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# Mixed Modified Fractional Merton model of the bear spread Basket put option using the multidimensional Mellin transform.

Eric Djeutcha<sup>1</sup>, Jules Sadefo Kamdem<sup>2</sup>, Louis Aimé Fono<sup>3</sup>

## Abstract

In this paper, The generalized Mixed-Modified-Fractional-Merton like partial differential equation with multi-assets under mixed modified fractional geometric Brownian motion was derived. The multidimensional Mellin transform was applied to derive the integral equation for the price of the European put option on a bear spread basket of multi-assets.

## 1 Introduction

Mandelbrot et al.[14] was introduced the Fractional Brownian Motion (FBM) as an extension of the Brownien motion. In recent decades, several researchers including, Willinger and Taqqu [21], Lin [12] et Rosteck [19] have focussed their studies on this process by studying self-similarity and the property of long-range dependency. Those authors use FBM to express the random part of financial models. However, the existence of this process in financial models can lead to some difficulties. We can show that except the case  $H = \frac{1}{2}$ , the FBM process is neither a Markov process, nor a semimartingale. Therefore, Roger [20] and Cheridito [5] have shown that there can be no equivalent probability under which the process becomes a local martingale, this raises the arbitrage problem. To overcome this problem, Cheridito [6] has applied the Mixed Fractional Brownian Motion (MFBM) and Zlli [23] has applied its generalisation to unveil the random part of the financial model. A MFBM process whose parameters  $a, b$  and  $H$  is a linear combination of Brownian Motion  $B_t$  and independent Fractional Brownian Motion  $B_t^H$ , defined on a probability space  $(\Omega, F, \mathbb{P})$  by:

$$M_t^{H,a,b} = M_t^H = aB_t + bB_t^H, \quad \forall(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+, \quad t \in [0, T]. \quad (1)$$

Where  $H \in (0, 1)$  is the Hurst parameter. Cheridito [6] showed that the MFBM process with  $H \in ]\frac{3}{4}, 1[$  equals to a martingale  $B_t$  and is therefore arbitrage-free. Nonetheless, there is no equivalent probability under which the MFBM for all  $H \in ]\frac{1}{2}, \frac{3}{4}[$  becomes a local martingale. Djeutcha et al.[7] suggest a new process defined on the same probability space  $(\Omega, F, \mathbb{P})$  by:

$$M_t^{H,\varepsilon} = aB_t + bB_t^{H,\varepsilon}, \quad \forall(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+, \quad H \in ]\frac{1}{2}, 1[, \quad (2)$$

with

$$B_t^{H,\varepsilon} = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dB_s, \quad \forall\varepsilon \in \mathbb{R}_+^*, \quad t \in [0, T]. \quad (3)$$

Where  $B_s$  is a Brownian Motion and  $\varepsilon$  is a positive constant.

The process (2) is an modification of process (1), framed MMFBM stands for "Mixed Modified Fractional Brownian Motion", its importance is linked to fact that it is a continous martingale. It is used in this paper to design the financial assets.The Black Scholes Model [1], assumes the price of heavily traded assets follows a geometric Brownian motion with constant drift and volatility. It model cannot capture many of the features of asset price returns in the absence of dividend payments, Merton [15] considered a jump-diffusion process that allows for the probability of the asset price to change at large magnitudes irrespective of the time interval between successive observations. Empirical studies have showed that the asset price is better described by a process with a discontinuous sample path (Rosenfeld [16], Jarrow and Rosenfeld [10], Ball and Torous [2], and Brown and Dybvig [3]). Cont and Tankov [4] showed that unlike standard diffusion models such as Black Scholes[1], jump-diffusion models produce rich structures of the distribution of asset returns and implied volatility surfaces. To account for the possibility of instantaneous jumps in the asset price, Merton [15]

proposed the following modification of Black Scholes[1] by assuming that the discontinuous jumps arrive as a Poisson process.

