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LORENTZ-SHIMOGAKI-ARAZY-CWIKEL THEOREM REVISITED

L. CADILHAC, F. SUKOCHEV, AND D. ZANIN

Abstract. We present a new approach to Lorentz-Shimogaki and Arazy-Cwikel Theorems which covers all range of $p, q \in (0, \infty]$ for function spaces and sequence spaces. As a byproduct, we solve a conjecture of Levitina and the last two authors.

1. Introduction

Descriptions of interpolation spaces for couples of L_p-spaces, $1 \leq p \leq \infty$ were extensively researched at the end of the 70's and in the 80's, providing satisfying answers to most problems which were considered relevant at the time.

However, new questions arising from noncommutative analysis recently highlighted some gaps in our knowledge of this subject, especially for the case of $p < 1$. In this paper, we revisit some important results of the literature ([20],[1], [25]), generalising them and thus filling some of the holes that were revealed in the theory. In particular, we answer a question asked in [19] by Levitina and the last two authors and already partially studied in [11] regarding the interpolation theory of sequence spaces (see Theorem 1.2). Besides this new result, this paper introduces a general approach which covers the range of all $0 \leq p \leq \infty$ and is self-contained. It puts an emphasis on the use of the space L_0 of all finitely supported measurable functions. As far as the authors know this space rarely appears in interpolation theory (however, see [15], [2] and [14]). We provide evidence that L_0 is a suitable "left endpoint" on the interpolation scale of L_p-spaces, despite its possessing an atypical structure (it is not even an F-space).

Recall that a function space E is an **interpolation space** for the couple (L_p, L_q) if any linear operator T bounded on L_p and L_q is also bounded on E (see Definition 1.6). This notion provides a way of transfering inequalities well-known in L_p-spaces to more exotic ones. To both understand the range of applicability of this technique and be able to check whether it applies to a given function space E, we are interested in simple descriptions of interpolation spaces for the couple (L_p, L_q).

This problem has a long history starting with seminal Calderón-Mityagin theorem [21] and [8] on the couple (L_1, L_∞) and followed by Lorentz and Shimogaki's [20] results on the couples (L_1, L_q) and (L_p, L_∞) with $1 \leq p, q \leq \infty$, which are stated in terms of various submajorizations, see also [17, Theorem 7.2].

We consider two orders: **head majorization** and **tail majorization**. Head majorization coincides with the usual notion of (sub)majorization and already appears in Calderón's work. It is defined on $L_1 + L_\infty$ by:

$$g \prec\prec_{\mathrm{hd}} f \Leftrightarrow \forall t > 0, \int_0^t \mu(s, g) ds \leq \int_0^t \mu(s, f) ds.$$

If moreover $f, g \in L_1$ and $\|f\|_1 = \|g\|_1$, then we write $g \prec_{\rm hd} f$. Above and in the remainder of the text $\mu(g) : t \to \mu(t, g)$ denotes the right-continuous decreasing rearangement of g. Note that $\mu(g)$ is well-defined if and only if g belongs to $L_0 + L_\infty$.

Tail majorization is defined on $L_0 + L_1$ by:

$$g \prec\prec_{\rm tl} f \Leftrightarrow \forall t > 0, \int_t^\infty \mu(s, g)ds \leq \int_t^\infty \mu(s, f)ds.$$

If moreover $f, g \in L_1$ and $\|f\|_1 = \|g\|_1$, then we write $g \prec_{\rm tl} f$. Remark that $g \prec_{\rm tl} f$ if and only if $f \prec_{\rm hd} g$.

It is well-known that the order $\prec\prec_{\rm hd}$ is strongly linked to the interpolation theory of the couples (L_p, L_∞). We show that similarly, the order $\prec\prec_{\rm tl}$ is linked to the couples (L_0, L_q). Combining these two tools, we recover in a self-contained manner characterisations of interpolation spaces for couples of arbitrary L_p-spaces, $0 < p \leq \infty$ earlier obtained in [7]. Note that tail majorization coincides with the weak supermajorization of [12].

Let $\mathcal{X}$ be the linear space of all measurable functions. If not precised otherwise, **the underlying measure space we are working on is** $(0, \infty)$ equipped with the Lebesgue measure m. We obtain the following:

Theorem 1.1. *Let $E \subset \mathcal{X}$ be a quasi-Banach function space (a priori, not necessarily symmetric). Let $p, q \in (0, \infty)$ such that $p < q$. Then:*

(a) *E is an interpolation space for the couple (L_p, L_∞) if and only if there exists $c_{p,E} > 0$ such that for any $f \in E$ and $g \in L_p + L_\infty$,*

$$|g|^p \prec\prec_{\rm hd} |f|^p \Rightarrow g \in E \text{ and } \|g\|_E \leq c_{p,E}\|f\|_E;$$

(b) *E is an interpolation space for the couple (L_0, L_q) if and only if there exists $c_{q,E} > 0$ such that for any $f \in E$ and $g \in L_0 + L_q$,*

$$|g|^q \prec\prec_{\rm tl} |f|^q \Rightarrow g \in E \text{ and } \|g\|_E \leq c_{q,E}\|f\|_E;$$

(c) *E is an interpolation space for the couples (L_0, L_q) and (L_p, L_∞) if and only if it is an interpolation space for the couple (L_p, L_q).*

This extends results of Lorentz-Shimogaki and Arazy-Cwikel to the quasi-Banach setting and contributes to the two first questions asked by Arazy in [9, p.232] in the particular case of L_p-spaces, $0 < p < \infty$. As mentioned before, our approach places L_0 as a left endpoint on the interpolation scale of L_p-spaces, in sharp contrast to earlier results which focused mostly on Banach spaces and had L_1 playing this part. An advantage of our approach is that it naturally encompasses every symmetric quasi-Banach space since they are all interpolation spaces for the couple (L_0, L_∞) (see [15], [2]). On the contrary, there exist some symmetric Banach spaces which are *not* interpolation spaces for the couple (L_1, L_∞) (see [24]). This led to some difficulties which were customarily be circumvented with the help of various technical conditions such as the Fatou property (as appears for example in [3]).

Our strategy in this paper is totally different from the techniques used in [21, 8, 20, 25, 10, 15, 1, 9, 2, 11, 17, 3, 7, 4] and is based on partition lemmas, which were originally developed in a deep paper due to Braverman and Mekler [6], which lies outside of the realm of interpolation theory. The approach of Braverman and Mekler was subsequently revised and redeveloped in [26] and precisely this revision consitutes the core of our approach in this paper.

We restate partition lemmas based on [26, Proposition 19] in Section 3. These lemmas allow us to restrict head and tail majorizations to very simple situations and reduce the problem to functions taking at most two values. Then, we deduce interpolation results from those structural lemmas.

Note that this scheme of proof is quite direct and in particular, does not involve at any point duality related arguments which are applicable only to Banach spaces ([20]) or more generally to L-convex quasi-Banach spaces ([16],[23]).

In Section 5, we pursue the same type of investigation, but in the setting of sequence spaces. The non-diffuse aspect of the underlying measure generates substantial technical difficulties. In particular, we require a new partition lemma which is not as efficient as those in Section 3 (compare Lemmas 5.2 and 3.8). This deficiency has been first pointed out to the authors by Cwikel. However, we are still able to resolve the conjecture of [19] (in the affirmative) by combining Lemma 5.1 with a Boyd-type argument which we borrow from Montgomery-Smith [22]. In particular, we substantially strengthen the results in [11]. Here is the precise statement that we obtain:

Theorem 1.2. *Let $E \subset \ell_\infty$ be a quasi-Banach sequence space and $q \geq 1$. The following conditions are equivalent:*

(a) *there exists $p < q$ such that E is an interpolation space for the couple (ℓ^p, ℓ^q);*

(b) *there exists $c > 0$ such that for any $u \in E$ and $v \in \ell_\infty$*

$$|v|^q \prec\prec_{\mathrm{tl}} |u|^q \Rightarrow v \in E \text{ and } \|v\|_E \leq c\|u\|_E;$$

(c) *for any $u \in E$ and $v \in \ell_\infty$,*

$$|v|^q \prec\prec_{\mathrm{tl}} |u|^q \Rightarrow v \in E.$$

We freely use results of Cwikel [10] and the first author [7] to avoid repeating too many similar arguments.

Preliminaries

Interpolation spaces. The reader is referred to [5] for more details on interpolation theory and [18] for an introduction to symmetric spaces. In the remainder of this section, p and q will denote two nonnegative reals such that $p \leq q$.

Let (Ω, m) be any measure space (in particular the following definitions apply to $\mathbb{N}$ equipped with the counting measure *i.e* sequence spaces). As previously mentioned, $L_0(\Omega) \subset \mathcal{X}(\Omega)$ denotes the set of functions whose supports have finite measures, it is naturally equipped with the group norm

$$\|f\|_0 = m(\text{supp } f), \ f \in L_0(\Omega).$$

The "norm" of a linear operator $T : L_0(\Omega) \to L_0(\Omega)$, is defined as follows:

$$\|T\|_{L_0 \to L_0} = \sup_{f \in L_0} \frac{m(\text{supp}(Tf))}{m(\text{supp}(f))}.$$

Definition 1.3. *A linear space $E \subset \mathcal{X}(\Omega)$ becomes a quasi-Banach function space when equipped with a complete quasi-norm $\|.\|_E$ such that*

- *if $f \in E$ and $g \in \mathcal{X}(\Omega)$ are such that $|g| \leq |f|$, then $g \in E$ and $\|g\|_E \leq \|f\|_E$.*

Definition 1.4. *A quasi-Banach function space $E \subset \mathcal{X}(\Omega)$ is called symmetric if*

- *if $f \in E$ and $g \in \mathcal{X}(\Omega)$ are such that $\mu(f) = \mu(g)$ then $g \in E$ and $\|g\|_E = \|f\|_E$;*

Definition 1.5 (Bounded operator on a couple of quasi-Banach function spaces)**.** *Let X and Y be quasi-Banach function spaces. We say that a linear operator T is bounded on (X, Y) if T is defined from $X + Y$ to $X + Y$ and restricts to a bounded operator from X to X and from Y to Y. Set:*

$$\|T\|_{(X,Y)\to(X,Y)} = \max\left(\|T\|_{X\to X}, \|T\|_{Y\to Y}\right).$$

Les us now recall the precise abstract definition of an interpolation space (see [5], [18]).

