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Propensity for hedging* and ambiguity aversion

Zaier Aouani[†], Alain Chateauneuf[‡], Caroline Ventura^{§¶}

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Abstract

We study ambiguity aversion by introducing some new notions of propensity for hedging that are less general than convexity of preferences. We therefore characterize the corresponding properties of the capacity and Choquet functional, and link them with actual observed behaviors under uncertainty (Fox et al., 1996; Tversky and Wakker, 1995).

Keywords: Capacity, ambiguity, anticomonicity, propensity for hedging, pseudo-convex, attraction for certainty, proper hedges

JEL: D81

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[†]Independent researcher, aouani@gmail.com

[‡]IPAG Business School and Paris School of Economics, Université de Paris 1, chateauneuf@univ-paris1.fr

[§]Université de Paris 1, caroline.ventura@univ-paris1.fr

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1 Introduction

The classical interpretation of convexity of preferences in terms of negative attitude towards the presence of ambiguity dates back to [Debreu \(1959\)](#). It turns out that in the Choquet Expected Utility (CEU) model, introduced by [Schmeidler \(1989\)](#), convexity of preferences is nicely characterized by a convex capacity or, equivalently, a superadditive Choquet functional. This straightforwardly remains true in the simplified version of Schmeidler’s model envisioned in this paper, also referred to as CEU, where the outcome space is the set of real numbers, \mathbb{R} , considered as the space of monetary payments, and where we assume that the decision maker (DM) displays a constant marginal utility of wealth, i.e., $u(x) = x$ for all $x \in \mathbb{R}$. The results of this paper can readily be extended to the popular Anscombe-Aumann model, where mixtures are considered instead of sums. As we are interested in behavioral properties that are reflected solely through the properties of the capacity, we adopt the simplified model.

Ambiguity aversion stipulates, roughly speaking, that mixing any two indifferent ambiguous acts X and Y reduces ambiguity and therefore allows to obtain a new act that is, from the perspective of the ambiguity averse DM, at least as desirable as X or Y . This means that any act may be a hedge against any other act, implying that ambiguity aversion cannot distinguish between separate attitudes towards hedging.

In fact, ambiguity aversion encompasses a wide spectrum of attitudes towards hedging, and this paper disentangles a variety of them (*propensity for hedging*, *propensity for minimum hedging*, and *propensity for proper hedging*) by focusing on the apparently pertinent attraction to hedging effects that result from the addition of *antimonotonic* acts, i.e., acts which vary in opposite directions. The choice of antimonotonic acts as hedging instruments is only natural for two reasons. First, in the extreme case of *perfect hedging* where mixing two acts leads to a constant act, those two acts are in fact antimonotonic, and second, adding *comonotonic* acts, i.e., acts which vary in the same direction, does not reduce variability hence, as emphasized by Schmeidler, two comonotonic acts cannot be considered hedges against each other.

This paper shows that mere inclination to hedging by antimonotonic acts, dubbed propensity for hedging, is characterized by a *pseudo-convex* capacity, or, equivalently, an antimonotonic superadditive Choquet functional. In addition to superadditivity, pseudo-convex capacities maintain convexity at the sure event ([Tversky and Wakker, 1995](#); [Fox et al., 1996](#)).

Propensity for minimum hedging, reflecting tendency to prefer hedges against the worst (minimum) outcome, is defined in a similar way to propensity for hedging, except that hedging instruments are further restricted to be *minimum-hedges*, that is, only improvement to only the lowest outcome of the act to be hedged. Of course, a minimum-hedge for an act is antimonotonic with the act. We show that propensity for minimum hedging is characterized by a superadditive capacity, or, equivalently, a minimum-hedge superadditive Choquet functional.

Finally, *proper hedges*, a notion introduced under risk by [Cheung et al. \(2014\)](#), are used as hedging instruments to define propensity for proper hedging in a similar way to propensity

for hedging. Proper hedges of an act are anticomonotonic with the act. We show that propensity for proper hedging is characterized by a capacity that is *superadditive at the sure event*, or, equivalently, a proper-hedge superadditive Choquet functional. We, furthermore, show that propensity for proper hedging coincides with preference for *perfect hedging*; a relaxed version of preference for sure diversification as defined in [Chateauneuf and Tallon \(2002\)](#) where now only two equivalent acts are used.

The paper is organized as follows. Section 2 presents the decision theoretic setup. Section 3 is concerned with the different notions of propensity for hedging. We first (Section 3.1) define pseudo-convex capacities and show (Theorem 1) that they characterize CEU preferences exhibiting propensity for hedging. An application to the elicitation of a lower bound for the reservation price of a European put option (see e.g. [Hull \(2018\)](#)) is provided. Clearly the payoff of a European put option is anticomonotonic with the payoff of the stock position. Hedging a stock position by purchasing put options on the stock is a very common hedging strategy.

In Section 3.2, we propose a new notion of ambiguity aversion, *minimum ambiguity aversion* (aversion to the lowest outcome of a distribution), and compare it to the notion of *strong ambiguity aversion*, which has been proved by [Chateauneuf \(1991\)](#) to characterize (under standard axioms) CEU preferences with a convex capacity, i.e., to characterize convexity of preferences. Proposition 3 provides a direct proof of this characterization. The new notion of propensity for minimum hedging is introduced and, for CEU preferences, is shown to be characterized by superadditive capacities.

In Section 3.3, we emphasize that pseudo-convex capacities, being convex at the sure event, exhibit attraction for certainty; a property that is supported by empirical evidence for options traders (see [Fox et al. \(1996\)](#)) and, along the way, confirmed by practitioners in the field (see [Tversky and Wakker \(1995\)](#), [Gonzalez and Wu \(1996\)](#), and [Wakker \(2001\)](#)).

Section 3.4 shows that proper hedges are particularly relevant under ambiguity (they are, for instance, ambiguity reducers; see Proposition 12). We use proper hedges as hedging instruments to define propensity for proper hedging. For CEU preferences, propensity for proper hedging is shown to be characterized by a capacity that is superadditive at the sure event; a property usually seen as a no-arbitrage condition and therefore considered a natural requirement. Concluding remarks are in Section 4 and proofs are gathered in Section 5.

2 Decision theoretic setup and preliminary results

We consider a nonempty set, S , of states of the world and a sigma-algebra, \mathcal{A} , of subsets (*events*) of S . A set function $P : \mathcal{A} \rightarrow \mathbb{R}$ is a *finitely additive measure* if $P(A) \geq 0$ for all $A \in \mathcal{A}$ and $P(A \cup B) = P(A) + P(B)$ for $A, B \in \mathcal{A}, A \cap B = \emptyset$. Furthermore, when $P(S) = 1$, P is a *finitely additive probability*. A set function $v : \mathcal{A} \rightarrow \mathbb{R}$ is a *capacity* if $v(\emptyset) = 0$ and v is *monotone*, i.e., $A, B \in \mathcal{A}, A \subseteq B \Rightarrow v(A) \leq v(B)$. The capacity v is *normalized* if $v(S) = 1$.¹ Each capacity v has a dual capacity \bar{v} defined by $\bar{v}(A) := v(S) - v(A^c)$ for all $A \in \mathcal{A}$, where

¹All the results in this paper hold for non-normalized capacities. However, proofs of (ii) \Rightarrow (iii) in Theorem 1 and 2 are written with normalized capacities in an effort to lighten notation.

A^c denotes the complement of A in S . The *core* of a capacity v is the set

$$C(v) = \left\{ \lambda : \mathcal{A} \rightarrow \mathbb{R}_+ : \lambda \text{ is a finitely additive measure, } \lambda(S) = v(S) \text{ and } \lambda \geq v \right\}.$$

A capacity v is said to be:

- (a) *balanced* if $C(v) \neq \emptyset$.
- (b) *exact* if for all $A \in \mathcal{A}$, there exists $\mu \in C(v)$, $v(A) = \mu(A)$.
- (c) *convex* if $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$, for all $A, B \in \mathcal{A}$.

The following implications are well-known (see, e.g., [Schmeidler \(1972\)](#)):

$$v \text{ is convex} \Rightarrow v \text{ is exact} \Rightarrow v \text{ is balanced.}$$

Denote by V the space of real-valued, bounded, and \mathcal{A} -measurable functions on S , i.e.,

$$V := \left\{ X : S \rightarrow \mathbb{R}, X \text{ is bounded and } \mathcal{A}\text{-measurable} \right\},$$

and let V_0 be the subset of V containing the step functions. For a capacity v and a function $X \in V$, the Choquet integral of X w.r.t. v is defined as

$$I(X) = \int X dv := \int_{-\infty}^0 (v(\{s : X(s) \geq t\}) - v(S)) dt + \int_0^{+\infty} v(\{s : X(s) \geq t\}) dt.$$

For standard results on Choquet integral ([Choquet \(1954\)](#)) we refer to [Schmeidler \(1986, 1989\)](#) and to [Marinacci and Montrucchio \(2004\)](#).

Throughout the paper, we consider only CEU preferences over V , that is, preferences \succsim that are representable through the Choquet integral with respect to a capacity v (for short, represented by v), in the sense that² for every $X, Y \in V$,

$$X \succsim Y \Leftrightarrow \int X dv \geq \int Y dv.$$

For a preference \succsim , denote by \sim (resp. \succ) its symmetric (resp. asymmetric) part.

Two real random variables are said to be comonotonic if they move in the same direction, and they are anticomonotonic if they move in opposite directions. The following definition of comonotonicity (and, symmetrically, that of anticomonotonicity) can be found in [Dellacherie \(1970\)](#) or [Schmeidler \(1986, 1989\)](#).

Definition 1. *Let $X, Y \in V$. We say that the pair (X, Y) is comonotonic (resp. anticomonotonic) or simply that X and Y are comonotonic (resp. anticomonotonic) if*

$$(X(s) - X(t))(Y(s) - Y(t)) \geq (\text{resp. } \leq) 0, \text{ for all } s, t \in S.$$

The following preliminary result, Lemma 1, a kind of folk theorem, appears to be very useful to illustrate the importance of comonotonicity (resp. anticomonotonicity) as a way to model the impossibility (resp. the possibility) of hedging effects when adding two acts X and Y . Therefore, we will give the proof of this result for sake of completeness.

²The general representation would involve an instantaneous von Neumann-Morgenstern utility function u (see, e.g., [Schmeidler \(1989\)](#)). As we are interested in behavioral properties that are reflected solely through the properties of the capacity v , we adopt this simplified model.

Lemma 1. *Let $X, Y \in V$. The following assertions are equivalent.*

- (i) (X, Y) are comonotonic (respectively anticomonotonic).
- (ii) $\text{cov}_P(X, Y) \geq 0$ (resp. $\text{cov}_P(X, Y) \leq 0$) for every probability P on \mathcal{A} .

Lemma 2 below explains why, throughout this paper, an act Z is considered to be a hedge against X if and only if $I(X + Z) \geq I(X) + I(Z)$.

Lemma 2. *Let \succsim be a CEU preference represented by the Choquet integral, I . For $X, Y, Z \in V$ with (Y, Z) comonotonic, the following assertions are equivalent.*

- (i) $X \succsim Y \Rightarrow X + Z \succsim Y + Z$.
- (ii) $I(X + Z) \geq I(X) + I(Z)$.

3 Propensity for hedging

For CEU preferences, ambiguity aversion, as defined by Axiom C below, is equivalent to convexity of preferences, Axiom C', which in turn is characterized by a convex capacity v or, equivalently, a superadditive Choquet functional, i.e., $I(X + Y) \geq I(X) + I(Y)$ for all $X, Y \in V$. See, e.g., [Schmeidler \(1986, 1989\)](#) and [Marinacci and Montrucchio \(2004\)](#).

Axiom C (Ambiguity aversion). *For $X, Y, Z \in V$:*

$$X \succsim Y, (Z, Y) \text{ comonotonic} \Rightarrow X + Z \succsim Y + Z.$$

Axiom C' (Convexity/Preference for diversification). *For $X, Y \in V$:*

$$X \succsim Y, \alpha \in (0, 1) \Rightarrow \alpha X + (1 - \alpha)Y \succsim Y.$$

Roughly speaking, Z is comonotonic with Y means that Z is “definitely not” a hedge against Y . Thus, Axiom C states that if a DM finds X at least as good as Y and Z is definitely not a hedge against Y (note that Z may or may not be a hedge against X) then adding Z to Y cannot sufficiently reduce ambiguity when compared to adding Z to X and, therefore, an ambiguity averse DM will not reverse preferences.

This clearly encompasses a rather wide spectrum of attitudes towards hedging effects. Consequently, in this section, we intend to characterize inclinations towards hedging effects resulting from combining anticomonotonic acts. In addition to the requirement that Z be “definitely not” a hedge against Y (as in Axiom C) we also require that Z “definitely” be some sort of hedge against X . Our first result characterizes CEU preferences satisfying the following axiom which depicts a weaker form of ambiguity aversion.

Axiom D (Propensity for hedging). *For $X, Y, Z \in V$:*

$$X \succsim Y, (Z, Y) \text{ comonotonic, and } (Z, X) \text{ anticomonotonic} \Rightarrow X + Z \succsim Y + Z.$$

For this purpose, we first define what appears to be (as far as we know) a new type of capacity, that we will call *pseudo-convex*. It turns out that these capacities exhibit some meaningful properties like consistency with hedging through anticomonotonic acts and attraction for certainty, while being weaker than the usual convex capacities that characterize the classical Axiom C or C'.

