

$$H_i^{\bar{p}}(|\beta * L_\beta|) = \begin{cases} H_i^{\bar{p}}(|L_\beta|) & \text{if } i \leq D\bar{p}(Q), \\ 0 & \text{if } i > D\bar{p}(Q), i \neq 0, \\ \mathbb{R} & \text{if } i = 0 \text{ and } D\bar{p}(Q) < -1, \\ \mathbb{R} & \text{if } i = 0, D\bar{p}(Q) = -1 \text{ and } H_0^{\bar{p}}(|L_\beta|) \neq 0, \\ 0 & \text{if } i = 0, D\bar{p}(Q) = -1 \text{ and } H_0^{\bar{p}}(|L_\beta|) = 0. \end{cases}$$

The first isomorphism is given by the inclusion $|L_\beta| \hookrightarrow |\beta * L_\beta|$. On the third and fourth lines, a generator of \mathbb{R} is any point in β or any \bar{p} -allowable point in L_β , respectively.

Proof. We write $q = \dim_v Q$. Since Q is singular, we have $D\bar{p}(Q) = n - q - 2 - \bar{p}(Q)$. The case $\dim \beta = 0$ is given by the classical cone formula, see [9, Theorem 4.2.1]. When $\dim \beta > 0$ we consider $\eta = \langle v_0 \rangle$ the 0-simplex given by the first vertex v_0 of β . It suffices to prove that the inclusion $\eta \hookrightarrow \beta$ induces the isomorphism $H_*^{\bar{p}}(|\eta * L_\beta|) \cong H_*^{\bar{p}}(|\beta * L_\beta|)$. Consider a simplicial homotopy $F: \beta \times [0, 1] \rightarrow \beta$ between the identity on β and the constant map $\beta \rightarrow \eta$. The realization map, still denoted $F: |\beta * L_\beta| \times [0, 1] \rightarrow |\beta * L_\beta|$, is a homotopy between the identity on $|\beta * L_\beta|$ and the map $\xi: |\beta * L_\beta| \rightarrow |\eta * L_\beta|$ given by $v * x \mapsto v_0 * x$.

From [9, Proposition 4.1.10], we get that both maps $F(-, 0), F(-, 1): |\beta * L_\beta| \rightarrow |\beta * L_\beta|$ induce the same morphism in $H_*^{\bar{p}}$ -homology. Since $F(-, 0)$ is the identity, then $(F(-, 1))_*: H_*^{\bar{p}}(|\beta * L_\beta|) \rightarrow H_*^{\bar{p}}(|\eta * L_\beta|)$ is the identity.

Finally, since the identity map is equal to the composition $|\eta * L_\beta| \hookrightarrow |\beta * L_\beta| \xrightarrow{\xi} |\eta * L_\beta|$ and the map $F(-, 1)$ to the composition $|\beta * L_\beta| \xrightarrow{\xi} |\eta * L_\beta| \hookrightarrow |\beta * L_\beta|$, we conclude that the inclusion $|\eta * L_\beta| \hookrightarrow |\beta * L_\beta|$ induces an isomorphism $H_*^{\bar{p}}(|\eta * L_\beta|) \cong H_*^{\bar{p}}(|\beta * L_\beta|)$. \clubsuit

When Q is a regular stratum we have $L_\beta = \emptyset$, $H_i^{\bar{p}}(|\beta * L_\beta|) = H_i^{\bar{p}}(|\beta|) = H_i(|\beta|) = 0$ if $i > 0$ and $H_0^{\bar{p}}(|\beta * L_\beta|) = H_0^{\bar{p}}(|\beta|) = H_0(|\beta|) = \mathbb{R}$.

4. SIMPLICIAL VERSUS SINGULAR

Let (K, \bar{p}) be a perverse filtered simplicial complex. After a reminder of the canonical inclusion map, $\iota: C_*^{\bar{p}}(K) \rightarrow C_*^{\bar{p}}(|K|)$, we prove in Theorem 4.2 that the map ι induces an isomorphism in homology when K is full. The result is no longer true if we remove the hypothesis “full,” as it is pointed out in [16, Remark 2].

Consider a perverse full filtered simplicial complex (K, \bar{p}) . With the “well ordering theorem” applied inductively to $K_0, K_1 \setminus K_0$, and so on, we can assume that the set of vertices of K is provided with an order \leq that restricts to a total order on each simplex and verifies

$$v \leq w \text{ and } w \in K_k \Rightarrow v \in K_k. \quad (24)$$

Let $\sigma \in K$ be an oriented simplex whose vertices are $\{v_0, \dots, v_i\}$ with $v_0 < v_1 < \dots < v_i$. If $\sigma = \langle v_0, \dots, v_i \rangle$, we say that σ is an *ordered simplex*. Taking in account the identification made in the definition of oriented simplices, we notice that $-\sigma$ is an ordered simplex if σ is not and vice-versa. So, the chain complex of oriented simplices $C_*(K)$ is generated by the ordered simplices. Associated to such simplicial simplex, we have the singular simplex $\iota(\sigma): \Delta \rightarrow |K|$ defined by:

$$\iota(\sigma) \left(\sum_{j=0}^i t_j a_j \right) = \sum_{j=0}^i t_j v_j, \quad (25)$$

where $\Delta = \langle a_0, \dots, a_i \rangle$ is the standard simplex. It is well known that $\iota(\sigma): \Delta \rightarrow |K|$ is a linear map and that $\iota: C_*^{\bar{p}}(K) \rightarrow C_*^{\bar{p}}(|K|)$ is a chain map.

Proposition 4.1. *Let (K, \bar{p}) be a perverse full filtered simplicial complex of virtual dimension n . The map $\iota: C_*^{\bar{p}}(K) \rightarrow C_*^{\bar{p}}(|K|)$, associated to an order verifying (24), is a chain map.*

Proof. Let $\sigma = \langle v_0, \dots, v_i \rangle \in K$, with $v_0 < v_1 < \dots < v_i$. It suffices to prove $\|\iota(\sigma)\|_S = \|\sigma\|_S$ for each singular stratum $S \in \mathcal{S}_{|K|}$ verifying $S \cap \iota(\sigma)(\Delta) = S \cap \sigma(\Delta) \neq \emptyset$. As we observed in Subsection 1.1, the canonical decomposition of σ is given by

$$\sigma \cap |K_\ell| = \sigma_0 * \dots * \sigma_\ell, \quad (26)$$

for each $\ell \in \{0, \dots, n\}$. The sets $\Delta_i = \iota(\sigma)^{-1}(\sigma_i)$ are empty-sets or faces of Δ . We have

$$\iota(\sigma)^{-1}(|K_\ell|) = \iota(\sigma)^{-1}(\sigma \cap |K_\ell|) = \iota(\sigma)^{-1}(\sigma_0 * \dots * \sigma_\ell) = \Delta_0 * \dots * \Delta_\ell.$$

If $j = \dim_v S$, the stratum S is the only j -dimensional stratum that meets σ and we deduce the claim from

$$\begin{aligned} \|\sigma\|_S &= \dim(\sigma \cap S) = \dim(\sigma \cap |K_j| \setminus |K_{j-1}|) = \dim(\sigma \cap |K_j|) \\ &= \dim(\sigma_0 * \dots * \sigma_j) = \dim(\Delta_0 * \dots * \Delta_j) = \|\iota(\sigma)\|_S. \end{aligned}$$

\clubsuit

The next statement is the existence of an isomorphism between the singular and the simplicial intersection homologies, for a full complex. This result was proven between the PL and the singular intersection homologies by M. Goresky and R. MacPherson in the Appendix of [16]. The key point of their proof is taken up in the proof of Proposition 6.5 below. The isomorphism

between the singular and the PL intersection homologies is set up by H. King in [15] for a CS-set. Let us also mention that these two isomorphisms are taken over with detail and comments in [9, Sections 3.3 and 5.4]. So, in the literature, the isomorphism between the singular and the simplicial intersection homologies is established by using the PL intersection homology as an intermediate. Here, the proof comes from a direct comparison between these two homologies and does not need a restriction to CS sets.

