

**27th International Congress  
on Sound and Vibration**

The annual congress of  
the International Institute  
of Acoustics and Vibration (IIAV)



11-16 July, 2021

**ICSV27**

# A low reduced frequency model for viscothermal wave propagation in conical tubes of arbitrary cross-section

Alexis Thibault, Juliette Chabassier,  
*École Normale Supérieure, Paris, France*

*MAKUTU team, Inria Bordeaux Sud-Ouest, Talence, France*

*e-mail: alexis.thibault@inria.fr*

Henri Boutin and Thomas Hélie

*STMS (UMR9912), CNRS/SU/Ircam, Paris, France*

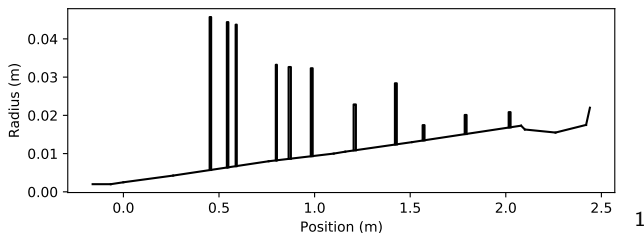
Does conicity influence boundary layer losses ?

# Context

## PhD about wind instruments

- Accurate models
- Stable and efficient numerical methods
- Applications : instrument design, preservation of ancient instruments, artistic creation. . .

→ 1D models



### 1. gam.

## Models of acoustic propagation

$\hat{U}(\ell)$  volume flow,  $\hat{p}(\ell)$  pressure.

$$Z_v(\ell)\hat{U}(\ell) + \frac{d\hat{p}}{d\ell}(\ell) = 0,$$

$$Y_t(\ell)\hat{p}(\ell) + \frac{d\hat{U}}{d\ell}(\ell) = 0,$$

## Models of acoustic propagation

$\hat{U}(\ell)$  volume flow,  $\hat{p}(\ell)$  pressure.

$$Z_v(\ell)\hat{U}(\ell) + \frac{d\hat{p}}{d\ell}(\ell) = 0,$$

$$Y_t(\ell)\hat{p}(\ell) + \frac{d\hat{U}}{d\ell}(\ell) = 0,$$

- Lossless acoustics

$$Z_v(\ell) = \frac{i\omega\rho_0}{S(\ell)},$$

$$Y_t(\ell) = \frac{i\omega S(\ell)}{\rho_0 c_0^2},$$

## Models of acoustic propagation

$\hat{U}(\ell)$  volume flow,  $\hat{p}(\ell)$  pressure.

$$Z_v(\ell)\hat{U}(\ell) + \frac{d\hat{p}}{d\ell}(\ell) = 0, \quad Y_t(\ell)\hat{p}(\ell) + \frac{d\hat{U}}{d\ell}(\ell) = 0,$$

- Lossless acoustics

$$Z_v(\ell) = \frac{i\omega\rho_0}{S(\ell)}, \quad Y_t(\ell) = \frac{i\omega S(\ell)}{\rho_0 c_0^2},$$

- Acoustics with viscous/thermal effects

$$Z_v(\ell) = \frac{1}{1 - K_v(\ell)} \frac{i\omega\rho_0}{S(\ell)}, \quad Y_t(\ell) = (1 + (\gamma - 1)K_t(\ell)) \frac{i\omega S(\ell)}{\rho_0 c_0^2},$$

with  $K_v, K_t$  known for *cylinders*.

## Models of acoustic propagation

$\hat{U}(\ell)$  volume flow,  $\hat{p}(\ell)$  pressure.

$$Z_v(\ell)\hat{U}(\ell) + \frac{d\hat{p}}{d\ell}(\ell) = 0, \quad Y_t(\ell)\hat{p}(\ell) + \frac{d\hat{U}}{d\ell}(\ell) = 0,$$

- Lossless acoustics

$$Z_v(\ell) = \frac{i\omega\rho_0}{S(\ell)}, \quad Y_t(\ell) = \frac{i\omega S(\ell)}{\rho_0 c_0^2},$$

- Acoustics with viscous/thermal effects

$$Z_v(\ell) = \frac{1}{1 - K_v(\ell)} \frac{i\omega\rho_0}{S(\ell)}, \quad Y_t(\ell) = (1 + (\gamma - 1)K_t(\ell)) \frac{i\omega S(\ell)}{\rho_0 c_0^2},$$

with  $K_v, K_t$  known for *cylinders*.