3.1 Pseudo-convex capacities

Definition 2. A capacity v on \mathcal{A} is:

(i) *superadditive if it satisfies*

$$v(A \cup B) \geq v(A) + v(B), \text{ for all } A, B \in \mathcal{A}, A \cap B = \emptyset. \quad (3.1)$$

(ii) *convex at the sure event, if it satisfies*

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B), \text{ for all } A, B \in \mathcal{A}, A \cup B = S. \quad (3.2)$$

(iii) *pseudo-convex if it is [superadditive](#) and [convex at the sure event](#).*³

Note that convexity of v at the sure event is equivalent to its dual, \bar{v} , being subadditive.⁴ Clearly, convex capacities are pseudo-convex. More precisely:

Proposition 1. *Let v be a capacity. If v is exact then it is pseudo-convex.*

As shown by the following example, the converse of Proposition 1 does not hold.

Example 1 (The converse of Proposition 1 does not hold). Let $S = \{1, 2, 3, 4\}$ and $v(\emptyset) = 0$, $v(S) = 6$ and for $A \subseteq S$,

$$v(A) = \begin{cases} 1 & \text{if } |A| = 1, \\ 3 & \text{if } |A| = 2, \\ 4 & \text{if } |A| = 3. \end{cases}$$

The following table lists the values of $v(A)$ and $\bar{v}(A)$ for $A \subseteq S$.

| A | \emptyset | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | S |
|--------------|-------------|---|---|---|---|----|----|----|----|----|----|-----|-----|-----|-----|-----|
| $v(A)$ | 0 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 6 |
| $\bar{v}(A)$ | 0 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 5 | 5 | 5 | 5 | 6 |

One can easily check that v is pseudo-convex as v is superadditive and \bar{v} is subadditive. On the other hand, it is easy to see that v is not exact. Indeed, it is enough to show that there does not exist $\lambda \in C(v)$ such that $\lambda(123) = v(123) = 4$. Since $v(A) = 3$ for all $A \subseteq S$ such that $|A| = 2$, for every $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in C(v)$, we must have $\lambda_i + \lambda_j \geq 3$ for all pairs $(i, j) \in S^2$ such that $i \neq j$. Thus, the equality $\lambda(123) = 4$, together with $\lambda_i \geq v(i) = 1$ for all $i \in S$ and $\lambda_i + \lambda_j \geq 3$ for all pairs $(i, j) \in S^2$ such that $i \neq j$, implies $\lambda_1 = \lambda_2 = \lambda_3 = 1$, which contradicts $\lambda(123) = 4$ and completes the proof. Note that, in this example, $C(v) \neq \emptyset$ since $\lambda = (3/2, 3/2, 3/2, 3/2) \in C(v)$. \diamond

We now state the main result of this section, Theorem 1. This theorem shows that, in the framework of CEU preferences, propensity for hedging is characterized by a pseudo-convex capacity, or, equivalently, an anticomonotonic superadditive Choquet functional.

Theorem 1. *Let \succsim be a CEU preference represented by the capacity v . The following assertions are equivalent.*

(i) \succsim *exhibits propensity for hedging, i.e., satisfies Axiom D.*

(ii) v *is pseudo-convex.*

³A similar concept in the literature is “likelihood insensitivity.” See (Wakker, 2010, Definition 10.4.1).

⁴A capacity v on \mathcal{A} is subadditive if it satisfies $v(A \cup B) \leq v(A) + v(B)$, for all $A, B \in \mathcal{A}, A \cap B = \emptyset$.

(iii) The functional $I(\cdot) = \int \cdot dv$ is anticomonotonic superadditive. That is, for all $X, Y \in V$ such that (X, Y) is anticomonotonic, we have

$$I(X + Y) \geq I(X) + I(Y).$$

Remark 1. Condition (iii) in Theorem 1 may be written as follows:

(iii') For all $X, Y \in V$ such that (X, Y) is anticomonotonic and $(X, X + Y)$ is comonotonic, we have $I(X + Y) \geq I(X) + I(Y)$.

Indeed, clearly (iii) \Rightarrow (iii') and, in order to establish the equivalence, it suffices to prove that (iii') \Rightarrow (ii) since, by Theorem 1, (ii) \Rightarrow (iii). To prove superadditivity, let $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$. Denoting by E^* the characteristic function of an event E , we have

| | A | B | $(A \cup B)^c$ |
|-------------|-----|-----|----------------|
| A^* | 1 | 0 | 0 |
| B^* | 0 | 1 | 0 |
| $A^* + B^*$ | 1 | 1 | 0 |

Note that (A^*, B^*) is anticomonotonic and $(A^*, A^* + B^*)$ is comonotonic and thus, from (iii'), we have $I(A^* + B^*) \geq I(A^*) + I(B^*)$, that is, $I((A \cup B)^*) \geq I(A^*) + I(B^*)$, or equivalently, $v(A \cup B) \geq v(A) + v(B)$, i.e. v is superadditive.

To prove convexity for the sure event, let $A, B \in \mathcal{A}$ be such that $A \cup B = S$. We have

| | $A \setminus B$ | $A \cap B$ | $B \setminus A$ |
|-------------|-----------------|------------|-----------------|
| A^* | 1 | 1 | 0 |
| B^* | 0 | 1 | 1 |
| $A^* + B^*$ | 1 | 2 | 1 |

Note that (A^*, B^*) is anticomonotonic⁵ and $(A^*, A^* + B^*)$ is comonotonic and thus, from (iii'), we have $I(A^* + B^*) \geq I(A^*) + I(B^*)$. Since $A^* + B^* = (A \cup B)^* + (A \cap B)^*$ and $(A \cap B)^*$ and $(A \cup B)^*$ are comonotonic, we get

$$I((A \cup B)^*) + I((A \cap B)^*) = I(A^* + B^*) \geq I(A^*) + I(B^*).$$

That is, $v(S) + v(A \cap B) \geq v(A) + v(B)$. Hence v is convex at the sure event. \diamond

The following example shows that if the capacity v is only convex for the sure event (that is convex for the sure event but not superadditive) then Condition (iii) of Theorem 1, or equivalently Condition (iii'), may not be satisfied.

Example 2 (A capacity that is convex at S , not superadditive, and CEU preferences do not satisfy Condition (iii')). Let v be defined on the subsets of $S := \{1, 2, 3\}$ as follows

| A | \emptyset | 1 | 2 | 3 | 12 | 13 | 23 | 123 |
|--------------|-------------|-----|-----|-----|-----|-----|-----|-----|
| $v(A)$ | 0 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1 |
| $\bar{v}(A)$ | 0 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1 |

⁵This would not be true if $A \cap B \neq \emptyset$ and $A \cup B \subsetneq S$.

The capacity v is convex at S as $\bar{v} = v$ is subadditive. As shown by the following example, the CEU preferences represented by v do not satisfy Condition (iii').

| | 1 | 2 | 3 | $f \cdot dv$ |
|---------|---|---|---|--------------|
| X | 1 | 1 | 5 | 3 |
| Y | 2 | 0 | 0 | 1 |
| $X + Y$ | 3 | 1 | 5 | 3 |

Note that

$$\left. \begin{array}{l} (X, Y) \text{ anticomonotonic} \\ (X, X + Y) \text{ comonotonic} \end{array} \right\} \text{ but we do not have } I(X + Y) \geq I(X) + I(Y). \quad \diamond$$

The following example shows that if the capacity v is only superadditive (that is superadditive but not convex for the sure event) then Condition (iii) of Theorem 1, or equivalently Condition (iii'), may not be satisfied.

Example 3 (A capacity that is superadditive, not convex at S , and CEU preferences do not satisfy Condition (iii')). Let v be defined on the subsets of $S := \{1, 2, 3\}$ as follows

| A | \emptyset | 1 | 2 | 3 | 12 | 13 | 23 | 123 |
|--------|-------------|---|---|---|----|----|----|-----|
| $v(A)$ | 0 | 1 | 1 | 1 | 3 | 3 | 3 | 4 |

The capacity v is superadditive and not convex at S . As shown by the following example, the CEU preferences represented by v do not satisfy Condition (iii').

| | 1 | 2 | 3 | $f \cdot dv$ |
|---------|---|---|---|--------------|
| X | 2 | 2 | 0 | 6 |
| Y | 1 | 2 | 2 | 7 |
| $X + Y$ | 3 | 4 | 2 | 12 |

Note that

$$\left. \begin{array}{l} (X, Y) \text{ anticomonotonic} \\ (X, X + Y) \text{ comonotonic} \end{array} \right\} \text{ but we do not have } I(X + Y) \geq I(X) + I(Y). \quad \diamond$$

Application. Eliciting a lower bound for the reservation price of a European put option:

We consider a DM who exhibits propensity for hedging, i.e., likes to hedge using anticomonotonic assets, and therefore, agrees with inequality (iii) in Theorem 1. Assume the DM holds one share of a stock. A common strategy, in order to reduce the variability of her position, is to purchase a European put option on this stock.

Actually, if X is the stock price at the put's maturity and K is the put's strike price, then the put's payoff at maturity is $Z = (K - X)^+ := \max(0, K - X)$ and the investor's new position $Y = X + Z$ is equal to $\max(X, K)$ which is less variable because the lower tail of the original position X is replaced by a constant. The question that arises then is to determine the reservation price of Z , i.e., the maximal price for Z at which the DM would agree to buy Z in order to hedge X with Z . The following Proposition 2 provides a lower bound for this reservation price.

Proposition 2. For a DM exhibiting propensity for hedging, $\int (K - X)^+ dv$ is a lower bound for the reservation price of a put $Z = (K - X)^+$.

We close this section with a geometric property of pseudo-convex capacities analogous to Axiom C'. A preference relation \succsim is said to exhibit *preference for anticomonotonic diversification* if it satisfies the following axiom.

Axiom D' (Preference for anticomonotonic diversification). For $X, Y \in V$:

$$X \succsim Y, (X, Y) \text{ anticomonotonic}, \alpha \in (0, 1) \Rightarrow \alpha X + (1 - \alpha)Y \succsim Y.$$

Corollary 1. A CEU preference \succsim exhibits propensity for hedging, i.e., satisfies Axiom D, if and only if it exhibits preference for anticomonotonic diversification.

3.2 Propensity for minimum hedging

In order to check for ambiguity aversion, Chateauneuf (1991) shows that one does not have to test the ambiguity aversion axiom, Axiom C, for all the acts Z that are comonotonic with Y . It suffices to verify that the preference is not reversed when adding acts that raise only the largest payoffs of Y and, therefore, increase the variability of Y . The following Axiom can be found in Chateauneuf (1991).

Axiom C'' (Strong ambiguity aversion). For $X, Y, Z \in V$:

$$X \succsim Y, Z \geq 0, \text{ and } \{Z > 0\} \subseteq \operatorname{argmax} Y \Rightarrow X + Z \succsim Y + Z.$$

Clearly, a CEU preference \succsim satisfying Axiom C also satisfies Axiom C''. The following proposition states that the two axioms are equivalent.

Proposition 3. A CEU preference \succsim represented by the capacity v satisfies Axiom C'' if and only if v is convex.

The following axiom⁶ clearly depicts a weaker form of ambiguity aversion (Axiom C or C'') and propensity for hedging (Axiom D).

Axiom D''. For $X, Y, Z \in V$:

$$X \succsim Y, Z \geq 0, \{Z > 0\} \subseteq \operatorname{argmax} Y, \text{ and } (Z, X) \text{ anticomonotonic} \Rightarrow X + Z \succsim Y + Z.$$

In fact, as shown by the following proposition, Chateauneuf (1991)'s idea of testing the ambiguity aversion axiom with only acts that raise only the largest payoffs of Y extends to the propensity for hedging axiom.

Proposition 4. A CEU preference \succsim exhibits propensity for hedging, i.e., satisfies Axiom D if and only if it satisfies Axiom D''.

One of the main motivations of our paper is to weaken ambiguity aversion as usually modeled through convexity of preferences as described in Axiom C, or equivalently owing to Proposition 3, Axiom C''. Axiom C'' states that if an act X is at least as desirable as another act Y , the preference is preserved by adding to the two acts another one that only raises the largest payoff of the worse act Y , therefore increasing its variability. We now study the effects of a reduction of variability of the better act X by adding to X another act that raises the smallest payoff. A new ambiguity aversion property is given by the following axiom.

⁶We thank an Associate Editor for suggesting this axiom.

Axiom B (Minimum ambiguity aversion). For $X, Y, Z \in V$:

$$X \succsim Y, Z \geq 0, \text{ and } \{Z > 0\} \subseteq \operatorname{argmin} X \Rightarrow X + Z \succsim Y + Z.$$

Axiom B requires that Z be explicitly a particular type of hedge against X : $Z \geq 0$, and $\{Z > 0\} \subseteq \operatorname{argmin} X$. Such an act Z will be called a *minimum hedge* for X . The following proposition implies that a CEU preference satisfying Axiom B, also satisfies the ambiguity aversion Axioms C and C''.

Proposition 5. A CEU preference \succsim represented by the capacity v satisfies Axiom B if and only if v is additive.

Employing only minimum hedges in Axiom D, we obtain the following weaker axiom.

Axiom E (Propensity for minimum hedging). For $X, Y, Z \in V$:

$$X \succsim Y, (Z, Y) \text{ comonotonic}, Z \geq 0, \text{ and } \{Z > 0\} \subseteq \operatorname{argmin} X \Rightarrow X + Z \succsim Y + Z.$$

Clearly, CEU preferences \succsim satisfying Axiom D also satisfy Axiom E since a minimum hedge for an act X is anticomonotonic with X . As a consequence from Theorem 1, a pseudo-convex capacity satisfies Axiom E. The following theorem shows that, in the framework of CEU preferences, propensity for minimum hedging is characterized by a superadditive capacity, or, equivalently, a minimum-hedge superadditive Choquet functional.