Theorem 4.2. *Let (K, \bar{p}) be a perverse full filtered simplicial complex, with an order on the set of vertices verifying (24). Then, the associated inclusion, $\iota: C_*^{\bar{p}}(K) \rightarrow C_*^{\bar{p}}(|K|)$, induces the isomorphism*

$$H_*^{\bar{p}}(K) \cong H_*^{\bar{p}}(|K|).$$

Proof. We proceed in two steps.

First Step: we suppose $K = K^{(\ell)}$ for some $\ell \in \mathbb{N}$.

Given a subcomplex $L \subset K$ we also have $L = L^{(\ell)}$. If $\text{comp } L = (a_L, b_L)$ then $b_L \leq \dim L < \infty$ and therefore $\text{comp } \mathcal{L}(L) < \text{comp } L$ (cf. Subsection 1.4). We use an induction on the complexity (a, b) of K . If $a = b = 0$ then $K = K_n$ is a discrete family of 0-dimensional simplices. So, $C_*^{\bar{p}}(K) = C_*^{\bar{p}}(|K|)$ and we get the claim. Let us suppose $(a, b) > (0, 0)$ and consider the following commutative diagram defining the relative homology,

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*^{\bar{p}}(\mathcal{L}(K)) & \hookrightarrow & C_*^{\bar{p}}(K) & \longrightarrow & C_*^{\bar{p}}(K, \mathcal{L}(K)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_*^{\bar{p}}(|\mathcal{L}(K)|) & \hookrightarrow & C_*^{\bar{p}}(|K|) & \longrightarrow & C_*^{\bar{p}}(|K|, |\mathcal{L}(K)|) \longrightarrow 0, \end{array}$$

where the vertical maps are induced by the inclusion map ι (cf. (11), (21)). From the induction hypothesis, we know that the left vertical arrow is a quasi-isomorphism. So, it suffices to prove that the right vertical arrow is a quasi-isomorphism. Using Propositions 2.3 and 3.3, this assertion is equivalent to the fact that, for each clot $\beta \in \mathcal{B}(K)$, the following map is a quasi-isomorphism

$$C_*^{\bar{p}}(\beta * L_\beta, \mathcal{L}(\beta * L_\beta)) \rightarrow C_*^{\bar{p}}(|\beta * L_\beta|, |\mathcal{L}(\beta * L_\beta)|).$$

Again we combine the short exact sequences (11) and (21) in a commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*^{\bar{p}}(\mathcal{L}(\beta * L_\beta)) & \hookrightarrow & C_*^{\bar{p}}(\beta * L_\beta) & \longrightarrow & C_*^{\bar{p}}(\beta * L_\beta, \mathcal{L}(\beta * L_\beta)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_*^{\bar{p}}(|\mathcal{L}(\beta * L_\beta)|) & \hookrightarrow & C_*^{\bar{p}}(|\beta * L_\beta|) & \longrightarrow & C_*^{\bar{p}}(|\beta * L_\beta|, |\mathcal{L}(\beta * L_\beta)|) \longrightarrow 0, \end{array}$$

where the vertical maps are induced by the inclusion map ι . As noticed at the beginning of this proof, we know that $\text{comp } \mathcal{L}(\beta * L_\beta) < \text{comp } (\beta * L_\beta) \leq \text{comp } K$. With the induction hypothesis, the left vertical arrow is a quasi-isomorphism. It remains to prove that the middle arrow is a quasi-isomorphism. For doing that, we distinguish two cases.

- *The clot β is included in a regular stratum.* Here, we have $L_\beta = \emptyset$ and the middle arrow becomes $\iota_*: C_*^{\bar{p}}(\beta) \rightarrow C_*^{\bar{p}}(|\beta|)$. The claim comes from the classical situation.

- *The clot β is included in a singular stratum.* From Subsection 1.3, we have $\text{comp } L_\beta < \text{comp } K$. From Propositions 2.5, 3.4 and the induction hypothesis, we deduce that the middle vertical arrow is a quasi-isomorphism.

Second Step: the general case. We consider the induced map $\iota_*: H_*^{\bar{p}}(K) \rightarrow H_*^{\bar{p}}(|K|)$ and decompose the proof in two points.

- *Claim: ι_* is an epimorphism.* Consider a cycle $c \in C_*^{\bar{p}}(|K|)$. The chain c being a finite sum, there exists an integer $\ell \in \mathbb{N}$ with $c \in C_*^{\bar{p}}(|K^{(\ell)}|)$. Applying the first step of the proof, there exist $f \in C_*^{\bar{p}}(|K^{(\ell)}|)$ and $e \in C_*^{\bar{p}}(K^{(\ell)})$ with $\partial e = 0$ and $\iota(e) = c + \partial f$. Since $f \in C_*^{\bar{p}}(|K|)$ and $e \in C_*^{\bar{p}}(K)$, we get $\iota_*([e]) = [c]$ and the claim.

• *Claim:* ι_* is a monomorphism. Consider a cycle $e \in C_*^{\overline{p}}(K)$ and a chain $f \in C_*^{\overline{p}}(|K|)$ with $\iota(e) = \partial f$. Since e and f are finite sums, there exists an integer $\ell \in \mathbb{N}$ with $e \in C_*^{\overline{p}}(K^{(\ell)})$ and $f \in C_*^{\overline{p}}(|K^{(\ell)}|)$. Applying the first step of the proof, there exists $g \in C_*^{\overline{p}}(K^{(\ell)})$ with $e = \partial g$. Since $g \in C_*^{\overline{p}}(K)$, we get $[e] = 0$ and the claim. \clubsuit

The intersection cohomology $H_p^*(-)$ of [10] is defined as the cohomology of the dual complex $\text{Hom}(C_*^{\overline{p}}(-); R)$. As the Universal Coefficient Theorem is true in this context ([9, Theorem 7.14]), the singular and simplicial intersection cohomologies are isomorphic for a full filtered simplicial complex.

5. BLOWN-UP INTERSECTION COHOMOLOGIES

Intersection homology and cohomology do not verify the Poincaré duality for any ring of coefficient (see [12, 10]). The blown-up intersection cohomology has been introduced in order to recover this property through a cap product with a fundamental class (see [5]). In this section, we extend the paradigm (simplicial, singular) to the blown-up intersection cohomology. After their definitions, we show in Propositions (5.10), (5.11) and (5.12) that the process used for intersection homology is still valid here. Finally, we prove the existence of an isomorphism between the simplicial and singular blown-up intersection cohomologies in Theorem 5.13, for a full filtered simplicial complex.

Definition 5.1. Let X be a filtered space of dimension n . A *filtered singular simplex* of X is a continuous map, $\sigma: \Delta \rightarrow X$, such that $\sigma^{-1}(X_i)$ is a face of Δ , for $i \in \{0, \dots, n\}$, or the empty set. The family of these simplices is denoted $\text{Sing}^{\mathcal{F}}X$. A filtered singular simplex σ is *regular* if $\sigma^{-1}(X_n \setminus X_{n-1}) \neq \emptyset$. The subfamily of regular simplices is denoted $\text{Sing}_+^{\mathcal{F}}X$.

Definition 5.2. A *filtered simplex* is a decomposition $\Delta = \Delta_0 * \dots * \Delta_n$, where each Δ_i is a face of the standard simplex Δ or the empty set. If $\Delta_n \neq \emptyset$ we say that the simplex is *regular*. The *blow up* of a regular simplex is the prism $\tilde{\Delta} = \mathbf{c}\Delta_0 \times \dots \times \mathbf{c}\Delta_{n-1} \times \Delta_n$.