Does conicity influence boundary layer losses ?

- 1 Overview of viscothermal models
  - Base model
  - Approaches for modeling losses
- 2 Low reduced frequency approach
  - Straight tubes (Stinson)
  - Conical tubes
- 3 Results
  - Causality and passivity of the immitances
  - Comparison with other models
- 4 Conclusion and prospects



## Base model (3D)

Linearized Navier–Stokes<sup>2</sup>

$$i\omega\hat{p} + \rho_0\nabla \cdot \hat{\mathbf{u}} = 0, \quad (1a)$$

$$\rho_0 i\omega\hat{\mathbf{u}} = -\nabla\hat{P} + \mu\nabla^2\hat{\mathbf{u}} + \left(\zeta + \frac{\mu}{3}\right)\nabla(\nabla \cdot \hat{\mathbf{u}}), \quad (1b)$$

$$\rho_0 C_p i\omega\hat{T} = \kappa\nabla^2\hat{T} + i\omega\hat{P}, \quad (1c)$$

$$\frac{\hat{P}}{\rho_0} = \frac{\hat{p}}{\rho_0} + \frac{\hat{T}}{T_0}, \quad (1d)$$

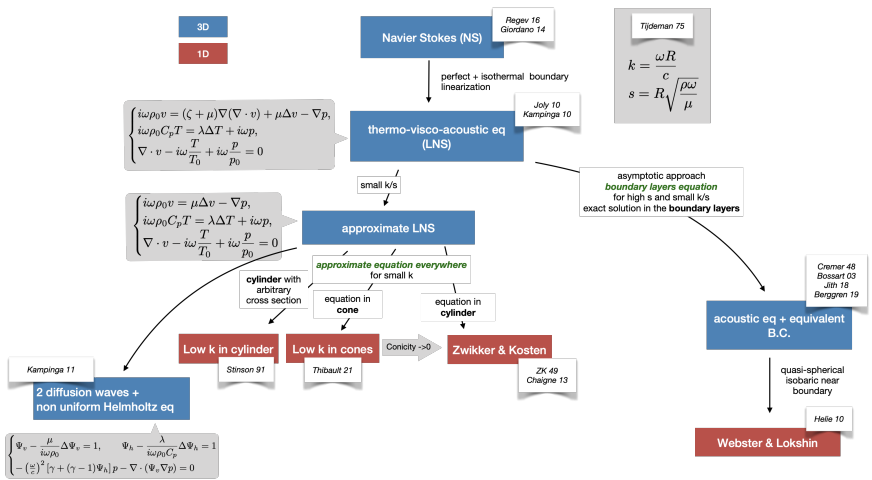
Small variations of : density  $\hat{\rho}$ , air speed  $\hat{\mathbf{u}}$ , pressure  $\hat{P}$ , temperature  $\hat{T}$ .

Air at rest :  $(\rho_0, \mathbf{u}_0 = 0, p_0, T_0)$ .

On the wall :

$$\hat{\mathbf{u}} = 0 \quad \text{and} \quad \hat{T} = 0. \quad (2)$$

3D  
1D



### 3. thibault2020viscothermal.

- 1 Overview of viscothermal models
  - Base model
  - Approaches for modeling losses
- 2 Low reduced frequency approach
  - Straight tubes (Stinson)
  - Conical tubes
- 3 Results
  - Causality and passivity of the immitances
  - Comparison with other models
- 4 Conclusion and prospects

## Simplifying assumptions<sup>4</sup>

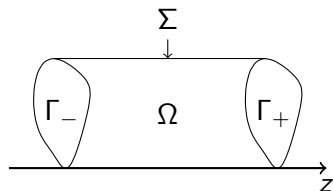


Figure – Straight tube

(A1) Planar pressure waves

$$\hat{P}(\mathbf{x}) = \hat{p}(z)$$

(A2) Longitudinal diffusion is neglected

$$\lambda \gg \delta$$

(A3) Losses due to dilation are neglected

$$\left| \left( \zeta + \frac{\mu}{3} \right) \nabla(\nabla \cdot \hat{\mathbf{u}}) \right| \ll |\nabla \hat{p}|.$$

