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A Solution of the 4th Clay Millennium Problem about the Navier-Stokes Equations

Konstantinos E. Kyritsis

AbstractIn this paper it is solved the 4th Clay Millennium problem about the Navier-Stokes equations, in the direction of regularity. It is done so by utilizing the hypothesis of finite initial energy The final key result to derive the regularity is that the pressures are bounded in finite time intervals, as proved after projecting the bounded by the conservation of energy, virtual work of the pressures forces on specially chosen bundles of instantaneous paths. It is proved that not only there is no Blow-up in finite time but not even at the time $T=+\infty$.

Index Terms—Incompressible flows, regularity, Navier-Stokes equations, 4th Clay millennium problem.
Mathematical Subject Classification: 76A02

I. INTRODUCTION

The famous problem of the 4th Clay Mathematical Institute as formulated in [19] FEFFERMAN C. L. 2006 CL 2006 is considered a significant challenge to the science of mathematical physics of fluids, not only because it has lasted the efforts of the scientific community for decades to prove it (or converses to it) but also because it is supposed to hide a significant missing perception about the nature of our mathematical formulations of physical flows through the Euler and Navier-Stokes equations.

When the 4th Clay Millennium Problem was formulated in the standard way, the majority was hoping that the regularity was also valid in 3 dimensions as it had been proven to hold in 2 dimensions.

The main objective of this paper is to prove the regularity of the Navier-Stokes equations with initial data as in the standard formulation of the 4th Clay Millennium Problem. (see PROPOSITION 5.2 (The solution of the 4th Clay Millennium problem).It is proved that not only there is no Blow-up in finite time but not even at the time $T=+\infty$.

The problem was solved in its present form during the spring 2017 and was uploaded as a preprint in February 2018 (see [28] KYRITSIS. K. Feb 2018).

The main core of the solution is the paragraphs 4, a new sufficient conditions of regularity is proved based on the pressures and paragraph 5, where it is proven that the pressures are bounded in finite time intervals, which leads after the previous sufficient conditions to the proof of the regularity of the Navier-Stokes equations. The paragraph 2 is devoted to reviewing the standard formulation of the 4th Clay Millennium problem, while the paragraph 3 is devoted in to collecting some well-known results that are good for the reader to have readily available to follow the later arguments.

According to[8] CONSTANTIN P. 2007 “..The blow-up problem for the Euler equations is a major open problem of PDE, theory of far greater physical importance that the blow-up problem of the Navier-Stokes equation, which is of course known to non-specialists because of the Clay Millennium problem...” For this reason, many of the propositions of this paper are stated for the Euler equations of inviscid flows as well.

II. THE STANDARD FORMULATION OF THE CLAY MATHEMATICAL INSTITUTE 4TH CLAY MILLENNIUM CONJECTURE OF 3D REGULARITY AND SOME DEFINITIONS.

In this paragraph we highlight the basic parts of the standard formulation of the 4th Clay millennium problem, together with some more modern, since 2006, symbolism, by relevant researchers, like T. Tao.

In this paper I consider the conjecture (A) of C. L. [19] FEFFERMAN 2006 standard formulation of the 4th Clay millennium problem , which I identify throughout the paper as the 4th Clay millennium problem.

The Navier-Stokes equations are given by (by R we denote the field of the real numbers, $\nu > 0$ is the density normalized viscosity coefficient)

$$\frac{\partial}{\partial t} u_i + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i \quad (x \in \mathbb{R}^3, t \geq 0, n=3) \quad (\text{eq.1})$$

$$\text{div} u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0 \quad (x \in \mathbb{R}^3, t \geq 0, n=3) \quad (\text{eq.2})$$

with initial conditions $u(x,0)=u^0(x) \quad x \in \mathbb{R}^3$ and $u^0(x) \in C^\infty$ divergence-free vector field on \mathbb{R}^3 (eq.3)

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \quad \text{is the Laplacian operator. The Euler equations are (eq1), (eq2), (eq3) when } \nu=0.$$

It is reminded to the reader, that in the equations of Navier-Stokes, as in (eq. 1) the as the density, is constant, it is custom to either normalised to 1, or it is divided out from the left side and it is included in the pressures and viscosity coefficient.

For physically meaningful solutions we want to make sure that $u^0(x)$ does not grow large as $|x| \rightarrow \infty$. This is set

by defining $u^0(x)$ and called in this paper **Schwartz initial conditions**, in other words

$$|\partial_x^\alpha u^0(x)| \leq C_{\alpha,K} (1+|x|)^{-K} \quad \text{on } \mathbf{R}^3 \text{ for any } \alpha \text{ and } K \quad \dots$$

(eq.4)

(Schwartz used such functions to define the space of Schwartz distributions)

We accept as physical meaningful solutions only if it satisfies

$$p, u \in C^\infty(\mathbf{R}^3 \times [0, \infty))$$

(eq.5)

and

$$\int_{\mathbf{R}^3} |u(x,t)| dx < C \quad \text{for all } t \geq 0 \quad \text{(Bounded or finite energy)} \quad \dots$$

(eq.6)

The conjecture (A) of the Clay Millennium problem (case of no external force, but homogeneous and regular velocities) claims that for the Navier-Stokes equations, $v > 0, n=3$, with divergence free, Schwartz initial velocities, there are for all times $t > 0$, smooth velocity field and pressure, that are solutions of the Navier-Stokes equations with bounded energy, **in other words satisfying the equations eq.1, eq.2, eq. 3, eq.4, eq.5, eq.6**. It is stated in the same formal formulation of the Clay millennium problem by C. L. Fefferman see C. L. [19] FEFERMAN 2006 (see page 2nd line 5 from below) that the conjecture (A) has been proved to hold locally. "...if the time interval $[0, \infty)$, is replaced by a small time interval $[0, T)$, with T depending on the initial data...". In other words there is $\infty > T > 0$, such that there is continuous and smooth solution $u(x,t) \in C^\infty(\mathbf{R}^3 \times [0, T))$. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^∞

Following [45] TAO, T 2013, we define some specific terminology, about the hypotheses of the Clay millennium problem, that **will not be used** in the next in the main solution of the 4th Clay Millennium problem, but we include it just for the sake of the state of the art that TAO, T, in 2013 (see references [45]) has created in studying this 4th Clay Millennium problem. For more details about the involved functional analysis norms, the reader should look in the above paper [45] TAO, T, in 2013 in the references.

We must notice that the definitions below can apply also to the case of inviscid flows, satisfying the **Euler** equations.

DEFINITION 2.1 (Smooth solutions to the Navier-Stokes system). A smooth set of data for the Navier-Stokes system up to time T is a triplet (u_0, f, T) , where $0 < T < \infty$ is a time, the initial velocity vector field $u_0 : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ and the forcing term $f : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ are assumed to be smooth on \mathbf{R}^3 and $[0, T] \times \mathbf{R}^3$ respectively (thus, u_0 is infinitely differentiable in space, and f is infinitely differentiable in space time), and u_0 is furthermore required to be divergence-free:

$$\nabla \cdot u_0 = 0.$$

If $f = 0$, we say that the data is *homogeneous*.

In the proofs of the main conjecture we will not consider any external force, thus the data will always be homogeneous. But we will state intermediate propositions with external forcing. Next we are defining simple

differentiability of the data by Sobolev spaces.

