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To cite this version:

L Cadilhac, B Collins. A METRIC CHARACTERISATION OF FREENESS. 2021. hal-03330346

HAL Id: hal-03330346

https://hal.science/hal-03330346v1

Preprint submitted on 31 Aug 2021

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arXiv:2103.02944v2 [math.OA] 6 Apr 2021

A METRIC CHARACTERISATION OF FREENESS

L. CADILHAC AND B. COLLINS

Abstract. Let $\mathcal{M}$ be a finite von Neumann algebra and $u_1, \dots, u_N$ be unitaries in $\mathcal{M}$. We show that $u_1, \dots, u_N$ generate $L(\mathbb{F}_N)$ if and only if

$$\left\| \sum_{i=1}^{N} u_i \otimes (u_i^{\mathrm{op}})^* + u_i^* \otimes u_i^{\mathrm{op}} \right\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\mathrm{op}}} = 2\sqrt{2N-1}.$$

1. Introduction

The von Neumann conjecture formulated by Day in 1957 says that a group is not amenable if and only if it contains a non amenable free group. It was first disproved by A. Ol'shanskii in 1980 [9] and since then, the family of counterexamples has been expanded. A similar question can be asked at the level of von Neumann algebras: if a finite factor is not amenable, does it necessarily contain a free group factor? Little is known in that direction except for a breakthrough of Gaboriau and Lyons [4], who show that for certain wreath product groups G (which may not contain $\mathbb{F}_2$), $L(\mathbb{F}_2) \subset L(G)$. Note that the authors of [4] are in fact mainly interested in a version of the von Neumann conjecture for measure preserving actions for which they provide a positive answer.

A difficulty in trying to tackle this problem is that there are no known abstract properties of $\mathcal{M}$ which would characterize the fact that $L(\mathbb{F}_2)$ embeds in $\mathcal{M}$. This remark is what first motivated us to write this note on a metric characterisation of freeness. Although it is not clear that the results obtained (see Corollary 1.2) can be used in the study of the von Neumann conjecture, we believe that they are of independent interest.

Indeed, they generalize at the operator level a well-known result of Kesten [6] who showed that given $g_1, \dots, g_N$ in a countable group G the freeness of the g_i's is characterized by the norm of the Markov operator associated to a random walk on G supported by the g_i's and their inverses. Let us denote by $\lambda : G \to L(G)$ the left-regular representation. In a von Neumann algebraic point of view, Kesten's result can be reformulated as follows:

$$g_1, \dots, g_N \text{ are free in } G \;\Leftrightarrow\; \left\| \sum_{i=1}^{N} \lambda(g_i) + \lambda(g_i)^* \right\|_{L(G)} = 2\sqrt{2N-1}. \tag{1.1}$$

We extend this result by replacing the $\lambda(g_i)$'s by any finite family of unitary operators in a finite von Neumann algebra $\mathcal{M}$. The notion of freeness and Haar unitaries (unitaries with null moments) will be considered with respect to a fixed normal faithful tracial state τ. We obtain the following:

Theorem 1.1. *Let $N \in \mathbb{N}$, $N > 1$. Let $u_1, \dots, u_N$ be unitaries in $\mathcal{M}$. Then, the following assertions are equivalent:*

(1) the operators $u_1, \dots, u_N$ are free Haar unitaries,
(2) we have the equality:

$$\left\| \sum_{i=1}^{N} u_i \otimes (u_i^{\mathrm{op}})^* + u_i^* \otimes u_i^{\mathrm{op}} \right\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\mathrm{op}}} = 2\sqrt{2N-1}.$$

Note that (1) $\Rightarrow$ (2) is a consequence of (1.1). Let us also mention that the inequality

$$\left\| \sum_{i=1}^{N} u_i \otimes (u_i^{\mathrm{op}})^* + u_i^* \otimes u_i^{\mathrm{op}} \right\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\mathrm{op}}} \geq 2\sqrt{2N-1},$$

is verified for any family of unitaries (see [10]). This leads to the following corollary:

Corollary 1.2. *Let $\mathcal{M}$ be a finite von Neumann algebra. Then the following are equivalent:*

(1) $L(\mathbb{F}_2)$ embeds in $\mathcal{M}$,
(2) $\inf_{u_1,u_2\in\mathcal{U}(\mathcal{M})} \|u_1 \otimes \bar{u_1}^* + u_1^* \otimes \bar{u_1} + u_2 \otimes \bar{u_2}^* + u_2^* \otimes \bar{u_2}\|_{\mathcal{M}\otimes\mathcal{M}} = \sqrt{3}$, and this infimum is achieved.

