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# A survey of the main fundamental European trigonometric tables printed in the 15th and 16th centuries <br> Denis Roegel 

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# A survey of the main fundamental European trigonometric tables printed in the 15th and 16th centuries 

Denis Roegel*

2 September 2021
(last version: 2 September 2021)


#### Abstract

This document is a survey of the main European fundamental trigonometric tables printed in the 15th and 16th centuries. After a review of the work done before the 15th century in Greece, India and the Arab world, the starting points in Europe are examined. The seminal work of Regiomontanus is carefully studied and the lineage of all later works is established.


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## 1 Introduction

The purpose of this survey is to sort out the many fundamental European purely (i.e., non astronomical) trigonometric tables published in the 15th and 16th centuries, and specifically to clarify their relationships. ${ }^{1}$ I am concerned here almost exclusively with tables of sines, tangents and secants, and not with more specialized trigonometric tables that might be used as auxiliary tables.

Although a study with a similar scope has been published by Glowatzki and Göttsche in 1990, ${ }^{2}$ I feel that it is necessary to review the tables in the light of their ready access, and to see whether their understanding can be improved. I believe that my study brings new information and corrects some earlier mistakes.

This new examination is also made in the context of the LOCOMAT project, ${ }^{3}$ where a number of historical tables have been reconstructed (computationally and typographically) and analyzed, enabling a better assessment of their accuracy and lineage. However, it must be stressed that the absolute accuracy of the historical tables under consideration here is less important than their relationships and the process that led to their computation or organization.

In the following sections, I first give a short review of the history of purely trigonometric tables before the 15th century, and follow their development through Greece, India, the Arab world, and finally Western Europe. I am then considering the work of four great innovators, Johannes von Gmunden, Giovanni Bianchini, Georg von Peuerbach and Johannes Regiomontanus. The latter was the one who greatly expanded the world of trigonometric tables, and set the background for almost all future work until the end of the 16th century. I am therefore examining what are Regiomontanus's seminal tables, and then journey through a century of tables, from Regiomontanus's Tabulæ directionum profectionumque of 1490 to Rheti-

[^1]cus's Opus palatinum of 1596, which was itself the start of a new era, but the end of this survey. In this journey, I am in particular examining the genealogy of the tables. In other words, I am trying to find out who copied on whom, and I am also trying to shed a new light on the computations that were made, whenever possible.

Finally, this survey is also a companion document to a number of modern reconstructions, that is, reconstructions usually giving the exact values, but also trying to reproduce the original layout of the tables, so as to make their comparison straightforward. These reconstructions are those of Regiomontanus's table of tangents (1490), ${ }^{4}$ Engel's table of sines (1490, but here reproduced from the 1504 edition), ${ }^{5}$ Peuerbach's arctangent table (1516), ${ }^{6}$ the tables of sines of Fine (1530), ${ }^{7}$ Apian (1533), ${ }^{8}$ Regiomontanus (1541), ${ }^{9}$ Rheticus (1542) ${ }^{10}$ and again Fine (1550), ${ }^{11}$ and eventually the trigonometric tables of Rheticus (1551), ${ }^{12}$ Reinhold (1554), ${ }^{13}$ Maurolico (1558), ${ }^{14}$ Viète (1579), ${ }^{15}$ Fincke (1583), ${ }^{16}$ Lansberge (1591), ${ }^{17}$ Rheticus \& Otho $(1596)^{18}$ and Pitiscus (1613). ${ }^{19}$

## 2 Before the 15th century

I give here a quick and rough sketch of the history of trigonometric tables before the 15th century, so as to serve as a background for the development of trigonometry in the 15th and 16th centuries. More detailed (although sometimes incorrect or dated) surveys can be found in the works of Braun-

[^2]mühl, ${ }^{20}$ Tropfke, ${ }^{21}$ Bond, ${ }^{22}$ Zeller ${ }^{23}$ and more recently of Brummelen. ${ }^{24}$ More general works on the history of mathematics may also sometimes be of interest, for instance those of Montucla, ${ }^{25}$ Kästner, ${ }^{26}$ Zeuthen, ${ }^{27}$ Katz $^{28}$, Boyer/Merzbach ${ }^{29}$ or Scriba/Schreiber, ${ }^{30}$ but they may at times be inaccurate.

### 2.1 Greek chord tables

Trigonometry started with triangles inscribed in circles of some radius $R$. This radius was typically taken to be 60, but other values were also used. Within such a circle, some quantities can then be defined. In particular chords are segments subtended by an arc (figure 1 ) and there is a simple relationship between chords in a circle of radius $R$ (which I denote Chd $_{R}$ or often merely Chd) and sines (which I assume to be defined in a unit circle). We have $\operatorname{Chd}_{R} \alpha=2 R \sin (\alpha / 2)$. In the case of sine tables, $R$ was later called the sinus totus. This radius was not made equal to unity before Abū al-Wafā ${ }^{\circ}$ in the 10th century (see § 2.3). ${ }^{31}$

We know that Hipparchus (c.190-c. 120 BC) and Menelaus of Alexandria (c.70-c.140) wrote treatises on chords, but these works are unfortunately lost. ${ }^{32}$ It is not known if they contained tables of chords. But we know that the use of tables in Greek mathematics apparently takes its roots in Babylonian sources. ${ }^{33}$

In 1974, Toomer suggested that Hipparchus may have had a table of

[^3]

Figure 1: Chords and sines. $A B$ is the chord of $\alpha$ and $B C$ is its sine, for a radius $R$. We have $\operatorname{Chd}_{R} \alpha=2 R \sin (\alpha / 2)$ and $B C=\operatorname{Sin}_{R} \alpha$.
chords with a circle of radius $R=3438$ and at intervals of $7.5^{\circ},{ }^{34}$ and that this radius was then copied by Indian mathematicians, but this is still debated, by Toomer himself, ${ }^{35}$ as well as by Klintberg in 2005 who believes that Hipparchus may have had instead a chord table with $R=3600 .{ }^{36}$ On the other hand, Duke, also in 2005, and using the analysis of two eclipse trios, concurs with Toomer's original suggestion. ${ }^{37}$

Later, Ptolemy (2nd century AD) gathered the earlier works and covered the computation and use of chords in the first book of the Almagest. His table gives the chords for every $30^{\prime}$ of the quadrant, using a circle of diameter 120 (figure 2). ${ }^{38}$

The way Ptolemy computed his table of chords was to find first the sides of the inscribed regular triangle, quadrilateral, pentagon, hexagon and decagon in a circle divided in 60 parts, that is, of radius $60 .{ }^{39}$ This gave

[^4]him the chords of $36^{\circ}, 60^{\circ}, 72^{\circ}, 90^{\circ}, 108^{\circ}, 120^{\circ}$, and $144^{\circ}$.
Using the theorem known as Ptolemy's theorem (a relation between the four sides and two diagonals of a cyclic quadrilateral), Ptolemy was able to compute the chord of the difference of two arcs, when the chords of these arcs are known, and also the chord of their sum. He also was able to compute the chord of the half arc from that of the arc. Eventually, Ptolemy computed the chords of $0.75^{\circ}$ and of $1.5^{\circ}$.

Then Ptolemy used an interpolation to find the chord of $1^{\circ}$ :

$$
\operatorname{Chd} 1^{\circ}=1^{\mathrm{p}} 2^{\prime} 50^{\prime \prime}
$$

This means that the chord of $1^{\circ}$ is a bit more than one part, given that the radius is equal to 60 parts. Of course, Chd $180^{\circ}=2 R=120$.

The above value for Chd $1^{\circ}$ is correct, since we actually have Chd $1^{\circ}=$ $2 \cdot 60 \cdot \sin 0.5^{\circ}=1.047184 \ldots \approx 1+2 / 60+50 / 60^{2}$. This value will also be written $1 ; 2,50$, following a convention used by many authors. ${ }^{40}$

After the computation of Chd $1^{\circ}$, Ptolemy obtained Chd $0.5^{\circ}$ and eventually all the other values in his table of chords. Glowatzki and Göttsche recomputed Ptolemy's table using the procedure he described in the Almagest. ${ }^{41}$

The beginning of Ptolemy's table of chords as given by Halma is shown in figure 2.

In the following excerpt of Ptolemy's table


236], [van Brummelen (2009), pp. 70-77], [Buscherini and Panaino (2010)], [Otero (2020)] and especially [van Brummelen (1993), pp. 46-73] for an extensive analysis of Ptolemy's chord table and its underlying mathematics.
${ }^{40}$ Throughout this document, I will count decimal places beyond this radius, and not including it, so that the value of Chd $1^{\circ}$ given here will be considered given to two (sexagesimal) places and not three.
${ }^{41}$ See [Glowatzki and Göttsche (1976)]. Glowatzki and Göttsche give the listings of the PL/I programs they used.
the values of $\operatorname{Chd} 0^{\circ} 30^{\prime}$, $\operatorname{Chd} 1^{\circ}$ and $\operatorname{Chd} 1^{\circ} 30^{\prime}$ are given, together with differences. These differences are given in thirtieth of the actual differences, so that $31^{\prime} 25^{\prime \prime}$ becomes $\frac{31^{\prime} 25^{\prime \prime}}{30}=\frac{2 \times 31^{\prime} 25^{\prime \prime}}{60}=\frac{62^{\prime} 50^{\prime \prime}}{60}=62^{\prime \prime} 50^{\prime \prime \prime}=1^{\prime} 2^{\prime \prime} 50^{\prime \prime \prime}$.

In Greek (right hand side), letters are used for numbers, in particular $\bar{o}$ for $0, \alpha$ for $1, \beta$ for $2, \iota$ for $10, \chi$ for $20, \lambda$ for $30, \nu$ for $50, \iota \varepsilon$ for $15, \chi \varepsilon$ for $25, \lambda \alpha$ for 31, $\lambda \delta$ for 34, etc. Note however that Halma uses $\varsigma^{\prime \prime}$ for $30^{\prime}$, when actually Ptolemy used a symbol for the half degree. ${ }^{42}$

As the chords are twice the sines of the half angles, Ptolemy's table would make it very easy to obtain the sines at intervals of $15^{\prime}$.

### 2.2 Indian tables

The history of mathematics in India is complex and a lot of details are shady or lost. ${ }^{43}$ As far as trigonometry is concerned, some elements of Greek chord tables were probably taken to India, but they were then converted to sines. ${ }^{44}$ It seems that it was for practical reasons that Indian astronomers replaced the chords (jyā) by the sines, that is by half-chords (jyā-ardha, eventually shortened to $j y \bar{a})$, with various values of the radius $R$ of the base circle. ${ }^{45}$

This move from chords to sines may seem to be a detail, but it had in fact far-reaching consequences, connecting trigonometric functions with right triangles and therefore to the Pythagorean theorem.

However, even though a transmission from Greece to India is compelling, there is no certainty about the origins of the calculations, and whether the values were borrowed from Greek sources or computed independently. ${ }^{46}$

In any case, once the sine ( $j y \bar{a}$ ) had been defined for radius $R$, we had

$$
\mathrm{jya}(\theta)=R \sin \theta=\operatorname{Sin} \theta .
$$

Among the oldest sine tables, Neugebauer and Pingree mention the Paitāmahasiddhānta, possibly of the 1st century AD, which had a table based

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| :--- | :--- | :--- |
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\hline  \& $$
\begin{gathered}
6 \\
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$x \alpha$
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48
5 \& - \& $a$
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$\alpha$ \& $\alpha$
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$\alpha$ \& $v x$
$\mu \sim$
$\mu \mu$ <br>

\hline | $x$ |  |
| :--- | :--- |
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| $x \beta$ | 0 |
| $x \beta$ | ${ }^{\prime \prime}$ | \& $x \beta$

$x \beta$
$x y$ \&  \& vn
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$\alpha$ \& $\mu \beta$
$\lambda \beta$
$\lambda_{2}{ }_{2}$ <br>
\hline
\end{tabular}

Figure 2: The beginning of Ptolemy's table of chords as retranscribed by Halma in 1813 [Ptolemaeus (1813-1816)]. The part on the right shows the numerical values as represented by Greek letters. The left part is the modern translation. One should take note that the layout of the table in the Greek manuscripts differs from that displayed here and half-degrees are marked with a special symbol [Ptolemaeus (1898-1903), v. 1, p. 48].
on $R=3438 .{ }^{47}$
As mentioned above, Toomer has suggested that this radius 3438, which is $60 \cdot 360$ divided by an approximation of $2 \pi$, was actually borrowed from Hipparchus, ${ }^{48}$ but this claim may now be questioned. In any case, the simultaneous choice in India of a radius of 3438 and a measure of the circumference of $360 \cdot 60^{\prime}$ means that the radius was actually measured in the same units as the circumference, thus anticipating the concept of radians. ${ }^{49}$

The Sūrya Siddhānta, a Sanskrit treatise on Indian astronomy, which in its original version goes back to the 4th century, may also have been one of the earliest texts giving a table of sine. The version now known of this work which had been heavily amended gives a table of sines with the same $R=3438$ and for every multiple of $3^{\circ} 45^{\prime} .^{50}$ This interval of $3^{\circ} 45^{\prime}$ may go back to an interval of $7^{\circ} 30^{\prime}$ for chords. ${ }^{51}$

Around 499 AD, Āryabhaṭa's Āryabhaț̄̄ya also used $R=3438$ and had tables of sines ( $\operatorname{Sin} x$ ) and versines (or versed sines, utkramajyā, $R-\operatorname{Cos} x$ ) for every $x$ multiple of $3^{\circ} 45^{\prime} . .^{52}$

Then in the sixth century AD, Varāhamihira (c505-c587) gave a table of sines with $R=120$ again for every multiple of $3^{\circ} 45^{\prime} \cdot{ }^{53}$ Neugebauer and Pingree write that this table uses a terminology derived from that of the Paitāmahasiddhānta mentioned above. In any case, Varāhamihira's table may be much older than the 6th century as his work, the Pañca-siddhāntik $\bar{a}$ is a summary of five earlier siddhāntas. And since, as observed by Bag, ${ }^{54}$ we have the chord of $7^{\circ} 30^{\prime}$ in a circle of radius $R=60$ which is equal to the sine of $3^{\circ} 45^{\prime}$ in a circle of radius $R=120$, and that therefore a table of chords can right away become a table of sines in a circle twice as large, it may be that Varāhamihira's table goes back to a table of chords with $R=60$.

In his Brāhma-sphuṭa-siddhānta, Brahmagupta (c598-668) ${ }^{55}$ computed a

[^6]table of sines but with the radius 3270 for every multiple of $3^{\circ} 45^{\prime 5} .^{5}$ Gupta suggested that this peculiar value of $R$ is rounded from 21600/6.6, where $6.6 / 2$ is an approximation of $\sqrt{10} .^{57}$
$\mathrm{Bag}^{58}$ gave a comparative view of the main early Indian tables of sines and examined how Varāhamihira and others may have computed their values.

Brahmagupta has also used the value $R=150$ in his Khandakhādyaka (665). ${ }^{59}$ This value was then used again by Al-Khwārizmī.

### 2.3 Arabic tables

I merely sketch here the main milestones in the developement of trigonometric tables between the 8th and 13th centuries, before their wider transmission to Western Europe. ${ }^{60}$

At the end of the 8th century, during the first years of the Abbasid Caliphate (750-1258), men of learning were gathered in Baghdad and they translated into Arabic the works of the Hindus and the Greeks. ${ }^{61}$ In particular, excerpts of Brahmagupta's Brāhma-sphuṭa-siddhānta were brought to the calif Al-Mansur (714-775) by a scholar named Kañka ${ }^{62}$ and a translation was made.

The Persian mathematician and astronomer Al-Khwārizmī (c780-850) wrote a revised edition of this translation, the $Z \bar{v} j$ al-Sindhind. ${ }^{63}$ The word $z \bar{\imath} j$ is a generic name used for tabular astronomical works in Arabic and Persian, and it is derived from a Persian word meaning "cord" or "string", the tables with their columns and lines bearing some similarity with strings. ${ }^{64}$

The Indian word $j y \bar{a}$ for the chord was translated to $j i b$ and was later probably incorrectly translated in the Latin sinus, based on the similar

[^7]unvocalized Arabic word jaib meaning "cavity". ${ }^{65}$
For the Z̄̄̄j al-Sindhind Al-Khwārizmī computed c820 a table of sines. In fact, according to McCarthy and Byrne, Al-Khwārizmī's treatise contained two sine tables. ${ }^{66}$ The main table used the radius $R=60$ and a step of $1^{\circ}{ }^{\circ}{ }^{67}$ and was likely based on Ptolemy's table of chords, ${ }^{68}$ but Al-Khwārizmī also used another simpler table with $R=150,{ }^{69}$ that is Brahmagupta's radius from the Khandakhādyaka. ${ }^{70}$ This simpler table only contained the sines at intervals of $15^{\circ}$ and was especially known from a commentary by Al-Biruni. ${ }^{71}$ In particular, McCarthy and Byrne convincingly discard Hogendijk's suggestion ${ }^{72}$ of a possible candidate for a full $R=150$ sine table that could be attributed to Al-Khwārizmī. ${ }^{73}$

Al-Khwārizmī's main table survives in Adelard of Bath (c1080-1152)'s Latin translation (1126) of Maslama al-Majriti (c950-c1007)'s late 10th century Cordova edition of the original table. It was reproduced by Suter in $1914{ }^{74}$ The table with $R=150$ is not found in Adelard of Bath's translation, but its radius $R=150$ made it to the tables of Toledo.

Al-Khwārizmī also had a table of shadows with a gnomon of $12,{ }^{75}$ following the Hindu custom. ${ }^{76}$ The shadows seem to have been viewed as apart from the cosines and they were gathered in the same category only by the Europeans in the 15th century. ${ }^{77}$

Al-Khwārizmī's Zī̀ al-Sindhind was brought to Al-Andalus, the Muslimruled area of the Iberian Peninsula, sometime between 821 and 852, that is only a short time after its conception. The Umayyad dynasty, after their replacement by the Abbasid dynasty in 750, had reestablished itself there, first as an emirate, then as a caliphate.

During the 9th century, Ptolemy's Almagest was also translated in

[^8]Arabic, so that his table of chords was then also known in Arabic. ${ }^{78}$
Around the year 860 the Iranian astronomer al-Marwazi (al-Hasib) (766-after 869) borrowed Ptolemy's table of chords and gave the sines for every 15 minutes. ${ }^{79}$ He also constructed the first systematic table of tangents/cotangents ${ }^{80}$ (from $0^{\circ} 30^{\prime}$ to $89^{\circ}$ at intervals of $30^{\prime}$ and to three places ${ }^{81}$ ), although the tangent function had been tabulated before without being identified as such. ${ }^{82}$ From then on, the tangent could have a place comparable to that of the sine. But in the West, tangents were only rediscovered in the 15th century by Bianchini and Regiomontanus.

It is interesting to note that a table equivalent to a table of tangents has appeared elsewhere before al-Hasib's table, namely in China. Indian mathematics had actually been exported to China and such a table was constructed there in the 8th century in the form of a shadow-list, but this table was a false start for Chinese trigonometry. ${ }^{83}$

In the $S_{a} b i Z \bar{i} \bar{j},{ }^{84}$ the Syrian astronomer Al-Battāni (Albategnius) (c858929) put forward the advantages of sines. His table gave the sines for $R=60$ and for every half-degree and to two sexagesimal places. ${ }^{85}$ Al-Battāni also computed a table of cotangents (table of shadows) for every degree. ${ }^{86}$

But the first original arabic constructions of sine tables were the works of Abū al-Wafā̄ and Ibn Yūnus. ${ }^{87}$

The Persian mathematician and astronomer Abū al-Wafā ${ }^{\circ}$ (940-998) gave a better method for the computation of trigonometric tables and his sine table for $R=60$ has a step of $15^{\prime}$ and the values were computed on four sexagesimal places. ${ }^{88}$ It should be observed, however, as already mentioned, that Ptolemy's table itself already gave the means to construct a

[^9]table of sines at intervals of $15^{\prime}$, since the sine of $15^{\prime}$ is half the chord of $30^{\prime}$. As an indication of Abū al-Wafá ${ }^{-3}$ s work, let us mention that he found ${ }^{89}$
$$
\operatorname{Sin} 30^{\prime}=0 ; 31,24,55,54,55 .
$$

The correct value is

$$
\operatorname{Sin} 30^{\prime}=0 ; 31,24,55,54,0,12, \ldots
$$

that is

$$
\operatorname{Sin} 30^{\prime}=0+31 / 60+24 / 60^{2}+55 / 60^{3}+54 / 60^{4}+0 / 60^{5}+12 / 60^{6}+\cdots
$$

Abū al-Wafā ${ }^{\overline{3}}$ was also the first to take the radius as unity. He constructed a table of tangents and cotangents with the radius 1, but it was still subdivided sexagesimally. ${ }^{90} \mathrm{He}$ seems also to have been the one who introduced the secant and cosecant. ${ }^{91}$

Ibn Yūnus (950-1009), astronomer of Cairo, wrote the Hakemite tables. He also recomputed a table of sines. According to Debarnot, Ibn Yūnus's table of sines is more directly based on the Almagest ${ }^{92}$ than that of Abū al-Wafā ${ }^{-}$. Ibn Yūnus gave the sines for every minute, for $R=60$ and to four sexagesimal places. An excerpt of that table is shown by Berggren and King. ${ }^{93}$

Ibn Yūnus obtained $\operatorname{Sin} 1^{\circ}=1 ; 2,49,43,4,{ }^{94}$ while the correct value is

$$
\operatorname{Sin} 1^{\circ}=1 ; 2,49,43,11,14,44, \ldots
$$

As observed by Glowatzki and Göttsche, the values of Ibn Yūnus's table were obtained by interpolation. ${ }^{95}$ Ibn Yūnus very likely computed the sines at intervals of $10^{\prime}$ and filled the intermediate values by interpolation. ${ }^{96}$ For instance, he gives ${ }^{97} \operatorname{Sin} 28^{\circ} 10^{\prime}=28 ; 19,20,11,0$ which is a rather good

[^10]approximation (the correct value is $\operatorname{Sin} 28^{\circ} 10^{\prime}=28 ; 19,20,12,0$ ), but the values for $\operatorname{Sin} 28^{\circ} 1^{\prime}, \operatorname{Sin} 28^{\circ} 2^{\prime}$, etc., $\operatorname{Sin} 28^{\circ} 9^{\prime}$, are all less accurate, with the least accurate being that for $\operatorname{Sin} 28^{\circ} 5^{\prime}$.

A table with such a small interval would not be available in Western Europe before the work of Regiomontanus in 1462 (see § 3.4). Incidentally, Regiomontanus's table was also obtained by interpolation, albeit certainly using a more elaborate scheme.

At about the same time as Ibn Yūnus, the Iranian scholar Al-Biruni (973c1050) had obtained the very accurate value $\operatorname{Sin} 1^{\circ}=1 ; 2,49,43,11,14 .^{98}$ His table ${ }^{99}$ gives the sines at intervals of $15^{\prime}$ and uses $R=1$.

In 1031, the Córdoban caliphate came to an end, the state was divided in a number of smaller kingdoms, and it is during the period 1031 to 1085 that Andalusian science flourished. ${ }^{100}$ In particular, around 1070 or 1080 a group of astronomers in Toledo, including Al-Zarqāl̄̄ (c1028-1087) and perhaps also S̄ā $^{\text {c }}$ id al-Andalusī $(1029-1070),{ }^{101}$ put together the "Toledan tables., ${ }^{102}$ The tables of Toledo were closely based on those of Al-Khwārizmī ${ }^{-103}$ and AlBattāni ${ }^{104}$ which had been available in Al-Andalus since the 10th century. ${ }^{105}$ Al-Zarqālī is often credited as the author of these tables, but this is not sure and he may also not be the author of their canons. ${ }^{106}$

The original Arabic Toledan tables are no longer extant, but they are known through many Latin editions from the 12th century onward and they had an important influence on Western European astronomy. ${ }^{107}$ The tables may have been organized (rather than translated) in Latin by Gerard of Cremona who died in $1187 .{ }^{108}$

The Toledan tables contained a sine table with $R=150$ (from $1^{\circ}$ to $180^{\circ}$ for every degree and with two sexagesimal places) ${ }^{109}$ (see figures 3 and 4

[^11]for Latin editions) and another sine table with $R=60$ (for every half degree of the quadrant and with two sexagesimal places) ${ }^{110}$ (see figures 5 and 6 for Latin editions).

As mentioned above, the radius 150 of this table ${ }^{111}$ (but not the values) possibly goes back to Al-Khwārizmī's table, and consequently to Brahmagupta's Khandakhādyaka. However, as observed by van Dalen, ${ }^{112}$ the idiosyncrasies of the table indicate that it was likely derived from a table with $R=60$ (probably by Al-Battāni ${ }^{113}$ ) by multiplying the values by 2.5 and McCarthy and Byrne ${ }^{114}$ believe that Al-Zarqālī was the one who made this transformation, perhaps in the hope of restoring a table which he thought to be that of Al-Khwārizmi. ${ }^{115}$

The second sine table in the Toledan tables, with $R=60$, originates neither in Al-Khwārizmī's treatise (because Al-Khwārizmī's table only gives the sines at intervals of one degree), nor in Al-Battāni's treatise (because of distinctive discrepancies). ${ }^{116}$ It may possibly be based on Ptolemy's table of chords.

The table of shadows of the Toledan tables (see figures 7 and 8 ) is the same as that in Al-Khwārizmī and Al-Battāni's tables. ${ }^{117}$

In the 12th century, the Christians, assisted by Jewish scholars, translated many Arabic works. In particular, Gerard of Cremona (c1114-1187) translated in Latin the canons of the tables of Toledo and, as mentioned above, Adelard of Bath made Al-Khwārizmī's astronomical tables accessible to the Latins.

Around 1272, the Alfonsine tables were constructed in Toledo under the guidance of King Alfonso X of Castile (1221-1284). ${ }^{118}$ They were the last major astronomical work by Spanish astronomers before the Renaissance. ${ }^{119}$

The canons of these Castilian Alfonsine tables are still extant in a unique

[^12]manuscript, but the original tables are not. These Alfonsine tables arrived in Paris in the early 14th century and they spread in a modified form in Latin, ${ }^{120}$ becoming the Parisian Alfonsine tables. These tables were only superseded in the 16th century by the Prutenic tables based on Copernicus's theory.

More accurate trigonometric tables were constructed in the Arabic world after those of Al-Khwārizmī and Al-Battāni. Chabás and Goldstein mention for instance a 14th century manuscript giving a table of sines for 2700 arguments, at $1^{\prime}$ intervals, ${ }^{121}$ so presumably up to $45^{\circ}$ and giving sines and cosines.

And during the next century in Samarkand (now in Uzbekistan), Ulugh Beg (1394-1449) also computed a table of sines for intervals of one minute. ${ }^{122}$

And finally, let's mention that at the beginning of the 15th century, the Persian mathematician Al-Kāshī (c1380-1429) was able to obtain

$$
\operatorname{Sin} 1^{\circ}=1 ; 2,49,43,11,14,44,16,19,16
$$

(correct value: $\operatorname{Sin} 1^{\circ}=1 ; 2,49,43,11,14,44,16,26,18, \ldots$ ) by solving numerically the equation $\sin 3 x=3 \sin x-4 \sin ^{3} x$ for $x=1^{\circ}{ }^{123}$

[^13]

Figure 3: The beginning of the table of sines to $R=150$ in a Latin edition of the tables of Toledo (BNF Ms. Latin 16211, $\mathrm{f}^{\circ} 26 \mathrm{v}$ ).


Figure 4: The beginning of the table of sines to $R=150$ in a Latin edition of the tables of Toledo (BNF Ms. Latin 16655, $\mathrm{f}^{\circ} 24 \mathrm{v}$ ).

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Figure 5：The beginning of the table of sines to $R=60$ in a Latin edition of the tables of Toledo（BNF Ms．Latin 16211， $\mathrm{f}^{\circ} 28 \mathrm{r}$ ）．


Figure 6: The beginning of the table of sines to $R=60$ in a Latin edition of the tables of Toledo (BNF Ms. Latin 16655, $\mathrm{f}^{\circ} 27 \mathrm{v}$ ).


Figure 7: The table of shadows in a Latin edition of the tables of Toledo (BNF Ms. Latin 16211, $\mathrm{f}^{\circ} 24 \mathrm{v}$ ).

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| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| gm8 alcutio | vombra |  | grix <br> alnum | vmbra |  | $\begin{aligned} & \text { grag } \\ & \text { aktits } \end{aligned}$ | vinbra |  |
| กท3 | piuca | が1 | vi้6 | pintr | ¢9）${ }_{1}$ | ni10 | prical | $\mathfrak{9}$ |
| 1 | 181 | 26 | 31 | 19 | 48 | का | 5 | 39 |
| $=$ | 383 | 39 | 32 | 19 | 12 | Gz | $\sigma$ | 39 |
| 3 | 203 | $\mathrm{IB}_{8}$ | 33 | 18 | 二の | 6 | $\sigma$ | 1 |
| 8 | 1.11 | $8=$ | 38 | 14 | 81 | 67 | 4 | 41 |
| 4 | 1311 | 10 | 34 | 11 | 8 | 04 | 4 | 36 |
| $\sigma$ | 114 | 10 | 36 | 5 | 30 | 0 | 4 | 21 |
| 1 | 9 n | 88 | 34 | 14 | 44 | 0.4 | 4 | $\sigma$ |
| 8 | 64 | 28 | 38 | 14 | 21 | G8 | 8 | 41 |
| 9 | 49 | 46 | 39 | 18 | 99 | 6 | 8 | 36 |
| 10 | $\square$ | 3 | 90 | 18 | 18 | no | 4 | $=2$ |
| 11 | O1 | 88 | 81 | 13 | 48 | 41 | 4 | 8 |
| 12 | 45 | 24 | $8=$ | 13 | 20 | ATV | 3 | 48 |
| 13 | 41 | 49 | 83 | に | $4=$ | 43 | 3 | 40 |
| 18 | 88 | 8 | 84 | に | z6 | 48 | 3 | 31 |
| 4 | 88 | 46 | 84 | に | $\bigcirc$ | 14 | 3 | 13 |
| 6 | 81 | 41 | 45 | 11 | 54 | 16 | 3 | 0 |
| 14 | 39 | 14 | 81 | 11 | $z$ | 11 | $\geq$ | 95 |
| 18 | 36 | 48 | 28 | 10 | 88 | 18 | $z$ | 33 |
| 10 | 38 | 41 | 40 | 10 | 36 | 49 | $=$ | $\geq 0$ |
| 20 | 32 | 49 | 40 | 10 | 4 | go | $z$ | $n$ |
| 21 | 31 | 16 | 41 | 9 | 73 | 81 | 1 | 48 |
| z | 20 | \％ | $4=$ | 9 | z2 | $8 z$ | 1 | 81 |
| 37 | 26 | 16 | 43 | 9 | 3 | 93 | 1 | $=8$ |
| 38 | $2 \pi$ | 41 | 49 | 9 | 85 | 88 | 1 | 16 |
| 34 | $\geq 4$ | 88 | 414 | 6 | I8 | 84 | 1 | 3 |
| 20 | z | 36 | 45 | 8 | $\sigma$ | 86 | 0 | 40 |
| 24 | $z 3$ | 33 | 44 | 4 | 88 | 84 | － | 39 |
| 28 | 27 | 38 | 419 | 1 | 31 | 8 g | 0 | 24 |
| 30 | $\geq 1$ | 40 | 49 | 4 |  | 89 | － | 13 |
| 30 | z0 | 81 | 10 | 5 | $4 \sigma$ | 90 | 0 | 0 |

Figure 8：The table of shadows in a Latin edition of the tables of Toledo （BNF Ms．Latin 16655， $\mathrm{f}^{\circ} 24 \mathrm{v}$ ）．

## 3 The starting point in Western Europe: from von Gmunden to Regiomontanus

In the 13th and 14th centuries, many writings appeared based on the canons of the Toledan tables, in particular the canons of John of Lignères (1322). These canons borrowed the details of the computation of the sines from the canons of the Toledan tables, but they also gave a sine table for $R=60$ and for every half degree, as well as a table of shadows. ${ }^{124}$

Moreover, in 1175 Gerard of Cremona (c1114-87) translated Ptolemy's Almagest from the Arabic in Latin. ${ }^{125}$ Hence, tables of sines and of chords were available to those willing to pick them up.

Several relatively independent works appeared in the following centuries, of which a few can be mentioned. For instance, in 1220, Leonardo of Pisa (c1170-c1250), known as Fibonacci, published his Practica Geometrix where he gave a table of chords with a radius of 21 perticæ and a circumference of 132 perticæ (figure 9). ${ }^{126}$ The pertica is a Roman length unit equal to 10 Roman feet or about 2.96 m . The ratio 132/21 corresponds to the approximation 22/7 for $\pi$. In Leonardo of Pisa's table, the arcs are measured with the circumference (first column), so that 90 degrees correspond to 33 perticae. In that case, the chord should have been $21 \sqrt{2}=29.69 \ldots$ but it is given as 29. For 180 degrees ( 66 perticae), the chord is 42 , corresponding to twice the radius.

In that same century, Campanus of Novara (c1220-1296) also supposedly constructed a table of tangents for each degree. ${ }^{127}$

In the 14th century, we should also note the work of Levi Ben Gershon (Gersonides) (1288-1344) who in 1342 independently constructed a table of sines for intervals of $15^{\prime}$ with a radius $R=60$ and two sexagesimal places. ${ }^{128}$

And in the fist quarter of the 15th century, Jean Fusoris (c1365-1436) has independently recomputed tables of sines and chords, also at intervals of $15^{\prime}$, with a radius $R=60$ and with three to six sexagesimal places. ${ }^{129}$

But the real starting point of new trigonometric computations in Europe were the investigations of Johannes von Gmunden and Giovanni Bianchini,

[^14]

Figure 9: Leonardo of Pisa's table of chords (1220) [Boncompagni (1862), p. 96].
which seem to have taken place independently at about the same time.

### 3.1 Johannes von Gmunden (c1384-1442)

Johannes von Gmunden (c1384-1442) founded the study of astronomy and trigonometry in Vienna in the early 1400 s. ${ }^{130}$ He had obtained his Master degree at the University in 1406. Johannes von Gmunden gave lectures on the construction of astronomical instruments and computed astronomical tables. ${ }^{131}$ A few years before his death, he bequeathed his books to the University and thereby founded its first library. ${ }^{132}$ Because very few of Johannes von Gmunden's works have been printed, he has been overshadowed by Georg Peuerbach and Regiomontanus. ${ }^{133}$

In 1437, he wrote a treatise De sinibus, chordis et arcubus. ${ }^{134}$ He described the computation of sines using the Arabic (in fact Indian) methods with the sines of multiples of 15 degrees, as well as the computation of chords using the methods given by Ptolemy in the Almagest. In particular, he described the computation of the sine of the half-angle $\alpha / 2$, as well as of the complementary angle $90^{\circ}-\alpha$, from the sine of $\alpha$. The formulas are given without proof, like in the canons of the Toledan tables and in John of Lignères's canons. ${ }^{135}$ This enabled von Gmunden to compute the sines for every multiple of $3^{\circ} 45^{\prime}$ for $R=150$ and $R=60$.

Johannes von Gmunden's treatise is accompanied by several tables which, according to Glowatzki and Göttsche, were only computed in 1437 or later. ${ }^{136}$

In the first part, a table of sines with $R=150$ is given for each degree, and for minutes and seconds of the unit of the sinus totus. It is attributed

[^15]to Al-Zarqālī and must come from the Toledan tables. I assume it does not originate in John of Lignères's canons as these canons only give the sines every 15 degrees for that sinus totus. ${ }^{137}$

Another table of sines, attributed to Ptolemy, with a sinus totus of 60 , is also given for each degree. This table may be the Toledan table restricted to degrees, or it may be the table borrowed from Ptolemy's Almagest, but restricted to degrees. The two original tables are reproduced by Glowatzki and Göttsche. ${ }^{138}$

In the second part, Johannes von Gmunden gives tables of chords and sines for every half-degree from $0^{\circ}$ to $180^{\circ}$, with a radius (sinus totus) of 60 . The two original tables are also reproduced by Glowatzki and Göttsche. ${ }^{139}$ Incidentally, Klug writes incorrectly ${ }^{140}$ that Johannes von Gmunden was the first to compute a table of sines at intervals of $30^{\prime}$.

The first of the tables in the second part is the table given by Ptolemy. ${ }^{141}$ The second table may be the result of Johannes von Gmunden's computation, as it goes slightly beyond the table found in the Toledan tables. As a matter of fact, the sines are given to three sexagesimal places, ${ }^{142}$ but the last place is always 0 (actually not shown at all) or 30. It may be derived from another table.

Glowatzki and Göttsche ${ }^{143}$ drew the attention of a number of incorrect statements on Johannes von Gmunden's tables, in particular by von Braunmühl. ${ }^{144}$ The latter, and later Bond ${ }^{145}$ and Zeller ${ }^{146}$ for instance incorrectly stated that Gmunden had a table with radius 600000. ${ }^{147}$ Cantor and Eneström also made the mistake. ${ }^{148}$ Some typos of Busard's transcription ${ }^{149}$ are also corrected by Glowatzki and Göttsche. ${ }^{150}$

[^16]Johannes von Gmunden's treatise was heavily used and Peuerbach borrowed much from it. Eventually, Peuerbach's own treatise made its way into Regiomontanus's works and was printed in $1541 .{ }^{151}$

Although Johannes von Gmunden's treatise did not contain any significant novelty, it brought the impetus for a new computation of sine tables, ${ }^{152}$ which would find its completion in Pitiscus' Thesaurus mathematicus in 1613. ${ }^{153}$

### 3.2 Giovanni Bianchini (c1410-c1469)

Giovanni Bianchini was a merchant and businessman, probably born in Bologna or Florence around 1410. He later went to Ferrara, but also visited other cities. He became interested in astronomical calculations at an early age. ${ }^{154}$ His scientific works were written between 1440 and 1460 and he is in particular the author of one of the few treatises of algebra written in the fifteenth century in Latin. ${ }^{155}$ He corresponded with Regiomontanus during the latter's stay in Italy. ${ }^{156}$

Rosińska was the first to describe Bianchini's purely trigonometric tables, which consist in two decimal and two sexagesimal tables. ${ }^{157}$ Earlier writers such as Boffito ${ }^{158}$ and Birkenmajer ${ }^{159}$ mentioned some of Bianchini's trigonometric tables, but did not describe them in detail.

Bianchini's table of sines for $R=60 \cdot 10^{3}$ appears in his Tabulae primi mobilis and was reproduced and transcribed by Glowatzki and Göttsche. ${ }^{160}$

[^17]They also reproduced his table of cotangents. ${ }^{161}$ For both tables, Bianchini appears to have been the first to use a step of $10^{\prime}$ and also the first to use a partly decimal radius $R=60 \cdot 10^{3}$. ${ }^{162}$ However, as mentioned earlier, in Cairo and in the 10th century, Ibn Yūnus had computed a table of sines at intervals of $1^{\prime}$.

Bianchini's table of cotangents uses $R=12000^{163}$ and this table may have been adapted from, or inspired by, a tabula umbræ found in the tables of Toledo, but with $R=12 \cdot 60 .{ }^{164}$ The table of shadows of the Toledan tables is itself the same as that in Al-Khwārizmī and Al-Battāni. ${ }^{165}$

Bianchini must have computed his tables of sines ab novo, at least in part, perhaps interpolating from the Toledan tables. Glowatzki and Göttsche ${ }^{166}$ observed that Bianchini computed a number of values in his table of shadows by interpolation, and not from his table of sines, thereby resulting in some inaccuracies.

Bianchini also computed decimal tables, that is tables not involving 60 and only based on powers of 10. These tables are found in the set of eight trigonometric tables named Tabulae magistrales. ${ }^{167}$ Some of these tables give the values of trigonometric functions multiplied by certain astronomical factors (for instance the cosine of the obliquity of the ecliptic), but two of the tables are decimal tables $(R=10000)$ for the tangent and cosecant. ${ }^{168}$

Among this set of tables, the Tabula magistralis quarta ${ }^{169}$ gives the tangents at $10^{\prime}$ intervals and with $R=10^{4}$.

This table may have been the incentive for Regiomontanus to construct his own table of tangents in his Tabulæ directionum profectionumque, ${ }^{170}$ for every degree and with $R=10^{5}$ (figure 15). He did not, however, use Bianchini's values, but computed his tangents using his large sexagesimal

[^18]table of sines.
Another of Bianchini's tables, the Tabula magistralis quinta ${ }^{171}$ gives the cosecants at $10^{\prime}$ intervals and with $R=10^{4}$.