The domain of a filtered singular simplex, $\sigma: \Delta \rightarrow X$, inherits a decomposition

$$\Delta_\sigma = \Delta_0 * \Delta_1 * \dots * \Delta_n,$$

with $\sigma^{-1}(X_i) = \Delta_0 * \dots * \Delta_i$, and Δ_i possibly empty, for each $i \in \{0, \dots, n\}$. We call it the σ -*decomposition* of Δ . This simplex σ is said regular if $\Delta_n \neq \emptyset$. Notice that each filtered singular simplex $\sigma: \Delta \rightarrow X$ (resp. regular) induces a filtered simplex (resp. regular) on Δ . The family $\text{Sing}^{\mathcal{F}}X$ is a Δ -set in the sense of Rourke and Sanderson ([18]) for the usual face maps δ_i . The blown-up construction, we describe above, is the key point in the construction of our cohomology. Let us begin with the local situation.

Definition 5.3. A *face* of $\tilde{\Delta}$ is represented by $(F, \varepsilon) = (F_0, \varepsilon_0) \times \dots \times (F_n, \varepsilon_n)$, with $\varepsilon_i \in \{0, 1\}$, $\varepsilon_n = 0$ and F_i a face of Δ_i for $i \in \{0, \dots, n\}$, or the empty set with $\varepsilon_i = 1$.

More precisely, we have:

- $\varepsilon_i = 0$ and $F_i \triangleleft \Delta_i$, that is, $(F_i, 0) = F_i$ is a face of Δ_i , or
- $\varepsilon_i = 1$ and $F_i \triangleleft \Delta_i$, that is, $(F_i, 1) = \mathbf{c}F_i$ is the cone of a face of Δ_i , or
- $\varepsilon_i = 1$ and $F_i = \emptyset$, that is, $(\emptyset, 1)$ is the apex of the cone $\mathbf{c}\Delta_i$, called the *virtual apex*.

Recall $\varepsilon_n = 0$. We also set

$$|(F, \varepsilon)|_{>j} = \dim(F_{j+1}, \varepsilon_{j+1}) + \dots + \dim(F_n, \varepsilon_n).$$

Let us denote $N^*(\Delta)$ the complex of simplicial cochains defined on the standard simplex Δ , with coefficients in \mathbb{R} .

Definition 5.4. The *blown-up complex* of a regular simplex $\Delta = \Delta_0 * \dots * \Delta_n$ is the tensor product

$$\tilde{N}^*(\Delta) = N^*(\mathbf{c}\Delta_0) \otimes \dots \otimes N^*(\mathbf{c}\Delta_{n-1}) \otimes N^*(\Delta_n).$$

Let $\mathbf{1}_{(F_i, \varepsilon_i)}$ be the cochain on $\mathbf{c}\Delta_i$, taking the value 1 on the simplex (F_i, ε_i) and 0 on the other simplices of $\mathbf{c}\Delta_i$, for $i \in \{0, \dots, n-1\}$. Similarly for $\mathbf{1}_{(F_n, \varepsilon_n)}$. A basis of $\tilde{N}^*(\Delta)$ is given by the family

$$\mathbf{1}_{(F, \varepsilon)} = \mathbf{1}_{(F_0, \varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_n, \varepsilon_n)},$$

where (F, ε) runs over the faces of $\tilde{\Delta}$. Each element of this basis owns an extra degree, coming from the filtration and called perverse degree.

Definition 5.5. Let $\ell \in \{1, \dots, n\}$. The ℓ -perverse degree of the cochain $\mathbf{1}_{(F, \varepsilon)} \in \tilde{N}^*(\Delta)$ is equal to

$$\|\mathbf{1}_{(F, \varepsilon)}\|_\ell = \begin{cases} -\infty & \text{if } \varepsilon_{n-\ell} = 1, \\ |(F, \varepsilon)|_{>n-\ell} & \text{if } \varepsilon_{n-\ell} = 0. \end{cases}$$

The ℓ -perverse degree of $\omega = \sum_\mu \lambda_\mu \mathbf{1}_{(F_\mu, \varepsilon_\mu)} \in \tilde{N}^*(\Delta)$, with each $\lambda_\mu \neq 0$, is equal to

$$\|\omega\|_\ell = \max_\mu \|\mathbf{1}_{(F_\mu, \varepsilon_\mu)}\|_\ell.$$

By convention, we set $\|0\|_\ell = -\infty$.

Remark 5.6. Let us consider a face (F, ε) of $\tilde{\Delta}$ with $F_0 = \cdots = F_{m-1} = \emptyset$ for some $m \in \{0, \dots, n-1\}$. From the definition, we observe that the perverse degrees $\|\mathbf{1}_{(F, \varepsilon)}\|_\ell$, for $\ell \in \{1, \dots, n\}$, do not depend on the face F_m .

Let X be a filtered space and \mathbf{P} be a subset of $\text{Sing}^{\mathcal{F}}X$ stable by the face operators. The subfamily of its regular elements is denoted \mathbf{P}_+ . In Example 5.9, we detail two examples of subsets \mathbf{P} of interest for this work.

Let us define the blown-up cochain complex associated to $\mathbf{P} \subseteq \text{Sing}^{\mathcal{F}}X$. First, to any regular simplex, $\sigma: \Delta = \Delta_0 * \cdots * \Delta_n \rightarrow X$ in \mathbf{P} , we associate the cochain complex defined by

$$\tilde{N}_\sigma^* = \tilde{N}^*(\Delta) = N^*(\mathbf{c}\Delta_0) \otimes \cdots \otimes N^*(\mathbf{c}\Delta_{n-1}) \otimes N^*(\Delta_n).$$

A face operator $\delta_\ell: \nabla = \nabla_0 * \cdots * \nabla_n \rightarrow \Delta = \Delta_0 * \cdots * \Delta_n$ is *regular* if $\nabla_n \neq \emptyset$. By restriction, such δ_ℓ induces

$$\delta_\ell^*: N^*(\mathbf{c}\Delta_0) \otimes \cdots \otimes N^*(\mathbf{c}\Delta_{n-1}) \otimes N^*(\Delta_n) \rightarrow N^*(\mathbf{c}\nabla_0) \otimes \cdots \otimes N^*(\mathbf{c}\nabla_{n-1}) \otimes N^*(\nabla_n).$$

Definition 5.7. Let X be a filtered space and \mathbf{P} be a sub Δ -set of $\text{Sing}^{\mathcal{F}}X$. The *blown-up complex*, $\tilde{N}^{*,\mathbf{P}}(X)$, is the cochain complex formed by the elements ω , associating to any $\sigma \in \mathbf{P}$ an element $\omega_\sigma \in \tilde{N}_\sigma^*$, so that $\delta_\ell^*(\omega_\sigma) = \omega_{\sigma \circ \delta_\ell}$, for any regular face operator, δ_ℓ , of \mathbf{P} . The differential of $\tilde{N}^{*,\mathbf{P}}(X)$ is defined by $(\delta\omega)_\sigma = \delta(\omega_\sigma)$. The *perverse degree of $\omega \in \tilde{N}^{*,\mathbf{P}}(X)$ along a singular stratum, $S \in \mathcal{S}_X$* , is equal to

$$\|\omega\|_S = \sup \{ \|\omega_\sigma\|_{\text{codim}_v S} \mid \sigma \in \mathbf{P}_+ \text{ with } \text{Im } \sigma \cap S \neq \emptyset \}.$$

We denote $\|\omega\|: \mathcal{S}_X \rightarrow \overline{\mathbb{Z}}$ the map associating to any singular stratum S of K the element $\|\omega\|_S$ and 0 to any regular stratum.

Definition 5.8. Let \bar{p} be a perversity on a filtered space X and \mathbf{P} be a sub Δ -set of $\text{Sing}^{\mathcal{F}}X$. A cochain $\omega \in \tilde{N}^{*,\mathbf{P}}(X)$ is \bar{p} -allowable if $\|\omega\| \leq \bar{p}$. A cochain ω is of \bar{p} -intersection if ω and its coboundary, $\delta\omega$, are \bar{p} -allowable.