Theorem 2. Let \succsim be a CEU preference represented by the capacity v . The following assertions are equivalent.

(i) \succsim exhibits Propensity for minimum hedging, i.e., satisfies Axiom E.

(ii) v is superadditive.

(iii) The functional $I(\cdot) = \int \cdot dv$ is minimum-hedge superadditive. That is, for all $X, Y \in V$ such that $Y \geq 0$ and $\{Y > 0\} \subseteq \operatorname{argmin} X$, we have

$$I(X + Y) \geq I(X) + I(Y).$$

A preference relation \succsim is said to exhibit preference for minimum-hedge diversification if it satisfies the following axiom.

Axiom E' (Preference for minimum-hedge diversification). For $X, Y \in V$:

$$X \succsim Y, Y \text{ is a minimum-hedge for } X, \alpha \in (0, 1) \Rightarrow \alpha X + (1 - \alpha)Y \succsim Y.$$

Corollary 2. A CEU preference \succsim exhibits propensity for minimum hedging, i.e., satisfies Axiom E, if and only if it exhibits preference for minimum-hedge diversification.

Following Chateauneuf (1991), there is no need to test Axiom E with all the acts Z that are comonotonic with Y , testing with acts that raise only the largest payoffs of Y suffices.

Axiom E''. For $X, Y, Z \in V$:

$$X \succsim Y, Z \geq 0, \{Z > 0\} \subseteq \operatorname{argmin} X \cap \operatorname{argmax} Y \Rightarrow X + Z \succsim Y + Z.$$

Proposition 6. A CEU preference \succsim exhibits propensity for minimum hedging, i.e., satisfies Axiom E if and only if it satisfies Axiom E''.

3.3 Convex capacities at the sure event

We now emphasize that pseudo-convex capacities, being convex at the sure event, exhibit attraction for certainty, a property pinpointed by empirical evidence for options traders (see [Fox et al. \(1996\)](#)) and confirmed by practitioners in the field (see [Tversky and Wakker \(1995\)](#), [Gonzalez and Wu \(1996\)](#), and [Wakker \(2001\)](#)). Note that under risk, attraction for certainty has been the subject of several thorough studies among which we quote [Cohen and Jaffray \(1988\)](#), [Chateauneuf \(1999\)](#), and more recently [Cerreiavioglio et al. \(2015\)](#). We first give two characterizations of convexity at the sure event. The following definition can be found in [Fox et al. \(1996\)](#).

Definition 3. A set function $v : \mathcal{A} \rightarrow \mathbb{R}$ is said to be upper subadditive if

$$v(S) - v(S \setminus A) \geq v(A \cup B) - v(B), \forall A, B \in \mathcal{A}, A \cap B = \emptyset. \quad (3.3)$$

In fact, this property is equivalent to convexity at the sure event as is shown in the following Proposition.

Proposition 7. A set function $v : \mathcal{A} \rightarrow \mathbb{R}$ is upper subadditive, i.e., satisfies (3.3), if and only if v is convex at the sure event, i.e., satisfies (3.2).

We now introduce some definitions of attraction for certainty which fit to our purpose.

Definition 4. A set function $v : \mathcal{A} \rightarrow \mathbb{R}$ exhibits attraction for certainty if

$$v(B \cup C) - v(B) \geq v(A \cup C) - v(A), \forall A, B, C \in \mathcal{A}, A \subseteq B, B \cap C = \emptyset, B \cup C = S. \quad (3.4)$$

This property is also equivalent to convexity at the sure event.

Proposition 8. A capacity v exhibits attraction for certainty, i.e., satisfies (3.4), if and only if v is convex at the sure event, i.e., satisfies (3.2).

For preference relations, we have the following attraction for certainty axiom.

Definition 5 (Attraction for certainty). The preference relation \succsim exhibits attraction for certainty if, for every partition $\{A_1, A_2, A_3\}$ of S and $(x_1, x_2, x_3) \in \mathbb{R}^3$ s.t. $x_1 \leq x_2 \leq x_3$,

$$x_1 A_1^* + x_2 A_2^* + x_3 A_3^* \sim x_1 A_1^* + x_2 A_2^* + x_3 A_3^* \quad (3.5)$$

implies

$$x_2 S^* \succsim x_1 A_2^* + x_2 A_1^* + x_3 A_3^*.^7 \quad (3.6)$$

Remark 2. Note that the simple acts appearing in (3.6) are deduced from those in (3.5) by merely transforming the common part $x_1 A_1^*$ into $x_2 A_1^*$, i.e., by adding $(x_2 - x_1) A_1^*$ to both simple acts in (3.5). Using the notation in the following table, Definition 5 says that $X \sim X' \Rightarrow X + Z \succsim X' + Z$.

| | A_1 | A_2 | A_3 |
|----------|-------------|-------|-------|
| X | x_1 | x_2 | x_2 |
| X' | x_1 | x_1 | x_3 |
| Z | $x_2 - x_1$ | 0 | 0 |
| $X + Z$ | x_2 | x_2 | x_2 |
| $X' + Z$ | x_2 | x_1 | x_3 |

⁷This definition is a restatement under ambiguity of ([Chateauneuf, 1999](#), Definition A.5, p.29), which was stated under risk.

Since Z is anticomonotonic with both X and X' it would be a hedge against X or X' . However, $X + Z$ could be preferred to $X' + Z$ precisely because Z ‘perfectly’ hedges X but might under or over hedge X' .

Clearly such a pattern of behavior is likely to be observed: an ambiguity averse individual could reasonably be attracted by the sure simple act x_2S^* , i.e., could reasonably prefer the sure simple act x_2S^* to the less certain one $x_1A_1^* + x_2A_2^* + x_3A_3^*$; of course in the case where all outcomes x_i ’s are not perceived as equivalent.

Such a behavior is consistent with the CEU model since the comonotonic sure-thing principle is not violated: the common outcome x_1 in first position in both simple acts in (3.5), is changed into a common outcome x_2 , which does not remain in first position in both simple acts in (3.6).

On the other hand such a behavior, though conceivable, cannot be explained by expected utility (EU) theory. Actually, attraction for certainty is inconsistent with EU theory (at least if strict preference occurs in (3.6)) since the sure-thing principle implies that equivalence between simple acts flows from (3.5) to (3.6).

Such remarks may serve as arguments in favor of the descriptive power of the CEU model when compared to the EU model. \diamond

Proposition 9. *If a capacity v exhibits attraction for certainty, i.e., satisfies (3.4), then the CEU preferences represented by v exhibit attraction for certainty.*

Example 4 (The converse of Proposition 9 does not hold). Let v be defined on the subsets of $S := \{1, 2, 3\}$ as follows

| | | | | | | | | |
|--------|-------------|---|---|---|----|----|----|-----|
| A | \emptyset | 1 | 2 | 3 | 12 | 13 | 23 | 123 |
| $v(A)$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |

Since $v(S) + v(\{1\}) = 1 + 0 < 1 + 1 = v(\{1, 2\}) + v(\{1, 3\})$, the capacity v is not convex at the sure event S (note that $C(v) = \emptyset$). However, CEU preferences represented by v exhibit attraction for certainty. Indeed, up to a permutation, the only 3-set partition of S is $\{A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}\}$ and if, for some $x_1 \leq x_2 \leq x_3$,

$$x_1A_1^* + x_2A_2^* + x_3A_3^* \sim x_1A_1^* + x_2A_2^* + x_3A_3^*,$$

then

$$x_1 + [x_2 - x_1]v(23) = x_1 + [x_3 - x_1]v(3),$$

and thus $x_2 = x_1$. Implying that $x_2 \geq x_2 = x_1 + [x_2 - x_1]v(13) + [x_3 - x_2]v(3)$, or equivalently, $x_2S^* \succsim x_1A_2^* + x_2A_1^* + x_3A_3^*$. \diamond

The converse of Proposition 9 holds if the capacity v satisfies $v(A) \neq 0$ whenever $A \neq \emptyset$. Note that if a capacity v is convex at the sure event then it satisfies, in particular, $v(S) \geq v(A) + v(A^c)$ for all $A \in \mathcal{A}$.

Proposition 10. *Let v be a capacity such that $v(A) \neq 0$ and $v(S) \geq v(A) + v(A^c)$ for all $A \neq \emptyset$. If the CEU preferences \succsim represented by v exhibit attraction for certainty, then v is convex at the sure event, i.e., satisfies (3.2).*

3.4 Proper hedges

In this section, we study the particular case of using specific anticomonotonic acts, namely proper hedges, introduced by [Cheung et al. \(2014\)](#) under risk, as hedging instruments.

Definition 6 (Proper hedge). *Let $X, Z \in V$. Then Z is said to be a proper hedge for X if $Z \in H(X) := \{Z' : (Z', X + Z') \text{ is anticomonotonic}\}$.*

The following proposition states that a proper hedge for X is a hedge against X whose “variation” does not exceed that of X .

Proposition 11. *Let $X, Z \in V$. The following statements are equivalent.*

- (i) $Z \in H(X)$, i.e., Z is a proper hedge for X .
- (ii) (Z, X) is anticomonotonic and $|Z(s) - Z(t)| \leq |X(s) - X(t)|$ for all $s, t \in S$.

Remark 3. It is worth noticing, with the help of (ii) in Proposition 11, that a European put option on a stock X is a proper hedge for X . \diamond

Definition 7 (Ambiguity reducer). *Let $X, Z \in V$. Then Z is said to be an ambiguity-reducer (or ambiguity-reducing) for X if*

$$\text{var}_P(X + Z) \leq \text{var}_P(X) \text{ for any probability } P \text{ on } (S, \mathcal{A}).$$

Under ambiguity, hedging using proper hedges appears to be meaningful since, according to the proposition below, every proper hedge is ambiguity-reducing.

Proposition 12. *Let $X, Z \in V$. If $Z \in H(X)$ then Z is ambiguity-reducing for X .*

For the sake of completeness we provide the following simple characterization of ambiguity-reducing acts, which are in fact anticomonotonic acts.

Corollary 3. *Let $X, Z \in V$. Then Z is an ambiguity reducer for X if and only if one of the following equivalent statements is satisfied.*

- (i) $Z \in H(2X)$.
- (ii) (Z, X) is anticomonotonic and $|Z(s) - Z(t)| \leq 2|X(s) - X(t)|$ for all $s, t \in S$.

3.4.1 Propensity for proper hedging

We now formulate the axiom describing attraction for proper hedges.

Axiom F (Propensity for proper hedging). *For $X, Y, Z \in V$:*

$$X \succsim Y, (Z, Y) \text{ comonotonic, and } Z \in H(X) \Rightarrow X + Z \succsim Y + Z.$$

Definition 8. *A capacity v is superadditive at the sure event if $v(S) \geq v(A) + v(A^c)$ for all $A \in \mathcal{A}$, i.e., if v satisfies $v \leq \bar{v}$.*

Note that superadditive capacities and capacities that are convex at the sure event, are superadditive at the sure event.

Proposition 13. Let $I_v : X \in V \rightarrow I_v(X) = \int X dv$ where v is a capacity on (S, \mathcal{A}) . The following statements are equivalent.

- (a) The capacity v is superadditive at the sure event.
- (b) For all $X \in V$ such that $I_v(X)$ exists: $-I_v(-X) \geq I_v(X)$, i.e., $I_{\bar{v}}(X) \geq I_v(X)$.
- (c) For all $X \in V$ such that $I_v(X)$ exists: $I_v(X) \geq 0 \Rightarrow I_{\bar{v}}(X) \geq 0$.

Superadditive capacities at the sure event characterize hedging through proper hedges.

Theorem 3. Let \succsim be a CEU preference represented by the capacity v . The following assertions are equivalent.

- (i) \succsim exhibits propensity for proper hedging, i.e., satisfies Axiom F.
- (ii) v is superadditive at the sure event.
- (iii) The functional $I(\cdot) = \int \cdot dv$ is proper-hedge superadditive. That is, for all $X, Y \in V$ such that $Y \in H(X)$, i.e., $(Y, X + Y)$ anticomonotonic, we have

$$I(X + Y) \geq I(X) + I(Y).$$

Remark 4. The following inclusions are straightforward:

$$\begin{aligned} & \{(X, Y) \in V^2 : (Y, X + Y) \text{ is anticomonotonic}\} \\ & \subseteq \{(X, Y) \in V^2 : (X, Y) \text{ is anticomonotonic and } (X, X + Y) \text{ is comonotonic}\} \\ & \subseteq \{(X, Y) \in V^2 : (X, Y) \text{ is anticomonotonic}\}. \end{aligned}$$

The reverse inclusions do not hold. Indeed,

- If $A, B \subsetneq S$ are strict subsets of S such that $A \cap B \neq \emptyset$ and $A \cup B = S$ then (A^*, B^*) is anticomonotonic, $(A^*, A^* + B^*)$ is comonotonic, and $(B^*, A^* + B^*)$ is not anticomonotonic.
- If $A, B \subsetneq S$ are strict subsets of S such that $A \cap B = \emptyset$ and $A \cup B = S$ then $(A^*, 2B^*)$ is anticomonotonic and $(A^*, A^* + 2B^*)$ is not comonotonic. \diamond

3.4.2 Preference for perfect hedging

As shown by the following Example 5, superadditivity at the sure event is too weak of a requirement to model a convexity property as “strong” as the one in Axiom D’ or E’.