Definition 1.6 (Interpolation space between quasi-Banach function spaces)**.** *Let X, Y and Z be quasi-Banach function spaces on Ω. We say that Z is an interpolation space for the couple (X, Y) if $X \cap Y \subset Z \subset X + Y$ and any bounded operator on (X, Y) restricts to a bounded operator on Z. Denote by $\mathrm{Int}(X, Y)$ the set of interpolation spaces for the couple (X, Y).*

Theorem 1.7. *Let $\mathcal{V}$ be a separable topological linear space and let $A, B, C \subset \mathcal{V}$ be quasi-Banach spaces. If C is an interpolation space for the couple (A, B), then there exists a constant $c(A, B, C) > 0$ such that for any bounded operator T on (A, B),*

$$\|T\|_{C\to C} \le c(A, B, C) \cdot \max\{\|T\|_{A\to A}, \|T\|_{B\to B}\}.$$

Proof. In [18, Lemma I.4.3], the assertion is proved for Banach spaces. The argument for quasi-Banach spaces is identical (because it relies on the closed graph theorem, which holds for F-spaces, hence, for quasi-Banach spaces). □

If, in particular, $0 < p < q < \infty$ and if E is an interpolation space for the couple $(L_p(\Omega), L_q(\Omega))$, then there exists a constant $c > 0$ called interpolation constant of E for $(L_p(\Omega), L_q(\Omega))$ such that for any bounded operator T on $(L_p(\Omega), L_q(\Omega))$,

$$\|T\|_{E\to E} \le c\|T\|_{(L_p(\Omega),L_q(\Omega))\to(L_p(\Omega),L_q(\Omega))}.$$

Finally, let us recall the definition of the K-functional associated to a couple of quasi-Banach spaces. For any $t > 0$, and $f \in X + Y$, let

$$K_t(f, X, Y) := \inf_{g+h=f} \|g\|_X + t\|h\|_Y.$$

In addition to that, we need to define an interpolation space between L_0 and a quasi-Banach space.

Definition 1.8 (Bounded operator on a couple $(L_0(\Omega), Y)$ for a quasi-Banach function space Y)**.** *Let Y be a quasi-Banach function space. We say that a linear operator T is bounded on $(L_0(\Omega), Y)$ if T is defined from $L_0(\Omega) + Y$ to $L_0(\Omega) + Y$ and restricts to a bounded operator from $L_0(\Omega)$ to $L_0(\Omega)$ and from Y to Y.*

Definition 1.9 (Interpolation space for a couple $(L_0(\Omega), Y)$ for a quasi-Banach function space Y)**.** *Let Y and Z be quasi-Banach function spaces on Ω. We say that Z is an interpolation space for the couple $(L_0(\Omega), Y)$ if $L_0(\Omega) \cap Y \subset Z \subset L_0(\Omega) + Y$ and any bounded operator on $(L_0(\Omega), Y)$ restricts to a bounded operator on Z. Denote by $\mathrm{Int}(L_0(\Omega), Y)$ the set of interpolation spaces for the couple $(L_0(\Omega), Y)$.*

Symmetry of interpolation spaces. In this subsection, we show that a quasi-Banach interpolation space for a couple of symmetric spaces can always be remormed into a symmetric space. Note that similar results can be found in the literature, see for example [18, Theorem 2.1].

As usual, we will use the term **measure preserving** for a map ω between measure spaces $(\Omega_1, \mathcal{A}_1, m_1)$ and $(\Omega_2, \mathcal{A}_2, m_2)$ verifying,

$$\forall A \in \mathcal{A}_1, \omega^{-1}(A) \in \mathcal{A}_2 \text{ and } m_2(\omega^{-1}(A)) = m_1(A).$$

Lemma 1.10. *Assume that Ω is $(0,1)$, $(0,\infty)$ or $\mathbb{N}$. Let $0 \le f, g \in L_0(\Omega) + L_\infty(\Omega)$ and let $\varepsilon > 0$. Assume that $\mu(f) = \mu(g)$. There exists a measure preserving map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ such that $(1+\varepsilon)(f \circ \omega) \ge \mu(f)$.*

Proof. **Case 1:** Suppose first that $\mu(\infty, f) = \mu(\infty, g) = 0$.

Define, for any $n \in \mathbb{Z}$,

$$F_n = \big\{t : (1+\varepsilon)^n < f(t) \le (1+\varepsilon)^{n+1}\big\}, \quad G_n = \big\{t : (1+\varepsilon)^n < g \le (1+\varepsilon)^{n+1}\big\}.$$

By assumption, $m(F_n) = m(G_n)$ for every $n \in \mathbb{Z}$. Let $\omega_n : G_n \to F_n$ be an arbitrary measure preserving bijection.

Define measure preserving map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ by concatenating $\omega_n : G_n \to F_n$, $n \in \mathbb{Z}$. For every $t \in G_n$, we have

$$f(\omega(t)) \ge (1+\varepsilon)^n, \quad g \le (1+\varepsilon)^{n+1}.$$

Thus,

$$(1+\varepsilon) f(\omega(t)) \ge g, \quad t \in \mathrm{supp}(g).$$

This completes the proof of Step 1.

Case 2. Let δ such that $(1+\delta)^2 = (1+\varepsilon)$. Let $a = \mu(\infty, f) = \mu(\infty, g) > 0$. Define for any $n \ge 1$

$$F_n = \big\{t : a(1+\delta)^n < f(t) \le a(1+\delta)^{n+1}\big\}, \quad G_n = \big\{t : a(1+\delta)^n < g \le a(1+\delta)^{n+1}\big\},$$

and

$$F_0 = \big\{t : (1+\delta)^{-1} a \le f(t) \le (1+\delta) a\big\}, \quad G_0 = \big\{t : 0 < g \le (1+\delta) a\big\}.$$

By asumption for any $n \ge 1$, $m(G_n) = m(F_n)$ and $m(G_0) = m(F_0) = \infty$. For any $n \ge 0$, choose a measure preserving bijection ω_n from G_n to F_n.

Define the measure preserving map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ by concatenating the ω_n's. For any $n \ge 0$ and any $t \in G_n$,

$$f(\omega(t)) \ge a(1+\delta)^{n-1}, \quad g \le a(1+\delta)^{n+1}.$$

Thus,

$$(1+\delta)^2 f(\omega(t)) = (1+\varepsilon) f(\omega(t)) \ge g, \quad t \in \mathrm{supp}(g).$$

□

Lemma 1.11. *Assume that Ω is $(0,1)$, $(0,\infty)$ or $\mathbb{N}$. Let $E, A, B \subset (L_0 + L_\infty)(\Omega)$ be quasi-Banach function spaces. Assume that A and B are symmetric and that E is an interpolation space for the couple (A, B). Then E admits an equivalent symmetric quasi-norm.*

Proof. Let $f \in E$ and $g \in L_0 + L_\infty$. Assume that $\mu(g) \le \mu(f)$. By Lemma 1.10, there exists a map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ such that for any $t \in \mathrm{supp}(g)$,

$$2\,|f \circ \omega(t)| \ge |g(t)|\,.$$

Define, for any $h \in \mathcal{X}(\Omega)$,

$$T(h) := \begin{cases} \dfrac{g}{f \circ \omega} h \circ \omega & \text{on } \operatorname{supp}(g) \\ 0 & \text{elsewhere.} \end{cases}$$

Since ω is measure preserving, T is bounded on A and B of norm less than 2. Let c_E be the interpolation constant of E for the couple (A, B) (as in Theorem 1.7). We know that $Tf = g \in E$ and

$$\|g\| \leq 2c_E \|f\|. \tag{1.1}$$

Define, for any $f \in E$

$$\|f\|_{E'} = \inf_{\mu(g) \geq \mu(f)} \|g\|_E.$$

By (1.1), $\|f\|_{E'} \leq \|f\|_E \leq 2c_E \|f\|_{E'}$ and $(E, \|\cdot\|_{E'})$ is a symmetric space. □

Remark 1.12. It is not difficult to see that if the underlying measure space Ω contains both a continuous part and atoms, Lemma 1.11 is no longer true for $A = L_p(\Omega)$, $B = L_q(\Omega)$ and $p < 1$. However, one can observe that if A and B are *fully symmetric* (*i.e.* interpolation spaces between $L_1(\Omega)$ and $L_\infty(\Omega)$), Lemma 1.11 remains valid for any Ω. This is reminiscent of the conditions required in [25, Section 4].

2. Interpolation for the couple (L_0, L_q)

In this section, $\Omega = (0, \infty)$ (for brevity, we omit Ω in the notations). We investigate some basic properties of the interpolation couple (L_0, L_q). First, we provide a statement analogous to Theorem 1.7 and applicable to L_0.

Since the closed graph theorem does not apply to L_0 (it is not an F-space), our proof uses a concrete constructions that relies on the structure of the underlying measure space.

For any $f \in \mathcal{X}$, denote by M_f the multiplication operator $g \mapsto f \cdot g$.

Theorem 2.1. *Let E be a quasi-Banach function space and $q \in (0, \infty]$. Assume that E is an interpolation space for the couple (L_0, L_q). Then, there exists a constant c such that for any contraction T on (L_0, L_q), $\|T\|_{E \to E} \leq c$.*

Proof. Let $(A_n)_{n \geq 1}$ be a partition of $(0, \infty)$ such that $m(A_n) = \infty$ for every $n \geq 1$.

Let $\gamma_n : A_n \to A_n^c$ be a measure preserving bijective transform. Set

$$(U_n x)(t) = \begin{cases} x(\gamma_n(t)), & t \in A_n \\ 0, & t \in A_n^c \end{cases}, \quad (V_n x)(t) = \begin{cases} x(\gamma_n^{-1}(t)), & t \in A_n^c \\ 0, & t \in A_n \end{cases}.$$

Obviously, U_n and V_n are bounded operators on the couple (L_0, L_q). By assumption, $U_n, V_n : E \to E$ are bounded mappings.

Note that

$$V_n U_n = M_{\chi_{A_n^c}}, \quad n \geq 1.$$

Let us argue by contradiction. For any $n \geq 1$, choose an operator T_n which is a contraction on (L_0, L_q) and such that

$$\|T_n\|_{E \to E} \geq 4^n \cdot \max\{\|U_n\|_{E \to E}, \|V_n\|_{E \to E}, 1\}^2. \tag{2.1}$$

It is immediate that

$$T_n = M_{\chi_{A_n}} T_n M_{\chi_{A_n}} + M_{\chi_{A_n^c}} T_n M_{\chi_{A_n}} + M_{\chi_{A_n}} T_n M_{\chi_{A_n^c}} + M_{\chi_{A_n^c}} T_n M_{\chi_{A_n^c}} =$$

$$= T_{1,n} + V_n T_{2,n} + T_{3,n} U_n + V_n T_{4,n} U_n,$$

where

$$T_{1,n} = M_{\chi_{A_n}} T_n M_{\chi_{A_n}}, \quad T_{2,n} = U_n T_n M_{\chi_{A_n}},$$
$$T_{3,n} = M_{\chi_{A_n}} T_n V_n, \quad T_{4,n} = U_n T_n V_n.$$

By quasi-triangle inequality, we have

$$\|T_n\|_{E\to E} \le C_E^2 \cdot \Big(\sum_{k=1}^{4} \|T_{k,n}\|_{E\to E}\Big) \cdot \max\{\|U_n\|_{E\to E}, \|V_n\|_{E\to E}.1\}^2.$$

Let $k_n \in \{1,2,3,4\}$ be such that

$$\|T_{k_n,n}\|_{E\to E} = \max_{1\le k\le 4} \|T_{k,n}\|_{E\to E}.$$

We, therefore, have

$$\|T_n\|_{E\to E} \le 4C_E^2 \|T_{k_n,n}\|_{E\to E} \cdot \max\{\|U_n\|_{E\to E}, \|V_n\|_{E\to E}.1\}^2. \tag{2.2}$$

Set $S_n = T_{k_n,n}$. Note that $\|S_n\|_{L_0\to L_0} \le 1$, $\|S_n\|_{L_q\to L_q} \le 1$. A combination of (2.1) and (2.2) yields

$$\|S_n\|_{E\to E} \ge 4^{n-1} C_E^{-2}, \quad n \ge 1.$$

Note that

$$S_n = M_{\chi_{A_n}} S_n M_{\chi_{A_n}}.$$

Set

$$S = \sum_{n\ge 1} S_n.$$

Since the S_n's are in direct sum, $\|S\|_{L_0\to L_0} = \sup_{n\ge1} \big\|S_n\big\|_{L_0\to L_0} \le 1$ and $\|S\|_{L_q\to L_q} = \sup_{n\ge1} \big\|S_n\big\|_{L_q\to L_q} \le 1$.

