



**HAL**  
open science

# Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world

Mitsuru Kawazoe

► **To cite this version:**

Mitsuru Kawazoe. Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world. 2021. hal-03330580v1

**HAL Id: hal-03330580**

**<https://hal.science/hal-03330580v1>**

Preprint submitted on 1 Sep 2021 (v1), last revised 17 Feb 2023 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world

Mitsuru Kawazoe

**Abstract.** For the use of embodied notions in teaching linear algebra, some studies indicate that it is helpful, whereas other studies indicate that it could be problematic or becomes an obstacle. Hence, additional research is needed. This study is focused on linear (in)dependence and basis, and investigates the relation between their understandings in the embodied and symbolic worlds. Moreover, we investigated the effectiveness of a teaching method that emphasized geometric images. The main results of the study were as follows: the conceptual understanding of linear dependence of four spatial vectors, such that any three of them do not lie on the same plane, was positively associated with the understanding of basis in the symbolic world; however, the understanding of linear dependence of such vectors was not been improved by a geometrical instruction. Although a further study is needed to examine whether the geometric instruction in this study can be improved, the result imply that a space spanned by three linearly independent spatial vectors, which is an object in the embodied world, becomes a learning object for students.

**Keywords.** linear algebra, understanding, visualization, APOS theory, Tall’s model of three worlds, metaphor theory

**Résumé.** En ce qui concerne l’utilisation de notions incarnées dans l’enseignement de l’algèbre linéaire, certaines études indiquent qu’elle est utile, tandis que d’autres études indiquent qu’elle pourrait être problématique ou devenir un obstacle. Des recherches supplémentaires sont donc nécessaires. Cette étude se concentre sur l’(in)dépendance linéaire et la base, et examine la relation entre leurs compréhensions dans les mondes incarné et symbolique. En outre, nous avons étudié l’efficacité d’une méthode d’enseignement qui met l’accent sur les images géométriques. Les principaux résultats de l’étude sont les suivants : la compréhension conceptuelle de la dépendance linéaire de quatre vecteurs spatiaux, telle que trois d’entre eux ne se trouvent pas sur le même plan, est positivement associée à la compréhension de la base dans le monde symbolique ; cependant, la compréhension de la dépendance linéaire de ces vecteurs n’a pas été améliorée par une instruction géométrique. Bien qu’une étude plus approfondie soit nécessaire pour examiner si l’enseignement géométrique dans cette étude peut être amélioré, les résultats impliquent qu’un espace couvert par trois vecteurs spatiaux linéairement indépendants, qui est un objet dans le monde incarné, devient un objet d’apprentissage pour les étudiants.

**Mots-Clés.** algèbre linéaire, compréhension, visualisation, théorie APOS, modèle des trois mondes de Tall, théorie des métaphores

## Contents

<b>1. Introduction</b>	<b>2</b>
<b>2. Results of our previous studies</b>	<b>3</b>
2.A. Results of the first study	3
2.B. Results of the second study	4
<b>3. Theoretical framework</b>	<b>5</b>
3.A. Tall’s model of three worlds	5
3.B. APOS theory	6
3.C. Combined framework of Tall’s model of three worlds and APOS theory	7
3.D. Metaphor theory by Lakoff and Núñez	8
<b>4. Context: the course, students, design of lessons and tasks</b>	<b>9</b>
4.A. Design of lessons	9
4.B. Design of tasks	10
4.C. A priori analysis of tasks	12
4.D. Methodology and data collection	12
<b>5. Results</b>	<b>13</b>
5.A. Results of each task and students’ groups	13
5.A.a. Results of Task 1	13
5.A.b. Results of Task 2	13
5.A.c. Results of Task 3	13
5.A.d. Results of Task 4	14
5.B. Relations between the results of each task	14

5.B.a. Relation between understandings in the embodied world and understanding of basis . . . . .	14
5.B.b. Relation between understandings in the embodied world and in the symbolic world . . . . .	14
5.B.c. Difference of understanding of linear (in)dependence between before and after of four-week lessons . . . . .	15
<b>6. Discussion</b> . . . . .	<b>15</b>
<b>References</b> . . . . .	<b>16</b>

## 1. Introduction

It is widely recognized that linear algebra is a difficult subject to learn because of its abstract and formal nature. Dorier et Sierpinska (2001) stated that “linear algebra remains a cognitively and conceptually difficult subject.” Therefore, overcoming the difficulties in teaching linear algebra has been a challenge. Despite many studies in the past two decades, “research results continue to show that students find it difficult to understand its main concepts” (Trigueros & Wawro, 2020). Hence, how to teach linear algebra remains an open research question.

Different approaches to teaching linear algebra have been investigated. A popular approach is to use embodied notions—namely visual images or geometry—to teach linear algebra concepts. Many existing linear algebra textbooks use geometry (Harel, 2018). It might seem natural to teach linear algebra by using visual images, and such way of teaching might seem to be helpful for students; however, this is not easy. Some researchers identified that visual images help students understand concepts in linear algebra: (Stewart & Thomas, 2007), (Stewart & Thomas, 2009), (Thomas & Stewart, 2011), (Hannah, Stewart, & Thomas, 2013), (Hannah, Stewart, & Thomas, 2014), and (Donevska-Todorova, 2018, p.268). The effectiveness of the use of dynamic geometry software in teaching of linear algebra was also reported: (Gol Tabaghi & Sinclair, 2013), (Dogan, 2018). However, other studies indicated that using visual images may be problematic in teaching linear algebra (Sierpinska, 2000, p.244) and become an obstacle in learning linear algebra concepts Hillel (2000). Therefore, these studies indicate that the use of visual images in teaching linear algebra and its effectiveness should be additionally investigated.

Despite two separate indications on the use of visual images in teaching linear algebra in the literature, as mentioned above, it seems that both studies are based on a belief that students can understand linear algebra concepts easily in the embodied world, that is, the world of geometric vectors, and discuss only whether students’ understanding of linear algebra concepts in the embodied world can be generalized to the understanding of abstract linear algebra concepts. For example, a question discussed by Gueudet-Chartier (2004) was whether it is “possible for students to learn linear algebra by abstracting from geometry,” related to the main question “should we teach linear algebra through geometry?”, which is the title of the paper. To the best of our knowledge, the belief itself has never been examined deeply in the literature. However, it remains unclear whether we can rely on such a naive belief. We consider that it should be examined before we discuss the effectiveness of visual images in teaching linear algebra. That is a motivation of our research to investigate students’ understanding of linear algebra concepts in the context of geometric vectors.

Our previous studies focused on the notion of linear independence and dependence, and we examined students’ understanding of this notion in the context of geometric vectors, especially spatial vectors. In a series of our studies, we observed the following results: (1) Many students failed to determine linear dependence of four spatial vectors such that any three of them do not lie on the same plane (Kawazoe, Okamoto, & Takahashi, 2014); (2) Some of those students required a longer time to imagine that three spatial vectors not lying on the same plane span the whole space (Kawazoe & Okamoto, 2016) and (Kawazoe, 2018). Details of our results are described in Section 2. Although our previous studies identified some limitations of students’ understanding in the context of geometric vectors, we have not investigated how these observations were related to understanding of concepts

and procedures in linear algebra. In this study, we examine the relation between understandings in the context of geometric vectors and in the general context.