Example 5 (A capacity that is convex at S , hence superadditive at S , and CEU preferences do not satisfy $X \succsim Y, Y \in H(X)$, and $\alpha \in (0, 1) \Rightarrow \alpha X + (1 - \alpha)Y \succsim Y$). Let v be defined on the subsets of $S := \{1, 2, 3\}$ as follows

| | | | | | | | | |
|--------|-------------|-----|-----|-----|-----|-----|-----|-----|
| A | \emptyset | 1 | 2 | 3 | 12 | 13 | 23 | 123 |
| $v(A)$ | 0 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1 |

The capacity v is convex at S as $\bar{v} = v$ is subadditive, hence v is superadditive at S . Define $X, Y \in V$ as follows.

| | | | | |
|-------------------|------|-----|------|-----------------|
| | 1 | 2 | 3 | $\int \cdot dv$ |
| X | 1 | 3 | 5 | 3 |
| Y | 4 | 2 | 2 | 3 |
| $(1/4)X + (3/4)Y$ | 13/4 | 9/4 | 11/4 | 11/4 |

Then $X \succsim Y, Y \in H(X)$, and $(1/4)X + (3/4)Y \not\sucsim Y$. Note that v is not superadditive, e.g., $v(12) \not\geq v(1) + v(2)$, hence v is not pseudo-convex. \diamond

However, superadditive capacities at the sure event do characterize, preference for perfect hedging; a much weaker convexity property. We say that \succsim exhibits attraction for perfect hedging if it satisfies the following Axiom **F'**.

Axiom **F'** (Attraction for perfect hedging). For $X, Y \in V$:

$$X \sim Y, \alpha \in (0, 1), \text{ and } \alpha X + (1 - \alpha)Y = aS^* \text{ (for some } a \in \mathbb{R}) \Rightarrow aS^* \succsim Y.$$

Note that Axiom **F'** is a relaxed version of the preference for sure diversification axiom introduced by [Chateauneuf and Tallon \(2002\)](#):

Let $X_i \in V (i = 1, 2, \dots, n)$, $X_1 \sim X_2 \sim \dots \sim X_n$, $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$, $\sum_{i=1}^n \alpha_i = 1$, and $\sum_{i=1}^n \alpha_i X_i = aS^*$ for some $a \in \mathbb{R}$ implies $aS^* \succsim X_i$ for all $i = 1, 2, \dots, n$.

For CEU preferences, preference for sure diversification was proved in [Chateauneuf and Tallon \(2002\)](#) to be characterized by a balanced capacity v , i.e., $C(v) \neq \emptyset$. Here, given that a perfect hedge is in fact a proper hedge, we have:

Proposition 14. Let \succsim be a CEU preference represented by the capacity v . The following assertions are equivalent.

- (a) \succsim exhibits attraction for perfect hedging, i.e., satisfies Axiom **F'**.
- (b) v is superadditive at the sure event, i.e., $v(S) \geq v(A) + v(A^c)$ for all $A \in \mathcal{A}$.

Again, there is no need to test the validity of Axiom **F** with all the acts Z that are comonotonic with Y , testing with acts that raise only the largest payoffs of Y is enough.

Axiom **F''**. For $X, Y, Z \in V$:

$$X \succsim Y, Z \geq 0, \{Z > 0\} \subseteq \text{argmax} Y, \text{ and } Z \in H(X) \Rightarrow X + Z \succsim Y + Z.$$

Proposition 15. A CEU preference \succsim exhibits propensity for proper hedging, i.e., satisfies Axiom **F** if and only if it satisfies Axiom **F''**.

4 Concluding remarks

One version of the Independence Axiom is the following.

Axiom **I** (Independence Axiom). For $X, Y, Z \in V$:

$$X \succsim Y \text{ and } \alpha \in (0, 1) \Rightarrow \alpha X + (1 - \alpha)Z \succsim \alpha Y + (1 - \alpha)Z.$$

This paper offers several new notions of ambiguity aversion. One, minimum ambiguity aversion, stronger than the usual notion of ambiguity aversion, turned out to be a new characterization of the Independence Axiom, while others (propensity for hedging, propensity for minimum hedging, and propensity for proper hedging) are proven to be weaker. In Axiom **B**, which mirrors Axiom **C''**, only minimum hedges are allowed as hedging instruments. If instead all anticomonotonic acts are available as hedges, we get the following requirement mirroring Axiom **C**, which is clearly stronger than Axiom **B**.

Axiom **A** (Strong minimum ambiguity aversion). For $X, Y, Z \in V$:

$$X \succsim Y \text{ and } (Z, X) \text{ anticomonotonic} \Rightarrow X + Z \succsim Y + Z.$$

The following proposition summarizes the different relationships between the axioms used in the paper.

Proposition 16. *For CEU preferences, the following hold true:*

- (a) $A \Leftrightarrow B \Leftrightarrow I \Leftrightarrow v$ is additive.
- (b) $C \Leftrightarrow C' \Leftrightarrow C'' \Leftrightarrow v$ is convex.
- (c) $D \Leftrightarrow D' \Leftrightarrow D'' \Leftrightarrow v$ is pseudo-convex.
- (d) $E \Leftrightarrow E' \Leftrightarrow E'' \Leftrightarrow v$ is superadditive.
- (e) $F \Leftrightarrow F' \Leftrightarrow F'' \Leftrightarrow v$ is superadditive at the sure event.
- (f) $B \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow F$.

The proof of Proposition 16 is immediate. The converses of all implications in (f) do not hold true. The interesting situation where only ambiguity reducers are available as hedging instruments is modeled through the following axiom.

Axiom R (Propensity for ambiguity reduction). *For $X, Y, Z \in V$:*

$$X \succsim Y, (Z, Y) \text{ comonotonic, and } Z \in H(2X) \Rightarrow X + Z \succsim Y + Z.$$

Axiom R falls between Axiom D and Axiom F and we would expect the corresponding Choquet functional to be ambiguity-reducer superadditive, i.e., for all $X, Y \in V$ such that $Y \in H(2X)$, we have $I(X + Y) \geq I(X) + I(Y)$. The question pertaining to the required properties for the corresponding capacity, beyond superadditivity at the sure event, is the subject of future research.

5 Proofs

Proof of Lemma 1. • [Proof of (i) \Rightarrow (ii)] Let μ be a probability on \mathcal{A} and fix $t \in S$. Integrating $(X(s) - X(t))(Y(s) - Y(t))$ with respect to s and using the linearity of the integral, we get:

$$\begin{aligned} \int_S (X(s) - X(t))(Y(s) - Y(t)) d\mu(s) &= \int_S (X(s)Y(s) + X(t)Y(t) - X(t)Y(s) - X(s)Y(t)) d\mu(s) \\ &= E_\mu(XY) + X(t)Y(t) - X(t)E_\mu(Y) - Y(t)E_\mu(X). \end{aligned}$$

Now, integrating with respect to t , we obtain:

$$\begin{aligned} \int_S (\int_S (X(s) - X(t))(Y(s) - Y(t)) d\mu(s)) d\mu(t) \\ &= \int_S (E_\mu(XY) + X(t)Y(t) - X(t)E_\mu(Y) - Y(t)E_\mu(X)) d\mu(t) \\ &= E_\mu(XY) + E_\mu(XY) - E_\mu(X)E_\mu(Y) - E_\mu(Y)E_\mu(X) \\ &= 2 \text{cov}_\mu(X, Y). \end{aligned}$$

Hence, the result follows from the fact that the above double integral is nonnegative (resp. nonpositive) if (X, Y) is comonotonic (resp. anticomonotonic).

• [Proof of (ii) \Rightarrow (i)] Let $s, t \in S$ such that $s \neq t$ and let $\alpha \in (0, 1)$. Consider the probability $P = \alpha\delta_s + (1 - \alpha)\delta_t$ where δ_s is the Dirac measure defined for $A \in \mathcal{A}$ by

$$\delta_s(A) := \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Writing that $\text{cov}_P(X, Y) \geq 0$ (resp. $\text{cov}_P(X, Y) \leq 0$), we obtain

$$\begin{aligned}
\text{cov}_P(X, Y) &= E(XY) - E(X)E(Y) \\
&= \alpha X(s)Y(s) + (1-\alpha)X(t)Y(t) - (\alpha X(s) + (1-\alpha)X(t))(\alpha Y(s) + (1-\alpha)Y(t)) \\
&= \alpha X(s)Y(s) + (1-\alpha)X(t)Y(t) - \alpha^2 X(s)Y(s) - (1-\alpha)^2 X(t)Y(t) \\
&\quad - \alpha(1-\alpha)X(s)Y(t) - \alpha(1-\alpha)X(t)Y(s) \\
&= \alpha(1-\alpha)(X(s)Y(s) + X(t)Y(t) - X(s)Y(t) - X(t)Y(s)) \\
&= \alpha(1-\alpha)(X(s) - X(t))(Y(s) - Y(t)).
\end{aligned}$$

From $\alpha \in (0, 1)$, we have $\alpha(1-\alpha) > 0$. Thus $\text{cov}_P(X, Y) \geq 0$ (resp. $\text{cov}_P(X, Y) \leq 0$) implies $(X(s) - X(t))(Y(s) - Y(t)) \geq 0$ (resp. $(X(s) - X(t))(Y(s) - Y(t)) \leq 0$), that is, (X, Y) is comonotonic (resp. anticomonotonic). \square

Proof of Lemma 2. • [Proof of (i) \Rightarrow (ii)] Let $a \in \mathbb{R}$ be such that $X \sim aS^*$. Then $X + Z \succsim aS^* + Z$. Therefore, $I(X + Z) \geq a + I(Z)$, i.e., $I(X + Z) \geq I(X) + I(Z)$.

• [Proof of (ii) \Rightarrow (i)] Let $X \succsim Y$. Then, by (ii), $I(X + Z) \geq I(X) + I(Z)$ and from $I(X) \geq I(Y)$, one gets $I(X + Z) \geq I(Y) + I(Z)$. Since (Y, Z) is comonotonic, $I(Y) + I(Z) = I(Y + Z)$, implying $I(X + Z) \geq I(Y + Z)$. Therefore $X + Z \succsim Y + Z$. \square

Proof of Proposition 1. Let v be an exact capacity on \mathcal{A} . First, we show that v is superadditive. Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$ and let $\lambda \in C(v)$ be such that $\lambda(A \cup B) = v(A \cup B)$. Then,

$$v(A \cup B) = \lambda(A \cup B) = \lambda(A) + \lambda(B) \geq v(A) + v(B),$$

where the last inequality follows from $\lambda \in C(v)$. Hence v is superadditive.

Second, we show that v is convex at the sure event, i.e., satisfies (3.2). Let $A, B \in \mathcal{A}$ be such that $A \cup B = S$ and let $\lambda \in C(v)$ be such that $\lambda(A \cap B) = v(A \cap B)$. Then,

$$\begin{aligned}
v(A \cup B) + v(A \cap B) &= v(S) + \lambda(A \cap B) = \lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B) \\
&\geq v(A) + v(B)
\end{aligned}$$

Where the last inequality follows from $\lambda \in C(v)$. Hence v is convex at the sure event. \square

Proof of Theorem 1. • [Proof of (i) \Rightarrow (ii)] First, we prove that v is superadditive. Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$. Denote $X = v(S)A^*$, $Y = v(A)S^*$, and $Z = v(S)B^*$. Then

| | A | B | $(A \cup B)^c$ | $\int \cdot dv$ |
|---------|--------|---------------|----------------|-----------------------|
| X | $v(S)$ | 0 | 0 | $v(S)v(A)$ |
| Y | $v(A)$ | $v(A)$ | $v(A)$ | $v(S)v(A)$ |
| Z | 0 | $v(S)$ | 0 | |
| $X + Z$ | $v(S)$ | $v(S)$ | 0 | $v(S)v(A \cup B)$ |
| $Y + Z$ | $v(A)$ | $v(S) + v(A)$ | $v(A)$ | $v(A)v(S) + v(B)v(S)$ |

Since $X \succsim Y$, (Z, Y) is comonotonic, and (Z, X) is anticomonotonic, from Axiom D, one gets $X + Z \succsim Y + Z$. Hence $v(A \cup B) \geq v(A) + v(B)$, i.e., v is superadditive.

Second, we prove that v is convex at the sure event. Let $A, B \in \mathcal{A}$ be such that $A \cup B = S$. Denote $X = v(S)A^*$, $Y = v(A)S^*$, and $Z = v(S)B^*$. Then

| | $A \setminus B$ | $A \cap B$ | $B \setminus A$ | $\int \cdot dv$ |
|---------|-----------------|---------------|-----------------|----------------------------|
| X | $v(S)$ | $v(S)$ | 0 | $v(A)$ |
| Y | $v(A)$ | $v(A)$ | $v(A)$ | $v(A)$ |
| Z | 0 | $v(S)$ | $v(S)$ | |
| $X + Z$ | $v(S)$ | $2v(S)$ | $v(S)$ | $v(S)(v(S) + v(A \cap B))$ |
| $Y + Z$ | $v(A)$ | $v(A) + v(S)$ | $v(A) + v(S)$ | $v(S)(v(A) + v(B))$ |

Note that $X \sim Y$, (Z, Y) is comonotonic, and (Z, X) is anticomonotonic. Hence, from Axiom D, we have $X + Z \succsim Y + Z$. Therefore $v(S) + v(A \cup B) \geq v(A) + v(B)$, that is, v is convex at the sure event.