Rosińska assumed that Bianchini's extant decimal tables were derived from a decimal table of sines for $R=10^{4}$, and this would make sense. Unfortunately, such a table is no longer extant. ${ }^{172}$ This table for $R=10^{4}$ may itself have been computed from Bianchini's sine table found in his Tabulae primi mobilis.

Bianchini's work did not stay confined in Italy but circulated until Krakow, as described by Walsh. ${ }^{173}$

### 3.3 Georg von Peuerbach (1423-1461)

Georg Aunpekh, known as Georg von Peuerbach (1423-1461), was an Austrian mathematician and astronomer. ${ }^{174}$ He was born in Peuerbach, Austria. In 1446 he registered at the University of Vienna and between 1448 and 1451, he travelled to Italy. There he met Nicholas of Cusa (14011464) who had been papal legate in Germany since 1446 and cardinal since 1448. In Ferrara, Peuerbach may also have met Giovanni Bianchini. ${ }^{175}$ The latter wanted to obtain positions for Peuerbach in Bologna or Padua, but Peuerbach did not accept them. ${ }^{176}$ He then returned to Vienna.

Peuerbach was first influenced by Johannes von Gmunden who had

[^19]died in 1442. He can thus be considered as Gmunden's spiritual student. It seems unlikely that he knew him, but he certainly studied his works. ${ }^{177}$

In 1454, after his return from Italy, Peuerbach completed a Theoricae Novae Planetarum which actually started as lectures on the theory of planetary motion. This work was published in $1473{ }^{178}$ by Regiomontanus (1436-1476), Peuerbach's student and successor, who had certainly attended these lectures. ${ }^{179}$ The Theoricae Novae Planetarum became a standard textbook of planetary theory for the next century. ${ }^{180}$ It contains solid sphere representations of Ptolemaic planetary models, and this work was of great importance until the solid sphere hypothesis was disproved by Tycho Brahe at the end of the 16th century. ${ }^{181}$

Peuerbach was acquainted with Cardinal Bessarion (1403-1472) who was then papal legate in Germany. In 1460, Bessarion spent more than one year in Vienna ${ }^{182}$ in order to gain imperial support for the war against the Turks and during this stay he became friends with Peuerbach. Bessarion, a Greek, wanted to produce a new translation of the Almagest, because he considered Trebizond's work to be flawed. George of Trebizond (c1395c1472) was of Greek origin and has translated many works from Antiquity in Latin. In particular in 1451 he composed a Commentary on the Almagest, which has never been printed. ${ }^{183}$ Bessarion had himself considered translating the Almagest from the Greek, but his duties didn't let him the time to. ${ }^{184}$

Bessarion asked Peuerbach (who did not know Greek) to write an Epitome (summary) of Ptolemy's Almagest. ${ }^{185}$ He also wanted him to accompany him to Italy for further investigations on the Almagest. Peuerbach certainly wanted to take Regiomontanus with him to Italy, but Peuerbach died in

[^20]1461 before the journey began. ${ }^{186}$ By that time, Peuerbach had written six chapters of his Epitome, and not based on the Greek text. ${ }^{187}$ Regiomontanus, who learned Greek, added the seven missing chapters to Peuerbach's work after Peuerbach's death. This Epitome was only printed in 1496 and was very influential, in particular on Copernicus. ${ }^{188}$

Together with Johannes von Gmunden, Peuerbach and Regiomontanus were in fact the most important members of the first Viennese mathematical school of the 15th century. ${ }^{189}$ Peuerbach's work on the Epitome led him to work on reforming Ptolemy's astronomy. Gassendi later wrote that Peuerbach resurrected an almost dying astronomy and that without him, we would have neither Copernicus nor Brahe. ${ }^{190}$ Or, as others have put it, Peuerbach and his pupil Regiomontanus ${ }^{191}$ woke up the study of astronomy and built the necessary tables. ${ }^{192}$ And Hellman and Swerdlow wrote that the "Epitome is the true discovery of ancient mathematical astronomy in the Renaissance because it gave astronomers an understanding of Ptolemy that they had not previously been able to achieve." ${ }^{193}$

But as Thorndike notes, "it very likely never occurred to Peurbach that his name would go down to posterity as the reviver of the mathematics of classical antiquity or as the reformer of the mathematics of his own time."194

For his work in trigonometry, Peuerbach was both influenced by Johannes von Gmunden and by Giovanni Bianchini. In particular, Peuerbach's treatise on sines and chords, ${ }^{195}$ printed in 1541, contains literal excerpts of Johannes von Gmunden's treatise. ${ }^{196}$ And Peuerbach copied Bianchini's sine table with $R=60000$ and a step of $10^{\prime} .{ }^{197}$

[^21]

Figure 10: The first page of Peuerbach's table of arctangents (1516) [von Peuerbach (1516)] (e-rara).

A table with $R=60000$ is again found in the 1490 edition of Regiomontanus's Tabulæ directionum profectionumque, ${ }^{198}$ but it happens to be a table derived from Regiomontanus's large sexagesimal table, and not Peuerbach's table. Moreover, the 1490 table gives the sines at intervals of $1^{\prime}$.

Around 1450, Peuerbach took $R=600000$ and a step of $10^{\prime}$ and went beyond what Johannes von Gmunden and Bianchini had done. But this table with $R=600000$ is no longer extant. ${ }^{199}$ We know of its existence because Peuerbach mentioned it in the Propositio prima of his Quadratum geometricum (or Canones gnomonis ${ }^{200}$ ) written in 1455 and published in 1516. ${ }^{201}$ And another work of Peuerbach confirms that the step of the table was $10^{\prime}$.

Gassendi ${ }^{202}$ also mentions a table of sines by Peuerbach with $R=$ 6000000 and a step of $10^{\prime}$, and that this table had been extended to a step of $1^{\prime}$ by Regiomontanus, but this is probably a typo, no such table with $R=6000000$ being known of Peuerbach. ${ }^{203}$

Peuerbach was the one who provided the impetus for the replacement of Ptolemy's chords with the sines from Arabic mathematics, and Regiomontanus computed tables of sines for every minute of arc for radiuses of 6000000 and 10000000 units.

Among Peuerbach's other works is also his Quadratum geometricum ${ }^{204}$ already mentioned, written in 1455 and published in 1516. This work describes the geometrical square, an instrument for measuring heights. A similar instrument was also described by Oronce Fine in his De re et praxi geometrica published in 1556.

Peuerbach's treatise contains what is basically a table of arctangents (figure 10). Peuerbach wrote that he used his now lost table with $R=600000$ for the computation of the table. The possible values of the tangents range

[^22]from 0 to 1200 and, for an entry $x$, Peuerbach's table actually gives the value $\arctan (x / 1200)$ in degrees. For instance, for $x=1200$, Peuerbach's table gives $45^{\circ}$. For $x=500$, Peuerbach's table gives $\arctan (5 / 12)=22^{\circ} 37^{\prime} 12^{\prime \prime}$. The value 1200 used in this table may have been influenced by the radius 12000 in Bianchini's table of cotangents. ${ }^{205}$

This table of arctangents was reprinted by Gemma Frisius ${ }^{206}$ in 1545 and a similar table was given by Magini in $1592 .{ }^{207}$

### 3.4 Johannes Regiomontanus (1436-1476)

Regiomontanus, or rather Hans Müller, was probably born in 1436 in Königsberg, near Bamberg in Germany (figure 11). ${ }^{208}$ He had latinized his name as Johannes de Monte Regio and it was only half a century after his death in 1476 that he became known as Regiomontanus. ${ }^{209}$

He established trigonometry as an independant field, separate from astronomy, in Western Europe, although the Persian al-Tūsī had already written a purely trigonometric treatise in the 13th century. Regiomontanus was the most famous Western mathematician of his time. ${ }^{210}$

[^23]

Figure 11: Regiomontanus's probable birthplace in Königsberg, Bavaria. (photographs by the author)

After having studied in Leipzig, he came to Vienna around 1450 and became a friend and pupil of Georg von Peuerbach. In 1457, this is where he took his Master's degree and was appointed to the faculty, hence a colleague of Peuerbach. ${ }^{211}$

Peuerbach was supposed to go to Italy with Cardinal Bessarion who had asked him to write an Epitome (summary) of Ptolemy's Almagest. But after Peuerbach's death in 1461, it was Regiomontanus who accompanied him to Italy. ${ }^{212}$ Regiomontanus completed the Epitome, probably in $1462 .{ }^{213}$ He also studied Greek and it was during the time of the completion of the Epitome that Regiomontanus studied the copy he had made of Trebizond's translation of the Almagest. ${ }^{214}$

In Italy, Regiomontanus also became associated with Giovanni Bianchini. Part of their correspondence still survives. ${ }^{215}$ Durand writes that

[^24]"Regiomontanus envisaged an exchange of problems and answers to be based on friendly emulation, but the older Italian was speedily scared away by the precocity of the enthusiastic German."216

It was during this time that Regiomontanus constructed his Tabula primi mobilis which was only published in $1514 .{ }^{217}$ This table gives the values of $\arcsin (\sin x \sin y)$ for $0 \leq x, y \leq 90^{\circ}$ and is useful for solving problems in spherical trigonometry. The table was computed using Regiomontanus's sine table with $R=6 \cdot 10^{6} .{ }^{218}$ Glowatzki and Göttsche gave a survey of similar tables or variants published until the 19th century. ${ }^{219}$

Regiomontanus returned from Italy around $1465,{ }^{220}$ he went to Pozsony (Pressburg, Bratislava) in 1467, at the invitation of Matthias Corvinus (14431490), King of Hungary, ${ }^{221}$ of whom he became an astronomical adviser. Some time later, he was called to Buda. ${ }^{222}$

It was during this time in Hungary that Regiomontanus worked with the Polish astronomer Marcin Bylica (c1433-1493) ${ }^{223}$ whom Regiomontanus met in Rome. Together they computed some tables, in particular Regiomontanus's Tabulæ directionum profectionumque. ${ }^{224}$

In 1471 Regiomontanus moved to Nuremberg. There he set up a printing press for the purpose of publishing the most important classical scientific works, ${ }^{225}$ as well as some of his own works. ${ }^{226}$ The first work to be published was Peuerbach's Theoricae Novae Planetarum. In 1475 Regiomontanus

[^25]returned to Rome at the invitation of Pope Sixtus IV in order to work on a reform of the Julian calendar, and this is where he died in 1476, probably from the plague. During all these years, Regiomontanus worked on a critique of Trebizond's translation of the Almagest, his Theonis Alexandrini Defensio in sex voluminibus contra Georgium Trapezuntium, a work which was probably only completed in the 1470s and still remains only in manuscript form. ${ }^{227}$

It seems that Regiomontanus started around 1460 to compute sines with a large radius in order to produce a table with $R=6 \cdot 10^{6}$ for his De triangulis (1462?) (figure 22). This table was certainly inspired by Peuerbach's table with $R=600000$, although Hallam claimed ${ }^{228}$ that Regiomontanus was ignorant of that table. Glowatzki and Göttsche ${ }^{229}$ give Regiomontanus's description of the computations, the Compositio tabularum sinuum rectorum, as well as a German translation. Regiomontanus's description is contained in the 1541 edition of Peuerbach's treatise on sines. ${ }^{230}$ In section 4 below I analyze how Regiomontanus may have computed his table.

Around 1468, Regiomontanus composed another table with a radius of 10000000 . Both the sexagesimal and the decimal tables were given at intervals of $1^{\prime}$. These tables were first printed in 1541 (figures 23 and 24). ${ }^{231}$ They were however not the first tables with such intervals, and they came after those of Ibn Yūnus and Ulugh Beg (see § 2.3).

Regiomontanus's table of sines with $R=10^{7}$ was accessible in Cracow at the end of the 15 th century ${ }^{232}$ and was undoubtly one of the sources of Copernicus's trigonometric tables.

The move from a sexagesimal division to a decimal division, initiated by Bianchini, but greatly developped by Regiomontanus, made it much simpler to use the tables. With the new decimal radius, there is therefore no longer any need to mix the bases 10 and 60, as was the case in the older tables.

Regiomontanus's Tabulæ directionum profectionuтque from 1467 and published in 1490 also contained a table of tangents (figure 15) which was probably inspired by Bianchini's table of tangents. ${ }^{233}$ Cardano consid-

[^26]ered that Regiomontanus's entire Tabulæ directionum profectionumque was largely drawn from Bianchini. ${ }^{234}$ Folkerts, ${ }^{235}$ however, considered that Regiomontanus's table of tangents was influenced by Al-Battāni. In fact, Regiomontanus's table of tangents was certainly computed using his large sexagesimal table as I shall show later. A modern reconstruction of this table of tangents is given separately. ${ }^{236}$

The Tabulæ directionum profectionumque also contains a table of sines with $R=60000$ and at $1^{\prime}$ intervals (figure 16). But contrary to what Bond, ${ }^{237}$ Delambre, ${ }^{238}$ or more recently Folkerts, ${ }^{239}$ Zinner, ${ }^{240}$ North, ${ }^{241}$ Brummelen, ${ }^{242}$ Husson, ${ }^{243}$ and Chabás and Goldstein wrote, ${ }^{244}$ this table is neither by Regiomontanus nor borrowed from Bianchini. It was appended to Regiomontanus's book, probably by Johannes Engel (or Johannes Angelus) (14531512), ${ }^{245}$ and was derived from Regiomontanus's table with $R=6000000$. Moreover, as observed by Glowatzki and Göttsche, the appended table was never used by Regiomontanus. ${ }^{246}$ A modern reconstruction of Engel's table is given separately. ${ }^{247}$

In fact, most of Regiomontanus's writings were only published after his death. His main work on trigonometry, De triangulis omnimodis, was completed about 1464 but only printed in 1533, without any table. ${ }^{248}$ It is the first systematic such treatise published in Europe and it was probably used by Copernicus. However, as observed by Stamm, ${ }^{249}$ it is unlikely that Copernicus had access to Regiomontanus's treatise in manuscript form and he probably only saw the 1533 edition in the 1530s.

[^27]Delambre was critical of Regiomontanus and wrote that except for his observations and trigonometrical work, Regiomontanus had hardly the time to do more than show his good intentions. ${ }^{250}$ Delambre stresses that Regiomontanus was less advanced as a calculator than Ibn Yūnus and Abū al-Wafā. However, this opinion may need to be revised in the light of my analysis of the construction of his tables.

Braunmühl ${ }^{251}$ considered that Regiomontanus's work on triangles was influential, even if it didn't contain anything original.

And as observed by Glowatzki and Göttsche, ${ }^{252}$ the tables computed by Regiomontanus are very modern and could still be used now, only the decimal point would have to be shifted.

Thorndike thought that Peuerbach and Regiomontanus's importance had perhaps been overestimated, among other things because Regiomontanus was more than a mathematician. He was a mathematical publisher, and he came at just the right time. ${ }^{253}$

[^28]
## 4 Regiomontanus's seminal tables

We can now pause and summarize the situation of Regiomontanus's tables at the end of the 15th century. There are four different trigonometric tables usually associated with Regiomontanus: a large table of sines with radius 6000000 , another one with radius $10^{7}$, a table of tangents with $R=10^{5}$ and a smaller table of sines with $R=60000$, but of which Regiomontanus is actually not the author. Most of the tables published during the 16th century are ultimately based on the table for $R=10^{7}$.

I also include in this section some tables which are not directly from Regiomontanus, for instance the tables of secants, but which are nevertheless based on Regiomontanus's other tables.

The following tables by Regiomontanus have been reconstructed in separate documents:

- the table of tangents, as published in 1490 (figure 15$)^{254}$;
- the table of sines with $R=6 \cdot 10^{6}$, as published in 1541 (figure 23) ${ }^{255}$;
- the table of sines with $R=10^{7}$, as published in 1541 (figure 24). ${ }^{256}$


### 4.1 Fundamental tables

When Regiomontanus set out to construct his new sine tables, he was certainly influenced by Peuerbach's work, and in particular by Peuerbach's sine table with $R=600000$ and at intervals of $10^{\prime 2}{ }^{257}$ This table is no longer extant, but it is likely that Regiomontanus used it as an inspiration for his further work. ${ }^{258}$

It seems that it was around 1460 that Regiomontanus first computed sines of values at $45^{\prime}$ intervals with $R=6 \cdot 10^{8}$ (figure 12), perhaps even before Peuerbach's death. ${ }^{259}$ This was to be the fundamental table from

[^29]which a more complete table for $R=6 \cdot 10^{6}$ could be computed. ${ }^{260}$ This auxiliary table is only partially extant.

Once he had his pivots, Regiomontanus computed the sines at intervals of $15^{\prime}$, dividing the sines at intervals of $45^{\prime}$ obtained earlier in three parts in such a way that the sines vary smoothly. ${ }^{261}$ Then Regiomontanus trisected each interval, again by ensuring that the differences vary smoothly. This gave him the sines at intervals of $5^{\prime 2} .{ }^{262}$ The same procedure was again applied to obtain the sines at intervals of $1^{1}{ }^{263}$

For the table with $R=10^{7}$, Regiomontanus possibly also first computed a number of pivot values with $R=10^{9}$, but these pivots have not been kept.

### 4.2 Sine table with $R=6000000$

Regiomontanus's first large complete sine table was for a radius of 6000000 and was probably computed around 1462 in Rome. ${ }^{264}$ It gives the sines for every minute. Figure 22 shows an excerpt of a manuscript of that table. This table is based on the computations made with $R=6 \cdot 10^{8}$ as described in the previous sections.

After Regiomontanus's death, Regiomontanus's table was long kept in manuscript form. It was only published in 1541 with Peuerbach's Tractatus super propositiones Ptolemæ etc., ${ }^{265}$ and together with the table for $R=10^{7}$ (figures 23 and 24). These two tables were then again published in 1561 in Regiomontanus's De triangulis. ${ }^{266}$ Glowatzki and Göttsche gave a facsimile of the 1541 sexagesimal table and listed its errors. ${ }^{267}$

Regiomontanus's sine table appears rather accurate, although it is probably slightly less accurate than the table for $R=10^{7}$. Sampling only the values for whole degrees, there are 25 last-place errors and one typo (for

[^30]

Figure 12: The list of pivots for Regiomontanus's large sexagesimal table [von Peuerbach and Regiomontanus (1541)] (source: Dresden).
$40^{\circ}, 3856796$ which should be 3856726 ). Of the last-place errors, all are of one unit, and one $\left(80^{\circ}\right)$ is of two units. I have given separately a modern reconstruction of this table with the exact values which can be used for comparison with Regiomontanus's table. ${ }^{268}$ And in $\S 5$, I am giving a more detailed analysis of Regiomontanus's errors and computation procedure.

In Regiomontanus's table, the column of differences does not give the actual difference $\Delta$, but the difference per second, in other words $\Delta / 60$. These differences are given to one decimal place which is separated by a space. ${ }^{269}$ For instance, the first difference is $\Delta=1745$ and it is given as 291 , because $1745 / 60=29.08 \ldots$.. But this value can also be read 291, in which case it is the sixth of the actual difference.

These differences follow a layout similar to those in Bianchini's table with $R=60 \cdot 10^{3}$, so that it is possible that Regiomontanus borrowed this layout. ${ }^{270}$

### 4.3 Sine table with $R=10000000$

Regiomontanus's second large sine table was for a radius of $10^{7}$ and was completed in 1468. ${ }^{271}$ It came shortly after the smaller decimal table of tangents which was computed in 1467.

This large decimal table is probably not the first decimal table of sines, although Folkerts claimed so. ${ }^{272}$ It has been assumed that Bianchini had a decimal table of sines, probably with a radius $R=10^{4}$ (see § 3.2), but this table is no longer extant.

Regiomontanus's table is also not based on his large sexagesimal table. ${ }^{273}$ Regiomontanus may have computed a number of pivot values, perhaps with $R=10^{9}$, or he may have reused the sexagesimal pivots by multiplying them by 10/6. In any case, these pivots have not been kept. Then, Regiomontanus must have proceeded by interpolation as in the sexagesimal table.

Like in the previous table, sines are given for every minute. This table was also published in 1541 and 1561 together with the sexagesimal table (figures 23 and 24). Glowatzki and Göttsche gave a facsimile of the entire

[^31]1541 edition of the table, and listed its typos. ${ }^{274}$
The differences are expressed like in Regiomontanus's sexagesimal table and the first difference is for instance $\Delta=2909$ and it is given as 485 , corresponding to $\Delta / 60=48.48 \ldots$

I have given separately a modern reconstruction of this table. ${ }^{275}$
It is interesting to note that Regiomontanus's table is slightly more accurate than the previous one with $R=6 \cdot 10^{6}$. ${ }^{276}$ Sampling only the sines for whole degrees, we can for instance see that there are only seven incorrect values, one of which (for $25^{\circ}$ ) being an obvious typo (4226583 which should be 4226183), and the other six values being only off by one unit of the last place. This suggests of course that the decimal table was not merely obtained from the sexagesimal table, but must have been obtained either from the pivots of the sexagesimal table, or from newly computed pivots, as described above.

### 4.4 Sine table with $R=60000$

The Tabulx directionum profectionumque published in 1490 contains a 30 pages long sine table with $R=60000$ giving the sines for every minute (figure 16), ${ }^{277}$ but this table was certainly computed by Johannes Engel for that edition (see § 6.1), and not by Regiomontanus.

### 4.5 Table of tangents

Regiomontanus's Tabulæ directionum profectionumque, ${ }^{278}$ from 1467 and printed in 1490, also contained a short table of tangents, which he called tabula fecunda (figure 15). ${ }^{279}$ The name "tangent" was as a matter of fact only introduced in 1583 by Fincke. ${ }^{280}$

Regiomontanus's table is only one page long and gives the tangents for every degree, and for a radius of 100000 . The tangents were computed from the 1462 table of sines (with $R=6000000$ ), by first dropping two digits and rounding the values, and then by mere division. ${ }^{281}$ This procedure

[^32]actually gives exactly Regiomontanus's values, except for the angles $43^{\circ}$, $73^{\circ}, 85^{\circ}$ and $89^{\circ}$. In these four cases, Regiomontanus very likely got the computations wrong, or these are typos. Incidentally, the same procedure fails miserably when using the decimal table of sines, and it is almost impossible to obtain the values of the table of tangents with this starting point.

Regiomontanus's table was not the first table of tangents, as tangents had already been used in eastern Islam, as mentioned above (see § 2.3).

Regiomontanus's table was reproduced in subsequent editions of his Tabulx directionum profectionumque, by Gemma Frisius in 1545 (but as cotangents) ${ }^{282}$ (also with Peuerbach's 1516 quadratum table ${ }^{283}$ ), by Gaurico in $1557{ }^{284}$ by Maurolico in 1558 (at least partially, and he called it umbra ver$s a),{ }^{285}$ by Schreckenfuchs in $1569,{ }^{286}$ and in subsequent editions of these works. ${ }^{287}$

Gaurico's table (1557) ${ }^{288}$ only goes up to $50^{\circ}$, and is attributed to Campanus. But neither Glowatzki and Göttsche, ${ }^{289}$ nor von Braunmühl, ${ }^{290}$ nor Zinner ${ }^{291}$ were able to understand this attribution.

Curiously, Gaurico also gives a sine table with the heading tabula fecunda and also only up to $50^{\circ}$.

And finally, mention should be made of Bendefy who mistakenly wrote in 1980 that Regiomontanus had constructed a table of tangents for a radius $R=10^{7}$ and for every minute, and that it was only Reinhold who published it in $1554{ }^{292}$
${ }^{282}$ See [Gemma Frisius (1545)]. It was also reprinted in 1557.
${ }^{283}$ [von Peuerbach (1516)]
${ }^{284}$ [Gaurico (1557)]
${ }^{285}$ [Maurolico (1558)]
${ }^{286}$ See [Glowatzki and Göttsche (1990), p. 180] and [Schreckenfuchs (1569), p. 153].
${ }^{287}$ [Glowatzki and Göttsche (1990), pp. 180-181]
${ }^{288}$ [Gaurico (1557)]
${ }^{289}$ [Glowatzki and Göttsche (1990), pp. 180-181]
${ }^{290}$ [von Braunmühl (1900, 1903), v. 1, p. 101]
${ }^{291}$ [Zinner (1968), p. 148]
${ }^{292}$ [Bendefy (1980), p. 248] Bendefy's statement seems based on Barna Szénássy's history of Hungarian mathematics (A magyarországi matematika története, 1970), but I was not able to check this source. Bendefy also cites Zinner's article on Regiomontanus in Hungary, published in Hungarian, and which does not seem to contain such a statement (Ernő Zinner, Regiomontanus Magyarországon, Matematikai és Természettudományi Értesitő, volume 55, 1936?, pp. 280-288).

### 4.6 Secant tables

Regiomontanus did not compute tables of secants, but the first tables of secants are based on his tables of sines. This is the case of Copernicus's table of secants, which might have been computed around 1530. Bianchini had computed a table of cosecants (§ 3.2). But I am not aware of earlier such tables, although, as mentioned before (§ 2.3), Abū al-Wafā̄ introduced the notion of secant in Baghdad in the 10th century.

The tables of secants published by Rheticus in 1551 [Rheticus (1551)] and by Maurolico in 1558 [Maurolico (1558)] also ultimately derive from Regiomontanus. ${ }^{293}$

It was Viète ${ }^{294}$ who in 1579 was the first to compute a table of secants with an interval of $1^{\prime}$ albeit with a variable radius between $R=10^{5}$ and $R=10^{9}$.

And the first table of secants with an interval of $1^{\prime}$ and $R=10^{7}$ was published by Fincke in $1583 .{ }^{295}$ Fincke was actually the one who named it secant. His secants were certainly computed from his tangents, which themselves go back to Regiomontanus, via Reinhold. ${ }^{296}$

[^33]
## 5 An analysis of Regiomontanus's great tables

One of my purposes has been to find out how Regiomontanus computed his two large tables of sines. We know rather well how he computed the sines at intervals of $45^{\prime}$, but we know little beyond that, and no one seems to have investigated this matter so far, not even Glowatzki and Göttsche. ${ }^{297}$

The first step in such an investigation is to clear the tables of the noise they contain, namely of the typos, both in the printed versions and in the manuscripts. Although I have not consulted manuscripts of Regiomontanus's table, I believe that it is possible to come very close to what Regiomontanus has actually computed.

### 5.1 Typos, accuracy and statistics

### 5.1.1 General principles

I have gone over each of the $2 \times 5401$ values of the sines (from $0^{\circ}$ to $90^{\circ}$ by steps of $1^{\prime}$ ) in the 1541 printing, trying to detect obvious typos. This work has been done independently of that of Glowatzki and Göttsche who had already reported a number of typos. ${ }^{298}$ I have consequently made two tables where I corrected a number of typos, such as wrong digits in the left figures, swapped figures, or swapped lines. In the resulting tables, I carefully examined all the cases where Regiomontanus's tables were in error by more than 2 units of the last place. Every such case which appeared isolated was removed. The justification for correcting these seemingly small errors was that they would have been very easy to detect by computing differences between consecutive terms, and that almost always these anomalies were isolated, and could not have been Regiomontanus's real values, at least not his intended values. These decisions may be objectionable, but I have only corrected errors which are easy to detect by anybody working on tables. I did not correct any more fundamental issue. And these corrections are necessary in order to get a better understanding of the underlying computations.

### 5.1.2 Corrections to the tables

Apart from the very conspicuous typos already reported by Glowatzki and Göttsche (mostly not repeated here), I made the following smaller

[^34]corrections to the sexagesimal table:

|  | value |  |  | values |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| angle | le | corrected | age | ble | cor |
| $1^{\circ}$ | 155 | 15531 |  | 94 |  |
|  | 1486 | 181487 | $49^{\circ} 34^{\prime}$ | 4566965 | 4566968 |
|  | 275668 | 275665 | $51^{\circ} 35^{\prime}$ | 4701078 | 77 |
| $4^{\circ} 58^{\prime}$ | 51945 | 519456 | $57^{\circ} 21^{\prime}$ | 5051893 | 5051891 |
|  | 524674 | 524673 | $61^{\circ} 26^{\prime}$ | 5269565 | 5269567 |
|  | 693009 | 693088 | 66 | 55 | 5504445 |
| $7^{\circ} 52^{\prime}$ | 82121 | 82121 | $70^{\circ} 2^{\prime}$ | 5639347 | 5639349 |
| $12^{\circ} 4^{\prime}$ | 12 | 1254297 | $70^{\circ} 37^{\prime}$ | 5659910 | 5659916 |
| $17^{\circ} 33^{\prime}$ | 1809221 | 180922 | $73^{\circ} 50^{\prime}$ | 5762737 | 762735 |
| $18^{\circ} 28^{\prime}$ | 1900518 | 19005 | $74^{\circ} 51^{\prime}$ | 5791465 | 5791470 |
| $18^{\circ} 40^{\prime}$ | 19 | 1920370 | $76^{\circ} 51^{\prime}$ | 5842661 | 5842667 |
| $19^{\circ} 29^{\prime}$ | 2001193 | 2001195 | $77^{\circ} 17^{\prime}$ | 5852821 | 823 |
| $19^{\circ} 50^{\prime}$ | 035 | 20357 | $79^{\circ} 58^{\prime}$ | 5908230 | 5908238 |
| $21^{\circ} 43^{\prime}$ | 222 | 2220102 | $81^{\circ} 10^{\prime}$ | 5928833 |  |
| $41^{\circ} 59^{\prime}$ | 4013488 | 401348 | $81^{\circ} 29^{\prime}$ | 5933835 | 933837 |
| $42^{\circ} 24^{\prime}$ | 4045818 | 404581 | $82^{\circ} 46^{\prime}$ | 5952258 | 952250 |
| $42^{\circ} 53^{\prime}$ | 4083045 | 083046 | $86^{\circ} 17^{\prime}$ | 5987385 | 987382 |
| $44^{\circ} 1^{\prime}$ | 4169203 | 416920 | $86^{\circ} 46^{\prime}$ | 5990440 | 990450 |
| $45^{\circ} 9^{\prime}$ | 4253736 | 425373 | $87^{\circ} 56^{\prime}$ | 59960 | 5996 |

Note that my corrections do not always replace the 1541 printed values by the exact ones, but by the values I believe should have been printed. For instance, for $6^{\circ} 38^{\prime}$, Glowatzki and Göttsche replaced 693009 by 693090, which does make sense as a typo. However, the value 693090 does not make much sense in its context (i.e., the surrounding values) and I believe that there was an error before that, and that Regiomontanus should have obtained 693088, which is the value I gave in my table. In this case, I believe that Regiomontanus accidently obtained the correct value 693090, and that the printer got it wrong by setting 693009 .

The only values with a deviation of 3 units of the last place are those from $6^{\circ} 50^{\prime}$ to $6^{\circ} 53^{\prime}$. I believe that the pivot $6^{\circ} 50^{\prime}$ was erroneously computed and should probably have been 713889 (with an error of 1). This has probably caused the sines of $6^{\circ} 51^{\prime}$ to $6^{\circ} 53^{\prime}$ to be also wrong by 3 units. I have however not fixed these deviations and these errors remain in the cleaned table, as they are not mere typos. But the truth is that these errors would not escape a close scrutiny by differencing.

The above table also contains corrections for a number of deviations of 2 units, when these were clearly isolated $\left(1^{\circ} 29^{\prime}, 1^{\circ} 44^{\prime}, 6^{\circ} 38^{\prime}, 17^{\circ} 33^{\prime}\right.$,
$41^{\circ} 59^{\prime}, 42^{\circ} 53^{\prime}, 45^{\circ} 42^{\prime}, 49^{\circ} 34^{\prime}, 51^{\circ} 35^{\prime}, 57^{\circ} 21^{\prime}, 70^{\circ} 2^{\prime}, 73^{\circ} 50^{\prime}, 77^{\circ} 17^{\prime}, 81^{\circ} 10^{\prime}, 81^{\circ} 29^{\prime}$, $82^{\circ} 46^{\prime}$ ).

I also corrected some suspicious transitions, where the error switched from 1 to -1 or from -1 to 1 . These errors would have been very easy to detect by differencing and concern $5^{\circ} 1^{\prime}, 7^{\circ} 52^{\prime}, 18^{\circ} 28^{\prime}, 18^{\circ} 40^{\prime}, 66^{\circ} 33^{\prime}$, and $79^{\circ} 58^{\prime}$.

In the case of the decimal table, I also made a number of corrections, including

|  | values |  | angle | values |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| angle | table | corrected |  | table | corrected |
| $4^{\circ} 19^{\prime}$ | 752688 | 752687 | $39^{\circ} 50^{\prime}$ | 6405569 | 6405566 |
| $13^{\circ} 14^{\prime}$ | 2289163 | 2289171 | $40^{\circ} 21^{\prime}$ | 6474556 | 6474550 |
| $13^{\circ} 28^{\prime}$ | 2328799 | 2328796 | $54^{\circ} 15^{\prime}$ | 8115746 | 8115740 |
| $17^{\circ} 14^{\prime}$ | 2962630 | 2962639 | $54^{\circ} 29^{\prime}$ | 8139469 | 8139466 |
| $19^{\circ} 39^{\prime}$ | 3362739 | 3362735 | $58^{\circ} 43^{\prime}$ | 8546096 | 8546099 |
| $20^{\circ} 24^{\prime}$ | 3485724 | 3485720 | $59^{\circ} 59^{\prime}$ | 8658793 | 8658799 |
| $35^{\circ} 21^{\prime}$ | 5785691 | 5785697 | $60^{\circ} 17^{\prime}$ | 8684873 | 8684875 |
| $37^{\circ} 42^{\prime}$ | 6115272 | 6115271 | $61^{\circ} 15^{\prime}$ | 8767267 | 8767269 |
| $38^{\circ} 20^{\prime}$ | 6202350 | 6202356 | $61^{\circ} 50^{\prime}$ | 8815783 | 8815781 |
| $38^{\circ} 53^{\prime}$ | 6277368 | 6277367 | $87^{\circ} 32^{\prime}$ | 9981731 | 9990734 |
| $39^{\circ} 24^{\prime}$ | 6347309 | 6347306 |  |  |  |

Among these corrections, the small ones for $37^{\circ} 42^{\prime}$ and $38^{\circ} 53^{\prime}$ have been made because their deviations appeared to be isolated. And like in the sexagesimal table, I have also corrected some suspicious transitions, where the error switched from 1 to -1 or from -1 to 1 . These errors concern the values $4^{\circ} 19^{\prime}, 13^{\circ} 14^{\prime}, 60^{\circ} 17^{\prime}, 61^{\circ} 15^{\prime}$, and $61^{\circ} 50^{\prime}$. Some of these corrections may appear larger than these small transitions, but that may be because there may have been both printer typos and earlier errors, and that I first corrected the large errors, for instance for $13^{\circ} 14^{\prime}$ whose sine value in Regiomontanus's manuscript may have been 2289173, but which still can't have been the right one.

Some of these typos/errors were reported by Glowatzki and Göttsche, but not all of them, and, as I explained above, Glowatzki and Göttsche reported other errors which I have corrected, but not included in the above tables. ${ }^{299}$ Moreover, my corrections do not always coincide with theirs, as I have tried to replace the incorrectly printed values by those that Regiomontanus has presumably computed, and not by the exact sines. I believe

[^35]however that all the typos reported by Glowatzki and Göttsche have been taken care of in my versions.

Eventually, we end up with two tables which must be very close to Regiomontanus's calculations, and which have been cleared of probably almost all typos, both in the printed versions and in the manuscripts. What I mean by this is that Regiomontanus would have found all these errors by mere differencing and that the resulting cleaned tables provide a better start for the analysis of Regiomontanus's actual computations.

These tables are provided separately ${ }^{300}$ as text files for others to analyze, should they wish to.

### 5.1.3 The sexagesimal pivots

The cleaned tables now make it possible to have a closer look at computational errors and in particular at the accuracy of the pivots. It first appears that the sexagesimal table contains about 2223 errors of one unit or more, and none of more than 3 units. This does agree with the count given by Glowatzki and Göttsche who came up with 2232, but with slightly different corrections. I am of course writing "about," because in some cases I made adjustments which may or may not be correct. The same remark applies to the decimal table.

The pivots at $45^{\prime}$ intervals (for $R=6 \cdot 10^{8}$ ) for the sexagesimal table appear very accurate. There are only 17 values which are not correct, and among them all are off by one unit of the last place, except those for $45^{\prime}$, $27^{\circ}, 57^{\circ} 75^{\prime}$, and $59^{\circ} 25^{\prime}$. In the case of $57^{\circ} 75^{\prime}$, there is an obvious typo, and the original value may have been correct. The value for $45^{\prime}$ may also be a typo. In any case, none of these small errors have any serious impact on the values in the sexagesimal table.

Consequently, the $45^{\prime}$ pivots in the sexagesimal table (for $R=6 \cdot 10^{6}$ ) are mostly correct. In fact, they should even all be correct. But there are three exceptions. The $6^{\circ} 45^{\prime}$ pivot is correctly given in the table for $R=6.10^{8}$, but there is a different value in the final table. The neighboring values would make things worse if I gave the correct value to $\operatorname{Sin} 6^{\circ} 45^{\prime}$, so that I suspect that Regiomontanus made an error when copying his own (correct) value of $\operatorname{Sin} 6^{\circ} 45^{\prime}$. The same observations apply to $\operatorname{Sin} 8^{\circ} 15^{\prime}$ and $\operatorname{Sin} 44^{\circ} 15^{\prime}$. These three sines are off by one in the sexagesimal table.

As far as the other pivots are concerned, two $15^{\prime}$ pivots are off by 2 and 78 are off by $1.3455^{\prime}$ pivots are off by 1 , and 11 by 2 or 3 .

[^36]
### 5.1.4 The decimal pivots

The decimal table contains about 1841 errors or one unit or more. Again, this is very close to Glowatzki and Göttsche's count which is 1833, but with slightly different corrections. There are also three $45^{\prime}$ pivots which are incorrect, but not the same ones as for the sexagesimal table. $2215^{\prime}$ pivots are off by 1 , and none by $2.2825^{\prime}$ pivots are off by 1 , and one is off by 2 .

It does therefore appear that the decimal table is somewhat more accurate than the sexagesimal table, but not by an order of magnitude.

### 5.1.5 Some general statistics

We can also observe that in the sexagesimal table the longest sequence without errors is of length 52 and starting at $56^{\circ} 4^{\prime}$ : once the typos are corrected, all the sines from $56^{\circ} 4^{\prime}$ to $56^{\circ} 55^{\prime}$ are correct. The longest sequence with a constant error of one unit of the last place (in the same direction) is of length 30 and starts at $3^{\circ} 52^{\prime}$. The longest sequence with a constant error of two units is of length 6 .

Similar results are obtained with the decimal table and the longest sequence without errors is of length 50 , starting at $27^{\circ} 45^{\prime}$.

The average errors are -0.10 for the sexagesimal table and 0.12 for the decimal table, but it is difficult to analyze errors in more depth without taking into account the structure of the computations, namely the two trisections and the possibly final linear interpolation.

We can now try to answer a number of questions on the computation of the pivots:

- For instance, assuming two $45^{\prime}$ pivots are correct, how often are the $15^{\prime}$ pivots correct?
The answer to this question is surprising, because there is a clear difference between the sexagesimal and decimal tables. In the first case, $7015^{\prime}$ pivots (for 114 ranges out of 120) are incorrect, but in the second case only 17 are incorrect (also for 114 ranges). The $15^{\prime}$ pivots of the decimal table appear clearly more accurate than in the sexagesimal table.
An example of incorrect $15^{\prime}$ pivot in the sexagesimal table is that of $45^{\circ} 15^{\prime}$, where the sines of $45^{\circ}$ and $45^{\circ} 45^{\prime}$ are correct.
One should actually distinguish the cases where the two twin/double $15^{\prime}$ pivots (in a $45^{\prime}$ interval) are wrong, and the cases where only one of
them is wrong. Surprisingly, there are 24 cases of wrong twin pivots in the sexagesimal table, and none in the decimal table.
Morevover, in the decimal table, 13 out of 17 wrong (non twin) $15^{\prime}$ pivots concern the second $15^{\prime}$ pivots. Things are more even in the sexagesimal table, where 13 out of 22 wrong (non twin) $15^{\prime}$ pivots concern the second $15^{\prime}$ pivots.
An example of incorrect $15^{\prime}$ pivot in the decimal table is that of $32^{\circ}$, where $31^{\circ} 30^{\prime}$ and $32^{\circ} 15^{\prime}$ are correct.
- And assuming two $15^{\prime}$ pivots are correct, how often are the $5^{\prime}$ pivots correct?
We find that for the sexagesimal table, $1195^{\prime}$ pivots are incorrect (for 231 ranges out of 360 ) when the $15^{\prime}$ pivots are correct, and that there are 30 incorrect twin pivots.
For the decimal table, $2175^{\prime}$ pivots are incorrect (for 318 ranges out of 360 ), including 52 twin pivots. Under this perspective, the sexagesimal table appears more accurate than the decimal table.
An example of an incorrect $5^{\prime}$ pivot in the sexagesimal table is that of $75^{\circ} 5^{\prime}$.
- Finally, how often are the $5^{\prime}$ interpolations correct?