Therefore, in this study, we focused on concepts of linear (in)dependence and basis, and we investigated the following research questions: (1) Is geometrical understanding of linear (in)dependence in the embodied world related to understanding of linear (in)dependence and basis in the symbolic world?; (2) Can geometrical understanding of linear (in)dependence in the embodied world, including the case of four vectors, be improved by an instruction emphasizing a geometric image of linear (in)dependence?

The structure of this article is as follows. Section 2. briefly summarizes of the results of our previous studies. In Section 3., we describe theoretical frameworks used in the present study. Section 4. presents the context, tasks, and the methodology of this study. In Section 5., we show the results of each task and analyzes the relations between them. In Section 6., we discuss our research questions based on the results of Section 5. and implications of the study regarding the use of geometric images in the teaching of linear algebra.

## 2. Results of our previous studies

For the convenience of the readers, we briefly summarize our two previous studies (Kawazoe et al., 2014) and (Kawazoe & Okamoto, 2016) closely related to the present study, because the details of (Kawazoe et al., 2014) have not been described in the paper and (Kawazoe & Okamoto, 2016) is written in Japanese.

### 2.A. Results of the first study

In our first study (Kawazoe et al., 2014), we examined students’ understanding of linear independence in the context of geometric vectors by using the following task. In the study, students’ understanding of subspace in the context of numerical vector spaces was also examined, but we omit it here to focus on the results related to the present study.

**Task A.** Determine whether vectors in the following pictures are linearly independent and explain the reasons. Moreover, when they are linearly dependent, give a maximal set of linearly independent vectors.

(1)

(2)

(3)

Participants of this study were 107 first-year university engineering students who attended a linear algebra course. The test was conducted in linear algebra classes. Results are shown in Table 1.

	correct	incorrect	no answer
(1)	85.05%	5.61%	9.35%
(2)	87.85%	3.74%	8.41%
(3)	62.62%	23.37%	14.02%

Table 1 – Results of (1), (2), (3)

The percentage of correct answers for (3) was lower than for (1) and (2). The following example of a student’s reasoning for (3) shows a typical error:

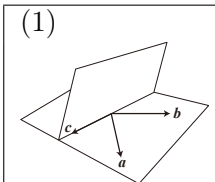
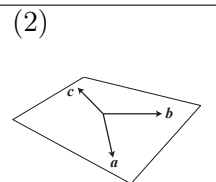
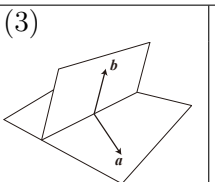
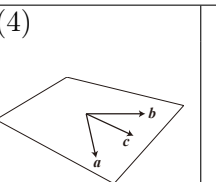
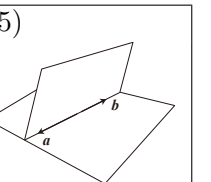
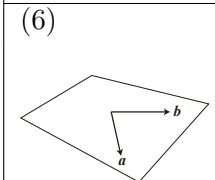
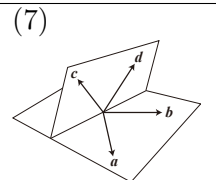
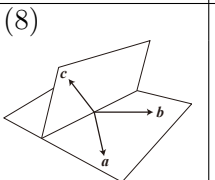
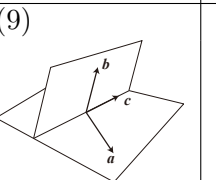
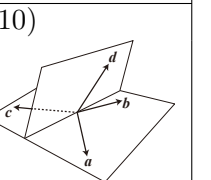
“As the space generated by three vectors does not contain the other vector, these vectors are linearly independent.” (This was originally written in Japanese and translated into English by the author.)

The error indicated that this student failed to understand that three linearly independent spatial vectors span the whole space. However, we could not determine the exact students’ difficulties in understanding the case of (3). Hence, a further study was needed to investigate students’ understanding of linear (in)dependence in the context of geometric vectors including the case of (3). For this aim, we conducted the second study (Kawazoe & Okamoto, 2016).

## 2.B. Results of the second study

The second study (Kawazoe & Okamoto, 2016) was designed as a two-round survey. Participants in this study were 71 first-year university engineering students attending a linear algebra course. In the first round, we conducted a paper-based test by using the following task.

**Task B.** Determine whether spatial vectors given in each picture are linearly independent. Answer “yes” or “no” for each case. Note that each vector lies on a line or a plane shown in the picture. (If there are multiple planes, each vector lies on one of them.)

(1) 	(2) 	(3) 	(4) 	(5) 
(6) 	(7) 	(8) 	(9) 	(10) 

The test items included the three pictures used in the first study. The test was conducted in the class after students learned the notions of abstract vector space, subspace, spanned space, and linear independence and dependence. The percentages of correct answers for the above question were as follows: (1) 84.5%, (2) 87.3%, (3) 90.1%, (4) 84.5%, (5) 94.4%, (6) 97.2%, (7) 45.1%, (8) 87.3%, (9) 88.7%, (10) 84.5%. Therefore, the percentage of correctness for (7) was significantly lower compared with the others. The picture in (7) is the same as the third picture in Task A in the first study. In this round, we again observed that the case of four spatial vectors, in which any three of them do not lie on the same plane, was quite difficult for students.

In the second round, we conducted a semi-structured interview. This round aimed to investigate why so many students fail to answer (7) correctly. We selected 17 participants among 71 students who attended the first round. The participants were students who had never missed a class until the time the interview was requested, and we distinguished the following two groups: (A) a group of 7 students who answered correctly in all cases, (B) a group of 10 students who failed (7) but answered correctly all the other cases. In the interview, the participants were first asked the following question.

**Q1.** Explain how you determined linear independency in each question.

If a participant did not use the fact that linear independence can be determined by checking whether some vector can be represented as a linear combination of the others, then the interviewer reminded the participant this fact. Subsequently, all participants were asked the following question.

**Q2.** What is the geometric figure of the set of all linear combinations of the vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  in (7)?

For the participants who felt difficulty in imagining the geometric figure asked in Q2, the following question was additionally asked.

**Q2'**. Why do you feel difficulty in imagining the set of all linear combinations?

In this study, additional two questions were asked for the participants who answered Q2', but we omit them here to focus on the main results. During the interview, the participants looked at the test paper used in the first round, and they answered by drawing diagrams or arrows on paper if necessary. Interviews were video-recorded.