• [Proof of (ii) \Rightarrow (iii)] Note that it suffices to prove the result for nonnegative functions X and Y . For, the result for arbitrary bounded functions may be deduced from the result for non negative functions using $\widehat{X} := \max(0, -\inf X) + X$ and $\widehat{Y} := \max(0, -\inf Y) + Y$ and noticing that \widehat{X} and \widehat{Y} satisfy the same condition as X and Y , i.e., \widehat{X} and \widehat{Y} are anticomonotonic.

◆ We first prove that (ii) \Rightarrow (iii) is true on V_0 . For $X, Y \in V_0$, it is immediate that X and Y are anticomonotonic if and only if there exists a finite partition $A_1, \dots, A_i, \dots, A_n$ ($n \in \mathbb{N}^*$) of S , such that $X = \sum_{i=1}^n x_i A_i^*$ and $Y = \sum_{i=1}^n y_i A_i^*$, with $A_i \in \mathcal{A}$ for all i , and $x_i \leq x_{i+1}$ and $y_i \geq y_{i+1}$ for all $i \in \llbracket 1, n-1 \rrbracket$.

We prove (ii) \Rightarrow (iii) by induction on n . We want to show that (iii), $I(X + Y) \geq I(X) + I(Y)$, is satisfied.

► If $n = 1$ this is immediate since in such a case $X = xS^*, Y = yS^*$, and thus

$$I(X + Y) = x + y = I(X) + I(Y).$$

► Assume $n = 2$, hence $X = x_1 A_1^* + x_2 A_2^*$ with $x_1 \leq x_2$ and $Y = y_1 A_1^* + y_2 A_2^*$ with $y_1 \geq y_2$ and $\{A_1, A_2\}$ partition of S with elements of \mathcal{A} . Note that $I(X) = x_1 + (x_2 - x_1)v(A_2)$ and $I(Y) = y_2 + (y_1 - y_2)v(A_1)$, hence

$$I(X) + I(Y) = (x_1 + y_2) + (x_2 - x_1)v(A_2) + (y_1 - y_2)v(A_1).$$

Thus, either $x_1 + y_1 \leq x_2 + y_2$ implying $I(X + Y) = (x_1 + y_1) + (x_2 - x_1 + y_2 - y_1)v(A_2)$ and hence, using v 's convexity at the sure event,

$$I(X + Y) - I(X) - I(Y) = (y_1 - y_2)(1 - v(A_1) - v(A_2)) \geq 0.$$

Or $x_2 + y_2 \leq x_1 + y_1$ which, using v 's convexity at the sure event, implies

$$I(X + Y) - I(X) - I(Y) = (x_2 - x_1)(1 - v(A_1) - v(A_2)) \geq 0$$

This completes the proof when $n = 2$.

► Suppose now that (ii) \Rightarrow (iii) is true on V_0 up to rank $n - 1$, where $n - 1 \geq 2$, we have to prove that it is also true for n . Let $X, Y \in V_0$ be anticomonotonic with respect to a n -partition as defined above, in such a case, we will say that X and Y are n -anticomonotonic. Denote $X' := x_2(A_1 \cup A_2)^* + \sum_{i=3}^n x_i A_i^*$, $Y' := y_2(A_1 \cup A_2)^* + \sum_{i=3}^n y_i A_i^*$, $X'' := (x_1 - x_2)A_1^*$, and $Y'' := (y_1 - y_2)A_1^*$. Note that $X = X' + X''$ and $Y = Y' + Y''$, X' and X'' are comonotonic and so are Y' and Y'' , X' and Y' are $(n - 1)$ -anticomonotonic, and X'' and Y'' are 2-anticomonotonic. Hence

$$\begin{aligned}
I(X) + I(Y) &= I(X' + X'') + I(Y' + Y'') \\
&= I(X') + I(X'') + I(Y') + I(Y'') \\
&= I(X') + I(Y') + I(X'') + I(Y''),
\end{aligned}$$

where the second equality follows from comonotonic additivity of the Choquet functional. From the induction hypothesis and the case $n = 2$, we obtain, respectively, $I(X') + I(Y') \leq I(X' + Y')$ and $I(X'') + I(Y'') \leq I(X'' + Y'')$. Therefore

$$I(X) + I(Y) \leq I(X' + Y') + I(X'' + Y'').$$

Denote $\alpha = I(X + Y)$, $\alpha' = I(X' + Y')$, and $\alpha'' = I(X'' + Y'')$ and let us prove that $\alpha' + \alpha'' \leq \alpha$ which would complete the proof for the case where $X, Y \in V_0$. Let $z_i = x_i + y_i$ and $z_{i_1} \leq \dots \leq z_{i_j} \leq \dots \leq z_{i_n}$ where $j \in \llbracket 1, n \rrbracket$. To simplify notations, we will write $v(i_s \dots i_n)$ instead of $v(A_{i_s} \cup \dots \cup A_{i_n})$ for $s \in \llbracket 2, n \rrbracket$ and so on. There are two cases to consider.

❶ **Case 1.** $i_k = 1$ and $i_\ell = 2$ where $k < \ell$. Then

$$\alpha = z_{i_1} + \sum_{s=2}^k (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n) + \beta + \sum_{s=\ell+1}^n (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n),$$

where

$$\beta = \sum_{s=k+1}^\ell (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n).$$

In the same way, but taking into account that $z_{i_k} \notin (X' + Y')(S)$, one gets:

$$\alpha' = z_{i_1} + \sum_{s=2}^{k-1} (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n) + \beta' + \sum_{s=\ell+1}^n (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n),$$

where

$$\beta' = (z_{i_{k+1}} - z_{i_{k-1}})v(i_k \ i_{k+1} \dots i_n) + \sum_{s=k+2}^\ell (z_{i_s} - z_{i_{s-1}})v(i_k \ i_s \dots i_n).$$

Noting that $(z_{i_{k+1}} - z_{i_{k-1}})v(i_k \dots i_n) = (z_{i_{k+1}} - z_{i_k})v(i_k \dots i_n) + (z_{i_k} - z_{i_{k-1}})v(i_k \dots i_n)$ and letting

$$\beta'' = \sum_{s=k+1}^\ell (z_{i_s} - z_{i_{s-1}})v(i_k \ i_s \dots i_n),$$

we have

$$\alpha' = z_{i_1} + \sum_{s=2}^k (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n) + \beta'' + \sum_{s=\ell+1}^n (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n).$$

Furthermore, since $i_k = 1$ and $i_\ell = 2$,

$$\alpha'' = \int -(z_2 - z_1)A_1^* dv = -(z_2 - z_1) + (z_2 - z_1)v(A_1^c) = (z_{i_\ell} - z_{i_k})(v(S \setminus i_k) - 1).$$

We have to prove that $\beta'' + \alpha'' \leq \beta$. Since obviously

$$\alpha'' = \sum_{s=k+1}^\ell (z_{i_s} - z_{i_{s-1}})(v(S \setminus i_k) - 1)$$

and, by v 's convexity at the sure event, $v(i_k \ i_s \dots i_n) + v(S \setminus i_k) - 1 \leq v(i_s \dots i_n)$, we successively have

$$\begin{aligned}
\beta'' + \alpha'' &= \sum_{s=k+1}^\ell (z_{i_s} - z_{i_{s-1}})(v(i_k \ i_s \dots i_n) + v(S \setminus i_k) - 1) \\
&\leq \sum_{s=k+1}^\ell (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n) = \beta.
\end{aligned}$$

❷ **Case 2.** $i_k = 2$ and $i_\ell = 1$ where $k < \ell$. Then

$$\alpha = z_{i_1} + \sum_{s=2}^k (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n) + \beta + \sum_{s=\ell+1}^n (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n),$$

where

$$\beta = \sum_{s=k+1}^\ell (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n).$$

In the same way as with Case 1,

$$\alpha' = z_{i_1} + \sum_{s=2}^k (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n) + \beta' + \sum_{s=\ell+2}^n (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n),$$

where

$$\beta' = \sum_{s=k+1}^{\ell-1} (z_{i_s} - z_{i_{s-1}})v((i_s \dots i_n) \setminus i_\ell) + (z_{i_{\ell+1}} - z_{i_{\ell-1}})v(i_{\ell+1} \dots i_n).$$

So that, letting

$$\beta'' = \sum_{s=k+1}^{\ell} (z_{i_s} - z_{i_{s-1}})v((i_s \dots i_n) \setminus i_\ell),$$

we have

$$\alpha' = z_{i_1} + \sum_{s=2}^k (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n) + \beta'' + \sum_{s=\ell+1}^n (z_{i_s} - z_{i_{s-1}})v(i_s \dots i_n).$$

Furthermore, $\alpha'' = (z_{i_\ell} - z_{i_k})v(i_\ell)$. We then have to prove that $\beta'' + \alpha'' \leq \beta$. Since

$$\alpha'' = \sum_{s=k+1}^{\ell} (z_{i_s} - z_{i_{s-1}})v(i_\ell),$$

and since v 's superadditivity implies $v((i_s \dots i_n) \setminus i_\ell) + v(i_\ell) \leq v(i_s \dots i_n)$, we get $\beta'' + \alpha'' \leq \beta$.

◆ It remains to show that $I(X + Y) \geq I(X) + I(Y)$ for all $X, Y \in V$ where X and Y are non-negative and anticomonotonic.⁸ Let $X_n, Y_n \in V_0$ be the usual uniform approximations of, respectively, X and Y . So

$$X_n = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \left\{ \frac{i}{2^n} < X \leq \frac{i+1}{2^n} \right\}^* \quad \text{and} \quad Y_n = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \left\{ \frac{i}{2^n} < Y \leq \frac{i+1}{2^n} \right\}^*.$$

It is easy to see that there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $X - \frac{1}{2^n} < X_n \leq X$ and $Y - \frac{1}{2^n} < Y_n \leq Y$. Thus, $X + Y - \frac{1}{2^{n-1}} < X_n + Y_n \leq X + Y$ and therefore

$$\int (X + Y - \frac{1}{2^{n-1}})dv \leq \int (X_n + Y_n)dv \leq \int (X + Y)dv.$$

Furthermore, since X and Y are anticomonotonic then it is the same for X_n and Y_n . Indeed, let $s, t \in S$ be such that $X_n(s) < X_n(t)$ then $X(s) < X(t) \Rightarrow Y(s) \geq Y(t) \Rightarrow Y_n(s) \geq Y_n(t)$, that is, X_n and Y_n are anticomonotonic. Thus, from above, $I(X_n + Y_n) \geq I(X_n) + I(Y_n)$. Taking the limit in the previous inequality, we obtain the desired inequality $I(X + Y) \geq I(X) + I(Y)$.

• [Proof of (iii) \Rightarrow (i)] Let $X, Y, Z \in V$ be such that $X \succsim Y$, (Z, Y) is comonotonic, and (Z, X) is anticomonotonic. Then

$$I(X + Z) \geq I(X) + I(Z) \geq I(Y) + I(Z) = I(Y + Z),$$

where the first inequality follows from anticomonotonic superadditivity of the Choquet functional (assumed in (iii)), the second inequality follows from $X \succsim Y$, and the equality follows from comonotonic additivity of the Choquet functional. Hence $X + Z \succsim Y + Z$. \square

Proof of Corollary 1. We show that a CEU preference \succsim represented by the capacity v satisfies Axiom D', i.e., preference for anticomonotonic diversification, if and only if v is pseudo-convex. By Theorem 1 Axiom D' is then equivalent to Axiom D.

• (*Only if part*) Assume that \succsim satisfies Axiom D'. We first prove that v is superadditive. Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$. If $v(A) = 0$ or $v(B) = 0$ the inequality $v(A \cup B) \geq$

⁸Note: since X and Y are bounded and I is constant additive, it is enough to prove this inequality for X and Y non-negative and anticomonotonic. Actually, for $a > 0$ sufficiently large $X + aS^*$ and $Y + aS^*$ are non-negative and are also anticomonotonic if X and Y are $(I(X + a + Y + a) \geq I(X + a) + I(Y + a) \Leftrightarrow I(X + Y) \geq I(X) + I(Y))$.

$v(A) + v(B)$ follows from monotonicity of v . Assume, then, that $v(A) > 0$ and $v(B) > 0$. There exist $\gamma, \delta > 0$ such that $\gamma v(A) = \delta v(B)$, i.e., $\gamma A^* \sim \delta B^*$. Since $(\gamma A^*, \delta B^*)$ is anticomonotonic, by Axiom **D'** with $\alpha = \delta/(\gamma + \delta)$ and thus $1 - \alpha = \gamma/(\gamma + \delta)$, we have

$$\frac{\delta}{\gamma + \delta}(\gamma A^*) + \frac{\gamma}{\gamma + \delta}(\delta B^*) \succsim \delta B^*, \text{ i.e., } \frac{\delta\gamma}{\gamma + \delta}v(A \cup B) \geq \delta v(B).$$

Taking into account the fact that $\delta v(B) = \gamma v(A)$, we get $v(A \cup B) \geq v(A) + v(B)$.