Again, we restrict ourselves to the cases where the two pivots are correct, as such a restriction is still representative. ${ }^{301}$ What is the most common outcome? Is it $0,0,0,0,0,0$ ? In other words, if two adjacent $5^{\prime}$ pivots are correct, are the four intermediate values also usually correct?
The number of ranges to consider (where the two $5^{\prime}$ pivots are correct) is similar in both tables: 576 ranges for the sexagesimal table and 604 ranges for the decimal table. Is the outcome the same? First, given that the table has been checked by differences, the only values which can appear between the two end 0 s are 0 and $\pm 1$. There are therefore $3^{4}=81$ different possible sequences, but the most common sequence is ( $0,0,0,0,0$ ) with 277 cases in the sexagesimal table and 258 cases in the decimal table. Again, under this perspective, the sexagesimal table is in fact slightly more accurate than the decimal one.

[^37]
### 5.2 A tentative analysis of Regiomontanus's construction

The procedure used by Regiomontanus to construct his large tables is a bit vague, but I believe that it can be clarified. As far as I know, no attempt has been made so far to explain this process. As mentioned above, Regiomontanus basically describes a subtabulation process, where from sine values at $45^{\prime}$ intervals he obtains values for every $15^{\prime}$, then for every $5^{\prime}$, and finally for every minute. Regiomontanus explicitely speaks of making the differences increase regularly, and it should be clear that the differences between values played a key role in this computation. It is also clear that what Regiomontanus has done was to interpolate values, more than merely to compute accurately thousands of sines.

Reading Regiomontanus's description, one can not avoid thinking of the works of Bürgi ${ }^{302}$ and Briggs ${ }^{303}$ and wonder if, perhaps, Regiomontanus had not anticipated them. I believe in fact that his computations were indeed forerunners of what Bürgi and Briggs did, a century or a century and a half later. Both Bürgi and Briggs analyzed how finite differences could be used not merely to find new values by adding differences, but also to subtabulate, and find intermediate values from larger differences. For instance, Briggs computed the logarithms of various integers as interpolations of logarithms given at larger intervals. Among the techniques he describes is the quinquisection, where he is able to divide an interval in five parts and obtain the intermediate logarithms.

### 5.2.1 The general setting

Here, I want briefly to test this hypothesis, which may be expanded later in the future. To be as general as possible, I will consider a sequence of sines $v_{0}, v_{1}, v_{2}, \ldots$, for angles $a_{0}, a_{1}, a_{2}, \ldots$, where $a_{i+1}-a_{i}$ is a constant interval, for instance $45^{\prime} . v_{i}=\operatorname{Sin}\left(a_{i}\right)$, with some radius $R$, which I will take here as $6 \cdot 10^{8}$, but which could be different.

These values are used to define the finite differences $\Delta_{0}^{1}=v_{1}-v_{0}$, $\Delta_{1}^{1}=v_{2}-v_{1}, \ldots, \Delta_{0}^{2}=\Delta_{1}^{1}-\Delta_{0}^{1}, \Delta_{1}^{2}=\Delta_{2}^{1}-\Delta_{1}^{1}, \ldots, \Delta_{0}^{3}=\Delta_{1}^{2}-\Delta_{0}^{2}$, etc.

What Regiomontanus sought to do was to find the sines $v_{1 / 3}, v_{2 / 3}$, etc., of the intermediate angles $a_{1 / 3}, a_{2 / 3}, a_{4 / 3}$, etc. In other words, he was working on a trisection. For instance, if $a_{0}=3^{\circ}, a_{1}=3^{\circ} 45^{\prime}$, etc., then $a_{1 / 3}=3^{\circ} 15^{\prime}$ and $v_{1 / 3}=\operatorname{Sin} 3^{\circ} 15^{\prime}$.

The subtabulated differences are $\delta_{0}^{1}=v_{1 / 3}-v_{0}, \delta_{0}^{2}=\delta_{1 / 3}^{1}-\delta_{0}^{1}$, and so on.

[^38]I believe that during the first stage of his procedure, Regiomontanus tried to compute the smaller differences, that is the differences for intervals of $15^{\prime}$, from the differences for intervals of $45^{\prime}$. In other words, I believe that he tried to compute $\delta_{0}^{1}$ and $\delta_{0}^{2}$, and these two values would then be sufficient to compute $v_{1 / 3}$ and $v_{2 / 3}$.

I invite those who are unconvinced by this suggestion to consider for instance the trisection of the $45^{\prime}$ interval between $\operatorname{Sin} 20^{\circ}$ and $\operatorname{Sin} 20^{\circ} 45^{\prime}$. The radius could be taken as $R=10^{5}$, and the sines to start with would then be $34202,35429,36650,37865$, etc. Merely manipulating these numbers without great thought leads to two approximations of the subtabulated first differences, namely 410 and 407 . We would have three 410 differences and three 407 differences. This is of course not satisfactorily, it is not an even decrease, and looking at the second differences, we find $0,0,-3,0,0$. This can be improved by starting with the first subtabulated difference 410 and spreading the -3 over five values, hence taking -0.6 instead of -3 for the second difference. We have here a very simple means to obtain the second differences. Adding up these differences, we end up with 36653 instead of 36650. It is not perfect, but it is not that bad. Since the second difference was correctly spred, we may want to improve the first difference 410, but it will actually be difficult to reach a better result with this radius. Such experiments are useful to convince oneself that it is practically unfeasible to get the differences to vary evenly merely by fiddling with the numbers, and at the same time they pave the way for the discovery of a relationship between certain values. And these are the key issues here.

The first key is to notice that the second differences are practically proportional to the sines. This had actually been discovered long before Regiomontanus, for instance in India by Āryabhaṭa in the 6th century ${ }^{304}$ And Wagner and Hunziker recently suggested ${ }^{305}$ that there was perhaps a transmission from India to Bürgi, although I am rather doubtful about such an assertion. In the above simplified example, one would readily find that the second differences are all equal to -6 , at least around $20^{\circ}$, and if this operation is done for other values, one can't be far from discovering that the second differences are proportional to the sines.

I will therefore assume that Regiomontanus first noticed that $\Delta_{0}^{2} \approx \frac{v_{1}}{C_{45}}$ where $C_{45}$ is some constant, and that this is true on the entire sine table. I am also guessing that Regiomontanus knew that the constant depends on

[^39]the size of the interval, hence my subscript. For intervals of $45^{\prime}$, we have $C_{45} \approx-5836$. The exact expression behind this value matters little here, ${ }^{306}$ but what is important is that by playing with differences, any serious table computer would eventually find out that there is some constant ratio involved, and perhaps think of using it backwards. By computing a few exact values of sines at $15^{\prime}$ intervals, Regiomontanus may have found that another constant is involved: ${ }^{307} \delta_{0}^{2} \approx \frac{v_{1 / 3}}{C_{15}}$ and that $C_{15} \approx-52525$. Regiomontanus may or may not have noticed that $C_{15} / C_{45} \approx 9$. But he must certainly have noticed that the third differences $\Delta^{3}$ vary only very slowly and that their variations can be neglected on small ranges.

At this stage, Regiomontanus could have had a means to compute $\delta_{0}^{2}$ using an approximation of $v_{1 / 3}$. Of course, $v_{1 / 3}$ is what we are looking for, but we can easily get an approximation of $v_{1 / 3}$ such as

$$
v_{1 / 3} \approx v_{0}+\frac{v_{1}-v_{0}}{3}
$$

and this is in fact sufficient to get a good approximation of $\delta_{0}^{2}$.
What now remains is to obtain an approximation of $\delta_{0}^{1}$. An obvious approximation is

$$
\frac{\Delta_{0}^{1}}{3}
$$

but Regiomontanus needed a better one.
The second key here is to see or guess that the second differences are involved in the approximations of the first differences. In any case, one may want to test whether

$$
\delta_{0}^{1} \approx \frac{\Delta_{0}^{1}}{3}+\alpha \delta_{0}^{2}
$$

for some value of $\alpha$. Although the constancy of $\delta^{2}$ makes this actually obvious, ${ }^{308}$ it is also possible to observe experimentally that $\alpha=-1$, and thus that

$$
\delta_{0}^{1} \approx \frac{\Delta_{0}^{1}}{3}-\delta_{0}^{2}
$$

Again, in the simplified example given above, where the first differences are 1227,1221 , and 1215 , and where the second differences are all about -6 ,

[^40]it should not be difficult to notice that the first first difference is equal to the average first difference minus the second difference, $\frac{37865-34202}{3}+6=1227$, or that the second (middle) first difference is also the mean first difference.

If these two observations are made, namely 1) the link between the second differences and the sines, and 2) the dependency of the subtabulated first differences on the subtabulated second differences, then it is possible to derive the values $v_{1 / 3}$ and $v_{2 / 3}$.

### 5.2.2 An example

Let me show how to put this in practice on a small example. Let's for instance interpolate the sines between $39^{\circ}$ and $39^{\circ} 45^{\prime}$. I will assume that all of Regiomontanus's values at $45^{\prime}$ intervals were exact, which, as mentioned above, is true except in a few instances. ${ }^{309}$ So, Regiomontanus must have had

$$
\begin{aligned}
& \operatorname{Sin} 39^{\circ}=377592235 \\
& \operatorname{Sin} 39^{\circ} 45^{\prime}=383663401
\end{aligned}
$$

Using the approximation $v_{1 / 3} \approx 379615957$ (the exact value is 379623197 ) we obtain

$$
\begin{aligned}
\delta_{0}^{2} & \approx-\frac{379615957}{52525}=-7227 \\
\Delta_{0}^{1} & =383663401-377592235=6071166 \\
\delta_{0}^{1} & \approx \frac{6071166}{3}+7227=2030949 \\
\delta_{1}^{1} & \approx \delta_{0}^{1}+\delta_{0}^{2}=2023722 \\
v_{1 / 3} & =\operatorname{Sin} 39^{\circ}+2030949=379623184 \\
v_{2 / 3} & =v_{1 / 3}+2023722=381646906
\end{aligned}
$$

and in fact Regiomontanus's table with $R=6 \cdot 10^{6}$ does have the values 3796232 and 3816469 which would have been obtained from the above computation.

This procedure does unfortunately not work on all $45^{\prime}$ intervals, and Regiomontanus's pivots sometimes differ from those obtained with this procedure, although the difference never exceeds one unit of the last place. This does not prove that Regiomontanus did not use such a procedure, but

[^41]it may be that some computations were lacking uniformity, and also that some errors were introduced in the computations. I also believe that the two guard digits, viz. those added when computing with $R=6 \cdot 10^{8}$, were used throughout the interpolation, and not merely for the pivotal values.

The same procedure used to obtain the sines at $15^{\prime}$ intervals can be used to obtain the sines at $5^{\prime}$ intervals. The only difference is that $\delta_{0}^{2}$ involves a new constant, which may have been guessed or computed by Regiomontanus, namely

$$
\delta_{0}^{2} \approx-\frac{v_{1 / 3}}{52525 \times 9}
$$

If for instance we want to compute $\operatorname{Sin} 39^{\circ} 5^{\prime}$, we find

$$
\begin{aligned}
\delta_{0}^{2} & \approx-\frac{378269218}{52525 \times 9}=-800 \\
\delta_{0}^{1} & \approx \frac{2030949}{3}+800=677783 \\
\delta_{1}^{1} & \approx 676983 \\
v_{1 / 3} & \approx 378270018 \\
v_{2 / 3} & \approx 378947001
\end{aligned}
$$

and these two values $v_{1 / 3}$ and $v_{2 / 3}$, when rounded to $R=6 \cdot 10^{6}$, are exactly the values given by Regiomontanus for the sines of $39^{\circ} 5^{\prime}$ and $39^{\circ} 10^{\prime}$. But again, I must stress that although this procedure works on this example, it does (slightly) fail to give Regiomontanus's values on others.

Anyway, if Regiomontanus proceeded along these lines, he now has obtained the sines for all multiples of $5^{\prime}$, using relatively simple techniques. In fact, the computations involved here (except those for the pivots) are more a matter of being clever than of being hard working.

What now remains is to divide the $5^{\prime}$ intervals in five parts. This is what Briggs called a quinquisection.

The same procedure could be applied here as for the trisection, but we would have

$$
\delta_{0}^{2} \approx-\frac{v_{1 / 5}}{52525 \times 9 \times 25}
$$

and ${ }^{310}$

$$
\delta_{0}^{1} \approx \frac{\Delta_{0}^{1}}{5}-2 \delta_{0}^{2}
$$

[^42]When applying this procedure (which is left as an exercise) to the interval from $39^{\circ}$ to $39^{\circ} 5^{\prime}$, one obtains

$$
\begin{aligned}
& \operatorname{Sin} 39^{\circ} 1^{\prime}=377727856 \\
& \operatorname{Sin} 39^{\circ} 2^{\prime}=377863445 \\
& \operatorname{Sin} 39^{\circ} 3^{\prime}=377999034 \\
& \operatorname{Sin} 39^{\circ} 4^{\prime}=378134623
\end{aligned}
$$

and Regiomontanus's table has $3777278,3778634,3779990$ and 3781345 , that is two values differ by one unit of the last place.

This suggests that Regiomontanus may perhaps not have used such an interpolation. If one performs a mere linear interpolation, with $\delta^{1}=135557$, we end up with the values $377727792,377863349,377998906$ and 378134463 , also with two differing values.

But if instead we interpolate with only one guard digit, that is between 37759224 and 37827002 and with $\delta^{1}=13556$, we end up with the values 37772780, 37786336, 37799892 and 37813448 , where only one value differs from that of Regiomontanus.

And if the guard digits are entirely discarded, we have an interpolation between 3775922 and $3782700, \delta^{1}=1356$, and we end up with the values $3777278,3778634,3779990$ and 3781346 , and again only one value differs from that of Regiomontanus.

Finally, if we interpolate with only one guard digit, but with $\delta^{1}=13557$, then we end up with exactly Regiomontanus's values. This does not prove that Regiomontanus did such an interpolation in every case, but it does at least make it plausible that he proceeded that way in some cases.

### 5.3 Conclusion

Looking at Regiomontanus's tables, it is pretty clear that he had the means to compute the $45^{\prime}$ pivots correctly. The $15^{\prime}$ and $5^{\prime}$ pivots are relatively accurate, but less than the $45^{\prime}$ pivots. In the previous section, I have given a procedure which may be close to the one used by Regiomontanus to find his pivots.

For the $15^{\prime}$ pivots, we have seen earlier that Regiomontanus's sexagesimal table has 70 pivot errors. Now, if we use my algorithm using finite differences, we end up with 42 errors on the all the $15^{\prime}$ pivots. However, if we compare my pivot values with those of Regiomontanus, there appears to be about 85 differences. Regiomontanus's values do not perfectly agree
with those of my algorithm for the first trisection, although the differences do not exceed one unit of the last place. This agreement can not be significantly improved even with a different constant $C_{15}$. It is still possible that Regiomontanus made use of an algorithm close to the one I sketched, but perhaps he did not always use two guard digits, in addition of having made a few computation errors here and there.

I also believe that the last step was a linear interpolation, but that glitches came into play and that the computations were not done totally uniformly and rigorously.

To sum up, and in the absence of other convincing theories, I believe that it is plausible that Regiomontanus applied two trisections, computed the first subtabulated first and second differences in each range, derived the missing values, and interpolated linearly in the $5^{\prime}$ intervals, perhaps using only one guard digit, and eventually rounding all values to $R=6 \cdot 10^{6}$. The same procedure could have been applied with the decimal table.

I believe that Regiomontanus's tables contain the germs of several innovations, and that it was the quality of workmanship underlying these tables which is the true reason why they endured so long. They did contain errors and typos, but they provided a solid foundation for others to build upon, and only Bürgi, Briggs and a few others were able to develop similar skills to renew the computation of tables.

## 6 After Regiomontanus

Most of the trigonometric tables printed in the 16th century actually use values or computations inherited from Regiomontanus's tables ${ }^{311}$ (see figures 13 and 14). Rheticus (1514-1574) was the only one to compute really new values which were eventually published in 1596 by Otho ${ }^{312}$ and in 1613 by Pitiscus. ${ }^{313}$ Bürgi also computed sines anew, but his table was not published and was not used by others.

Among all these tables, Glowatzki and Göttsche distinguished those which retain the radius $R=10^{7}$ and those for which $R=10^{5}{ }^{314}$ But we should also consider separately the few sexagesimal tables based on Regiomontanus's tables, namely those of Engel, Fine, Schreckenfuchs and Bressieu.

The tables with radius $10^{7}$ include those of Rheticus (1542 and 1551), ${ }^{315}$ Reinhold (1554), ${ }^{316}$ Eisenmenger (1562), ${ }^{317}$ Viète (1579), ${ }^{318}$ Fincke (1583), ${ }^{319}$ Clavius (1586), ${ }^{320}$ Lansberge (1591), ${ }^{321}$ Magini (1592), ${ }^{322}$ Blundeville (1594), ${ }^{323}$ and Ceulen (1596). ${ }^{324}$ Glowatzki and Göttsche also considered the 17th century tables of Sems/Dou (1600, 1612, 1616 and 1620), Stevin (1608 and 1628), Roomen (1609), Crüger (1612), Napier (1614, 1616 and 1620), Blebel (1616 and 1629), Ursinus (1618), Alsted (1620, 1630 and 1649), Muller (1621) and Tonski (1640 and 1645), ${ }^{325}$ which all go back to Regiomontanus, but which are outside the limited scope of this survey. The fact that all these tables use Regiomontanus's values is asserted by several checks, including on the typographical and last digit errors stemming from Regiomontanus, but also on mere layout considerations. Many table makers did actually not

[^43]

Figure 13: The interrelationships between the main 15th and 16th century fundamental trigonometric tables. Corner squares ( $\square$ ) indicate no longer extant tables, unfilled corner circles $(\bigcirc)$ indicate new computations, and filled corner circles ( ) indicate computations based on earlier tables. Tables marked " $(\mathrm{m})$ " in the lower part are manuscript tables. See figure 14 for details on the links.

```
means entirely copied from, or printed at a later date.
inspired or influenced by.
probable link.
Regiomontanus (tan, 1467) obtained by computation from the 1462 table.
Engel (1490) obtained from the }1462\mathrm{ table by truncation (and not rounding) (no computation).
Gaurico (1524): table obtained from Engel by multiplication by 10/6.
Fine (1530): computed from Regiomontanus's table with R = 6000000.
Copernicus (c1530): computed from Regiomontanus's table with R=10
Apian (1533): obtained from Regiomontanus (10}\mp@subsup{)}{}{7})\mathrm{ by mere truncation, without rounding.
Copernicus (1543): probably obtained from a combination of Regiomontanus's tables.
Rheticus (1551) (tan, sec): computed from Regiomontanus (or Rheticus 1542).
Reinhold (1554): tangents computed from the sines.
Maurolico (1558): computed from Regiomontanus.
Rheticus (c1560): computed from Regiomontanus.
Schreckenfuchs (1569): one table computed from the 1541/1561 table.
Viète (1579): computed from Regiomontanus.
Bressieu (1581) (tan, sec): from Regiomontanus and interpolation.
Fincke (1583): sines based on Reinhold's sines, but with slight adaptations.
Fincke (1583): tangents based on Reinhold, but with corrections.
Fincke (1583): secants computed from Fincke's tangents.
Rheticus (1596): 10 table was probably excerpted from an earlier table from c1560.
Rheticus (1596): }1\mp@subsup{0}{}{10}\mathrm{ table obtained from a 10 }\mp@subsup{}{}{15}\mathrm{ table.
```

Figure 14: The interrelationships between the main 15th and 16th century fundamental trigonometric tables (cont'd, see figure 13). The number of places of sexagesimal tables is shown as $60 ; 60^{n}$, the first 60 being the value of $R$, and $n$ being the number of additional sexagesimal places. Note that in Rheticus's 1551 table, the sines were copied from Regiomontanus (or Rheticus 1542); in Reinhold's table (1554), the sines were copied from Regiomontanus (or Rheticus 1551); in Schreckenfuchs's table (1569), one table was copied from Engel (in an edition of the Tabulæ directionum profectionumque) and another was copied from the 1490 table of tangents (or another edition of the Tabulæ directionum profectionumque); in Bressieu (1581), the sines were copied from Fine (1530 or 1550).
bother checking the values using the differences, and Clavius in 1586 was apparently the first to get rid of the typographical errors of earlier tables. Note however that Glowatzki and Göttsche do not always give the direct predecessor of a table, and do seldom consider the layouts of the tables as an indication for their source.

Among the tables with radius $10^{5}$ are the tables of Bassantin (1557), ${ }^{326}$ Witekind (1576) ${ }^{327}$ Peucer (1579) ${ }^{328}$, Giuntini (1581) ${ }^{329}$, Padovani (1582) ${ }^{330}$ and Fale (1593) ${ }^{331}$ which were taken directly or indirectly from Apian (1533), ${ }^{332}$ which itself goes back to Regiomontanus's table of 1468, merely by dropping two digits and no rounding. ${ }^{333}$

Engel's table with $R=60000$ (figure 16) is derived from Regiomontanus's large sexagesimal table and was used by Schreckenfuchs in 1569 (see § 6.14).

Immediately following the table of sines for radius 60000 published in $1524,{ }^{334}$ there is an additional table of sines with radius 100000 and for every $10^{\prime}$ (figure 17). As mentioned by Delambre, ${ }^{335}$ this table was added by Gaurico (see § 6.2).

Fine's tables from 1530 and 1550 are not mentioned by Glowatzki and Göttsche. ${ }^{336}$ Fine's table published in 1530 and reprinted in 1550 gives the sines for a radius $R=60$, at intervals of $1^{\prime}$ and to two sexagesimal places.

Fine's tables are the only fully sexagesimal tables based on Regiomontanus's tables, apart from those of Schreckenfuchs published in 1569 and of Bressieu published in 1581.

Mention should also be made of Bürgi's sexagesimal sine table from c1587, which seems to be a totally independent and very accurate recomputation of sines, paralleling to some extent Rheticus's efforts that led to the Opus palatinum (1596) and the Thesaurus mathematicus (1613).

To sum up, the main new computations based on Regiomontanus's values are the following, which are detailed in the subsequent sections:
${ }^{326}$ See [Bassantin (1557)] and [Glowatzki and Göttsche (1990), p. 176].
${ }^{327}$ See [Witekind (1576)] and [Glowatzki and Göttsche (1990), p. 176]
${ }^{328}$ See [Peucer (1579)] and [Glowatzki and Göttsche (1990), p. 177]
${ }^{329}$ [Giuntini (1581)]
${ }^{330}$ [Padovani (1582)]
${ }^{331}$ See [Fale (1593)] and [Glowatzki and Göttsche (1990), p. 177].
${ }^{332}$ [Apian (1533)]
${ }^{333}$ [Glowatzki and Göttsche (1990), p. 169]
${ }^{334}$ [Regiomontanus (1524)]
${ }^{335}$ [Delambre (1819), p. 292]
${ }^{336}$ [Glowatzki and Göttsche (1990)]

- in 1551, Rheticus published his computations of tangents and secants at intervals of $10 ;$; 337
- in 1554, Reinhold published his computations of tangents at intervals of $1^{\prime}$ (and $10^{\prime \prime}$ for the last degree); ${ }^{338}$
- in 1579, Viète published his computations of tangents and secants at intervals of 1 '; ${ }^{\prime} 339$
- in 1583, Fincke published his computations of secants at intervals of $1^{1} .340$

In the following sections, I go into more detail for each of these tables copied from Regiomontanus's tables, or based on them. This list tries to be as complete as possible, but it is possible that some lesser known work containing a sine table or a more complete canon still escaped my attention.

### 6.1 Engel (1490)

Johannes Engel (or Johannes Angelus) (1453-1512) was an astronomer and astrologer from Aichach, near Augsburg. He published many almanachs and astronomical tables. ${ }^{341}$

The 1490 edition of Regiomontanus's Tabulx directionum profectionum$q u e^{342}$ contains corrections by Johannes Engel and in particular a 30 pages long sine table with $R=60000$ giving the sines for every minute (figure 16). ${ }^{343}$

Folkerts ${ }^{344}$ assumed that this table had been computed before 1463-1464, but in fact the table was certainly added by Johannes Engel who obtained it by truncating (not rounding) Regiomontanus's table for $R=6 \cdot 10^{6}{ }^{345}$ Here is a sample of Regiomontanus's values (R) and Engel's values (E):

[^44]| Angle | R | E |
| :---: | :---: | :---: |
| $60^{\circ} 0^{\prime}$ | 5196152 | 51961 |
| $60^{\circ} 1^{\prime}$ | 5197024 | 51970 |
| $60^{\circ} 2^{\prime}$ | 5197896 | 51978 |
| $60^{\circ} 3^{\prime}$ | 5198768 | 51987 |
| $60^{\circ} 4^{\prime}$ | 5199639 | 51996 |
| $60^{\circ} 5^{\prime}$ | 5200510 | 52005 |
| $60^{\circ}{ }^{\prime}{ }^{\prime}$ | 5201380 | 52013 |
| $60^{\circ} 7^{\prime}$ | 5202350 | 52022 |
| $60^{\circ} 8^{\prime}$ | 5203119 | 52031 |

In this sample, we can also see that Engel introduced an additional error for $60^{\circ} 7^{\prime}$.

Moreover, Regiomontanus's table contains a column of differences, whose values can be interpreted as the sixths of the differences of the sines. For instance, $(5197024-5196152) / 6=145.333 \ldots$ and that column starts with 145 . Engel's table contains exactly the same value 145 , although it is basically meaningless in that reduced table.

By analyzing the correspondence of Regiomontanus with Bianchini and others, Glowatzki and Göttsche have also shown that Regiomontanus actually did not himself use the table $R=60000$ printed in 1490, and whenever he used this radius, he drew the values by dropping two digits from his large table and rounding the value. ${ }^{346}$ This is an additional proof for the fact that the table for $R=60000$ actually did not exist before it was prepared for printing in 1490. Instead, Glowatzki and Göttsche ${ }^{347}$ assume that Regiomontanus had made a table with $R=60000$ for himself, but different from the printed one. Such a table may have been held at the Seitenstetten Abbey until 1924, but it was then sold and had not been located by the authors. If this table surfaces again, one should check whether its values are truncated or rounded.

Engel's table (figure 16) is found again in the 1504 edition, ${ }^{348}$ where the title of the work explicitely mentions this sine table. It is also found in later editions of the Tabulx directionum profectionumque, where it is often attributed to Regiomontanus. My modern reconstruction ${ }^{349}$ is based on the 1504 edition which has less idiosyncrasies than the 1490 version.

[^45]Engel's table was used by Gaurico in 1524 (see below, § 6.2), in 1569 by Schreckenfuchs ( $\$ 6.14$ ) and reprinted in 1588 by Gallucci ( $£ 6.24$ ).

### 6.2 Gaurico (1524)

Luca Gaurico (in Latin, Lucas Gauricus, in French Luc Gauric) (1475-1558) was an Italian astrologer, astronomer, and mathematician. ${ }^{350}$

In 1524, as an appendix to Regiomontanus's Tabulæ directionum profectionumque, ${ }^{351}$ Gaurico published a table of sines with $R=100000$ and at intervals of $10^{\prime}$ (figure 17). This table was reprinted in $1557,{ }^{352}$ together with Regiomontanus's table of tangents but the latter only up to $50^{\circ}$.

It is tempting to consider that Gaurico took his sines from a manuscript of Regiomontanus's table for $R=10^{7}$ and truncated or rounded the values (this was also suggested by Glowatzki and Göttsche ${ }^{353}$ ), but this is actually not the case. Gaurico's values differ both from the truncated values and from the rounded ones of the $10^{7}$ table.

In fact, it seems that Gaurico took the values in Engel's table for $R=$ 60000 , and merely multiplied them by $10 / 6$, although this procedure will in a few cases give values that differ from those in Gaurico's table. ${ }^{354}$

Gaurico's table was also certainly not the basis of Copernicus's table of sines (or semi-chords) published in 1543, although it uses the same radius and interval.

### 6.3 Copernicus (c1530?)

The earliest known decimal table of secants is a handwritten table by Nicolaus Copernicus (1473-1543), ${ }^{355}$ included in his copy of Regiomontanus's Tabulx directionum profectionumque published in $1490 .{ }^{356}$ There Copernicus

[^46]gave the values of the secant for each degree ${ }^{357}$ and for $R=10000$, overlayed to Regiomontanus's table of tangents. A reproduction of Copernicus's table was given by Glowatzki and Göttsche. ${ }^{358}$

Curtze considered that Copernicus had computed the secants from the cosines ${ }^{359}$ but Birkenmajer ${ }^{360}$ thought that the secants were computed using the formula $\operatorname{Sec} x=\sqrt{\operatorname{Tan}^{x}+R^{2}}$. Rosińska also observed that Copernicus's table of secants is not copied from Bianchini's table of cosecants. ${ }^{361}$ Rosińska concluded therefore that Copernicus used neither Bianchini's nor Regiomontanus's tables for his table of secants. ${ }^{362}$

But still, Glowatzki and Göttsche observed that since Copernicus's values are very accurate, he must have used a manuscript of Regiomontanus's tables with $R=6 \cdot 10^{6}$ or $R=10^{7}$, later published in 1541 . ${ }^{363}$

Now, according to my experiments, using either of Regiomontanus's tables, computing the exact secants, by mere division, as Curtze suggested, and rounding, will give almost always Copernicus's values, except for $88^{\circ}$ and $89^{\circ}$ where Copernicus probably tried to obtain more accurate values. The discrepancy of these two values is not, in my opinion, a sufficient reason to look for a different source or a different computation for Copernicus's entire table of secants.

### 6.4 Fine (1530)

Oronce Fine (1494-1555) was a French mathematician and cartographer. After having learned his first lessons of mathematics from his father in Briançon, he matriculated at the University of Paris and from about 1531 until his death he occupied the chair of mathematics of the Collège Royal in Paris. ${ }^{364}$

[^47]In 1530, Fine published his De geometria ${ }^{365}$ which contains a sexagesimal table of sines. Fine's table gives the sines for a radius $R=60$, at intervals of $1^{\prime}$ and to two sexagesimal places (figure 18).

This table is not based on Engel's 1490 table $^{366}$ as one might think. Instead it must be based directly on a manuscript of Regiomontanus's large sexagesimal table, without truncating.

In 1542, Fine published his De sinibus libri II which was the first treatise solely on trigonometry to be printed in France. ${ }^{367}$ This work is an appendix of Fine's De mundi sphaera. ${ }^{368}$ Ross is very critical of this work and considers that it was unoriginal and out of date, ${ }^{369}$ because it does not contain any contributions to trigonometric mathematics, and because it lags behind the developments soon introduced by Rheticus for the unification of sines, shadows, etc., in a same framework. Fine also appears to be unaware of Regiomontanus's De triangulis omnimodis ${ }^{370}$ published in 1533 and which laid the foundations of modern trigonometry. But on the other hand, Fine's purpose with this book was pedagogical and he succeeded in contributing to the revival of mathematics in Paris.

The 1542 De sinibus libri II reprinted the table of sines published in 1530, with only minor variations. The layout is the same, although the table was obviously reset, as can be observed on the last lines of each page.

The second edition of De sinibus libri II was published in $1550,{ }^{371}$ but the sine table now uses a different layout (figure 30). In the two editions of this work (1542 and 1550), Fine's introduction gives the sines up to $90^{\circ}$ at intervals of $3^{\circ} 45^{\prime}$. In a second table, he gives the sines up to $7^{\circ} 30^{\prime}$ at intervals of $15^{\prime}$. These two tables were not given in the 1530 De geometria ${ }^{372}$ and were presumably not used for the computation of the table published in 1530. None of these tables are mentioned by Glowatzki and Göttsche. ${ }^{373}$

Fine's table is the only sexagesimal table based on Regiomontanus's tables, apart from those of Schreckenfuchs ${ }^{374}$ published in 1569 and of
${ }^{365}$ [Fine (1530)]
${ }^{366}$ [Regiomontanus (1490)]
${ }^{367}$ [Ross (1975), p. 379]
${ }^{368}$ [Fine (1542)]
${ }^{369}$ [Ross (1975), pp. 385-386]
${ }^{370}$ [Regiomontanus (1533)]
${ }^{371}$ [Fine (1550)]
${ }^{372}$ [Fine (1530)]
${ }^{373}$ [Glowatzki and Göttsche (1990)]
${ }^{374}$ [Schreckenfuchs (1569)]

Bressieu published in 1581. ${ }^{375}$ Bressieu has actually copied Fine's table, from either of the three editions I have mentioned.

Certainly in order to help for the work with sexagesimal numbers, Fine had also published a sexagesimal multiplication table, his tabula proportionalis. ${ }^{376}$

I am giving separately a modern reconstruction of Fine's 1530 and 1550 tables. ${ }^{377}$

### 6.5 Apian (1533)

Peter Apian (1495-1552), also known as Petrus Apianus, was actually born Peter Bennewitz, or Peter Bienewitz, in Leisnig, Germany. ${ }^{378} \mathrm{He}$ was active in astronomy and geography and was a popularizer of astronomical and geographical instrumentation. ${ }^{379}$ Apian studied at the University of Leipzig from 1516 to 1519 and then for two years in Vienna. His first major work was his Cosmographia (1524), which was later revised by Gemma Frisius (1508-1555), Apian's student.

Apian's second major work was his Astronomicum Caesareum (1540) which displayed an elaborate typography and the use of sophisticated volvelles.

From 1526 until his death he occupied the chair of mathematics and astronomy at the University of Ingoldstadt.

Apians's mathematical work is linked to Regiomontanus's writings. ${ }^{380}$ He published his work on sines in 1533.

In his Introductio geographica published in 1533, ${ }^{381}$ Apian provides a table of sines with $R=10^{5}$ and for every minute of the quadrant (figure 19). The same table was reprinted in 1534 in Apian's Instromentom primi mobilis ${ }^{382}$ (figure 20) and in 1541 in his Instrumentum sinuum ${ }^{383}$ (figure 21).

[^48]This table appears to have been obtained by merely dropping the last two digits of Regiomontanus's table for $R=10^{7}$, without any rounding. ${ }^{384}$ Regiomontanus's table having only been printed in 1541, Apian must have had access to one of the manuscripts of the 1468 table. Moreover, as observed by Glowatzki and Göttsche, the manuscript table used by Apian is in fact the same as the one used in the 1541 printed edition of Regiomontanus's table. ${ }^{385}$

And, as remarked by Kish, ${ }^{386}$ Apian's sine table is the first printed table giving sines every minute and divided decimally, Regiomontanus's table having only been printed in 1541.

I am giving separately a modern reconstruction of Apian's 1533 table. ${ }^{387}$
Apian's tables seem to have been copied by directly or indirectly by Bassantin in $1557,{ }^{388}$ by Witekind in $1576,{ }^{389}$ by Peucer in $1579{ }^{390}$, by Giuntini in $1581,{ }^{391}$, by Padovani in 1582, ${ }^{392}$ and indirectly by Fale ${ }^{393}$ in 1593.

### 6.6 Rheticus (1542)

Georg Joachim Rheticus (1514-1574) was born in Feldkirch (Austria). ${ }^{394}$ In 1539, while he was professor of mathematics in Wittenberg, he set out to meet Copernicus in Frombork (Poland). He stayed with Copernicus for two years, published a first account of Copernicus's theory as Narratio Prima in Gdansk (1540), and was instrumental in the publication of Copernicus's De revolutionibus orbium coelestium ${ }^{395}$ in 1543 . When Rheticus returned from

[^49]his visit, he also made it possible for Erasmus Reinhold to become closely acquainted with Copernicus's theory, leading to the publication of the Prutenic tables in 1551.

But in 1542, before the publication of Copernicus's De revolutionibus, Rheticus published its trigonometrical chapters under the title De lateribus et angulis triangulorum. ${ }^{396}$ This work contains a table of sines at intervals of $1^{\prime}$ and for a radius of $10^{7}$ (figure 25). The sines were in fact not called sines, but half-chords. And the table is actually by Rheticus and not by Copernicus. ${ }^{397}$ More precisely, Rheticus took the sines from Regiomontanus's table ${ }^{398}$ published in 1541 (or from a common manuscript source). This is in particular confirmed by printing errors found in both editions. ${ }^{399}$ Some of the typos that remain in Rheticus's table are in fact so conspicuous that they should have been corrected by Rheticus. Rheticus however did not carry over Regiomontanus's differences, but introduced the actual differences.

In her article on Copernicus's tables, Rosińska ${ }^{400}$ hypothesizes that Copernicus had first planned to append to his work a table of sines with $R=10^{6}$ but that this table was eventually replaced by Rheticus's one with $R=10^{7}$.

In the past von Braunmühl, ${ }^{401}$ Cantor, ${ }^{402}$ Busard, ${ }^{403}$ Rosen ${ }^{404}$ and Folkerts ${ }^{405}$ were of the opinion that Rheticus was the real author (computer) of the sine table. And Zinner thought that the author of the table was Copernicus himself and that he may have been inspired to construct a table with $R=10^{7}$ by a glimpse of Regiomontanus's table. ${ }^{406}$

Rheticus's table seems to be the first table of sines where a value can easily and explicitely be read in two different ways. This novelty was

[^50]observed by Stamm, ${ }^{407}$ Rosen ${ }^{408}$ and more recently by Husson. ${ }^{409}$ For each value, there is both a reading using the first line and the first column (giving the sines), and a second reading using the last line and the last column (giving the cosines). Similar features are found again in the tables of Rheticus (1551), Reinhold (1554), Viète (1579), Clavius (1583), Magini (1592) and Rheticus/Otho (1596). Of course, earlier tables, including those of Regiomontanus, can also be read that way, but not explicitely, and it is necessary to perform a (simple) computation to find the cosine of an angle, for instance.

I am giving separately a modern reconstruction of Rheticus's table. ${ }^{410}$

### 6.7 Copernicus (1543)

As we have seen earlier, Copernicus (1473-1543) ${ }^{411}$ had computed a small table of secants, perhaps around 1530, and in 1542 the trigonometric chapters of De revolutionibus orbium coelestium were published separately by Rheticus, together with a sine table. But in Copernicus's famous De revolutionibus orbium coelestium published in 1543 shortly before his death, ${ }^{412}$ Copernicus included another table of sines, with an interval of $10^{\prime}$ and a radius $R=10^{5}$ (figure 26). Like in the excerpt published in 1542, the sines were actually not called sines, but half-chords.

Copernicus's table shows a few deviations from the values obtained from Regiomontanus's table with $R=10^{7}$ when the values are rounded to $R=10^{5}$. For instance there are three errors in the first 36 values (from $0^{\circ}$ to $6^{\circ}$ ) and Copernicus gives $\operatorname{Sin} 0^{\circ} 40^{\prime}=1163$ instead of $1164, \operatorname{Sin} 1^{\circ} 30^{\prime}=$ 2617 instead of 2618 , and $\operatorname{Sin} 4^{\circ}=6975$ instead of 6976 . However, there are more deviations when Copernicus's table is compared to Gaurico's table published in 1524 (also with $R=10^{5}$ ), with eight errors in the same interval. The most likely basic explanation would then be that Copernicus used Regiomontanus's table for $R=10^{7}$ and made a few rounding errors.

[^51]Glowatzki and Göttsche were of the same opinion and concluded that Copernicus must have used a manuscript version of Regiomontanus's table of sines with $R=10^{7},{ }^{413}$ as he certainly did for his table of secants. ${ }^{414}$

Copernicus probably did not take the sines directly from the table published in 1542, although it is almost identical to that of Regiomontanus, because Copernicus's manuscript must have been ready long before its publication. In any case, if Copernicus took his values from Regiomontanus, he also made some corrections to that table, as Regiomontanus's typos for the sines of $36^{\circ} 10^{\prime}$ and $51^{\circ} 20^{\prime}$, reprinted in 1542 , have been corrected in the De revolutionibus orbium coelestium. ${ }^{415}$

However, I think that it is possible to get a somewhat better understanding of the elaboration of Copernicus's table. I have said above that among the first 36 values of Copernicus's table, there are three obvious errors when comparing them to the rounded values from Regiomontanus's table with $R=10^{7}$. For instance, for $0^{\circ} 40^{\prime}$, Regiomontanus's table gives 116353, and Copernicus has 1163, which looks like a truncation, but for almost every other angle Copernicus's sine is the rounded and not truncated value obtained from Regiomontanus's table.