Moreover, E is an interpolation space for the couple (L_0, L_q), it follows that $S : E \to E$ is bounded.

For any $n \ge 1$, choose $f_n \in E$ such that $\|f_n\|_E \le 1$ and $\|S_n f_n\|_E \ge 4^{n-2} C_E^{-2}$. Recall that $S_n = S_n M_{\chi_{A_n}}$. Hence, we may assume without loss of generality that f_n is supported on A_n. Thus, $S(f_n) = S_n(f_n)$ and

$$\big\|S(f_n)\big\|_E = \big\|S_n(f_n)\big\|_E \ge 4^{n-2} C_E^{-2}.$$

This contradicts the boundedness of S.

□

Remark 2.2. Theorem 2.1 above remains true for other underlying measure spaces:

- *for sequence spaces*. Indeed, in the proof of Theorem 2.1, we only use properties of the underlying measure space in the first sentence, namely when we consider a partition of $(0,\infty)$ into countably many sets, each of them isomorphic to $(0,\infty)$. Since a partition satisfying the same property exists for $\mathbb{Z}_+$, Theorem 2.1 remains true for interpolation spaces between ℓ_0 and ℓ_q.
- *for* $(0,1)$. The same general idea applies in this case but some modification have to be made because the maps γ_n introduced in the proof cannot be assumed to be measure preserving. The details are left to the reader.

Lemma 2.3. *Let $E, Y \subset L_0 + L_\infty$ be quasi-Banach function spaces. Assume that Y is symmetric and that E is an interpolation space for the couple (L_0, Y). Then E admits an equivalent symmetric quasi-norm.*

Proof. The argument follows that in Lemma 1.11 *mutatis mutandi.* □

The following assertion is a special case Theorem 1.1 and an important ingredient in the proof of the latter theorem.

Theorem 2.4. *Let E be a quasi-Banach function space and $q \in (0, \infty)$. Assume that $L_0 \cap L_q \subset E \subset L_0 + L_q$ and that for any $f \in E$ and $g \in L_0 + L_q$,*

$$|g|^q \prec\prec_{\mathrm{tl}} |f|^q \Rightarrow g \in E.$$

Then E is an interpolation space for the couple (L_0, L_q).

The rest of this section is occupied with the proof of Theorem 2.4.

Let C_E be the concavity modulus of E, that is, (the minimal) constant such that

$$\|x_1 + x_2\|_E \le C_E(\|x_1\|_E + \|x_2\|_E), \quad x_1, x_2 \in E.$$

It allows to write a triangle inequality with infinitely many summands. This inequality should be understood in the usual sense: if scalar-valued series in the right hand side converges, then the series on the left hand side converges in E and the inequality holds.

Lemma 2.5. *Let E be a quasi-Banach space. We have*

$$\|\sum_{n\ge1} x_n\|_E \le \sum_{n\ge1} C_E^n \|x_n\|_E, \quad (x_n)_{n\ge1} \subset E.$$

Proof. The proof is standard and is, therefore, omitted. □

Lemma 2.6. *Let $E \subset L_0 + L_q$ be a symmetric quasi-Banach function space. If $\|f\|_E \le 1$, then $\|\mu(f)\chi_{(1,\infty)}\|_q \le c_E$.*

Proof. Assume the contrary. Choose $(f_n)_{n\ge1} \in E$ such that $\|f_n\|_E \le 1$ and $\|\mu(f_n)\chi_{(1,\infty)}\|_q \ge n$. Set

$$h_n = \sum_{k\ge0} \mu(k+1, f_n)\chi_{(k,k+1)}, \quad n \ge 1.$$

It is immediate that $\|h_n\|_E \le 1$ and $\|h_n\|_q \ge n$.

Let C_E be the concavity modulus of the space E and let $m \ge 4C_E$ be a natural number. Set

$$h = \sum_{n\ge1} (2C_E)^{-n} h_{m^n}.$$

By Lemma 2.5, we have

$$\|h\|_E \le \sum_{n\ge1} C_E^n \|(2C_E)^{-n} h_{m^n}\|_E \le \sum_{n\ge1} 2^{-n} = 1.$$

Since $h \in E$, it follows that $h \in L_0 + L_q$. Since $h = \mu(h)$ is constant on the interval $(0, 1)$, it follows that $h \in L_q$. Thus,

$$\|h\|_q \ge (2C_E)^{-n} \|h_{m^n}\|_q \ge (2C_E)^{-n} \cdot m^n \ge 2^n, \quad n \ge 1.$$

It follows that $h \notin L_q$. This contradiction completes the proof. □

Lemma 2.7. *Let E be as in Theorem 2.4. If T is a contraction on (L_0, L_q), then $T : E \to E$.*

Proof. Take $f \in E$. Fix $t > 0$ and let $A = A_1 \bigcup A_2$, where

$A_1 = \{|f| > \mu(t, f)\}$ and $A_2 \subset \{|f| = \mu(t, f)\}$ is such that $m(A_2) = t - m(A_1)$.

Obviously, $m(A) = t$ and

$$\mu(s, f\chi_{A^c}) = \mu(s + t, f), \quad s > 0.$$

Now, we write

$$Tf = T(f\chi_A) + T(f\chi_{A^c}).$$

Since $T : L_q \to L_q$ is a contraction, it follows that

$$\|T(f\chi_{A^c})\|_q^q \le \|f\chi_{A^c}\|_q^q = \int_t^\infty \mu^q(s + t, f)ds = \int_t^\infty \mu^q(s, f)ds,$$

$$\|T(f\chi_A)\|_0 \le \|f\chi_A\|_0 \le t.$$

Now,

$$\int_t^\infty \mu^q(s, Tf)ds = \inf\{\|Tf - h\|_q^q : \ \|h\|_0 \le t\}.$$

Taking $h = T(f\chi_A)$, we write

$$\int_t^\infty \mu^q(s, Tf)ds \le \|Tf - T(f\chi_A)\|_q^q = \|T(f\chi_{A^c})\|_q^q \le \int_t^\infty \mu^q(s, f)ds.$$

Since $t > 0$ is arbitrary, it follows that

$$|Tf|^q \prec\prec_{\mathrm{tl}} |f|^q.$$

By assumption, $Tf \in E$. Since $f \in E$ is arbitrary, it follows that $T : E \to E$. □

Our next lemma demonstrates the boundedness of the mapping $T : E \to E$ established in Lemma 2.7.

Lemma 2.8. *Let E be as in Theorem 2.4. If T is a contraction on (L_0, L_q), then the mapping $T : E \to E$ is bounded.*

Proof. Using Lemma 2.3, we may assume without loss of generality that E is symmetric.

Let us show that the graph of $T : E \to E$ is closed. Assume the contrary. Choose a sequence $(f_n)_{n\ge1} \subset E$ such that $f_n \to f \in E$ and $Tf_n \to g \ne Tf$ in E. Replacing $(f_n)_{n\ge1}$ with $(f_n - f)_{n\ge1}$ if necessary, we may assume without loss of generality that $f = 0$. That is, $f_n \to 0$ in E and $Tf_n \to g \ne 0$ in E.

Passing to a subsequence, if needed, we may assume without loss of generality that

$$\frac{\|f_n\|_E}{\|\chi_{(0,\frac{1}{n})}\|_E} \to 0, \quad n \to \infty.$$

Clearly,

$$\|f_n\|_E = \|\mu(f_n)\|_E \ge \|\mu(f_n)\chi_{(0,\frac{1}{n})}\|_E \ge \mu(\frac{1}{n}, f_n)\|\chi_{(0,\frac{1}{n})}\|_E.$$

It follows that $\mu(\frac{1}{n}, f_n) \to 0$ as $n \to \infty$.

Set

$$f_{1n} = f_n\chi_{\{|f_n|>\mu(\frac{1}{n},f_n)\}}, \quad n \ge 1,$$

$$f_{2n} = f_n\chi_{\{\mu(1,f_n)<|f_n|\le\mu(\frac{1}{n},f_n)\}}, \quad n \ge 1,$$

$$f_{3n} = f_n \chi_{\{|f_n| \le \mu(1, f_n)\}}, \quad n \ge 1.$$

Since $T : L_0 \to L_0$ is a contraction, it follows that

$$\|T(f_{1n})\|_0 \le \|f_{1n}\|_0 \le \frac{1}{n}, \quad n \ge 1.$$

In particular, $T(f_{1n}) \to 0$ in measure.

Since $T : L_q \to L_q$ is a contraction, it follows that

$$\|T(f_{2n})\|_q \le \|f_{2n}\|_q.$$

Since f_{2n} is supported on a set of measure 1, it follows that

$$\|f_{2n}\|_q \le \|f_{2n}\|_\infty = \mu(\frac{1}{n}, f_n) \to 0, \quad n \to \infty.$$

In particular, $T(f_{2n}) \to 0$ in measure.

Since $T : L_q \to L_q$ is a contraction, it follows that

$$\|T(f_{3n})\|_q \le \|f_{3n}\|_q \le \| \min\{\mu(f_n), \mu(1, f_n)\}\|_q.$$

By Lemma 2.6, we have

$$\| \min\{\mu(f_n), \mu(1, f_n)\}\|_q \le c_{q,E} \|f_n\|_E \to 0, \quad n \to \infty.$$

In particular, $T(f_{3n}) \to 0$ in measure.

Combining the last 3 paragraphs, we conclude that $Tf_n \to 0$ in measure. However, $Tf_n \to g$ in E and, therefore, in measure. It follows that $g = 0$, which contradicts our choice of the sequence $(f_n)_{n \ge 1}$. This contradiction shows that the graph of $T : E \to E$ is closed. By the closed graph theorem, $T : E \to E$ is bounded. □

Proof of Theorem 2.4. Let T be a bounded operator on (L_0, L_q). Fix $n \in \mathbb{N}$ such that

$$\|T\|_{L_0 \to L_0} \le n, \quad \|T\|_{L_q \to L_q} \le n.$$

Let $\sigma_{\frac{1}{n}}$ be a dilation operator defined by the usual formula

$$(\sigma_{\frac{1}{n}} f)(t) = f(nt), \quad n \ge 0.$$

Set

$$S = \frac{1}{n} \sigma_{\frac{1}{n}} \circ T \text{ so that } T = n\sigma_n \circ S.$$

Obviously,

$$\|S\|_{L_0 \to L_0} \le 1, \quad \|S\|_{L_q \to L_q} \le 1.$$

By Lemma 2.8, $S : E \to E$ is a bounded mapping. Since E is a symmetric quasi-Banach function space, it follows that $\sigma_n : E \to E$ is bounded; this implies $T : E \to E$. □

Theorem 2.4 applies in particular to L_p-spaces, $p \le q$. We decided to add more precise statement and to provide a direct proof of the latter.