The obtained data were qualitatively analyzed to investigate the difference between Group A and Group B. We observed the following results. No difference was found in the conceptions of linear (in)dependence of two to three geometric vectors, and spanned spaces with one or two dimensions. However, the difference was found in the conceptions of linear (in)dependence of four geometric vectors and of spanned space generated by three linearly independent geometric vectors. Although all students in Group B could answer for Q2 correctly, six of them showed some difficulties. Among them, four students answered that they need a longer time for imagining that three linearly independent geometric vectors span the whole space. It was observed that their imagining processes were common. Their image starts from some shape in a space, then it gradually extends, and finally fills the whole space. There were three types of a starting object. One is a parallelepiped created by three vectors. The second one is a triangular pyramid created by three vectors. And the third one is a plane spanned by two vectors with a cylindrical object along the third vector. Though a starting object is different, those students imagine a space spanned by linearly independent three spatial vectors as a gradually expanding three-dimensional object, which finally fills the whole space.

The second study identified some limitations of students' understanding in the context of geometric vectors. However, it was not investigated how these observations were related to understanding of concepts and procedures in linear algebra. The present study aims to investigate this relation.

### 3. Theoretical framework

We describe the present study starting from this section. As a theoretical framework, we use three theories by (Tall, 2013), (Arnon et al., 2014), and (Lakoff & Núñez, 2000), together with a combined framework of (Tall, 2013) and (Arnon et al., 2014) used by Stewart et Thomas (2007).

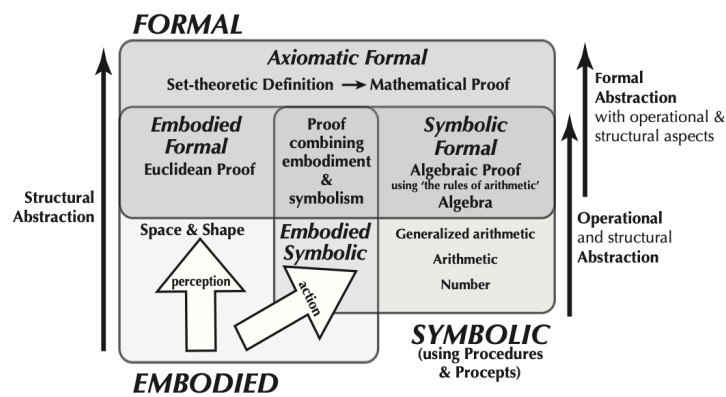
#### 3.A. Tall's model of three worlds

Tall (2013) described the development of mathematical thinking in terms of three worlds: embodied world, symbolic world, and formal world. He proposed these three worlds based on his observation of three different ways in which mathematical thinking can develop: conceptual embodiment, operational symbolic, and axiomatic formalism. Tall (2013) illustrated the outline of the development of the three worlds of mathematics as Figure 1.

According to Tall's model, a mathematical concept is a blend of embodiment, symbolism, and formalism. For example, the real number system is a blend of a number line in the embodied world, decimal symbols in the symbolic world, and a complete ordered field  $\mathbf{R}$  with operations  $+$  and  $\times$  in the formal world.

“The whole development of number—from whole number to fraction, to positive and negative numbers, to finite and infinite decimals represented as points on a number line—is a succession of extensional blends, broadening one number system to a larger one with richer properties.” (Tall, 2013, p.25)

For higher (university) levels of pure mathematics, Tall stated that “the combination of embodied and symbolic mathematics can be seen as a preliminary stage to the axiomatic formal presentation of mathematics” (Tall, 2013, p.18). In this study, we consider the three worlds in the context of linear



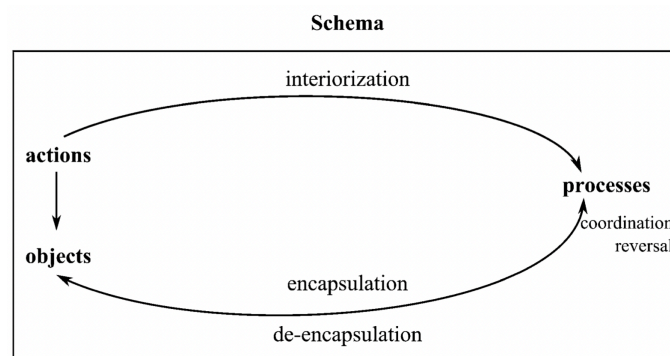
(Tall, 2013, p.17, Figure 1.5)

Figure 1 – Preliminary outline of the development of the three worlds of mathematics

algebra. In linear algebra, the embodied world is a world of geometric vectors (e.g., arrows), the symbolic world is a world of numerical vectors, matrices, polynomials, and operations represented by symbols, and the formal world is a world of axiomatic vector spaces.

### 3.B. APOS theory

APOS theory is “a theory of how mathematical concepts can be learned” (Arnon et al., 2014, p.1). It is a constructivist theory rooted in the work of Piaget, which gives us a tool for describing a mental structure of a student’s conception. With APOS theory, we can distinguish student’s conceptions into four levels: Action, Process, Object, and Schema (hence, the acronym APOS). From the viewpoint of APOS theory, conceptions of mathematical knowledge develop as  $A \rightarrow P \rightarrow O \rightarrow S$  (Figure 2), though it is noted that “the development does not always proceed linearly” (Arnon et al., 2014, p.9).



(Arnon et al., 2014, p.18, Fig. 3.1)

Figure 2 – Mental structures and mechanisms for the construction of mathematical knowledge

In general, a student who needs to manipulate explicit expressions to think about the concept and cannot skip any step is considered to have an Action conception. When some actions has been interiorized in a student’s mind, the student is considered to have a Process conception. When some process has been encapsulated into a cognitive object in a student’s mind, the student is considered to have an Object conception. In (Arnon et al., 2014, p.25), Schemas are described as “structures that contain the descriptions, organization, an exemplifications of the mental structures that an individual



has constructed regarding a mathematical concept.”

Action, Process, Object, and Schema depend on the context. In (Arnon et al., 2014), their examples in linear algebra are given as follows:

**Action:** “The construction of the concept of  $n$ -tuple may begin by performing the Action that consists in taking a specified quantity of numbers and placing them in a particular order.” (Arnon et al., 2014, p.20)

**Process:** “When the Actions involved in the construction of an  $n$ -tuple are interiorized into a Process, the subject can construct an  $n$ -tuple mentally even when  $n$  is not specified; he or she can also consider the construction of  $n$ -tuples in any vector space, including infinite dimensional spaces. It is also possible for the individual to think about the elements of the tuple, considering that the elements may repeat, but the order in which they appear cannot be changed.” (Arnon et al., 2014, p.21)

**Object (Encapsulation of Process):** “Comparing  $n$ -tuples or performing binary operations on  $n$ -tuples are Actions on  $n$ -tuples. For these Actions to be applied successfully, the Process of forming an  $n$ -tuple must be encapsulated into an Object.” (Arnon et al., 2014, p.22)

**Schema:** “a Schema for vector space may include  $n$ -tuples and matrices as Objects and polynomials and functions as Processes. All these structures may be related by the fact that they share some properties, such as satisfying a set of axioms that define a vector space. Coherence of this Schema lies in the mathematical definition of vector space which the individual uses to determine whether or not the Schema is applicable to a given situation.” (Arnon et al., 2014, p.25)

In this study, we use Action, Process, and Object in the context of linear (in)dependence and of spanned space. For linear (in)dependence, Action, Process, and Object are considered as follows.