DEFINITION 2.2 We define the H^1 norm (or enstrophy norm) $H^1(u_0, f, T)$ of the data to be the quantity

$$H^1(u_0, f, T) := \|u_0\|_{H_x^1(\mathbf{R}^3)} + \|f\|_{L_t^\infty H_x^1(\mathbf{R}^3)} < \infty \quad \text{and say that } (u_0, f, T) \text{ is } H^1 \text{ if}$$

$$H^1(u_0, f, T) < \infty.$$

DEFINITION 2.3 We say that a smooth set of data (u_0, f, T) is *Schwartz* if, for all integers $\alpha, m, k \geq 0$, one has

$$\sup_{x \in \mathbf{R}^3} (1+|x|)^k |\nabla_x^\alpha u_0(x)| < \infty$$

$$\text{and } \sup_{(t,x) \in [0,T] \times \mathbf{R}^3} (1+|x|)^k |\nabla_x^\alpha \partial_t^m f(x)| < \infty$$

Thus, for instance, the solution or initial data having Schwartz property implies having the H^1 property.

DEFINITION 2.4 A smooth solution to the Navier-Stokes system, or a smooth solution for short, is a quintuplet (u, p, u_0, f, T) , where (u_0, f, T) is a smooth set of data, and the velocity vector field $u : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ and pressure field $p : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}$ are smooth functions on $[0, T] \times \mathbf{R}^3$ that obey the Navier-Stokes equation (eq. 1) but with external forcing term f ,

$$\frac{\partial}{\partial t} u_i + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i + f_i \quad (x \in \mathbf{R}^3, t \geq 0, n=3)$$

and also the incompressibility property (eq.2) on all of $[0, T] \times \mathbf{R}^3$, but also the initial condition $u(0, x) = u_0(x)$ for all $x \in \mathbf{R}^3$

DEFINITION 2.5 Similarly, we say that (u, p, u_0, f, T) is H^1 if the associated data (u_0, f, T) is H^1 , and in addition one has

$$\|u\|_{L_t^\infty H_x^1([0,T] \times \mathbf{R}^3)} + \|u\|_{L_t^2 H_x^2([0,T] \times \mathbf{R}^3)} < \infty$$

We say that the solution is *incomplete* in $[0, T)$, if it is defined only in $[0, t]$ for every $t < T$.

We use here the notation of *mixed norms* (as e.g. in TAO, T 2013). That is if $\|u\|_{H_x^k(\Omega)}$ is the classical Sobolev norm of smooth function of a spatial domain Ω , $u : \Omega \rightarrow \mathbf{R}$, I is a time interval and $\|u\|_{L^p(I)}$ is the classical L^p -norm, then the mixed norm is defined by

$$\|u\|_{L_t^p H_x^k(I \times \Omega)} := \left(\int_I \|u(t)\|_{H_x^k(\Omega)}^p dt \right)^{1/p}$$

and

$$\|u\|_{L_t^\infty H_x^k(I \times \Omega)} := \text{ess sup}_{t \in I} \|u(t)\|_{H_x^k(\Omega)}$$

Similar instead of the Sobolev norm for other norms of function spaces.

We also denote by $C_x^k(\Omega)$, for any natural number

$k \geq 0$, the space of all k times continuously differentiable functions $u : \Omega \rightarrow R$, with finite the next norm

$$\|u\|_{C_x^k(\Omega)} := \sum_{j=0}^k \|\nabla^j u\|_{L_x^\infty(\Omega)}$$

We use also the next notation for *hybrid norms*. Given two normed spaces X, Y on the same domain (in either space or time), we endow their intersection $X \cap Y$ with the norm

$$\|u\|_{X \cap Y} := \|u\|_X + \|u\|_Y.$$

In particular in the we will use the next notation for intersection functions spaces, and their hybrid norms.

$$X^k(I \times \Omega) := L_t^\infty H_x^k(I \times \Omega) \cap L_x^2 H_x^{k+1}(I \times \Omega).$$

We also use the *big O notation*, in the standard way, that is $X=O(Y)$ means

$X \leq CY$ for some constant C . If the constant C depends on a parameter s , we denote it by C_s and we write $X=O_s(Y)$.

We denote the difference of two sets A, B by $A \setminus B$. And we denote Euclidean balls

by $B(a, r) := \{x \in R^3 : |x - a| \leq r\}$, where $|x|$ is the Euclidean norm.

With the above terminology the target Clay millennium conjecture in this paper can be restated as the next proposition

The 4th Clay millennium problem (Conjecture A)

(Global regularity for homogeneous Schwartz data).

Let $(u_0, 0, T)$ be a homogeneous Schwartz set of data. Then there exists a smooth finite energy solution $(u, p, u_0, 0, T)$ with the indicated data (notice it is for any $T > 0$, thus global in time).

III. SOME KNOWN OR DIRECTLY DERIVABLE, USEFUL RESULTS THAT WILL BE USED.

In this paragraph I state, some known theorems and results, that are to be used in this paper, or is convenient for the reader to know, so that the reader is not searching them in the literature and can have a direct, at a glance, image of what already holds and what is proved.

A review of this paragraph is as follows:

Propositions 3.1, 3.2 are mainly about the uniqueness and existence locally of smooth solutions of the Navier-Stokes and Euler equations with smooth Schwartz initial data. Proposition 3.3 are necessary or sufficient or necessary and sufficient conditions of regularity (global in time smoothness) for the Euler equations without viscosity. Equations 8-13 are forms of the energy conservation and finiteness of the energy loss in viscosity or energy dissipation. Equations 14-16 relate quantities for the conditions of regularity. Proposition 3.4 is the equivalence of smooth Schwartz initial data with smooth compact support initial data for the formulation of the 4th Clay millennium problem. Propositions 3.5-3.9 are necessary and sufficient conditions for regularity, either for the Euler or Navier-Stokes equations, while Propositions 4.10 is a necessary and sufficient condition of regularity for only the

Navier-Stokes with non-zero viscosity.

In the next I want to use, the basic local existence and uniqueness of smooth solutions to the Navier-Stokes (and Euler) equations, that is usually referred also as the well-posedness, as it corresponds to the existence and uniqueness of the physical reality causality of the flow. The theory of well-posedness for smooth solutions is summarized in an adequate form for this paper by the Theorem 5.4 in [45]TAO, T. 2013.

I give first the definition of **mild solution** as in [45] TAO, T. 2013 page 9. Mild solutions must satisfy a condition on the pressure given by the velocities. Solutions of smooth initial Schwartz data are always mild, but the concept of mild solutions is a generalization to apply for non-fast decaying in space initial data, as the Schwartz data, but for which data we may want also to have local existence and uniqueness of solutions.

DEFINITION 3.1

We define a H^1 mild solution (u, p, u_0, f, T) to be fields $u, f : [0, T] \times R^3 \rightarrow R^3$,

$p : [0, T] \times R^3 \rightarrow R, u_0 : R^3 \rightarrow R^3$, with $0 < T < \infty$, obeying the regularity hypotheses

$$u_0 \in H_x^1(R^3)$$

$$f \in L_t^\infty H_x^1([0, T] \times R^3)$$

$$u \in L_t^\infty H_x^1 \cap L_t^2 H_x^2([0, T] \times R^3)$$

with the pressure p being given by (Poisson)

$$p = -\Delta^{-1} \partial_i \partial_j (u_i u_j) + \Delta^{-1} \nabla \cdot f \tag{eq. 7}$$

(Here the summation conventions is used, to not write the Greek big Sigma).

which obey the incompressibility conditions (eq. 2), (eq. 3) and satisfy the integral form of the Navier-Stokes equations

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-t')\Delta} (-(u \cdot \nabla)u - \nabla p + f)(t') dt'$$

with initial conditions $u(x,0)=u^0(x)$.

We notice that the definition holds also for the in viscid flows, satisfying the Euler equations. The viscosity coefficient here has been normalized to $\nu=1$.