Let us make a few remarks in relation to this result. Firstly, amenability can also be characterized via the consideration of the same quantity. Indeed, in the spirit, of [3, Theorem 5.1], we know that a factor $\mathcal{N}$ is hyperfinite if and only if for every finite family of unitaries $u_1, \ldots, u_n$ in $\mathcal{N}$,

$$\left\| \sum_{i=1}^{n} u_i \otimes (u_i^*)^{\mathrm{op}} \right\|_{\mathcal{N}\otimes^{\min}\mathcal{N}^{\mathrm{op}}} = n.$$

In this sense, it is at the extreme opposite of freeness. Secondly, it is worth pointing out that the problem considered in this manuscript complements results of [7], and also of [5, 2], who consider generators of a group instead of general unitaries. Thirdly, in view of the above papers, it is natural to wonder what are the possible values of

$$\left\| \sum_{i=1}^{N} u_i \otimes (u_i^{\mathrm{op}})^* + u_i^* \otimes u_i^{\mathrm{op}} \right\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\mathrm{op}}}$$

when the unitaries range on all possible choices in any tracial von Neumann algebra. This is clearly a subset of the interval $[2\sqrt{2N-1}, 2N]$, and it can easily be seen that this is the whole interval. Although many other approaches seem to be possible, let us just outline one way to prove this assertion: take N free unitary Brownian motions $\{t \mapsto u_i(t), i \in \{1, \ldots, N\}\}$ as defined in [11]. Using explicit descriptions of the free unitary Brownian motion (see [1]) one can show that it is norm continuous, converges to free Haar unitaries, and that this convergence holds in norm for $t \mapsto \sum_{i=1}^{N} u_i(t) \otimes (u_i^{\mathrm{op}}(t))^* + u_i^*(t) \otimes u_i^{\mathrm{op}}(t)$, therefore its norm is a continuous function taking value $2N$ at $t = 0$ and tending to $2\sqrt{2N-1}$. It follows that the whole range $[2\sqrt{2N-1}, 2N]$ is attained. Finally, it was brought to our attention by Franz Lehner that the traciality condition in Theorem 1.1 is necessary as illustrated by a counterexample appearing in his PhD thesis [8, p.51].

In section 2, we introduce our combinatorial approach to Theorem 1.1. Section 3 contains the core technicalities: we use free group combinatorics in order to obtain a suitable lower bound on positive characters of $\mathbb{F}_N$ which allows us to conclude in Section 4.

Acknowledgements: This work was initiated during the visit of LC to Kyoto University early 2020, after preliminary discussions at MF Oberwolfach in 2018. BC was supported by JSPS KAKENHI 17K18734 and 17H04823. The authors are grateful to Mikael de la Salle, Cyril Houdayer, Eric Ricard, Adam Skalski and Narutaka Ozawa for inspiring comments and discussions.

2. A combinatorial approach

2.1. **Reformulation of Theorem 1.1.** Let $N \in \mathbb{N}$. Let φ be a positive definite function on the free group $\mathbb{F}_N$. We extend φ linearly to $\mathbb{C}[\mathbb{F}_N]$ and keep the same notation, *i.e.* for any finitely supported family $(a_g)_{g\in\mathbb{F}_N} \in \mathbb{C}$,

$$\varphi\left(\sum_{g\in\mathbb{F}_N} a_g \cdot g\right) = \sum_{g\in\mathbb{F}_N} a_g \cdot \varphi(g).$$

Let $s_1, \dots, s_N \in \mathbb{F}_N$ be free generators of $\mathbb{F}_N$ and set

$$a := \sum_{i=1}^{N} s_i + s_i^{-1} \in \mathbb{C}[\mathbb{F}_N].$$

We aim to prove the following:

Theorem 2.1. *Assume that:*

- *φ is constant on the conjugacy classes of $\mathbb{F}_N$ (it is a character),*
- *$\varphi(e) = 1$,*
- *$\exists g \in \mathbb{F}_N, g \neq e, \varphi(g) \neq 0$,*
- *$\forall g \in \mathbb{F}_N, \varphi(g) \geq 0$.*