Now, if we start with Regiomontanus's sexagesimal table, that is, the table with $R=6 \cdot 10^{6}$, the decimal values can be obtained by dividing Regiomontanus's values by 6. Considering only the first 36 values in Copernicus's table (from $0^{\circ}$ to $6^{\circ}$ ), it appears that until $3^{\circ} 50^{\prime}$, one obtains Copernicus's values by dropping one digit of Regiomontanus's table and rounding, then dividing by 6 , then rounding. ${ }^{416}$ For instance, for $0^{\circ} 50^{\prime}$, one obtains $8726 / 6=1454.333 \ldots$ which is rounded to 1454 . In case the result is a half integer, the rounding occurs to the integer below, except if the first rounding was by default, although there may be exceptions (such as $2^{\circ} 40^{\prime}$ ) taking account of how the first rounding was performed.

This procedure fails after $3^{\circ} 50^{\prime}$ and it seems that a different operation was involved. In fact, between $4^{\circ}$ and $5^{\circ} 20^{\prime}$, there was apparently a truncation of the last two digits of Regiomontanus's table, the resulting value was multiplied by 10 , divided by 6 and rounded. Between $5^{\circ} 40^{\prime}$ and $6^{\circ}$, the initial procedure was again applied. These two procedures give a slightly better outcome than merely using Regiomontanus's table with $R=10^{7}$.

[^52]If we look at the last page of Copernicus's table, also containing 36 values, a comparison with Regiomontanus's table with $R=10^{7}$ reveals five rounding errors $\left(84^{\circ} 30^{\prime}, 85^{\circ}, 86^{\circ} 40^{\prime}, 87^{\circ} 40^{\prime}\right.$ and $89^{\circ} 50^{\prime}$ ). But if we start with the great sexagesimal table as above, there are in fact even more errors, 16 altogether. For instance, for $86^{\circ}$ one obtains 99757, and not Copernicus's 99756.

I have of course only sampled the first and last 36 values of Copernicus's table, and this should be further investigated. It suggests however that different computations may have been involved in the making of Copernicus's sine table, and probably that some parts of Copernicus's table are based on Regiomontanus's sexagesimal table, whereas others are based on the table with $R=10^{7}$, in addition of involving different rounding schemes from the same source. It is also possible that some values were based on other tables. But among the 16 values that are incorrectly rounded on the last page of Copernicus's table when starting with Regiomontanus's great sexagesimal table, only 10 of Copernicus's values are identical with those published by Apian in $1534 .{ }^{417}$ It is therefore not possible to conclude that Copernicus used Apian's table. Perhaps for some comparisons, but not for all values.

Given this somewhat confused situation, it is understandable that Copernicus's table led to other opinions or conclusions. For instance, Stamm ${ }^{418}$ wrote that Copernicus probably compared his values to those published by Apian in $1534,{ }^{419}$ but I have just shown that this is not conclusive. Folkerts ${ }^{420}$ thought that Copernicus had computed the table himself, since he could not have been able to use Regiomontanus's table for $R=10^{7}$ which was only published in 1541. Looking for Copernicus's source, Swerdlow and Neugebauer ${ }^{421}$ excluded most sources, including Regiomontanus's tables printed in 1541, but they did not conclude further. And according to Rosińska, ${ }^{422}$ Copernicus did not use Regiomontanus's table for his table of sines, although she did not provide another theory for the origin or calculation of the table.

[^53]
### 6.8 Gemma Frisius (1545)

Gemma Frisius (1508-1555) (Jemme Reinerszoon, or Rainer Gemma) was a Dutch physician, mathematician, cartographer, philosopher, and instrument maker. He was born in Dokkum in the Netherlands. ${ }^{423}$

Gemma Frisius first practiced medicine in Louvain, but his real interests seem to have been geography and mathematics. In 1529 he revised Peter Apian's Cosmographia. He also designed globes and astronomical instruments. He died in Louvain.

In 1545, he published his De radio astronomico et geometrico ${ }^{424}$ in which he included a table of cotangents (figure 27) which was copied from Regiomontanus. ${ }^{425}$

This work also contained a table of arctangents (figures 28 and 29) obviously copied from that of Peuerbach published in $1516{ }^{426}$

### 6.9 Rheticus (1551)

In 1551, Rheticus (1514-1574) ${ }^{427}$ published his Canon doctrinæ triangulorum. ${ }^{428}$ There, he gave the sines, cosines, tangents, cotangents, secants and cosecants at intervals of $10^{\prime}$ and for a radius $R=10^{7}$. Rheticus's table was in fact the first table to give all six possible ratios in a right triangle (figure 31). ${ }^{429}$

The sines in Rheticus's table were copied from Regiomontanus's table for $R=10^{7} .{ }^{430}$ Most of the values were not changed, but some of the typos were corrected, for instance the cosine of $38^{\circ} 40^{\prime}$ whose value was still incorrect (as Sin $51^{\circ} 20^{\prime}$ ) in Rheticus's 1542 table.

Rheticus made new computations for the tangents and the secants using

[^54]these sines. According to Glowatzki and Göttsche, ${ }^{431}$ Rheticus merely computed the ratios for the tangents, but things are actually a bit more complicated.

First, it appears that the secants and cosecants were computed by dividing 1 (or rather $10^{14}$ ) by the values of the cosines or sines, and truncating the results. This can readily be observed on the secants of $17^{\circ} 20^{\prime}, 29^{\circ}, 29^{\circ} 50^{\prime}$, $43^{\circ}$, etc., and practically every ratio whose decimal part is greater than 0.5. This is also true for the cosecants, an example being $35^{\circ} 30^{\prime}$.

But the tangents and cotangents are another story. I don't know exactly how Rheticus computed these values, but a close examination of Rheticus's values reveals that the tangents are more accurate than the cotangents and consequently one cannot have been computed from the other. They must have been computed differently. The tangents may have been computed by dividing the sines by the cosines, but this cannot have been the case for the cotangents. ${ }^{432}$

It appears that the values of the cotangents are close to those obtained when computing

$$
\operatorname{Cot} x=\sqrt{\operatorname{Csc}^{2} x-R^{2}}
$$

but they are not totally identical. The agreement is however much better than that obtained by merely dividing the values of the cosines by the sines of Regiomontanus, and it may even be a little better if $\operatorname{Csc}^{2} x$ is rounded to seven or eight significant digits. This hypothesis may need to be tested further, but it parallels a suggestion by van Brummelen and Byrne for the computation of secants by Maurolico, ${ }^{433}$ although I argue below that their suggestion is in fact not applicable to Maurolico's computations. However, I also suggest below that Fincke used a similar procedure to compute his secants in 1583.

In any case, Rheticus's work remains based on Regiomontanus's tables, and although he was the first to construct a table giving all six triangle ratios, he did not compute the cosecants and cotangents sufficiently accurately for small angles, and seems to have not yet understood that more accurate sines were needed. He had no problems giving cosecants and cotangents to 10 figures, when the sines were only given to 5 figures. This understanding of the need for more accurate sines only came later, and even the Opus

[^55]palatinum ${ }^{434}$ published in 1596 is still marred by this problem which will only fully be solved by Pitiscus in the early 17 th century. ${ }^{435}$

I am giving separately a modern reconstruction of Rheticus's table. ${ }^{436}$

### 6.10 Reinhold (1554)

Erasmus Reinhold (1511-1553) was a German astronomer and mathematician. He was born in Saalfeld, Germany. In 1536 he became professor of mathematics at the university of Wittenberg. ${ }^{437}$ In 1542, Reinhold published a commentary on Peuerbach's Theoricae Novae Planetarum. When Rheticus came back from his visit to Copernicus, Reinhold studied Copernicus's theory closely and after the publication of Copernicus's De revolutionibus orbium coelestium, Reinhold made detailed annotations of this work. ${ }^{438}$

Between 1544 and 1551, Reinhold worked on recasting Copernicus's theory in handier tables and in 1551 he finally published his Tabulæ prutenicæ coelestium motuиm (Prutenic tables).

In his Primus liber tabularum directionum published in 1554 after his death, ${ }^{439}$ Reinhold gave a table of sines (figure 34) and a table of tangents (figures 32 and 33), both with radius $R=10^{7}$ and at intervals of $1^{\prime}$. ${ }^{440}$

The sines were copied from Regiomontanus's sines, ${ }^{441}$ probably from the 1541 printing, but the tangents were recomputed at intervals of $1^{\prime}$, using these sines. ${ }^{442}$ Moreover, in the range from $89^{\circ}$ to $89^{\circ} 59^{\prime} 50^{\prime \prime}$, Reinhold gave the tangents at intervals of $10^{\prime \prime}$.

The tangents seem to have been computed in a non systematic way. For the angles which are found in Regiomontanus's table, Reinhold has apparently mostly taken the ratio of the sines given by Regiomontanus, but sometimes the result was truncated (for instance for Tan $1^{\circ}$ where Reinhold gives 174550 instead of 174551), and sometimes the sines were rounded to the tens (for instance for Tan $10^{\circ}$ where 173648/984808 was computed

[^56]instead of $1736482 / 9848078$ ). In the case of $\operatorname{Tan} 80^{\circ}$, there must have been a computation error, as Reinhold has actually computed 9848085/1736482 instead of $9848078 / 1736482$. This error is not present in Rheticus's table. ${ }^{443}$ My samples may or may not be representative of the entire table, and it would be useful to conduct a thorough analysis of Reinhold's table of tangents. ${ }^{444}$ It seems in particular that Reinhold did not copy Rheticus's values given in 1551 at intervals of $10^{\prime}$. Reinhold's error on Tan $80^{\circ}$, incidentally, is found again in Fincke's table, ${ }^{445}$ as well as in Clavius's table. ${ }^{446}$

In the last part of the table, Reinhold added the tangents at intervals of $10^{\prime \prime}$. The values themselves are not as accurate as one might wish, but this matters little here. What does interest us is to find out how Reinhold computed these values. These computations seem so far not to have been analyzed, not even by Glowatzki and Göttsche. ${ }^{447}$ At first, this part of the table suggests a new computation of the sines and cosines of $10^{\prime \prime}, 20^{\prime \prime}$, etc., up to $59^{\prime} 50^{\prime \prime}$, but this was most certainly not the case. It is in fact very easy to see what Reinhold has done, because the ratios behind each tangent value can be reconstructed. I am giving here only some samples:

| Angle | Fraction | Value |
| :---: | :---: | ---: |
| $89^{\circ} 0^{\prime} 10^{\prime \prime}$ | $\frac{9998370}{174038}$ | $57.44935014 \ldots$ |
| $89^{\circ} 0^{\prime} 30^{\prime \prime}$ | $\frac{999802}{173070}$ | $57.77143352 \ldots$ |
| $89^{\circ} 0^{\prime} 50^{\prime \prime}$ | $\frac{9998519}{172101}$ | $58.09680943 \ldots$ |
| $89^{\circ} 30^{\prime} 10^{\prime \prime}$ | $\frac{9999625}{86780}$ | $115.22960359 \ldots$ |
| $89^{\circ} 30^{\prime} 20^{\prime \prime}$ | $\frac{9996928}{86296}$ | $115.87591543 \ldots$ |
| $89^{\circ} 59^{\prime} 10^{\prime \prime}$ | $\frac{9999999}{2424}$ | $4125.41212871 \ldots$ |
| $89^{\circ} 59^{\prime} 20^{\prime \prime}$ | $\frac{9999999}{1939}$ | $5157.29706034 \ldots$ |
| $89^{\circ} 59^{\prime} 30^{\prime \prime}$ | $\frac{10000000}{1455}$ | $6872.85223367 \ldots$ |
| $89^{\circ} 59^{\prime} 40^{\prime \prime}$ | $\frac{10000000}{970}$ | $10309.27835051 \ldots$ |
| $89^{\circ} 59^{\prime} 50^{\prime \prime}$ | $\frac{10000000}{485}$ | $20618.55670103 \ldots$ |

[^57]The "Fraction" column gives the ratios actually used by Reinhold for the tangents, and the column on the right gives the values of these fractions. These can be compared with those in Reinhold's table.

It turns out that the values given by these fractions are almost exactly those of Reinhold, with the occasional rounding errors or typos. For instance, Reinhold table has $\operatorname{Tan} 89^{\circ} 0^{\prime} 10^{\prime \prime}=574493507$ which must be a typo for 574493501 . In most cases, the fractions seem to have been truncated (and for instance for $89^{\circ} 30^{\prime} 10^{\prime \prime}$ this resulted in the incorrect value), but in some cases they were rounded (for instance for $89^{\circ} 59^{\prime} 30^{\prime \prime}$ ).

We can see in this sample that Reinhold uses

$$
\begin{array}{llc}
\operatorname{Sin} 10^{\prime \prime}=485 & \operatorname{Sin} 20^{\prime \prime}=970 & \operatorname{Sin} 30^{\prime \prime}=1455 \\
\operatorname{Sin} 40^{\prime \prime}=1939 & \operatorname{Sin} 50^{\prime \prime}=2424 & \ldots
\end{array}
$$

and these values are obtained by linear interpolation of Regiomontanus's sines. There may again be some slight inaccuracies, and the value of Sin $59^{\prime} 50^{\prime \prime}$ would for instance have been better at 174039 than 174038.

The numerators used by Reinhold were also obtained by interpolation from Regiomontanus's values. For instance, for $89^{\circ} 30^{\prime \prime}, 9998502$ is just halfway between Regiomontanus's sine values 9998477 and 9998527. But for $\operatorname{Tan} 89^{\circ} 10^{\prime \prime}$, something obviously went wrong, because the value of $\operatorname{Sin} 89^{\circ} 10^{\prime \prime}$ used is smaller than that for $\operatorname{Sin} 89^{\circ}$, although the resulting value is still acceptable. Of course, given the limited number of significant digits for the sines, especially at the end of the range, most of the figures in the tangents end up being meaningless. It doesn't make much sense to give $\operatorname{Tan} 89^{\circ} 59^{\prime} 50^{\prime \prime}$ to 12 places, when the value of $\operatorname{Sin} 10^{\prime \prime}$ used only has three places...

We can also see that at the end of the range Reinhold moved to cosine values of $10^{7}$, but that he did not try to do a finer interpolation. In any case, such an analysis can be made for all $60 \times 5=300$ values which are not multiples of $1^{\prime}$, but this is left as an exercise. There may be other errors such as the one mentioned for $89^{\circ} 10^{\prime \prime}$ and it might be interesting to do some detailed statistics about these errors.

It is precisely for these reasons that Viète's tangents and secants published in 1579 are much more accurate than those of Reinhold, because Viète used sufficiently accurate sines for the number of figures he was trying to compute for the tangents and secants.

Reinhold's table of tangents was the first table of tangents at intervals of $1^{\prime}$. Secants at this interval would only be published 25 years later by

Viète. ${ }^{448}$ Reinhold's tangents were reused by Fincke in 1583, and Fincke added the secants.

A modern reconstruction of Reinhold's tables is provided separately. ${ }^{449}$

### 6.11 Bassantin (1557)

James Bassantin (c1504-1568) was a Scottish astronomer and mathematician who came to France under the reign of Henri II who was King of France from 1547 to $1559 .{ }^{450}$

Bassantin owes his fame to the publication of his Astronomique Discours ${ }^{451}$ in 1557 in Lyon. This book contains many volvelles based on the system of Ptolemy. ${ }^{452}$ It also contains a table of sines with $R=10^{5}$ and at intervals of $1^{\prime}$ (figure 35) which was presumably copied from Apian, since both the values and the layout agree with Apian's 1533, ${ }^{453} 1534^{454}$ and $1541^{455}$ tables. Glowatzki and Göttsche came to the same conclusion. ${ }^{456}$ Possibly the only altered value is the sine of $89^{\circ} 59^{\prime}$ which Apian had put at 100000, but which Bassantin put at its correct value, 99999. However, many other values are wrong, since Apian truncated and did not round Regiomontanus's values.

### 6.12 Maurolico (1558)

Francesco Maurolico (1494-1575) was a mathematician and astronomer born in Messina, Sicily, but of Greek lineage. He lived almost all of his life in Sicily and made contributions to the fields of geometry, optics, conics, mechanics, music, and astronomy. ${ }^{457}$ He was ordained a priest in 1521 and

[^58]later became a Benedictine.
Maurolico edited the works of classical authors including Archimedes, Apollonius, Autolycus, Theodosius and Serenus. He also composed his own unique treatises on mathematics and mathematical science.

In 1558, Maurolico published his commentary on the spherics of Theodosius. ${ }^{458}$ It contains short tables of sines (figure 39), tangents (figure 40) and secants (figure 41), all using the radius $R=10^{5}$ and only giving values every degree. Maurolico also gave the tangents and secants for $89^{\circ} 15^{\prime}$, $89^{\circ} 30^{\prime}, 89^{\circ} 45^{\prime}, 89^{\circ} 55^{\prime}$ and $89^{\circ} 59^{\prime}$.

The values of the sines differ from those of the earlier tables with $R=10^{5}$, namely those of Gaurico (1524) ${ }^{459}$ and Apian (1533). ${ }^{460}$ Instead, Maurolico seems to have taken his values from Regiomontanus's table by dropping two digits and rounding. ${ }^{461}$ And, contrary to what von Braunmühl wrote, ${ }^{462}$ Maurolico was unaware of Rheticus's Canon doctrinæ triangulorum ${ }^{463}$ published in $1551 .{ }^{464}$

As far as the tangents are concerned, Glowatzki and Göttsche wrote that Maurolico took his values from Regiomontanus's Tabulæ directionum profectionumque ${ }^{465}$ and recomputed those above $45^{\circ}$ using Regiomontanus's sines. ${ }^{466}$

But according to Brummelen, ${ }^{467}$ the tangents were copied from Regiomontanus's 1490 table up to about $60^{\circ}$. Above $60^{\circ}$, the values of the tangents seem to have been recomputed from Regiomontanus's sines and Brummelen implies (his table 3) that they have been recomputed that way until $89^{\circ} 15^{\prime}$ inclusive. Beyond $89^{\circ} 15^{\prime}$ the difference between Maurolico's values and those computed from Regiomontanus's sines becomes much larger. The last four values are more accurate than the values that could have been obtained from Regiomontanus's table with $R=10^{7}$. Van Brum-

[^59]melen therefore suggested some kind of independent computation. ${ }^{468}$
I believe however that the threshhold for that "independent" computation occurs earlier than $89^{\circ} 15^{\prime}$. In fact, it is easy to see what Maurolico has done for the last values of the table. He basically recomputed the required sines with one more digit and used them for the tangents. For instance,

- for $89^{\circ} 30^{\prime}$, Maurolico used $\frac{9999619.2}{87265.3}=114.588721 \ldots$ (and printed 11458872), when Regiomontanus had only given $\operatorname{Sin} 89^{\circ} 30^{\prime}=9999619$ and $\operatorname{Sin} 30^{\prime}=87265$;
- for $89^{\circ} 45^{\prime}$, Maurolico used $\frac{9999904.8}{43633.1}=229.1816258 \ldots$ (and printed 22918163), when Regiomontanus had only given $\operatorname{Sin} 89^{\circ} 45^{\prime}=9999904$ and $\operatorname{Sin} 15^{\prime}=43632$ (and not 43633);
- for $89^{\circ} 55^{\prime}$, Maurolico used $\frac{9999989.4}{14544.5}=687.544391 \ldots$ (and printed 68754439), when Regiomontanus had only given $\operatorname{Sin} 89^{\circ} 55^{\prime}=9999989$ and $\operatorname{Sin} 5^{\prime}=14544$;
- for $89^{\circ} 59^{\prime}$, Maurolico used $\frac{9999999.6}{2908.9}=3437.725463 \ldots$ (and printed 343772546), when Regiomontanus had only given $\operatorname{Sin} 89^{\circ} 59^{\prime}=9999999$ and $\operatorname{Sin} 1^{\prime}=2909$.
The values taken by Maurolico are all correct, except for $\operatorname{Sin} 5^{\prime}$ which should have been 14544.4 and $\operatorname{Sin} 30^{\prime}$ which should have been 87265.4. In summary, Maurolico must have recomputed the sines of eight angles $\left(1^{\prime}, 5^{\prime}\right.$, $15^{\prime}, 30^{\prime}, 89^{\circ} 30^{\prime}, 89^{\circ} 45^{\prime}, 89^{\circ} 55^{\prime}$ and $89^{\circ} 59^{\prime}$ ), perhaps using $\operatorname{Sin} 1^{\prime}$ as a basis. In fact, if Maurolico has started with $\operatorname{Sin} 1^{\prime}=2908.9$, he could have computed $\operatorname{Sin} 5^{\prime}$ and found 14544.5 or even 14544.4, depending how he computed the value. Continuing with 14544.4, Maurolico could have found $\operatorname{Sin} 15^{\prime}$ to be 43633.1, and eventually $\operatorname{Sin} 30^{\prime}$ to be 87265.3.

This procedure was applied at least as early as $85^{\circ}$ and we can see for instance that the ratio 9961947/871557.4 gives exactly Maurolico's value, but that 9961947/871557 (Regiomontanus's values) does not.

Maurolico's accurate computation of the last tangents then boils down to a one digit more accurate value of $\sin 1^{\prime}$ than that provided by Regiomontanus. I don't know how Maurolico obtained that value, but there is a very simple way, which is to observe that the sine of a small angle measured on the circumference (in radians) is almost equal to the angle itself. Therefore, with $R=10^{7}$,

$$
\operatorname{Sin} 1^{\prime} \approx \frac{\pi}{180 \cdot 60} \cdot 10^{7}=2908.882 \ldots
$$

[^60]and all that Maurolico needed was to take 2908.9 instead of Regiomontanus's 2909. There was no need to resort to bisections, trisections, etc., and to recompute the sines of small angles. There was also no need to interpolate as Regiomontanus did to construct his tables. And proceeding the above way does not require a very accurate value of $\pi$, as Ptolemy's value $3.1416 \ldots$ is sufficient.

Maurolico's third table is his table of secants, which he called tabella benefica. ${ }^{469}$ It was actually Fincke who first named that function secant. ${ }^{470}$ Fincke thought that Maurolico had copied his secants from Rheticus. ${ }^{471}$ But Brummelen recently gave an edition of Maurolico's short manual of his tabella benefica. ${ }^{472}$ There Maurolico claims that he worked on this matter in 1550, and it would then very likely be a work independent of that of Rheticus. Magini had also assumed that Maurolico's work was an independent one, not influenced by Rheticus. ${ }^{473}$ One might therefore assume that it was directly computed from Regiomontanus, probably from the 1541 edition.

Maurolico's table of secants is in fact very accurate, and more accurate than what would have been obtained by a mere use of Regiomontanus's tables. ${ }^{474}$ In order to explain this accuracy, Brummelen and Byrne claim that Maurolico computed the secants from the tangents, and not directly from Regiomontanus's sines, as claimed by Glowatzki and Göttsche. ${ }^{475}$

According to van Brummelen and Byrne, Maurolico used the relation

$$
\sec ^{2} \theta=\tan ^{2} \theta+1
$$

which transcribes into

$$
\operatorname{Sec}^{2} \theta=\operatorname{Tan}^{2} \theta+R^{2}
$$

when the radius is $R$. For instance, for $\theta=81^{\circ}, \operatorname{Tan} \theta=631375$ and $\sqrt{631375^{2}+100000^{2}}=639245.172 \ldots$ and Maurolico gives $\operatorname{Sec} \theta=639245$. This appears to work for $81^{\circ}$, but merely computing $1 / 1564345$ would have given the correct result too.

[^61]In fact, the procedure suggested by van Brummelen and Byrne (which echoes Birkenmajer's suggestion for Copernicus's table of secants) does not always work, and fails to give Maurolico's values for $86^{\circ}, 87^{\circ}$, or $89^{\circ} 30^{\prime}$. The solution is actually much simpler, and Maurolico must have proceeded like for the tangents, using an additional digit for a number of sines. Doing so from $85^{\circ}$ to the last angle gives the values in Maurolico's table, the only exception being $86^{\circ}$. But in that case, computing the secant from the tangent also does not yield the value in Maurolico's table. It is possible that Maurolico only used one additional digit for the secants from $87^{\circ}$, or that he made a mistake, or perhaps that the table contains a typo there.

Incidentally, Fincke seems to have used the procedure suggested by van Brummelen and Byrne in order to compute his secants in 1583 (and in fact van Brummelen and Byrne claim so, ${ }^{476}$ but with no references).

I have also given a modern reconstruction of Maurolico's tables in a separate document. ${ }^{477}$

### 6.13 Eisenmenger (1562)

Samuel Eisenmenger (1534-1585), known as Siderocrates, was a German physician, theologian and astronomer. He was professor of astronomy at the University of Tübingen in 1557-1568.

In 1562, Eisenmenger published his Libellus geographicus ${ }^{478}$ in which he gave a table of sines with $R=10^{7}$ and at intervals of $1^{\prime}$ (figure 42). This table was certainly also copied from Regiomontanus, ${ }^{479}$ and probably from the 1541 printing. ${ }^{480}$ The layout and headings of Eisenmenger's table are practically identical to those of Regiomontanus's published table, except that Eisenmenger put only half a degree in each column.

### 6.14 Schreckenfuchs (1569)

Erasmus Oswald Schreckenfuchs (1511-1579) was an Austrian humanist, astronomer and Hebraist. ${ }^{481}$ In 1551 he produced a commentary to the Al-
${ }^{476}$ [van Brummelen and Byrne (2021), p. 206]
${ }^{477}$ [Roegel (20211)]
${ }^{478}$ [Eisenmenger (1562)]
${ }^{479}$ [Glowatzki and Göttsche (1990), pp. 153-154]
${ }^{480}$ [von Peuerbach and Regiomontanus (1541)]
${ }^{481}$ For a summary of Schreckenfuchs's life and works, see [von Khauz (1755), pp. 184203]. Note in passing that Gessner briefly mentions Schreckenfuchs [Gessner and Simmler (1574), p. 184].
magest of Ptolemy and in 1556, he published a commentary on Peuerbach's Theoricae Novae Planetarum. ${ }^{482}$

In 1569, in his Commentaria in Sphaeram Ioannis de Sacrobusto, ${ }^{483}$ he reprinted Regiomontanus's table of tangents from one of the editions of the Tabulæ directionum profectionumque (figure 43).

But Schreckenfuchs also gave two tables of sines. His first table of sines covers two pages and uses the radius $R=60000$ and an interval of $15^{\prime}$ (figure 44). This table is likely based on the table of sines found in Regiomontanus's Tabulx directionum profectionumque (1490, 1504, 1550, 1552 or 1559), namely Johannes Engel's table (figure 16), as the values are truncated and not rounded from the large table with $R=6 \cdot 10^{6}$.

Schreckenfuchs's second table spans six pages, also with $15^{\prime}$ intervals, and uses a radius $R=60$ and three sexagesimal places (figure 45). However, the last place is always given as 0 or 30 . This second table cannot have been obtained from the first one. For instance, for $15^{\prime}$, Schreckenfuchs gives the sine as $261\left(60000 \times \sin 15^{\prime}=261.7 \ldots\right)$, but 261 would give a sine of $0^{\mathrm{p}} 15^{\prime} 39^{\prime \prime} 36^{\prime \prime \prime}$, not $0^{\mathrm{p}} 15^{\prime} 42^{\prime \prime} 30^{\prime \prime \prime}$. It is therefore to assume that now Schreckenfuchs used the sines in Regiomontanus 1541 (or 1561). For 15', Regiomontanus gave 26180, and this then would lead to $0^{\text {p }} 15^{\prime} 42^{\prime \prime} 28.8^{\prime \prime \prime}$ that Schreckenfuchs could have rounded to $0^{\mathrm{p}} 15^{\prime} 42^{\prime \prime} 30^{\prime \prime \prime}$.

### 6.15 Witekind (1576)

Hermann Witekind (or Wilken) (1522-1603), a student of Melanchthon, was a German humanist and mathematician. In 1585, under the pseudonym of Augustine Lercheimer he published a book against the persecution of witches. ${ }^{484}$

In 1576, he published his work Conformatio horologiorum sciotericorum etc. ${ }^{485}$ in which he included a table of sines for a radius of 100000 and for every minute of the quadrant (figure 46). This table was presumably copied from one of Apian's tables (1533, 1534 or 1541), ${ }^{486}$ as the values all seem to agree. ${ }^{487}$ The layout, however, is different. Each page has six columns for degrees and 30 rows for 30 minutes. Six degrees therefore span two pages.

[^62]Moreover, repeated leading digits are not printed (for instance the value 145 following 116 is shown as 45).

Among Witekind's other scientific publications, I mention only his De sphaera mundi published in 1574 (second edition in 1590).

### 6.16 Peucer (1579)

Caspar Peucer (1525-1602) was a German reformer, physician, and scholar from Bautzen, Germany. ${ }^{488}$ He wrote on mathematics, astronomy, geometry, and medicine, and edited some of Philip Melanchthon (1497-1560)'s letters, having married one of his daughters. He also became professor of mathematics in Wittenberg in 1554, the successor of Erasmus Reinhold after his untimely death in 1553. In 1560, he was appointed to the medical faculty of Wittenberg.

In his De dimensione terrex etc. ${ }^{489}$ published in 1579, and reprinted in 1587, Peucer included a table of sines for $R=10^{5}$ and at intervals of $1^{\prime}$ (figure 47). The situation parallels that of Witekind, and Peucer's table was presumably also copied from Apian ( 1533,1534 or 1541 ), ${ }^{490}$ since the values agree, ${ }^{491}$ with only minor alterations. However Peucer adapted Apian's layout and put only five degrees and 30 minutes per page. Therefore one page of Apian's table corresponds to four pages of Peucer's table.

### 6.17 Viète (1579)

François Viète (1540-1603) was a French mathematician whose work on the new algebra was an important step towards modern algebra. According to Zeller, Viète "was the foremost mathematician of France in the sixteenth century." ${ }^{492}$ Viète received a bachelor's degree in law in 1560 and held a number of official positions. In 1573, the King Charles IX made him counselor to the parlement of Brittany. He came back to Paris in 1580. Among his many works is his In artem analyticem isagoge, the earliest work on symbolic algebra (1591).

[^63]Also inspired by Rheticus' 1551 Canon doctrinæ triangulorum, ${ }^{493}$ Viète constructed a new table, which he called the Canon mathematicus. ${ }^{494}$ This work contained a typographically sophisticated table of the six trigonometric functions for every minute of the quadrant and with a radius of 100000 , with sometimes one or more additional figures ${ }^{495}$ (see figure 48). The printing of the table was started in 1571 but it was only completed in $1579 .{ }^{496}$

This was the first published canon giving the trigonometric functions every minute, but on the other hand it gave them to less places than Rheticus' 1551 table (which however only had an interval of $10^{\prime}$ ).

The sines from $0^{\circ}$ to $45^{\circ}$ and the cosines from $0^{\circ} 30^{\prime}$ to $45^{\circ}$ were taken from Regiomontanus (or Reinhold) and rounded or not, depending on the range of the table. Glowatzki and Göttsche had observed that Viète had recomputed the sines from $89^{\circ} 61^{\prime}$ to $90^{\circ},^{497}$ but in fact the cosines from $0^{\circ}$ to $30^{\prime}$ have been computed from Regiomontanus's (or Reinhold's) sines, probably with $\cos x \approx 1-\frac{\sin ^{2} x}{2}$, which, with a radius $R$ other than 1, becomes $\operatorname{Cos} x \approx R-\frac{\operatorname{Sin}^{2} x}{2 R}$. For instance, for $6^{\prime}, R=10^{8}$ and Regiomontanus's sine value 17453:

$$
\operatorname{Cos} 6^{\prime} \approx 10^{8}-\frac{174530^{2}}{2 \cdot 10^{8}}=10^{8}-\frac{17453^{2}}{2 \cdot 10^{6}}=99999847.69 \ldots
$$

and Viète gives the value 99999848. Viète did not use the values of the sines in his table for this purpose, and using 175 (Viète's value for Sin $6^{\prime}$ ) in the previous example would not produce a sufficiently accurate cosine.

For the tangents and secants, Glowatzki and Göttsche wrote that they were recomputed from Regiomontanus's values. ${ }^{498}$ But we can actually tell a bit more.

First, we can see that Viète computed the secants from $0^{\circ}$ to $45^{\circ}$ by inverting his cosines (and not those of Regiomontanus). The tangents between $0^{\circ}$ to $45^{\circ}$ were computed by using Regiomontanus's full values.

[^64]Then, the cotangents from about $5^{\circ}$ to $45^{\circ}$ were also computed from the ratios $\cos / \sin$ using Viète's values (and not by inverting the tangents). The cosecants from about $5^{\circ}$ to $45^{\circ}$ were computed by inverting Viète's values of the sines.

But for the cotangents and cosecants between $0^{\circ}$ and about $5^{\circ}$, Viète used more accurate values of the sines than those printed in his table, with 1 to 5 more figures than Regiomontanus. For instance, for Cot 1', Viète used $\operatorname{Sin} 1^{\prime}=29.0888204$ (here with $R=10^{5}$ ), that is 5 more figures than Regiomontanus. This enabled him to obtain (reducing the cosine and sine to $R=10^{9}$ )

$$
\cot 1^{\prime}=\frac{999999958}{290888.204}=3437.74668
$$

or (with $R=10^{5}$ )

$$
\operatorname{Cot} 1^{\prime}=\frac{999999958}{290888.204} \times 10^{5}=343774668
$$

and this is precisely the value given in the Canon mathematicus (the exact value being 343774667).

The same applies for Csc $1^{\prime}$. For Sin 2', Viète took 58.1776385 , whereas Regiomontanus only has 5818. And so on.

These values are much more accurate than the tangents and secants given by Reinhold in 1554 and Fincke in 1583 for large angles, and obviously Viète had a much better understanding of the requirements for exact computations.

In his treatise on angular sections, ${ }^{499}$ Viète describes a way to compute the sine of $1^{\prime}$ and other values he needed. This sine can be obtained as follows. First, like Ptolemy before, one can compute the sines of $18^{\circ}$ and $60^{\circ}$. Trisecting $60^{\circ}$ twice, we obtain $\operatorname{Sin} 20^{\circ}$ and then $\operatorname{Sin} 6^{\circ} 40^{\prime}$. Using quinquisection with $18^{\circ}$, we obtain $\operatorname{Sin} 3^{\circ} 36^{\prime}$. Bisecting $6^{\circ} 40^{\prime}$ we find $\operatorname{Sin} 3^{\circ} 20^{\prime}$. Using the two values $\operatorname{Sin} 3^{\circ} 36^{\prime}$ and $\operatorname{Sin} 3^{\circ} 20^{\prime}$, we obtain the sine of the difference of the angles, namely $\sin 16^{\prime}$. And bisecting $16^{\prime}$ four times, we obtain $\operatorname{Sin} 1^{\prime}$. But as a matter of fact Viète seems to have proceeded slightly differently for his table. He actually found two approximations of $\operatorname{Sin} 1^{\prime}$, one greater and one smaller than the sought value. An interpolation between these two values then gave a better approximation of $\operatorname{Sin} 1^{1} .^{500}$

As observed by Tannery, Viète's tables are rare because of the success of Rheticus' Opus palatinum (1596), ${ }^{501}$ of Pitiscus' Thesaurus mathematicus

[^65](1613), ${ }^{502}$ and because of the introduction of logarithms in 1614. They all made Viète's tables obsolete.

A persistent legend is also that the Canon mathematicus contained many errors, and that Viète consequently withdrew or re-purchased all the copies he could find and had them destroyed. This would then explain why this book is of great rarity. ${ }^{503}$ But according to Ritter's biography of Viète, ${ }^{504}$ this legend rests on the editor of Viète's 1646 Opera omitting the Canon mathematicus, on the grounds that the computations would have to be redone. ${ }^{505}$ Moreover, as I have shown, Viète's table is actually very accurate. ${ }^{506}$

The Canon mathematicus was also published with a London imprint in 1589 (Opera mathematica, London: Bouvier) and there is an edition dated 1609, but Bosmans showed that it is not a reprint. It is the 1579 edition rebound. ${ }^{507}$ Cantor and von Braunmühl had mistakenly thought that it was a new edition, ${ }^{508}$ probably after Eneström led them to think so. ${ }^{509}$

A modern reconstruction of Viète's table is given separately. ${ }^{510}$

### 6.18 Bressieu (1581)

Maurice Bressieu (c1546-1617) was a French mathematician and humanist. ${ }^{511}$ In 1576, Bressieu won a position of mathematics professor founded in Paris by Petrus Ramus (1515-1572), which he kept until 1608. ${ }^{512}$

In his Metrices astronomicx published in 1581, ${ }^{513}$ Bressieu first gives a sexagesimal table of sines, with an unusual layout (figure 49). The sines and sines of the complementary angle (cosines) are given in two adjacent columns and the table therefore only runs up to $45^{\circ}$. But Bressieu's layout is in fact very unusual, in that it doesn't use a footer line. In figure 49, the first column (headed 18) gives from top to bottom the sines from $18^{\circ}$ to $19^{\circ}$. The second column (headed 71) gives the sines from $71^{\circ}$ to $72^{\circ}$, but from

[^66]bottom to top. Consequently, the second column actually also gives the cosines from $18^{\circ}$ to $19^{\circ}$, from top to bottom.

The values are given in degrees (or parts) with a radius of 60 . For instance, the sine of $45^{\circ}$ is given as $42 ; 25,35$ as $\operatorname{Sin} 45^{\circ}=42+25 / 60+$ $35 / 60^{2}+\cdots$. This table contains exactly the same values as in Fine's tables. ${ }^{514}$

Bressieu also gives a second table (figure 50), which actually contains values of the tangents and secants, also with a radius of 60 . For instance,

$$
\begin{array}{ll}
\operatorname{Tan} 45^{\circ}=60 & \text { is given as } 1,0 ; 0,0 \\
\operatorname{Sec} 60^{\circ}=120 & \text { is given as } 2,0 ; 0,0 \\
\operatorname{Tan} 89^{\circ} 3^{\prime}=60^{2}+18+20 / 60+\cdots & \text { is given as } 1,0,18 ; 20
\end{array}
$$

and so on. Bressieu's tables of tangents and secants are the only known printed fully sexagesimal tables of tangents and secants. Their layout follows the style used by Rheticus in 1542 and not that used in Bressieu's table of sines.

It appears that these tangents and secants have not been computed from Bressieu's table of sines which is not sufficiently accurate. Bressieu could have taken another table giving the tangents and secants for every minute, but the only such table available in 1581 was Viète's table ${ }^{515}$ and the last values of Bressieu's tangents do not agree with Viète's values. Yet another possibility is that Bressieu used Reinhold's values for the tangents. ${ }^{516}$ But this appears again not to be the case.

I believe instead that Bressieu used Regiomontanus's table of sines (or a derivative thereof) for $R=10^{7}$ and computed a number of sexagesimal values of the tangents and secants in his second table, but not all of them. For the last values of the table, Bressieu may have done a number of special computations, but for the other gaps, I believe that Bressieu interpolated the missing values. In fact, if Bressieu would have used Regiomontanus's values in each case, he would have obtained more accurate values for the tangents and secants. The deviations do not occur only for the last values around $90^{\circ}$, but also for smaller values. For instance, for Tan $75^{\circ}$, Bressieu gives 3,$43 ; 55,18$, and working with Regiomontanus's values would have given him 3, 43; 55,23 which is the correct value. For Tan $89^{\circ} 59^{\prime}$, Bressieu gives $57,17,42 ; 26$, when the correct value is $57,17,44 ; 48,1, \ldots$, which he would have obtained using Viète's table. Regiomontanus's values instead would have given $57,17,36 ; 25$.

[^67]In figure 13, the accuracy of Bressieu's tables is indicated as $60 ; 60^{2}$, by which I mean a radius of 60 and two sexagesimal places. However, for $89^{\circ}$ and above, the values of the tangents and secants are given to only one sexagesimal place.

Bressieu is mentioned by Zeller ${ }^{517}$ but not by Glowatzki and Göttsche.

### 6.19 Giuntini (1581)

Francesco Giuntini (1523-1590) was an Italian theologian and one of the most famous astrologer of the second half of the 16th century. ${ }^{518}$

In his Speculum astrologie ${ }^{519}$ published in 1581, Giuntini included a table of sines with $R=10^{5}$ and an interval of $1^{\prime}$ (figure 51). This table was most certainly copied from one of Apian's tables (1533, 1534 or 1541), ${ }^{520}$ or perhaps from one of its derivatives.

The 1573 edition of the Speculum astrologix does not contain this sine table.

### 6.20 Padovani (1582)

Giovanni Padovani (b. c1512) was an Italian mathematician and astronomer. ${ }^{521}$ He was from Verona and a student of the astronomer and mathematician Pietro Pitati.