## Expressing $v$ and $T$ as functions of $p$

Solutions to a diffusion equation with oscillating source :

$$\begin{cases} \psi(\alpha, \mathbf{x}) + \frac{R^2}{\alpha} \nabla_c^2 \psi(\alpha, \mathbf{x}) = 1 & \text{inside the domain } \mathbf{x} \in \Omega, \\ \psi(\alpha, \mathbf{x}) = 0 & \text{on the wall } \mathbf{x} \in \Sigma, \end{cases} \quad (3)$$

where  $\nabla_c^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$  Laplacian on cross-section.

## Expressing $v$ and $T$ as functions of $p$

Solutions to a diffusion equation with oscillating source :

$$\begin{cases} \psi(\alpha, \mathbf{x}) + \frac{R^2}{\alpha} \nabla_c^2 \psi(\alpha, \mathbf{x}) = 1 & \text{inside the domain } \mathbf{x} \in \Omega, \\ \psi(\alpha, \mathbf{x}) = 0 & \text{on the wall } \mathbf{x} \in \Sigma, \end{cases} \quad (3)$$

where  $\nabla_c^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$  Laplacian on cross-section.

Viscous and thermal effects use respectively

$$\alpha_v = -is^2 = -R^2 \frac{\rho_0 i \omega}{\mu} \quad \alpha_t = -i\sigma^2 s^2 = -R^2 \frac{\rho_0 i \omega C_p}{\kappa}. \quad (4)$$

## Expressing $v$ and $T$ as functions of $p$

Solutions to a diffusion equation with oscillating source :

$$\begin{cases} \psi(\alpha, \mathbf{x}) + \frac{R^2}{\alpha} \nabla_c^2 \psi(\alpha, \mathbf{x}) = 1 & \text{inside the domain } \mathbf{x} \in \Omega, \\ \psi(\alpha, \mathbf{x}) = 0 & \text{on the wall } \mathbf{x} \in \Sigma, \end{cases} \quad (3)$$

where  $\nabla_c^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$  Laplacian on cross-section.

Viscous and thermal effects use respectively

$$\alpha_v = -is^2 = -R^2 \frac{\rho_0 i \omega}{\mu} \quad \alpha_t = -i\sigma^2 s^2 = -R^2 \frac{\rho_0 i \omega C_p}{\kappa}. \quad (4)$$

Separation of variables :

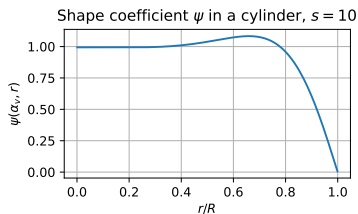
$$\hat{v}(x, y, z) = -\frac{1}{\rho_0 i \omega} \frac{d\hat{p}}{dl}(z) \psi(\alpha_v, x, y),$$
$$\hat{T}(x, y, z) = \frac{1}{\rho_0 C_p} \hat{p}(z) \psi(\alpha_t, x, y).$$

## Case of cylinders

In a circular tube of radius  $R$  the solution is

$$\psi(\alpha, r) = 1 - \frac{J_0(\sqrt{\alpha}r/R)}{J_0(\sqrt{\alpha})}$$

where  $\alpha_v = -is^2 = -R^2 \rho_0 i \omega / \mu$ .





## Case of cylinders

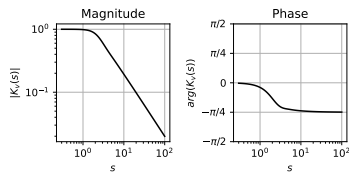
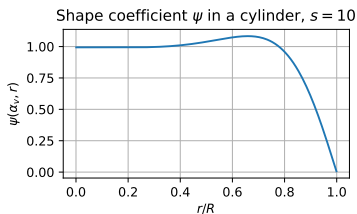
In a circular tube of radius  $R$  the solution is

$$\psi(\alpha, r) = 1 - \frac{J_0(\sqrt{\alpha}r/R)}{J_0(\sqrt{\alpha})}$$

where  $\alpha_v = -is^2 = -R^2 \rho_0 i \omega / \mu$ .