Then,

$$\lim_{n\to\infty} \varphi(a^{2n})^{\frac{1}{2n}} > 2\sqrt{2N-1}.$$

Lemma 2.2. *Theorem 2.1 implies Theorem 1.1.*

Proof. Consider the representation π of $\mathbb{F}_N$ determined by $\pi(s_i) = u_i \otimes (u_i^{\mathrm{op}})^*$ in $\mathcal{U}(\mathcal{M}\overline{\otimes}\mathcal{M}^{\mathrm{op}})$ for any $i \in \{1, \dots, N\}$. Define $\varphi := (\tau \otimes \tau^{\mathrm{op}}) \circ \pi$. Note that φ is a positive character on $\mathbb{F}_N$ and that

$$\begin{aligned}\left\|\sum_{i=1}^{N} u_i \otimes (u_i^{\mathrm{op}})^* + u_i^* \otimes u_i^{\mathrm{op}}\right\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\mathrm{op}}} &= \lim_{n\to\infty} \left\|\sum_{i=1}^{N} u_i \otimes (u_i^{\mathrm{op}})^* + u_i^* \otimes u_i^{\mathrm{op}}\right\|_{2n} \\ &= \lim_{n\to\infty} \varphi(a^{2n})^{\frac{1}{2n}}.\end{aligned}$$

Assume that the u_i's are not free Haar unitaries. This means that there exists $g \in \mathbb{F}_N$ such that $g \neq e$ and $\varphi(g) \neq 0$. So φ satisfies the conditions for Theorem 2.1 and hence

$$\left\|\sum_{i=1}^{N} u_i \otimes (u_i^{\mathrm{op}})^* + u_i^* \otimes u_i^{\mathrm{op}}\right\|_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\mathrm{op}}} > 2\sqrt{2N-1}.$$

□

Remark 2.3. To prove Theorem 2.1, it suffices to show that

$$\exists n, \varphi(a^{2n})^{\frac{1}{2n}} > 2\sqrt{2N-1},$$

by Hölder's inequality.

2.2. **Some notations.** We always assume $\mathbb{F}_N$ to be equipped with a set of distinguished generators $S = \{s_1, \dots, s_N\}$. Denote by $\mathbb{W}_N$ the set of words on the alphabet $S \cup S^{-1}$. For any $w \in \mathbb{W}_N$, denote by $l(w)$ its length. We identify elements of $\mathbb{F}_N$ with their writing as a reduced word.

Define, for any $k \in \mathbb{N}$ and $w = w_1 \dots w_k \in W_N$ of length k, the circular permutation of w by

$$\sigma_1(w) = w_k w_1 \dots w_{k-1}.$$

For any $t \in \mathbb{Z}$, let $\sigma_t := \sigma_1^t$. We say $w \in \mathbb{W}_N$ is *cyclically reduced* if $\sigma_t(w)$ is reduced for any t (note that it is equivalent to w and $\sigma_1(w)$ being reduced). If w is a reduced word, w can uniquely be written as

$$w = uvu^{-1},$$

where v is cyclically reduced and u is any reduced word. The element v will be referred to as the *root* of w an denoted by $r(w)$.

3. Lower bound for positive characters of $\mathbb{F}_N$

3.1. **Overview of the section.** Let φ be as in Theorem 2.1 and $g_0 \neq e$ such that $\varphi(g_0) = \alpha > 0$. Using Lemma 3.2 (in the following subsection), without loss of generality, g_0 can be chosen to be of even length $2l_0$, $l_0 \in \mathbb{N}$. This section is devoted to proving the following proposition:

Proposition 3.1. *There exists a constant $C > 0$ such that for any $k \geq l_0$,*

$$\sum_{l(g)=2k} \varphi(g) \geq Ck^2(2N-1)^k.$$

Note that the constant C that we obtain above may be explicitly computed from N, l_0 and α but its precise value is inconsequential. Our goal is to exhibit a large enough family of words in $\mathbb{F}_N$ on which we can bound φ from below. To do so, we combine two different operations:

- The first operation consists in conjugating elements thanks to the tracial property. However, by a simple calculation using the estimate given in Lemma 4.1, it can be seen that considering only conjugates of g_0 is not enough to obtain a sufficiently good lower bound on $\varphi(a^{2n})$.
- The second operation is to multiply elements for which we already have a lower bound and apply Lemma 3.2. This is however a bit cumbersome because simplifications may occur and estimating the length of the products obtained this way (to be able to use Lemma 4.1) requires some caution.