In his De compositione, $\mathcal{E}$ vsu multiformium horologiorum solarium, ${ }^{522}$ a work on sundials published in 1582, Padovani included a table of sines with $R=10^{5}$ and an interval of $1^{\prime}$ (figure 52). Like Giuntini's table (1581), Padovani's table was also most certainly copied from one of Apian's tables (1533, 1534 or 1541), ${ }^{523}$ or perhaps from one of its derivatives, but Giuntini and Padovani's tables do not share the same layout.

An earlier edition of Padovani's work on sundials was published in 1570, but I have not seen it. It possibly lacks the table of sines.

[^68]
### 6.21 Fincke (1583)

Thomas Fincke (1561-1656) was born in Flensburg, Germany, now at the border with Denmark. From 1577 to 1582, he studied mathematics, astrology, rhetoric and philosophy, in particular with Conrad Dasypodius, a teacher at the Strasbourg University and one of the authors of the second astronomical clock of the Strasbourg cathedral. ${ }^{524}$

In 1581, Fincke published an ephemeris based on the prutenic tables (Ephemeris coelestium motuum anni 1582, supputata ex Tabulis Prutenicis). He returned from Strasbourg to Heidelberg and Leipzig, and moved to Basel in 1583. This is where, at the age of 22 , he published his most famous work, his Geometrix Rotundi, ${ }^{525}$ an influential work on plane and spherical trigonometry based on Ramus's Geometria (1569). ${ }^{526}$

This book does in particular contain tables of sines (figure 54), tangents (figure 53) and secants (figure 55) with $R=10^{7}$ and intervals of $1^{1.527}$ And it was precisely Fincke who coined the names "tangent" and "secant" which had not been used before. Incidentally, Viète did apparently not approve of these names. ${ }^{528}$

Fincke's sines do slightly differ from those of Reinhold, hence from those of Regiomontanus. It seems that Fincke made a number of small last figure adjustments to either Reinhold's or Regiomontanus's tables. ${ }^{529}$ Given that the tangents were certainly taken from Reinhold (1554), ${ }^{530}$ I assume that this was also the case for the sines.

As far as the tangents are concerned, we can see for instance that the last values agree with those of Reinhold, except for $89^{\circ} 53^{\prime}, 89^{\circ} 56^{\prime}$, and $89^{\circ} 57^{\prime}$. In the first case, Fincke's tangent is less accurate than Reinhold's, but in the two other cases the tangents are slightly more accurate.

Finally, Fincke's secants are the result of new computations. The val-

[^69]ues differ from those of Rheticus's Canon doctrinæ triangulorum. ${ }^{531}$ Fincke, however, did not use Regiomontanus's sines, nor his own version to compute the secants. Instead, it seems that he computed the secants using his tangents. Fincke most certainly used the formula
$$
\operatorname{Sec} x=\sqrt{\operatorname{Tan}^{2} x+R^{2}}
$$
to compute the secants, and when computing $\operatorname{Tan}^{2} x$, he must have kept only seven or eight significant figures and rounded, although the procedure may not have been systematic. This is reminiscent of the computation of cotangents by Rheticus in 1551, and also echoes a recent suggestion by van Brummelen and Byrne for the computation of secants by Maurolico. ${ }^{532}$ In fact, it is only after I concluded the above that I noticed that van Brummelen and Byrne claimed that Fincke used this formula to compute the secant. ${ }^{533}$

Fincke's tangents and secants, as well as Reinhold's tangents, are less accurate than those published by Viète in 1579. For instance, Fincke and Reinhold gave $\operatorname{Tan} 89^{\circ} 59^{\prime}=\operatorname{Cot} 1^{\prime}=34376070815\left(\right.$ for $\left.R=10^{7}\right)$ where only the first four figures are correct. Instead, Viète gives a value whose error is about 10000 times smaller. This is so because Viète took more accurate values for the sines and understood that this was necessary in order to obtain tangents with such an accuracy.

I am giving separately a modern reconstruction of Fincke's tables. ${ }^{534}$
After the publication of his Geometrix Rotundi, Fincke began to study medicine in Basel, Padua, Siena and Pisa. He became MD in 1587. He then returned to Denmark where he held the chair of mathematics at the University of Copenhagen from 1591 until 1602, but afterwards was more active as a physician and his mathematical activity never reached again the level of his 1583 book.

### 6.22 Clavius (1586)

Christopher Clavius (1537 or 1538-1612) was a German mathematician and astronomer. He was born in Bamberg and entered the Jesuit order in Rome in $1555 .{ }^{535}$ He published his Euclidis elementorum libri XV (The elements of

[^70]Euclid) in 1574 and was a supporter of the Ptolemaic system, and at the same time a friend of Galileo. He also helped develop algebra in Italy and introduced Stifel's symbols "+" and " - ."

He was also a member of the Vatican commission that accepted the proposed calendar invented by Aloysius Lilius, that is known as Gregorian calendar.

In his last years he was probably the most respected astronomer in Europe and his textbooks were used for astronomical education for over fifty years in and even out of Europe.

In 1586, Clavius published an edition of Theodosius's sphaerics, ${ }^{536}$ in which he included tables of sines (figure 56), tangents (figure 57) and secants (figures 58 and 59) with $R=10^{7}$ and at intervals of $1^{\prime}$. ${ }^{537}$

Clavius's sines and tangents were taken from Reinhold (1554), ${ }^{538}$ as they do not show the alterations made by Fincke. ${ }^{539}$ But the secants instead were taken from Fincke's work (1583). ${ }^{540}$ And in fact Clavius used the new names "tangent" and "secant" coined by Fincke. Clavius, however, corrected all the typos in the earlier editions (but not the last digit deviations). ${ }^{541}$

On the other hand, Clavius's table has at least one typo, namely for $\sin 89^{\circ} 30^{\prime}$ which he gives as 9999616 instead of the correct 9999619 . This error was corrected by Magini in 1592, and by Clavius himself in 1593.

Clavius's 1586 table, without the corrections of the typos, was copied by Blundeville in 1594.

### 6.23 Bürgi (1587)

This survey of 15th and 16th century trigonometrical tables based on Regiomontanus's work would not be complete without mentioning Jost Bürgi (1552-1632). Bürgi is well known as a (very) skillful mechanician, clockmaker and instrument maker, and also as an inventor of a table of progressions which could be used for the same purpose as logarithms. ${ }^{542}$

[^71]Around 1587, Bürgi devised a new way (his so-called "Kunstweg") to compute sines iteratively, without any geometrical construction ${ }^{543}$ and he constructed at least two tables, one giving the sines at intervals of $2^{\prime \prime}$ and another giving the sines at intervals of $1^{\prime}$. However, I believe that Bürgi did not use his new algorithm to construct these tables, and instead built up the tables by finite differences. Although Bürgi's work represents a new computation of sines, it is therefore possible that he reinvented some techniques already used by Regiomontanus, and even before in India, as mentioned earlier (see §5).

The $2^{\prime \prime}$ table does not seem to have survived, but the $1^{\prime}$ table resurfaced a few years ago. At that time, I made modern reconstructions of both tables. ${ }^{544}$

Bürgi's surviving sine table (figure 60) gives the sines at intervals of $1^{\prime}$, with a radius $R=60$, and to four sexagesimal places, except for the last two degrees where they are given to five and six sexagesimal places. These four sexagesimal places correspond to a radius of $10^{9}$ with a sine usually correct to 9 decimal places.

For instance, $\operatorname{Sin} 75^{\circ}$ is given as $57 ; 57,19,58,43$ which corresponds to the decimal value 0.965925827 , the correct value being $0.96592582628906 \ldots$.

In contrast, Rheticus and Otho's Opus palatinum (1596) ${ }^{545}$ gives the value 9659258263 for $\operatorname{Sin} 75^{\circ}$, and this is correct to 10 places. Rheticus also gives the sines every $10^{\prime \prime}$.

And in 1613 Pitiscus ${ }^{546}$ gave $96592,58262,89067$, instead of the correct $96592,58262,89068$. Rheticus must have had such accurate values already in the 1570s, before Bürgi, but with the exception of Rheticus, Bürgi's table was probably the most accurate sine table constructed at the end of the 16th century.

Bürgi's table can be compared to those of Fine, Schreckenfuchs and Bressieu which are also sexagesimal tables, but which are less accurate and based on Regiomontanus's sines.

[^72]
### 6.24 Gallucci (1588)

Giovanni Paolo Gallucci (1538-1621) was an Italian astronomer and translator. ${ }^{547}$ Among his notable translations, Gallucci published in 1591 his Della simmetria dei corpi humani, a translation of Dürer's "Four books on human proportion" (Vier bücher von menschlicher Proportion, 1528). He was also a private teacher to the Venetian nobility and a founding member of the second Venetian Academy.

Gallucci's most famous works are probably his Theatrum mundi, et temporis ${ }^{548}$ published in 1588, and his Speculum Uranicum published in 1593, both featuring some volvelles. In the Theatrum mundi Gallucci also included a table of sines with a radius $R=60000$ and an interval of $1^{\prime}$ (figure 61). This table was most certainly copied from Engel's table (§ 6.1), in one of the editions of the Tabulæ directionum profectionumque where it appears, not necessarily the 1490 edition. Gallucci uses exactly the same layout, with six half-degrees per page, but he has dropped the differences. The values seem to agree, with the exception of a few transcription errors.

### 6.25 Lansberge (1591)

Philip van Lansberge (1561-1632) was born in Ghent, Belgium, but in 1566 his parents moved to France and then to England, because of the religious troubles. There, he studied mathematics and theology. ${ }^{549}$ He became a protestant minister in Antwerp in 1580 and then established himself in the Netherlands.

In 1591 he published his Triangulorum geometrix ${ }^{550}$ which is closely based on Fincke's Geometrix Rotundi. ${ }^{551}$ Lansberge did in particular include Fincke's tables of sines (figure 62), tangents (figure 63) and secants (figure 64) with $R=10^{7}$ and at intervals of $1^{1}{ }^{552}$ These tables are therefore ultimately based on those of Regiomontanus. ${ }^{553}$ Lansberge also used the new names "tangent" and "secant" coined by Fincke. I am giving

[^73]separately a modern reconstruction of Lansberge's tables. ${ }^{554}$
In 1632, Lansberge published his best known work, his Tabulae motuum coelestium perpetuæ, for the prediction of planetary positions. Lansberge was a follower of Copernicus and his work is based on an epicyclic theory, but he did not accept Kepler's theories.

Lansberge died that same year in Middelburg in the Netherlands.

### 6.26 Magini (1592)

Giovanni Antonio Magini (1555-1617) was an Italian astronomer, astrologer, cartographer, and mathematician. He was born in Padua and studied in Bologna where in 1588 he obtained one of the chairs of mathematics. ${ }^{555}$

Magini's chief scholarly interest was astrology and he adhered to the Ptolemaic principles. He was much more skilled in calculations than in theory and his ephemerides were useful. In 1592, he published his work De planis triangulis. ${ }^{556}$ This work also contained a Tabula tetragonica ${ }^{557}$ which could be used to compute the products of two numbers.

The De planis triangulis also contains tables of sines (figure 65), tangents (figure 66) and secants (figure 67) with $R=10^{7}$ and at $1^{\prime}$ intervals. ${ }^{558}$ These tables are copied from those of Clavius (and borrow Fincke's new names), ${ }^{559}$ and thus are ultimately based on those of Regiomontanus. ${ }^{560}$ But contrary to Clavius, Magini has adopted a semi-quadrantal arrangement and only runs the angles up to $45^{\circ}$. The sines are the sinus primus, the cosines are the sinus secundus, and similarly with the tangents and secants. The value of $\operatorname{Sin} 89^{\circ} 30^{\prime}$ is given by Magini as 9999619 , which is correct, but Clavius had 9999616. It therefore appears that Magini has corrected Clavius's typo.

Magini's De planis triangulis also contains a Tabula gnomonica which is a table of arctangents similar to that of Peuerbach, ${ }^{561}$ but where the entries vary between 0 and 1000.

In his Primum mobile duodecim libris contentum ${ }^{562}$ published in 1609,

[^74]Magini gives another table with $R=10^{7}$ and at $1^{\prime}$ intervals, with sines, versines, tangents and secants, but the values are not those of the 1592 table. Instead, Magini took the values from Rheticus and Othos' Opus palatinum (1596). ${ }^{563}$

The later years of Magini's life were devoted to cartography and geography. He worked in particular on an atlas of Italy.

### 6.27 Clavius (1593)

In 1593, Clavius published his work Astrolabium ${ }^{564}$ which contained a sine table with $R=10^{7}$ and at intervals of $1^{\prime}$ (figure 68). This table was copied from Clavius's earlier tables published in $1586,{ }^{565}$ but with some corrections. For instance, as mentioned previously, the value of $\operatorname{Sin} 89^{\circ} 30^{\prime}$ was given incorrectly in Clavius's 1586 table, and was corrected here, perhaps after the discovery of the typo by Magini.

### 6.28 Fale (1593)

Thomas Fale (born c1560?) was an English mathematician. Very little is known of him. ${ }^{566}$

In 1593, Fale published his Horologiographia. ${ }^{567}$ This work, which is the only one known of him, appears to be the first book in English on sundials. ${ }^{568}$ It contains in particular a table of sines (figure 69) which was presumably copied from Witekind's Conformatio horologiorum sciotericorum etc. ${ }^{569}$ published in 1576 and with which it shares the values and the layout. ${ }^{570}$ There are however some slight differences, and Fale gives for instance $\operatorname{Sin} 5^{\circ} 3^{\prime}=8803$, when Witekind gave the correct 8802 (compare figures 46 and 69).

As observed by De Morgan and Goodwin, Fale's table may be the earliest sine table printed in England. ${ }^{571}$

[^75]
### 6.29 Blundeville (1594)

Thomas Blundeville (c1522-c1606) was an English writer and mathematician, who wrote in particular on horsemanship and cartography. ${ }^{572}$

In 1594, he published his Exercises, containing sixe Treatises, etc. ${ }^{573}$ which contain tables of sines, tangents and secants for $R=10^{7}$ and at intervals of $1^{1.574}$ These tables are based on those published by Clavius ${ }^{575}$ in 1586 and Blundeville explicitely mentions Clavius. They use the new names coined by Fincke in 1583. It is possible that Blundeville's tables are the first complete (that is not merely of sines) trigonometric tables published in England. ${ }^{576}$

Interestingly, Blundeville carried Clavius's incorrect value for $\sin 89^{\circ} 30^{\prime}$ given in the 1586 table.

Thus Blundeville's sines are ultimately based on Regiomontanus's tables. ${ }^{577}$

### 6.30 Ceulen (1596)

Ludolph van Ceulen (1540-1610) was a German-Dutch mathematician born in Hildesheim. At some point he settled in Holland. In the 1580s and 1590s he was a fencing master as well as a mathematics teacher. He died in 1610 in Leiden. ${ }^{578}$

In 1596 he published his main work, Vanden circkel etc. ${ }^{579}$ where he gave among other things a 20-place approximation of $\pi$.

Ceulen's book also contains tables of sines, tangents and secants for $R=10^{7}$ and at intervals of $1^{\prime}$ (figure 71). Ceulen's tables are certainly based on those of Lansberge ${ }^{580}$ who is mentioned by Ceulen. ${ }^{581}$ Ceulen uses the

[^76]new names introduced by Fincke in 1583.
Ceulen departed somewhat from the previous tables, in that he did not separate sines, tangents and secants in different tables, but put them together, for a range of two degrees, on each page.

### 6.31 Rheticus/Otho (1596)

After the publication of his Canon doctrinæ triangulorum in 1551 which was based on Regiomontanus's tables, ${ }^{582}$ Rheticus (1514-1574) ${ }^{583}$ continued to work on a more extensive project, where the six trigonometric functions would be given every $10^{\prime \prime}$ and for a larger radius. As observed by Zeller, ${ }^{584}$ "Rheticus built his trigonometry on the foundation established by Regiomontanus."

Rheticus embarked on totally new computations, but his work was only completed after his death by Lucius Valentinus Otho (c1545-1603) and published in 1596 in the Opus palatinum ${ }^{585}$ (figure 72). Otho had met Rheticus in 1573 and Rheticus had asked him to complete his work.

With the exception of Bürgi's work, this was the first new computation of trigonometric values in the 16th century, since most of the trigonometric tables printed in the 16th century actually use values or computations inherited from Regiomontanus's tables ${ }^{586}$ (see figure 13).

However, even a cursory examination of the Opus palatinum reveals that it contains two overlapping tables. On one hand, there is a table giving all six functions with a radius $R=10^{10}$ and an interval of $10^{\prime \prime}$. This table spans 540 pages. On the other hand, there is a table giving only the cosecants and cotangents, with a radius $R=10^{7}$ and the same interval of $10^{\prime \prime}$. This second table spans 180 pages. One might expect the second table to be an abridgement of the first, but this is not the case, as is apparent when comparing the first values of the cosecants and cotangents. These two tables obviously correspond to two different computations. This has actually been noticed before, and Glaisher wrote that "there seems no reason why it should have been printed at all, as the great ten-decimal canon completely supersedes it., ${ }^{587}$

[^77]I have therefore assumed that the shorter table is in fact an older table, perhaps computed by Rheticus around $1560 .{ }^{588}$ I believe that after his Canon doctrinæ triangulorum (1551), which already used $R=10^{7}$ and an interval of $10^{\prime}$, Rheticus decided first to compute the functions with an interval 60 times smaller, that is $10^{\prime \prime}$, but with the same radius. This is what I have shown in figure 13.

It is in fact easy to see what were the computations in this first attempt at a $10^{\prime \prime}$ table (and it would consequently be rather straightforward to complete this table with those for the sines, cosines, tangents and secants, which presumably existed). We can observe that the cosecants at $1^{\prime}$ intervals were merely obtained by the fractions $10^{14} / 2909,10^{14} / 5818,10^{14} / 8727$, $10^{14} / 11635,10^{14} / 14544$, etc. In other words, Rheticus merely used the sines found in Regiomontanus's table, apparently sometimes with slight adjustments (as for Csc $4^{\prime}$ or Csc $10^{\prime}$ ), but adjustments that did not always produce more accurate results (as for Csc 4 '). It is possible that some of these "adjustments" were in fact typos. Rheticus did the same for the cotangents, taking the sines from Regiomontanus. It seems that the adjustments made for the sines in the case of cosecants were also used for the cotangents, but this should be checked throughout the table.

For the $10^{\prime \prime}$ intervals, Rheticus merely interpolated the sines. For instance, Csc $10^{\prime \prime}$ is obtained using $\operatorname{Sin} 10^{\prime \prime}=485, \operatorname{Csc} 20^{\prime \prime}$ uses $\operatorname{Sin} 20^{\prime \prime}=970$, and so on. There may be the usual typos, such as for $\operatorname{Cot} 10^{\prime \prime}$ which is given as 206085546390, but should be 206185546392, and was merely obtained by dividing 9999999 by 485.

Sometime after that first computation, Rheticus must have realized that the cosecants and cotangents could not be computed accurately with such a scheme, because Regiomontanus's sines were not accurate enough for small angles. He must therefore have decided to construct a larger table, and he computed this time the sines and cosines with a radius of $10^{15}$ and an interval of $10^{\prime \prime}$. This was probably done around 1570. This work was used to produce the table for $R=10^{10}$ published in 1596. However, the cosecants and cotangents were not computed using these accurate values of the sines, but those from the Opus palatinum itself. For instance, for Csc 1', Rheticus (or Otho) used the sine value 2908882, instead of Regiomontanus's 2909, but not the more accurate 290888204563 in the $R=10^{15}$ table.

When the Opus palatinum was published, Otho must have decided to

[^78]include Rheticus's earlier computation of cosecants and cotangents, but the reason for publishing it remains unclear, as Otho must have realized that these first calculations were inadequate. On the other hand, it was much more difficult for him than for us to realize $i t$, and he perhaps decided to include these tables in case they contained some valuable results.

Of course, computing the cosecants and cotangents with the sines given in the Opus palatinum is still not enough for small angles, as the sines are still not sufficiently accurate. This led Bartholomaeus Pitiscus (1561-1613) to correct the Opus palatinum and to publish Rheticus's sine table with $R=10^{15}$ (figure 73) as well as other tables that he computed himself in his Thesaurus mathematicus. ${ }^{589}$ Incidentally, Pitiscus was the one who first coined the word "trigonometry."

## 7 Conclusion

This marks the end of our journey through 15th and 16th century fundamental trigonometric tables. But this end is also a beginning. Rheticus's Opus palatinum and its amendments by Pitiscus were the start of a new era and these tables would themselves last until the 20th century. And the first years of the 17th century were the place of a bifurcation. On one hand trigonometric tables would continue their path, with little changes beyond Rheticus's masterpiece, ${ }^{590}$ and on the other hand they made their foray into the world of logarithms, as if logarithms naturally absorbed the trigonometric functions. ${ }^{591}$

Logarithms first appeared in public in 1614, and they started in association with sines. Indeed, when Napier published ${ }^{592}$ the first table of logarithms in 1614, it was a table of logarithms of sines, and these sines were either those of Fincke ${ }^{593}$ or those of Lansberge. ${ }^{594}$ Napier's work was therefore based again on that of Regiomontanus, and not yet on Rheticus's work.

[^79]Three years later, the decimal logarithms were introduced by Briggs, ${ }^{595}$ and expanded in 1624 and $1628 .{ }^{596}$ They were however unrelated to trigonometric tables.

Edmund Gunter was the first to compute and publish tables of decimal logarithms of sines and tangents in $1620 .{ }^{597}$ His tables gave the logarithms to 8 places and were probably based on Rheticus's Opus palatinum or Pitiscus's Thesaurus mathematicus.

In 1633, Henry Gellibrand completed and published Henry Briggs's Trigonometria Britannica ${ }^{598}$ which was a large table of trigonometric functions and decimal logarithms of trigonometric functions. Briggs's table was in fact the result of a new computation of sines, tangents and secants, ${ }^{599}$ in which he divided the degree in 100 parts. The sines were computed with $R=10^{15}$ and the tangents and secants with $R=10^{10}$. Briggs's trigonometric functions are not based on earlier tables, not even on those of Rheticus's Opus palatinum.

The same year 1633, Adriaan Vlacq independently published his Trigonometria artificialis. ${ }^{600}$ This work gives only the logarithms of sines, cosines, tangents and cotangents, and not the trigonometric functions themselves. But contrary to Briggs, Vlacq computed his logarithms using the values given by Rheticus in his Opus palatinum. It was Vlacq's table and not Briggs's table which had the greatest offspring, and was many times reprinted, simplified and adapted until the 20th century.

[^80]
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Figure 15: Excerpt of Regiomontanus's table of tangents [Regiomontanus (1490)] (source: The Budapest University of Technology and Economics, 85.211, www.manuscriptorium.com).

Tabella.

| g | 0 |  | 1 |  | 2 |  | 3 |  | 4 |  | 5 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| m | ptes |  | ptes |  | ptes |  | ptes |  | Rtes |  | Ptes |  |
| $1)$ | 171 | 291 | 1064 | 291 | 2111 | 291 | 31571 | 2901 | 4202 | $290 \mid$ | 5246 | 290 |
| 2 | 341 |  | 82 |  | 281 |  | 751 |  | 201 |  | 64 |  |
| 3 | 52 |  | 991 |  | 46 |  | 921 |  | 371 |  | 5281 |  |
| 4 | 69 |  | 1116 |  | 2.1631 |  | 32091 |  | 4255 |  | 98 |  |
| 5 | 87 |  | 34 |  | 81 |  | 27\| |  | 721 |  | 5316 |  |
| 6 | 104\| |  | 1151 |  | 981 | 1 | 44 ! |  | 891 |  | 33 |  |
| 7 | 221 |  | 69 |  | 22161 |  | 3262 |  | 43071 |  | 51 |  |
| 81 | 39 |  | 861 |  | 331 |  | 791 |  | 24 |  | 5368 |  |
| 9 | 1571 |  | 1204 |  | 501 |  | 971 |  | 42 |  | 85 |  |
| 10 | 74 |  | 21 |  | 2268 \| |  | 33141 |  | 4359 |  | 5403 |  |
| 11 | 91 |  | 391 |  | 851 |  | 31 |  | 761 |  | 20 |  |
| 12 | 209 |  | 1256 |  | 23031 |  | 491 |  | 94 |  | 37 |  |
| 131 | 261 |  | 741 |  | 201 |  | 33661 |  | 4411 |  | 5455 |  |
| 14 | 44 ! |  | 91 |  | 381 |  | 84 |  | 291 |  | 72 |  |
| 151 | 61 |  | 1308 |  | 23551 |  | 3401) |  | 46 |  | 90 |  |
| 161 | 279 |  | 26 |  | 731 |  | 18) |  | 44631 |  | 55071 |  |
| 17 | 961 |  | 43 |  | $90 \mid$ |  | 361 |  | SII |  | 24 |  |
| 181 | 3141 |  | 1361 |  | 2407 |  | 34531 |  | 981 |  | 42 |  |
| 191 | 311 |  | 781 |  | 25 |  | 71 |  | 45161 |  | 5559 |  |
| 201 | 491 |  | 961 |  | 42 |  | 881 |  | 331 |  | 77 |  |
| 21 | 3691 |  | 14131 |  | 24601 |  | 35061 |  | 50 |  | 94 |  |
| 22 | 831 |  | 3! |  | 77 |  | 23) |  | 4568 |  | 5611 |  |
| 231 | 401) |  | 48 |  | 951 |  | 40\| |  | 851 |  | 29 |  |
| 24 | 181 |  | 1465 ! |  | 25121 |  | 3558 |  | 4603 |  | 46 |  |
| 29 | 361 |  | 831 |  | 291 |  | 751 |  | $20 \mid$ |  | 56631 |  |
| $26!$ | 4531 |  | 15001 |  | 471 |  | 931 |  | 37 |  | $8: 1$ |  |
| 271 | 71 |  | 18 |  | 2564 |  | 36101 |  | 4655 |  | 98 |  |
| 281 | 881 |  | 35 |  | 821 |  | 281 |  | 721 |  | 5716 |  |
| 29 | 5061 |  | 15531 |  | 991 |  | 451 |  | 90\| |  | 331 |  |
| 301 | 23 |  | 70\| |  | 26171 |  | 62 |  | 4707 |  | 501 |  |

Figure 16: Excerpt of Engel's table of sines with $R=60000$ [Regiomontanus (1490)] (source: The Budapest University of Technology and Economics, 85.211, www.manuscriptorium.com).


Figure 17: Excerpt of Gaurico's table of sines, with $R=100000$ [Regiomontanus (1524)] (source: Google books).


Figure 18: Excerpt of Fine's table of sines [Fine (1530)] (source: Google books).


Figure 19: An excerpt of Apian's table of sines [Apian (1533)] (source: Bayerische Staatsbibliothek).

\footnotetext{
Tabuila Sinuü rectoriri fiue femichordarü minuttim extenfa.


Figure 20: An excerpt of Apian's table of sines [Apian (1534)].


Figure 21: An excerpt of Apian's table of sines [Apian (1541)].


Figure 22: The first page of a manuscript of Regiomontanus's first great table of sines $\left(R=6 \cdot 10^{6}\right)$ (source: Kislak Center for Special Collections, Rare Books and Manuscripts, University of Pennsylvania, LJS 172, ca1476).

|  |  |  |  |  |  |  |  | portio unius z 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | ol ol | $\mid 29$ 1/104715\| | \|29 1/209397/29 | 29 1/314016\| | 129 d | $418540 / 2$ |  |
|  |  | 7845 <br> $2 \|$17491 | [ $\left\|\begin{array}{l}100450 \\ 108205\end{array}\right\|$ | $\left\|\begin{array}{ll}21 & 1 \\ 2 & 141 \\ 21 & 2885\end{array}\right\|$ | $\cdots$ |  | [420281 |  |
|  |  | 3 5336 <br> 4 $69^{82}$ <br>   | 109950 111695 | 214530 | $\begin{array}{\|} 319244 \\ 3209^{5} 7 \\ \hline \end{array}$ |  | $\begin{aligned} & 423563 \\ & 425504 \end{aligned}$ |  |
|  |  | 5727  <br> 6 10472 <br> 6 1 | 113440 115159 | 218118 219873 | 322730 324473 |  | $\left\|\begin{array}{l} 427245 \\ 428996 \end{array}\right\|$ |  |
|  |  | 7 1228 <br> 8 139631 |  | $\left\|\begin{array}{l} 221600 \\ 223351 \end{array}\right\|$ | $\left\|\begin{array}{l} 326216 \\ 32795^{8} \end{array}\right\|$ |  | $\left\lvert\, \begin{aligned} & 430727 \\ & 432457 \end{aligned}\right.$ |  |
|  |  | 9 15709 <br> 10 17454 <br> 10 1959 | $\left\|\begin{array}{l}120420 \\ 122165\end{array}\right\|$ | $\left.\begin{aligned} & 225095 \\ & 2268_{39} \end{aligned} \right\rvert\,$ | $\begin{array}{\|l\|} 329701 \\ 331444 \end{array}$ |  | $\begin{array}{\|l\|} 434208 \\ 435949 \end{array}$ |  |
|  |  | 11 19199 <br> 12 20944 <br> 13 2209 | $\begin{aligned} & 123910 \\ & 125655 \end{aligned}$ | $\begin{aligned} & 2285^{8} 3 \\ & 230327 \end{aligned}$ | $\begin{aligned} & 333187 \\ & 334929 \end{aligned}$ |  | $\left\|\begin{array}{l} 437690 \\ 439430 \end{array}\right\|$ |  |
|  |  |  | 127400 129145 | $\left.\begin{aligned} & 232071 \\ & 233515 \end{aligned} \right\rvert\,$ | $\begin{array}{\|} 336672 \\ 338414 \end{array}$ |  | $\left\|\begin{array}{l} 441171 \\ 442911 \end{array}\right\|$ |  |
|  |  | $\begin{array}{\|l\|l\|} \hline 15 & 26180 \\ 166 & 27925 \\ \hline \end{array}$ | $\begin{aligned} & 130890 \\ & 132635 \end{aligned}$ | $\begin{aligned} & 235559 \\ & 237303 \end{aligned}$ | $\left\|\begin{array}{l} 340157 \\ 341899 \end{array}\right\|$ |  | $\left\|\begin{array}{l} 444652 \\ 446392 \end{array}\right\|$ |  |
|  |  | 17 29671 <br> 18 314161 <br> 1 33169 | 134380 136124 | $\begin{aligned} & 239047 \\ & 2407911 \end{aligned}$ | $\left\|\begin{array}{l} 343 \sigma 42 \\ 345384 \end{array}\right\|$ |  | $\begin{array}{\|l\|} 448133 \\ 449873 \end{array}$ |  |
|  |  | 19 33162 <br> 20 34907 <br> 9 3605 | $\left[\begin{array}{l} 137869 \\ 139614 \end{array}\right]$ | $\begin{aligned} & 242535 \\ & 244279 \end{aligned}$ | $\left\|\begin{array}{l} 347187 \\ 343369 \end{array}\right\|$ |  | $\left\|\begin{array}{l} 451614 \\ 453354 \end{array}\right\|$ |  |
|  |  |  | $\left\lvert\, \begin{aligned} & 141359 \\ & 143104 \end{aligned}\right.$ | $\begin{aligned} & 246023 \\ & 247767 \end{aligned}$ | $\left\|\begin{array}{ll} 350511 \\ 352354 \end{array}\right\|$ |  | $\left\|\begin{array}{l} 455094 \\ 4569_{34} \end{array}\right\|$ |  |
|  |  | 23 40143 <br> 24 41888 <br> 23 4383 | $\left\|\begin{array}{l}144848 \\ 146593\end{array}\right\|$ | $\left\lvert\, \begin{aligned} & 249519 \\ & 251254\end{aligned}\right.$ | $\left\|\begin{array}{l} 354096 \\ 355^{8} 39 \end{array}\right\|$ |  | $\left\|\begin{array}{ll} 458575 \\ 4503 & 15 \end{array}\right\|$ |  |
|  |  | 25 43633 <br> 26 45375 | \|14833 19 | $\left\|\begin{array}{ll} 25^{2} & 299 \\ 254742 \end{array}\right\|$ | $\begin{aligned} & 3575^{80} \\ & 359322 \end{aligned}$ |  | $\begin{aligned} & 462055 \\ & 463795 \end{aligned}$ |  |
|  |  | 27 47123 <br> 28 48869 | 151828 153572 |  | 361064 <br> 362807 |  | $\begin{aligned} & 465535 \\ & 467275 \end{aligned}$ |  |
|  |  | 29 50614 <br> 30 52359 | 155315 157062 | $\begin{aligned} & 259972 \\ & 2617101 \end{aligned}$ | $\begin{array}{\|} 364549 \\ 365291 \end{array}$ |  | $\left\|\begin{array}{l} 469015 \\ 470755 \end{array}\right\|$ |  |
|  |  | 31 54104 <br> 32 $55^{8} 50$ <br> 3 5759 | 158807 160551 | 263460 <br> 265203$\|$ | $\left\|\begin{array}{l}368033 \\ 309775\end{array}\right\|$ |  | $\left\lvert\, \begin{aligned} & 472495 \\ & 474235 \end{aligned}\right.$ |  |
|  |  | $\begin{array}{\|l\|l\|} \hline 33 & 57595 \\ \hline 44 & 59341 \\ \hline \end{array}$ | $\begin{aligned} & 162296 \\ & 164040 \end{aligned}$ | $\left.\begin{array}{\|l\|} 266947 \\ 268690 \end{array} \right\rvert\,$ | $\begin{array}{\|l\|} 371517 \\ 3732599 \end{array}$ |  | $\left.\begin{array}{\|l\|} \hline 475974 \\ 477714 \end{array} \right\rvert\,$ |  |
|  |  | 35 61086 <br> 36 62834 | $\begin{aligned} & 165755 \\ & 167530 \end{aligned}$ | $\begin{aligned} & 270434 \\ & 272178 \end{aligned}$ | $\left\|\begin{array}{l}375001 \\ 376743\end{array}\right\|$ |  | $\begin{aligned} & 479454 \\ & 451194 \\ & \hline \end{aligned}$ |  |
|  |  | 37 64576 <br> 38 66322 | 169274 171019 | $\left.\begin{aligned} & 273921 \\ & 275660\end{aligned} \right\rvert\,$ | $\left\|\begin{array}{l}378485 \\ 3 \text { S0226 }\end{array}\right\|$ |  | $\left\|\begin{array}{l} 482933 \\ 484673 \end{array}\right\|$ |  |
|  |  | $\begin{array}{c\|c\|c\|} 39 & 68067 \\ 40 & 69812 \end{array}$ | $\begin{aligned} & 172763 \\ & 174508 \end{aligned}$ | $\left\|\begin{array}{l} 277408 \\ 279152 \end{array}\right\|$ | $\left\|\begin{array}{l} 381968 \\ 38_{3710} \end{array}\right\|$ |  | $\begin{aligned} & 486412 \\ & 488152 \end{aligned}$ |  |
|  |  | $\begin{array}{\|l\|l\|} \hline 41 & 71557 \\ 42 & 73302 \\ \hline \end{array}$ | $\begin{aligned} & 176253 \\ & 177997 \end{aligned}$ | $\left\|\begin{array}{l} 288395 \\ 282639 \end{array}\right\|$ | $\left\|\begin{array}{l}385452 \\ 387194\end{array}\right\|$ |  | $\begin{aligned} & 439871 \\ & 491631 \end{aligned}$ |  |
|  |  | $\begin{array}{\|l\|l\|} \hline 43 & 75048 \\ 44 & 76793 \\ \hline \end{array}$ | $\left.\begin{array}{\|l\|} 179742 \\ 181486 \end{array} \right\rvert\,$ | $\left\|\begin{array}{l} 284382 \\ 286126 \end{array}\right\|$ | $\left.\begin{array}{r} 388935 \\ 390677 \end{array} \right\rvert\,$ |  | $\begin{aligned} & 493370 \\ & 495110 \\ & \hline \end{aligned}$ |  |
|  |  | $\begin{array}{\|c\|c\|c} 45 & 785 & 8 \\ 46 & 80283 \\ \hline \end{array}$ | $\begin{aligned} & 183231 \\ & 184975 \end{aligned}$ | $\begin{aligned} & 287869 \\ & 289612 \\ & \hline \end{aligned}$ | $\left\|\begin{array}{l} 392419 \\ 394161 \end{array}\right\|$ |  | $\begin{array}{\|} 496849 \\ 498585 \end{array}$ |  |
|  |  | 47 82028 <br> 43 83774 | 186720 1 | $\left\|\begin{array}{l}291355 \\ 293 \\ 009\end{array}\right\|$ | $\left.\begin{aligned} & 395602 \\ & 397644 \end{aligned} \right\rvert\,$ |  | $\begin{aligned} & 500327 \\ & 502067 \end{aligned}$ |  |
|  |  | $\left\lvert\, \begin{array}{ll} 491 & 85519 \\ 501 & 57264 \end{array}\right.$ | $\left\|\begin{array}{l} 190209 \\ 191953 \end{array}\right\|$ | $\left\|\begin{array}{l} 294842 \\ 295585 \end{array}\right\|$ | $\left\|\begin{array}{l\|l\|} 3993^{8} 5 \\ 401 & 127 \end{array}\right\|$ |  | $\left[\left.\begin{array}{l} 503806 \\ 505545 \end{array} \right\rvert\,\right.$ |  |
|  |  |  | $\left\|\begin{array}{l}193697 \\ 195442\end{array}\right\|$ | $\left[\begin{array}{l}298323 \\ 300071\end{array}\right]$ | [ 4028681 |  | $\left\|\begin{array}{c} 507284 \\ 509023 \end{array}\right\|$ |  |
|  |  | 53 92500 <br> 54 94245 <br> 5  | $\left[\left.\begin{array}{l} 197186 \\ 198931 \end{array} \right\rvert\,\right.$ | $\left\|\begin{array}{ll} 301 & 15 \\ 303558 \end{array}\right\|$ | $\left\|\begin{array}{l} 406351 \\ 408093 \end{array}\right\|$ |  | $\begin{aligned} & 510762 \\ & 512501 \end{aligned}$ |  |
|  |  | 55 95990 <br> 56 97735 <br> 50 99 | 200875 202419 | $\left\|\begin{array}{l}305301 \\ 307044\end{array}\right\|$ | $\left\|\begin{array}{l}4 \times 9834 \\ 411575\end{array}\right\|$ |  | $\begin{array}{\|l\|} 514240 \\ 515979 \end{array}$ |  |
|  |  |  | $\begin{aligned} & 204164 \\ & 205908 \end{aligned}$ | $\left\|\begin{array}{l} 303737 \\ 310530 \end{array}\right\|$ | $\left[\begin{array}{l} 413316 \\ 415058 \end{array}\right]$ |  | $\begin{aligned} & 517718 \\ & 519454 \end{aligned}$ |  |
|  |  | $\begin{gathered} 59102970 \\ 60 \mid 104715 \end{gathered}$ | $\left\|\begin{array}{l} 207553 \\ 209397 \end{array}\right\|$ | $\begin{aligned} & 312273 \\ & 314016 \mid 29 \end{aligned}$ | 29 <br> 29 <br> 1410799 <br> $0 / 418540 \mid$ |  | 521195 522934 |  |
| $\begin{array}{\|c\|} \hline \text { SB } \\ \hline \end{array}$ |  |  |  |  |  |  |  |  |

Figure 23: The first page of the first printing of Regiomontanus's first great table of sines $\left(R=6 \cdot 10^{6}\right.$ ) [von Peuerbach and Regiomontanus (1541)] (source: Dresden).


Figure 24: The first page of the first printing of Regiomontanus's second great table of sines ( $R=10^{7}$ ) [von Peuerbach and Regiomontanus (1541)] (source: Dresden).


Figure 25: The first page of Rheticus's table of sines in Copernicus's De lateribus [Copernicus (1542)] (source: Dresden).