We denote $\tilde{N}_{\bar{p}}^{*,\mathbf{P}}(X)$ the complex of \bar{p} -intersection cochains and $\mathcal{H}_{\bar{p}}^{*,\mathbf{P}}(X)$ its homology, called the *blown-up intersection cohomology of X with coefficients in \mathbb{R} , for the perversity \bar{p}* .

Example 5.9. Let X be a filtered space and \bar{p} be a perversity on X . In the sequel, we consider the two following cases of subsets $\mathbf{P} \subset \text{Sing}^{\mathcal{F}}X$.

- a) If $\mathbf{P} = \text{Sing}^{\mathcal{F}}X$, we recover the complex of blown-up \bar{p} -intersection cochains of X and the blown-up intersection cohomology defined in [3]. We denote them $\tilde{N}_{\bar{p}}^*(X)$ and $\mathcal{H}_{\bar{p}}^*(X)$. If there is an ambiguity with the simplicial situation introduced in the next item, we will call them *the singular blown-up intersection complex and the singular blown-up intersection cohomology*.

- b) Let $X = |K|$ be the geometric realization of a full filtered simplicial complex K . The family of singular simplices induced by the simplices of K is denoted $\text{Simp } K$. Since K is full, this is a sub Δ -set of $\text{Sing}^{\mathcal{F}}|K|$ and we can apply the previous process with $\mathbf{P} = \text{Simp } K$. We denote $\tilde{N}_{\bar{p}}^*(K)$ and $\mathcal{H}_{\bar{p}}^*(K)$ the associated complex and cohomology and call them *the simplicial blown-up intersection complex* and *the simplicial blown-up intersection cohomology*.

In Theorem 5.13, we prove the existence of an isomorphism between the singular and the simplicial blown-up intersection cohomologies, $\mathcal{H}_{\bar{p}}^*(K) \cong \mathcal{H}_{\bar{p}}^*(|K|)$.

The next result allows the existence of relative blown-up intersection cohomologies.

Proposition 5.10. *Let (K, \bar{p}) be a perverse full filtered simplicial complex. Then, the two following restrictions,*

- a) $\gamma: \tilde{N}_{\bar{p}}^*(|K|) \rightarrow \tilde{N}_{\bar{p}}^*(|\mathcal{L}(K)|)$, and
b) $\gamma: \tilde{N}_{\bar{p}}^*(K) \rightarrow \tilde{N}_{\bar{p}}^*(\mathcal{L}(K))$,

are onto maps.

Proof. Let $\text{comp } K = (a, b)$. When $b = \infty$, we have $\mathcal{L}(K) = K$ and the result is clear. Suppose now $b < \infty$, which implies (see Subsection 1.4) $\text{comp } \mathcal{L}(K) < \text{comp } K$.

a) It suffices to prove that the extension, $\eta \in \tilde{N}_{\bar{p}}^*(|K|)$, by 0 of a cochain $\omega \in \tilde{N}_{\bar{p}}^*(|\mathcal{L}(K)|)$ belongs to $\tilde{N}_{\bar{p}}^*(|K|)$. We clearly have $\|\eta\| \leq \|\omega\|$. Now, we want to bound $\|\delta_{|K|}\eta\|$. For that, we write

$$\delta_{|K|}\eta = \delta_{|\mathcal{L}(K)|}\eta + (\delta_{|K|} - \delta_{|\mathcal{L}(K)|})(\eta)$$

and notice first that $\|\delta_{|\mathcal{L}(K)|}\eta\| \leq \|\delta_{|\mathcal{L}(K)|}\omega\|$. As the perverse degree of a linear combination is the maximum of the perverse degrees of its terms (see Definition 5.7), we are reduced to study $\|(\delta_{|K|} - \delta_{|\mathcal{L}(K)|})(\eta)\|$. We claim that

$$\|(\delta_{|K|} - \delta_{|\mathcal{L}(K)|})(\eta)\| \leq \|\omega\|, \quad (27)$$

which gives the result. Without loss of the generality, it suffices to prove that

$$\|\delta_{|\mathcal{L}(K)|}\mathbf{1}_{(F,\varepsilon)} - \delta_{|K|}\mathbf{1}_{(F,\varepsilon)}\| \leq \|\mathbf{1}_{(F,\varepsilon)}\|,$$

where $\sigma: \Delta \rightarrow |K|$ is a regular singular simplex and (F, ε) is a face of the blown-up $\tilde{\Delta}$ such that $\sigma(F) \subset |\mathcal{L}(K)|$.

Since $K_{n-a-1} = \emptyset$ and $\text{comp } \mathcal{L}(K) < \text{comp } K$, we have $F_0 = \dots = F_{n-a-1} = \emptyset$ and $\sigma(F_{n-a}) \subset |K_{n-a}^{(b-1)}|$. An element of $\delta_{|\mathcal{L}(K)|}\mathbf{1}_{(F,\varepsilon)} - \delta_{|K|}\mathbf{1}_{(F,\varepsilon)}$ is of the form $\pm\mathbf{1}_{(H,\tau)}$ and we can suppose, without loss of generality, that (H, τ) is a face of $\tilde{\Delta}$. This face verifies $(F, \varepsilon) \triangleleft (H, \tau)$, $\dim(H, \tau) = \dim(F, \varepsilon) + 1$, and $\sigma(H) \not\subset |\mathcal{L}(K)|$ (i.e., $\sigma(H_{n-a}) \not\subset K_{n-a}^{(b-1)}$). This implies $H_{n-a} \neq F_{n-a}$, $H_i = F_i$, for $i \neq n-a$, and $\tau = \varepsilon$. Since $\|\mathbf{1}_{(H,\tau)}\| = \|\mathbf{1}_{(F,\varepsilon)}\|$ (cf. Remark 5.6), we get the claim.

- b) The proof of the singular situation is similar. ♣

From Proposition 5.10, we deduce the two following exact sequences,

$$\begin{aligned} 0 &\longrightarrow \tilde{N}_{\bar{p}}^*(|K|, |\mathcal{L}(K)|) \hookrightarrow \tilde{N}_{\bar{p}}^*(|K|) \xrightarrow{\gamma} \tilde{N}_{\bar{p}}^*(|\mathcal{L}(K)|) \longrightarrow 0, \\ 0 &\longrightarrow \tilde{N}_{\bar{p}}^*(K, \mathcal{L}(K)) \hookrightarrow \tilde{N}_{\bar{p}}^*(K) \xrightarrow{\nu} \tilde{N}_{\bar{p}}^*(\mathcal{L}(K)) \longrightarrow 0. \end{aligned} \quad (28)$$

The homology of $\tilde{N}_{\bar{p}}^*(|K|, |\mathcal{L}(K)|)$ is isomorphic to the relative blown-up \bar{p} -intersection cohomology introduced in [3]. We denote it $\mathcal{H}_{\bar{p}}^*(|K|, \mathcal{L}(|K|))$. The homology of the complex $\tilde{N}_{\bar{p}}^*(K, \mathcal{L}(K))$ is denoted $\mathcal{H}_{\bar{p}}^*(K, \mathcal{L}(K))$. In the next statement, we show that the relative intersection cohomology can be decomposed, as does the relative homology of $(X^{(\ell)}, X^{(\ell-1)})$ in the case of a CW-complex X .