**Action:** Students having an Action conception need to draw a linear combination explicitly in a discussion of linear (in)dependence.

**Process:** Students having a Process conception can use a partial set of linear combinations in a discussion of linear (in)dependence, but they cannot use a spanned space correctly.

**Object:** Students having an Object conception understand any two non-parallel geometric vectors and any three spatial vectors not lying on the same plane are linearly independent. They also understand any four spatial vectors are always linearly dependent. Moreover, they can consider linear (in)dependence of a sum of two sets of linearly (in)dependent vectors.

For spanned space, Action, Process, and Object are considered as follows.

**Action:** Students need to draw (or calculate) linear combinations of given vectors explicitly, or they can only consider a partial set of linear combinations of given vectors.

**Process:** Students can consider all linear combinations of given vectors as a set.

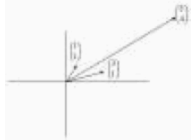
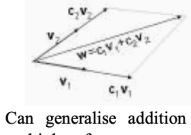
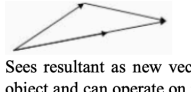
**Object:** Students having Object conception can completely understand a spanned space generated by any set of vectors correctly. For example, they understand any two non-parallel geometric vectors span a plane, any three spatial vectors not lying on the same plane span the whole space. Moreover, students having an Object conception can consider a sum of spanned spaces as a vector space.

### 3.C. Combined framework of Tall’s model of three worlds and APOS theory

As Action, Process, Object, and Schema in APOS theory depend on the context, their descriptions may differ in each of three worlds in Tall’s model. Stewart et Thomas (2007) and Stewart et Thomas (2010) used a combined framework of Tall’s model of three worlds and APOS theory in their study.



In (Stewart & Thomas, 2007) and (Stewart & Thomas, 2010), they described Action, Process, and Object conceptions of some notions in linear algebra in each of the three worlds (Figure 3).

APOS	Embodied World	Symbolic World		Formal World
		Algebra	Matrix	
Action	 <p>Can add multiples of two given vectors</p>	<p>Can create a new vector <math>w</math> by, say addition, e.g.  <math>w = 3u + 5v = 8u</math></p>	<p>Can calculate with linear combinations, e.g.</p> $2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 3 \\ 5 \end{bmatrix}$ <p>Can determine whether a vector <math>w</math> is a linear combination of <math>u</math> and <math>v</math> using row reduction</p>	
Process	 <p>Can generalise addition of multiples of vectors</p>	<p>Can think of linear combinations of vectors e.g. <math>w = 3u + 5v</math> without having to perform operations</p>	<p>Can consider operations on vectors without performing them e.g.</p> $k_1 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + k_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$	
Object	 <p>Sees resultant as new vector object and can operate on it</p>	<p>Can operate on a linear combination e.g.  <math>T(3u + 5v)</math></p>	<p>Can operate on a linear combination e.g.</p> $\mathbf{M} \cdot k_1 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + k_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$	$w = c_1v_1 + c_2v_2 + \dots + c_kv_k$ <p>Sees a general linear combination as an element of a vector space <math>\mathbf{V}</math></p>

(Stewart & Thomas, 2007)

Figure 3 – A part of framework for linear algebra concepts

In this study, we consider how students' understanding of linear algebra concepts in the embodied world are related to their understanding in the symbolic world. To do this, we have to investigate students' understanding in each world. Hence, the combined framework used by Stewart et Thomas (2007) is helpful for our study. We use this framework to analyze students' understanding in each world.

### 3.D. Metaphor theory by Lakoff and Núñez

We use the metaphor theory by Lakoff et Núñez (2000) in designing linear algebra lessons implemented in this study. In general, it is important to design lessons according to students' cognitive process of understanding new concepts. Although Tall's model of three worlds and APOS theory give some insight about it, we think that metaphor theory (Lakoff & Núñez, 2000) is helpful for designing lessons in this study.

For example, the "Basic Metaphor of Infinity (BMI)" proposed in (Lakoff & Núñez, 2000) can be applied to students' cognitive process of imagining a spanned space that we observed in the second study (Kawazoe & Okamoto, 2016). Lakoff et Núñez (2000) explain this metaphor as follows:

"We hypothesize that all cases of actual infinity—infinite sets, points at infinity, limits of infinite series, infinite intersections, least upper bounds—are special cases of a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result. We call this metaphor the *Basic Metaphor of Infinity*, or the BMI for short." (Lakoff & Núñez, 2000, p.158)

As we stated in Section 2.B., we observed students' cognitive process of imagining a space spanned by three linearly independent spatial vectors as a gradually expanding three-dimensional object which finally fills the whole space. By applying the BMI to this case, a spanned space in such process can be explained as the "final resultant state" obtained by BMI from an infinite sequence of three-dimensional objects, which is getting larger step by step. In the case of a space spanned by three linearly independent spatial vectors, each three-dimensional object is a subset of the whole space, and

difference between the whole space and the object is getting smaller when the object becomes larger. This leads to a cognition that a space spanned by three linearly independent spatial vectors as the final resultant state obtained by BMI coincides with the whole space.

The metaphor theory can also be applied to the concept of basis of vector space. With the viewpoint of Lakoff et Núñez (2000), space is naturally continuous, but due to Descartes's invention of analytic geometry, it is also conceptualised as "a set of points." Using the word of Lakoff et Núñez (2000), the latter conceptualization is called "discretization" of space. Lakoff et Núñez (2000) identified a central metaphor at the heart of the discretization. The metaphor is called "A Space Is a Set of Points." With this metaphor, a space is regarded as a set of  $n$ -tuples of numbers, where a point in the set corresponds to a location in the space. To obtain a location in the space, axes are needed, and each axis is a number line. From the viewpoint of Lakoff et Núñez (2000), discretization in the case of three-dimensional space is obtained by a "conceptual blend" of three number lines and the Euclidean space. Discretization of space can be easily applied to vector spaces, because a coordinate system of a vector space is given by a basis of it, where a basis determines axes of the space.

Therefore, the metaphor theory by Lakoff et Núñez (2000) seems to be useful in designing linear algebra lessons in which students naturally understand concepts from a geometric viewpoint. In this study, we use the above two metaphors in designing and implementing linear algebra lessons on spanned space, linear (in)dependence, basis, and dimension in which geometrical viewpoint is emphasized.