In reviewing the local well-posedness theory of H^1 mild solutions, the next can be said. The content of the theorem 5.4 in [45] TAO, T. 2013 (that I also state here for the convenience of the reader and from which derive our PROPOSITION 3.2) is largely standard (and in many cases it has been improved by more powerful current well-posedness theory). I mention here for example the relevant research by [38] PRODIG 1959 and [42] SERRIN, J 1963, The local existence theory follows from the work of [24] KATO, T. PONCE, G. 1988, the regularity of mild solutions follows from the work of [30] LADYZHENSKAYA, O. A. 1967. There are now a number of advanced local well-posedness results at regularity, especially that of [25] KOCH, H., TATARU, D.2001.

There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions with

different methods. As it is referred in C. L. [19] FEFFERMAN 2006 I refer too the reader to the [34] MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4,

I state here for the convenience of the reader the summarizing theorem 5.4 as in TAO T. 2013. I omit the part (v) of Lipchitz stability of the solutions from the statement of the theorem. I use the standard $O()$ notation here, $x=O(y)$ meaning $x \leq cy$ for some absolute constant c . If the constant c depends on a parameter k , we set it as index of $O_k()$.

It is important to remark here that the existence and uniqueness results locally in time (well-posedness), hold also not only for the case of viscous flows following the Navier-Stokes equations, but also for the case of inviscid flows under the Euler equations. There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions both for the Navier-Stokes and the Euler equation with the same methodology, where the value of the viscosity coefficient $\nu=0$, can as well be included. I refer e.g. the reader to the [34] MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4, paragraph 3.2.3, and paragraph 4.1 page 138.

PROPOSITION 3.1 (Local well-posedness in H^1). Let (u_0, f, T) be H^1 data.

(i) (Strong solution) If (u, p, u_0, f, T) is an H^1 mild solution, then

$$u \in C_t^0 H_x^1([0, T] \times R^3)$$

(ii) (Local existence and regularity) If

$$(\|u_0\|_{H_x^1(R^3)} + \|f\|_{L_t^1 H_x^1(R^3)})^4 T < c$$

for a sufficiently small absolute constant $c > 0$, then there exists

a H^1 mild solution (u, p, u_0, f, T) with the indicated data, with

$$\|u\|_{X^k([0, T] \times R^3)} = O_k(\|u_0\|_{H_x^1(R^3)} + \|f\|_{L_t^1 H_x^1(R^3)})$$

and more generally

$$\|u\|_{X^k([0, T] \times R^3)} = O_k(\|u_0\|_{H_x^k(R^3)}, \|f\|_{L_t^1 H_x^k(R^3)}, 1)$$

for each $k \geq 1$. In particular, one has local existence whenever

T is sufficiently small, depending on the norm $H^1(u_0, f, T)$.

(iii) (Uniqueness) There is at most one H^1 mild solution (u, p, u_0, f, T)

with the indicated data.

(iv) (Regularity) If (u, p, u_0, f, T) is a H^1 mild solution, and (u_0, f, T)

is (smooth) Schwartz data, then u and p is smooth solution; in fact, one has

$$\partial_t^j u, \partial_t^j p \in L_t^\infty H^k([0, T] \times R^3) \text{ for all } j, K \geq 0.$$

For the proof of the above theorem, the reader is referred to the [45] TAO, T. 2013 theorem 5.4, but also to the papers and books, of the above mentioned other authors.

Next I state the local existence and uniqueness of smooth solutions of the Navier-Stokes (and Euler) equations with smooth Schwartz initial conditions, that I will use in this

paper, explicitly as a PROPOSITION 4.2 here.

PROPOSITION 3.2 Local existence and uniqueness of smooth solutions or smooth well posedness. Let $u_0(x), p_0(x)$ be smooth and Schwartz initial data at $t=0$ of the Navier-Stokes (or Euler) equations, then there is a finite time interval $[0, T]$ (in general depending on the above initial conditions) so that there is a unique smooth local in time solution of the Navier-Stokes (or Euler) equations

$$u(x), p(x) \in C^\infty(R^3 \times [0, T])$$

Proof: We simply apply the PROPOSITION 3.1 above and in particular, from the part (ii) and the assumption in the PROPOSITION 3.2, that the initial data are smooth Schwartz, we get the local existence of H^1 mild solution $(u, p, u_0, 0, T)$. From the part (iv) we get that it is also a smooth solution. From the part (iii), we get that it is unique.

As an alternative we may apply the theorems in [34] MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4, paragraph 3.2.3, and paragraph 4.1 page 138, and get the local in time solution, then derive from the part (iv) of the PROPOSITION 4.1 above, that they are also in the classical sense smooth. QED.

Remark 3.1 We remark here that the property of smooth Schwartz initial data, is not known in general if is conserved in later times than $t=0$, of the smooth solution in the Navier-Stokes equations, because it is a very strong fast decaying property at spatially infinity. But for lower rank derivatives of the velocities (and vorticity) we have the (global and) local energy estimate, and (global and) local enstrophy estimate theorems that reduce the decaying of the solutions at later times than $t=0$, at spatially infinite to the decaying of the initial data at spatially infinite. See e.g. TAO, T. 2013, Theorem 8.2 (Remark 8.7) and Theorem 10.1 (Remark 10.6).

Furthermore in the same paper of formal formulation of the Clay millennium conjecture, L. [19] FEFFERMAN 2006 (see page 3rd line 6 from above), it is stated that the 3D global regularity of such smooth solutions is controlled by the bounded accumulation in finite time intervals of the vorticity (Beale-Kato-Majda). I state this also explicitly for the convenience of the reader, for smooth solutions of the Navier-Stokes equations with smooth Schwartz initial conditions, as the PROPOSITION 3.6 **When we say here bounded accumulation** e.g. of the deformations D , **on finite internals**, we mean in the sense e.g. of the proposition 5.1 page 171 in the book [34] MAJDA A.J-BERTOZZI A. L. 2002, which is a definition designed to control the existence or not of finite blowup times. In other words for any finite time interval

$[0, T]$, there is a constant M such that

$$\int_0^t |D|_{L^\infty}(s) ds \leq M$$

I state here for the convenience of the reader, a well known proposition of equivalent necessary and sufficient conditions of existence globally in time of solutions of the Euler equations, as inviscid smooth flows. It is the proposition 5.1 in [34] MAJDA A.J-BERTOZZI A. L. 2002 page 171.

The stretching is defined by

$$S(x, t) =: D\xi \cdot \xi \text{ if } \xi \neq 0 \text{ and } S(x, t) =: 0 \text{ if } \xi = 0$$

$\xi = \frac{\omega}{|\omega|}$
where ω being the vorticity.

PROPOSITION 3.3 *Equivalent Physical Conditions for Potential Singular Solutions of the Euler equations . The following conditions are equivalent for smooth Schwartz initial data:*

(1) *The time interval, $[0, T^*)$ with $T^* < \infty$ is a maximal interval of smooth H^s*

existence of solutions for the 3D Euler equations.

(2) *The vorticity ω accumulates so rapidly in time that*

$$\int_0^t \|\omega\|_{L^\infty}(s) ds \rightarrow +\infty$$

as t tends to T^*

(3) *The deformation matrix D accumulates so rapidly in time that*

$$\int_0^t \|D\|_{L^\infty}(s) ds \rightarrow +\infty$$

as t tends to T^*

(4) *The stretching factor $S(x, t)$ accumulates so rapidly in time that*

$$\int_0^t [\max_{x \in R^3} S(x, s)] ds \rightarrow +\infty$$

as t tends to T^*

The next theorem establishes the equivalence of smooth connected compact support initial data with the smooth Schwartz initial data, for the homogeneous version of the 4th Clay Millennium problem. It can be stated either for local in time smooth solutions or global in time smooth solutions. The advantage assuming connected compact support smooth initial data, is obvious, as this is preserved in time by smooth functions and also integrations are easier when done on compact connected sets.