Corollary 2.9. *Let $p, q \in (0, \infty)$ such that $p < q$. Then, L_p is an interpolation space for the couple (L_0, L_q). More precisely, if T is a contraction on (L_0, L_q), then T is a contraction on L_p.*

Proof. Let us first consider characteristic functions. Let E be a set with finite measure. Since T is a contraction on L_0, the measure of the support of $T(\chi_E)$ is less than $m(E)$. So by Hölder's inequality, setting $r = (p^{-1} - q^{-1})^{-1}$, we have:

$$\|T(\chi_E)\|_p \leq \|T(\chi_E)\|_q \cdot m(E)^{1/r} \leq \|\chi_E\|_q \cdot m(E)^{\frac{1}{r}} = \|\chi_E\|_p.$$

First, consider the case $p \leq 1$. Let $f \in L_p$ be a step function, i.e.

$$f = \sum_{i\in\mathbb{N}} a_i \chi_{E_i},$$

where $a_i \in \mathbb{C}$ and the sets E_i are disjoint sets with finite measure. By the p-triangular inequality:

$$\|Tf\|_p^p \leq \sum_{i\in\mathbb{N}} |a_i|^p \|T(\chi_{E_i})\|_p^p \leq \sum_{i\in\mathbb{N}} |a_i|^p \|\chi_{E_i}\|_p^p = \|f\|_p^p.$$

Since $T : L_p \to L_p$ is bounded by Theorem 2.4 and since step functions are dense in L_p, it follows that $T : L_p \to L_p$ is a contraction (for $p \leq 1$).

Now consider the case $p > 1$. Since $p < q$, it follows that $q > 1$. By the preceding paragraph, $T : L_1 \to L_1$ is a contraction. By complex interpolation, $T : L_p \to L_p$ is also contraction. □

3. Construction of contractions on (L_0,L_q) and (L_p,L_∞)

Let $p, q \in (0, \infty)$. In this section, we are interested in the following question. Given functions f and g in L_0+L_q (resp. L_p+L_∞), does there exist a bi-contraction T on (L_0,L_q) (resp. (L_p,L_∞)) such that $T(f) = g$? We show that such an operator exists provided that $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$ (resp. $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$. This directly implies a necessary condition for a symmetric space to be an interpolation space for the couple (L_0, L_q) (resp. (L_p, L_∞)) which will be exploited in the next section.

Our method of proof is very direct. We construct the bi-contraction T as direct sums of very simple operators. This is made possible by three partition lemmas that enable us to understand the orders $\prec\prec_{\mathrm{tl}}$ and $\prec\prec_{\mathrm{hd}}$ as direct sums of simple situations.

3.1. **Partition lemmas.**

We state our first lemma without proof since it essentially repeats that of Proposition 19 in [26].

Lemma 3.1. *Let $f, g \in L_1$ be positive decreasing step functions. Assume that $g \prec_{\mathrm{hd}} f$. There exists a partition $\{I_k, J_k\}_{k\geq 0}$ of $(0, \infty)$ such that*

(i) *for every $k \geq 0$, I_k and J_k are disjoint intervals of finite length;*
(ii) *$(I_k \cup J_k) \cap (I_l \cup J_l) = \varnothing$ for $k \neq l$;*
(iii) *f and g are constant on I_k and on J_k;*
(iv) *$g|_{I_k\cup J_k} \prec_{\mathrm{hd}} f|_{I_k\cup J_k}$ for every $k \geq 0$;*

Remark 3.2. Note that in [26], Proposition 19 is proved for couples of functions f, g of the form

$$f = \sum_{i\in\mathbb{Z}} f_i \mathbf{1}_{[a_i,a_{i+1})},\ g = \sum_{i\in\mathbb{Z}} g_i \mathbf{1}_{[a_i,a_{i+1}]},$$

with (a_i) an increasing sequence in $(0, \infty)$ and $f_i, g_i \in \mathbb{C}$ for any $i \in \mathbb{N}$. However, the proof applies with very little modification to couples of decreasing positive step functions as in Lemma 3.1 above.

Scholium 3.3. *Let $f, g \in \mathcal{X}$ be positive decreasing functions. Let $\Delta \subset (0,\infty)$ be an arbitrary measurable set.*

(i) *if $f, g \in L_1 + L_\infty$ are such that*
$$\int_{[0,t]\cap\Delta} g \le \int_{[0,t]\cap\Delta} f, \quad t > 0,$$
then $g\chi_\Delta \prec\prec_{\rm hd} f\chi_\Delta$.

(ii) *if $f, g \in L_0 + L_1$ are such that*
$$\int_{(t,\infty)\cap\Delta} g \le \int_{(t,\infty)\cap\Delta} f, \quad t > 0,$$
then $g\chi_\Delta \prec\prec_{\rm tl} f\chi_\Delta$.

Our second partition lemma shows that the order $\prec\prec_{\rm hd}$ can be reduced for our purpose to a direct sum of $\prec_{\rm hd}$ and $\le$.

Lemma 3.4. *Let $f, g \in L_1 + L_\infty$ be such that $f = \mu(f)$, $g = \mu(g)$ and $g \prec\prec_{\rm hd} f$. There exists a collection $\{\Delta_k\}_{k\ge 0}$ of pairwise disjoint sets such that*

(i) *$g|_{\Delta_k} \prec_{\rm hd} f|_{\Delta_k}$ for every $k \ge 0$;*
(ii) *$g \le f$ on the complement of $\bigcup_{k\ge 0} \Delta_k$;*

Proof. Consider the set $\{g > f\}$. Since $g - f$ is right-continuous, it follows that, for every $t \in \{g > f\}$, there exists an $\epsilon > 0$ (which depends on t) such that $[t, t+\epsilon) \subset \{g > f\}$. Hence, connected components of the set $\{g > f\}$ are intervals (closed or not) not reduced to points. Let us enumerate these intervals as (a_k, b_k), $k \ge 0$ (it does not really matter for us if boundary points of these intervals belong to $\{g > f\}$).

We have
$$\int_0^t (f-g)_+ - \int_0^t (f-g)_- = \int_0^t (f-g) \ge 0.$$
Let
$$H(t) = \inf\Big\{u : \int_0^u (f-g)_+ = \int_0^t (f-g)_-\Big\}.$$
Obviously, H is a monotone function, $H(t) \le t$ for all $t > 0$ and
$$\int_0^{H(t)} (f-g)_+ = \int_0^t (f-g)_-.$$

Set
$$\Delta_k = (a_k, b_k) \cup \Big((H(a_k), H(b_k)) \cap \{f \ge g\}\Big).$$

Note that
$$\int_0^{a_k} (f-g)_+ = \int_0^{b_k} (f-g)_+ \ge \int_0^{b_k} (f-g)_-$$
and, therefore, $H(b_k) \le a_k$.

We claim that $\Delta_k \cap \Delta_l = \varnothing$ for $k \ne l$. Indeed, let $a_k < b_k \le a_l < b_l$. We have $H(a_k) \le H(b_k) \le H(a_l) \le H(b_l)$. Thus, $(H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) = \varnothing$. We now have
$$\Delta_k \cap \Delta_l = \Big((\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\})\Big) \bigcup \Big((\Delta_k \cap \{f \ge g\}) \cap (\Delta_l \cap \{f \ge g\})\Big).$$
Obviously,
$$(\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\}) = (a_k, b_k) \cap (a_l, b_l) = \varnothing,$$

$(\Delta_k \cap \{f \geq g\}) \cap (\Delta_l \cap \{f \geq g\}) = (H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) \cap \{f \geq g\} = \varnothing$. This proves the claim.

We now claim that

$$\int_{[0,t]\cap\Delta_k} (f-g) \geq 0.$$

If $t \leq a_k$, then, taking into account that $H(b_k) \leq a_k$, we infer that $[0,t] \cap \Delta_k \subset \{f \geq g\}$ and the claim follows immediately. If $t \in (a_k, b_k)$, then

$$\int_{[0,t]\cap\Delta_k} (f-g) = \int_{(H(a_k),H(b_k))} (f-g)_+ - \int_{a_k}^{t} (f-g)_- \geq$$

$$\geq \int_{(H(a_k),H(b_k))} (f-g)_+ - \int_{a_k}^{b_k} (f-g)_- = 0.$$

This proves the claim.

It follows from the claim and Scholium 3.3 that $g\chi_{\Delta_k} \prec\prec_{\mathrm{hd}} f\chi_{\Delta_k}$. Since

$$\int_{\Delta_k} g = \int_{\Delta_k} f,$$

the first assertion follows.

By construction, $(a_k, b_k) \subset \Delta_k$. Thus,

$$\{g > f\} = \bigcup_{k\geq 0} (a_k, b_k) \subset \bigcup_{k\geq 0} \Delta_k.$$

The second assertion is now obvious. □

Finally, the third partition lemma deals with describing the order $\prec\prec_{\mathrm{tl}}$ in terms of $\prec_{\mathrm{tl}}$ and $\leq$.

Lemma 3.5. *Let $f, g \in L_0 + L_1$ be such that $f = \mu(f)$, $g = \mu(g)$ and $g \prec\prec_{\mathrm{tl}} f$. There exists a collection $\{\Delta_k\}_{k\geq 0}$ of pairwise disjoint sets such that*

(i) *$f|_{\Delta_k} \prec_{\mathrm{hd}} g|_{\Delta_k}$ for every $k \geq 0$;*
(ii) *$g \leq f$ on the complement of $\bigcup_{k\geq 0} \Delta_k$;*

Proof. Consider the set $\{g > f\}$. Similarly to the previous proof, connected components of the set $\{g > f\}$ are intervals (closed or not) not reduced to points. Let us enumerate these intervals as (a_k, b_k), $k \geq 0$.

We have

$$\int_t^\infty (f-g)_+ - \int_t^\infty (f-g)_- = \int_t^\infty (f-g) \geq 0.$$

Let

$$H(t) = \sup\Big\{u : \int_u^\infty (f-g)_+ = \int_t^\infty (f-g)_-\Big\}.$$

Obviously, H is a monotone function, $H(t) \geq t$ for all $t > 0$ and

$$\int_{H(t)}^\infty (f-g)_+ = \int_t^\infty (f-g)_-.$$

Set

$$\Delta_k = (a_k, b_k) \cup \Big((H(a_k), H(b_k)) \cap \{g \leq f\}\Big).$$

Note that

$$\int_{b_k}^\infty (f-g)_+ = \int_{a_k}^\infty (f-g)_+ \geq \int_{a_k}^\infty (f-g)_-$$

and, therefore, $H(a_k) \geq b_k$.

We claim that $\Delta_k \cap \Delta_l = \varnothing$ for $k \neq l$. Indeed, let $a_k < b_k \leq a_l < b_l$. We have $H(a_k) \leq H(b_k) \leq H(a_l) \leq H(b_l)$. Thus, $(H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) = \varnothing$. We now have

$$\Delta_k \cap \Delta_l = \Big((\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\})\Big) \bigcup \Big((\Delta_k \cap \{f \geq g\}) \cap (\Delta_l \cap \{f \geq g\})\Big).$$

Obviously,

$$(\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\}) = (a_k, b_k) \cap (a_l, b_l) = \varnothing,$$

$$(\Delta_k \cap \{f \geq g\}) \cap (\Delta_l \cap \{f \geq g\}) = (H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) \cap \{f \geq g\} = \varnothing.$$

This proves the claim.