Figure 26: The first page of Copernicus's table of sines in the De revolutionibus [Copernicus (1543)] (source: e-rara).

| II Tabula foecunda. |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Gra.\|Part. ${ }^{\text {a }}$ g.\|l |  | Gra.\|Par.x̆q.||Gra.|Par.xq̆. |  |  |  |
| 1\| | 5729799 | 311 | 166429 | 61 | 55432 |
| 21 | 2863563 | ; 21 | 160035 | 621 | 53170 |
| 31 | 1908217 | 331 | 153987 \|| |  | 50952 |
| 41 | 1430203 \|| | 341 | 148253 II |  | 48772 |
| 5 | 1145131 | 351 | 142813 |  | 46631 |
| 61 | 951387 \|| | 361 | 137639 \|1 |  | 44522 |
| 71 | 81445611 | 371 | 132704 \|| |  | 42448 |
| 81 | 71156911 | 381 | 127994 | 681 | 40402 |
| 91 | 631377 \|1 | 391 | 123491 |  | 38387 |
| 101 | 567118 \|| | 401 | 119197 \|| | 701 | 36396 |
| 11] | \$14438 \|1 | 411 | 115037 \|1 | $71 \mid$ | 34433 |
| 121 | 470453 \|| | 421 | 111062 \|| | 721 | 32492 |
| 19 | 433148 | 431 | 107236 | 731 | 30573 |
| 141 | 401089 \|| | 441 | 10355111 | 741 | 28674 |
| 151 | $3732111 \mid$ | 451 | 100000 \|| |  | 26794 |
| 161 | 348748 \|| | 461 | 9657111 | 761 | 24932 |
| 171 | 327088 II | 471 | 9;294 11 | 771 | 23087 |
| 181 | 30776711 | 481 | 9004011 | 781 | 21256 |
| 191 | 290422 \|| | 491 | 86929 \|| | 791 | 19439 |
| 2 Cl | 274753.11 | 501 | 8390911 | 801 | 17633 |
| 21. | 260511 | 51 | 80978 |  | 15838 |
| 221 | 247513 | 52 | 78129 II | 8.21 | 14053 |
| 231 | 239583 | 531 | 753561 | $8^{\prime} 31$ | 12278 |
| 24 | 224607 II | 541 | 72654 H | 841 | 10511 |
| 251 | 2144.50 \|l | 55. | . 7002211 | 8.51 | 8748 |
| 26. | 20503411 | 561 | 67452 If | 861 | 6992 |
| 271 | 196263 \|| | 57 | 6494011 | 87 | 5240 |
| 281 | $198075 \cdot 11$ | 581 | 6248611 | 881 | 3492 |
| 29. | 180402 \|| | 591 | 60086 | 891 | 1745 |
| 301 | 173207 | 601 | 57734 11 | 901 | 0 |

Figure 27: Excerpt of Frisius's table of cotangents [Gemma Frisius (1545)].

## s. Tabula Gnomonica



Figure 28: Excerpt of Frisius's table of arctangents [Gemma Frisius (1545)]. (continued on next page)

## Georgï Peurbacbij.



Figure 29: Excerpt of Frisius's table of arctangents [Gemma Frisius (1545)] (cont'd).
Continuatio paginæ ANTECEDENTIS.


Figure 30: Excerpt of Fine's table of sines [Fine (1550)] (source: Google books).


Figure 31: Excerpt of Rheticus's table of the six trigonometric functions [Rheticus (1551)] (source: Dresden).


Figure 32: An excerpt from Reinhold's table of tangents [Reinhold (1554)] (source: e-rara).


Figure 33: The end of Reinhold's table of tangents, with values every 10 seconds [Reinhold (1554)] (source: e-rara).


Figure 34: An excerpt from Reinhold's table of sines [Reinhold (1554)] (source: e-rara).

|  | - |  | 2 | 3 | 4 |  |  |  | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| m. | Sinus. | Sinus. | Sinus. | Sinus | Sinus. | sinus. |  |  |  |  |
| 0 | 00 | 17 | 3489 | 52 | 697 | 8715 | 10 | 12186 | 13 | 156 |
| 1 | 29 | 1774 | 3519 | 5 | 7004 | 8744 | 1048 | 1221 | 13946 |  |
| 2 | 58 | 1803 | 3548 | 5291 |  | 87 | 10510 | 12 | [ 397 |  |
| 3 | 87 | 1832 | 3577 | 53 | 70 | 8802 | 10639 | 12273 | 14003 | 15 |
| 4 | 16 | 1861 | 3606 | 5349 | 70 | 883 f | 1056 | 12 | 1403 |  |
| 5 | 45 | 1890 | 3635 |  | 71 | 8860 | ros9 | 12331 | 14061 |  |
| 6 | 17 | 19 | 3664 | 54 | 71 | 8889 | 10626 | 12 |  |  |
|  | 20 | 19 | 36 | 54 | 71 | 8918 | 10655 | 123 | 14118 |  |
| 8 | 232 | 19 | 3722 | 546 | 7207 | 8947 | 10684 | 12417 | 14147 |  |
| 9 | 261 | 200 | 3751 | 549 | 7236 | 89 | 10713 | 12446 | 14176 | 15 |
| 10 |  | 203 | 3780 | 55 | 726 | 9005 | 10742 | 12,47 ${ }^{\text {d }}$ |  |  |
| 11 | 319 | 206 |  | 5553 | 7294 | 90 | 10771 | 12504 | 14 | 15 |
| 12 | 349 | 20 | 3838 | 5582 | 7 | 9063 | 1079 | 125 |  |  |
| 13 | 378 | 21 | 3867 |  | 73 ¢2 | 9092 | 108 | 12 | 14291 | 16016 |
| 14 | 407 |  | 3896 | 56 | 7381 | 91 | 10857 | 12591 | 14320 |  |
| 15 | 43 | 218 | 3925 | 5669 | 7.410 | 915 | 10886 | 12619 | 14349 |  |
| 16 | 46 | 2210 | 39 | 5698 | 74 | 9179 | 10915 | 12648 | I4 |  |
| 17 | 494 | 22 | 39 | 5727 | 74 | 9208 | 109 | 126 | 14406 |  |
| 18 | ¢23 |  | 4013 | 575 | 749 |  |  | 12706 | 14435 |  |
| $\underline{19}$ | 552 | 2297 | 40 | 5785 | 7526 | 92 | 110 | 127 | 14 |  |
| 20 | 581 | 232 | 407 |  | 75 | 294 |  | 12764 | , |  |
| 2 I | 6 | 23 | 4 | 5843 | 758 | 932 | 11 | 12793 | 1452I | 16246 |
| 22 | 639 | 238 |  | 5872 | 76 | 9352 |  |  | 14550 | 16371 |
| 23 | 669 | 24 | 4158 | 5901 | 7642 | 93 | 11117 | 12850 | 14579 | 16 |
| 24 | 69 | 24 | 4187 | 5930 | 76 |  | 1114 |  |  | 16 |
| 25 | 727 | 24 | 4216 | 5959 | 7700 | 9439 | 1 I |  | 1463. |  |
| 26 | 7 |  | 4245 | 598 |  | 946 | 11204 |  |  |  |
| 27 | 785 | 253 | 427 | 601 | 7758 | 9497 | 1123 | 129 | 14 |  |
| 28 | 814 | 255 |  | 6046 |  | 9526 | 11262 | 12994 |  |  |
| 29 | 843 | 25 | 4332 | 607 | 781 | 9555 | 1129 | 13023 | 14 |  |
| 30 | 873 |  | 436 | 04 |  | 9584 |  | 13052 |  |  |
| 31 | 901 | 2646 | 4391 | 6133 | 787 | 96 | 11 | 130 | 148 |  |
| 32 | 930 |  |  | 6162 | 79 | 9642 | 11 |  |  |  |
| 33 | 959 | 2704 | 4449 | 6191 | 7932 | 96 | 11407 | 13139 | 14867 | 16 |
| 34 | 989 | 2734 | 44 |  | 796 |  | 1143 |  | 8 |  |
| 35 | 1018 | 2763 | 4507 | 62 | 7990 | 9729 | 11464 | 13196 | 492 |  |
| 361 | 1047 |  | 453 |  | 801 |  | 11493 |  |  |  |
| 37 | 1076 | 282 | 45 | 630 | 804 | 9787 | 522 | 13254 | 14982 |  |
| 38 |  |  | 45 | 63 |  | 9816 | 1 | 13283 |  |  |
| 39 | 11 | 2879 | 4622 | 6366 | 810 | 9845 | 1158 | 13312 | 150 |  |
| 40 |  |  |  |  |  | 9874 |  | $1334{ }^{\circ}$ |  |  |
| 41 | 119 | 2937 | 46 | 6424 | 8164 | 9903 | 1163 | 13;69 | 1509 | 1682 c |
| 42 |  |  |  |  |  |  |  | 13398 |  |  |
| 43 | 12 | 299 | 47 | 6482 | 8222 | 9960 | 1169 | 1;427 | 15154 | 1687 |
| 44 |  |  | 47 |  |  |  |  | 13456 |  |  |
| 45 | 1308 | 3053 | 4797 | 6540 | 828 | 10 | 11753 | 1.3485 | 1521 | 16938 |
| 46 | 1338 | 308 | 4826 |  |  | 10 |  | 13513 |  |  |
| 47 | 1367 | 3112 | 4855 | 6598 | 8338 | 10 | 1181 | 13942 | 15269 |  |
| 48 |  |  | 488 | 6627 |  |  | 118 |  |  |  |
| 49 | 1425 | 3170 | 4914 | 6656 | 8396 | 10134 | 11869 | 13600 | 15327 |  |
| 50 |  |  |  |  | 84 | 1016 |  | 1 |  |  |
| 51 | 1483 |  | 4972 | 6714 | 8454 | 10 | 11927 | 13658 | 1538 |  |
| $5{ }^{2}$ |  |  |  | 6743 | 8483 |  |  | 13686 | I 541 |  |
| 53 | 1541 | 32 | 5030 | 6772 | 8512 | 10250 | 1198 | 13715 | 15442 | 17164 |
| 54 | 1570 |  |  | 680 | 853 | 102 | 120 | 13744 | 15471 |  |
| 55 | 1599 | 3344 | 5088 | 6830 | 8570 | 1030 | 1204 | 13773 | 15499 |  |
| 56 | 1628 | 3373 | $5 \times 1$ |  | 8 | 10337 | 1207 | 13802 | 15528 |  |
| 59 | 1657 | 3402 | 5146 | 6888 | 8628 | 10366 | 121 | 13830 | 15557 | 17278 |
| 58 | 1687 | 3431 | 5175 | 6917 | 865 | 10394 | 1212 | 13859 | 15585 |  |
| 59 | 1716 | 3460 | 5204 | 6946 | 8686 | 10423 | 1215 | 13888 | 15614 | 17336 |
| 60 | 1745 | 3489 | 52331 | 6975 | 8715 | 10452 |  | 13917 | 156 | 1736 |

Figure 35: An excerpt of Bassantin's table of sines [Bassantin (1557)].


Figure 36: An excerpt of Gaurico's sine table with the heading tabula fecunda [Gaurico (1557)].

# Tabella fęcünda per Campinaūum 

 Nouarienfem iamdiu fupputata

Figure 37: Gaurico's table of tangents [Gaurico (1557)].


Figure 38: An excerpt of Gaurico's sine table [Gaurico (1557)].


Figure 39: Maurolico's table of sines [Maurolico (1558)] (source: Österreichische Nationalbibliothek).


Figure 40: Maurolico's table of tangents [Maurolico (1558)] (source: Österreichische Nationalbibliothek).

TABELLA BENEFICA 66


Figure 41: Maurolico's table of secants [Maurolico (1558)] (source: Österreichische Nationalbibliothek).

Tabula Sinuumad 10000000 .particulateomputater $T$

| \% | Sinus | portio unig 2 10 |  | portio uni9 2 10 | $\left\lvert\, \begin{gathered} 2 \\ \text { Sinus } \end{gathered}\right.$ | $\left\lvert\, \begin{array}{r} \text { portio } \\ \text { uaig } 2 \\ 10 \\ \hline \end{array}\right.$ | $\begin{array}{c\|} 3 \\ \text { Sinus } \end{array}$ |  | Sinus | Yportio |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ol | ol |  | 174524\| | 485 | \|348995| | 484 | \|523360| | 484 | 697565 | 48.4 |
| 1 1 1 | $\begin{array}{r} 2909 \\ 58.8 \end{array}$ | 485 | $\left.\begin{array}{\|} 177433 \\ 180341 \end{array} \right\rvert\,$ |  | $\left\|\begin{array}{l} 351902 \\ 354809 \end{array}\right\|$ |  | $\left\lvert\, \begin{aligned} & 526265 \\ & 52927\end{aligned}\right.$ |  | $\left\|\begin{array}{l} 700467 \\ 703369 \end{array}\right\|$ |  |
| 3 4 | $\begin{array}{r} 8727 \\ 11636 \\ \hline \end{array}$ |  | $\left\lvert\, \begin{aligned} & 183250 \\ & 186158 \end{aligned}\right.$ |  | \| $\begin{aligned} & 357716 \\ & 360623\end{aligned}$ |  | $\left\|\begin{array}{l} 532075 \\ 534980 \end{array}\right\|$ |  | $\left\lvert\, \begin{aligned} & 706270 \\ & 709172 \end{aligned}\right.$ |  |
| 5 | $\begin{aligned} & 14544 \\ & 17453 \end{aligned}$ |  | $\left\|\begin{array}{l} 189066 \\ 191975 \end{array}\right\|$ |  | $\left[\begin{array}{l} 3635301 \\ 3664371 \end{array}\right.$ |  | $\left\|\begin{array}{l} 537884 \\ 540789 \end{array}\right\|$ |  | $\begin{aligned} & 712073 \\ & 1714975 \end{aligned}$ |  |
| $\begin{aligned} & 7 \\ & 8 \\ & 8 \end{aligned}$ | $\begin{aligned} & 20362 \\ & 23271 \end{aligned}$ |  | $\left\|\begin{array}{l} 494883 \\ 197792 \end{array}\right\|$ |  | $\left\|\begin{array}{l}369344 \\ 3>2251\end{array}\right\|$ |  | $\left\|\begin{array}{l}543694 \\ 546598\end{array}\right\|$ |  | $\left\|\begin{array}{l}717876 \\ 720777\end{array}\right\|$ |  |
| $\begin{array}{r} 9 \\ 101 \end{array}$ | $\begin{aligned} & 26180 \\ & 29088 \\ & \hline \end{aligned}$ |  | $\begin{aligned} & 200700 \\ & 203608 \end{aligned}$ |  | $\left\|\begin{array}{l}375158 \\ 378064\end{array}\right\|$ |  | $\left\|\begin{array}{c}549503 \\ 552407\end{array}\right\|$ |  | $\left\|\begin{array}{l}723678 \\ 726579\end{array}\right\|$ |  |
| 121 | $\begin{aligned} & 319971 \\ & 34906 \end{aligned}$ |  | $\left\lvert\, \begin{aligned} & 206527 \\ & 209425 \end{aligned}\right.$ |  | $\left\|\begin{array}{l}380971 \\ 388878\end{array}\right\|$ |  | $\left\|\begin{array}{l}555312 \\ 558216\end{array}\right\|$ |  | $\binom{729480}{732381}$ |  |
| $\begin{array}{r} 13 \\ 3 \\ \hline \end{array}$ | $\begin{aligned} & 37815 \\ & 40724 \end{aligned}$ |  | $\left\lvert\, \begin{aligned} & 212333 \\ & 215 \\ & 2\end{aligned} 1\right.$ |  | $\left\|\begin{array}{l}386785 \\ 389692\end{array}\right\|$ |  | $\left\|\begin{array}{l}561120 \\ 564924\end{array}\right\|$ |  | $\left\|\begin{array}{l} 735282 \\ 738183 \end{array}\right\|$ | $1483$ |
| $\begin{aligned} & 15 \\ & 16 \end{aligned}$ | $\begin{aligned} & 43632 \\ & 46541 \\ & \hline \end{aligned}$ |  | $\left\lvert\, \begin{aligned} & 218149 \\ & 221057\end{aligned}\right.$ |  | $\left\|\begin{array}{l}392598 \\ 395505\end{array}\right\|$ |  | $\left\|\begin{array}{l}566928 \\ 569832\end{array}\right\|$ |  | $\left[\begin{array}{l} 741084 \\ 743985 \end{array}\right.$ |  |
| $\begin{aligned} & 27 \\ & 28 \end{aligned}$ | $\begin{aligned} & 49450 \\ & 52359 \end{aligned}$ |  | $\left\|\begin{array}{l} 223969 \\ 226873 \end{array}\right\|$ |  | $\left\|\begin{array}{l}398412 \\ 401318\end{array}\right\|$ |  | $\left\|\begin{array}{l} 572736 \\ 575640 \end{array}\right\|$ |  | $\left\|\begin{array}{l} 746886 \\ 749787 \end{array}\right\|$ |  |
| $\begin{array}{r} 19 \\ 20 \end{array}$ | 55268 <br> 58177 |  | $\left\|\begin{array}{l}2 \times 9781 \\ 232689\end{array}\right\|$ |  | $\left\|\begin{array}{l}404225 \\ 407131\end{array}\right\|$ |  | $\left\|\begin{array}{l}578,544 \\ 58.448\end{array}\right\|$ |  | $\mid 7526881$ |  |
| $\begin{aligned} & 21 \\ & 22 \end{aligned}$ | 61086 63995 |  | $\left[\begin{array}{l} 235597 \\ 238505 \end{array}\right]$ |  | $\left\|\begin{array}{l}410038 \\ 412944\end{array}\right\|$ |  | $\left\|\begin{array}{l}584352 \\ 587256\end{array}\right\|$ |  | $\left\|\begin{array}{l} 758489 \\ 761389 \end{array}\right\|$ |  |
| 23 <br> 24 <br> 21 | $\begin{array}{r} 66904 \\ 69813 \\ \hline \end{array}$ |  | $\begin{array}{\|l\|} 241423 \\ 244321 \end{array}$ |  | $\left\|\begin{array}{l}415852 \\ 418757\end{array}\right\|$ |  | $\left\|\begin{array}{l}590160 \\ 593064\end{array}\right\|$ |  | $\left\|\begin{array}{l} 764290 \\ 767180 \end{array}\right\|$ |  |
| 25 <br> 26 <br> 27 | $\begin{aligned} & 72722 \\ & 75630 \\ & \hline \end{aligned}$ |  | $\begin{array}{\|l\|} 247229 \\ 250137 \end{array}$ |  | $\left\|\begin{array}{l}421663 \\ 424 \\ 4\end{array}\right\|$ |  | $\left\lvert\, \begin{aligned} & 595967 \\ & 598872\end{aligned}\right.$ |  | $\begin{aligned} & 770090 \\ & 772990 \end{aligned}$ |  |
| 27 <br> 28 <br> 2 | $\begin{array}{\|} 78539 \\ 8.448 \\ \hline \end{array}$ |  | $\left\|\begin{array}{l} 253045 \\ 255953 \end{array}\right\|$ |  | $\left\|\begin{array}{l}427476 \\ 430382\end{array}\right\|$ |  | $\left\|\begin{array}{l} 601775 \\ 604678 \end{array}\right\|$ |  | $\left\|\begin{array}{l} 775891 \\ 778791 \end{array}\right\|$ |  |
| $\begin{array}{r} 29 \\ 30 \\ \hline \end{array}$ | $\begin{aligned} & 84357 \\ & 87265 \\ & \hline \end{aligned}$ |  | $\begin{array}{\|l\|} 258861 \\ 262769 \end{array}$ |  | $\left\|\begin{array}{l} 433288 \\ 436194 \end{array}\right\|$ |  | $\left\lvert\, \begin{aligned} & 607582 \\ & 610485 \end{aligned}\right.$ |  | $\left\|\begin{array}{l} 781691 \\ 784591 \end{array}\right\|$ |  |

Figure 42: Eisenmenger's table of sines [Eisenmenger (1562)].

TABVLA YOECVNDA.

| $\therefore$ Numerus |  | \| Numerus | |  | Numerus |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| G | \| G | |  | G | 1 |  |
| $0 \mid 000001$ | 131 | 60086 | \|61] | 1804021 |  |
| 1 1745 <br> 2 3492 | $\left\|\begin{array}{l}32 \\ 3\end{array}\right\|$ | 62486 64940 | $\left.\begin{aligned} & 62 \\ & 63\end{aligned} \right\rvert\,$ | 1880751 |  |
| 3 5240 <br> 4 6992 | $\left\|\begin{array}{l}34 \\ 35\end{array}\right\|$ | $1 \begin{aligned} & 67452 \\ & 70022\end{aligned}$ | $\left\|\begin{array}{l}64 \\ 65\end{array}\right\|$ | 205034 214450 |  |
| 5\|r|r|r|r| | $\left\|\begin{array}{l}36 \\ 37\end{array}\right\|$ | $1 \begin{aligned} & 72654 \\ & 75356\end{aligned}$ | 661 6 | 224697 235583 |  |
| $7\left\|\begin{array}{l\|l\|l\|}12278 \\ 8 & 14053\end{array}\right\|$ | $\cdot\left\|\begin{array}{l}38 \\ 39\end{array}\right\|$ | 78129 <br> 80978 | $\left\|\begin{array}{l}68 \\ 69\end{array}\right\|$ | 247513 260511 |  |
| 9 15838  <br> 10 1 7633 | $\left\|\begin{array}{l}40 \\ 41\end{array}\right\|$ | 83909 86929 | $\left\|\begin{array}{l}70 \\ 78\end{array}\right\|$ | 274753 290422 |  |
| 11 19439 <br> 12 21256 | $\left\|\begin{array}{l}42 \\ 43\end{array}\right\|$ | $\left\lvert\, \begin{aligned} & 90040 \\ & 93254\end{aligned}\right.$ | $\left\|\begin{array}{l}7.2 \\ 73\end{array}\right\|$ | 307767 327088 |  |
| 13 2087 <br> $14\|24932\|$  | $\left\|\begin{array}{l}44 \\ 45\end{array}\right\|$ | $\left\|\begin{array}{l}96571 \\ 100000\end{array}\right\|$ | 74 <br> 75$\|$ | 348748 373211 |  |
| 156794  <br> 16 28674 | $\left\|\begin{array}{l}46 \\ 47\end{array}\right\|$ | $\left\|\begin{array}{l}103551 \\ 107236\end{array}\right\|$ | $\left\|\begin{array}{l}76 \\ 77\end{array}\right\|$ | 401089 <br> 433148 |  |
| 17 30573 <br> 18 32492 | 48 <br> 49 | $\left\|\begin{array}{l}1115062 \\ 115037\end{array}\right\|$ | $\left\|\begin{array}{l}78 \\ 79\end{array}\right\|$ | 470453 <br> 514438 |  |
| 19 34433 <br> 20 363961 | $\left\|\begin{array}{l}50 \\ 51\end{array}\right\|$ | $119177 \mid$ <br> 123491 | $\left\|\begin{array}{l}80 \\ 81\end{array}\right\|$ | 567118 <br> 631377 |  |
| 21 8387 <br> 22 $40402 \mid$ | $\left\|\begin{array}{l}5 \\ 53\end{array}\right\|$ | $\left\lvert\, \begin{aligned} & 127994 \\ & 132704\end{aligned}\right.$ | $\left\|\begin{array}{l}82 \\ 83\end{array}\right\|$ | 711569 <br> 814456 |  |
| 23 $\|$4248  <br> 24 44522 | $\left\|\begin{array}{l}54 \\ 55\end{array}\right\|$ | 137639 <br> 142813$\|$ | 84 <br> 85 | $\left\|\begin{array}{r}951387 \\ 1143138\end{array}\right\|$ |  |
| $25 \mid 46631$ $26\left\|\begin{array}{l\|l\|}46772\end{array}\right\|$ | $\left\|\begin{array}{l}56 \\ 57\end{array}\right\|$ | $\left\|\begin{array}{l}1488253 \\ 1539\end{array}\right\|$ | 86 87 | 1430203 <br> 190821$\|$ | $\cdots$ |
| 27 50952 <br> 28 53470 | $\left\|\begin{array}{l}58 \\ 59\end{array}\right\|$ | $\left\|\begin{array}{l}160035 \\ 166429\end{array}\right\|$ | 88 89 | $\left\|\begin{array}{l}2863563 \\ 5729796\end{array}\right\|$ |  |
| $\begin{array}{l\|l\|l\|} \hline 29543^{2} \\ 30 & 57734 \\ \hline \end{array}$ | 160] | 173207 |  | $\left\|\begin{array}{c}\text { unfinituma } \\ 1\end{array}\right\|$ |  |

Figure 43: Excerpt of Schreckenfuchs's table of tangents [Schreckenfuchs (1569)].

| Arcts. 1 Sinus. |  |
| :---: | :---: |
| G. $\mathrm{M}^{\text {. }}$ | par. |
| 0.1251261 |  |
| - \|301523 |  |
| -1451785 |  |
| $\underline{1} 10.11047$ |  |
| $\pm \mid 1511308$ |  |
| 13011570 |  |
| 1.1451;832 |  |
| 2\|012093 |  |
| 211512355 |  |
| 213012617 |  |
| $2\|45\| 287^{8}$ |  |
| 310 /3140 |  |
| 3\|15|3,401 |  |
| 313013652 |  |
| 314513924 |  |
| 41014185 |  |
| 411514446 |  |
| 413014707 |  |
| 414514968 |  |
| 51015229 |  |
| 511915490 |  |
| 513015750 |  |
| 914916016 |  |
| 61016271 |  |
| 611516532 |  |
| 613016792 |  |
| 614517052 |  |
| 71017312 |  |
| 711917571 |  |
| 713017831 |  |
|  |  |
| 81018350 |  |
| $8\|15\| 8609$ |  |
| $8 \mathrm{~B} / 301886$ |  |
| 8 45197 <br> 9 191938 |  |
| 910 9386 |  |
| 9.\|15|9644 |  |
| 9 13.019902. |  |
| 9145110160 |  |
| 1010 10418 |  |
| 10119110676 |  |
| cotz0110934 |  |
| 10\|45|11591 |  |
| 1110\|11448 |  |
| 1115111075 |  |


| Arate I | Simus. |
| :---: | :---: |
| G. $\|\mathrm{M} \cdot\|$ | par. |
| 11/30\| | 11962 |
| $\underline{1515}$ | 12218 |
| 12101 | $1{ }^{12474}$ |
| 12(15) | 12730 |
| 121301 | 12086 |
| $12 \mid 451$ | 113241 |
| 3310 | 18397 |
| $13\|15\|$ | \|13752 |
| 13\|301 | 114006 |
| 1410 | 14515 |
| 14\|15! | 14769 |
| 1413.01 | 15022 |
| 14 \| 49.1 | 15276 |
| 1510 | 15529 |
| 15\|15| | 15781 |
| $15130 \mid$ | 16034 |
| 15145 | 16286 |
| 1610 | 16538 |
| 161151 | 16789 |
| 161301 | 17040 |
| 16\| 451 | 17291 |
| 1710 | 117541 |
| ${ }_{17} 1151$ | 17792 |
| 171301 | 18042 |
| $17 / 451$ | 18291 |
| 18101 | 18545 |
| 18\|151 | 18789 |
| 181301 | 19038 |
| 18\|451 | 19286 |
| 1910 | 119534 |
| 19\|151 | 19781 |
| 19\|301 | 20028 |
| 191451 | 20275 |
| 2010 | 20521 |
| 20\|151 | 20767 |
| 201301 | 21012 |
| 201451 | 21297 |
| 210 | 21502 |
| 2115 | 21746 |
| 211301 | 21990 |
| 211451 | 22233 |
| 2210 | 22476 |
| 22\|15|z | 22718 |
| 221301 | 2.2961 |
| $22 \mid 451$ | 23202 |


| Arcm. \| Sinut. | Arcun. \| Simus. |
| :---: | :---: |
| G. [M. \| par. | G. \|M. | par. |
| 2310 23443 <br> 2315  | 34115133768 |
| 23\|15|23684 | 34\|30133934 |
| 23\|30|23934 | 34\|45|34199 |
| 23\|45|24164 | 3510\|34414 |
| $2410 \mid 24404$ | 35\|151.3628 |
| 24\|15124643 | 35130134842 |
| 24/30\|2+881 | 35145135054 |
| 24\|45|25119 | 3610135267 |
| 2510 125357 | 36145135478 |
| 25115129594 | 36130135689 |
| 25130\|25830 | 36145135899 |
| $25 \mid 45126066$ | 3710136189 |
| 2610126302 | 3715136317 |
| 26\|15126537 | .37130136525 |
| 26\|30|26771 | 37145136733 |
| 26145127005 | $3 8 1 0 \longdiv { 3 6 9 3 9 }$ |
| 2710 [27239 | 38145137145 |
| 27115127472 | 38130137350 |
| 27130127704 | 38145137555 |
| 27\|45|27936 | 3910137759 |
| 2810.28168 | 39115137962 |
| 28\|1512839.9 | 39130138164 |
| 28\|30128629 | 39149138366 |
| 28/45/28858 | 4010138567 |
| 2910\|29088 | 40119\|38767 |
| 29\|15129317 | 40/30138966 |
| 29130129545 | 40149139165 |
| 29145129742 | 410139363 |
| 3010130000 | 41115139560 |
| 30175130226 | 4130139757 |
| 30130130452 | 41145139952 |
| 30145\|30677 | 4210140147 |
| 3110130902 | 42115140342 |
| 31\|15|31126 | 42\|30140535 |
| 31\|30|31349 | 42145140728 |
| [31\|45|31972 | 4310140919 |
| 3210 131799 | 43115141110 |
| 32\|15132016 | 43130/44301 |
| 32\|30|32237 | 43145141449 |
| $32 \mid 45132458$ | 4410 141679 |
| 3310132678 | 44\|15141867 |
| 33\|15|328971 | 49\|30142024 |
| 33/30133116 | 44145142240 |
| 33\|45|33334 | 4510142426 |
| 34\|0|33541 | 45/15\|42611 |

Figure 44: Excerpt of Schreckenfuchs's first table of sines [Schreckenfuchs (1569)].

## Tabula finuum rectorum.

|  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | 1 s | 101 | \|151 | 142130 |
| 110 | 3.11 | 10 | 1311 | 12510 |
|  | 45 | 1 |  | 17130 |
| $\frac{11}{11}$ |  | 1 | 12 | 15010 |
|  | 15 | I 1 | 181 | 13210 |
| 11 | 301 | 1. | 1341 | 41410 |
| 11. | \|4511 | $1 \cdot$ |  | $156 \%$ |
| 112 | 1011 | 121 | Is | 13810 |
|  | 151 | 2 | [21\| | 12010 |
| $\frac{112}{112}$ | 30 | 12 |  | 1210 |
| 12 | 441 | 12 |  | /43/30 |
| 13 | - | 13 | 18 | 124130 |
| 13 | \|isil | 13 |  | 15130 |
| 13 | 130 | 13 | 1391 | 146130 |
| 113 | 145 | 13 | \|ssi | I271 |
| 114 | 1 11 | 4. | \| 11 | 17130 |
| $\underline{114}$ | 1.51 | 14 | 126 | 147130 |
| 114 | las 11 | 14 | 1581 | 17130 |
| IIs | 10 | Is | \| 231 | 146130 |
| 15 | 19 | \% | $\|29\|$ | 12410 |
| 115 | 13011 | Is | 1451 | 12 |
|  | 1411 | 16 | 10 | 140 |
| 116 |  | 16 |  | 18810 |
| 116 | 1.51 |  |  | Issio |
| 116 | 130 | 161 |  | 132\% |
| 116 | 1451 | 17 | 131 | 1810 |
| 117 | 1011 | 17 | \|181 | 143130 |
| 17 | 1 s | 171 | 1341 | 41910 |
| $\frac{17}{17}$ | 301 | 17 |  | 153130 |
| 117 | 45111 | 18 | Is 12 | 12810 |
| $\frac{118}{118}$ | $1 \cdot$ | 18 | [21\| | 12130 |
|  | 1511 | 81 | 136] | 134130 |
| $\frac{118}{118}$ | 13011 |  | 1521 | 17 |
| 118 | 4511 |  | 17 | 138130 |


|  |  |
| :---: | :---: |
| 119 10 119 12319 |  |
| 1.5 | 38140 |
| 1191301 | \|s4100 |
|  |  |
| Il1010 11 | 1012¢18 |
|  |  |
| 11.01301 | 1015613 |
|  |  |
| 11.1011 | 11]26\|54| |
|  |  |
| $\frac{\|11\| 30\|\|11\| 57\| 43 \mid 30}{\|14\| 4\|\|12\| 13\| 6 \mid 30}$ |  |
|  |  |
| $\frac{115210}{112122] 281290^{\circ}}$ |  |
|  |  |
| \|122|30|122159|rio |  |
| $\underline{112\|45\| \mid 12314430130}$ |  |
| IIrglo II | 13129149 |
|  |  |
| $\underline{123\|30\|\|14\| 0\|24\|^{\circ}}$ |  |
| $11314511 \leq 41 \times 14010$ |  |
| 11.410 11 | 14130155 |
| 11.415 511441469910 |  |
| [14 43011251 12210 |  |
| \|144|4s|1151.613410 |  |
|  |  |
|  |  |
|  |  |
|  |  |
| [11610 \| $116132 \mid 17130$ |  |
| $\\|\left. 26\|15\| 126\right\|_{4712310}$ |  |
|  |  |
| \|16|45|12717130130 |  |
| $11170\|127132132\| 30$ |  |
| 1771.5\|1 | 1747133 |

Figure 45: Excerpt of Schreckenfuchs's second table of sines [Schreckenfuchs (1569)].


Figure 46: Excerpt of Witekind's table of sines [Witekind (1576)].
44. Tabula finuum refforum fiue femichorda* rum minutiom extenfa.

|  | 0 | I | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| m. | simus. | Sinus. | Sinus. | Sinus. | sinus. |
| 0 | 00 | 1745 | 3489 | 5233 | 6975 |
| 1 | 29 | 1774 | 3519 | 5262 | 7004 |
| 2 | 88 | 1803 | 3548 | 5291 | 7037 |
| 3 | 87 | 18*2 | 3577 | 5320 | 7062 |
| 4 | $1 \times 6$ | 1861 | 3606 | 5349 | 7091 |
| 5 | 145 | 1890 | 3635 | r378 | 7120 |
| 6 | 174 | 1919 | 1664 | 5407 | 7149 |
| 2 | 203 | 1948 | 3693 | [436 | 7178 |
| 8 | 232 | 19.77 | 3722 | 5465 | 7207 |
| 9 | 261 | 2007 | 3751 | 1405 | 72;6 |
| 10 | 290 | 2036 | 3780 | TI 84 | 7265 |
| 11 | 319 | 2065 | \$809 | T5S | 7294 |
| 12 | 349 | 2094 | 3838 | 5582 | 7.23 |
| 13 | 7.83 | 2123 | 3867 | 56, | 7352 |
| 14 | 407 | 2452 | 3896 | 5640 | 7381 |
| 45 | 436 | 2181 | 39.25 | 5669 | 7410 |
| 46 | 465 | 2210 | 3955 | 5698 | 7439 |
| 47. | 494 | 2239 | 3984 | 5727 | 7468 |
| 18 | 523 | 2268 | 4013 | 5756 | 7.497 |
| 19 | 552 | 2297 | 4042 | 5785 | 7526 |
| 20 | 581 | 2326 | 4071 | 5814 | 7555 |
| 27 | 610 | 2355 | 4100 | 5843 | 7584 |
| 22 | 639 | 2385 | 4129 | 5872 | 7613 |
| 23 | 669 | 2414 | 4158 | 5901 | 76.42 |
| 2 | 698 | 2443 | 4187 | 5930 | 7671 |
| 25 | 747 | 2472 | 4216 | 5959. | 77.00 |
| 26. | 756 | 2501 | 4245 | 5988 | 7729 |
| 2 | 785 | 2530 | 4274 | 50.17 | 7758 |
| 28 | 814 | 2515 | 4303 | 6046 | 7787 |
| 29 | 843 | 2588 | 43.32 | 6075 | 7816 |
| 30 | 872 | 2617 | 4361 | 6104 | 7845 |

Figure 47: Excerpt of Peucer's table of sines [Peucer (1579)].


Figure 48: Excerpt of Viète's Canon mathematicus [Viète (1579)] (source: e-rara).
C. ANONSINVVM.


Figure 49: Excerpt of Bressieu's table of sines [Bressieu (1581)] (source: Google Books).


Figure 50: Excerpt of Bressieu's table of tangents (odd columns) and secants (even columns) [Bressieu (1581)] (source: Google Books). Note that the faded parts are artefacts of the way Google Books stores images.


Figure 51: An excerpt from Giuntini's table of sines [Giuntini (1581)] (source: Google books).

## Pars Prima.

## 23

Sinuum reGorutabula.m


Sinuum

Figure 52: An excerpt from Padovani's table of sines [Giuntini (1581)] (source: Google books).


Figure 53: An excerpt from Fincke's table of tangents [Fincke (1583)] (source: e-rara).


Figure 54: An excerpt from Fincke's table of sines [Fincke (1583)] (source: e-rara).


Figure 55: An excerpt from Fincke's table of secants [Fincke (1583)] (source: e-rara).


Figure 56: An excerpt from Clavius's table of sines [Clavius (1586)].


Figure 57: An excerpt from Clavius's table of tangents [Clavius (1586)].


Figure 58: An excerpt from Clavius's table of secants [Clavius (1586)].
arcuum eiufdem Quadrantis


Figure 59: Another excerpt from Clavius's table of secants [Clavius (1586)].


Figure 60: An excerpt of Bürgi's table of sines (1587). Bürgi's Fundamentum Astronomix manuscript is kept at the Biblioteka Uniwersytecka Wrocław, under call number IV Qu 38a. This excerpt of Bürgi's sine table was provided by Dieter Launert and is included in LOCOMAT (http://locomat.loria.fr) with permission.
Incipit tabula.

|  | 0 | 1 | 2 |  | $4$ | $5$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| M. |  |  |  |  |  |  |
| 2 | 17 34 | 1064 1082 | 2111 2128 | 3157 3175 | 4202 4220 | 5246 5264 |
| 3 | 52 | 1099 | 2146 | 3192 | 4237 | 5281 |
| 4 | 69 | 1116 | 2163 | 3209 | 4255 | 5298 |
| 5 | 87 | 1134 | 2181 | 3227 | 4272 | 5316 |
| 6 | 104 | 1 I | 3198 | 3244 | 4289 | 5333 |
| 7 | 122 | $\pm 169$ | 2216 | 3262 | 4307 | 5351 |
| 8 | 139 | 1186 | 2233 | 3279 | 4324 | 5368 |
| 9 | 157 | 12 | 2250 | 3297 | 4342 | 5385 |
| 10 | 174 | 12 | 2268 | 3314 | 4359 | 5403 |
| 11 | 191 | 1239 | 2285 | 3331 | 4376 | 5420 |
| 12 | 209 | 1256 | 2203 | 3349 | 4394 | 5437 |
| 13 | 226 | 1274 | 2320 | 3366 | 4411 | 5455 |
| 14 | 244 | 1291 | 2338 | 3384 | 4429 | 5472 |
| 15 | 261 | 1308 | 2355 | 3401 | 4446 | 5490 |
| 16 | 279 | 1326 | 2373 | 3418 | 4463 | 5507 |
| 17 | 296 | 1343 | 2390 | 3436 | $44^{81}$ | 5524 |
| 18 | 314 | $\underline{361}$ | 2407 | 3453 | 4498 | 5542 |
| 19 | 331 | $\times 378$ | 2425 | 3471 | 4516 | 5559 |
| 20 | 349 | 1396 | 2442 | 3488 | 4533 | 5577 |
| 21 | 366 | 1413 | 2460 | 3506 | 4550 | 5594 |
| 22 | 383 | 1431 | 2477 | 3523 | 4568 | 5611 |
| 23 | 401 | 1448 | 2495 | $354{ }^{\circ}$ | 4585 | 5629 |
| 24 | 418 | 1465 | 2512 | 3558 | 4603 | 5646 |
| 25 | 436 | 1483 | 2529 | 3575 | 4620 | 5663 |
| 26 | 453 | 1500 | 2547 | 3593 | 4637 | 5681 |
| 27 | 471 | 1518 | 2564 | 3610 | 4655 | 5698 |
| 28 | 488 | 1535 | $2582$ | 3628 | 4672 | 5716 |
| 29 | 506 | 1553 | 2599 | 3645 | 4690 | 5733 |
| 30 | 523 | 1570 | 2617 | 3662 | 4707 | 5750 |
|  |  |  |  |  |  |  |

Figure 61: An excerpt from Gallucci's table of sines [Gallucci (1588)] (source: Google books).


Figure 62: An excerpt from Lansberge's table of sines [van Lansberge (1591)] (source: e-rara).


Figure 63: An excerpt from Lansberge's table of tangents [van Lansberge (1591)] (source: e-rara).