Proposition 5.11. *Let (K, \bar{p}) be a perverse full filtered simplicial complex. The restriction map induces the isomorphisms*

- i) $\mathcal{H}_{\bar{p}}^*(|K|, |\mathcal{L}(K)|) \xrightarrow{\cong} \prod_{\beta \in \mathcal{B}(K)} \mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|, |\mathcal{L}(\beta * L_{\beta})|)$, and
- ii) $\mathcal{H}_{\bar{p}}^*(K, \mathcal{L}(K)) \xrightarrow{\cong} \prod_{\beta \in \mathcal{B}(K)} \mathcal{H}_{\bar{p}}^*(\beta * L_{\beta}, \mathcal{L}(\beta * L_{\beta}))$.

Proof. i) We proceed as in the proof of Proposition 3.3 keeping the same notations. Applying [3, Proposition 11.3], we get that the inclusion induces the isomorphism $\mathcal{H}_{\bar{p}}^*(W) \cong \mathcal{H}_{\bar{p}}^*(|\mathcal{L}(K)|)$ which gives an isomorphism

$$\mathcal{H}_{\bar{p}}^*(|K|, W) \cong \mathcal{H}_{\bar{p}}^*(|K|, |\mathcal{L}(K)|).$$

By excision (cf. [3, Proposition 12.9]) we have

$$\mathcal{H}_{\bar{p}}^*(|K|, W) \cong \mathcal{H}_{\bar{p}}^*(|K| \setminus |\mathcal{L}(K)|, W \setminus |\mathcal{L}(K)|).$$

From the decompositions made in (23), we get

$$\mathcal{H}_{\bar{p}}^*(|K| \setminus |\mathcal{L}(K)|, W \setminus |\mathcal{L}(K)|) \cong \prod_{\beta \in \mathcal{B}(K)} \mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}| \setminus |\partial\beta * L_{\beta}|, |\beta * L_{\beta}| \setminus (\{b_{\beta}\} \cup |\partial\beta * L_{\beta}|)).$$

Using excision relatively to the closed subset $|\partial\beta * L_{\beta}|$, we obtain

$$\mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}| \setminus |\partial\beta * L_{\beta}|, |\beta * L_{\beta}| \setminus (\{b_{\beta}\} \cup |\partial\beta * L_{\beta}|)) \cong \mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|, |\beta * L_{\beta}| \setminus \{b_{\beta}\}).$$

Finally, applying [3, Theorem D] to the homeomorphism(22), we deduce

$$\mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|, |\beta * L_{\beta}| \setminus \{b_{\beta}\}) \cong \mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|, |\partial\beta * L_{\beta}|)$$

and therefore the isomorphism i).

ii) The complex $\tilde{N}_{\bar{p}}^*(K, \mathcal{L}(K))$ is made up of the cochains of K vanishing on $\mathcal{L}(K)$. From (6) we get $\tilde{N}_{\bar{p}}^*(K, \mathcal{L}(K)) \cong \prod_{\beta \in \mathcal{B}(K)} \tilde{N}_{\bar{p}}^*(\beta * L_{\beta}, \mathcal{L}(\beta * L_{\beta}))$ and therefore,

$$\mathcal{H}_{\bar{p}}^*(K, \mathcal{L}(K)) \cong \prod_{\beta \in \mathcal{B}(K)} \mathcal{H}_{\bar{p}}^*(\beta * L_{\beta}, \mathcal{L}(\beta * L_{\beta})),$$

by restriction. ♣

The next result specifies the blown-up intersection cohomology of some pieces of the decomposition brought by Proposition 5.11.

Proposition 5.12. *Consider a perverse full filtered simplicial complex (K, \bar{p}) . Let $\beta \in \mathcal{B}(K)$ be a clot such that the stratum $Q \in \mathcal{S}_K$ containing β is singular. We have*

- i) $\mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|) \cong \begin{cases} \mathcal{H}_{\bar{p}}^*(|L_{\beta}|) & \text{if } * \leq \bar{p}(Q), \\ 0 & \text{if not,} \end{cases}$
- ii) $\mathcal{H}_{\bar{p}}^*(\beta * L_{\beta}) \cong \begin{cases} \mathcal{H}_{\bar{p}}^*(L_{\beta}) & \text{if } * \leq \bar{p}(Q), \\ 0 & \text{if not,} \end{cases}$

where the isomorphisms are induced by the natural inclusion $L_{\beta} \hookrightarrow \beta * L_{\beta}$.

Proof. If $L_{\beta} = \emptyset$ then $N_{\bar{p}}^*(\beta * L_{\beta}) = N_{\bar{p}}^*(\beta) = 0$ since there are no regular simplices (Q is singular). So, we can suppose $L_{\beta} \neq \emptyset$. Let $\text{comp } K = (a, b)$.

i) We proceed as in the proof of Proposition 3.4 keeping the same notations. The case $\dim \beta = 0$ is given by [3, Theorem E], since $L_{\beta} \neq \emptyset$. When $\dim \beta > 0$ by using [3, Proposition 11.3], we get that $(F(-, 1))^*: \mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|) \rightarrow \mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|)$ is the identity. Finally, since $\text{id}: |\eta * L_{\beta}| \hookrightarrow |\beta * L_{\beta}| \xrightarrow{\xi} |\eta * L_{\beta}|$ and $F(-, 1): |\beta * L_{\beta}| \xrightarrow{\xi} |\eta * L_{\beta}| \hookrightarrow |\beta * L_{\beta}|$, we conclude that the inclusion $|\eta * L_{\beta}| \hookrightarrow |\beta * L_{\beta}|$ induces the isomorphism $\mathcal{H}_{\bar{p}}^*(|\beta * L_{\beta}|) \cong \mathcal{H}_{\bar{p}}^*(|\eta * L_{\beta}|)$.

ii) By definition, we have

$$\tilde{N}^*(\beta * L_{\beta}) = N^*(c\beta) \otimes \tilde{N}^*(L_{\beta}),$$

and any cochain $\omega \in \tilde{N}^*(\beta * L_\beta)$ can be written as a sum

$$\omega = \sum_{F \triangleleft \beta} \mathbf{1}_{(F,0)} \otimes \omega_F + \sum_{F \triangleleft \beta} \mathbf{1}_{(F,1)} \otimes \tau_F + \mathbf{1}_{(\emptyset,1)} \otimes \tau_\emptyset,$$

with $\omega_F, \tau_F, \tau_\emptyset \in \tilde{N}^*(L_\beta)$ if $F \triangleleft \beta$. The perverse degrees are computed as follows,

$$\begin{aligned} - \|\mathbf{1}_{(F,0)} \otimes \tau\|_\ell &= \begin{cases} \deg \tau & \text{if } \ell = a, \\ \|\tau\|_\ell & \text{if } \ell < a, \end{cases} \\ - \|\mathbf{1}_{(F,1)} \otimes \tau\|_\ell &= \begin{cases} -\infty & \text{if } \ell = a, \\ \|\tau\|_\ell & \text{if } \ell < a, \end{cases} \end{aligned}$$

for each $\ell \in \{1, \dots, n\}$. The condition $\|\omega\| \leq \bar{p}$ is equivalent to $\max(\|\omega_F\|, \|\tau_F\|, \|\tau_\emptyset\|) \leq \bar{p}$ and $\deg \omega_F \leq \bar{p}(Q)$, for each $F \triangleleft \beta$. For $\|\delta\omega\| \leq \bar{p}$, this becomes $\max(\|\delta\omega_F\|, \|\delta\tau_F\|, \|\delta\tau_\emptyset\|) \leq \bar{p}$, and $\deg(\delta\omega_F) \leq \bar{p}(Q)$. In particular, we have, for each $F \triangleleft \beta$,

$$\omega_F \in \tilde{N}_{\bar{p}}^{\leq \bar{p}(Q)}(L_\beta) \oplus \left(\tilde{N}_{\bar{p}}^{\bar{p}(Q)}(L_\beta) \cap \delta^{-1}(0) \right) \text{ and } \tau_F, \tau_\emptyset \in \tilde{N}_{\bar{p}}^*(L_\beta).$$