PROPOSITION 3.4. (3D global smooth compact support non-homogeneous regularity implies 3D global smooth Schwartz homogeneous regularity) *If it holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time $[0, T]$, finite energy, flow-solutions with smooth compact support (connected with smooth boundary) initial data of velocities and pressures (thus finite initial energy) and smooth compact support (the same connected support with smooth boundary) external forcing for all times $t > 0$, exist also globally in time $t > 0$ (are globally regular) then it also holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time $[0, T]$, finite energy, flow-solutions with smooth Schwartz initial data of velocities and pressures (thus finite initial energy), exist also globally in time for all $t > 0$ (are regular globally in time).*

Proof: see [26] KYRITSIS, K. June 2017, or [29] KYRITSIS, K. February 2019, PROPOSITION 6.4)

Remark 3.2 Finite initial energy and energy conservation equations:

When we want to prove that the smoothness in the local in

time solutions of the Euler or Navier-Stokes equations is conserved, and that they can be extended indefinitely in time, we usually apply a “reduction ad absurdum” argument: Let the maximum finite time T^* and interval $[0, T^*)$ so that the local solution can be extended smooth in it. Then the time T^* will be a blow-up time, and if we manage to extend smoothly the solutions on $[0, T^*]$. Then there is no finite Blow-up time T^* and the solutions holds in $[0, +\infty)$. Below are listed necessary and sufficient conditions for this extension to be possible. Obviously not smoothness assumption can be made for the time T^* , as this is what must be proved. But we still can assume that at T^* the energy conservation and momentum conservation will hold even for a singularity at T^* , as these are universal laws of nature, and the integrals that calculate them, do not require smooth functions but only integrable functions, that may have points of discontinuity.

A very well known form of the energy conservation equation and accumulative energy dissipation is the next:

$$\frac{1}{2} \int_{R^3} \|u(x, T)\|^2 dx + \nu \int_0^T \int_{R^3} \|\nabla u(x, t)\|^2 dx dt = \frac{1}{2} \int_{R^3} \|u(x, 0)\|^2 dx$$

(eq. 8)

where

$$E(0) = \frac{1}{2} \int_{R^3} \|u(x, 0)\|^2 dx$$

(eq. 9)

is the initial finite energy

$$E(T) = \frac{1}{2} \int_{R^3} \|u(x, T)\|^2 dx$$

(eq. 10)

is the final finite energy

$$\Delta E = \nu \int_0^T \int_{R^3} \|\nabla u(x, t)\|^2 dx dt$$

(eq. 11)

and

is the accumulative finite energy dissipation from time 0 to time T, because of viscosity in to internal heat of the fluid.

For the Euler equations it is zero. Obviously

$$\Delta E = E(0) - E(T) \quad (\text{eq. 12})$$

The rate of energy dissipation is given by

$$\frac{dE}{dt}(t) = -\nu \int_{R^3} \|\nabla u\|^2 dx < 0$$

(eq. 13)

(ν , is the density normalized viscosity coefficient. See e.g. [34] MAJDA, A.J-BERTOZZI, A. L. 2002 Proposition 1.13, equation (1.80) pp. 28)

Remark 3.3 The next are 3 very useful inequalities for the unique local in time $[0, T]$, smooth solutions u of the Euler and Navier-Stokes equations with smooth Schwartz initial data and finite initial energy (they hold for more general conditions on initial data, but we will not use that):

By $\|\cdot\|_m$ we denote the Sobolev norm of order m . So if $m=0$ it is essentially the L_2 -norm. By $\|\cdot\|_{L^\infty}$ we denote the supremum norm, u is the velocity, ω is the vorticity, and c_m are constants.

$$\|u(x, T)\|_m \leq \|u(x, 0)\|_m \exp\left(\int_0^T c_m \|\nabla(u(x, t))\|_{L^\infty} dt\right)$$

1)

(eq. 14)

(see e.g. [34] MAJDA, A.J-BERTOZZI, A. L. 2002, proof of Theorem 3.6 pp117, equation (3.79))

$$\|\omega(x,t)\|_0 \leq \|\omega(x,0)\|_0 \exp\left(c \int_0^t \|\nabla u(x,s)\|_{L_\infty} ds\right)$$

2)
(eq. 15)

(see e.g.[34] MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp117, equation (3.80))

$$\|\nabla u(x,t)\|_{L_\infty} \leq \|\nabla u(x,0)\|_0 \exp\left(\int_0^t \|\omega(x,s)\|_{L_\infty} ds\right)$$

3)
(eq. 16)

(see e.g.[34] MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp118, last equation of the proof)

The next are a list of well know necessary and sufficient conditions , for regularity (global in time existence and smoothness) of the solutions of Euler and Navier-Stokes equations, under the standard assumption in the 4th Clay Millennium problem of smooth Schwartz initial data, that after theorem Proposition 4.4 above can be formulated equivalently with smooth compact connected support data. We denote by T* be the maximum Blow-up time (if it exists) that the local solution u(x,t) is smooth in [0,T*].

DEFINITION 3.2

When we write that a quantity Q(t) of the flow ,in general depending on time, is uniformly in time bounded during the flow, we mean that there is a bound M independent from time , such that Q(t) <= M for all t in [0, T*].

PROPOSITION 3.5 (Necessary and sufficient condition for regularity)

The local solution u(x,t) , t in [0,T*) of the Euler or Navier-Stokes equations, with smooth Schwartz initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the Sobolev norm ||u(x,t)||_m , m>=3/2+2, remains bounded , by the same bound in all of [0,T*), then , there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)

Remark 3.4 See for a proof e.g.[34]MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, line 10 from below)

1) PROPOSITION 3.6 (Necessary and sufficient condition for regularity. Beale-Kato-Majda)

The local solution u(x,t) , t in [0,T*) of the Euler or Navier-Stokes equations, with smooth compact support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the vorticity has bounded by M, accumulation in [0,T*]:

$$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M$$

(eq17)

Then there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)

Remark 3.5 See e.g.[34]MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, Theorem 3.6. Also page 171 theorem 5.1 for the case of inviscid flows. . See also [32] LEMARIE-RIEUSSET P.G. 2002 . Conversely if regularity holds, then in any interval from the smoothness in a compact

connected set, the vorticity is supremum bounded. The above theorems in the book [34] MAJDA A.J-BERTOZZI A. L. 2002 guarantee that the above conditions extent the local in time solution to global in time , that is to solutions (u, p, u₀, f, T) which is H¹ mild solution, for any T. Then applying the part (iv) of the PROPOSITION 4.1 above, we get that this solutions is also smooth in the classical sense, for all T>0, thus globally in time smooth.

2) PROPOSITION 3.7 (Necessary and sufficient condition of vorticity for regularity)

The local solution u(x,t) , t in [0,T*) of the Euler or Navier-Stokes equations, with smooth compact support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the vorticity is bounded by M, in the supremum norm L[∞] in [0,T*] and on any compact set:

$$\|\omega(x,t)\|_{L_\infty} \leq M \text{ for all } t \text{ in } [0,T^*) \text{ (eq. 18)}$$

Then there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)

Remark 3.6 Obviously if $\|\omega(x,t)\|_{L_\infty} \leq M$, then also the

$$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M_1$$

integral exists and is bounded: and the previous proposition 3.6 applies. Conversely if regularity holds, then in any interval from smoothness in a compact connected set, the vorticity is supremum bounded.