Figure 64: An excerpt from Lansberge's table of secants [van Lansberge (1591)] (source: e-rara).

| $\bigcirc$ |  |  | I | 2 |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Primus | dus | Primus Secandus | Primus | Scundus |
| 1 | -0, 0 | 100300,00 | 1745, $4 \div 99984.77$ | 3899959 | 99939,08 60 |
| $\cdots$ | -29rog | \%99099. $99^{\circ}$ | 17743999084.26 | 351908 | 99 $3^{8}$ o6 139 |
| 3. | ${ }^{5} 8 \times 18$ | 994099, 981 | $1803 \cdot 4 \pm 99983 \cdot 7+1$ | $35+8, \mathrm{cb} 9$ |  |
| $\frac{3}{4}$ | 87, 27 | 99999,96 | 1832. $50,99933.21$ | .3577, 16.4 | 2935,99 ${ }^{57}$ |
| 4 | 41513 | 999999, 93 | 1861,58999332,67 | \%600, 2j 9 | 993+ 95 |
| 5 | 145.44 | 99999.89 |  | 1635,3019 | 9933.90 |
|  | 174, 53 | 9 906 | To407599981. $\chi^{2}$ | $13684 \times 379$ | 99332, $8+$ |
| 7 | 203,62 | 29999 | $19.88 ; 83.99981,01$ | 3693,4t9 | 99931,77 |
|  | 232,74 | 99999.7 ${ }^{1}$ | 1977,92.99980,44 | 3722, 56 | 99930,69 |
| $\frac{9}{10}$ | 361, 88 | 99999, 65 | 2007 $00.98979: 86$ | 375, 58 | 29999:6 |
|  | 290, 88 | \$9999, 57 | 2036.08;99979.27, | . 7780.64 | 99938, $0^{\circ}$ |
|  | 319,97 | 99999. 48 | 2061 17 99978, 67 | 3809, 719 | 99927 ${ }^{+0}$ |
| 12 | 549, 06 | 99999.38: | 2094, 25:99978,06 | 38,8878 | 99926, 29 |
| 13 | 378,35 | 99898:27: | :123, $23.399977,45$ | 3867,85 | 98925, 1787 |
|  | - $407{ }^{15} 4$ | 99299, 19 | $2152,41759978,83$ | 3876, 5 \% | 999220, 0 ¢ 96 |
| $\frac{15}{16}$ | 4 $4 \sqrt{613} 3$ | 299999194 |  | : 3924.989 |  |
| 16 | 469, 41 | 99498, 91 | 2210, 57 99923, 56 | 3955,05 | 99921,75 + + |
| $\|17\|$ | 49+, 50 | \%9998, 71 | 2239.65, $9997+.98$ | 3987\% | 999:0, 0 |
|  | 523.19 55268 | 99998, 63 | $2268,73499874,35$ | 4013,18 | 99915,4 4 |
|  | 552,68 | 99998, 46 | 2297. 81.99973 .59 | 40, 2 2, 25 | 99918, 27 |
|  | 581,77 <br> 610.86 | 99998,30 | 2326, 89,99972,92 | 4071, 31 | 99917,09 |
|  |  | 99998, 13 | 2355,97 99973,24 | 4100,38 | 99915,90 39 |
| 23 | 639, 95 | 99997, 95 | 2585,0; 99971,55 | 4129,44 | 99914, 701318 |
|  | 698,13 | 99997, 989 |  | $\frac{41 ; 8,51}{4^{18} 7,57}$ | 99913, 49.37 |
| $25$ | 727.21 | 999997. 35 | 2472, 29 999969, +3 | ${ }_{4}+16,63$ | 99911,06 |
| 26 | 756, ${ }^{\circ}$ | 99997, 13 | 2501,3799968,7 | $44^{245} .70$ | [999a9, 83 3 $3+$ |
|  | 785, 39 | 99996,91 | 2530, 4599967,98 | +274 70 | 99908. 59. |
| 28 | 814,48 8 8 | 99996,68 | 2559, 53999067,2 | 4303, 82 | 99907, $3+$ |
| 29 | 8+3,57 | 99996, 4 + | 25886199956,44 | 4332, 88 | 99906,08 |
| 30 | 872,65 | 99996. 19, | 2617,69 99965,7 | 4365,9 | 99904,82330 |
|  | Secandus | s Primus | securdies Pimus | secunda | Primus ${ }^{\text {a }}$ |
|  | 49 |  | 88 | 87 |  |

Figure 65: An excerpt from Magini's table of sines [Magini (1592)].


Figure 66: An excerpt from Magini's table of tangents [Magini (1592)].
TABELA


Figure 67: An excerpt from Magini's table of secants [Magini (1592)].


Figure 68: An excerpt from Clavius's table of sines [Clavius (1593)].

| $\left\|\frac{D}{M}\right\|$ | $\frac{\mathrm{O}}{\text { Parts }}$ | $\frac{\mathrm{T}}{\text { Pavts }}$ | $\frac{2}{\text { Parts }}$ | $\frac{3}{P_{a \gamma t s}}$ | $\frac{4}{P_{\text {arts }}}$ | $\frac{5}{\text { Parts }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 29 | 1774 | 3519 | 5262 | 7004 | 8744 |
| 2 | 58 | 1803 | 48 | 91. | 33 | 73 |
| 3 | 87 | 32 | 77 | 5320 | 62 | 8803 |
| 4 | 116 | 61 | 3606 | 49 | 19 | 131 |
| 5 | 45 | 90 | 35 | 78 | 7120 | 60 |
| $\sigma$ | 74 | 1919 | 64 | 5407 | 49 | 89 |
| 7 | 203 | 48 | 93 | 35 | 78 | 8918 |
| 8 | 32 | 77 | 3722 | 65 | $7 \pm 07$ | 47 |
| 9 | 61 | 2007 | 51 | 95 | 36 | 76 |
| 10 | 90 | 36 | 80 | 5524 | 65 | 9005 |
| 11 | 319 | 65 | 3809 | 53 | 94 | 34 |
| 12 | 49 | 94 | 38 | 82 | 7323 | 63 |
| 13 | -78 | 2123 | 67 | 5611 | 52 | 92 |
| 14 | $44^{407}$ | 52 | 96 | 40 | 81 | 9121 |
| I5 | $5{ }^{6}$ | 8 I | 3925 | 69 | 7410 | 50 |
| 16 | 65 | 2210 | 55 | 98 | 39 | 79 |
| 17 | $7 \quad 94$ | 39 | 84 | 5727 | 68 | 9208 |
| 18 | 853 | 68 | 4013 | 56 | 97 | 37 |
| 1.9 | 92 | 97 |  | 85 | 7526 | -66 |
| 20 | - 81 | 2326 | 7 t | 5814 | 55 | 94 |
| 2 I | 1 610 | 55 | 4100 | 43 | 84 | 9323 |
| 22 | 2.39 | 85 | 29 | 72 | 7513 | 52 |
| 23 | 3.69 | 2414 | 58 | 5901 | 42 | 81 |
| 24 | 4 - 98 | 43 | 87 | $3{ }^{\circ}$ | 71 | 9410 |
| 25 | 5727 | 72 | 4216 | 59 | 17700 | 39 |
| 26 | 656 | 2501 | 45 | 88 | 29 | 68 |
| 27 | 71.85 | 30 | 74 | 6017 | 58 | 97 |
|  | 88.814 | 59 | 4303 | 46 | 87 | 9520 |
|  | 2943 | 88 | 32 | 75 | 7816 | 5 |
|  | 30173 | 26.17 | 61 | 6104 | 45 | - 84 |

Figure 69: Excerpt of Fale's table of sines [Fale (1593)]. (source: https://archive.org/details/b30333106)

|  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Cafelen van Sinuum, Tangentium, emSecantium, tegen 20000000 hen Dia. |  |  |  |  |  |  |  |
|  | Sinus. | Perpen. | Snijder. |  | Sinus. | Perpen. | Snijder. |
| 寿 | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | Gradē | 1 | 1 | 1 |
| 이 | 이 | 00001 | 100000001 | 101 | 1745241 | 1745501 | 10001524 |
|  | 2909 5818 | 2909 5818 | $\left.\begin{aligned} & 10000001 \\ & 10000002\end{aligned} \right\rvert\,$ | 1 | $\begin{aligned} & 177433 \\ & 180341 \end{aligned}$ | $\begin{aligned} & 177455 \\ & 180365 \end{aligned}$ | $\begin{aligned} & 10001574 \\ & 10001656 \\ & \hline \end{aligned}$ |
| 3 <br> 4 | 8727 <br> 11636 | $\begin{array}{r} 8727 \\ 11636 \end{array}$ | 10000004 1000008 | 3 | $\begin{aligned} & 183250 \\ & 186158 \end{aligned}$ | $\begin{array}{r} 183279 \\ 186189 \end{array}$ | $\begin{aligned} & 10001679 \\ & 10001733 \end{aligned}$ |
| 56 | 14544 17453 | $\begin{aligned} & 14544 \\ & 17452 \end{aligned}$ | 10000010 10000014 | 5 | 189066 191975 | $\begin{aligned} & 189100 \\ & 192010 \\ & \hline \end{aligned}$ | $\begin{aligned} & 10001788 \\ & 10001844 \\ & \hline \end{aligned}$ |
| 7 <br> 8 | 20362 23271 | 20361 23270 | 10000020 10000027 | 7 | 194883 197792 | $\begin{aligned} & 194920 \\ & 19783 \mathrm{c} \\ & \hline \end{aligned}$ | $\begin{aligned} & 10001900 \\ & 10001957 \end{aligned}$ |
| 9 <br> 10 | 26180 29088 | $26 r 79$ 29088 | 10000034 10000042 | 9 10 | 200700 203608 | $\begin{aligned} & 20074 \mathrm{C} \\ & 2036 \mathrm{cc} \\ & \hline \end{aligned}$ | $\begin{aligned} & 10002015 \\ & 10002074 \end{aligned}$ |
| 11 12 | 39197 34906 | 31996 34905 37 | $\begin{aligned} & 100000{ }^{1} 1 \\ & 10000060 \end{aligned}$ | 11 12 | 206517 209425 | 206961 209471 | 10002134 <br> 10002195 <br> 1002256 |
| 13 14 14 | 37815 40724 | 37814 40723 | $\begin{aligned} & 10000071 \\ & 10000083 \end{aligned}$ | 13 <br> 14 | $\begin{aligned} & 2123331 \\ & 215241 \\ & \hline \end{aligned}$ | 212381 215291 | $\begin{array}{r} 10002256 \\ 10002319 \end{array}$ |
| 15 16 | 43632 47531 | 43632 46941 | $1000009 \%$ 10000108 | 156 16 | 218149 221057 | 818201 221111 | 10001381 10002445 |
| 17 18 | 49450 52359 | 49450 52359 | 10000122 10000137 | 17 | $\begin{aligned} & 223965 \\ & 226873 \\ & \hline \end{aligned}$ | $\begin{aligned} & 224022 \\ & 226932 \\ & \hline \end{aligned}$ | $\begin{aligned} & 10002510 \\ & 10002576 \end{aligned}$ |
| 19 20 | 55268 58177 | 55268 58177 | 10000152 10000168 | 19 <br> 20 <br> 12 | $\begin{aligned} & 22978 \mathrm{I} \\ & 232689 \end{aligned}$ | 229842 232752 | $\begin{aligned} & 10002642 \\ & 10002709 \\ & \hline \end{aligned}$ |
| 21 22 21 | $\begin{array}{r} 61086 \\ 63995 \\ \hline \end{array}$ | 61086 63995 | $\begin{aligned} & \hline 10000186 \\ & 10000204 \end{aligned}$ | 21 22 | 235597 2385051 | $\begin{aligned} & 23\{6631 \\ & 238574 \end{aligned}$ | $\begin{aligned} & 10002777 \\ & 10002846 \end{aligned}$ |
| 23 24 24 | $\begin{aligned} & 66904 \\ & 69813 \end{aligned}$ | 66904 69813 | 10000223 10000243 | 23 <br> 24 <br> 25 | 241413 244321 | 2414851 $244298 \mid$ | $\begin{aligned} & 10002916 \\ & 10002987 \\ & \hline \end{aligned}$ |
| $\begin{aligned} & 25 \\ & 26 \\ & \hline \end{aligned}$ | 72721 75630 | $\begin{aligned} & 72722 \\ & 75631 \end{aligned}$ | $\begin{aligned} & 10000264 \\ & 10000285 \end{aligned}$ | 25 26 | $\begin{aligned} & 247229 \\ & 250137 \\ & \hline \end{aligned}$ | 247306 250217 | $\begin{aligned} & 10003058 \\ & 10003130 \end{aligned}$ |
| 27 28 28 | 78539 81448 | 78540 81450 | $\begin{aligned} & 10000308 \\ & 10000332 \end{aligned}$ | 27 28 | 253045 255953 | 253128 256038 | $\begin{aligned} & 10003203 \\ & 10003277 \\ & \hline \end{aligned}$ |
| 29 <br> 30 <br> 11 | 84357 87265 | 84359 87268 | $\begin{aligned} & 100003\{7 \\ & 10000381 \end{aligned}$ | 29 30 | $\begin{aligned} & 258861 \\ & 261769 \end{aligned}$ | $\begin{aligned} & 258949 \\ & 26189 \end{aligned}$ | $\begin{aligned} & 10003352 \\ & 10003428 \\ & \hline \end{aligned}$ |
| 31 32 | 90174 93083 | 90177 93086 | $\begin{aligned} & 10000407 \\ & 1000043 . \end{aligned}$ | 31 32 | $\begin{aligned} & 264677 \\ & 267585 \end{aligned}$ | $\begin{aligned} & 264770 \\ & 26768 \mathrm{I} \end{aligned}$ | $\begin{aligned} & 10003505 \\ & 10003582 \\ & \hline \end{aligned}$ |
| $\begin{aligned} & 23 \\ & 34 \end{aligned}$ | $\begin{aligned} & 95992 \\ & 98901 \end{aligned}$ | $\begin{aligned} & 95995 \\ & 98904 \end{aligned}$ | $\begin{aligned} & 10000461 \\ & 10000489 \end{aligned}$ | 33 34 | $\begin{aligned} & 270493 \\ & 273401 \end{aligned}$ | $\begin{aligned} & 270592 \\ & 273503 \end{aligned}$ | $\begin{aligned} & 10003660 \\ & 10003739 \end{aligned}$ |
| 38 35 36 | 101809 104718 | 101814 104723 | $\begin{aligned} & 10000{ }^{18} \\ & 10000548 \\ & \hline \end{aligned}$ | 35 | $\begin{aligned} & 276308 \\ & 279216 \\ & \hline \end{aligned}$ | 276414 279325 | $\begin{aligned} & 10003819 \\ & 10003900 \\ & \hline \end{aligned}$ |
| 37 38 | $\begin{aligned} & 107027 \\ & 110536 \end{aligned}$ | 107632 110541 | $\begin{aligned} & 10000579 \\ & 10000611 \end{aligned}$ | 37 38 | 282124 285032 | $\begin{aligned} & 282237 \\ & 285148 \end{aligned}$ | $\begin{aligned} & 10003982 \\ & 10004065 \end{aligned}$ |
| 39 40 | $\begin{aligned} & 113445 \\ & 116353 \\ & \hline \end{aligned}$ | $\begin{aligned} & 113450 \\ & 116360 \\ & \hline \end{aligned}$ | $\begin{aligned} & 10000643 \\ & 10000677 \\ & \hline \end{aligned}$ | 39 40 | $\begin{aligned} & 287940 \\ & 2.90847 \end{aligned}$ | $\begin{aligned} & 2880591 \\ & 290970 \mid \end{aligned}$ | $\begin{aligned} & 10004148 \\ & 10004232 \end{aligned}$ |
| 41 42 | $\begin{aligned} & 112262 \\ & 122171 \end{aligned}$ | 119269 122178 | $\begin{aligned} & 10000711 \\ & 10000746 \end{aligned}$ | 41 | $\begin{aligned} & 293755 \\ & 296663 \end{aligned}$ | $\begin{aligned} & 293882 \\ & 296794 \end{aligned}$ | $\begin{aligned} & 10004317 \\ & 10004403 \\ & \hline \end{aligned}$ |
| 43 44 4 | $\begin{aligned} & 125078 \\ & 127988 \\ & \hline \end{aligned}$ | 125088 127297 | $\begin{aligned} & 100007^{82} \\ & 10000819 \\ & \hline \end{aligned}$ | 43 44 | $\begin{aligned} & 299570 \\ & 302478 \end{aligned}$ | $\begin{aligned} & 299705 \\ & 302617 \end{aligned}$ | $\begin{aligned} & 10004490 \\ & 10004578 \\ & \hline \end{aligned}$ |
| 45 | $\begin{aligned} & 130896 \\ & 133805 \\ & \hline \end{aligned}$ | $\begin{aligned} & 130006 \\ & 133816 \\ & \hline \end{aligned}$ | $\begin{aligned} & 10000857 \\ & 10000895 \end{aligned}$ | 45 | $\begin{aligned} & 305385 \\ & 308293 \end{aligned}$ | $\begin{aligned} & 305528 \\ & 308439 \end{aligned}$ | $\begin{aligned} & 10004666 \\ & 10004755 \end{aligned}$ |
| 478 | 136714 139622 | $\begin{array}{r}136725 \\ \hline 139635 \\ \hline\end{array}$ | 10000934 10000975 | 478 | $\begin{aligned} & 311200 \\ & 314108 \end{aligned}$ | $\begin{aligned} & 311351 \\ & 314262 \end{aligned}$ | $\begin{aligned} & 10004^{845} \\ & 10004926 \end{aligned}$ |
| 49 50 | 142531 145439 | 142544 145454 | $\begin{aligned} & 10001016 \\ & 1000105^{8} \end{aligned}$ | 49 50 | 317015 <br> 319922 | $\begin{aligned} & 317174 \\ & 320085 \end{aligned}$ | $\begin{aligned} & 10005028 \\ & 10005122 \end{aligned}$ |
| 51 52 | $\left.\begin{aligned} & 148348 \\ & 151257\end{aligned} \right\rvert\,$ | 148363 151273 | 10001100 10001144 | 51 <br> 52 | 322830 <br> 325737 | $\begin{array}{r} 322.997 \\ 325909 \end{array}$ | $\begin{aligned} & 1000 \int 216 \\ & 1000 \$ 310 \end{aligned}$ |
| 53 54 | 154165 157074 | 154182 157092 | 10001188 10001233 | 53 54 | 328645 351552 | 328821 | $\begin{aligned} & 10005405 \\ & 10005501 \\ & \hline \end{aligned}$ |
| 55 | 159982 162891 | 160001 162911 | 10001280 10001327 | 55 | 334459 337367 | $\begin{aligned} & 334645 \\ & 33758 \end{aligned}$ | $\begin{aligned} & 1000559^{8} \\ & 10005696 \end{aligned}$ |
| 57 58 | 165790 168708 | 168820 168730 | 10001375 10001423 | $\|$57 <br> 58 <br> 8 | 340274 343181 | $\begin{aligned} & 340470 \\ & 343382 \end{aligned}$ | $\begin{aligned} & 10005795 \\ & 10005894 \\ & \hline \end{aligned}$ |
| 59 60 | $\begin{aligned} & 171616 \\ & 174524 \end{aligned}$ | 171640 174550 | $\begin{aligned} & 10001473 \\ & 10001524 \end{aligned}$ | $\left\lvert\, \begin{aligned} & 59 \\ & 60\end{aligned}\right.$ | $\begin{aligned} & 346088 \\ & 348995 \end{aligned}$ | $\begin{aligned} & 346295 \\ & 349207 \end{aligned}$ | $\begin{aligned} & 10005994 \\ & 10006095 \end{aligned}$ |

Figure 70: The first page from Ceulen's table of trigonometric functions [Ceulen (1596)].


Figure 71: The last page from Ceulen's table of trigonometric functions [Ceulen (1596)].


Figure 72: Excerpt of Rheticus's Opus palatinum [Rheticus and Otho (1596)] (source: e-rara).


Figure 73: Excerpt of Pitiscus's Thesaurus mathematicus [Pitiscus (1613)] (source: École des Ponts ParisTech, Paris, photograph by the author).


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[^1]:    ${ }^{1}$ There is a vast literature on numerical tables, and I am directing the reader to a number of general surveys, such as [Hutton (1785)], [De Morgan (1842)], [De Morgan (1851)], [Glaisher (1873)], [Davis (1933), pp. 1-40], [Campbell-Kelly et al. (2003)], etc. This document also mentions many people, and I am not always directing to specific biographical information for each of them. Valuable informations can in particular be found in the notices of [Hockey (2014)], in particular on Al-Battāni, Abū al-Wafā ${ }^{3}$, Al-Khwārizmī, Al-Zarqālī, Apian, Bürgi, Clavius, Copernicus, Engel, Fine, Gemma Frisius, Lansberge, Magini, Maurolico, Peucer, Peuerbach, Regiomontanus, Reinhold, Rheticus, and many others.
    ${ }^{2}$ [Glowatzki and Göttsche (1990)]
    ${ }^{3}$ https://locomat.loria.fr, see [Roegel (2012)].

[^2]:    ${ }^{4}$ [Regiomontanus (1490)]
    ${ }^{5}$ [Regiomontanus (1490), Regiomontanus (1504)]
    ${ }^{6}$ [von Peuerbach (1516)]
    ${ }^{7}$ [Fine (1530)]
    ${ }^{8}$ [Apian (1533)]
    ${ }^{9}$ [von Peuerbach and Regiomontanus (1541)]
    ${ }^{10}$ [Copernicus (1542)]
    ${ }^{11}$ [Fine (1550)]
    ${ }^{12}$ [Rheticus (1551)]
    ${ }^{13}$ [Reinhold (1554)]
    ${ }^{14}$ [Maurolico (1558)]
    ${ }^{15}$ [Viète (1579)]
    ${ }^{16}$ [Fincke (1583)]
    ${ }^{17}$ [van Lansberge (1591)]
    ${ }^{18}$ [Rheticus and Otho (1596)]
    ${ }^{19}$ [Pitiscus (1613)]

[^3]:    ${ }^{20}$ [von Braunmühl $\left.(1900,1903)\right]$
    ${ }^{21}$ [Tropfke (1902-1903), v. 2, pp. 189-221, and 296-306]
    ${ }^{22}$ [Bond (1921)]
    ${ }^{23}$ [Zeller (1944)]
    ${ }^{24}$ See [van Brummelen (2009)] and [van Brummelen (2021)].
    ${ }^{25}$ [Montucla (1758)]
    ${ }^{26}$ [Kästner (1796)]
    ${ }^{27}$ [Zeuthen (1903)]
    ${ }^{28}$ [Katz et al. (2007)]
    ${ }^{29}$ [Merzbach and Boyer (2010)]
    ${ }^{30}$ [Scriba and Schreiber (2015)]
    ${ }^{31}$ In the sequel, sines will often be defined in non-unit circles, and I will use Sin for this purpose, often leaving the radius implicit. We have of course $\operatorname{Sin}_{R} \alpha=R \sin \alpha$. Some authors call this Sin the $R$-sine, but I will always use "sine" alone, as the context should be unambiguous. I will also use other variants such as Tan, Sec, etc., when needed.
    ${ }^{32}$ [Bond (1921), pp. 297-298]
    ${ }^{33}$ [Sidoli (2014), p. 13]

[^4]:    ${ }^{34}$ [Toomer (1974), p. 7] See [van Brummelen (2009), pp. 41-45] for a recent discussion on this topic.
    ${ }^{35}$ [Ptolemaeus (1984), p. 215]
    ${ }^{36}$ [Klintberg (2005)]
    ${ }^{37}$ [Duke (2005)]
    ${ }^{38}$ See [Ptolemaeus (1813-1816), v. 1, pp. 38-45], [Ptolemaeus (1898-1903), v. 1, pp. 48-63], [Ptolemaeus (1984), pp. 57-60].
    ${ }^{39}$ Besides Toomer's edition of the Almagest [Ptolemaeus (1984), pp. 57-60], see [Neugebauer (1975), pp. 21-24], [Pedersen (2011), ch. 3], [Bond (1921), pp. 301-303], [Clagett (1957), pp. 200-205], [Kneale (1965)], [Glowatzki and Göttsche (1976)], [Thurston (1996), pp. 235-

[^5]:    ${ }^{42}$ [Ptolemaeus (1898-1903), v. 1, p. 48]
    ${ }^{43}$ For summaries of the history of mathematics and astronomy in India and extensive discussions on trigonometry or tables, see [Srinivasiengar (1967)], [Pingree (1978)], [Bag (1979)], [Katz et al. (2007)], [Plofker (2009)], [González-Velasco (2011), pp. 25-34], [van Brummelen (2009), pp. 94-134], [Puttaswamy (2012), pp. 108-116], [Divakaran (2018)], [Montelle and Plofker (2018)] (especially page 57) and [Ramasubramanian (2019)].
    ${ }^{44}$ [van Brummelen (2009), p. 99]
    ${ }^{45}$ [van Brummelen (2009), p. 96]
    ${ }^{46}$ [van Brummelen (2009), p. 99]

[^6]:    ${ }^{47}$ [Neugebauer and Pingree (1970-1971), part 2, p. 37]
    ${ }^{48}$ See [Toomer (1974), p. 6] and [Divakaran (2018), p. 198].
    ${ }^{49}$ [Neugebauer (1956)]
    ${ }^{50}$ [Burgess (1860), pp. 58-60]
    ${ }^{51}$ [van Brummelen (2009), p. 97]
    ${ }^{52}$ See [Clark (1930), p. 19], [Srinivasiengar (1967), pp. 40-54], [Filliozat (1988)], and [Mazars (1974)].
    ${ }^{53}$ See [Neugebauer and Pingree (1970-1971), part 2, pp. 37-38] and [Plofker (2009), p. 51]
    ${ }^{54}$ [Bag (1969), p. 84]
    ${ }^{55}$ [Bhattacharyya (2011)]

[^7]:    ${ }^{56}$ [Plofker (2009), p. 81, 157]
    ${ }^{57}$ [Gupta (1978)]
    ${ }^{58}$ [Bag (1969)]
    ${ }^{59}$ See [Chatterjee (1970), p. 206], [Gupta (1978)], [Pingree (1996), p. 43], and [Pingree (2003)].
    ${ }^{60}$ For more extensive descriptions of Arabic mathematics and astronomy, see in particular the surveys of [von Braunmühl (1900, 1903), v. 1, pp. 42-86] and [Rashed and Morelon (1996)]. Heydari-Malayeri's short survey may also be of interest [HeydariMalayeri (2007)]. On trigonometric tables in the Islamic world, see [Berggren (1986), p. 144] and [van Brummelen (2009), pp. 135-222].
    ${ }^{61}$ [Bond (1921), p. 307]
    ${ }^{62}$ [Bag (1969), p. 84]
    ${ }^{63}$ See [Kennedy (1956), pp. 148-154], [Pingree (1996), p. 41] and [van Dalen (1996)].
    ${ }^{64}$ [Heydari-Malayeri (2007)]

[^8]:    ${ }^{65}$ See [Folkerts (2006), pp. 75-76], [Goldstein (2019), p. 132] and [Filliozat (1988), p. 261]
    ${ }^{66}$ [McCarthy and Byrne (2003), p. 247]
    ${ }^{67}$ [Neugebauer (1962), p. 104]
    ${ }^{68}$ [McCarthy and Byrne (2003), pp. 265-266]
    ${ }^{69}$ See [Neugebauer (1962), p. 104] and [Chabás Bergón and Goldstein (2012), p. 19].
    ${ }^{70}$ [Pingree (1996), p. 43]
    ${ }^{71}$ [McCarthy and Byrne (2003), p. 246]
    ${ }^{72}$ [Hogendijk (1991)]
    ${ }^{73}$ [McCarthy and Byrne (2003), pp. 264-265]
    ${ }^{74}$ [Suter (1914), tab. 58 and 58a]
    ${ }^{75}$ [Suter (1914), tab. 60]
    ${ }^{76}$ [Bond (1921), p. 307]
    ${ }^{77}$ [Bond (1921), p. 308]. See also [Moussa (2010)] who considers the process by which the tangent and cotangent functions became more abstract, especially with Abū al-Wafä ${ }^{\square}$.

[^9]:    ${ }^{78}$ See [Glowatzki and Göttsche (1976), pp. 12-13] and [Folkerts (2006), p. 76].
    ${ }^{79}$ [Debarnot (1996), p. 524]
    ${ }^{80}$ [Joseph (2011), p. 497]
    ${ }^{81}$ [Debarnot (1996), p. 512]. These "three places" probably include the radius.
    ${ }^{82}$ [Debarnot (1996), p. 509]
    ${ }^{83}$ See [Cullen (1982)], [Gupta (1987), p. 241], [Qu Anjing (2002)] and [Divakaran (2018), p. 209]. A recent summary of Indian and Islamic trigonometry in China is given in [van Brummelen (2021), pp. 185-191].
    ${ }^{84}$ See [Al-Battāni (1899-1907)] and [Kennedy (1956), pp. 154-156].
    ${ }^{85}$ [Al-Battāni (1899-1907), vol 2, pp. 55-56]
    ${ }^{86}$ [Al-Battāni (1899-1907), vol 2, p. 60]
    ${ }^{87}$ [Debarnot (1996), p. 524]
    ${ }^{88}$ This is what Folkerts writes [Folkerts (2006), p. 76], but it may mean four places including the integer part, which would then mean three sexagesimal places with our conventions.

[^10]:    ${ }^{89}$ [Debarnot (1996), p. 527]
    ${ }^{90}$ [Bond (1921), p. 311]
    ${ }^{91}$ [Joseph (2011), p. 497]
    ${ }^{92}$ [Debarnot (1996), p. 524]
    ${ }^{93}$ See [Berggren (1986), p. 150], [Berggren (2016), p. 181] and [King (1975), p. 43].
    ${ }^{94}$ See [Debarnot (1996), p. 525] and [Schoy (1923), pp. 382-383].
    ${ }^{95}$ [Glowatzki and Göttsche (1990), p. 9]
    ${ }^{96}$ Note however that [Debarnot (1996), p. 524] misleadingly states that Ibn Yūnus gave his sines only every $10^{\prime}$.
    ${ }^{97}$ [Schoy (1923), p. 394]

[^11]:    ${ }^{98}$ [Schoy (1923), p. 386]
    ${ }^{99}$ [Schoy (1923), p. 396]
    ${ }^{100}$ [Samsó Moya (2020)]
    ${ }^{101}$ [Richter-Bernburg (1987)]
    ${ }^{102}$ See [Pingree (1996), p. 46], [Chabás Bergón (2019), pp. 47-75], and [Samsó Moya (2020)].
    ${ }^{103}$ [Suter (1914)]
    ${ }^{104}$ [Toomer (1968)]
    ${ }^{105}$ [van Brummelen (2018), p. 547]
    ${ }^{106}$ See [Busard (1971a), p. 74]. The canons of the tables were published by Curtze in 1900 [Curtze (1900), p. 337].
    ${ }^{107}$ [Zinner (1936), Toomer (1968), Pedersen (2002)]
    ${ }^{108}$ [Zinner (1936), p. 747]
    ${ }^{109}$ See [Zinner (1936), table 25, p. 749], [Toomer (1968), table 12, pp. 27-28], [Pedersen (2002), pp. 946-952], [Millás Vallicrosa (1950), pp. 62-63] and [Kennedy (1956), p. 128].

[^12]:    ${ }^{110}$ See [Zinner (1936), table 135, p. 757], [Toomer (1968), table 13, p. 29] and [Pedersen (2002), pp. 954-959].
    ${ }^{111}$ [Toomer (1968), table 12, pp. 27-28]
    ${ }^{112}$ [van Dalen (1996), p. 206]
    ${ }^{113}$ [McCarthy and Byrne (2003), p. 266]
    ${ }^{114}$ [McCarthy and Byrne (2003), pp. 252-253]
    ${ }^{115}$ [McCarthy and Byrne (2003), p. 264]
    ${ }^{116}$ [McCarthy and Byrne (2003), p. 265] Pedersen, however, attributes this table to AlBattāni [Pedersen (2002), p. 954].
    ${ }^{117}$ [Toomer (1968), table 15, p. 32]
    ${ }^{118}$ See [Dreyer (1920)], [Poulle (1988)], [Chabás Bergón (2002)], [Chabás Bergón and Goldstein (2003)], [Swerdlow (2004)] and [Chabás Bergón (2019), pp. 125-132].
    ${ }^{119}$ [Heydari-Malayeri (2007), p. 10]

[^13]:    ${ }^{120}$ See [Goldstein and Chabás Bergón (2004), p. 455] and [Chabás Bergón (2019), pp. 237276].
    ${ }^{121}$ [Chabás Bergón and Goldstein (2012), p. 20]
    ${ }^{122}$ See [Bond (1921), p. 304], [Schoy (1923), pp. 398-399], [Archibald (1949), p. 31], and [Gloden (1950), p. 10]. For the development of table literature in Indian and Arabic mathematics, especially after Ulugh Beg's tables, see for instance the surveys of [Ghori (1985)] and [Plofker (2009)]. Gloden's text just cited, as well as a number of others, should be taken cautiously, as they contain many approximations.
    ${ }^{123}$ [Aaboe (1954)]

[^14]:    ${ }^{124}$ See [Curtze (1900), pp. 411-412] and [Glowatzki and Göttsche (1990), pp. 73-79].
    ${ }^{125}$ See [Haskins (1924)] and [Glowatzki and Göttsche (1976), p. 15].
    ${ }^{126}$ See [Boncompagni (1862), p. 96] and [Hughes (2008), p. 355].
    ${ }^{127}$ [Bond (1921)]
    ${ }^{128}$ See [Goldstein (1974), pp. 153-155], [Goldstein (1985), pp. 134-140], and [Goldstein (2019), p. 133].
    ${ }^{129}$ See [Gassendi (1654), pp. 340-342] and [Poulle (1963), pp. 75-80].

[^15]:    ${ }^{130}$ For summaries of Johannes von Gmunden's life and works, see [von Khauz (1755), pp. 27-32], [Aschbach (1865), pp. 455-467], [Klug (1943)], [Vogel (1973a)], [Grössing (1983), pp. 73-78], [Firneis (1988)], [Kaiser (1988)], [Shank (1997)], [Grössing (2002)], [Folkerts (2006)], [van Brummelen (2009), pp. 248-249], and [Simek and Klein (2012)]. For a survey of his tables, see [Porres de Mateo (2003)] and [Chabás Bergón (2019), pp. 321-336]. See also [Durand (1952), pp. 54-56], and [Duhem (1959), pp. 349-367], especially for the scientific context in Vienna. Gessner mentions von Gmunden very briefly [Gessner and Simmler (1574), p. 375].
    ${ }^{131}$ [Schmeidler (1977), p. 315]
    ${ }^{132}$ [von Khauz (1755), p. 29]
    ${ }^{133}$ [Sperl (1971a)]
    ${ }^{134}$ This treatise was published in [Busard (1971a)]. See also [Kaiser (1988), pp. 91-96] and [Folkerts (2006), p. 71].
    ${ }^{135}$ See [Busard (1971a), p. 78] and [Folkerts (2006), p. 81].
    ${ }^{136}$ [Glowatzki and Göttsche (1990), pp. 79-92]

[^16]:    ${ }^{137}$ See [Bond (1920), p. 319] and [Curtze (1900)].
    ${ }^{138}$ [Glowatzki and Göttsche (1990), p. 81]
    ${ }^{139}$ [Glowatzki and Göttsche (1990), pp. 85 and 89]
    ${ }^{140}$ [Klug (1943), p.57]
    ${ }^{141}$ [Ptolemaeus (1984), pp. 57-60]
    ${ }^{142}$ In figure 13, the number of places of sexagesimal tables is shown as $60 ; 60^{n}$, the first 60 being the value of $R$, and $n$ being the number of additional sexagesimal places.
    ${ }^{143}$ [Glowatzki and Göttsche (1990), p. 92]
    ${ }^{144}$ [von Braunmühl (1900, 1903), v. 1, pp. 110-111]
    ${ }^{145}$ [Bond (1921), p. 320]
    ${ }^{146}$ [Zeller (1944), p. 16]
    ${ }^{147}$ [Glowatzki and Göttsche (1990), p. 72]
    ${ }^{148}$ [Eneström (1913-1914)].
    ${ }^{149}$ [Busard (1971a)]
    ${ }^{150}$ [Glowatzki and Göttsche (1990), p. 92]

[^17]:    ${ }^{151}$ [Folkerts (2006), p. 87]
    ${ }^{152}$ [Busard (1971a), p. 76]
    ${ }^{153}$ [Pitiscus (1613)]
    ${ }^{154}$ See [Barotti (1792), vol. 1, p. 119-132], [Birkenmajer (1911)], [Federici Vescovini (1968)], [Goldstein and Chabás Bergón (2004)] and [Chabás Bergón and Goldstein (2009), p. 13] For a survey of Bianchini's tables, see [van Brummelen (2018)] and [Chabás Bergón (2019), pp. 337-364]. See also [Gruyer (1897), v. 2, pp. 428-430] for some background on his astronomical tables. Gessner mentions Bianchini very briefly [Gessner and Simmler (1574), p. 346].
    ${ }^{155}$ [Rosińska (1994a), Rosińska (1997-1998)]
    ${ }^{156}$ See [von Murr (1786), vol. 1, p. 74-205], [Curtze (1902)] and [Gerl (1989)].
    ${ }^{157}$ See [Rosińska (1981a)], [Rosińska (1981b)], [Rosińska (1987)], and [Rosińska (2006)]. A more complete summary of Bianchini's trigonometric tables was recently given by Chabás [Chabás Bergón (2016)].
    ${ }^{158}$ [Boffito (1908)]
    ${ }^{159}$ [Birkenmajer (1911), p. 273]
    ${ }^{160}$ See [Chabás Bergón and Goldstein (2012), p. 20] and [Chabás Bergón (2019), p. 361].

[^18]:    ${ }^{161}$ [Glowatzki and Göttsche (1990), pp. 95-114]
    ${ }^{162}$ [Glowatzki and Göttsche (1990), p. 94]
    ${ }^{163}$ [Chabás Bergón (2019), p. 361]
    ${ }^{164}$ See for instance BNF, Manuscrit Latin 16655, $\mathrm{f}^{\circ} 31$ r, here reproduced in figure 8.
    ${ }^{165}$ [Toomer (1968), table 15, p. 32]
    ${ }^{166}$ [Glowatzki and Göttsche (1990), p. 105]
    ${ }^{167}$ See [Rosińska (1984), pp. 476-477] and [Chabás Bergón (2019), p. 349].
    ${ }^{168}$ In 1981, [Rosińska (1981a)] wrote mistakenly that the tangents are given with $R=10^{3}$. This error was repeated by Rosińska in 1987 [Rosińska (1987)] and 2002 [Rosińska (2002), p. 12], by Chabás and Goldstein [Chabás Bergón and Goldstein (2009), p. 20] and Brummelen in 2009 [van Brummelen (2009), p. 262], but it was corrected by Chabás in 2016 [Chabás Bergón (2016)].
    ${ }^{169}$ [Chabás Bergón (2019), p. 351]
    ${ }^{170}$ [Regiomontanus (1490)]

[^19]:    ${ }^{171}$ [Chabás Bergón (2019), p. 351]
    ${ }^{172}$ As mentioned above, Rosińska actually wrote that the decimal table of tangents used $R=10^{3}$ and therefore also posited a sine table with that radius.
    ${ }^{173}$ [Walsh (1996), pp. 289-291]
    ${ }^{174}$ For summaries of Peuerbach's life and works, see in particular [Gassendi (1654), pp. 335-373], [von Khauz (1755), pp. 33-57], [Montucla (1758), v. 1, pp. 443445], [Martin (1764), pp. 157-158], [Aschbach (1865), pp. 479-493], [Gallois (1890a), pp. 1-11], [Thorndike (1929), ch. 8], [Sperl (1971b)], [Vogel (1973b)], [Rose (1975)], [Hellman and Swerdlow (1978)], [Grössing (1983), pp. 79-116], [Shank (1997)], [Samhaber (2000)], [Grössing (2002)], [Kaunzner (2006)], [van Brummelen (2009), pp. 249-252], [Malpangotto (2020), pp. 19-34], and [Horst (2019)]. Several other references not cited here are given in the Geschichtsquellen des deutschen Mittelalters (https://www.geschichtsquellen.de/autor/749). One of the first biographical notices on Peuerbach was that of Tannstetter, when he published Peuerbach's table of eclipses [von Peuerbach and Regiomontanus (1514)]. Gessner, on the other hand, only briefly mentions Peuerbach [Gessner and Simmler (1574), p. 231]. For the scientific context in Vienna, see [Durand (1952)] and [Duhem (1959), pp. 349-367].
    ${ }^{175}$ See [Hellman and Swerdlow (1978), p. 473], [Grössing (1983), p. 80] and [Malpangotto (2020), p. 24].
    ${ }^{176}$ See [von Khauz (1755), p. 38] and [Hellman and Swerdlow (1978), p. 473].