The complex $N^*(\mathbf{c}\beta, \beta)$ is made up of the cochains on $\mathbf{c}\beta$ vanishing on β . It is generated by the family $\{\mathbf{1}_{(F,1)} \mid F = \emptyset \text{ or } F \triangleleft \beta\}$. Let us consider the two short exact sequences,

$$0 \rightarrow N^*(\mathbf{c}\beta, \beta) \hookrightarrow N^*(\mathbf{c}\beta) \xrightarrow{\nu} N^*(\beta) \rightarrow 0,$$

where $\nu \left(\sum_{F \triangleleft \beta} n_F \mathbf{1}_{(F,0)} + \sum_{F \triangleleft \beta} m_F \mathbf{1}_{(F,1)} + n_\emptyset \mathbf{1}_{(\emptyset,1)} \right) = \sum_{F \triangleleft \beta} u_F \mathbf{1}_F$, with $n_F, m_F, n_\emptyset \in \mathbf{R}$, and

$$0 \rightarrow N^*(\mathbf{c}\beta, \beta) \otimes \tilde{N}_{\bar{p}}^*(L_\beta) \hookrightarrow \tilde{N}_{\bar{p}}^*(\beta * L_\beta) \xrightarrow{v} N^*(\beta) \otimes \tau_{\bar{p}(Q)} \tilde{N}_{\bar{p}}^*(L_\beta) \rightarrow 0,$$

where $v \left(\sum_{F \triangleleft \beta} \mathbf{1}_{(F,0)} \otimes \omega_F + \sum_{F \triangleleft \beta} \mathbf{1}_{(F,1)} \otimes \tau_F + \mathbf{1}_{(\emptyset,1)} \otimes \tau_\emptyset \right) = \sum_{F \triangleleft \beta} \mathbf{1}_{(F,0)} \otimes \omega_F$.

Since the map ν is a quasi-isomorphism, the complex $N^*(\mathbf{c}\beta, \beta)$ is acyclic. So, the map v is a quasi-isomorphism. The result comes from $H^j(\beta) = 0$ if $j > 0$ and $H^0(\beta) = \mathbf{R}$. \clubsuit

When Q is a regular stratum, we have $L_\beta = \emptyset$, $\mathcal{H}_{\bar{p}}^i(|\beta * L_\beta|) = \mathcal{H}_{\bar{p}}^i(|\beta|) = H^i(|\beta|) = 0$ if $i > 0$ and $H^0(|\beta|) = \mathbf{R}$. Similarly, the simplicial blown-up intersection cohomology verifies $\mathcal{H}_{\bar{p}}^i(\beta * L_\beta) = \mathcal{H}_{\bar{p}}^i(\beta) = H^i(\beta) = 0$ if $i > 0$ and $\mathcal{H}_{\bar{p}}^0(\beta) = H^0(\beta) = \mathbf{R}$.

The second main result of this work establishes an isomorphism between the singular and the simplicial blown-up intersection cohomologies. These cohomologies are related by the cochain map $\rho: \tilde{N}_{\bar{p}}^*(|K|) \rightarrow \tilde{N}_{\bar{p}}^*(K)$ induced by the natural inclusion $\text{Simp } K \subset \text{Sing}^{\mathcal{F}}|K|$ (cf. (25)).

Theorem 5.13. *Let (K, \bar{p}) be a perverse full filtered simplicial complex. Then, the map ρ induces an isomorphism*

$$\mathcal{H}_{\bar{p}}^*(|K|) \cong \mathcal{H}_{\bar{p}}^*(K). \quad (29)$$

Proof. We proceed in two steps.

First Step : Suppose $\text{comp } K = (a, b)$ with $b < \infty$. Given a subcomplex $L \subset K$, we also have $b_L \leq \dim L < \infty$, with $\text{comp } L = (a_L, b_L)$, and therefore $\text{comp } \mathcal{L}(L) < \text{comp } L$ (cf. Subsection 1.4). We use an induction on the complexity (a, b) of K . When $a = b = 0$, the complex K is a discrete family of 0-dimensional simplices, so we have $N_{\bar{p}}^*(K) = N_{\bar{p}}^*(|K|)$ and the claim. For the inductive step, we consider the following commutative diagram deduced from (28),

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_{\bar{p}}^*(|K|, |\mathcal{L}(K)|) & \hookrightarrow & N_{\bar{p}}^*(|K|) & \xrightarrow{\gamma} & N_{\bar{p}}^*(|\mathcal{L}(K)|) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_{\bar{p}}^*(K, \mathcal{L}(K)) & \hookrightarrow & N_{\bar{p}}^*(K) & \xrightarrow{\gamma} & N_{\bar{p}}^*(\mathcal{L}(K)) \longrightarrow 0 \end{array}$$

where the vertical maps are induced by the map ρ . From the induction hypothesis, we know that the right arrow is a quasi-isomorphism. So, it suffices to prove that the left arrow is a

quasi-isomorphism. Using Proposition 5.11, this assertion is equivalent to the fact that the map ρ induces a quasi-isomorphism, for each $\beta \in \mathcal{B}(K)$,

$$N_{\bar{p}}^*(|\beta * L_{\beta}|, |\mathcal{L}(\beta * L_{\beta})|) \rightarrow N_{\bar{p}}^*(\beta * L_{\beta}, \mathcal{L}(\beta * L_{\beta})).$$

From (28), we deduce the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_{\bar{p}}^*(|\beta * L_{\beta}|, |\mathcal{L}(\beta * L_{\beta})|) & \hookrightarrow & N_{\bar{p}}^*(|\beta * L_{\beta}|) & \xrightarrow{\gamma} & N_{\bar{p}}^*(|\mathcal{L}(\beta * L_{\beta})|) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_{\bar{p}}^*(\beta * L_{\beta}, \mathcal{L}(\beta * L_{\beta})) & \hookrightarrow & N_{\bar{p}}^*(\beta * L_{\beta}) & \xrightarrow{\gamma} & N_{\bar{p}}^*(\mathcal{L}(\beta * L_{\beta})) \longrightarrow 0 \end{array}$$

where the vertical maps are induced by ρ . We have $\text{comp } \mathcal{L}(\beta * L_{\beta}) < \text{comp } (\beta * L_{\beta}) \leq \text{comp } K$ (cf. Subsection 1.2) and the induction hypothesis implies that the right arrow is a quasi-isomorphism. It remains to prove that the middle arrow is a quasi-isomorphism. We distinguish two cases.

- β is included in a regular stratum. We have $L_{\beta} = \emptyset$ and this case is resolved in the paragraph following the proof of Proposition 5.12.

- β is included in a singular stratum. We have $\text{comp } L_{\beta} < \text{comp } K$ (cf. Subsection 1.3). From Proposition 5.12 and the induction hypothesis, the middle arrow is a quasi-isomorphism.

Second Step : Suppose $\text{comp } K = (a, \infty)$. We proceed by induction on $a \in \{0, \dots, n\}$. When $a = 0$, K has no singular part, that is, $K_{n-1} = \emptyset$. So, from the classical situation, the map ρ induces the isomorphism $H^*(|K|) \cong H^*(K)$. Let us consider the inductive step with $a > 0$.

Given $k \in \mathbb{N}$, we define $K^k = \{\sigma \in K \mid \dim \sigma_{n-a} \leq k\}$ and $K^{-1} = \{\sigma \in K \mid \sigma_{n-a} = \emptyset\}$. They are simplicial subcomplexes of K with $K = \bigcup_{k \geq -1} K^k$ and there exists an infinite sequence

$$K^{-1} \subset K^0 \subset \dots \subset K^k \subset K^{k+1} \subset \dots$$

Each of these complexes is endowed with the induced structure defined in Subsection 1.2. We prove that the filtered simplicial complex K^k , with $k \in \mathbb{N} \cup \{-1\}$, verifies (29).