## 4. Context: the course, students, design of lessons and tasks

The study was conducted in a linear algebra course for engineering students at our university, but in a special class for students who failed to pass it in their first year. The course consists of a spring semester class and a fall semester class. The former is a 2-credit class, meeting for 90 minutes each week for 15 weeks. The latter is a 4-credit class, meeting for 180 minutes each week for 15 weeks. Each of them is followed by an examination period. The course covers usual linear algebra topics: numerical vector space, matrix, Gaussian elimination, system of linear equations, determinant, etc., in the spring semester; formal vector space, spanned space, linear (in)dependence, basis, dimension, linear map, inner product, orthogonal basis, eigenvalue, eigenvector, and diagonalization, etc., in the fall semester. This study was conducted during the first five weeks in the fall semester. In these weeks, students learned the concepts of formal vector space, spanned space, linear (in)dependence, basis, and dimension.

### 4.A. Design of lessons

Each lesson included a lecture part and an exercise part. Lectures and exercises were given in the first and second half of the lesson, respectively. In the design of lessons, we regarded the notion of spanned space as a key notion in understanding linear (in)dependence, basis, and dimension. Hence, the instruction of spanned space was carefully implemented so that all students could have a geometric image of it in the embodied world, by considering our observations from the second study (Kawazoe & Okamoto, 2016) and by applying the metaphor theory of Lakoff et Núñez (2000). The lecture part was designed to emphasize geometric images of linear algebra concepts, especially by using the image of a spanned space in the embodied world. In the lecture part, the teacher introduced linear algebra concepts the following way.

First, the notions of linear combination and spanned space were introduced. A space spanned by three linearly independent spatial vectors was shown to students by using teacher's fingers, and it was emphasized that linear combinations with negative coefficients were contained in the spanned space. The teacher stressed the importance of imagining a part of the space consisting of linear combinations with some (or all) coefficients being negative to grasp the correct concept of the spanned space.

The notions of linear independence and dependence were introduced by using usual formal definitions. The reason was as follows: the students in the class were students who failed to pass the course in the previous years and hence they already learned the notions. Although we used usual formal definitions, the meaning of linear independence and dependence of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in a vector space were explained in terms of spanned space as follows:

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly dependent if and only if one of the  $n$  vectors can be represented by a linear combination of the other  $n - 1$  vectors, that is, one of the  $n$  vectors is contained in the space spanned by the other  $n - 1$  vectors.

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly independent if and only if none of the  $n$  vectors can be represented by a linear combination of the other  $n - 1$  vectors, that is, none of the  $n$  vectors is contained in the space spanned by the other  $n - 1$  vectors.

Such explanations emphasized geometric images of the notions. In the above explanations, the teacher used the following drawings (Figure 4):

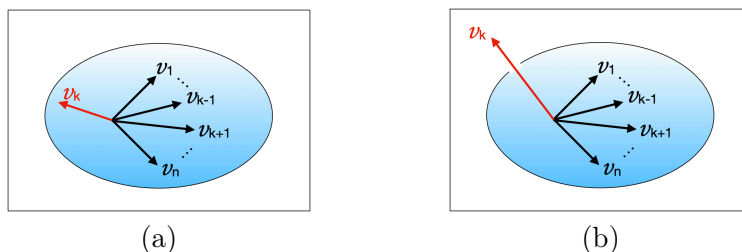


Figure 4 – Drawings used in the explanations of (a) linear dependence and (b) independence

Moreover, it was also explained that linearly independent vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  give an ascending sequence of vector spaces  $V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n$ , where  $V_k$  ( $k = 1, 2, \dots, n$ ) is the space spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ . Subsequently, the notion of basis was introduced by a usual formal definition:

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in a vector space  $V$  form a basis of  $V$  if and only if they are linearly independent and any vector in  $V$  can be represented as their linear combination.

It was explained that the second condition is equivalent to that  $V$  is spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . In the introduction of basis, the role of basis was explained as to give a coordinate system, and a basis was explained as a set of “axes.” Furthermore, it was explained that the second condition means that it contains a sufficient number of axes to represent the whole space, and that the first condition means that there is no extra axis in the set.

In the exercise part, students worked on paper-based exercises given by the teacher. Exercises mainly included questions in the symbolic world. However, some of them can be viewed as questions in the embodied world through the correspondence between  $\mathbf{R}^2$  and a plane or between  $\mathbf{R}^3$  and a three-dimensional space: determining linear (in)dependence of vectors in  $\mathbf{R}^n$  ( $n = 2, 3, 4$ ) or in polynomial spaces, determining whether a given set of vectors in  $\mathbf{R}^n$  ( $n = 2, 3, 4$ ) or in polynomial spaces is a basis, finding a basis and the dimension of given subspaces in  $\mathbf{R}^n$  ( $n = 2, 3, 4$ ) or in polynomial spaces, etc. Many questions were computational. Some of them were related to the geometric instruction given in the lecture part, and they can be answered with geometrical reasoning.

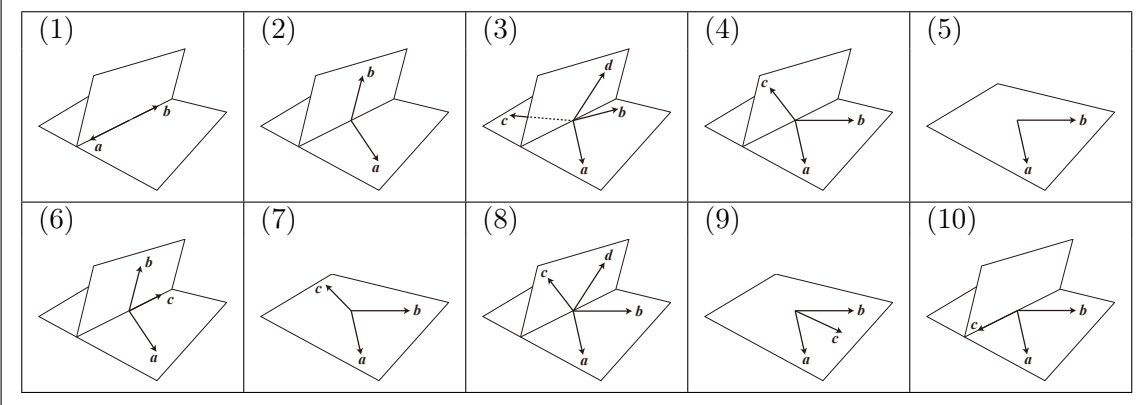
#### 4.B. Design of tasks

The following four tasks, which were translated from Japanese, were used in this study. They were designed to investigate students’ understanding of dimension, linear (in)dependence, and basis. As the same items are used in Task 2 and Task 4, correct answers for Task 2 were not shown to the students during the lessons.

**Task 1.** Answer the following questions. If you do not know (or if you have not learned), write your answer as “I don’t know.”

- (1) What examples of one dimension, two dimension, and three dimension come to your mind? Describe the image you have for each of them using figures and words freely.
- (2) For vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}$ , assume that vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  span an  $n$ -dimensional space  $V$ , and that  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}$  span an  $(n + 1)$ -dimensional space  $W$ . When you draw a picture showing this situation, what kind of picture do you draw? Draw a picture of your image.