The proof is divided into 4 different steps:

- Step 0 - we show that we can obtain new elements of positive trace by multiplication (Lemma 3.2).
- Step 1 - we fix an integer i and apply Lemma 3.2 to the conjugates of g_0 of length $2i$ to obtain a set $\mathcal{C}_i^{(2)}$ of new elements of quantified positive trace which have length approximately $4i$ and are roots. The control on the length is essential to guarantee that our sets $\mathcal{C}_i^{(2)}$ do not overlap too much when i varies. This will be key during step 3.
- Step 2 - we apply circular permutations to the elements of $\mathcal{C}_i^{(2)}$ to construct a bigger set $\mathcal{D}_i$ of roots with positive traces.
- Step 3 - we consider the sets $D_i(k)$ of elements of length $2k$ with roots in D_i, show that they are essentially disjoint and big enough (in terms of trace) to conclude.

To conclude the introduction to the section, fix an integer $k_0 \geq 6$ such that

$$k_0 \geq l_0 + \frac{\ln(4) - \ln(\alpha^2)}{\ln(2N-1)}. \tag{3.1}$$

The main reason for this k_0 is linked to step 1 above. When multiplying a family of elements of $\mathbb{F}_N$ (as in Lemma 3.2) of length $2i$, we would like to keep only the products obtained of length $4i$. Unfortunately, by doing so, we would not obtain enough elements

for the argument to work and hence we have to allow for a certain number of simplifications to occur. This number of simplifications will be given by k_0 which, crucially (see the proof of Lemma 3.4), does not depend on i.

3.2. **Step 0: locating more positivity by multiplication.** The following lemma allows to obtain new elements of positive trace through multiplication.

Lemma 3.2. *Let $k \in \mathbb{N}$ and $a > 0$. Let $h_1, \dots, h_k$ be such that $\varphi(h_i) = \alpha$ for any $j \in \{1, \dots, k\}$. Then,*

$$\sum_{1 \leq i,j \leq k} \varphi(h_i^{-1} h_j) \geq k^2 \alpha^2.$$

Proof. Set $h_0 = e$ and consider the matrix

$$A = (\varphi(h_i^{-1} h_j))_{0 \leq i,j \leq k}.$$

Since φ is positive definite, A is positive. Let $\varepsilon > 0$. Consider the vector $v = (1, -\varepsilon, \dots, -\varepsilon) \in \mathbb{C}^k$. Denote by b the mean of $(\varphi(h_i^{-1} h_j))_{0 < i,j \leq k}$. Note that

$$v^{\perp} A v = 1 - 2k\alpha\varepsilon + k^2 b \varepsilon^2.$$

Set $\varepsilon = \frac{1}{k\alpha}$. By positivity of A, we have:

$$0 \leq 1 - 2 + \frac{b}{\alpha^2}.$$

Hence, $b \geq \alpha^2$. □

3.3. **Step 1: products of conjugates of g_0.** Let $i \in \mathbb{N}$, $i > l_0 + 6k_0$ and consider the set $\mathcal{C}_i$ of all conjugates of g_0 or g_0^{-1} of length $2i$. Denote by R_0 the set of conjugates of g_0 or g_0^{-1} of length $2l_0$. Note that:

$$|\mathcal{C}_i| = |R_0| \, (2N-2)(2N-1)^{i-l_0} =: c_0 (2N-1)^i. \tag{3.2}$$

In the course of the proof, we have to apply circular permutations σ_j to the words we obtain to generate new words. In order to guarantee at that future step that we indeed obtain new words, we need to remove from the beginning some pathological elements of $\mathcal{C}_i$. For any $j \in \mathbb{N}$, define:

$$\mathcal{C}_i(j) := \{g \in \mathcal{C}_i : \forall n, m \in \{k_0 + 1, \dots, i - l_0\} \, (|n - m| = 2j) \Rightarrow g_n = g_m\}, \tag{3.3}$$

where g_n denotes the n-th letter of g. When $j << i$, these are the words of length $2i$ which have been obtained by conjugating g_0 by a word which is $2j$-periodic except maybe on a small part (at the beginning of the word). This means that

$$|C_i(j)| \leq c_0 (2N-1)^{2j + k_0}.$$

Define

$$\mathcal{C}_i' := \mathcal{C}_i \setminus \bigcup_{j \leq \frac{i - l_0}{k_0}} \mathcal{C}_i(j). \tag{3.4}$$