Let us begin with the case $k = -1$. By construction, we have $\text{comp } (K^{-1}) = (a_-, b_-) < (a, 0)$. If $b_- < \infty$, the First Step gives the claim. If $b_- = \infty$, the inductive step assures us that K^{-1} verifies (29). Let $k \in \mathbb{N}$ with $K^k \neq K^{k-1}$, we have $\text{comp } K^k = (a, k)$. Following the First Step, we conclude that K^k verifies (29). We also have $\mathcal{L}(K^k) = K^{k-1}$ by definition. By using Proposition 5.10, we know that the morphisms,

$$\rho^* : N_{\bar{p}}^*(K^k) \rightarrow N_{\bar{p}}^*(K^{k-1}) \quad \text{and} \quad \rho^* : N_{\bar{p}}^*(|K^k|) \rightarrow N_{\bar{p}}^*(|K^{k-1}|), \quad (30)$$

induced by the inclusion $K^{k-1} \hookrightarrow K^k$, are onto maps.

Associated to the directed set $K = \bigcup_{k \geq -1} K^k = \varinjlim K^k$, we have the towers $(N_{\bar{p}}^*(K^k))_{k \geq -1}$ and $(N_{\bar{p}}^*(|K^k|))_{k \geq -1}$. As they are Mittag-Leffler (see (30)), we have the commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varprojlim^1 \mathcal{H}_{\bar{p}}^{*-1}(|K^k|) & \longrightarrow & \mathcal{H}_{\bar{p}}^*(|K|) & \longrightarrow & \varprojlim \mathcal{H}_{\bar{p}}^*(|K^k|) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varprojlim^1 \mathcal{H}_{\bar{p}}^{*-1}(K^k) & \longrightarrow & \mathcal{H}_{\bar{p}}^*(K) & \longrightarrow & \varprojlim \mathcal{H}_{\bar{p}}^*(K^k) \longrightarrow 0, \end{array}$$

where the vertical maps are induced by ρ . From the first step, the left and right maps are isomorphisms. So the Five's Lemma ends the proof. \clubsuit

6. SIMPLICIAL AND SINGULAR VERSUS PL

Let X be a PL space relatively to a family \mathcal{T} of triangulations, endowed with a PL filtration. In this section, we first define the PL blown-up intersection cohomology of X . In Theorem 6.7, we prove that this cohomology is isomorphic to the singular blown-up intersection cohomology of X and to the simplicial blown-up cohomology of any full triangulation belonging to \mathcal{T} .

Let us begin with basic recalls on PL spaces.

Definition 6.1. A *triangulation* of X is a pair (K, h) of a simplicial complex, K , and a homeomorphism, $h: |K| \rightarrow X$. A *subdivision* of (K, h) is a pair (L, h) where L is a subdivision of K , denoted $L \triangleleft K$. Two triangulations, (K, h) and (L, f) , of X are *equivalent* if $f^{-1}h: |K| \rightarrow X \rightarrow |L|$ is induced by a simplicial isomorphism $K \rightarrow L$.

Definition 6.2. A *PL space* is a second countable, Hausdorff topological space, X , endowed with a family, \mathcal{T} , of triangulations of X , called admissible and such that the following properties are satisfied.

- (a) If (K, h) in \mathcal{T} and $L \triangleleft K$, then $(L, h) \in \mathcal{T}$.
- (b) If $(K, h) \in \mathcal{T}$ and $(L, f) \in \mathcal{T}$, they have a common subdivision in \mathcal{T} .

Definition 6.3. Let (X, \mathcal{T}) and (Y, \mathcal{S}) be two PL spaces. A *PL map*, $\psi: (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$, is a continuous map, $\psi: X \rightarrow Y$, such that for any $(K, h) \in \mathcal{T}$ and any $(L, f) \in \mathcal{S}$, there is a subdivision K' of K for which $j^{-1}\psi h$ takes each simplex of K' linearly *into* a simplex of L . A *PL subspace* of (X, \mathcal{T}) is a PL space (X', \mathcal{T}') , such that X' is a subspace of X and the inclusion map, $X' \hookrightarrow X$, is a PL map.

Definition 6.4. A *filtered PL space* is a PL space (X, \mathcal{T}) , filtered by a sequence of closed PL subspaces,

$$X = X_n \supsetneq X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset.$$

From [9, Subsection 2.5.2], we may suppose that there is a triangulation (K, h) of X with respect to which each of the X_i is the image under h of a subcomplex of K .

If L is a full subcomplex of a simplicial complex K and K' is a subdivision of K , we denote L' the subdivision of L induced by K' . If L is full in K , then ([17, Lemma 3.3]) L' is full in K' . Therefore the fullness property adapts to PL spaces: any filtered PL space admits an admissible full triangulation, see [9, Lemma 3.3.19].

6.1. Blown-up and subdivision. Let us connect the blown-up cochain complex of a simplicial complex and of one of its subdivisions.

Proposition 6.5. *Let K' be a subdivision of a full filtered simplicial complex K and \bar{p} be a perversity. There exist cochain maps*

$$j_{K'K}: \tilde{N}_{\bar{p}}^*(K') \rightarrow \tilde{N}_{\bar{p}}^*(K) \quad \text{and} \quad \varphi_{KK'}: \tilde{N}_{\bar{p}}^*(K) \rightarrow \tilde{N}_{\bar{p}}^*(K')$$

such that $j_{K'K} \circ \varphi_{KK'} = \text{id}$.

Proof. Let $\sigma: \Delta_\sigma \rightarrow K$ be an oriented, regular, filtered simplex of K . The filtration on K induces a decomposition in join product, $\Delta_\sigma = \Delta_{\sigma,0} * \cdots * \Delta_{\sigma,n}$, where each $\Delta_{\sigma,i} \setminus \Delta_{\sigma,i-1}$ is included in a stratum S_i . The simplex σ is subdivided in some oriented simplices $(\sigma\{\ell\})_{1 \leq \ell \leq k}$ of K' , of equal dimension. Let $\ell \in \{1, \dots, k\}$. We can suppose that the orientations of σ and $\sigma\{\ell\}$ are compatible. Each simplex of the subdivision is obtained by adding new vertices to a subset of $\text{Vert}(K)$. These new vertices belongs to a stratum and we can write $\Delta_{\sigma\{\ell\}}$ as a join product $\Delta_{\sigma\{\ell\}} = \Delta_{\sigma\{\ell\},0} * \cdots * \Delta_{\sigma\{\ell\},n}$.

- *Construction of $j_{K'K}$.* For any ε as in Definition 5.3 and any $\ell \in \{1, \dots, k\}$, we set

$$j \left(\mathbf{1}_{(\Delta_{\sigma\{\ell\}}, \varepsilon)} \right) = \mathbf{1}_{(\Delta_\sigma, \varepsilon)}.$$

We get a cochain map $j: \tilde{N}^*(K') \rightarrow \tilde{N}^*(K)$. Concerning the perverse degrees, let us notice that $\dim(\Delta_{\sigma\{\ell\}} \cap S) \leq \dim(\Delta_\sigma \cap S)$. Thus, we have $\|\mathbf{1}_{(\Delta_\sigma, \varepsilon)}\| \leq \|\mathbf{1}_{(\Delta_{\sigma\{\ell\}}, \varepsilon)}\|$ for any $\ell \in \{1, \dots, k\}$. Defined on a basis, the association j extends linearly in a map

$$j_{K'K}: \tilde{N}_{\bar{p}}^*(K') \rightarrow \tilde{N}_{\bar{p}}^*(K).$$

On each factor of the join product, this map is the transposed map of the subdivision map. From [9, Lemmas 3.3.1, 3.3.15], it follows that $j_{K'K}$ is a chain map.