We now claim that

$$\int_{(t,\infty)\cap\Delta_k} (f - g) \geq 0.$$

If $t \geq b_k$, then, taking into account that $H(a_k) \geq b_k$, we infer that $(t, \infty) \cap \Delta_k \subset \{f \geq g\}$ and the claim follows immediately. If $t \in (a_k, b_k)$, then

$$\int_{(t,\infty)\cap\Delta_k} (f - g) = \int_{(H(a_k),H(b_k))} (f - g)_+ - \int_t^{b_k} (f - g)_- \geq$$

$$\geq \int_{(H(a_k),H(b_k))} (f - g)_+ - \int_{a_k}^{b_k} (f - g)_- = 0.$$

This proves the claim.

It follows from the claim and Scholium 3.3 that $g\chi_{\Delta_k} \prec\prec_{\mathrm{tl}} f\chi_{\Delta_k}$. Since

$$\int_{\Delta_k} g = \int_{\Delta_k} f,$$

it follows that $g\chi_{\Delta_k} \prec_{\mathrm{tl}} f\chi_{\Delta_k}$, which immediately implies the first assertion.

By construction, $(a_k, b_k) \subset \Delta_k$. Thus,

$$\{g > f\} = \bigcup_{k\geq 0} (a_k, b_k) \subset \bigcup_{k\geq 0} \Delta_k.$$

The second assertion is now obvious. □

3.2. **Construction of operators.** We repeat the same structure as in the previous subsection, proving four lemmas, each one dealing with a certain order : $\prec_{\mathrm{hd}}$, $\prec_{\mathrm{tl}}$, $\prec\prec_{\mathrm{hd}}$ and finally $\prec\prec_{\mathrm{tl}}$.

Lemma 3.6. *Let $p \in (0, \infty)$. Let $f, g \in L_p(0, \infty)$, assume that $|g|^p \prec_{\mathrm{hd}} |f|^p$, $f = \mu(f)$ and $g = \mu(g)$. There exists a linear operator $T : \mathcal{X}(0, \infty) \to \mathcal{X}(0, \infty)$ such that $g = T(f)$ and*

$$\|T\|_{L_p\to L_p} \leq 2 \cdot 3^{\frac{1}{p}}, \quad \|T\|_{L_\infty\to L_\infty} \leq 2 \cdot 2^{\frac{1}{p}}.$$

Proof. Step 1: First, let us assume that f and g are step functions

Apply Lemma 3.1 to the functions f^p and g^p and let I_k and J_k be as in Lemma 3.1. Without loss of generality, the interval I_k is located to the left of the interval J_k.

For every $k \geq 0$, define the mapping $S_k : \mathcal{X}(I_k \cup J_k) \to \mathcal{X}(I_k \cup J_k)$ as below. The construction of this mapping will depend on whether $f^p|_{J_k} \leq \frac{1}{2} g^p|_{J_k}$ or $f^p|_{J_k} > \frac{1}{2} g^p|_{J_k}$.

If $f^p|_{J_k} \le \frac{1}{2} g^p|_{J_k}$, then

$$g^p|_{J_k} \cdot m(J_k) \le g^p|_{I_k} \cdot m(I_k) + g^p|_{J_k} \cdot m(J_k) =$$

$$= f^p|_{I_k} \cdot m(I_k) + f^p|_{J_k} \cdot m(J_k) \le f^p|_{I_k} \cdot m(I_k) + \frac{1}{2} g^p|_{J_k} \cdot m(J_k).$$

Therefore,

$$g^p|_{J_k} \cdot m(J_k) \le 2 f^p|_{I_k} \cdot m(I_k).$$

Let l_k be a linear bijection from J_k to I_k. We set

$$S_k x = \frac{g|_{I_k}}{f|_{I_k}} \cdot x\chi_{I_k} + \frac{g|_{J_k}}{f|_{I_k}} \cdot (x \circ l_k)\chi_{J_k}.$$

Clearly, S_k is a contraction in the uniform norm.

Let $x \in L_p$. We have

$$\|S_k x\|_p^p \le \frac{g^p|_{I_k}}{f^p|_{I_k}} \cdot \|x\chi_{I_k}\|_p^p + \frac{g^p|_{J_k}}{f^p|_{I_k}} \cdot \|(x \circ l_k)\chi_{J_k}\|_p^p \le$$

$$\le \frac{g^p|_{I_k}}{f^p|_{I_k}} \cdot \|x\|_p^p + \frac{g^p|_{J_k}}{f^p|_{I_k}} \cdot \frac{m(J_k)}{m(I_k)} \cdot \|x\|_p^p \le 3\|x\|_p^p.$$

Also, we have

$$\|S_k x\|_\infty \le \|x\|_\infty.$$

If $f^p|_{J_k} > \frac{1}{2} g^p|_{J_k}$, then we set $S_k = M_{gf^{-1}}$. Clearly, $\|S_k x\|_\infty \le 2^{\frac{1}{p}} \|x\|_\infty$ and $\|S_k x\|_p \le 2^{\frac{1}{p}} \|x\|_p$.

We define $S : \mathcal{X} \to \mathcal{X}$ by:

$$S = \bigoplus_{k \ge 0} S_k.$$

Remark that for any $r \in [0, \infty]$, $\|S\|_{r \to r} = \sup_{k \ge 0} \|S_k\|_{L_r \to L_r}$. Hence,

$$\|S\|_{L_p \to L_p} \le 3^{\frac{1}{p}}, \quad \|S\|_{L_\infty \to L_\infty} \le 2^{\frac{1}{p}}.$$

It remains only to note that $Sf = g$.

Step 2: Now, only assume that f and g are positive and non-increasing. Define for any $n \in \mathbb{Z}$,

$$a_n = \sup\left\{t \in (0, \infty) : f(t) \ge 2^{\frac{n}{2}}\right\} \text{ and } b_n = \sup\left\{t \in (0, \infty) : g(t) \ge 2^{\frac{n}{2}}\right\}.$$

Let $\mathcal{A}$ be the σ-algebra generated by the intervals (a_n, a_{n+1}) and (b_n, b_{n+1}). Define

$$f_0^p = \mathbb{E}[f^p \mid \mathcal{A}] \text{ and } g_0^p = \mathbb{E}[g^p \mid \mathcal{A}].$$

Note that f_0 and g_0 are step functions such that

$$g_0^p \prec_{\mathrm{hd}} f_0^p, \quad 2^{-\frac{1}{2}} f \le f_0 \le 2^{\frac{1}{2}} f \quad \text{and} \quad 2^{-\frac{1}{2}} g \le g_0 \le 2^{\frac{1}{2}} g.$$

Apply Step 1 to f_0 and g_0 to obtain an operator S and set

$$T = M_{ff_0^{-1}} \circ S \circ M_{g_0 g^{-1}}.$$

Clearly, $Tf = g$ and

$$\|T\|_{L_p \to L_p} \le 2\|S\|_{L_p \to L_p} \le 2 \cdot 2^{\frac{1}{p}} \quad \text{and } \|T\|_{L_\infty \to L_\infty} \le 2\|S\|_{L_\infty \to L_\infty} \le 2^2 \cdot 2^{\frac{1}{p}}.$$

□

Lemma 3.7. *Let $f, g \in L_q(0,\infty)$ be positive non-increasing functions such that $g^q \prec_{\rm tl} f^q$. Let $d > 1$. There exists a linear operator $T : \mathcal{X}(0,\infty) \to \mathcal{X}(0,\infty)$ such that $g = T(f)$ and*

$$\|T\|_{L_0\to L_0} \le 4, \quad \|T\|_{L_q\to L_q} \le 2\cdot 3^{\frac{1}{q}}.$$

Proof. Following step 2 of Lemma 3.6, we are reduced to dealing with step functions.

Apply Lemma 3.1 to the functions g^q and f^q and let $(I_k)_{k\ge1}$ and $(J_k)_{k\ge1}$ be as in Lemma 3.1. Without loss of generality, the intervals I_k is located to the left of the intervals J_k.

Let $k \ge 1$. Define the mappings $S_k : \mathcal{X}(I_k \cup J_k) \to \mathcal{X}(I_k \cup J_k)$ as below.

Note that since $f^q|_{I_k\cup J_k} \prec_{\rm hd} g^q|_{I_k\cup J_k}$,

$$g|_{I_k} \ge f|_{I_k} \ge f|_{J_k} \ge g|_{J_k}.$$

The construction of S_k will depend on whether $\|f\chi_{I_k}\|_q \le \|f\chi_{J_k}\|_q$ or $\|f\chi_{I_k}\|_q > \|f\chi_{J_k}\|_q$.

If $\|f\chi_{I_k}\|_q \le \|f\chi_{J_k}\|_q$, then $m(I_k) \le m(J_k)$ and

$$g_0^q|_{I_k}\cdot m(I_k) \le 2f^q|_{J_k}\cdot m(J_k).$$

Let $l_k : I_k \to J_k$ be a linear bijection. We set

$$S_k x = \frac{g|_{I_k}}{f|_{J_k}}(x\circ l_k)\chi_{I_k} + \frac{g}{f}x\chi_{J_k}.$$

Let $x \in L_q$.

$$\|S_k x\|_0 \le \|x\chi_{J_k}\|_0 + \|(x\circ l_k)\chi_{I_k}\|_0 \le \left(1 + \frac{m(I_k)}{m(J_k)}\right)\|x\|_0.$$

Thus, $\|S_k\|_{L_0\to L_0} \le 2$. We have

$$\|S_k x\|_q^q = \frac{g^q|_{I_k}}{f^q|_{J_k}}\|(x\circ l_k)\chi_{I_k}\|_q^q + \frac{g^q|_{J_k}}{f^q|_{J_k}}\|x\chi_{J_k}\|_q^q \le$$

$$\le \frac{g^q|_{I_k}}{f^q|_{J_k}}\cdot\frac{m(I_k)}{m(J_k)}\cdot\|x\|_q^q + \frac{g^q|_{J_k}}{f^q|_{J_k}}\|x\|_q^q \le 3\|x\|_q^q.$$

If $\|f\chi_{I_k}\|_q > \|f\chi_{J_k}\|_q$, then

$$g^q|_{I_k}\cdot m(I_k) \le 2f^q|_{I_k}\cdot m(I_k) \text{ and, therefore, } g_0 \le 2^{\frac{1}{q}} f_0.$$

We set $S_k = M_{gf^{-1}}$. Obviously, $\|S_k\|_{L_0\to L_0} \le 1$ and $\|S_k\|_{L_q\to L_q} \le 2^{\frac{1}{q}}$.