PROPOSITION 3.8 (Necessary and sufficient condition for regularity)

The local solution u(x,t) , t in [0,T*) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the space partial derivatives or Jacobean is bounded by M, in the supremum norm L[∞] in [0,T*]:

$$\|\nabla u(x,t)\|_{L_\infty} \leq M \text{ for all } t \text{ in } [0,T^*) \text{ (eq. 19)}$$

Then there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)

Remark 3.7 Direct from the inequality (eq.14) and the application of the proposition 3.5. Conversely if regularity holds, then in any finite time interval from smoothness, the space derivatives are supremum bounded.

PROPOSITION 3.9 ([19] FEFFERMAN C. L. 2006. Velocities necessary and sufficient condition for regularity)

The local solution u(x,t) , t in [0,T*) of the Euler or Navier-Stokes equations, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if

the velocities ||u(x,t)|| do not get unbounded as t->T*.

Then there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞).

Remark 3.8. This is mentioned in the Standard formulation of the 4th Clay Millennium problem FEFFERMAN C. L. 2006 pp.2 , line 1 from below: quote "...For the Navier-Stokes equations ($\nu > 0$) , if there is a solution with a finite blow-up time T , then the velocities $u_i(x,t)$, $1 \leq i \leq 3$ become unbounded near the blow-up time." The converse-negation of this is that if the velocities remain bounded near the T^* , then there is no blow-up at T^* and the solution is regular or global in time smooth. Conversely of course , if regularity holds, then in any finite time interval, because of the smoothness, the velocities, in a compact set are supremum bounded.

I did not find a dedicated such theorem in the books or papers that I studied, but I take it for granted as the official formulation of the problem too.

A probable line of arguments so as to prove it might goes follows:

We want to prove that it cannot be that a blow-up occurs only at the spatial partial derivatives of the velocities and not in the velocities themselves. If such a strange blow-up occurs, then as in the PROPOSITION 3.8 , the Jacobean of the velocities blows-up. This gives that the convective acceleration

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \bullet \nabla u$$

also blows-up (as the term in the convective acceleration of the partial derivative of the velocity remains either bounded by the hypothesis of not blowing-up velocities or oscillates wildly)

$$\frac{Du}{Dt} \rightarrow +\infty \text{ will blows-up as } t \rightarrow T^*$$

Thus by integrating on a path trajectory $\int_0^T \frac{Du}{Dt} dt$ we

deduce that the velocities on the trajectories blow-up which is contradiction from the initial hypothesis. Therefore the flow is regular in $[0, T^*]$ as claimed By Fefferman C.L. in the proposition 3.9.

QED

We notice that Fefferman C.L. states this condition only for the viscous flows, but since PROPOSITION 3.7 holds for the inviscid flows under the Euler equations, this necessary and sufficient condition holds also for the inviscid flows too.

Remark 3.9.

Similar results about the local smooth solutions, hold also for the non-homogeneous case with external forcing which is nevertheless space-time smooth of bounded accumulation in finite time intervals. Thus an alternative formulation to see that the velocities and their gradient , or in other words up to their 1st derivatives and the external forcing also up to the 1st derivatives , control the global in time existence is the next proposition. See [45] TAO. T. 2013 Corollary 5.8

PROPOSITION 3.10 (Maximum Cauchy development)

Let (u_0, f, T) be H^1 data. Then at least one of the following two statements hold:

1) There exists a mild H^1 solution (u, p, u_0, f, T) in $[0, T]$,with the given data.

2) There exists a blowup time $0 < T^* < T$ and an incomplete mild H^1 solution

(u, p, u_0, f, T^*) up to time T^* in $[0, T^*]$, defined as complete on every $[0, t]$, $t < T^*$ which blows up in the enstrophy H^1 norm in the sense that

$$\lim_{t \rightarrow T^*, t < T^*} \|u(x, t)\|_{H^1_x(\mathbb{R}^3)} = +\infty$$

Remark 3.10 The term "almost smooth" is defined in [45] TAO, T. 2013, before Conjecture 1.13. The only thing that almost smooth solutions lack when compared to smooth solutions is a limited amount of time differentiability at the starting time $t = 0$;

The term *normalized pressure*, refers to the symmetry of the Euler and Navier-Stokes equations to substitute the pressure, with another that differs at, a constant in space but variable in time measurable function. In particular normalized pressure is one that satisfies the (eq. 7) except for a measurable at a, constant in space but variable in time measurable function. It is proved in [45] TAO, T. 2013, at Lemma 4.1, that the pressure is normalizable (exists a normalized pressure) in almost smooth finite energy solutions, for almost all times. The viscosity coefficient in these theorems of the above TAO paper has been normalized to $\nu = 1$.

PROPOSITION 3.11 (Differentiation of a potential)

Let a sub-Newtonian kernel $K(x, y)$, and f a bounded and integral function on the open set Ω , of \mathbb{R}^n , then for all x in $\Omega \cup \Sigma$, where Σ is a relatively open subset of $\partial\Omega$, the $\int_{\Omega} K(x, y)F(y)dy$ is in $C^1(\Omega \cup \Sigma)$ and

$$D_{x_i} \int_{\Omega} K(x, y)F(y)dy = \int_{\Omega} D_{x_i} K(x, y)F(y)dy$$

Proof: By D_{x_i} we denote the partial derivative relative to x_i . For the definition of sub-Newtonian kernel and a proof of the above theorem, see [18] HELMS L.L. (2009) paragraph 8.2 pp 303 and Theorem 8.2.7 pp 306. QED.

PROPOSITION 3.12 (Estimates of partial derivatives of harmonic functions)

Assume u is harmonic function in the open set Ω of \mathbb{R}^n . Then

$$|D^a u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0, r))}$$

For each ball $B(x_0, r) \subset \Omega$ and each multi-index a of order $|a| = k$.

$$\text{Here } C_0 = \frac{1}{a(n)}, C_k = \frac{(2^{n+1}nk)^k}{a(n)}, k=1, \dots$$

In particular, for the fundamental harmonic function $u = 1/(|x-y|)$ the next estimates for the partial derivatives hold:

$$\frac{\partial}{\partial x_i} \frac{1}{\|x-y\|} \leq \frac{1}{\|x-y\|^2} \tag{eq. 20}$$

$$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{\|x-y\|} \leq \frac{4}{\|x-y\|^3} \tag{eq.21}$$

And in general there is constant $C(n, \beta)$ such that



$$|D^\beta u| \leq C(n, \beta) \frac{1}{\|x - y\|^{n-2+k}} \quad \text{if } |\beta| = k > 1 \quad (\text{eq.22})$$

Proof: See [9] EVANS L. C. (2010) chapter 2, Theorem 7, pp 29. And for the fundamental harmonic function also see [18] HELMS L.L. (2009), pp 317 equations 8.18, 8.19, 8.20 QED.

PROPOSITION 3.13 (The well-known divergence theorem in vector calculus)

Let an non-empty bounded open set Ω , of R^n with C^1 boundary $\partial\Omega$, and let

$F: \bar{\Omega} \rightarrow R^n$ be vector field that is continuously differentiable in Ω and continuous up to the boundary. Then the divergence theorem asserts that

$$\int_{\Omega} \nabla \cdot F = \int_{\partial\Omega} F \cdot \nu \quad (\text{eq. 23})$$

where ν is the outward pointing unit normal to the boundary $\partial\Omega$.