Since $k_0 \geq 6$, $jk_0 \leq i - l_0$ and $i - l_0 \geq 6k_0$, we have $2j + k_0 \leq (i - l_0)/2$. Hence,

$$\sum_{k_0 \leq \frac{i - l_0}{k_0}} |\mathcal{C}_i(j)| \leq c_0 (i - l_0)(2N-1)^{(i-l_0)/2} \leq \frac{c_0}{2} (2N-1)^{i-l_0}.$$

Then, by (3.2),

$$|\mathcal{C}_i'| \geq \frac{c_0}{2} (2N-1)^{i-l_0}. \tag{3.5}$$

Now that we have selected suitable conjugates of g_0, let us multiply them to obtain new elements. Let

$$\mathcal{C}_i^{(2)} := \{r(g) : g \in \mathcal{C}_i' \cdot \mathcal{C}_i', l(g) > 4i - 2k_0\}.$$

Remark 3.3. Let us keep in mind the form of the elements that we are dealing with. Let $h, h' \in \mathcal{C}_i'$. This means that there exists r_0 and r_0' in R_0 and u, u' in $\mathbb{F}_N$ such that $h = ur_0u^{-1}$ and $h' = u'r_0'(u')^{-1}$. Write $u = wv$ and $u' = wv'$ such that no cancellation occur in the product $v^{-1}v'$. Then $r(hh')$ belongs to $\mathcal{C}_i^{(2)}$ if and only if $l(w) < k_0$ and $r(hh') = vr_0v^{-1}v'r_0'v'^{-1}$.

Lemma 3.4. *The following estimate holds:*

$$\sum_{g \in \mathcal{C}_i^{(2)}} \varphi(g) \geq \frac{c_0}{8}\alpha^2(2N-1)^{2i-2l_0-k_0}.$$

Proof. For any $g \in \mathcal{C}_i'$, there are at most $c_0(2N-1)^{i-k_0}$ elements g' of $\mathcal{C}_i'$ for which $l(gg') \leq 4i - 2k_0$. Indeed, the first k_0 letters of g' must coincide with the first k_0 letters of g in order for at least k_0 simplifications to occur. This means that:

$$\sum_{g,g' \in \mathcal{C}_i'} \varphi(gg') \leq c_0(2N-1)^{i-k_0}\,|\mathcal{C}_i'| + \sum_{\substack{g,g' \in \mathcal{C}_i' \\ l(gg') > 4i-2k_0}} \varphi(gg').$$

Then, by Lemma 3.2 (note that considering gg' or $g^{-1}g'$ does not modify the following sum since $\mathcal{C}_i' = (\mathcal{C}_i')^{-1}$):

$$\begin{aligned}\sum_{\substack{g,g' \in \mathcal{C}_i' \\ l(gg') > 4i-2k_0}} \varphi(gg') &\geq |\mathcal{C}_i'|^2\,\alpha^2 - c_0(2N-1)^{i-k_0}\,|\mathcal{C}_i'| \\ &\geq \frac{c_0^2\alpha^2}{4}(2N-1)^{2i-2l_0} - \frac{c_0^2}{2}(2N-1)^{2i-l_0-k_0} \\ &\geq \frac{c_0^2\alpha^2}{8}(2N-1)^{2i-2l_0},\end{aligned}$$

where we used the fact that by definition of k_0,

$$(2N-1)^{-k_0} \leq \frac{\alpha^2}{4}(2N-1)^{-l_0}.$$

Finally, note that at most $(2N-1)^{k_0}$ factors $h, h' \in \mathcal{C}_i'$ can give rise to the same element $r(hh') \in \mathcal{C}_i^{(2)}$. This means that

$$(2N-1)^{k_0} \sum_{g \in \mathcal{C}_i^{(2)}} \varphi(g) \geq \sum_{\substack{g,g' \in \mathcal{C}_i' \\ l(gg') > 4i-2k_0}} \varphi(gg'),$$

which is the desired estimate. □

3.4. **Step 2: circular permutations.** Define:

$$\mathcal{D}_i := \left\{\sigma_j(g) : 0 \leq j \leq \frac{i-l_0}{k_0}, g \in \mathcal{C}_i^{(2)}\right\}.$$