Loss coefficients  $K_v, K_t$  are obtained by averaging on a disk :

$$K_{v,t} = 1 - \langle \psi(\alpha_{v,t}) \rangle = \frac{2J_1(\sqrt{\alpha_{v,t}})}{\sqrt{\alpha_{v,t}} J_0(\sqrt{\alpha_{v,t}})}$$



- 1 Overview of viscothermal models
  - Base model
  - Approaches for modeling losses
- 2 Low reduced frequency approach
  - Straight tubes (Stinson)
  - **Conical tubes**
- 3 Results
  - Causality and passivity of the immitances
  - Comparison with other models
- 4 Conclusion and prospects

## Generalization to cones

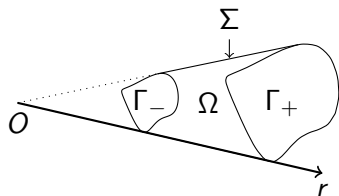


Figure – Conical tube

(A1) Spherical pressure waves

$$\hat{P}(\mathbf{x}) = \hat{p}(r)$$

(A2) Longitudinal diffusion is neglected

$$\nabla^2 - \frac{\partial^2}{\partial r^2} \approx \nabla^2$$

(A3) Losses due to dilation are neglected

$$\left| \left( \zeta + \frac{\mu}{3} \right) \nabla(\nabla \cdot \hat{\mathbf{u}}) \right| \ll |\nabla \hat{p}|.$$

## Generalization to cones

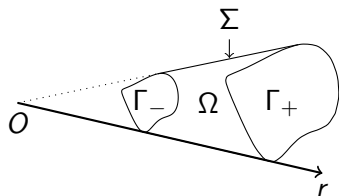


Figure – Conical tube

(A1) Spherical pressure waves

$$\hat{P}(\mathbf{x}) = \hat{p}(r)$$

(A2) Longitudinal diffusion is neglected

$$\nabla^2 - \frac{\partial^2}{\partial r^2} \approx \nabla^2$$

(A3) Losses due to dilation are neglected

$$\left| \left( \zeta + \frac{\mu}{3} \right) \nabla(\nabla \cdot \hat{\mathbf{u}}) \right| \ll |\nabla \hat{p}|.$$

Calculate  $\psi$  on a spherical cap, and average to obtain  $K_{v,t} = 1 - \langle \psi_{v,t} \rangle$ .

Express  $v$  and  $T$  as functions of  $p$ .

The shape function is now defined on a spherical cap. For circular cones of half-angle  $\Theta$  :

$$\begin{cases} \psi(\alpha, r, \theta) + \frac{R^2}{\alpha r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi(\alpha, r, \theta)}{\partial \theta} \right) = 1 & \text{for } 0 \leq \theta \leq \Theta, \\ \psi(\alpha, r, \Theta) = 0. \end{cases}$$

## Express $v$ and $T$ as functions of $p$ .

The shape function is now defined on a spherical cap. For circular cones of half-angle  $\Theta$  :

$$\begin{cases} \psi(\alpha, r, \theta) + \frac{R^2}{\alpha r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi(\alpha, r, \theta)}{\partial \theta} \right) = 1 & \text{for } 0 \leq \theta \leq \Theta, \\ \psi(\alpha, r, \Theta) = 0. \end{cases}$$

Equation of spherical harmonics. The solution uses Legendre function  $P_\eta$  of complex order :