**Task 2.** Determine whether spatial vectors given in each picture are linearly independent. Note that each vector lies on a line or a plane shown in the picture. (If there are multiple planes, each vector lies on one of them.)



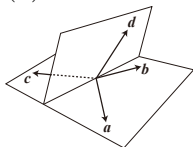
**Task 3.** (Q1) For vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in a vector space  $V$  over  $K$ , describe two conditions (in the definition of basis) for  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  to be a basis of  $V$ . Write your answer in the answer columns (A) and (B).

(Q2) Determine whether the following set of vectors is a basis. If it is not a basis, answer which condition that you described in Q1 is not satisfied. In the latter case, write your answer by using a symbol A or B, and write “A, B” in both cases. (Vector spaces are as follows: (1)  $\mathbf{R}^4$ , (2)  $\mathbf{R}^3$ , (3)  $\mathbf{R}^2$ , (4)  $\mathbf{R}^3$ , (5) the space of polynomials  $f(x)$  with degree less than 3 whose coefficients are in  $\mathbf{R}$ , (6) the space of polynomials  $f(x)$  with degree less than 2 whose coefficients are in  $\mathbf{R}$ .)

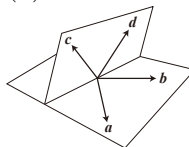
- |  |  |  |  |
|--|--|--|--|
| (1) $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ | (2) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ | (3) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |  |
| (4) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix}$             | (5) $x + 1, x^2$   | (6) $x - 1, x + 1$   |  |

**Task 4.** (Q1) Determine whether spatial vectors given in each picture are linearly independent and describe the reason.

(1)



(2)



(Q2) Determine whether the given vectors in  $\mathbf{R}^3$  are linearly independent and describe the reason.

$$(1) \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(2) \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

#### 4.C. A priori analysis of tasks

Task 1 and Task 2 are pretests conducted at the beginning of the semester. Task 1(1) can be answered as “line,” “plane,” and “space.” Task 1(2) is a non-routine task to examine whether students have an understanding that  $V$  is contained in  $W$ , or  $W$  extends outside of  $V$  as a space. Task 2 includes all of the important cases of at most four spatial vectors regarding linear (in)dependence. Items in Task 2 are the same ones that we used in our previous study (Kawazoe & Okamoto, 2016). According to the previous results, Task 2 (8) was expected to be difficult for the participants. Task 2 (8) contains four vectors, and any three of them do not lie on the same plane; hence, it cannot be reduced to the case of at most three vectors. Task 2 (3) also contains four vectors, but it can be reduced to the case of three vectors because vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  lie on the same plane. The terms “dimension,” “span,” and “linearly independent” were used in the texts in these tasks. As the participants were in their second year or higher, they had already learned these terms in their first year.

The aim of Task 3 is to investigate students’ understanding of the definition of basis. For any set of vectors listed in (1)–(6), one can determine their linear (in)dependence without computation. Only (2) and (6) are bases, and the others are not.

In Task 4, Q1 is a task in the embodied world, and Q2 is a task in the symbolic world. The two pictures in Q1 were taken from Task 2. According to the result of our previous study (Kawazoe & Okamoto, 2016), determining linear (in)dependence of four spatial vectors is problematic. Q1(1) and Q2(2) present essentially the same situation, and Q1(2) and Q2(1) likewise. Q1(1) and Q1(2) can be answered by drawing vectors representing linear combinations, or by using the fact on vector subspaces spanned by two or three vectors. Q2(1) and Q2(2) can be answered by using numerical computation (with or without the use of the Gaussian elimination), but they also can be answered with geometrical reasoning.

#### 4.D. Methodology and data collection

We implemented four-week lessons whose design is described in Section 4.A.. Task 1 and Task 2 were offered at the beginning of the first lesson. Task 3 was at the third week, and Task 4 was at the beginning of the fifth week. Participants’ answers for Task 1 were qualitatively analyzed whether they understood the dimension at most three and the increment of dimension. Participants’ reasoning for Task 4 Q1 was qualitatively analyzed using APOS theory. Participants’ reasoning for Task 4 Q2 was classified into two types: algebraical and geometrical reasoning. For other tasks, participants’ answers were evaluated depending on their correctness. Subsequently, the relations between the results of these tasks were investigated by using a statistical analysis.

The study was conducted in the fall semester in the academic year 2018. All data were collected during the first five weeks in the linear algebra class for engineering students who had failed in the

previous year or before. The number of students in the class was 58. Among the 58 students, 38 of them completed all the tasks, from Task 1 to Task 4. In this study, the data of the 38 participants was qualitatively and statistically analyzed.

## 5. Results

First, we present the results of each task and distinguish several groups of students depending on the results. Second, we statistically analyze the relations between the results of each task.

### 5.A. Results of each task and students' groups

#### 5.A.a. Results of Task 1

For Task 1 (1), almost all participants described their understanding of dimension 1, 2, 3, as a “line,” “plane,” and “space,” respectively. For Task 1 (2), only 11 (28.9%) could draw their image of increment of dimension as extending outside the space. We set two groups according to the result of Task 1 (2): *GI+* is the group of 11 participants who provided a geometric image of increment of dimension (Figure 5), *GI-* is the group of the others.

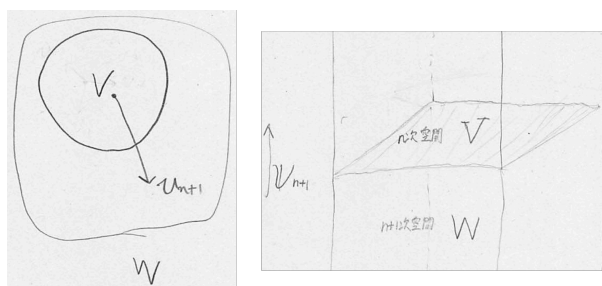


Figure 5 – Examples of students' geometric images of increment of dimension

#### 5.A.b. Results of Task 2

The percentages of correct answers for Task 2 were as follows: (1) 97.4%, (2) 94.7%, (3) 65.8%, (4) 97.4%, (5) 89.5%, (6) 94.7%, (7) 86.8%, (8) 52.6%, (9) 89.5%, (10) 86.8%. Therefore, the percentages of correctness for (3) and (8) were significantly lower than for the others. The pictures of (3) and (8) contain four vectors. The number of vectors in the others is less than four. The result of Task 2 was almost the same as the one in our previous study (Kawazoe & Okamoto, 2016), except for the result of (3). In the previous study, the percentage of correct answers for (3) was 84.5%. The median of the number of correct answers per participant was 9. We set two groups according to the result of Task 2: *GV+* is the group of participants who answered correctly to more than 8 questions, and *GV-* is the group of the others.