Second, we prove that v is convex for the sure event. Let $A, B \in \mathcal{A}$ be such that $A \cup B = S$. If $v(A) = 0$ or $v(B) = 0$ the inequality $v(S) + v(A \cap B) \geq v(A) + v(B)$ follows from monotonicity of v . Assume, then, that $v(A) > 0$ and $v(B) > 0$. There exist $\gamma, \delta > 0$ such that $\gamma v(A) = \delta v(B)$, i.e., $\gamma A^* \sim \delta B^*$. Since $(\gamma A^*, \delta B^*)$ is anticomonotonic, by Axiom **D'** with $\alpha = \delta/(\gamma + \delta)$ and thus $1 - \alpha = \gamma/(\gamma + \delta)$, we have

$$\frac{\delta}{\gamma + \delta}(\gamma A^*) + \frac{\gamma}{\gamma + \delta}(\delta B^*) \succsim \delta B^*, \text{ i.e., } \frac{\delta\gamma}{\gamma + \delta}(v(A \cup B) + v(A \cap B)) \geq \delta v(B).$$

Taking into account the fact that $\delta v(B) = \gamma v(A)$, we get $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$.

• (If part) Assume now that v is pseudo-convex and let $X, Y \in V$ be such that $X \succsim Y$ and (X, Y) anticomonotonic, and let $\alpha \in (0, 1)$. Then $(\alpha X, (1 - \alpha)Y)$ is anticomonotonic. Hence

$$I(\alpha X + (1 - \alpha)Y) \geq I(\alpha X) + I((1 - \alpha)Y) = \alpha I(X) + (1 - \alpha)I(Y) \geq I(Y),$$

where the inequality follows from Theorem 1 and the last inequality follows from $X \succsim Y$. \square

Proof of Proposition 2. Denote by Z the put's payoff, i.e., $Z := (K - X)^+$. If π_Z is the put price, a DM ranking prospects through the Choquet integral with respect to v will prefer $X + Z - \pi_Z$ to X if $I(X + Z - \pi_Z) \geq I(X)$. It is immediate that Z and X are anticomonotonic. Thus, from Theorem 1, if the DM exhibits propensity for hedging, $I(X + Z - \pi_Z) \geq I(X) + I(Z) - \pi_Z$. Hence, for $\pi_Z \leq I(Z) := \int Z dv$, we have $I(X + Z - \pi_Z) \geq I(X) + I(Z) - \pi_Z \geq I(X)$. Therefore the DM will purchase the put to reduce variability, which completes the proof. \square

Proof of Proposition 3. • We prove that if the CEU preference \succsim satisfies Axiom **C''** then v is convex. Let $A, B \in \mathcal{A}$. If $v(A) = 0$ or $v(B) = 0$ then the inequality

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$$

follows from monotonicity of v . Assume that $v(A) > 0$ and $v(B) > 0$. Let $X = v(S)A^*$, $Y = v(A)S^*$, and $Z = v(S)((A \cap B)^* + (B \setminus A)^*)$. Then

| | $A \setminus B$ | $A \cap B$ | $B \setminus A$ | $(A \cup B)^c$ | note or $\int \cdot dv$ |
|---------|-----------------|---------------|-----------------|----------------|---------------------------------------|
| X | $v(S)$ | $v(S)$ | 0 | 0 | $v(A)$ |
| Y | $v(A)$ | $v(A)$ | $v(A)$ | $v(A)$ | $v(A)$ |
| Z | 0 | $v(S)$ | $v(S)$ | 0 | $\{Z > 0\} \subseteq \text{argmax} Y$ |
| $X + Z$ | $v(S)$ | $2v(S)$ | $v(S)$ | 0 | $v(S)(v(A \cup B) + v(A \cap B))$ |
| $Y + Z$ | $v(A)$ | $v(S) + v(A)$ | $v(S) + v(A)$ | $v(A)$ | $v(S)(v(A) + v(B))$ |

Since $X \succsim Y$, $Z \geq 0$, and $\{Z > 0\} \subseteq \text{argmax} Y$, from Axiom **C''**, one gets $X + Z \succsim Y + Z$, that is, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$.

• Conversely, assume that v is convex. We first claim that $Z \geq 0$ and $\{Z > 0\} \subseteq \text{argmax} Y$ implies that (Y, Z) is comonotonic, that is,

$$(Y(s) - Y(t))(Z(s) - Z(t)) \geq 0 \quad \forall (s, t) \in S^2.$$

Indeed, it is enough to consider the case where s is such that $Z(s) > 0$ and $t \notin \operatorname{argmax} Y$, the other cases being trivial. From $Z(s) > 0$ and $\{Z > 0\} \subseteq \operatorname{argmax} Y$, we get $Y(s) = \max_{s' \in S} Y(s')$. From $t \notin \operatorname{argmax} Y$, we get $Z(t) = 0$ and $Y(t) < Y(s) = \max_{s' \in S} Y(s')$. Thus $(Y(s) - Y(t))(Z(s) - Z(t)) > 0$, which ends the proof of the claim.

Now, since v is convex, we successively have

$$\int (X + Z)dv \geq \int Xdv + \int Zdv \geq \int Ydv + \int Zdv = \int (Y + Z)dv,$$

where the first inequality follows from superadditivity of the Choquet functional for convex capacities (v is convex), the second inequality follows from $X \succsim Y$, and the last equality follows from comonotonic additivity of the Choquet functional ((Z, Y) is comonotonic). Hence $X + Z \succsim Y + Z$. \square

Proof of Proposition 4. Axiom D clearly implies Axiom D''. To prove they are equivalent, we just need to show that if \succsim is a CEU preference represented by the Choquet integral w.r.t. v and satisfies Axiom D'' then v is pseudo-convex. The latter property being equivalent, by Theorem 1, to Axiom D.

First, we prove that v is superadditive. Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$. If $v(A) = 0$ or $v(B) = 0$ then the inequality follows from monotonicity of v . Assume that $v(A) > 0$ and $v(B) > 0$. There exist $\gamma, \delta > 0$ such that $\gamma v(A) = \delta v(B)$, i.e., $\gamma A^* \sim \delta B^*$. Define X, Y, Z as follows

| | A | B | $(A \cup B)^c$ | note or $\int \cdot dv$ |
|---------|----------|-------------------|----------------|---|
| X | γ | 0 | 0 | $\gamma v(A)$ |
| Y | 0 | δ | 0 | $\delta v(B) = \gamma v(A)$ |
| Z | 0 | γ | 0 | Z satisfies LHS in Axiom D'' |
| $X + Z$ | γ | γ | 0 | $\gamma v(A \cup B)$ |
| $Y + Z$ | 0 | $\gamma + \delta$ | 0 | $(\gamma + \delta)v(B) = \gamma(v(A) + v(B))$ |

Since \succsim satisfies Axiom D'', we have $X + Z \succsim Y + Z$, i.e., $v(A \cup B) \geq v(A) + v(B)$.

Second, we prove that v is convex at S . Let $A, B \in \mathcal{A}$ be such that $A \cup B = S$. If $v(A) = 0$ or $v(B) = 0$ then the inequality follows from monotonicity of v . Assume that $v(A) > 0$ and $v(B) > 0$. There exist $\gamma, \delta > 0$ such that $\gamma v(A) = \delta v(B)$, i.e., $\gamma A^* \sim \delta B^*$. Define X, Y, Z ⁹ as follows

| | $A \setminus B$ | $A \cap B$ | $B \setminus A$ | note or $\int \cdot dv$ |
|---------|-----------------|-------------------|-------------------|---|
| X | γ | γ | 0 | $\gamma v(A)$ |
| Y | 0 | δ | δ | $\delta v(B) = \gamma v(A)$ |
| Z | 0 | γ | γ | Z satisfies LHS in Axiom D'' |
| $X + Z$ | γ | 2γ | γ | $\gamma(v(S) + v(A \cap B))$ |
| $Y + Z$ | 0 | $\gamma + \delta$ | $\gamma + \delta$ | $(\gamma + \delta)v(B) = \gamma(v(A) + v(B))$ |

Hence, from Axiom D'', $X + Z \succsim Y + Z$, i.e., $v(S) + v(A \cap B) \geq v(A) + v(B)$. \square

⁹Note that, as defined, Z would not satisfy LHS of Axiom D'' if $A \cup B \subsetneq S$. More specifically, (Z, X) would not be anticomontonic if $A \cup B \subsetneq S$.

Proof of Proposition 5. If v is additive then the corresponding Choquet functional is additive and, thus, it is immediate that the CEU preference represented by v satisfies Axiom B. Conversely, let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$. Define $X, Y, Z \in V$ as follows

| | A | B | $(A \cup B)^c$ | $\int \cdot dv$ |
|---------|--------|---------------|----------------|---------------------|
| X | $v(S)$ | 0 | 0 | $v(S)v(A)$ |
| Y | $v(A)$ | $v(A)$ | $v(A)$ | $v(A)v(S)$ |
| Z | 0 | $v(S)$ | 0 | |
| $X + Z$ | $v(S)$ | $v(S)$ | 0 | $v(S)v(A \cup B)$ |
| $Y + Z$ | $v(A)$ | $v(S) + v(A)$ | $v(A)$ | $v(S)(v(A) + v(B))$ |

Note that $\{Z > 0\} \subseteq \operatorname{argmin} X$ and $\{Z > 0\} \subseteq \operatorname{argmin} Y$. Since $X \sim Y$, from Axiom B we get $X + Z \sim Y + Z$, i.e., $v(A \cup B) = v(A) + v(B)$. \square

Proof of Theorem 2. • [Proof of (i) \Rightarrow (ii)] Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$. Denote $X = v(S)A^*$, $Y = v(A)S^*$, and $Z = v(S)B^*$. Then

| | A | B | $(A \cup B)^c$ | $\int \cdot dv$ |
|---------|--------|---------------|----------------|---------------------|
| X | $v(S)$ | 0 | 0 | $v(S)v(A)$ |
| Y | $v(A)$ | $v(A)$ | $v(A)$ | $v(S)v(A)$ |
| Z | 0 | $v(S)$ | 0 | |
| $X + Z$ | $v(S)$ | $v(S)$ | 0 | $v(S)v(A \cup B)$ |
| $Y + Z$ | $v(A)$ | $v(S) + v(A)$ | $v(A)$ | $v(S)(v(A) + v(B))$ |

Since $X \succsim Y$, (Z, Y) is comonotonic, $Z \geq 0$, and $\{Z > 0\} \subseteq \operatorname{argmin} X$, from Axiom E, one gets $X + Z \succsim Y + Z$. Hence $v(A \cup B) \geq v(A) + v(B)$, i.e., v is superadditive.

• [Proof of (ii) \Rightarrow (iii)] \blacklozenge We first prove that (ii) \Rightarrow (iii) is true on V_0 . We will use the following lemma.

Lemma 3. Let $X := \sum_{i=1}^n x_i A_i^*$ where $(A_i)_{1 \leq i \leq n}$ is a partition of S with $A_i \in \mathcal{A}$ for all i and $x_i \leq x_{i+1}$ for all $i \in \llbracket 1, n-1 \rrbracket$. Let $Y := \alpha B^*$ where $\alpha \geq 0$ and $B \subseteq A_k$ for some $k \in \llbracket 1, n \rrbracket$.

If v is superadditive then $I(X + Y) \geq I(X) + I(Y)$.

Note that it suffices to prove that the result holds for $B = A_k$. Indeed, if $B \subsetneq A_k$ then one could rewrite X using the partition $\{B, A_k \setminus B, A_i : i \neq k\}$ instead of the partition $(A_i)_i$.

Proof of Lemma 3. We prove the lemma for $B = A_k$. If $\alpha = 0$ or X is constant, the result is immediate.

• If $x_n < x_k + \alpha$, then

$$X + Y = \sum_{\ell=1}^{k-1} x_\ell A_\ell^* + \sum_{\ell=k+1}^n x_\ell A_\ell^* + (x_k + \alpha) A_k^*.$$

Thus

$$\begin{aligned} I(X + Y) &= x_1 + \sum_{\ell=2}^{k-1} (x_\ell - x_{\ell-1}) v(\ell \dots n) \\ &\quad + (x_{k+1} - x_{k-1}) v(k \dots n) \\ &\quad + \sum_{\ell=k+2}^n (x_\ell - x_{\ell-1}) v(k, \ell \dots n) \\ &\quad + (x_k + \alpha - x_n) v(k). \end{aligned}$$

Note that

$$(x_{k+1} - x_{k-1})v(k \dots n) = (x_k - x_{k-1})v(k \dots n) + (x_{k+1} - x_k)v(k \dots n).$$

Moreover, since v is superadditive, we have

$$v(k, \ell \dots n) \geq v(k) + v(\ell \dots n) \text{ for } \ell \in \{k+1, \dots, j+1\}.$$

Hence

$$\begin{aligned} I(X+Y) &\geq I(X) + v(k)((x_{k+1} - x_k) + \sum_{\ell=k+2}^n (x_\ell - x_{\ell-1}) + (x_k + \alpha - x_n)) \\ &\geq I(X) + \alpha v(k) = I(X) + I(Y). \end{aligned}$$

- Otherwise, there exists $j \geq k$ such that $x_j < x_k + \alpha \leq x_{j+1}$. Then

$$X+Y = \sum_{\ell=1}^{k-1} x_\ell A_\ell^* + \sum_{\ell=k+1}^j x_\ell A_\ell^* + (x_k + \alpha)A_k^* + \sum_{\ell=j+1}^n x_\ell A_\ell^*.$$

Thus

$$\begin{aligned} I(X+Y) &= x_1 + \sum_{\ell=2}^{k-1} (x_\ell - x_{\ell-1})v(\ell \dots n) \\ &\quad + (x_{k+1} - x_{k-1})v(k \dots n) \\ &\quad + \sum_{\ell=k+2}^j (x_\ell - x_{\ell-1})v(k, \ell \dots n) \\ &\quad + (x_k + \alpha - x_j)v(k, (j+1) \dots n) \\ &\quad + (x_{j+1} - x_k - \alpha)v((j+1) \dots n) \\ &\quad + \sum_{\ell=j+2}^n (x_\ell - x_{\ell-1})v(\ell \dots n). \end{aligned}$$