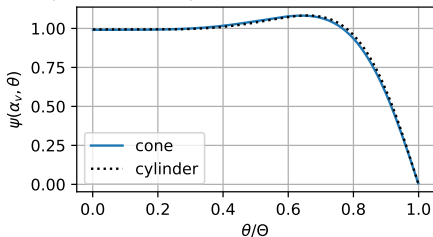
$$\psi(\theta) = 1 - \frac{P_\eta(\cos \theta)}{P_\eta(\cos \Theta)}.$$

where

$$\eta_v(\eta_v + 1) = -ir^2 \frac{\rho_0 \omega}{\mu},$$

$$\text{and } \eta_t(\eta_t + 1) = -ir^2 \frac{\rho_0 \omega C_p}{\kappa}.$$

Shape coefficient  $\psi$  in a cone,  $\Theta = \pi/4$ ,  $s = 10$



## Complex coefficients for the cone

By averaging on the spherical cap we obtain :

$$K_{v,t,\Theta}(\ell, \omega) = \frac{\sin^2 \Theta}{1 - \cos \Theta} \frac{P'_{\eta_{v,t}}(\cos \Theta)}{\eta_{v,t}(\eta_{v,t} + 1)P_{\eta_{v,t}}(\cos \Theta)}, \quad (5)$$

## Complex coefficients for the cone

By averaging on the spherical cap we obtain :

$$K_{v,t,\Theta}(\ell, \omega) = \frac{\sin^2 \Theta}{1 - \cos \Theta} \frac{P'_{\eta_{v,t}}(\cos \Theta)}{\eta_{v,t}(\eta_{v,t} + 1)P_{\eta_{v,t}}(\cos \Theta)}, \quad (5)$$

Stokes number

$$s = R \sqrt{\frac{\omega \rho_0}{\mu}}$$



## Complex coefficients for the cone

By averaging on the spherical cap we obtain :

$$K_{v,t,\Theta}(\ell, \omega) = \frac{\sin^2 \Theta}{1 - \cos \Theta} \frac{P'_{\eta_{v,t}}(\cos \Theta)}{\eta_{v,t}(\eta_{v,t} + 1)P_{\eta_{v,t}}(\cos \Theta)}, \quad (5)$$

Stokes number

$$s = R \sqrt{\frac{\omega \rho_0}{\mu}}$$

Usual choice : radius of the circle at distance  $\ell$  from the apex.

$$R_{circle} = \ell \sin \Theta$$

## Complex coefficients for the cone

By averaging on the spherical cap we obtain :

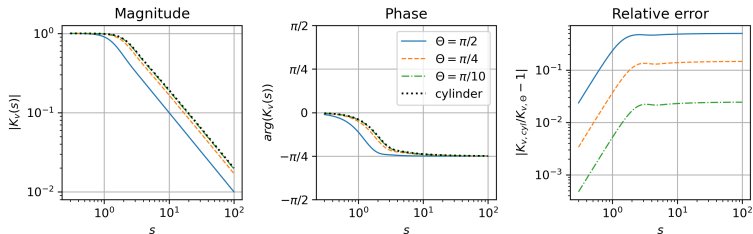
$$K_{v,t,\Theta}(\ell, \omega) = \frac{\sin^2 \Theta}{1 - \cos \Theta} \frac{P'_{\eta_{v,t}}(\cos \Theta)}{\eta_{v,t}(\eta_{v,t} + 1)P_{\eta_{v,t}}(\cos \Theta)}, \quad (5)$$

Stokes number

$$s = R \sqrt{\frac{\omega \rho_0}{\mu}}$$

Usual choice : radius of the circle at distance  $\ell$  from the apex.

$$R_{circle} = \ell \sin \Theta$$



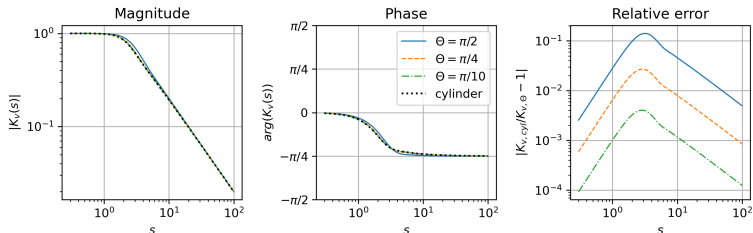
## High frequency correction : hydraulic radius

$|\omega| \rightarrow \infty \implies$  losses depend on surface-to-volume ratio<sup>5</sup>.