- *Construction of $\varphi_{KK'}$.* We use a simplicial map, $\nu: K' \rightarrow K$, built by Goresky and MacPherson in [16, Appendix]. Let us recall their construction, using [9, Subsection 3.3.4], assuming

that K is full and that the set $\text{Vert}(K)$ is well ordered. The map ν is defined from a map $\bar{\nu}: \text{Vert}(K') \rightarrow \text{Vert}(K)$ between the set of vertices of K and K' , that is extended linearly on simplices.

If the vertex $v' \in K'$ is already in K , we set $\bar{\nu}(v') = v'$. Otherwise, v' is in the interior of a simplex σ of K . Denote S the stratum of $|K|$ containing the interior of σ . One knows (see [16] or [9, Lemma 3.3.25]) that the interior of a simplex is contained in the stratum S if, and only if, all the vertices of σ are contained in the closure \bar{S} and at least one vertex of σ is in S . We define $\bar{\nu}(v')$ as the vertex of σ in S that is greatest in the selected order. As the vertices v' and $\bar{\nu}(v')$ are in the same stratum (see [9, Proof of Lemma 3.3.21]), the map $\bar{\nu}$ is compatible with the strata decomposition. Also, from the same proof, it is explicit that *only one* of the simplices $\Delta_{\sigma\{\ell\}}$ of the subdivision has an image by ν which is of the same dimension than Δ_{σ} . Let us denote it $\Delta_{\sigma\{\nu\}}$ and set

$$\varphi_{KK'}(\mathbf{1}_{(\Delta_{\sigma,\varepsilon})}) = \mathbf{1}_{(\Delta_{\sigma\{\nu\},\varepsilon})}.$$

As $\nu: K' \rightarrow K$ is a simplicial map, compatible with the strata, we have $\|\mathbf{1}_{(\Delta_{\sigma,\varepsilon})}\| = \|\mathbf{1}_{(\Delta_{\sigma\{\nu\},\varepsilon})}\|$ and a map

$$\varphi_{KK'}: \tilde{N}_{\bar{\nu}}^*(K) \rightarrow \tilde{N}_{\bar{\nu}}^*(K').$$

As the association $\bar{\nu}$ gives a chain map (see [16, Appendix] or [9, Lemma 3.3.21]), by duality, $\varphi_{KK'}$ is a chain map. The equality $j_{K'K} \circ \varphi_{KK'} = \text{id}$ follows directly from the definitions of the two maps. \clubsuit

6.2. PL blown-up cohomology and simplicial cohomology. Let (X, \mathcal{T}) be a PL filtered space and K be any triangulation of \mathcal{T} . We denote $\text{sd } K$ the barycentric subdivision of K and $(\text{sd}^i K)_{i \in \mathbb{N}}$ the family of iterated barycentric subdivisions, with the convention $\text{sd}^0 = \text{id}$. The subdivision maps give a direct system $\text{Sub}_i: \text{sd}^{i+1} K \rightarrow \text{sd}^i K$, for $i \in \mathbb{N}$. Let \bar{p} be a perversity on X . In [9, Lemma 5.4.1], Friedman proves that the PL homology is obtained as the homology of the inductive limit of this direct system, $H_{*,\text{PL}}^{\bar{p}}(X) \cong H_*(\varinjlim_i C_*^{\bar{p}}(\text{sd}^i K))$.

Denote $j_i: \tilde{N}_{\bar{p}}^*(\text{sd}^i K) \rightarrow \tilde{N}_{\bar{p}}^*(\text{sd}^{i-1} K)$ and $\varphi_i: \tilde{N}_{\bar{p}}^*(K) \rightarrow \tilde{N}_{\bar{p}}^*(\text{sd}^i K)$ from the maps of Proposition 6.5, for $i \in \mathbb{N}$. More specifically, we set $j_i = j_{\text{sd}^i K \text{ sd}^{i-1} K}$ and, by induction, $\varphi_i = \varphi_{\text{sd}^{i-1} K \text{ sd}^i K} \circ \varphi_{i-1}$ and $\varphi_0 = \text{id}$. By construction and Proposition 6.5, it follows $j_i \circ \varphi_i = \varphi_{i-1}$. The maps j_i define a projective system which allows the following definition.

Definition 6.6. Let (X, \mathcal{T}) be a PL filtered space and \bar{p} be a perversity on X . The complex of \bar{p} -intersection PL cochains is the inverse (projective) limit,

$$\tilde{N}_{\bar{p}}^{*,\text{PL}}(X) = \varprojlim_i \tilde{N}_{\bar{p}}^*(\text{sd}^i K).$$

We denote $\mathcal{H}_{\bar{p}}^{*,\text{PL}}(X)$ the corresponding cohomology, and call it the PL *blown-up \bar{p} -intersection cohomology*.

From Lemma 1.3, we can suppose that K is full. Let's now prove that the PL blown-up \bar{p} -intersection cohomology is isomorphic to the blown-up singular and the blown-up simplicial ones, and thus is independent of the choice of K .

Theorem 6.7. *Let (X, \mathcal{T}) be a PL filtered space and \bar{p} be a perversity on X . Then for a pure $K \in \mathcal{T}$, there are isomorphisms,*

$$\mathcal{H}_{\bar{p}}^{*,\text{PL}}(X) \cong \mathcal{H}_{\bar{p}}^*(K) \cong \mathcal{H}_{\bar{p}}^*(X) \text{ and } H_{*,\text{PL}}^{\bar{p}}(X) \cong H_*^{\bar{p}}(K) \cong H_*^{\bar{p}}(X).$$

The last part recovers [9, Theorem 5.4.2] without the hypothesis of CS set structure.

Proof. Let $j_{\leq i}$ be the map defined by $j_{\leq 1} = j_1$ and $j_{\leq i} = j_{\leq i-1} \circ j_i$. We also denote $p_i: \tilde{N}_{\bar{p}}^{*,\text{PL}}(X) \rightarrow \tilde{N}_{\bar{p}}^*(\text{sd}^i(K))$ the projection given by the projective limit.

$$\begin{array}{ccccc}
 & & \tilde{N}_{\bar{p}}^*(K) & & \\
 & \nearrow p_0 & \uparrow j_{\leq i-1} & \searrow \text{id} & \\
 \tilde{N}_{\bar{p}}^{*,\text{PL}}(X) & \xrightarrow{p_{i-1}} & \tilde{N}_{\bar{p}}^*(\text{sd}^{i-1}K) & \xleftarrow{\varphi_{i-1}} & \tilde{N}_{\bar{p}}^*(K) \\
 & \searrow p_i & \uparrow j_i & \swarrow \varphi_i & \\
 & & \tilde{N}_{\bar{p}}^*(\text{sd}^iK) & & \\
 & & \Psi & &
 \end{array} \tag{31}$$

From the equalities $j_i \circ \varphi_i = \varphi_{i-1}$ and the universal property of inverse limits, we get a cochain map $\Psi: \tilde{N}_{\bar{p}}^*(K) \rightarrow \tilde{N}_{\bar{p}}^{*,\text{PL}}(X)$ such that $p_0 \circ \Psi = \text{id}$ and $p_i \circ \Psi = \varphi_i$ for any $i \in \mathbb{N}$. From Proposition 6.5 applied to $\text{sd}^i K$ and $\text{sd}^{i-1} K$, we deduce that each j_i and each $H_*(j_i)$ is surjective. Thus [13, Proposition 13.2.3] and Theorem 5.13 give the existence of isomorphisms,

$$\mathcal{H}_{\bar{p}}^{*,\text{PL}}(X) = H_*(\varprojlim_i \tilde{N}_{\bar{p}}^*(\text{sd}^i K)) \cong \varprojlim_i \mathcal{H}_{\bar{p}}^*(\text{sd}^i K) \cong \mathcal{H}_{\bar{p}}^*(K) \cong \mathcal{H}_{\bar{p}}^*(X).$$

The existence of the isomorphisms in \bar{p} -intersection homology follows directly from Theorem 4.2 and the commutativity of homology with inductive limits. \clubsuit

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