PROPOSITION 3.14 (Representation formula of the bounded solutions of the Poisson equation.) Let

$f \in C_c^2(R^n), n \geq 3$. In other words f is with continuous second derivatives, and of compact support. Then any bounded solution of the scalar Poisson equation

$$-\Delta u = f \quad \text{in } R^n \text{ has the form}$$

$$u(x) = \int_{R^n} \Phi(x - y) f(y) dy + C \quad x \in R^n \text{ for some}$$

constant C , and $\Phi(x)$ is the fundamental harmonic function.

$$\Phi(x) = \frac{1}{4\pi \|x\|}$$

For $n=3$,

Proof: See [9] EVANS L. C. (2010) §2.2 Theorem 1 pp23 and mainly Theorem 9 pp 30. The proof is a direct consequence of the Liouville's theorem of harmonic functions. There is a similar representation described e.g. in as in [34] MAJDA, A.J-BERTOZZI, A. L. 2002 §1.9.2 Lemma 1.12 pp 38, where f is defined on all of R^n and not only on a compact support. Notice also that solutions of the above Poisson equation that are also of compact support are included in the representation. More general settings of the Poisson equation with solutions only on bounded regions and with prescribed functions on the boundary of the region, do exist and are unique but require correction terms and Green's functions as described again in [9] EVANS L. C. (2010) §2.2 Theorem 5 pp28 QED.

Remark 3.11

Such a more general form of the solution of the Poisson equations as in [34] MAJDA, A.J-BERTOZZI, A. L. 2002 §1.9.2 Lemma 1.12 pp 38, and in particular when smooth bounded input data functions lead to smooth bounded output solutions, could allow us to state the new necessary and sufficient conditions of the next paragraph 4, with the more general hypothesis of the smooth Schwartz initial data, rather than compact support initial data Nevertheless, it holds the equivalence of the smooth Schwartz initial data with compact support initial data holds after PROPOSITION 3.4. and

[26] KYRITSIS, K. June 2017, PROPOSITION 6.4.. In other words, we could proceed and try to prove the 4th Clay Millennium problem without utilizing the PROPOSITION 3.4. of the equivalence of smooth Schwartz initial data and smooth compact support initial data for the 4th Clay Millennium problem. Still it is simpler when thinking about the phenomena to have in mind simpler settings like compact support flows and that is the mode in which we state our results in the next in this paper.

IV. THE PRESSURES SUFFICIENT CONDITIONS FOR REGULARITY.

In this paragraph we utilize one part of our main strategy to solve the 4th Clay Millennium problem, which is to derive a pressures sufficient condition of regularity The second part of the strategy is to integrate over trajectories, and derive integral equations of the velocities and their partial spatial derivatives. Since integrals of velocities may turn out to involve the finite energy which is invariant we hope so to bound the supremum norm of the special partial derivatives of the velocities and use the very well-known necessary and sufficient condition for regularity as in PROPOSITION 3.8

PROPOSITION 4.1. (The pressures, sufficient condition for regularity)

Let the local solution $u(x,t)$, t in $[0, T^*)$ of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, then it can be extended to $[0, T^*]$, where T^* is the maximal time that the local solution $u(x,t)$ is smooth in $[0, T^*)$, and thus to all times $[0, +\infty)$, in other words the solution is regular, if and only if there is a time uniform bound M for the pressures p , in other words such that

$p \leq M$ for all t in $[0, T^*)$ Still in other words smoothness and boundedness of the pressures p on the compact support $V(t)$ and in finite time intervals $[0, T]$ is a characteristic condition for regularity.

Proof: Let us start from this characteristic smoothness and boundedness of the pressures p on the compact support $V(t)$ and in finite time intervals $[0, T]$ to derive regularity.

We notice that in the Navier-Stokes equations of incompressible fluids the pressure forces defines a **conservative force-field**, as it is the gradient of a scalar-field that of the pressures p , which play the role of scalar potential. And this property, of being a **conservative force-field**, is an invariant during the flow. It is an invariant even for viscous flows, compared to other classical invariants, the Kelvin circulation invariant and the Helmholtz vorticity-flux invariant which hold only for inviscid flows. That the force-field F_p is a conservative field, means that if we take two points $x_1(0), x_2(0)$, and any one-dimensional path $P(x_1(0), x_2(0))$, starting and ending on them, then for any test particle of mass m , the integral of the work done by the forces is independent from the particular path, and depends only on the two points $x_1=x_1(0), x_2=x_2(0)$, and we denote it here by $W(x_1, x_2)$.

$$W(x_1, x_2) = \int_{P(x_1, x_2)} F_p ds \quad (\text{eq. 24})$$



In particular, it is known by the **gradient theorem**, that this work equals, the difference of the potential at these points, and here, it is the pressures:

$$W(x_1, x_2) = (1/c) |p(x_2(0)) - p(x_1(0))|. \quad (\text{eq. 25})$$

(The constant (1/c) here is set, because of the normalization of the constant density in the equations of the Navier-Stokes, and accounts for the correct dimensions of units of measurements of the pressure, force, and work).

Similarly, if we take a test-flow, of test particles instead of one test particle, in the limit of points, again the work density, depends only on the two points x_1, x_2 .

In the next arguments we will not utilize the path invariance. From the hypothesis that the pressure are time-uniformly bounded by the same constant M in $[0, T^*]$ we deduce that the integral on a trajectory in (eq24) is also time-uniformly bounded by the same constant M in $[0, T^*]$.

$$\int_{P(x_0, x_1)} F_p ds \leq M$$

We may re-write this integral by changing integration parameter to be the time as

$$\int_{P(t_0, t_1)} F_p \frac{ds}{dt} dt \leq M \text{ in } [0, T^*]. \text{ Or as it is on a trajectory}$$

$$\int_{P(t_0, t_1)} F_p u_m dt \leq M \text{ in } [0, T^*]. \quad (\text{eq26})$$

Where u_m is the (material) velocity on the trajectory.

If the $u_m \rightarrow +\infty$ blows-up as $t \rightarrow T^*$, then also the (material) convective acceleration

$$\frac{Du}{Dt} \rightarrow +\infty \text{ will blowup as } t \rightarrow T^*,$$

And from the Navier-Stokes equations

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u_i \quad (\text{eq. 27})$$

The pressure forces $F_p = -\nabla p \rightarrow +\infty$ will blowup as $t \rightarrow T^*$,

as the friction term only subtracts from the pressure forces.

Nevertheless if both F_p and u_m will blow-up, so also it will, the integral in

(eq 26) which is a contradiction. Thus the (material) velocities do not blowup!

Thus we may apply the necessary and sufficient condition for regularity as in PROPOSITION 3.6 (FEFFERMAN C. L. 2006. **Velocities necessary and sufficient condition for regularity**) and we derive the regularity QED.

PROPOSITION 4.2. (Smooth particle Trajectory mapping and Trajectories finite length, necessary condition for regularity)

Let the local solution $u(x,t)$, t in $[0, T^*)$ of the Euler or Navier-Stokes equations of inviscid or viscous flows correspondingly, and with smooth compact connected support initial data, that it can be extended to $[0, T^*]$, where

T^* is the maximal time that the local solution $u(x,t)$ is smooth in $[0, T^*)$, and thus to all times $[0, +\infty)$, in other words the solution is regular, then the particle trajectory mapping is smooth in finite time intervals and the trajectories-paths are smooth and of length $l(a,t) \leq M$ that remains bounded by a constant M for all t in $[0, T^*)$.

Proof:

The particle trajectory mapping is the representation of the spatial flow in time of the fluid per trajectories-paths. For a definition see MAJDA, A.J-BERTOZZI, A. L. 2002 § 1.3 Equation 1.13 pp 4. Here we apply this mapping on the compact support V initial data.

Let us assume now that the solutions is regular. Then also for all finite time intervals $[0, T]$, the velocities and the accelerations are bounded in the L_∞ , supremum norm, and this holds along all trajectory-paths too. Then also the length of the trajectories, as they are given by the formula

$$l(a_0, T) = \int_0^T \|u(x(a_0, t))\| dt \quad (\text{eq. 28})$$

are also bounded and finite (see e.g. APOSTOL T. 1974, theorem 6.6 p128 and theorem 6.17 p 135). Thus if at a trajectory the lengths becomes unbounded as t converges to T^* , then there is a blow-up. QED.