Lemma 3.5. *The following estimate holds:*

$$\sum_{g \in \mathcal{D}_i} \varphi(g) \geq \left\lfloor \frac{i-l_0}{k_0} \right\rfloor \frac{c_0}{8}\alpha^2(2N-1)^{2i-2l_0-k_0}.$$

Proof. Given the estimate obtained in Lemma 3.4, it suffices to prove that for any $0 \leq j, j' \leq i/k_0$ and $g, g' \in \mathcal{C}_i^{(2)}$,

$$\sigma_j(g) = \sigma_{j'}(g') \Rightarrow (g = g' \text{ and } j = j').$$

Let us assume that $\sigma_j(g) = \sigma_{j'}(g')$. If $j = j'$ then immediately $g = g'$ and we get the expected conclusion. So assume by contradiction that $j \neq j'$. We can also assume without loss of generality that $j = 0$.

According to Remark 3.3, g and g' can be written as follows:

$$g = ur_0u^{-1}vs_0v^{-1} \quad g' = u'r_0'(u')^{-1}v's_0'(v')^{-1}$$

where r_0, s_0, r_0', s_0' belong to R_0 and u, u', v, v' are words of the same length j with $i - l_0 - k_0 < j \leq i - l_0$. Write $u = u_1 \dots u_j$ and $u' = u_1' \dots u_j'$. Since $g = \sigma_t(g')$, we obtain by looking at the occurrences of u and u' in g and g',

$$u_n' = u_{n+t} \quad \forall n \in [1, j - t].$$

And, looking at the occurrences of u^{-1} and $(u')^{-1}$,

$$u_n = u_{n+t}' \quad \forall n \in [1, j - t].$$

This means that u is $2t$-periodic. Since g is an element of $\mathcal{C}_i^{(2)}$, g comes from a product hh', $h, h' \in \mathcal{C}_i'$. Write $h = wv_0w^{-1}$. We have $w = w_1 \dots w_{i-l_0-j}u$. By construction of $\mathcal{C}_i'$, w cannot be $2t$-periodic starting from its k_0-th letter, which is a contradiction (see (3.4), (3.3), and recall that by assumption $t \leq \dfrac{i - l_0}{k_0}$). □

3.5. **Step 3: conjugation.** For any $k \geq 2i$, define:

$$\mathcal{D}_i(k) := \{g \in \mathbb{F}_N : |g| = 2k, r(g) \in \mathcal{D}_i\}.$$

Lemma 3.6. *Let $k \in \mathbb{N}$ and $i, i' \geq l_0 + 6k_0$. Assume that $2i, 2i' \leq k$ and $|i - i'| \geq k_0$. Then,*

- *$\mathcal{D}_i(k)$ and $\mathcal{D}_{i'}(k)$ are disjoint,*
- *There exists $C' > 0$ independent of k or i such that:*

$$\sum_{g \in \mathcal{D}_i(k)} \varphi(g) \geq C' i (2N - 1)^k.$$

Proof. For the first point, simply remark that elements in $\mathcal{D}_i$ have length between $4i$ and $4i - 4k_0 + 4$. So if $|i - i'| \geq k_0$ then $\mathcal{D}_i$ and $\mathcal{D}_{i'}$ are disjoint so $\mathcal{D}_i(k)$ and $\mathcal{D}_{i'}(k)$ are disjoint.

Let g be a cyclically reduced element of length $4i$ in $\mathbb{F}_N$. Note for $k > 2i$, there are $(2N - 2)(2N - 1)^{k-2i-1}$ elements of $\mathbb{F}_N$ of length $2k$ and root g. This means that any element of $\mathcal{D}_i$ is the root of at least $(2N - 2)(2N - 1)^{k-i-1}$ elements in $\mathcal{D}_i(k)$. Hence,

$$\sum_{g \in \mathcal{D}_i(k)} \varphi(g) \geq (2N - 2)(2N - 1)^{k-2i-1} \sum_{g \in \mathcal{D}_i} \varphi(g).$$