[^20]:    ${ }^{177}$ [Vogel (1973a), p. 120]
    ${ }^{178}$ See [Malpangotto (2020), pp. 116-119 \& 678-679]. Many sources give the date of publication as 1472, but I follow Malpangotto here. Note that Khauß wrote that the Theoricae were published in 1460 [von Khauz (1755), p. 46].
    ${ }^{179}$ [Shank (1996), p. 124]
    ${ }^{180}$ [Schmeidler (1977), p. 315]
    ${ }^{181}$ [Hellman and Swerdlow (1978), p. 475]
    ${ }^{182}$ [Malpangotto (2020), p. 33]
    ${ }^{183}$ [Glowatzki and Göttsche (1976), p. 16]
    ${ }^{184}$ Meskens writes that Bessarion had started the translation, but doesn't give any substantial proof of this statement [Meskens (2010), p. 136]. Meskens's statement may rest on [Glowatzki and Göttsche (1976), p. 17].
    ${ }^{185}$ See [Malpangotto (2020), p. 20] and [Shank (2002), p. 183].

[^21]:    ${ }^{186}$ [von Khauz (1755), p. 42]
    ${ }^{187}$ Peuerbach actually followed closely the Almagestum minor, a textbook from the late thirteenth century [Hellman and Swerdlow (1978), p. 477]. Bendefy incorrectly stated that Peuerbach's Epitome was translated from the Greek [Bendefy (1980), p. 244].
    ${ }^{188}$ [Rosen (1975a), p. 349]
    ${ }^{189}$ [Grössing (1983), p. 146]
    ${ }^{190}$ [Gassendi (1658), p. 518]
    ${ }^{191}$ [Folkerts (1977), Kaunzner (1980), Zinner (1968)]
    ${ }^{192}$ [Gerhardt (1877), p. 87]
    ${ }^{193}$ [Hellman and Swerdlow (1978), p. 477]
    ${ }^{194}$ [Thorndike (1929), p. 143]
    ${ }^{195}$ [von Peuerbach and Regiomontanus (1541)]
    ${ }^{196}$ [Busard (1971a), p. 75]
    ${ }^{197}$ This table is reproduced by Glowatzki and Göttsche [Glowatzki and Göttsche (1990), pp. 116-123]. They draw the attention to incorrect statements by [Cantor (1900), p. 182]

[^22]:    and [Zinner (1968), p. 36], [Zinner (1990), p. 23] about the radius of the table.
    ${ }^{198}$ [Regiomontanus (1490)]
    ${ }^{199}$ See [Glowatzki and Göttsche (1990), pp. iii and 115]. Earlier, Hellman and Swerdlow had mentioned a manuscript table with $R=600000$, but this is in fact a table with $R=60000$ [Hellman and Swerdlow (1978), p. 478]. Brummelen also seems to mention this no longer extant table [van Brummelen (2009), p. 249].
    ${ }^{200}$ [Hellman and Swerdlow (1978), p. 477]
    ${ }^{201}$ [von Peuerbach (1516)]
    ${ }^{202}$ See [Gassendi (1658), p. 520] and [von Khauz (1755), p. 54].
    ${ }^{203}$ This incorrect statement is also found in [Martin (1764), p. 158] (and in [Lublink and Meijer (1763), pp. 183-198] which must have the same source), and it was more recently repeated by [Bendefy (1980), p. 245].
    ${ }^{204}$ [von Peuerbach (1516)]. See [Roegel (2021a)] for a modern reconstruction.

[^23]:    ${ }^{205}$ [Glowatzki and Göttsche (1990), pp. 124-125]
    ${ }^{206}$ [Gemma Frisius (1545)]
    ${ }^{207}$ [Magini (1592)]
    ${ }^{208}$ [Schmeidler (1977), p. 315]. Some authors, for instance recently [Meskens (2010)], have incorrectly confused this Königsberg with the modern Kaliningrad.
    ${ }^{209}$ According to some sources, the name Regiomontanus was coined by Philip Melanchthon. It does indeed appear in his De capta Constantinopoli, Anno 1453 (1556). However, the earliest appearance I found of "Regiomontanus" (or rather Regiomontano) is that in Marcus Beneventanus's Apologeticum opusculum (1521). For summaries of Regiomontanus's life and works, see mainly [Zinner (1968)], which can be supplemented by [Gassendi (1654), pp. 335-373], [Doppelmayr (1730)], [Montucla (1758), v. 1, pp. 445-453], [Martin (1764), pp. 146-157], [Aschbach (1865), pp. 537-557], [Ziegler (1874)], [Günther (1885)], [Gallois (1890a), pp. 1-11], [Thorndike (1929), ch. 8], [Vogel (1973b)], [Rosen (1975a)], [Rose (1975)], [Hamann (1978)], [Hamann (1980)], [Grössing (1983), pp. 117-126], [Glowatzki and Göttsche (1990), p. 1-8], [Mett (1996)], [Grössing (2002)], [Malpangotto (2008)], [van Brummelen (2009), pp. 251-263], and [van Brummelen (2021), pp. 2-5]. One of the first biographical notices on Regiomontanus was that of Tannstetter, when he published Peuerbach's table of eclipses and Regiomontanus's table of the first mobile [von Peuerbach and Regiomontanus (1514)]. There are also many smaller articles of interest, some more specialized, some more introductory, such as [Shank (2017)], [Horst (2019)], [Götz (2003)], etc., but which are not all cited here. Gessner also mentions Regiomontanus [Gessner and Simmler (1574), p. 397]. And many sources on Peuerbach, not cited in the previous list, contain some information on Regiomontanus. For the scientific context in Vienna, see [Durand (1952)] and [Duhem (1959), pp. 349-367].
    ${ }^{210}$ [Glowatzki and Göttsche (1990), p. i]

[^24]:    ${ }^{211}$ See [Rosen (1975a), p. 348] and [Schmeidler (1977), p. 316].
    ${ }^{212}$ See [Rose (1975), pp. 90-117], [Schmeidler (1977), p. 316], [Grössing (1980)], [Mett (1989)] and [Moos (2020)]. On Regiomontanus's knowledge of Latin and Greek, see [Ben-Tov (2009), pp. 195-196] and [Jensen (1996), p. 65] who theorizes that Regiomontanus may not have mastered Latin as well as the Italian scholars.
    ${ }^{213}$ See [Zinner (1990), p. 52] and [Shank (1996), p. 125]. It was however only printed in 1496.
    ${ }^{214}$ [Zinner (1990), p. 59]
    ${ }^{215}$ See [von Murr (1786), vol. 1, p. 74-205], [Curtze (1902)] and [Gerl (1989)]. See

[^25]:    also [Swerdlow (1990)].
    ${ }^{216}$ [Durand (1943), p. 13]
    ${ }^{217}$ See [Mett (1996), pp. 96-97], [Swerdlow (1999), p. 1], [van Brummelen (2009), p. 263], and [Chabás Bergón (2019), pp. 378-379].
    ${ }^{218}$ [Glowatzki and Göttsche (1990), p. 199]
    ${ }^{219}$ [Glowatzki and Göttsche (1990), pp. 197-207]
    ${ }^{220}$ [Hayton (2010), p. 33]
    ${ }^{221}$ [Schmeidler (1977), p. 317]
    ${ }^{222}$ [Orbán (2015), p. 118]
    ${ }^{223}$ [Domonkos (1968), Vargha and Both (1987), Hayton (2007), Hayton (2010), Orbán (2015)]
    ${ }^{224}$ There are several editions of the Tabulæ directionum profectionumque, in particular in 1490, 1504, 1550, 1552, 1559, 1584 and 1606. A French edition was published by Henrion in 1626 [Henrion (1626)]. For the collaboration between Bylica and Regiomontanus, see [Hayton (2007), p. 188] and [Chabás Bergón (2019), pp. 380-387]. On the relations between the astronomical schools of Vienna and Cracow, see [Markowski (1978), p. 268]. Bylica sent works from Peuerbach and Regiomontanus to the University of Cracow. See also [Walsh (1996)] and [Bendefy (1980)] on Regiomontanus's stay in Hungary.
    ${ }^{225}$ [Folkerts (1996), pp. 91-92]
    ${ }^{226}$ [Schmeidler (1977), p. 318]

[^26]:    ${ }^{227}$ See [Shank (2007)] for some excerpts.
    ${ }^{228}$ [Hallam (1837), p. 259]
    ${ }^{229}$ [Glowatzki and Göttsche (1990), pp. 11-24]
    ${ }^{230}$ [von Peuerbach and Regiomontanus (1541)]
    ${ }^{231}$ [von Peuerbach and Regiomontanus (1541), Roegel (2021b)]
    ${ }^{232}$ See [Rosińska (1984), pp. 503-504] and [Rosińska (1987), pp. 421-422].
    ${ }^{233}$ [van Brummelen (2018)]

[^27]:    ${ }^{234}$ [Thorndike (1929), p. 148]
    ${ }^{235}$ [Folkerts (1977), p. 235]
    ${ }^{236}$ [Roegel (2021c)]
    ${ }^{237}$ [Bond (1921), p. 321]
    ${ }^{238}$ [Delambre (1819), p. 365]
    ${ }^{239}$ See [Folkerts (1977), p. 234], [Folkerts (1995), p. 224] and [Folkerts et al. (2016), p. 136].
    ${ }^{240}$ See [Zinner (1968), p. 345] and [Zinner (1990), p. 236].
    ${ }^{241}$ [North (2008), p. 275]
    ${ }^{242}$ [van Brummelen (2009), p. 262]
    ${ }^{243}$ [Husson (2014), p. 116]
    ${ }^{244}$ [Chabás Bergón and Goldstein (2012), p. 20]
    ${ }^{245}$ [Glowatzki and Göttsche (1990), p. 48] On Johannes Engel, see [Dobrzycki and Kremer (1996)].
    ${ }^{246}$ [Glowatzki and Göttsche (1990), p. iii]
    ${ }^{247}$ [Roegel (2021d)]
    ${ }^{248}$ [Regiomontanus (1533)], edited om [Regiomontanus (1967)].
    ${ }^{249}$ [Stamm (1933)]

[^28]:    ${ }^{250}$ [Delambre (1819), p. 365]
    ${ }^{251}$ [von Braunmühl (1900, 1903), v. 1, pp. 124-133]
    ${ }^{252}$ [Glowatzki and Göttsche (1990), p. i]
    ${ }^{253}$ [Thorndike (1929), p. 150]

[^29]:    ${ }^{254}$ [Regiomontanus (1490)]
    ${ }^{255}$ [von Peuerbach and Regiomontanus (1541)]
    ${ }^{256}$ [von Peuerbach and Regiomontanus (1541)]
    ${ }^{257}$ There have been some incorrect statements about the tables constructed by Regiomontanus and a table with $R=600000$ is sometimes attributed to him, for instance by Günther in 1885 [Günther (1885), p. 573].
    ${ }^{258}$ In a long chapter, Glowatzki and Göttsche try to find the forerunners of Regiomontanus's large sexagesimal table and which may have influenced him [Glowatzki and Göttsche (1990), pp. 72-125].
    ${ }^{259}$ [Glowatzki and Göttsche (1990), pp. 10, 16, 22]

[^30]:    ${ }^{260}$ However, as observed by Glowatzki and Göttsche, an error in the computation of $\sin 45^{\prime}$ caused other (small) errors, in particular in the interpolation leading to $\sin 1^{\circ}$ [Glowatzki and Göttsche (1990), pp. 26-27].
    ${ }^{261}$ [Glowatzki and Göttsche (1990), p. 23]
    ${ }^{262}$ [Glowatzki and Göttsche (1990), p. 23]
    ${ }^{263}$ [van Brummelen (2009), p. 263] gives Regiomontanus's implied value of $\operatorname{Sin} 1^{\circ}$, but does not describe the actual interpolation process. See also [van Brummelen (2021), pp. 1821], who hints at a procedure below $15^{\prime}$ but without detailing it. Kästner gives also only a cursory description [Kästner (1796), pp. 540-560].
    ${ }^{264}$ See [Glowatzki and Göttsche (1990), p. 71] and [Mett (1996), p. 65].
    ${ }^{265}$ [von Peuerbach and Regiomontanus (1541)]
    ${ }^{266}$ [Regiomontanus (1561)]
    ${ }^{267}$ [Glowatzki and Göttsche (1990), pp. 28-47]

[^31]:    ${ }^{268}$ [Roegel (2021b)]
    ${ }^{269}$ [Glowatzki and Göttsche (1990), p. 27]
    ${ }^{270}$ [Glowatzki and Göttsche (1990), p. 94]
    ${ }^{271}$ See [Folkerts (1977), p. 234] and [Mett (1996), p. 96].
    ${ }^{272}$ [Folkerts et al. (2016), p. 136]
    ${ }^{273}$ [Glowatzki and Göttsche (1990), p. 126]

[^32]:    ${ }^{274}$ [Glowatzki and Göttsche (1990), pp. 127-147]
    ${ }^{275}$ [Roegel (2021b)]
    ${ }^{276}$ [Glowatzki and Göttsche (1990), p. 147]
    ${ }^{277}$ [Regiomontanus (1490)]
    ${ }^{278}$ [Regiomontanus (1490)]
    ${ }^{279}$ [Chabás Bergón (2019), p. 383]
    ${ }^{280}$ [Fincke (1583)]
    ${ }^{281}$ [Glowatzki and Göttsche (1990), p. 183]

[^33]:    ${ }^{293}$ [Glowatzki and Göttsche (1990), p. 193] Incidentally, there have also been surprising statements, such as the one of Davis [Davis (1933), p. 21] who wrote that the first table of secants was that of Maurolico, and that Lansberge was wrong in ascribing this fact to Rheticus, when in fact Lansberge was right, and still is if one ignores manuscript tables.
    ${ }^{294}$ [Viète (1579)]
    ${ }^{295}$ [Fincke (1583)]
    ${ }^{296}$ See [Reinhold (1554)] and [Glowatzki and Göttsche (1990), p. 193].

[^34]:    ${ }^{297}$ [Glowatzki and Göttsche (1990)]
    ${ }^{298}$ [Glowatzki and Göttsche (1990), p. 46-47 and 145-147]

[^35]:    ${ }^{299}$ The errors which have not been reported can easily be found either by a careful comparison of my cleaned tables with Regiomontanus's tables, or by checking the tables given by Glowatzki and Göttsche.

[^36]:    ${ }^{300}$ See the files roegel2021regio6.txt and roegel2021regio10.txt.

[^37]:    ${ }^{301}$ We could also consider the computation of 5 ' pivots from incorrect $15^{\prime}$ pivots, for instance by shifting these pivots, but I don't think we would reach significantly different results.

[^38]:    ${ }^{302}$ [Roegel (2016a)]
    ${ }^{303}$ [Roegel (2010a)]

[^39]:    ${ }^{304}$ See [Hayashi (1997)], [Bressoud (2002)], [Raju (2007), p. 132], [Lefort (2007)] and [Gupta (2008)] for some references (among many others) describing Āryabhaṭa's computation of sines and how the second differences are used.
    ${ }^{305}$ [Wagner and Hunziker (2019)]

[^40]:    ${ }^{306}$ See [Roegel (2010b), § 2.4] for the computation of the exact values of $\Delta \sin x$. The value 5836 is actually about $1 / \sin ^{2} \Delta x$, that is $1 / \sin ^{2} 45^{\prime}$.
    ${ }^{307}$ This constant is given by $1 / \sin ^{2} 15^{\prime} \approx 52525$.
    ${ }^{308}$ The three first differences are then $x-\delta^{2}, x$, and $x+\delta^{2}$, and the average first difference is necessarily the median first difference.

[^41]:    ${ }^{309}$ The two values I am using are in fact given in [von Peuerbach and Regiomontanus (1541)].

[^42]:    ${ }^{310}$ This is in fact also pretty obvious, because like in the case of the trisection, we have a sequence of values of which the median is necessarily equal to the average, and the first first difference is obtained by subtracting twice the second difference from the median value.

[^43]:    ${ }^{311}$ [Glowatzki and Göttsche (1990), p. i]
    ${ }^{312}$ [Rheticus and Otho (1596)]
    ${ }^{313}$ [Pitiscus (1613)]
    ${ }^{314}$ [Glowatzki and Göttsche (1990), p. 148]
    ${ }^{315}$ [Copernicus (1542)] and [Rheticus (1551)].
    ${ }^{316}$ [Reinhold (1554)]
    ${ }^{317}$ [Eisenmenger (1562)]
    ${ }^{318}$ [Viète (1579)]
    ${ }^{319}$ [Fincke (1583)]
    ${ }^{320}$ [Clavius (1586)]
    ${ }^{321}$ [van Lansberge (1591)]
    ${ }^{322}$ [Magini (1592)]
    ${ }^{323}$ [Blundeville (1594)]
    ${ }^{324}$ [Ceulen (1596)]
    ${ }^{325}$ [Glowatzki and Göttsche (1990), pp. 161-168]

[^44]:    ${ }^{337}$ [Rheticus (1551)]
    ${ }^{338}$ [Reinhold (1554)]
    ${ }^{339}$ [Viète (1579)]
    ${ }^{340}$ [Fincke (1583)]
    ${ }^{341}$ On Johannes Engel, see [Knobloch (1983)] and [Dobrzycki and Kremer (1996)]. He is also mentioned by Gessner [Gessner and Simmler (1574), p. 336].
    ${ }^{342}$ [Regiomontanus (1490)]
    ${ }^{343}$ Not all editions seem to contain this sine table, and it is for instance absent from the copy at ULB Darmstadt (Inc II 357).
    ${ }^{344}$ [Folkerts (1977), p. 234]
    ${ }^{345}$ [Glowatzki and Göttsche (1990), pp. 48-49]

[^45]:    ${ }^{346}$ [Glowatzki and Göttsche (1990), pp. 65-71]
    ${ }^{347}$ [Glowatzki and Göttsche (1990), p. 71]
    ${ }^{348}$ [Regiomontanus (1504)]
    ${ }^{349}$ [Roegel (2021d)]

[^46]:    ${ }^{350}$ For a summary of Gaurico's life and works, see [Gessner and Simmler (1574), p. 455] as well as [Moréri (1733), p. 243-244].
    ${ }^{351}$ [Regiomontanus (1524)]
    ${ }^{352}$ [Gaurico (1557)]
    ${ }^{353}$ [Glowatzki and Göttsche (1990), p. 178]
    ${ }^{354}$ This procedure anticipates what Copernicus has probably done in some places in the sine table included in his 1543 opus, although on the basis of Regiomontanus's full sexagesimal table.
    ${ }^{355}$ For a summary of Copernicus's life and works, see [Rosen (1971)]. Note in passing that in 1574 Gessner only briefly mentions Copernicus [Gessner and Simmler (1574), p. 518].
    ${ }^{356}$ See [Curtze (1875), pp. 34-37], [Glowatzki and Göttsche (1990), pp. 190-192] and [Folkerts et al. (2019)].

[^47]:    ${ }^{357}$ Stamm mistakenly wrote that the secants are given for every minute, but this is surely a typo [Stamm (1933)].
    ${ }^{358}$ [Glowatzki and Göttsche (1990), p. 191]
    ${ }^{359}$ See [Curtze (1875), pp. 34-37] and [Rosińska (2002), pp. 15-16].
    ${ }^{360}$ [Birkenmajer (1900), pp. 62-63]
    ${ }^{361}$ [Rosińska (2002), p. 16]
    ${ }^{362}$ [Rosińska (1987), p. 422]
    ${ }^{363}$ [Glowatzki and Göttsche (1990), p. 192]
    ${ }^{364}$ For summaries of Fine's life and works, see [Gallois (1890b)], Poulle [Poulle (1978)], [Marr (2009)], [Pantin (2013)] and [Axworthy (2016), Axworthy (2020)]. See also the accounts given by [Lindgren (2007)] (on land surveys) and [Fréchet (2009)], as well as the early notice by Gessner [Gessner and Simmler (1574), p. 534]. I have chosen to spell his name "Fine," in accordance with Poulle, but it is also sometimes spelled "Finé."

[^48]:    ${ }^{375}$ [Bressieu (1581)]
    ${ }^{376}$ [Roegel (2021e), Roegel (2021f)]
    ${ }^{377}$ See [Roegel (2021g)] and [Roegel (2021h)].
    ${ }^{378}$ For surveys of Apian's life and works, see in particular [Günther (1882)], [Gallois (1890a), pp. 102-116], [North (1966)], [Kish (1970)] and [Röttel (1995)]. See also the early notice by Gessner [Gessner and Simmler (1574), p. 552].
    ${ }^{379}$ See in particular [Lindgren (2007)] for some background on land surveys.
    ${ }^{380}$ See [Kaunzner (1995)], [Folkerts (1995)] and [Lindgren (2007), p. 501].
    ${ }^{381}$ [Apian (1533)]
    ${ }^{382}$ [Apian (1534)]
    ${ }^{383}$ [Apian (1541)]

[^49]:    ${ }^{384}$ Delambre had written that the table was "computed by Apian," but this is a bit excessive [Delambre (1819), p. 395].
    ${ }^{385}$ See [von Peuerbach and Regiomontanus (1541)] and [Glowatzki and Göttsche (1990), pp. 173-174].
    ${ }^{386}$ [Kish (1970)]
    ${ }^{387}$ [Roegel (2021i)]
    ${ }^{388}$ [Bassantin (1557)]
    ${ }^{389}$ [Witekind (1576)]
    ${ }^{390}$ [Peucer (1579)]
    ${ }^{391}$ [Giuntini (1581)]
    ${ }^{392}$ [Padovani (1582)]
    ${ }^{393}$ [Fale (1593)]
    ${ }^{394}$ For summaries of Rheticus's life and works, see in particular [Kästner (1796), 561564], [De Morgan (1841)], [Burmeister (1967-1968)], [Bernleithner (1973)], [Rosen (1975b)], [Kraai (2003)], [Danielson (2006)] [Wanner and Schöbi-Fink (2010)], [Schöbi-Fink and Sonderegger (2014)], and [van Brummelen (2021), pp. 7-9]. Note also Gessner's description of Rheticus's work [Gessner and Simmler (1574), p. 228].
    ${ }^{395}$ [Copernicus (1543)]

[^50]:    ${ }^{396}$ [Copernicus (1542)] An edition of this work is given in [Folkerts et al. (2019)].
    ${ }^{397}$ See [Swerdlow and Neugebauer (1984), pp. 27-28] and [Rosińska (2002), pp. 18-20].
    ${ }^{398}$ See [von Peuerbach and Regiomontanus (1541)], [Zinner (1988), pp. 193-194], [Glowatzki and Göttsche (1990), p. 150] and [Rosińska (1994b)].
    ${ }^{399}$ [Rosińska (1987), p. 423]
    ${ }^{400}$ [Rosińska (2002)]
    ${ }^{401}$ [von Braunmühl (1900, 1903), v. 1, pp. 140-141]
    ${ }^{402}$ [Cantor (1900), p. 474]
    ${ }^{403}$ [Busard (1971a), p. 76]
    ${ }^{404}$ [Rosen (1975b), p. 396]
    ${ }^{405}$ [Folkerts (1977), p. 235]
    ${ }^{406}$ [Zinner (1990), p. 183]

[^51]:    ${ }^{407}$ [Stamm (1933), p. 2]
    ${ }^{408}$ [Rosen (1975b)]
    ${ }^{409}$ [Husson (2014)]
    ${ }^{410}$ [Roegel (2021j)]
    ${ }^{411}$ For a first summary of Copernicus's life and works, see [Rosen (1971)]. A recent biography of Copernicus is that by [Freely (2014)]. For further study, one might turn to [Swerdlow and Neugebauer (1984)], to Owen Gingerich's works as well as to Copernicus's complete works. On the connections between Italy and Krakow before Copernicus, see [Walsh (1996)]. For Copernicus's trigonometric tables, see [Rosińska (2002)].
    ${ }^{412}$ [Copernicus (1543)]

[^52]:    ${ }^{413}$ [Glowatzki and Göttsche (1990), pp. 178-179]
    ${ }^{414}$ [Glowatzki and Göttsche (1990), p. 192]
    ${ }^{415}$ [Glowatzki and Göttsche (1990), p. 150]
    ${ }^{416}$ This procedure is reminiscent from that probably used by Gaurico in 1524, although Gaurico started with Engel's table.

[^53]:    ${ }^{417}$ [Apian (1534)]
    ${ }^{418}$ [Stamm (1933)]
    ${ }^{419}$ [Apian (1534)]
    ${ }^{420}$ [Folkerts (1977), p. 234]
    ${ }^{421}$ [Swerdlow and Neugebauer (1984), pp. 100-101]
    ${ }^{422}$ [Rosińska (1987), p. 422]

[^54]:    ${ }^{423}$ For summaries on Gemma Frisius's life and works, see [Cantor (1878)], [Hallyn (1996), Hallyn (1998), Hallyn (2004), Hallyn (2008)] and [Kish (1972)]. [Lindgren (2007)] gives some background on Gemma Frisius's work on land surveys. Note in passing that Gessner briefly mentions Gemma Frisius [Gessner and Simmler (1574), p. 221].
    ${ }^{424}$ [Gemma Frisius (1545)]
    ${ }^{425}$ [Glowatzki and Göttsche (1990), p. 181]
    ${ }^{426}$ [von Peuerbach (1516)]
    ${ }^{427}$ [Rosen (1975b)]
    ${ }^{428}$ [Rheticus (1551)] See [De Morgan (1845a), De Morgan (1845b)].
    ${ }^{429}$ However, in the first treatise of trigonometry independent of astronomical applications, the Treatise on the Quadrilateral, the Persian al-Tūsī (1201-1274), already in the 13th century, had used all six trigonometric functions [Archibald (1949), p. 31].
    ${ }^{430}$ See [Rosińska (1994b)] and [Glowatzki and Göttsche (1990), p. 152]

[^55]:    ${ }^{431}$ [Glowatzki and Göttsche (1990), p. 185]
    ${ }^{432}$ It will be interesting to see to what conclusions came [Pritchard (2021)] who seems to have conducted a similar investigation, but whose result is not yet published at the time I am writing this.
    ${ }^{433}$ [van Brummelen and Byrne (2021)]

[^56]:    ${ }^{434}$ [Rheticus and Otho (1596)]
    ${ }^{435}$ [Pitiscus (1613)]
    ${ }^{436}$ [Roegel (2010c)]
    ${ }^{437}$ For a summary of Reinhold's life and works, see [Gingerich (1975)]. Note in passing that Gessner briefly mentions Reinhold [Gessner and Simmler (1574), p. 184].
    ${ }^{438}$ [Gingerich (1973)]
    ${ }^{439}$ [Reinhold (1554)]
    ${ }^{440}$ See [van Brummelen (2021), pp. 5-7].
    ${ }^{441}$ [Glowatzki and Göttsche (1990), pp. 152-153]
    ${ }^{442}$ [Glowatzki and Göttsche (1990), p. 185]

[^57]:    ${ }^{443}$ [Rheticus (1551)]
    ${ }^{444}$ The forthcoming study [Pritchard (2021)] may contain some interesting clues on this matter.
    ${ }^{445}$ [Fincke (1583)]
    ${ }^{446}$ [Clavius (1586)]
    ${ }^{447}$ [Glowatzki and Göttsche (1990), p. 185]

[^58]:    ${ }^{448}$ [Viète (1579)]
    ${ }^{449}$ [Roegel (2021k)]
    ${ }^{450}$ See [de Chaufepié (1750), p. 112], [Delambre (1821), v. 1, p. 308-309], [Hoefer (1873), p. 314] and [Henderson (1885)] for some biographical elements on Bassantin. In France, he was called Jacques Bassantin.
    ${ }^{451}$ [Bassantin (1557)]
    ${ }^{452}$ For some interesting information on the physical structure of this work, see [Vaucher (2020)].
    ${ }^{453}$ [Apian (1533)]
    ${ }^{454}$ [Apian (1534)]
    ${ }^{455}$ [Apian (1541)]
    ${ }^{456}$ [Glowatzki and Göttsche (1990), p. 169]
    ${ }^{457}$ For a summary of Maurolico's life and works, see [Masotti (1974)] and [van Brummelen (2021), pp. 12-13]. Note in passing that Gessner briefly mentions Maurolico [Gessner

[^59]:    and Simmler (1574), p. 204].
    ${ }^{458}$ [Maurolico (1558)]
    ${ }^{459}$ [Regiomontanus (1524)]
    ${ }^{460}$ [Apian (1533)]
    ${ }^{461}$ [Glowatzki and Göttsche (1990), pp. 178-179]
    ${ }^{462}$ [von Braunmühl (1900, 1903), vol. 1, p. 151]
    ${ }^{463}$ [Rheticus (1551)]
    ${ }^{464}$ See [van Brummelen and Byrne (2021), p. 200]. In 1944, Zeller had already considered the different opinions of Fincke and Magini, but without settling with any [Zeller (1944), p. 72].
    ${ }^{465}$ [Regiomontanus (1490)]
    ${ }^{466}$ [Glowatzki and Göttsche (1990), pp. 181, 185]
    ${ }^{467}$ [van Brummelen and Byrne (2021), p. 202]

[^60]:    ${ }^{468}$ [van Brummelen and Byrne (2021), p. 205]

[^61]:    ${ }^{469}$ [Delambre (1819), p. 440]
    ${ }^{470}$ [Fincke (1583)]
    ${ }^{471}$ [van Brummelen and Byrne (2021), p. 200]
    ${ }^{472}$ See [van Brummelen and Byrne (2021)] as well as [van Brummelen (2021), p. 22].
    ${ }^{473}$ See the second page of the preface of Magini's Primum mobile [Magini (1609)], of which an excerpt is translated in [van Brummelen and Byrne (2021)], but mistakenly attributed to Magini's De planis triangulis.
    ${ }^{474}$ [van Brummelen and Byrne (2021), p. 206]
    ${ }^{475}$ [Glowatzki and Göttsche (1990), p. 193]

[^62]:    ${ }^{482}$ [Malpangotto (2020), pp. 221-232]
    ${ }^{483}$ [Schreckenfuchs (1569)]
    ${ }^{484}$ [Binz (1888)]
    ${ }^{485}$ [Witekind (1576)]
    ${ }^{486}$ [Apian (1533)], [Apian (1534)] and [Apian (1541)].
    ${ }^{487}$ [Glowatzki and Göttsche (1990), p. 169]

[^63]:    ${ }^{488}$ For a summary of Peucer's life and works, see [Kolb (1976)].
    ${ }^{489}$ [Peucer (1579)]. Earlier editions from 1550 [Chassagnette (2006)] and 1554 do not include the table of sines.
    ${ }^{490}$ [Apian (1533)], [Apian (1534)] and [Apian (1541)].
    ${ }^{491}$ [Glowatzki and Göttsche (1990), p. 169]
    ${ }^{492}$ [Zeller (1944), p. 73]. For summaries of Viète's life and works, see [De Morgan (1843)], [Ritter (1895)], [Busard (1976)], and [van Brummelen (2021), pp. 9-11].

[^64]:    ${ }^{493}$ See [Rheticus (1551)] and [Hunrath (1899)].
    ${ }^{494}$ [Viète (1579)]
    ${ }^{495}$ See [Hunrath (1899)] and [Delambre (1819), pp. 455-456]. The cosines and secants are given more accurately than the other values throughout the table. They are given to four more places from $0^{\circ}$ to $0^{\circ} 2^{\prime}$, to three more places from $0^{\circ} 3^{\prime}$ to $0^{\circ} 24^{\prime}$, to two more places from $0^{\circ} 25^{\prime}$ to $4^{\circ} 5^{\prime}$, and to one more place from $4^{\circ} 6^{\prime}$ to $45^{\circ}$.
    ${ }^{496}$ [Tannery (1896), p. 205]
    ${ }^{497}$ [Glowatzki and Göttsche (1990), pp. 154-155]
    ${ }^{498}$ [Glowatzki and Göttsche (1990), pp. 189, 196]

[^65]:    ${ }^{499}$ See [Viète (1615)] and [Zeller (1944), pp. 79-80].
    ${ }^{500}$ See [Viète (1579), pp. 62-67] and [van Brummelen (2021), pp. 22-23].
    ${ }^{501}$ [Rheticus and Otho (1596)]

[^66]:    ${ }^{502}$ [Pitiscus (1613)]
    ${ }^{503}$ See [Eneström (1892)] and [Cantor (1900), pp. 583-584].
    ${ }^{504}$ [Ritter (1895), p. 54]
    ${ }^{505}$ See [Tannery (1896), p. 208] and [Tannery (1900)].
    ${ }^{506}$ [Roegel (2011)]
    ${ }^{507}$ See [Bosmans (1901-1902), pp. 111-114] and [Bosmans (1901), pp. 297-298].
    ${ }^{508}$ See [von Braunmühl (1900, 1903), v. 1, p. 158] and [Cantor (1900), p. 583].
    ${ }^{509}$ [Eneström (1892)]
    ${ }^{510}$ [Roegel (2011)]
    ${ }^{511}$ See [de Mérez (1880)] for a summary of Bressieu's life and works.
    ${ }^{512}$ [Waddington (1855), p. 337]
    ${ }^{513}$ [Bressieu (1581)]

[^67]:    ${ }^{514}$ [Fine (1530), Fine (1550)]
    ${ }^{515}$ [Viète (1579)]
    ${ }^{516}$ [Reinhold (1554)]

[^68]:    ${ }^{517}$ [Zeller (1944), pp. 86-88]
    ${ }^{518}$ For a summary of Giuntini's life and works, see [Ernst (2001)].
    ${ }^{519}$ [Giuntini (1581)]
    ${ }^{520}$ [Apian (1533)], [Apian (1534)] and [Apian (1541)].
    ${ }^{521}$ For a summary of Padovani's life and works, see [Pizzamiglio (2004), p. 58-59].
    ${ }^{522}$ [Padovani (1582)]
    ${ }^{523}$ [Apian (1533)], [Apian (1534)] and [Apian (1541)].

[^69]:    ${ }^{524}$ For summaries of Fincke's life and works, see [Thorndike (1958), p. 140], [Verdonk (1971)], [Moesgaard (1972), p. 119-120] and [van Brummelen (2021), pp. 13-16]. Some authors have wrongly attributed some works of Kaspar Finck (1578-1631), who was a German theologian, to Thomas Fincke. In particular, the Methodica tractatio doctrinae sphaericae published in 1626, and cited by Moesgaard, is not by Thomas Fincke.
    ${ }^{525}$ [Fincke (1583)]
    ${ }^{526}$ See [Schönbeck (2004)] and [Zeller (1944), pp. 88-90].
    ${ }^{527}$ See also [Glaisher (1873), p. 42].
    ${ }^{528}$ [Zeller (1944), p. 88]
    ${ }^{529}$ [Glowatzki and Göttsche (1990), pp. 157-158]
    ${ }^{530}$ [Reinhold (1554)]

[^70]:    ${ }^{531}$ [Rheticus (1551)]
    ${ }^{532}$ [van Brummelen and Byrne (2021)]
    ${ }^{533}$ [van Brummelen and Byrne (2021), p. 206]
    ${ }^{534}$ [Roegel (2021m)]
    ${ }^{535}$ For summaries of Clavius's life and works, see [Busard (1971b)], [Naux (1983)], [Knobloch (1988)], [Lattis (1994)] and [Sasaki (2003), p. 45-93].

[^71]:    ${ }^{536}$ [Clavius (1586)]
    ${ }^{537}$ [Zeller (1944), pp. 91-94]
    ${ }^{538}$ [Reinhold (1554)]
    ${ }^{539}$ [Fincke (1583)]
    ${ }^{540}$ [Fincke (1583)]
    ${ }^{541}$ [Glowatzki and Göttsche (1990), p. 158]
    ${ }^{542}$ On Bürgi's table of progressions, see [Roegel (2010d)]. The most recent overview of Bürgi's work, which contains many other references, is [Staudacher (2018)]. For reasons explained in [Roegel (2017)], I do not view Bürgi as a coinventor of logarithms.

[^72]:    ${ }^{543}$ [Roegel (2015), Roegel (2016b), Roegel (2016a)]
    ${ }^{544}$ [Roegel (2016c), Roegel (2016d)]
    ${ }^{545}$ [Rheticus and Otho (1596)]
    ${ }^{546}$ [Pitiscus (1613)]

[^73]:    ${ }^{547}$ For a description of some of Gallucci's works, see [Delambre (1821), v. 1, pp. 711-714] and [Ernst (1998)].
    ${ }^{548}$ [Gallucci (1588)]
    ${ }^{549}$ For a summary of Lansberge's life and works, see [Busard (1973)].
    ${ }^{550}$ [van Lansberge (1591)]
    ${ }^{551}$ [Fincke (1583)]
    ${ }^{552}$ [Zeller (1944), pp. 94-97]
    ${ }^{553}$ [Glowatzki and Göttsche (1990), p. 159]

[^74]:    ${ }^{554}$ [Roegel (2021n)]
    ${ }^{555}$ For a summary of Magini's life and works, see [Campedelli (1974)].
    ${ }^{556}$ [Magini (1592)]
    ${ }^{557}$ [Magini (1593)]
    ${ }^{558}$ [Zeller (1944), pp. 97-100]
    ${ }^{559}$ [Clavius (1586)]
    ${ }^{560}$ [Glowatzki and Göttsche (1990), pp. 159-160]
    ${ }^{561}$ [von Peuerbach (1516)]
    ${ }^{562}$ [Magini (1609)]

[^75]:    ${ }^{563}$ See [Rheticus and Otho (1596)] and [Glowatzki and Göttsche (1990), p. 160].
    ${ }^{564}$ [Clavius (1593)]
    ${ }^{565}$ [Clavius (1586)]
    ${ }^{566}$ [Goodwin (1889)]
    ${ }^{567}$ [Fale (1593)] Later editions were printed in 1626, 1627, 1633, 1652 and perhaps other years. A facsimile was published in 1971.
    ${ }^{568}$ [Turner (1989)]
    ${ }^{569}$ [Witekind (1576)]
    ${ }^{570}$ [Glowatzki and Göttsche (1990), p. 177]
    ${ }^{571}$ See [De Morgan (1851), p. 598] and [Goodwin (1889)].

[^76]:    ${ }^{572}$ For summaries of Blundeville's life and works, see [Bullen (1886)], [Jacquot (1953)], [Taylor (1954), p. 173, 331], and [de Smet (1979)].
    ${ }^{573}$ [Blundeville (1594)]
    ${ }^{574}$ [Zeller (1944), p.101]
    ${ }^{575}$ [Clavius (1586)]
    ${ }^{576}$ See [De Morgan (1851), p. 598] and [van Brummelen (2021), p. 53].
    ${ }^{577}$ [Glowatzki and Göttsche (1990), p. 160]
    ${ }^{578}$ For a summary of Ceulen's life and works, see [Struik (1971)].
    ${ }^{579}$ [Ceulen (1596)]
    ${ }^{580}$ [van Lansberge (1591)]
    ${ }^{581}$ See [Ceulen (1596), $\mathrm{f}^{\mathrm{o}} 25$ ] which mentions Regiomontanus, Reinhold, Rheticus, Clavius and Lansberge, but not Fincke. Glowatzki and Göttsche only relate Ceulen's tables to Regiomontanus [Glowatzki and Göttsche (1990), pp. 160-161].

[^77]:    ${ }^{582}$ [Rheticus (1551)]
    ${ }^{583}$ [Rosen (1975b)]
    ${ }^{584}$ [Zeller (1944), p. 62]
    ${ }^{585}$ See [Rheticus and Otho (1596)] and [Roegel (2010e)]. See also [Glaisher (1873), p. 43] and [van Brummelen (2009), pp. 273-282] for descriptions of the Opus palatinum.
    ${ }^{586}$ [Glowatzki and Göttsche (1990), p. i]
    ${ }^{587}$ [Glaisher (1873), p. 43]

[^78]:    ${ }^{588}$ This concurs with De Morgan who considered that "it is clearly nothing but a previous attempt made before the larger plan was resolved on."([De Morgan (1851), p. 599] and [Glaisher (1873), p. 43])

[^79]:    ${ }^{589}$ See [Pitiscus (1613)] and [Roegel (2010f)].
    ${ }^{590}$ De Morgan wrote that "There have been no trigonometrical tables of note published since the invention of logarithms, except those which contain logarithms" [De Morgan (1842), p. 497].
    ${ }^{591}$ See [van Brummelen (2021), pp. 62-109] for a recent survey of the development of logarithms as a continuation of trigonometry.
    ${ }^{592}$ See [Napier (1614)] and [Roegel (2010g)].
    ${ }^{593}$ [Fincke (1583)]
    ${ }^{594}$ [van Lansberge (1591)]

[^80]:    ${ }^{595}$ [Briggs (1617)]
    ${ }^{596}$ [Briggs (1624), Vlacq (1628)]
    ${ }^{597}$ [Gunter (1620)]
    ${ }^{598}$ [Briggs and Gellibrand (1633)]
    ${ }^{599}$ [Glowatzki and Göttsche (1990), p. ii]
    ${ }^{600}$ [Vlacq (1633)]