We set

$$S = \bigoplus_{k\ge0} S_k.$$

Since $S : \mathcal{X}(0,\infty) \to \mathcal{X}(0,\infty)$ is defined as a direct sum:

$$\|S\|_{L_0\to L_0} = \sup_{k\ge1}\left\|S_k\right\|_{L_0\to L_0} \le 2, \quad \|S\|_{L_q\to L_q} = \sup_{k\ge1}\left\|S_k\right\|_{L_q\to L_q} \le 3^{\frac{1}{q}}.$$

It remains only to note that $Sf = g$. □

Lemma 3.8. *Let $f, g \in (L_p + L_\infty)(0,\infty)$ be such that $|g|^p \prec\prec_{\rm hd} |f|^p$, $f = \mu(f)$ and $g = \mu(g)$. There exists a linear operator $T : \mathcal{X} \to \mathcal{X}$ such that $g = T(f)$ and*

$$\|T\|_{L_p\to L_p} \le 2\cdot 3^{\frac{1}{p}}, \quad \|T\|_{L_\infty\to L_\infty} \le 2\cdot 2^{\frac{1}{p}}.$$

Proof. Let $(\Delta_k)_{k\geq 0}$ be as in Lemma 3.4 and let Δ_∞ be the complement of $\bigcup_{k\geq 0}\Delta_k$. By Lemma 3.6, there exists $T_k : \mathcal{X}(\Delta_k) \to \mathcal{X}(\Delta_k)$ such that $T_k(f) = g$ on Δ_k and such that

$$\|T_k\|_{L_p(\mathcal{X}_k)\to L_p(\mathcal{X}_k)} \leq 2\cdot 9^{\frac{1}{p}}, \quad \|T_k\|_{L_\infty(\mathcal{X}_k)\to L_\infty(\mathcal{X}_k)} \leq 2\cdot 4^{\frac{1}{p}}.$$

Set $T_\infty = M_{\frac{g}{f}}$ on $\mathcal{X}(\Delta_\infty)$. We now set

$$T = T_\infty \bigoplus \Big(\bigoplus_{k\geq 0} T_k\Big).$$

Obviously, $Tf = g$ on $(0,\infty)$ and

$$\|T\|_{L_p\to L_p} \leq 2\cdot 9^{\frac{1}{p}}, \quad \|T\|_{L_\infty\to L_\infty} \leq 2\cdot 4^{\frac{1}{p}}.$$

□

Lemma 3.9. *Let $f, g \in (L_0 + L_q)(0,\infty)$ be such that $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$, $f = \mu(f)$ and $g = \mu(g)$. There exists a linear operator $T : \mathcal{X} \to \mathcal{X}$ such that $g = T(f)$ and*

$$\|T\|_{L_0\to L_0} \leq 4, \quad \|T\|_{L_q\to L_q} \leq 2\cdot 3^{\frac{1}{q}}.$$

Proof. Without loss of generality, $g = \mu(g)$ and $f = \mu(f)$. Let $(\Delta_k)_{k\geq 0}$ be as in Lemma 3.5 and let Δ_∞ be the complement of $\bigcup_{k\geq 0}\Delta_k$. By Lemma 3.7, there exists $T_k : \mathcal{X}(\Delta_k) \to \mathcal{X}(\Delta_k)$ such that $T_k(f) = g$ on Δ_k and such that

$$\|T_k\|_{L_0(\mathcal{X}_k)\to L_0(\mathcal{X}_k)} \leq 8, \quad \|T_k\|_{L_q(\mathcal{X}_k)\to L_q(\mathcal{X}_k)} \leq 2\cdot 9^{\frac{1}{q}}.$$

Set $T_\infty = M_{\frac{g}{f}}$ on $\mathcal{X}(\Delta_\infty)$. We now set

$$T = T_\infty \bigoplus \Big(\bigoplus_{k\geq 0} T_k\Big).$$

Obviously, $Tf = g$ on $(0,\infty)$ and

$$\|T\|_{L_0\to L_0} \leq 4, \quad \|T\|_{L_q\to L_q} \leq 2\cdot 3^{\frac{1}{q}}.$$

□

4. Interpolation spaces for the couple (L_p, L_q)

In this section, we obtain characterizations of interpolation spaces for the couple (L_p, L_q), in terms of the majorization notions studied earlier. The necessity of the condition we consider is a direct consequence of the constructions explained in the previous section. The fact that it is sufficient is shown by linking majorization to the K-functional.

Theorem 4.1. *Let $0 \leq p < q \leq \infty$. Let E be a quasi-Banach function space such that $E \in \mathrm{Int}(L_p, L_q)$. There exist $c_{p,E}$ and $c_{q,E}$ in $\mathbb{R}_{>0}$ such that:*

(i) *suppose $p \neq 0$: for any $f \in E$ and $g \in L_p + L_\infty$ if $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$, then $g \in E$ and $\|g\|_E \leq c_{p,E}\|f\|_E$;*
(ii) *suppose $q \neq \infty$: for any $f \in E$ and $g \in L_0 + L_q$ if $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$, then $g \in E$ and $\|g\|_E \leq c_{q,E}\|f\|_E$.*

Proof. By Lemma 1.11 (for $p > 0$) or Lemma 2.3 (for $p = 0$), we may assume without loss of generality that E is a symmetric function space.

Assume that $p \neq 0$. Let $f \in E$ and let $g \in L_p + L_\infty$ be such that $|g|^p \prec\prec_{\rm hd} |f|^p$. Since E is symmetric, we may assume without loss of generality that $f = \mu(f)$ and $g = \mu(g)$. By Lemma 3.8, there exists an operator T such that $T(f) = g$ and

$$\|T\|_{(L_p,L_\infty)\to(L_p,L_\infty)} \leq 2 \cdot 3^{\frac{1}{p}}.$$

Recall that L_q is an interpolation space for the couple (L_p, L_∞) (one can take, for example, real or complex interpolation method). Let $c_{p,q}$ be the interpolation constant of L_q for the couple (L_p, L_∞). We have

$$\|T\|_{L_q\to L_q} \leq c_{p,q} \cdot 2 \cdot 3^{\frac{1}{p}}.$$

Let c_E be the interpolation constant of E for (L_p, L_q) (see Theorem 1.7). Then,

$$\|T\|_{E\to E} \leq c_E \cdot \max\{1, c_{p,q}\} \cdot 2 \cdot 3^{\frac{1}{p}}.$$

Thus,

$$\|g\|_E \leq \|T\|_{E\to E}\|f\|_E \leq c_E \cdot \max\{1, c_{p,q}\} \cdot 2 \cdot 3^{\frac{1}{p}}\|f\|_E.$$

This proves the first assertion. The proof of the second one follows *mutatis mutandi* using Lemma 2.9 instead of complex interpolation and (for $p = 0$) Theorem 2.1 instead of Theorem 1.7. □

Lemma 4.2. *Assume that $0 < p < q < \infty$. Let $f, g \in L_p + L_q$ such that $f = \mu(f)$ and $g = \mu(g)$. Suppose that at every $t > 0$, one of the following inequalities holds*

$$\int_0^t g^p ds \leq \int_0^t f^p ds \text{ or } \int_t^\infty g^q ds \leq \int_t^\infty f^q ds.$$

Then, there exist $g_1, g_2 \in (L_p + L_q)^+$ satisfying: $g = g_1 + g_2$, $g_1^p \prec\prec_{\rm hd} f^p$ and $g_2^q \prec\prec_{\rm tl} f^q$.

Proof. Set:

$$A = \Big\{t > 0 : \int_0^t g(s)^p ds \leq \int_0^t f(s)^p ds\Big\},$$
$$B = \Big\{t > 0 : \int_t^\infty g(s)^q ds \leq \int_t^\infty f(s)^q ds\Big\}.$$

Let

$$u_+(t) = \inf\{s \in A : \ s \geq t\}, \quad u_-(t) = \sup\{s \in A : s \leq t\},$$
$$v_+(t) = \inf\{s \in B : \ s \geq t\}, \quad v_-(t) = \sup\{s \in B : s \leq t\}.$$

Note that, for $t \notin A$, $f^p\chi_{(u_-(t),u_+(t))} \prec_{\rm hd} g^p\chi_{(u_-(t),u_+(t))}$ and, therefore, $g(u_+(t) - 0) \leq f(u_+(t) - 0)$.

Set $h_1(t) = g(u_+(t) - 0)$, $t > 0$. By definition, $u_+(t) \geq t$ for all $t > 0$. Since g is decreasing, it follows that $h_1 \leq g$. Set $h_2 = g\chi_B$. Since $u_+(t) = t$ for $t \in A$, it follows that $h_1 = g$ on A. Thus, $h_1 + h_2 \geq g\chi_A + g\chi_B \geq g$.

We claim that

$$\int_0^t \mu(s, h_1)^p ds \leq \int_0^t f(s)^p ds.$$

Indeed, for $t \in A$, we have

$$\int_0^t h_1(s)^p ds \leq \int_0^t g(s)^p ds \leq \int_0^t f(s)^p ds.$$

For $t \notin A$, we have $h_1(s) = g(u_+(t))$ for all $s \in (u_-(t), u_+(t))$. Thus,

$$\int_0^t h_1(s)^p ds = \int_0^{u_-(t)} h_1(s)^p ds + \int_{u_-(t)}^t h_1(s)^p ds \leq$$

$$\leq \int_0^{u_-(t)} g(s)^p ds + \int_{u_-(t)}^t g(u_+(t))^p ds.$$

Since

$$\int_0^{u_-(t)} g(s)^p ds = \int_0^{u_-(t)} f(s)^p ds, \quad g(u_+(t)) \leq f(u_+(t)),$$

it follows that

$$\int_0^t h_1(s)^p ds \leq \int_0^{u_-(t)} f(s)^p ds + \int_{u_-(t)}^t f(u_+(t))^p ds \leq \int_0^t f(s)^p ds.$$

Since $h_1 = \mu(h_1)$, the claim follows.

We claim that

$$\int_t^\infty \mu(s, h_2)^q ds \leq \int_t^\infty f(s)^q ds.$$

For $t \in B$, we have

$$\int_t^\infty h_2(s)^q ds \leq \int_t^\infty g(s)^q ds \leq \int_t^\infty f(s)^q ds.$$

For $t \notin B$, we have

$$\int_t^\infty h_2(s)^q ds = \int_{v_+(t)}^\infty h_2(s)^q ds \leq$$

$$\leq \int_{v_+(t)}^\infty g(s)^q ds = \int_{v_+(t)}^\infty f(s)^q ds \leq \int_t^\infty f(s)^q ds.$$

In either case,

$$\int_t^\infty \mu(s, h_2)^q ds \leq \int_t^\infty h_2(s)^q ds \leq \int_t^\infty f(s)^q ds.$$

This proves the claim.

Setting

$$g_1 = \frac{h_1}{h_1 + h_2} g, \quad g_2 = \frac{h_2}{h_1 + h_2} g,$$

we complete the proof. □

Theorem 4.3. *Let $0 \leq p < q \leq \infty$ with either $p \neq 0$ or $q \neq \infty$. Let E be a quasi-Banach function space. Assume that there exist $c_{p,E}$ and $c_{q,E}$ in $\mathbb{R}_{>0}$ such that:*

(i) *if $p \neq 0$: for any $f \in E$ and $g \in L_0 + L_p$, if $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$, then $g \in E$ and $\|g\|_E \leq c_{p,E} \|f\|_E$;*

(ii) *if $q \neq \infty$: for any $f \in E$ and $g \in L_0 + L_q$, if $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$, then $g \in E$ and $\|g\|_E \leq c_{q,E} \|f\|_E$;*

Then E belongs to $\mathrm{Int}(L_p, L_q)$.