By Lemma 3.5, we obtain the expected estimate. □

Proof of Proposition 3.1. Let $k \in \mathbb{N}$, k large enough for the sums below to be non-empty (for example $k \geq 20k_0 + 2l_0$). By Lemma 3.6, we have:

$$\begin{aligned}\sum_{|g|=2k} \varphi(g) &\geq \sum_{\substack{6k_0+l_0 \leq i \leq k/2 \\ k_0|i}} \sum_{g \in \mathcal{D}_i(k)} \varphi(g) \\ &\geq \sum_{\substack{6k_0+l_0 \leq i \leq k/2 \\ k_0|i}} C' i (2N-1)^k \\ &\geq C k^2 (2N-1)^{2k},\end{aligned}$$

where C is a small enough constant independent of k. □

4. Proof of the main theorem

For any $w \in \mathbb{W}_N$, denote by $g(w)$ the corresponding element of $\mathbb{F}_N$.

Lemma 4.1. *Let $n \in \mathbb{N}$ and $k < n$. Let $g \in \mathbb{F}_N$ such that $l(g) = 2k$. Then:*

$$|\{w \in \mathbb{W}_N : l(w) = 2n, g(w) = g\}| \geq \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) (2N-1)^{n-k} =: N_{n,k}.$$

Proof. We interpret words as paths on the Cayley graph of $\mathbb{F}_N$. Note that the quantity we consider only depends on $l(g)$. A way to generate paths going from e to *any* element of length $2k$ is to first choose at which times the path is going to go away from e and at which times the path is going to come back towards e. Since the path is going to an element of length $2k$, it has to go $n+k$ times away from e and $n-k$ back to e. The number of possible choices there, for a path of length $2n$, is given by the Catalan triangle

$$C_{n,k} = \binom{2n}{n-k} - \binom{2n}{n-k-1}.$$

Moreover, when we chose to go away from e, there are *at least* $2N-1$ possible directions ($2N$ if the path is at e and $2N-1$ otherwise), and $2N$ possible directions for the first time, thus obtaining

$$C_{n,k} 2N (2N-1)^{n+k-1}$$

paths. Finally, since for now all we have fixed is the length of the target of the path and not a particular point, we have to divide this result by the number of elements of length $2k$ in $\mathbb{F}_N$ *i.e.* $2N(2N-1)^{2k-1}$, to get the desired estimate. □

We are now ready to prove our main theorem.

Proof of Theorem 2.1. Let $n \in \mathbb{N}$. First remark that by Lemma 4.1 and Proposition 3.1,

$$\begin{aligned}\varphi(a^{2n}) &= \sum_{w \in \mathbb{W}_N} \varphi(g(w)) \\ &\geq \sum_{k \leq n} C_{n,k} (2N-1)^{n-k} \sum_{|g|=2k} \varphi(g) \\ &\geq C \sum_{l_0 \leq k \leq n} C_{n,k} k^2 (2N-1)^n.\end{aligned}$$

Note that given the expression of $C_{n,k}$,

$$\sum_{l_0 \leq k \leq n} k^2 C_{n,k} \geq \sup_{l_0 \leq k \leq n} k^2 \binom{2n}{n-k}.$$

Now chose $k = n^{1/3}$. For n large,

$$\begin{aligned}\binom{2n}{n-k} &\sim \frac{2^{2n}}{\sqrt{\pi n}}\left(\frac{n}{n+k}\right)^{n+k}\left(\frac{n}{n-k}\right)^{n-k}\\ &\sim \frac{2^{2n}}{\sqrt{\pi n}}e^{-k+o(1)}e^{k+o(1)}\\ &\sim \frac{2^{2n}}{\sqrt{\pi n}}.\end{aligned}$$

Hence, for n large enough,

$$\varphi(a^{2n}) \geq \frac{C}{\sqrt{\pi}}n^{1/6}2^{2n}(2N-1)^n > 2^{2n}(2N-1)^n.$$

Which concludes the proof by Remark 2.3. □

Proof of Corollary 1.2. Let τ be a faithful trace on $\mathcal{M}$ and consider $L^2(\mathcal{M},\tau)$. It follows from Theorem 1.1 that all reduced words in u_i and their inverses form an orthonormal family. Since the GNS representation of the von Neumann subalgebra generated by u_1, u_2 is faithful, it allows to conclude that is is isomorphic to $L(\mathbb{F}_2)$. □

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