Note that

$$(x_{k+1} - x_{k-1})v(k \dots n) = (x_k - x_{k-1})v(k \dots n) + (x_{k+1} - x_k)v(k \dots n).$$

Moreover, since v is superadditive, we have

$$v(k, \ell \dots n) \geq v(k) + v(\ell \dots n) \text{ for } \ell \in \{k+1, \dots, j+1\}.$$

Hence

$$\begin{aligned} I(X+Y) &\geq I(X) + v(k)((x_{k+1} - x_k) + \sum_{\ell=k+2}^j (x_\ell - x_{\ell-1}) + (x_k + \alpha - x_j)) \\ &\geq I(X) + \alpha v(k) = I(X) + I(Y). \end{aligned} \quad \square$$

Let $X, Y \in V_0$ be such that $\text{supp}Y := \{Y > 0\} \subseteq \text{argmin}X$. Then there exists a finite partition $A_1, \dots, A_i, \dots, A_n$ ($n \in \mathbb{N}^*$) of S , such that

$$\begin{aligned} X &= \sum_{i=1}^n x_i A_i^* \text{ with } A_i \in \mathcal{A} \text{ for all } i \text{ and } x_1 \leq x_2 \leq \dots \leq x_n, \text{ and} \\ Y &= \sum_{j=1}^m y_j B_j^* \text{ with } B_j \in \mathcal{A}, \cup_{j=1}^m B_j = A_1, \text{ and } 0 \leq y_1 < y_2 < \dots < y_m. \end{aligned}$$

The act Y may be written, with the convention $y_0 = 0$, as

$$Y = \sum_{k=1}^m (y_k - y_{k-1})(\cup_{j=k}^m B_j)^* = \sum_{k=1}^m Y_k \text{ where } Y_k := (y_k - y_{k-1})(\cup_{j=k}^m B_j)^*.$$

Note that, for $k \geq 1$, Y_k is comonotonic with $Y_{k+1} + \dots + Y_m$.¹⁰ Hence

$$\begin{aligned} I(Y_1 + \dots + Y_m) &= I(Y_1) + I(Y_2 + \dots + Y_m) \\ I(Y_2 + \dots + Y_m) &= I(Y_2) + I(Y_3 + \dots + Y_m) \\ &\vdots \end{aligned}$$

¹⁰Note that, for $1 \leq k, \ell \leq m$, Y_k is comonotonic with Y_ℓ , but it is not enough to conclude that $I(Y) = I(Y_1) + \dots + I(Y_m)$.

$$I(Y_{m-2} + \cdots + Y_m) = I(Y_{m-2}) + I(Y_{m-1} + Y_m)$$

$$I(Y_{m-1} + Y_m) = I(Y_{m-1}) + I(Y_m).$$

Summing up the above equalities, we obtain

$$I(Y) = I(Y_1) + \cdots + I(Y_m).$$

Note that (X, Y_1) satisfies the conditions of Lemma 3. Indeed,

$$X = x_1A_1 + x_2A_2 + \cdots + x_nA_n \text{ and } Y_1 = y_1A_1.$$

Applying Lemma 3 to (X, Y_1) , we get $I(X + Y_1) \geq I(X) + I(Y_1)$.

We claim that $(X + Y_1, Y_2)$ satisfies the conditions of Lemma 3. Indeed,

$$Y_2 = (y_2 - y_1)(B_2 \cup \cdots \cup B_m)$$

$$X + Y_1 = x_2A_2 + \cdots + x_{\ell_1}A_{\ell_1} + (x_1 + y_1)A_1 + x_{\ell_1+1}A_{\ell_1+1} + \cdots + x_nA_n \text{ (for some } \ell_1)$$

$$= x_2A_2 + \cdots + x_{\ell_1}A_{\ell_1}$$

$$+ (x_1 + y_1)B_1 + (x_1 + y_1)(B_2 \cup \cdots \cup B_m)$$

$$+ x_{\ell_1+1}A_{\ell_1+1} + \cdots + x_nA_n.$$

Applying Lemma 3 to $(X + Y_1, Y_2)$, we get $I(X + Y_1 + Y_2) \geq I(X + Y_1) + I(Y_2)$.

Similarly, $(X + Y_1 + Y_2, Y_3)$ satisfies the conditions of Lemma 3. Indeed,

$$Y_3 = (y_3 - y_2)(B_3 \cup \cdots \cup B_m)$$

$$X + Y_1 + Y_2 = x_2A_2 + \cdots + x_{\ell_1}A_{\ell_1}$$

$$+ (x_1 + y_1)B_1$$

$$+ x_{\ell_1+1}A_{\ell_1+1} + \cdots + x_{\ell_2}A_{\ell_2}$$

$$+ (x_1 + y_2)(B_2 \cup \cdots \cup B_m)$$

$$+ x_{\ell_2+1}A_{\ell_2+1} + \cdots + x_nA_n$$

$$= x_2A_2 + \cdots + x_{\ell_1}A_{\ell_1}$$

$$+ (x_1 + y_1)B_1$$

$$+ x_{\ell_1+1}A_{\ell_1+1} + \cdots + x_{\ell_2}A_{\ell_2}$$

$$+ (x_1 + y_2)B_2 + (x_1 + y_2)(B_3 \cup \cdots \cup B_m)$$

$$+ x_{\ell_2+1}A_{\ell_2+1} + \cdots + x_nA_n.$$

Applying Lemma 3 to $(X + Y_1 + Y_2, Y_3)$, we get $I(X + Y_1 + Y_2 + Y_3) \geq I(X + Y_1 + Y_2) + I(Y_3)$.

Similarly, for $k \in \llbracket 1, m-1 \rrbracket$, we apply Lemma 3 to $(X + Y_1 + \cdots + Y_k, Y_{k+1})$, and we obtain

$$I(X + Y_1 + \cdots + Y_k + Y_{k+1}) \geq I(X + Y_1 + \cdots + Y_k) + I(Y_{k+1}).$$

Summing up the above inequalities, we get

$$I(X + Y_1 + \cdots + Y_m) \geq I(X) + I(Y_1) + \cdots + I(Y_m) = I(X) + I(Y),$$

where the last equality follows from $I(Y) = I(Y_1) + \cdots + I(Y_m)$ (shown above).

◆ Second, we prove that $(ii) \Rightarrow (iii)$ is true on V .¹¹ Let $X_n, Y_n \in V_0$ be the usual uniform approximations of, respectively, X and Y . That is

$$X_n = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \left\{ \frac{i}{2^n} < X \leq \frac{i+1}{2^n} \right\}^* \text{ and } Y_n = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \left\{ \frac{i}{2^n} < Y \leq \frac{i+1}{2^n} \right\}^*.$$

¹¹Let X, Y satisfy the assumptions of (iii) . Note that for $a > 0$ sufficiently large, $X + a$ and $Y + a$ are non-negative and satisfy the assumptions of (iii) . So, we may assume X and Y are non-negative.

Clearly, $\{Y_n > 0\} \subseteq \{Y > 0\}$ for all n . Moreover, since X is bounded, there exists N such that $X(S) \subseteq [0, n]$ for all $n \geq N$. Thus $\operatorname{argmin} X \subseteq \operatorname{argmin} X_n$ for $n \geq N$. Hence, for $n \geq N$,

$$\{Y_n > 0\} \subseteq \{Y > 0\} \subseteq \operatorname{argmin} X \subseteq \operatorname{argmin} X_n.$$

Therefore, from above, $I(X_n + Y_n) \geq I(X_n) + I(Y_n)$. By passing to the limit in the previous inequality, we obtain the desired inequality $I(X + Y) \geq I(X) + I(Y)$.

• [Proof of (iii) \Rightarrow (i)] Let $X, Y, Z \in V$ be such that $X \succsim Y$, (Z, Y) is comonotonic, $Z \geq 0$, and $\{Z > 0\} \subseteq \operatorname{argmin} X$. Then

$$I(X + Z) \geq I(X) + I(Z) \geq I(Y) + I(Z) = I(Y + Z),$$

where the first inequality follows from minimum-superadditivity superadditivity of the Choquet functional (assumed in (iii)), the second inequality follows from $X \succsim Y$, and the equality follows from comonotonic additivity of the Choquet functional. Hence $X + Z \succsim Y + Z$. \square

Proof of Corollary 2. We show that a CEU preference \succsim represented by the capacity v satisfies Axiom E', i.e., preference for anticomonotonic diversification, if and only if v is superadditive. By Theorem 2, Axiom E' is then equivalent to Axiom E.

• (*Only if part*) Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$. If $v(A) = 0$ or $v(B) = 0$ the inequality $v(A \cup B) \geq v(A) + v(B)$ follows from monotonicity of v . Assume, then, that $v(A) > 0$ and $v(B) > 0$. There exist $\gamma, \delta > 0$ such that $\gamma v(A) = \delta v(B)$, i.e., $\gamma A^* \sim \delta B^*$. Since δB^* is a minimum hedge for γA^* , by Axiom E' with $\alpha = \delta/(\gamma + \delta)$ and thus $1 - \alpha = \gamma/(\gamma + \delta)$, we have

$$\frac{\delta}{\gamma + \delta}(\gamma A^*) + \frac{\gamma}{\gamma + \delta}(\delta B^*) \succsim \delta B^*, \text{ that is, } \frac{\delta\gamma}{\gamma + \delta}v(A \cup B) \geq \delta v(B).$$

Taking into account the fact that $\delta v(B) = \gamma v(A)$, we get $v(A \cup B) \geq v(A) + v(B)$.

• (*If part*) Assume now that v is superadditive and let $X, Y \in V$ be such that $X \succsim Y$ and Y a minimum hedge for X , and let $\alpha \in (0, 1)$. Then $(1 - \alpha)Y$ is a minimum hedge for αX . Hence

$$I(\alpha X + (1 - \alpha)Y) \geq I(\alpha X) + I((1 - \alpha)Y) = \alpha I(X) + (1 - \alpha)I(Y) = I(Y),$$

where the inequality follows from Theorem 2 and the last equality follows from $X \succsim Y$. \square

Proof of Proposition 6. Axiom E clearly implies Axiom E''. To prove they are equivalent, we just need to show that if \succsim is a CEU preference represented by the Choquet integral w.r.t. v and satisfies Axiom E'' then v is superadditive. The latter property being equivalent, by Theorem 2, to Axiom E.

Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$. If $v(A) = 0$ or $v(B) = 0$ then the inequality follows from monotonicity of v . Assume that $v(A) > 0$ and $v(B) > 0$. There exist $\gamma, \delta > 0$ such that $\gamma v(A) = \delta v(B)$, i.e., $\gamma A^* \sim \delta B^*$. Define X, Y, Z as follows

| | A | B | $(A \cup B)^c$ | note or $\int \cdot dv$ |
|---------|----------|-------------------|----------------|---|
| X | γ | 0 | 0 | $\gamma v(A)$ |
| Y | 0 | δ | 0 | $\delta v(B) = \gamma v(A)$ |
| Z | 0 | γ | 0 | Z satisfies LHS in Axiom E'' |
| $X + Z$ | γ | γ | 0 | $\gamma v(A \cup B)$ |
| $Y + Z$ | 0 | $\gamma + \delta$ | 0 | $(\gamma + \delta)v(B) = \gamma(v(A) + v(B))$ |

Since \succsim satisfies Axiom E'', we have $X + Z \succsim Y + Z$, i.e., $v(A \cup B) \geq v(A) + v(B)$. \square

Proof of Proposition 7. • (Only if part) Assume v is upper subadditive. Let $A_1, A_2 \in \mathcal{A}$ be such that $A_1 \cup A_2 = S$. Denote $B = A_1 \cap A_2$ and $A = A_2 \setminus B$. Note that $S \setminus A = A_1$ and $A \cup B = A_2$. From upper subadditivity, we have $v(S) - v(S \setminus A) \geq v(A \cup B) - v(B)$. Thus $v(A_1 \cup A_2) - v(A_1) \geq v(A_2) - v(A_1 \cap A_2)$, that is v is convex at the sure event.

• (If part) Assume v is convex at the sure event. Let $A, B \in \mathcal{A}, A \cap B = \emptyset$. Note that condition (3.3) is trivially satisfied if $A \cup B = S$. Assume then that $A \cap B = \emptyset$. Denote $A_1 = A \cup B$ and $A_2 = S \setminus A$. Then $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = B$. From convexity at the sure event, $v(A_1 \cup A_2) + v(A_1 \cap A_2) \geq v(A_1) + v(A_2)$, that is, $v(S) + v(B) \geq v(A \cup B) + v(S \setminus A)$. Hence v is upper subadditive. \square

Proof of Proposition 8. The proof is similar to the one showing that the Shapley property (i.e. (3.4) without the restriction $C \cup B$ equal to the sure event) is equivalent to convexity of v (i.e. (3.2) without the restriction $A \cup B = S$).

• v satisfies (3.2) $\Rightarrow v$ satisfies (3.4): Let $A, B, C \in \mathcal{A}$ be such that $A \subseteq B, C \cap B = \emptyset$, and $C \cup B = S$. Denote $A_1 = C \cup A$ and $A_2 = B$. Then $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = A$. By (3.2), $v(S) + v(A_1 \cap A_2) \geq v(A_1) + v(A_2)$, that is, $v(B \cup C) - v(B) \geq v(A \cup C) - v(A)$, i.e., (3.4) holds.