## High frequency correction : hydraulic radius

$|\omega| \rightarrow \infty \implies$  losses depend on surface-to-volume ratio<sup>5</sup>.  
Using the hydraulic radius corrects this effect.

$$R_{hydrau} = \frac{2 \text{ area}}{\text{perimeter}} = \ell \frac{2(1 - \cos \Theta)}{\sin \Theta}$$



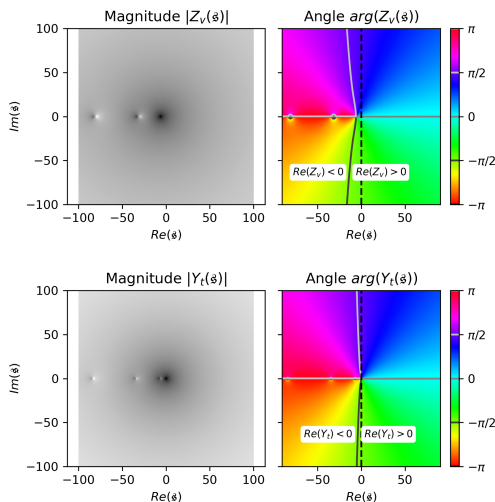
### 5. allard2009propagation.

# Causality and passivity of the immittances

In the complex plane, with Laplace parameter  $\mathfrak{s} = i\omega \in \mathbb{C}$ , operators  $Z_V$  and  $Y_t$  are *positive real* :

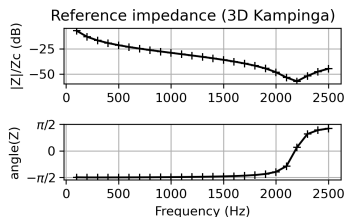
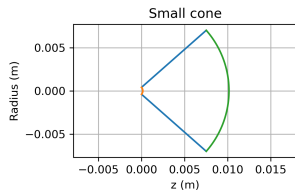
- $\operatorname{Re}(Z_V(\mathfrak{s})) \geq 0$  for  $\operatorname{Re}(\mathfrak{s}) \geq 0$ .
  - $Z_V(\bar{\mathfrak{s}}) = \overline{Z_V(\mathfrak{s})}$
- Causal and passive operators.

Cone,  $\Theta = \pi/4$



# Comparison with other models

Comparison on short, closed cone  
(small Stokes number).



- 1 Reference : 3D Kampinga <sup>6</sup>
- 2 Zwicker-Kosten <sup>7</sup> (with length correction)
- 3 Proposed model :  $K_{V,t}$  of spherical caps

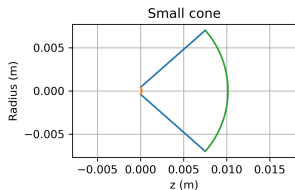
---

6. [kampinga2011efficient](#).

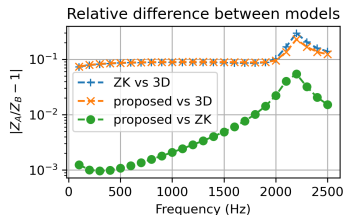
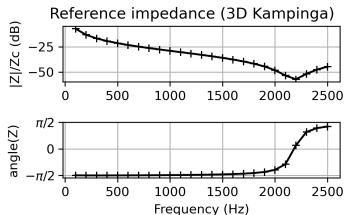
7. [chaigne2016acoustics](#).

# Comparison with other models

Comparison on short, closed cone (small Stokes number).



- 1 Reference : 3D Kampinga<sup>6</sup>
- 2 Zwicker-Kosten<sup>7</sup> (with length correction)
- 3 Proposed model :  $K_{V,t}$  of spherical caps



- 
6. [kampinga2011efficient](#).
  7. [chaigne2016acoustics](#).

# Conclusion and prospects

## Does conicity influence acoustic losses ?

- Generalization of Zwicker–Kosten model to cones
- Formula for  $K_v, K_t$  parametrized by  $\Theta$
- Similar coefficients, especially when using hydraulic radius
- Passive operators
- Correction amplitude 10 to 100 times smaller than difference with 3D

## Possible further research

- More numerical comparisons
- Influence of wall porosity/roughness