#### 5.A.c. Results of Task 3

For Q1, the number of participants who could describe two conditions in the definition of basis correctly was 23 (60.5%). Although 34 (89.5%) of the participants described linear independence of the vectors correctly as one of the conditions, 24 (63.2%) of them described correctly that the vectors span  $V$  or that any vector in  $V$  can be represented as a linear combination of the vectors. 8 (21.1%) of them described  $\dim V = n$  as one of the conditions, which is a wrong answer because  $\dim V$  is defined after the definition of basis is introduced.

For Q2, although the percentages of correct answers for (2), (3), and (4) were high, those of (1), (5), and (6) were relatively low: (1) 78.9%, (2) 97.4%, (3) 94.7%, (4) 94.7%, (5) 78.9%, (6) 65.8%.



As for reasoning in (1), (3), (4), and (5), we evaluated whether a participant could answer correctly based on the necessary and sufficient conditions to be a basis. Hence, for a participant who described  $\dim V = n$  in Q1, we evaluated whether his/her answer for Q2 was logically correct based on his/her answer in Q1. The percentages of correct answers for reasoning were as follows: (1) 65.8%, (3) 63.2%, (4) 36.8%, (5) 65.8%. The median of the number of errors in Q2 (including errors in reasoning in the case of non-basis) per participant was 2. We set two groups according to the number of incorrect answers for Task 3 Q2:  $B+$  is the group of participants whose incorrect answers were at most 2, and  $B-$  is the group of the others.

#### 5.A.d. Results of Task 4

The percentages of correct answers for Task 4 were as follows: Q1(1) 89.5%, Q1(2) 55.3%, Q2(1) 86.8%, Q2(2) 89.5%. The pictures in Q1(1) and Q1(2) are the same as in Task 2 (3) and Task 2 (8), respectively. Although the percentage of correct answers for Q1(2) remained low, the one for Q1(1) was much improved from the result of Task 2 (3). Though Q1(2) is essentially the same as Q2(1) from a geometrical viewpoint, their results were different. According to the reasoning in Q1, we set the following groups: For  $j = 1, 2$ ,  $O_{j+}$  is the group of participants showing Object conceptions in the reasoning for Q1(j),  $O_{j-}$  is the group of participants showing Action/Process conceptions or giving no reason in the reasoning for Q1(j). According to the reasoning in Q2(j), we set the following groups: For  $j = 1, 2$ ,  $GR_{j+}$  is the group of participants using geometrical reasoning for Q2(j),  $GR_{j-}$  is the group of the others.

### 5.B. Relations between the results of each task

In the following statistical analysis, we used Fisher's exact test instead of the Chi-square test because of small numbers in cross-tabulation.

#### 5.B.a. Relation between understandings in the embodied world and understanding of basis

Fisher's exact test indicated that having a geometric image of increment of dimension (Task 1 (2)) and the result of Task 3 Q2 were positively associated ( $p < 0.05$ , Table 2). Moreover, showing an Object conception in reasoning for Task 4 Q1(2) and the result of Task 3 Q2 were positively associated ( $p < 0.05$ , Table 3). However, we could not find any significant relation between  $O_{1+/-}$  and  $B+/-$ .

	$B+$	$B-$
$GI+$	9	2
$GI-$	11	16

Table 2 – Relation between the results of Task 1(2) and Task 3 Q2

	$B+$	$B-$
$O_{2+}$	9	2
$O_{2-}$	11	16

Table 3 – Relation between having an Object conception and the result of Task 3 Q2

#### 5.B.b. Relation between understandings in the embodied world and in the symbolic world

Fisher's exact test indicated that showing an Object conception in reasoning for Task 4 Q1(2) and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ( $p < 0.01$ , Table 4), where  $NC$  means the number of correct answers in determining linear (in)dependence in Task 4. However, we could not find any significant relation between  $O_{1+/-}$  and the result of Task 4. Fisher's exact test also indicated that the use of geometrical reasoning for Task 4 Q2 and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ( $p < 0.05$ , Table 5), where  $GR+ = GR_1+ \cup GR_2+$ ,  $GR- = GR_1- \cap GR_2-$ , and  $NC$  is the same as in Table 4. Moreover, it indicated significant correlations for  $GR_1+/-$  ( $p < 0.05$ ) and for  $GR_2+/-$  ( $p < 0.05$ ).



	$NC = 4$	$NC < 4$
$O_2+$	8	1
$O_2-$	9	20

Table 4 – Relation between having an Object conception and the result of Task 4

	$NC = 4$	$NC < 4$
$GR+$	10	5
$GR-$	7	16

Table 5 – Relation between the use of geometrical reasoning and the result of Task 4

### 5.B.c. Difference of understanding of linear (in)dependence between before and after of four-week lessons

The images in Task 4 Q1(1) and Q1(2) are the same as the ones in Task 2 (3) and (8), respectively. McNemar’s test indicated a significant difference between the results of Task 2 (3) and Task 4 Q1(1) ( $p < 0.05$ , Table 6), where the participants were divided into two groups depending on whether their answers for Task 2(3) were correct ( $T_{2(3)+}$ ) or not ( $T_{2(3)-}$ ), and they were divided into two groups depending on whether their answers for Task 4 Q1(1) were correct ( $T_{4Q1(1)+}$ ) or not ( $T_{4Q1(1)-}$ ). However, Fisher’s exact test indicated that the result of Task 2 and the number of correct answers in determining linear (in)dependence in Task 4 Q1 were positively associated ( $p < 0.01$ , Table 7), where  $NC_{Q1}$  means the number of correct answers in determining linear (in)dependence in Task 4 Q1.

	$T_{4Q1(1)+}$	$T_{4Q1(1)-}$
$T_{2(3)+}$	23	2
$T_{2(3)-}$	11	2

Table 6 – Relation between the results of Task 2 (3) and Task 4 Q1(1)

	$NC_{Q1} = 2$	$NC_{Q1} < 2$
$GV+$	16	7
$GV-$	3	12

Table 7 – Relation between the results of Task 2 and Task 4 Q1

## 6. Discussion

In this section, we discuss our research questions, based on the results from the previous section. First, we recall our research questions stated in Section 1.:

- (1) Is geometrical understanding of linear (in)dependence in the embodied world related to understanding of linear (in)dependence and basis in the symbolic world?
- (2) Can geometrical understanding of linear (in)dependence in the embodied world, including the case of four vectors, be improved by an instruction emphasizing a geometric image of linear (in)dependence?