V. THE FINITE ENERGY, BOUNDED PRESSURE VARIANCE THEOREM FOR INVISCID AND VISCOUS FLOWS AND THE SOLUTION OF THE 4TH CLAY MILLENNIUM PROBLEM.

Remark 5.1 This paragraph utilizes two simple techniques

a) **Energy conservation in various alternative forms and formulae.**

b) **The property of the pressures forces being conservative in the present situation of incompressible flows (gradient theorem).**

The 4th Clay Millennium problem is not just a challenging exercise of mathematical calculations. It is an issue of the standard modelling the physical reality, and therefore we may utilize all our knowledge of the underlying physical reality.

In the strategy that this paper has adopted here to solve the 4th Clay Millennium problem, in a short and elegant way, we will involve as much as possible intuitive physical ideas that may lead us to choose the correct and successful mathematical formulae and techniques, still everything will be within strict and exact mathematics. As T. Tao has remarked in his discussion of the 4th Clay Millennium problem, to prove that the velocity remains bounded (regularity) for all times, by following the solution in the general case, seems hopeless due to the vast number of flow-solution cases. And that the energy conservation is not of much help. And it seems that it is so! But we need more smart and shortcut ideas, through invariants of the flow. In particular, we need clever techniques to calculate in alternative ways part of the energy of the flow, with virtual-test flows, and alternative integrals of virtual work of the pressure forces on instantaneous paths, and that still have the physical units' dimensions of energy. We will base our strategy to the next three factors

- 1) The conservation of energy and the hypothesis of finite initial energy. Then as by proposition 3.8, we have from this necessary and sufficient condition of regularity that we need to have that the partial derivatives of the

Jacobian are bounded: $\|\nabla u(x,t)\|_{L^\infty} \leq M$, and are uniformly in time bounded in the maximal time interval $[0, T^*)$ that a solution exists, then we need to highlight a formula that computes the partial derivatives of the velocities from integrals of the velocities in space and time till then, because the bounded energy invariant is in the form of integrals of velocities.

- 2) The shortcut of physical magnitude with physical units' dimensions of energy as indeed calculating energy: In other words if we reach in the calculations to an expression which has as physical magnitude the physical dimensions units of energy then the expression calculates indeed energy.
- 3) The technique of virtual-test flows on instantaneous paths, to find special formulas for the calculation of energy from alternative magnitudes. Instead of having to recalculate the energy starting from the classical formulae based on the velocities and transform it as the fluid flows, we may use shortcuts to calculate parts of the energy of the fluid based on alternative perceptions, like virtual test-particles flows, and work of the pressure forces on instantaneous paths. Of course the alternative formulae must always have the physical units' dimensions of energy.
- 4) Meanwhile one smart idea to start is to think of alternative ways that forms of energy and projections of them on to bundle of paths, can be measured, even at single time moment and state of the fluid and relate it with its total energy which is finite and remains bounded throughout the flow. Such alternative measurements of parts of the energy as projected on to a bundle of paths, can be done by integrating the conservative pressure forces F_p of the fluid (gradient of the pressures) on paths AB, of space, and relate the resulting theoretical work of them with the pressure differences $p(A)-p(B)$ since the pressures are a potential to such conservative pressure forces.

PROPOSITION 5.1. (Thefinite energy, uniformly in time bounded pressure-variance, theorem).

Let a local in time , t in $[0,T)$, smooth flow solution with velocities $u(x,t)$, with pressures $p(x,t)$, of the Navier-Stokes equations of viscous fluids or of Euler equations of inviscid fluids, with smooth Schwartz initial data, and finite initial energy $E(0)$, as in the standard formulation of the 4th Clay Millennium problem.,Then the pressure differences $|p(x_2(t))-p(x_1(t))|$ for any two points $x_1(t)$, $x_2(t)$, for times that the solution exists, remain bounded by $kE(0)$, where k is a constant depending on the initial conditions, and $E(0)$ is the finite initial energy.

Proof: Let us look again at theNavier-Stokes equations as in (eq. 1) that we bring them here

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u; \quad (\text{eq. 29})$$

Where $\frac{Du}{Dt}$ is the material acceleration, along the

trajectory path.

(It is reminded to the reader, that in the equations of Navier-Stokes, as in (eq. 29) as the density, is constant, it is custom to either normalised to 1, or it is divided out from the left side and it is included in the pressures and viscosity coefficient).

We may separate the forces (or forces multiplied by a constant mass density) , that act at a point, by the two terms of the right side as

$$F_p = -\nabla p \quad (\text{eq. 30})$$

which is the force-field due the pressures and the

$$F_v = \nu \Delta u \quad (\text{eq. 31})$$

which is the force-field due to the viscosity.

We notice that (eq. 30) defines a **conservative force-field**, as it is the gradient of a scalar-field that of the pressures p , which play the role of scalar potential. And this property, of being a **conservative force-field**, is **an invariant** during the flow. It is an invariant **even for viscous flows**, compared to other classical invariants, the Kelvin circulation invariant and the Helmholtz vorticity-flux invariant which hold only for inviscid flows. That the force-field F_p is a conservative field, means that if we take two points $x_1(0)$, $x_2(0)$, and any one-dimensional path $P(x_1(0), x_2(0))$, starting and ending on them, then for any test particle of mass m , the integral of the work done by the forces is independent from the particular path, and depends only on the two points $x_1=x_1(0)$, $x_2=x_2(0)$, and we denote it here by $W(x_1,x_2)$.

$$W(x_1, x_2) = \int_{P(x_1,x_2)} F_p ds \quad (\text{eq. 32})$$

In particular, it is known by the **gradient theorem**, that this work equals, the difference of the potential at these points, and here, it is the pressures:

$$W(x_1, x_2) = (1/c) [p(x_2(0)) - p(x_1(0))]. \quad (\text{eq. 33})$$

(The constant $(1/c)$ here is set, because of the normalization of the constant density in the equations of the Navier-Stokes, and accounts for the correct dimensions of units of measurements of the pressure, force, and work).

Similarly, if we take a test-flow , of test particles instead of one test particle, in the limit of points, again the work density, depends only on the two points x_1, x_2 .

Let now again the two points $x_1(0)$, $x_2(0)$, at the initial conditions of the flow then as we assume Schwartz smooth initial conditions (and not connected compact smooth initial conditions) , there is at least one double circular cone denoted by $DC(x_1(0),x_2(0))$, made by two circular cones united at their circular bases C and with vertices $x_1(0)$, $x_2(0)$ opposite to the plane of the common circular base C . And let us take a bundle of paths, that start from $x_1(0)$, and end at $x_2(0)$ and fill all the double cone DC . We may assume now a test-fluid (a flow of test-particles), inside this double cone which has volume V , that flows from $x_1(0)$, to $x_2(0)$ along these paths. Let us now integrate the work-density on paths done by the pressure forces F_p of the original fluid, as they act on the test-fluid, and inside this 3 dimensional double cone $DC(x_1(0),x_2(0))$. **This will give an instance of a spatial distribution of work done by the pressure forces in the fluid as projected to the assumed paths.** This energy is from the instant action of the pressure forces spatially

distributed, and depends not only on the volume of integration but also on the chosen bundle of paths. It is a double integral, 1 dimensional and 2 dimensional (say on the points of the circular base C), covering all the interior of the double cone DC. Because the work-density per path is constant on each such path, by utilizing the Fubini's theorem (e.g. seeSPIVAK, M. 1965 pp 56), the final integral is:

$$W = \int_C \int_{x_1}^{x_2} F_p dx ds$$

$$W(0) = c \cdot V \cdot \| p(x_2(0)) - p(x_1(0)) \|. \tag{eq. 34}$$

On the other hand this work that would be done by the pressure forces of the original fluid at any time t, is real energy, it is an instance of a spatial distribution of work done by the pressure forces in the fluid as projected to the assumed paths and it would be subtracted from the finite initial energy E(0). Although this energy is only an instance at fixed time t as distributed in space of the action of pressure forces as projected on the assumed bundle of paths, it still has to be finite as calculated in the 3-dimensional double cone. This therefore translates in to that the instance in time of energy flow due to pressure forces as projected on to the assumed bundle of paths, of the original fluid is uniformly in time bounded or in other words bounded in every finite time interval. Therefore:

$$W \leq E(0). \tag{eq. 35}$$

And after combining the (eq. 35), with (eq. 34), we get

$$c \cdot V \cdot \| p(x_2(0)) - p(x_1(0)) \| \leq E(0) \tag{eq. 36}$$

As we remarked that the force field F_d due to pressures is conservative and is an invariant of the flow, and so is the volumes, therefore we can repeat this argument for later times t in [0,T], so that we also have

$$W(t) = c \cdot V \cdot \| p(x_2(t)) - p(x_1(t)) \| \leq E(t) \tag{eq. 37}$$

But since due to energy conservation we have $E(t) \leq E(0)$ (for inviscid fluids $E(t) = E(0)$), then also it holds

$$c \cdot V \cdot \| p(x_2(t)) - p(x_1(t)) \| \leq E(0) \tag{eq. 38}$$

Which is what it is required to prove for x_1, x_2 and $k=1/(cV)$. As an alternative line of arguments, probably simpler, we could have started with a 3-ball and instead of instantaneous paths on a cone we could have all the trajectories of the points of this 3-ball, and again from the bounded of the initial energy and thus of the work of the pressure forces on the 3D trajectory of this 3-ball, we could have concluded the uniform in time boundedness of the pressures at start and end points of the trajectories.

In particular, we notice that if there is a supremum $\sup(p)$ and infimum $\inf(p)$ of pressures at time t, so that $|\sup(p) - \inf(p)|$ is a measure of the variance of the pressures at time t, then this variance is bounded up to a constant, by the initial finite energy, justifying the title of the theorem. For the case of fluid with smooth compact connected support initial data, the infimum of the pressures is zero, which occurs at the boundary of the compact support. So the pressures, in general, are uniformly bounded by the same constant throughout the time interval $[0, T^*]$. (which includes the case $T^* = +\infty$) QED.

Remark 5.2. It is interesting to analyse if we could prove the same proposition as the above 5.1, not for smooth Schwartz initial data, on all the 3 dimensional space, but for connected

compact C_p region smooth Schwartz initial data.

The arguments with the finite energy and the pressures is the same, except we must be able to find for any two points $x_1(0), x_2(0)$ in the compact smooth region C_p , a double circular cone denoted by $DC(x_1(0), x_2(0))$, made by two circular cones united at their circular bases C and with vertices $x_1(0), x_2(0)$ opposite to the plane of the common circular base C. This is actually a matter of geometric topology. If the compact connected region C_p is also simply connected it is known that there is a smooth homeomorphism F that it sends it to a sphere S_3 in the 3 dimensional space. Then of course for the points $F(x_1(0)), F(x_2(0))$, there is such a double circular cone denoted by $DC(F(x_1(0)), F(x_2(0)))$, and the inverse image $F^{-1}(DC)$ is a curvilinear such double cone in the compact connected region C_p . Then we apply the 3-dimensional integrations of eq(30) on it. If the compact connected region C_p is not simply connected it is known that there is a smooth homeomorphism F that it sends it to a sphere S_3 , with n-handles H_n in the 3 dimensional space, or in symbols. $S_3 \cup H_1 \cup \dots \cup H_n$. Then again there is a choice of the F (a matter of geometric 3-dimensional topology) such that the images of the initial points $F(x_1(0)), F(x_2(0))$, are in the interior of the sphere S_3 . And thus there is again a double circular cone denoted by $DC(F(x_1(0)), F(x_2(0)))$, and the inverse image $F^{-1}(DC)$ is a curvilinear such double cone in the compact connected region C_p , to make the integrations as in eq(30). Therefore we may remark that we may have the PROPOSITION 5.1 to hold not for smooth Schwartz initial data, on all the 3 dimensional space, but for connected compact C_p region smooth Schwartz initial data.

PROPOSITION 5.2 (The solution of the 4th Clay Millennium problem). *Let a local in time, t in $[0, T)$, smooth flow solution with velocities $u(x, t)$, of the Navier-Stokes equations of viscous fluids with smooth Schwartz initial data, and finite initial energy $E(0)$, as in the standard formulation of the 4th Clay Millennium problem. Then the solution is regular, in other words it can be extended as smooth solution for all times t in $[0, +\infty)$.*

Proof: From the necessary and sufficient condition of regularity that we have stated in the paragraphs 3 and 4 we just need to apply any one of them. In addition, we use here the equivalence of the smooth Schwartz initial data with compact support initial data holds after PROPOSITION 3.4. and [26] KYRITSIS, K. June 2017, PROPOSITION 6.4. because the necessary and sufficient conditions for regularity of the paragraphs 4 and 5 are stated mainly for smooth compact support initial data. As we mentioned in Remark 3.11 we could avoid using the above equivalence of smooth Schwartz and smooth compact support initial data and still prove the 4th Clay Millennium problem, but we preferred for reasons of simplicity of intuitive physical thinking to state our necessary and sufficient conditions of regularity for smooth compact support initial data.

From the previous proposition 5.1, we have that the pressures are smooth and bounded in finite time intervals and therefore we apply the pressures necessary and sufficient condition of regularity as in PROPOSITION 4.1. **(The pressures, sufficient condition for regularity).** Hence the solution of the 4th Clay Millennium problem in its original formulation. All the 5 new necessary and sufficient conditions of

regularity in paragraph 4, show a clear pattern: Once one of the basic magnitudes of the flow (like pressures, velocities, trajectories lengths, pressure forces, viscosity forces, vorticity etc.) turns out to be bounded in finite time intervals, then immediately regularity follows and cascades the same boundedness for all the other magnitudes. The magnitudes of the flows are no ordinary smooth functions but are smooth functions interrelated with Poisson equations through harmonic functions.

We had mentioned this phenomenon in [27] KYRITSIS, K. November 2017 in Remark 6.2 as “Homogeneity of smoothness relative to a property” QED.

Remark 5.3. As we mentioned above and also in Remark 3.11, it was our choice to prefer to use rather than not use, the PROPOSITION 3.4. and [26] KYRITSIS, K. June 2017, PROPOSITION 6.4. in other words, the equivalence of the smooth Schwartz initial data with smooth compact support initial data for the 4th Clay Millennium problem. But as PROPOSITION 4.4 is stated only for the Navier-Stokes equations and viscous flows, and not for the Euler equations. So we missed to prove the regularity of the Euler equations with the previous method. It will be left for the future the investigation of a different line of statements that might as well prove the regularity of the Euler equations under the standard hypotheses for initial data as in the 4th Clay Millennium problem.

VI. EPILOGUE.

In this paper it is has been proved the regularity of the Navier-Stokes equations and therefore it has been solved the 4th Clay Millennium problem. To do so it was utilized mainly that the initial energy was finite, the conservation of the energy, with alternative ways to compute parts of it, and that many of the magnitudes of the flow are interrelated through the very well-studied and regular Poisson equation through harmonic functions. **Finite initial energy, conservation of energy and the regularity of the pressures gave finally the regularity of the Navier-Stokes equations with the standard hypotheses for initial data as in the corresponding Clay Millennium problem.**

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