$$dS(t) = S(t)[r - \lambda E(Y - 1)]dt + \sigma S(t)dW(t) + \sigma S(t)(Y - 1)dN(t) \quad (4)$$

where  $Y(t)$  is a nonnegative continuous random variable with  $Y - 1$  denoting the impulse change in the asset price from  $S(t)$  to  $YS(t)$  as a result of the jump and  $Y - 1 \sim \mathcal{N}(\mu, \delta^2)$ ,  $\mathbb{E}$  is the expectation operator, and  $N = (N(t), t \geq 0)$  is a Poisson process with constant intensity  $\lambda$  and such that  $dN(t) = 1$  (respectively,  $dN(t) = 0$ ) with probability  $\lambda dt$  (respectively,  $1 - \lambda dt$ ). Using Mellin transform techniques adapted from Rodrigo and Mamon [17], Li and Rodrigo [13] studied the Partial Integro Differential Equation (PIDE) and found exact pricing formulas for European options with general payoffs such as those in many types of option. A PIDE analogue of Dupire's PDE was also derived and used to find an explicit formula for the implied volatility. In the same vein, Rodrigo and Goard [18] considered PIDE with a time varying D and obtained exact pricing formulas for European options on discrete dividend-paying assets.

An option is a financial derivative that grants its holder the right without obligation to buy a specific asset on or before given date in the future for an agreed price called the exercise price. A multi-asset option is path dependent option that depends on more than one underlying asset with the payoff defined by a function of the asset prices. A basket option is a financial contract with a portfolio of several underlying equity assets. The payoff function for a basket put option is of the form:

$$(K - \sum_{i=1}^n \alpha_i S_i)^+ \quad (5)$$

where  $\alpha_i$  is the number of shares of asset  $i$  in the basket.  $S_i$  is the price of asset  $i$  in the basket,  $K$  is the exercise price and  $\sum_{i=1}^n \alpha_i = 1$ . Several authors have studied the Basket option, among which Xu and Zheng [22] who have studied the Basket options valuation for a local volatility jump-diffusion model with asymptotic expansion method, Kamdem[11], use Mellin transform to get the expression for the free boundary an price of an American finite-lived option, when the underlying is govern by the Levy process and show how to compute the price of an American option on a basket of stocks using Mellin transform of several variables. A bear spread option is a vertical spread. It consists of buying one put in hopes of profiting from a decline in the underlying stock, and writing another put with the same expiration but with a lower strike price, as a way to offset some of the cost. The payoff function of the bear spread option is given by

$$(K_2 - x)^+ - (K_1 - x)^+ \quad (6)$$

Using Eq.(5) and Eq.(6), we define the Bear spread Basket put option as follows

$$(K_2 - \sum_{i=1}^n \alpha_i S_i)^+ - (K_1 - \sum_{i=1}^n \alpha_i S_i)^+ \quad (7)$$

This paper add a long-range dependency property to the model given by Merton [15] as far as the non-arbitrage condition is concerned.

Our approach will consist firstly of describing the random part of the Merton model by using the Mixed Modified Fractional Brownian Motion and secondly to evaluate the basket option coupled with the bear spread option using the Mellin transform method with several variables and drawing inspiration from research of Kamdem [11], Frontczak [8] and Fadugba [9].

Our motivation comes from the fact that the study of the markets with the MMFBM process is recent and the evaluation of the bear spread basket put option price whose the payoff function is given by Eq.(7) seems to be not studied in the recent litterature.

The main concern in this paper will be centered around finding the partial differential equation of the pricing Basket options under MMFBM model (Mixed Modified Fractional Merton model). We

also determine the closed formula for the bear spread Basket put option price. The section 2 of this paper presents the Mixed Modified Fractional Merton Model, the section 3 defined the Bear spread Basket put options in a MMFM model, the section 4 establish the Partial Integro Differential Equation (PIDE) under the MMFM model and the section 5 gives the Multi-dimensional Mellin transform for the tarification of the european Put option on a bear spread basket of multi-assets.

## 2 Presentation of the MMFM model

We recall that the Merton model is defined by Eq.(4). However, to resolve the arbitrage problem in such a market when  $H \in ]\frac{1}{2}, 1[$ , we plan to substitute in (4) the process  $W_t$  by the MMFBM defined by equation (2). The latter is thus called "Mixed Modified Fractional Merton model" in short MMFM model and is a modification of model (4) defined by Eq.(8) below.