As for the first research question, we observed some relations between understanding in the embodied world and understanding in the symbolic world. The analysis of Table 2 indicated that having a geometric image of increment of dimension was positively associated with understanding the basis in the symbolic world. The analysis of Table 5 indicated that the use of geometrical reasoning in the symbolic world was positively associated with understanding linear (in)dependence in both embodied and symbolic worlds. The analysis of Table 3 and Table 4 indicated that having an Object conception for linear (in)dependence in the embodied world, especially for the case of four spatial vectors such that any three of them do not lie on the same plane (as in the picture of Task 2(8) and Task 4 Q1(2)), was positively associated with understanding the basis in the symbolic world (Table 3), and also positively associated with understanding linear independence in both embodied and symbolic worlds (Table 4). Therefore, all these results indicate positive correlations between understanding in the embodied world and understanding in the symbolic world. Note that these results only show correlations, which does not imply causal relationship. Hence, we should be careful in describing the effect of using geometric images for teaching linear algebra. The analysis of Table 2 might seem to support that students

having a geometric image have some advantage in learning a linear algebra concept, because it shows a significant positive correlation between the result of pretest and the understanding of the notion of basis taught after the pretest. However, we should remind the reader that it was not the first time for the participants of this study to learn the notion of basis because they were students who failed to pass the course in their first year. Hence, it remains uncertain whether the understanding in the embodied world accelerates the understanding in the symbolic world or whether both types of understanding simultaneously develop by affecting each other. Thus, for the first research question, this study shows that geometrical understanding of linear (in)dependence in the embodied world is positively related to the understanding of linear (in)dependence and basis in the symbolic world, but we need more studies to investigate how the understanding in the embodied world affects to the learning of linear algebra concepts.

As for the second research question, we observed a limited effectiveness of the instruction that emphasized geometric images. The analysis of Table 6 indicated that the understanding of linear dependence of four spatial vectors in Task 2 (3) had been improved during the four-week lessons. However, the result of Task 4 and the analysis of Table 7 indicated that the understanding of linear dependence of four spatial vectors in Task 2 (8) had not been improved. The latter was surprising and also disappointing. Improving students' understanding of Task 2 (8) was more important because, as shown in Table 3 and Table 4, conceptual understanding of linear dependence in the case of Task 2 (8) was related to the understanding of basis and linear independence in the symbolic world. These results can be interpreted considering the following two possibilities: one is that the geometrical instruction implemented in this study was insufficient and can be improved; the other is that there is a limitation of students' perception even in the embodied world and it is cognitively hard to overcome such limitation. In the latter case, we should consider such limitation in teaching linear algebra, and it may lead us to reconsider how to design a linear algebra course using Tall's model of three worlds, especially to reconsider the balance and integration between geometric and algebraic presentation. However, the two possibilities need to be carefully examined in the future study.

Next, we determine whether the results on this study can provide any recommendations regarding the use of geometric images in teaching linear algebra. From the above discussion, we can say that it is difficult for students to view a space spanned by three linearly independent spatial vectors as the whole space. On the contrary, it is rather easy for students to view a spanned space that is a line or a plane as an object in the whole space. Hence, when we want to use geometric images of spanned space in a linear algebra class, we cannot assume that students would have a solid geometric understanding of a space spanned by three linearly independent spatial vectors as the whole space. This suggests that we should restrict using a spanned space that becomes a line or a plane when we use a geometric image of spanned space to help students learn concepts of linear algebra. We should remember that when we use a geometric image of a space spanned by three linearly independent spatial vectors, such space itself becomes a learning object for students.

Finally, we should mention the limitations of the study. First, the sample size was small. Second, the participants were not ordinary because they failed to pass the subject in the earlier years. Hence, further studies with a larger number of first-year students are needed.

## References

- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*. New York, NY: Springer.
- Dogan, H. (2018). Mental Schemes of: Linear Algebra Visual Constructs. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and Strategies in Teaching Linear Algebra. ICME-13 Monographs* (pp. 219–239). Cham, Switzerland : Springer International Publishing.
- Donevska-Todorova, A. (2018). Fostering students' competencies in linear algebra with digital resources. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and Strategies in Teach-*

- ing *Linear Algebra. ICME-13 Monographs* (pp. 261–276). Cham, Switzerland: Springer International Publishing.
- Dorier, J.-L., & Sierpiska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 255–273). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Gol Tabaghi, S., & Sinclair, N. (2013). Using dynamic geometry software to explore eigenvectors: The emergence of dynamic-synthetic-geometric thinking. *Technology, Knowledge and Learning*, 18(3), 149–164.
- Gueudet-Chartier, G. (2004). Should we teach linear algebra through geometry? *Linear Algebra and its Applications*, 379, 491–501.
- Hannah, J., Stewart, S., & Thomas, M. (2013). Emphasizing language and visualization in teaching linear algebra. *International Journal of Mathematics Education in Science and Technology*, 44(4), 475–489.
- Hannah, J., Stewart, S., & Thomas, M. O. J. (2014). Teaching linear algebra in the embodied, symbolic and formal worlds of mathematical thinking: Is there a preferred order? In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36 (Vol. 3)* (pp. 241–248). Vancouver, Canada: PME.
- Harel, G. (2018). Varieties in the use of geometry in the teaching of linear algebra. *ZDM*, 1–12.
- Hillel, J. (2000). Modes of Description and the Problem of Representation in Linear Algebra. In J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 191–207). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Kawazoe, M. (2018). Students' conception of spanned space and its relation to conception of linear independence. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education (Vol. 5)* (p. 82). Umeå, Sweden: PME.
- Kawazoe, M., & Okamoto, M. (2016). The influence of students' mental representation on understanding of linear independency of spatial vectors. *Journal of Japan Society of Mathematical Education*, 98, 17–24. (in Japanese)
- Kawazoe, M., Okamoto, M., & Takahashi, T. (2014). Students' mental representations of geometric vectors and sets related to linear algebra. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36 (Vol. 6)* (p. 126). Vancouver, Canada: PME. (Presentation Slides: <http://www.las.osakafu-u.ac.jp/~kawazoe/slides/PME38-SO-kawazoe.pdf>)
- Lakoff, G., & Núñez, R. (2000). *Where Mathematics Comes From: The Embodied Mind Brings Mathematics into Being*. New York, NY: Basic Books.
- Sierpiska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 209–246). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Stewart, S., & Thomas, M. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematics Education in Science and Technology*, 40, 951–961.
- Stewart, S., & Thomas, M. O. J. (2007). Embodied, symbolic and formal aspects of basic linear algebra concepts. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education (Vol. 4)* (pp. 201–208). Seoul, Korea: PME.
- Stewart, S., & Thomas, M. O. J. (2010). Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematics Education in Science and Technology*, 41(2), 173–188.
- Tall, D. (2013). *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*. New York, NY: Cambridge University Press.
- Thomas, M. O. J., & Stewart, S. (2011). Eigenvalues and eigenvectors: embodied, symbolic, and formal thinking. *Mathematics Education Research Journal*, 23, 275–296.
- Trigueros, M., & Wawro, M. (2020). Linear algebra teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 474–478). Cham, Switzerland : Springer International Publishing.

Mitsuru Kawazoe

Faculty of Liberal Arts and Sciences, Osaka Prefecture University

1-1 Gakuen-cho Naka-ku Sakai Osaka 599-8531 Japan

e-mail: kawazoe@las.osakafu-u.ac.jp