Proof. Assume that $p \neq 0$. Let us show that the first condition implies $E \subset L_p + L_\infty$. Indeed, assume the contrary and choose $f \in E$ such that $\mu(f)\chi_{(0,1)} \notin L_p$. Let

$$f_n = \min\{\mu(\frac{1}{n}, f), \mu(f)\chi_{(0,1)}\}, \quad n \geq 1.$$

Obviously, $\|f_n\|_E \leq \|\mu(f)\chi_{(0,1)}\|_E \leq \|f\|_E$. On the other hand, $\|f_n\|_p^p \chi_{(0,1)} \prec\prec_{\rm hd} f_n^p$. By the first condition on E, we have $\|f_n\|_p \|\chi_{(0,1)}\|_E \leq c_{p,E}\|f\|_E$. However $\|f_n\|_p \uparrow \|\mu(f)\chi_{(0,1)}\|_p = \infty$. This contradiction shows that our initial assumption was incorrect. Thus, $E \subset L_p + L_\infty$.

A similar argument shows that the second condition implies $E \subset L_0 + L_q$. Thus, a combination of both conditions implies $E \subset L_p + L_q$.

Let T be a contraction on (L_p, L_q) and $f \in E$. To conclude the proof, it suffices to show that Tf belongs to E. First, note that

$$K(t, Tf, L_p, L_q) \leq K(t, f, L_p, L_q).$$

Assume that $p > 0$ and $q < \infty$. Let $\alpha^{-1} = \frac{1}{p} - \frac{1}{q}$. By Holmstedt formula for the K-functional (see [13]), there exists a constant $c_{p,q} > 0$ such that for any $t \in \mathbb{R}_{>0}$:

$$\Big(\int_0^{t^\alpha} \mu(s, Tf)^p ds\Big)^{\frac{1}{p}} + t\Big(\int_{t^\alpha}^\infty \mu(s, Tf)^q ds\Big)^{\frac{1}{q}} \leq$$

$$\leq c_{p,q}\Big(\Big(\int_0^{t^\alpha} \mu(s, f)^p ds\Big)^{\frac{1}{p}} + t\Big(\int_{t^\alpha}^\infty \mu(s, f)^q ds\Big)^{\frac{1}{q}}\Big).$$

Hence, for any given $t > 0$, we either have

$$\int_0^{t^\alpha} \mu(s, Tf)^p ds \leq \int_0^{t^\alpha} \mu(s, c_{p,q} f)^p ds$$

or

$$\int_{t^\alpha}^\infty \mu(s, Tf)^q ds \leq \int_{t^\alpha}^\infty \mu(s, c_{p,q} f)^q ds.$$

By Lemma 4.2, one can write

$$\mu(Tf) = g_1 + g_2, \quad g_1^p \prec\prec_{\rm hd} (c_{p,q}\mu(f))^p, \quad g_2 \prec\prec_{\rm tl} (c_{p,q}\mu(f))^q. \tag{4.1}$$

By assumption, we have

$$\|g_1\|_E \leq c_{p,E}\|f\|_E, \quad \|g_2\|_E \leq c_{q,E}\|f\|_E.$$

By triangle inequality, we have

$$\|Tf\|_E \leq c_{p,q,E}\|f\|_E.$$

Assume now that $p > 0$ and $q = \infty$. This case is simpler since by the Holmstedt formula (see [13]), there exists $c_p \in \mathbb{R}_{>0}$ such that for any $t \in \mathbb{R}_{>0}$:

$$\Big(\int_0^{t^p} \mu(s, Tf)^p ds\Big)^{\frac{1}{p}} \leq c_p\Big(\int_0^{t^p} \mu(s, f)^p ds\Big)^{\frac{1}{p}}.$$

This means that $|Tf|^p \prec\prec_{\rm hd} |c_p f|^p$ so by assumption (1), Tf belongs to E and

$$\|Tf\|_E \leq c_p c_{p,E}\|f\|_E.$$

The case of $p = 0$ and $q < \infty$ is given by Theorem 2.4. □

Theorem 1.1 claimed in the introduction compiles some results of this section.

Proof of Theorem 1.1. The assertion (a) is obtained by combining Theorem 4.1 and Theorem 4.3 with $q = \infty$.

The assertion (b) is derived similarly from Theorem 4.1 and Theorem 4.3 by applying them with $p = 0$.

Finally, using (a), (b), Theorem 4.1 and 4.3, for $0 < p < q < \infty$, one obtains (c). □

Remark 4.4. In the spirit of Corollary 2.4, we could have used a non-quantitative condition to deal with the case of $q = \infty$ in Theorem 4.3. Let E be a quasi-Banach function space and $p, q \in (0, \infty)$. This means that the two following conditions are equivalent:

(i) for any $f \in E$, $g \in L_p + L_\infty$,
$$|g|^p \prec\prec_{\mathrm{hd}} |f|^p \Rightarrow g \in E.$$
(ii) there exists $c > 0$ such that for any $f \in E$, $g \in L_p + L_\infty$,
$$|g|^p \prec\prec_{\mathrm{hd}} |f|^p \Rightarrow g \in E \text{ and } \|g\|_E \leq c\|f\|_E.$$

Similarly, the two following conditions are equivalent:

(i) for any $f \in E$, $g \in L_0 + L_q$,
$$|g|^q \prec\prec_{\mathrm{tl}} |f|^q \Rightarrow g \in E.$$
(ii) there exists $c > 0$ such that for any $f \in E$, $g \in L_0 + L_q$,
$$|g|^q \prec\prec_{\mathrm{tl}} |f|^q \Rightarrow g \in E \text{ and } \|g\|_E \leq c\|f\|_E.$$

5. Interrpolation spaces for couples of ℓ^p-spaces

In this section, we show that our approach to the Lorentz-Shimogaki and Arazy-Cwikel theorems also applies to sequence spaces. We follow a structure similar to the previous sections, proving partition lemmas, then constructing bounded operators on couples (ℓ^p, ℓ^q) with suitable properties to finally draw conclusions on the interpolation spaces of the couple (ℓ^p, ℓ^q). Additional arguments involving Boyd indices will be required to prove Theorem 1.2.

We identify sequences with bounded functions on $(0, \infty)$ which are almost constant on intervals of the form $(k, k+1)$, $k \in \mathbb{Z}^+$ by
$$i\colon \ell^\infty \to L_\infty$$
$$(u_k)_{k\in\mathbb{Z}^+} \mapsto \sum_{k=0}^{\infty} u_k \mathbf{1}_{(k,k+1)}.$$

5.1. An interpolation theorem for the couple (ℓ^p, ℓ^q).

We start with a partition lemma playing, for sequence spaces, the role of Lemma 3.5.

Lemma 5.1. *Let $a = (a_n)_{n\in\mathbb{Z}^+}$, $b = (b_n)_{n\in\mathbb{Z}^+}$ be two positive decreasing sequences such that $b \prec\prec_{\mathrm{tl}} a$. There exists a sequence $(\Delta_n)_{n\in\mathbb{Z}^+}$ of subsets of $\mathbb{Z}^+$ such that:*

(i) *for every $k \in \mathbb{Z}_+$, we have $|\{n \in \mathbb{Z}^+ : k \in \Delta_n\}| \leq 3$.*
(ii) *$\sum_{k\in\Delta_n} a_k \geq b_n$ for any $n \in \mathbb{Z}^+$.*

Proof. Define $I = \{n \in \mathbb{Z}^+ : b_n > a_n\}$. For $n \notin I$, set $\Delta_n = \{n\}$.

For any $n \in \mathbb{Z}^+$, define:

$$i_n = \sup\left\{ i : \sum_{k=i}^{\infty} a_k \geq \sum_{k=n}^{\infty} b_k \right\},$$

and for $n \in I$, $\Delta_n = \{i_n, \ldots, i_{n+1}\}$.

From the definition of i_n, we have

$$\sum_{k \geq i_n} a_k \geq \sum_{k \geq n} b_k.$$

From the definition of i_{n+1}, we have

$$\sum_{k > i_{n+1}} a_k < \sum_{k \geq n+1} b_k.$$

Taking the difference of these inequalities, we infer that

$$\sum_{k \in \Delta_n} a_k \geq b_n.$$

This proves the second condition.

Note that since $b \prec\prec_{\mathrm{tl}} a$, $i_n \geq n$ for any $n \in \mathbb{Z}^+$. Hence, if $n \in I$, $b_n > a_n \geq a_{i_{n+1}}$. Furthermore, by definition of i_{n+1},

$$\sum_{k > i_{n+1}} a_k < \sum_{k=n+1}^{\infty} b_k \text{ so } \sum_{k=i_{n+1}}^{\infty} a_k < \sum_{k=n}^{\infty} b_k.$$

Hence, by definition of i_n, $i_{n+1} > i_n$ for $n \in I$.

Let us now check the first condition. Suppose there exist distinct numbers $n_1, n_2, n_3 \in I$ such that $k \in \Delta_{n_1}, \Delta_{n_2}, \Delta_{n_3}$. Without loss of generality, $n_1 < n_2 < n_3$. Since $k \in \Delta_{n_1}$, it follows that $k \leq i_{n_1+1} \leq i_{n_2}$. Since $k \in \Delta_{n_3}$, it follows that $k \geq i_{n_3} \geq i_{n_2+1}$. Hence, $i_{n_2+1} \leq k \leq i_{n_2}$ and, therefore, $i_{n_2+1} = i_{n_2}$. Since $n_2 \in I$, it follows $i_{n_2+1} > i_{n_2}$. This contradiction shows that $|\{n \in I : k \in \Delta_n\}| \leq 2$. By definition, k also belongs to at most one set Δ_n, $n \notin I$. Consequently, $|\{n \in \mathbb{Z}^+ : k \in \Delta_n\}| \leq 3$. □

From the partition lemma, we deduce an operator lemma similar to Lemma 3.8. It extends Proposition 2 in [4], which is established there for the special case $p = 1$ by completely different method.

Lemma 5.2. *Let $p \geq 1$. Let $a, b \in \ell^p$ such that $|b|^p \prec\prec_{\mathrm{tl}} |a|^p$. Then there exists an operator $T : \ell^p \to \ell^p$ such that:*

(i) $T(a) = b$.
(ii) $\|T\|_{p \to p} \leq 3^{\frac{1}{p}}$ *and* $\|T\|_{0 \to 0} \leq 3$.

Proof. We can assume that both sequences are non-negative and decreasing. Apply Lemma 5.1 to $|a|^p$ and $|b|^p$. For every $n \in \mathbb{Z}^+$, choose a linear form φ_n on ℓ^p of norm less than 1, supported on Δ_n and such that $\varphi_n(a) = \varphi_n(a\mathbf{1}_{\Delta_n}) = b_n$. Define

$$T : x \in \ell^p \mapsto (\varphi_n(x))_{n \in \mathbb{Z}^+}.$$

By construction, $T(a) = b$. Let us check the norm estimates. Let $x \in \ell^p$,

$$\begin{aligned}\big\|T(x)\big\|_p^p &= \sum_{n\in\mathbb{Z}^+} |\varphi_n(x)|^p = \sum_{n\in\mathbb{Z}^+} \big|\varphi_n(x\mathbf{1}_{\Delta_n})\big|^p \\ &\le \sum_{n\in\mathbb{Z}^+}\sum_{k\in\Delta_n} |x_k|^p \\ &= \sum_{k\in\mathbb{Z}^+} |\{n : k\in\Delta_n\}|\,|x_k|^p \\ &\le 3\big\|x\big\|_p^p.\end{aligned}$$

The second estimate is clear, using once again the fact that an integer k belongs to at most three Δ_n's. □

The following remarks were communicated to the authors by Cwikel and Nilsson.