• v satisfies (3.4) $\Rightarrow v$ satisfies (3.2): Let $A_1, A_2 \in \mathcal{A}$ be such that $A_1 \cup A_2 = S$. Set $A = A_1 \cap A_2, B = A_1$, and $C = A_2 \setminus A_1$. Then $A \subseteq B, B \cup C = S$, and $C \cap B = \emptyset$. Thus, by (3.4), $v(B \cup C) - v(B) \geq v(A \cup C) - v(A)$, that is, $v(A_1 \cup A_2) + v(A_1 \cap A_2) \geq v(A_1) + v(A_2)$, which completes the proof. \square

Proof of Proposition 9. Let $\{A_1, A_2, A_3\}$ be a partition of S and $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 \leq x_2 \leq x_3$. First, note that (3.5) is equivalent to $(x_2 - x_1)v(A_2 \cup A_3) = (x_3 - x_1)v(A_3)$, that is,

$$(x_2 - x_1)(v(A_2 \cup A_3) - v(A_3)) = (x_3 - x_2)v(A_3). \quad (5.1)$$

Second, (3.6) is equivalent to:

$$(x_2 - x_1)(v(S) - v(A_1 \cup A_3)) \geq (x_3 - x_2)v(A_3). \quad (5.2)$$

Therefore, assuming that (3.4) (which is equivalent to (3.2) according to Proposition 8) holds, we must prove that (5.1) implies (5.2).

Since $A_3 \subseteq A_1 \cup A_3, (A_1 \cup A_3) \cup A_2 = S$, and $(A_1 \cup A_3) \cap A_2 = \emptyset$, from (3.4) we have

$$v(S) - v(A_1 \cup A_3) \geq v(A_2 \cup A_3) - v(A_3),$$

and since $x_1 \leq x_2$, we obtain

$$(x_2 - x_1)(v(S) - v(A_1 \cup A_3)) \geq (x_2 - x_1)(v(A_2 \cup A_3) - v(A_3)).$$

Since, according to (5.1), the term on the right hand side of the last inequality is equal to $(x_3 - x_2)v(A_3)$, we obtain the desired inequality (5.2), that is,

$$(x_2 - x_1)(1 - v(A_1 \cup A_3)) \geq (x_3 - x_2)v(A_3). \quad \square$$

Proof of Proposition 10. Let $A, B \in \mathcal{A}$ be such that $A \cup B = S$. If $A \cap B = \emptyset$ then $B = A^c$ and $v(S) \geq v(A) + v(B) = v(A) + v(A^c)$ by assumption. Assume then $A \cap B \neq \emptyset$,

and thus $v(A \cap B) \neq 0$ (since we're assuming $v(A) \neq 0$ for all $A \neq \emptyset$). Define X, X' , and Z as follows:

| | $A \setminus B$ | $B \setminus A$ | $A \cap B$ | $\int \cdot dv$ |
|----------|-----------------|-----------------|---------------|-------------------|
| X | 0 | $v(A \cap B)$ | $v(A \cap B)$ | $v(A \cap B)v(B)$ |
| X' | 0 | 0 | $v(B)$ | $v(B)v(A \cap B)$ |
| Z | $v(A \cap B)$ | 0 | 0 | |
| $X + Z$ | $v(A \cap B)$ | $v(A \cap B)$ | $v(A \cap B)$ | $v(A \cap B)v(S)$ |
| $X' + Z$ | $v(A \cap B)$ | 0 | $v(B)$ | |

Since $0 \leq v(A \cap B) \leq v(B)$, by Definition 5, $X + Z \succsim X' + Z$, that is,

$$v(A \cap B)v(S) \geq \int (X' + Z)dv = v(A \cap B)v(A) + (v(B) - v(A \cap B))v(A \cap B).$$

Since $v(A \cap B) \neq 0$, we get $v(S) \geq v(A) + v(B) - v(A \cap B)$. Hence v is convex at S . \square

Proof of Proposition 11. • [Proof of (i) \Rightarrow (ii)] From $(Y, X + Y)$ anticomonotonic we get, for $s, t \in S$,

$$(Y(s) - Y(t))(X(s) - X(t)) + (Y(s) - Y(t))^2 \leq 0.$$

Hence

$$(Y(s) - Y(t))(X(s) - X(t)) \leq -(Y(s) - Y(t))^2.$$

Which implies both that (Y, X) is anticomonotonic and $|Y(s) - Y(t)| \leq |X(s) - X(t)|$ for all $s, t \in S$.

• [Proof of (ii) \Rightarrow (i)] From (X, Y) is anticomonotonic and $|Y(s) - Y(t)| \leq |X(s) - X(t)|$ for all $s, t \in S$, we get

$$(Y(s) - Y(t))^2 \leq -(Y(s) - Y(t))(X(s) - X(t)) \text{ for all } s, t \in S.$$

That is, $(Y, X + Y)$ is anticomonotonic. \square

Proof of Proposition 12. Let P be a probability on (S, \mathcal{A}) . We have

$$\text{var}_P(X + Z) = \text{var}_P(X) + \text{var}_P(Z) + 2\text{cov}_P(X, Z).$$

Hence, it is enough to prove that $\text{var}_P(Z) + 2\text{cov}_P(X, Z) \leq 0$. Using the bilinearity of the covariance, we have

$$\text{cov}_P(Z, X + Z) = \text{cov}_P(Z, X) + \text{cov}_P(Z, Z) = \text{cov}_P(Z, X) + \text{var}_P(Z).$$

Since $(Z, X + Z)$ anticomonotonic, by Proposition 1, $\text{cov}_P(Z, X + Z) \leq 0$, i.e., $\text{cov}_P(Z, X) + \text{var}_P(Z) \leq 0$. Since $Z \in H(X)$, by Proposition 11, (Z, X) is anticomonotonic. Hence, again by Proposition 1, $\text{cov}_P(X, Z) \leq 0$. Therefore $\text{var}_P(Z) + 2\text{cov}_P(X, Z) \leq 0$. \square

Proof of Corollary 3. From the proof of Proposition 12, Z is ambiguity reducer for X if and only if, for every probability P , $\text{var}_P(Z) + 2\text{cov}_P(X, Z) \leq 0$, i.e., $\text{cov}_P(Z, 2X + Z) \leq 0$. By Lemma 1, this is equivalent to $(Z, 2X + Z)$ anticomonotonic, i.e., $Z \in H(2X)$.

The equivalence between assertions (i) and (ii) of Corollary 3 follows immediately from Proposition 11 (note that (Z, X) is anticomonotonic if and only if $(Z, 2X)$ is anticomonotonic). \square

Proof of Proposition 13. • [Proof of (a) \Rightarrow (b)] Note that $\int -Xdv = -\int Xd\bar{v}$ where \bar{v} denotes the conjugate capacity of v . Hence,

$$\int Xdv + \int -Xdv = \int Xdv - \int Xd\bar{v} = \int_{\mathbb{R}} (v(X \geq t) + v(X < t) - v(S))dt.$$

Since v is superadditive at S , $v(X \geq t) + v(X < t) - v(S) \leq 0$. Therefore $\int -X dv \leq -\int X dv$.

• [Proof of (b) \Rightarrow (c)] Immediate.

• [Proof of (c) \Rightarrow (a)] Let $A \in \mathcal{A}$. Denote $X := A^* - v(A)S^*$. Then $I(X) = 0 \geq 0$. By Assertion (c), $I_{\bar{v}}(X) \geq 0$. Since $I_{\bar{v}}(X) = -v(A) + \bar{v}(A)$, we obtain $\bar{v}(A) \geq v(A)$, for all $A \in \mathcal{A}$, which is equivalent to superadditivity of v at the sure event. \square

Proof of Theorem 3. • [Proof of (i) \Rightarrow (ii)] Let $A \in \mathcal{A}$. If $v(A^c) = 0$ then $v(S) \geq v(A) + v(A^c)$ follows from monotonicity of v . Assume then that $v(A^c) \neq 0$ and define $(X, Y, Z) := (v(S)A^*, v(A)S^*, \alpha(A^c)^*)$ where $0 < \alpha < v(S)$. Then

| | A | A^c | $\int \cdot dv$ |
|---------|--------|-----------------|-------------------------------------|
| X | $v(S)$ | 0 | $v(S)v(A)$ |
| Y | $v(A)$ | $v(A)$ | $v(S)v(A)$ |
| Z | 0 | α | |
| $X + Z$ | $v(S)$ | α | $\alpha v(S) + (v(S) - \alpha)v(A)$ |
| $Y + Z$ | $v(A)$ | $v(A) + \alpha$ | $v(A)v(S) + \alpha v(A^c)$ |

Then X, Y , and Z satisfy the conditions (LHS) of Axiom F. Indeed, $X \succsim Y$, (Z, Y) comonotonic and, because $0 < \alpha < v(S)$, $Z \in H(X)$. Thus $X + Z \succsim Y + Z$, that is, $I(X + Z) \geq I(Y + Z)$, i.e., $\alpha v(S) + (v(S) - \alpha)v(A) \geq v(A)v(S) + \alpha v(A^c)$. Since $\alpha > 0$, we have $v(S) \geq v(A) + v(A^c)$.

• [Proof of (ii) \Rightarrow (iii)] Let $X, Y \in V$ be such that $(Y, X + Y)$ is anticomonotonic. Since $(-Y, X + Y)$ is comonotonic, we have

$$I(X) = I(X + Y - Y) = I(X + Y) + I(-Y).$$

That is, $I(X + Y) = I(X) - I(-Y)$. From assertion (ii), $\bar{v} \geq v$, hence by Proposition 13, we have $I(Y) \leq -I(-Y)$. Therefore $I(X + Y) \geq I(X) + I(Y)$.

• [Proof of (iii) \Rightarrow (i)] Let $X, Y, Z \in V$ satisfy the conditions of Axiom F. Then

$$I(X + Z) \geq I(X) + I(Z) \geq I(Y) + I(Z) = I(Y + Z),$$

where the first inequality follows from proper-hedge superadditivity of the Choquet functional, the second inequality follows from $X \succsim Y$, and the last equality follows from comonotonic additivity of the Choquet functional. Hence $X + Z \succsim Y + Z$. \square

Proof of Proposition 14. • [Proof of (a) \Rightarrow (b)] Assume that v satisfies Axiom F'. Let $A \in \mathcal{A}$. If $v(A) = 0$ or $v(A^c) = 0$ then monotonicity of v implies $v(S) \geq v(A) + v(A^c)$. Suppose then that $v(A) > 0$ and $v(A^c) > 0$. Denote

$$X := \frac{v(A) + v(A^c)}{v(A)}A^* \quad \text{and} \quad Y := \frac{v(A) + v(A^c)}{v(A^c)}(A^c)^*.$$

Then $I(X) = I(Y) = v(A) + v(A^c)$, that is, $X \sim Y$, and we have

$$\alpha X + (1 - \alpha)Y = S^* \quad \text{for} \quad \alpha = \frac{v(A)}{v(A) + v(A^c)}.$$

Thus, by Axiom F', $S^* \succsim Y$. Therefore $v(S) \geq v(A) + v(A^c)$, i.e., v is superadditive at S .

• [Proof of (b) \Rightarrow (a)] Let $X, Y \in V$ and $\alpha \in (0, 1)$ be such that $X \sim Y$ and $\alpha X + (1 - \alpha)Y = aS^*$. Denote $Z := (1 - \alpha)Y = aS^* - \alpha X$. Then $(Z, \alpha X)$ is anticomonotonic. Moreover,

$|Z(s) - Z(t)| = |\alpha X(s) - \alpha X(t)|$ for all $s, t \in S$. Thus, from Proposition 11, $Z \in H(\alpha X)$ and from Theorem 3, $I(Z + \alpha X) \geq I(Z) + \alpha I(X)$. Hence $I(aS^*) \geq I((1 - \alpha)Y) + \alpha I(X)$, i.e., $I(aS^*) \geq I(Y) = I(X)$ or $aS^* \succsim Y$, which completes the proof. \square

Proof of Proposition 15. Axiom F clearly implies Axiom F''. To prove they are equivalent, we need to show that if \succsim is a CEU preference represented by v and satisfies Axiom F'' then v is superadditive at S . The latter property being equivalent, by Theorem 3, to Axiom F.

Let $A, B \in \mathcal{A}$ be such that $A \cap B = \emptyset$ and $A \cup B = S$. If $v(A) = 0$ or $v(B) = 0$ then the inequality $v(S) \geq v(A) + v(B)$ follows from monotonicity of v . Assume that $v(A) > 0$ and $v(B) > 0$. There exist $\gamma, \delta > 0$ such that $\gamma v(A) = \delta v(B)$, i.e., $\gamma A^* \sim \delta B^*$. Define X, Y, Z as follows

| | A | B | note or $\int \cdot dv$ |
|---------|----------|-------------------|---|
| X | γ | 0 | $\gamma v(A)$ |
| Y | 0 | δ | $\delta v(B) = \gamma v(A)$ |
| Z | 0 | γ | Z satisfies LHS in Axiom F'' |
| $X + Z$ | γ | γ | $\gamma v(S)$ |
| $Y + Z$ | 0 | $\gamma + \delta$ | $(\gamma + \delta)v(B) = \gamma(v(A) + v(B))$ |

Since \succsim satisfies Axiom F'', we have $X + Z \succsim Y + Z$, i.e., $v(S) \geq v(A) + v(B)$. \square

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