Remark 5.3. *A bounded linear operator on ℓ^p, $p \le 1$ extends automatically to a bounded linear operator on ℓ^1.*

Proof. Indeed, let $(e_n)_{n\in\mathbb{Z}^+}$ be the canonical basis of ℓ^∞. Let T be a contraction on ℓ^p, $p < 1$. Then by Hölder's inequality $\big\|T(e_n)\big\|_1 \le \big\|T(e_n)\big\|_p \le \|T\|_{p\to p}$. By the triangle inequality, for any finite sequence $a = (a_n)_{n\in\mathbb{Z}^+}$,

$$\big\|T(a_n)\big\|_1 \le \sum_{n\in\mathbb{Z}^+} |a_n|\,\big\|T(e_n)\big\|_1 \le \big\|a\big\|_1\|T\|_{p\to p}.$$

Hence, T extends to a contraction on ℓ^1. □

Remark 5.4. *The condition $p \ge 1$ in Lemma 5.2 is necessary.*

Proof. Let us show that Lemma 5.2 cannot be true for $p < 1$. Assume by contradiction that there exists $c > 0$ such that for any finite sequences a and b in ℓ^p such that $|b|^p \prec\prec_{\mathrm{tl}} |a|^p$ there exists T with $\big\|T\big\|_{p\to p} \le c$ and $T(a) = b$. By Remark 5.3 above, we also have $\big\|T\big\|_{1\to 1} \le c$. In particular, $\big\|b\big\|_1 \le c\big\|a\big\|_1$. By considering $b = e_1$ and $a = \frac{1}{N^{1/p}}\sum_{i=1}^N e_i$ for N large enough, one obtains a contradiction. □

We will not prove a sequence version of Lemma 3.7 to avoid the repetition of too many similar arguments. Fortunately, the expected result already appears in the literature, see [10, Theorem 3]:

Lemma 5.5. *Let $p > 0$. Let $a, b \in \ell^\infty$ such that $|b|^p \prec\prec_{\mathrm{hd}} |a|^p$. Then there exists an operator $T : \ell^p \to \ell^p$ such that:*

(i) $T(a) = b$.
(ii) $\big\|T\big\|_{p\to p} \le 8^{1/p}$ *and* $\big\|T\big\|_{\infty\to\infty} \le 2^{1/p}$.

We conclude this subsection with a new interpolation theorem.

Theorem 5.6. *Let $p < q \in (0,\infty]$ such that $q \ge 1$. Let E be a quasi-Banach sequence space. Then E belongs to $Int(\ell^p, \ell^q)$ if and only if there exists $c_{p,E}$ and $c_{q,E}$ in $\mathbb{R}_{>0}$ such that:*

(i) *for any $u \in E$ and $v \in \ell^\infty$, if $|v|^p \prec\prec_{\mathrm{hd}} |u|^p$, then $v \in E$ and $\|v\|_E \le c_{p,E}\|u\|_E$;*
(ii) *if $q < \infty$: for any $u \in E$ and $v \in \ell^\infty$, if $|v|^q \prec\prec_{\mathrm{tl}} |u|^q$, then $v \in E$ and $\|v\|_E \le c_{q,E}\|u\|_E$.*

Proof. The proof of the "only if" implication is identical to the proof of Theorem 4.1 using Lemmas 5.2 and 5.5 instead of Lemmas 3.7 and 3.6. The "if" implication is given by [7, Theorem 4.7].

□

5.2. **Upper Boyd index.** Let us now recall the definition of the upper Boyd index, in the case of sequence spaces. For any $n \in \mathbb{N}$ define the dilation operator:

$$D_n \colon \ell^\infty \to \ell^\infty$$
$$(u_k)_{k\in\mathbb{Z}^+} \mapsto \left(u_{\lfloor \frac{k}{n} \rfloor}\right)_{k\in\mathbb{Z}^+}.$$

Let E be a symmetric function space. Define the Boyd index associated to E by:

$$\beta_E = \lim_{k\to\infty} \frac{\log \left\|D_k\right\|_{E\to E}}{\log k}.$$

Note that since E is a quasi-Banach space, $\beta_E < \infty$.

In this next proposition, we relate the upper Boyd index to an interpolation property. We follow [22, Theorem 2].

Proposition 5.7. *Let E be a quasi-Banach symmetric sequence space. Let $p < \frac{1}{\beta_E}$. There exists a constant C such that for any $u \in E$ and $v \in \ell^\infty$, satisfying $|v|^p \prec\prec_{\mathrm{hd}} |u|^p$, we have $v \in E$ and $\left\|v\right\|_E \le C\left\|u\right\|_E$.*

Define the map $V : \ell_\infty \to \ell_\infty$ by setting

$$Vu = \sum_{n=0}^{\infty} 2^{-n} D_{2^n} u,$$

and the map $C : \ell_\infty \to \ell_\infty$ by

$$(Cu)(n) = \frac{1}{n+1}\sum_{i=0}^{n} u_n.$$

Lemma 5.8. *If $p < \frac{1}{\beta_E}$, then*

$$\left\|(V(u^p))^{\frac{1}{p}}\right\|_E \le c_{p,E}\|u\|_E, \quad 0 \le u \in E.$$

Proof. Let E_p be the $p-$concavification of E, that is,

$$E_p = \{f : \ |f|^{\frac{1}{p}} \in E\}, \quad \|f\|_{E_p} = \||f|^{\frac{1}{p}}\|_E^p.$$

Obviously, E_p is a quasi-Banach space. Apply Aoki-Rolewicz theorem to the space E_p and fix $q = q_{p,E} > 0$ such that

$$\|\sum_{n\ge 0} x_n\|_{E_p}^q \le C_{p,E}\sum_{n\ge 0}\|x_n\|_{E_p}^q.$$

For every $u \in E$, we have

$$\left\|(V(u^p))^{\frac{1}{p}}\right\|_E^{qp} = \|V(u^p)\|_{E_p}^q = \|\sum_{n=0}^{\infty}\frac{1}{2^n}(D_{2^n}u)^p\|_{E_p}^q \le$$

$$\le C_{p,E}\sum_{n=0}^{\infty}\|\frac{1}{2^n}(D_{2^n}u)^p\|_{E_p}^q = C_{p,E}\sum_{n=0}^{\infty}2^{-nq}\|D_{2^n}u\|_E^{qp}.$$

Let $r \in (p, \beta_E^{-1})$. By the definition of β_E, there exists $c_{p,E} > 0$ such that $\|D_n\|_{E\to E} \le c_{p,E} n^{\frac{1}{r}}$ for any $n \in \mathbb{N}$. Therefore,

$$\|(V(u^p))^{\frac{1}{p}}\|_E^{qp} \le C_{p,E} \cdot c_{p,E}^q \cdot \sum_{n=0}^{\infty} 2^{-nq} 2^{\frac{nqp}{r}} \|u\|_E^{qp} = C_{p,E} \cdot c_{p,E}^q \cdot \frac{2^q}{2^q - 2^{\frac{qp}{r}}} \cdot \|u\|_E^{qp}.$$

□

Lemma 5.9. *If $x = \mu(x)$, then $Cx \le 3Vx$ for every $x \in \ell_\infty$.*

Proof. Let $k \ge 0$. Since x is decreasing, it follows that

$$(Cx)(2^k - 1) = \frac{1}{2^k}\Big(x(0) + \sum_{i=0}^{k-1} \sum_{j=2^i}^{2^{i+1}-1} x(j)\Big) \le \frac{1}{2^k}\Big(x(0) + \sum_{i=0}^{k-1} 2^i x(2^i)\Big).$$

On the other hand, we have

$$(Vx)(2^{k+1} - 1) = \sum_{n\ge 0} 2^{-n} x\Big(\Big\lfloor \frac{2^{k+1}-1}{2^n} \Big\rfloor\Big) =$$

$$= \sum_{n=0}^{k} 2^{-n} x(2^{k+1-n} - 1) + \sum_{n=k+1}^{\infty} 2^{-n} x(0) =$$

$$= \frac{1}{2^k}\Big(x(0) + \sum_{i=0}^{k} 2^i x(2^{i+1} - 1)\Big).$$

Again using the fact that x is decreasing, we obtain

$$\sum_{i=0}^{k-1} 2^i x(2^i) = x(1) + \sum_{i=0}^{k-2} 2^{i+1} x(2^{i+1}) \le$$

$$\le x(1) + 2\sum_{i=0}^{k-2} 2^i x(2^{i+1} - 1) \le 3\sum_{i=0}^{k} 2^i x(2^{i+1} - 1).$$

Combining the three previous inequalities, we have just shown that for any $k \ge 0$,

$$(Cx)(2^k - 1) \le 3(Vx)(2^{k+1} - 1).$$

Now let $n \ge 0$ and choose k such that $n \in [2^k - 1, 2^{k+1} - 1]$. Since Cx and Vx are decreasing, we have:

$$(Cx)(n) \le (Cx)(2^k - 1) \le 3(Vx)(2^{k+1} - 1) \le 3(Vx)(n).$$

□

Proof of Proposition 5.7. Without loss of generality, $u = \mu(u)$ and $v = \mu(v)$. Since $v^p \prec\prec_{\mathrm{hd}} u^p$, it follows that

$$|v|^p \le C(|v|^p) \le C(|u|^p) \le 3V(|u|^p),$$

where we used Lemma 5.9 to obtain the last inequality. By Lemma 5.8, we have

$$\|v\|_E \le 3^{\frac{1}{p}} \|(V(|u|^p))^{\frac{1}{p}}\|_E \le c_{p,E} \|u\|_p.$$

□

We are now ready to deliver a complete resolution of the conjecture stated by Levitina and two of the authors in [19].

Proof of Theorem 1.2. Let E be a quasi-Banach sequence space. Let $q \geq 1$. Recall that Theorem 1.2 states that the two following conditions are equivalent:

(a) there exists $p < q$ such that E is an interpolation space for the couple (ℓ^p, ℓ^q);
(b) there exists $c > 0$ such that for any $u \in E$ and $|v|^q \prec\prec_{\mathrm{tl}} |u|^q$, then $v \in E$ and $\|v\|_E \leq c\|u\|_E$.

Note that by Lemma 1.11, we may assume that E is a symmetric space.

$(a) \Rightarrow (b)$. This is immediate by Theorem 5.6.

$(b) \Rightarrow (a)$. Let $p < 1/\beta_E$. By Proposition 5.7, for any sequence $u \in E$ and $v \in \ell^\infty$, if $|v|^p \prec\prec_{\mathrm{hd}} |u|^p$, $v \in E$ and $\big\|v\big\|_E \leq c_{p,E}\big\|u\big\|_E$. Applying Theorem 5.6 for indices p and q, we obtain that E belongs to $\mathrm{Int}(\ell^p, \ell^q)$. □

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