**Definition 1.** *The MMFM model is one in which the dynamic underlying asset price  $S_t^\varepsilon$  verify the following stochastic equation:*

$$\frac{dS_t^\varepsilon}{S_t^\varepsilon} = [r - \lambda\mathbb{E}(Y - 1)]dt + \sigma dM_t^{H,\varepsilon} + \sigma(Y - 1)dN(t), S_0^\varepsilon = S_0 > 0, H \in ]\frac{1}{2}, 1[ \quad (8)$$

By using the equations (2)-(3) and taking into consideration that :

$$dM_t^{H,\varepsilon} = b\phi_t^\varepsilon dt + \left(a + b\varepsilon^{H-\frac{1}{2}}\right) dB_t \quad (9)$$

with

$$\phi_t^\varepsilon = \left(H - \frac{1}{2}\right) \int_0^t (t - s + \varepsilon)^{H-\frac{3}{2}} dB_s, \quad (10)$$

Eq.(8) can also be written as

$$\frac{dS_t^\varepsilon}{S_t^\varepsilon} = [r - \lambda\mathbb{E}(Y - 1) + b\sigma\phi_t^\varepsilon]dt + \sigma(a + b\varepsilon^{H-\frac{1}{2}})dW_t + \sigma(Y - 1)dN(t). \quad (11)$$

In the remainder of this paper, the MMFM Model given by Eq.(8) is used for the tarification of the bear spread Basket options. The solution of Eq.(8) is

$$S_t^\varepsilon = S_0^\varepsilon \left\{ \left( r - \lambda\mathbb{E}(Y - 1) - \frac{\sigma^2}{2} \left( a + b\varepsilon^{H-\frac{1}{2}} \right)^2 \right) t + \sigma M_t^{H,\varepsilon} + \sum_{i=1}^{N(t)} Y_i \right\} \quad (12)$$

## 3 Bear spread Basket options in a MMFM model

In this section, we now shall concentrate on the pricing of bear spread basket options in the MMFM model. Denoting by  $S_{i,t}^\varepsilon$  the price of the  $i$ -th asset in the basket at time  $t$ , the basket is given by

$$S_t^\varepsilon = \sum_{i=1}^n \alpha_j S_{i,t}^\varepsilon, \quad (13)$$

where  $\alpha_j$ , for all  $j$  is deterministic, positive and constant weights specified by the option contract. We assume that the risk neutral measure

$$\frac{dS_{i,t}^\varepsilon}{S_{i,t}^\varepsilon} = (r - \lambda\mathbb{E}(Y - 1))dt + \sigma_i dM_{i,t}^{H,\varepsilon} + (Y - 1)dN(t), \quad (14)$$

where  $(M_{i,t}^{H,\varepsilon})_{t \geq 0}$  is a Mixed Modified Fractional Brownian motion associated with the price process of asset  $i$ . Given the above dynamics, the  $i$ -th asset price at time  $t$  equals

$$S_{i,t}^\varepsilon = S_{i,0}^\varepsilon \left\{ \left( r - \lambda\mathbb{E}(Y - 1) - \frac{\sigma_i^2}{2} \left( a + b\varepsilon^{H-\frac{1}{2}} \right)^2 \right) t + \sigma_i M_{i,t}^{H,\varepsilon} + \sum_{i=1}^{N(t)} N(t_i) \right\} \quad (15)$$

Further, we assume the different asset prices to be instantaneously correlated according to

$$\text{corr}(dM_{i,t}^{H,\varepsilon}, dM_{j,t}^{H,\varepsilon}) = \rho_{ij}(a + b\varepsilon^{H-\frac{1}{2}})^2 dt, \forall i \neq j, \quad (16)$$

with  $\rho \in [-1, 1]$ . Using Eq.(17) the Mixed Modified fractional Bear spread Basket put option is defined as follows

$$\phi(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon) = (K_2 - \sum_{i=1}^n \alpha_i S_i^\varepsilon)^+ - (K_1 - \sum_{i=1}^n \alpha_i S_i^\varepsilon)^+ \quad (17)$$

where  $K_1$  and  $K_2$  are respectively the strike price of the option such that  $K_1 < K_2$ .

## 4 PDE of the option price under the MMFM model

We consider an option on  $n$  stocks, with each the stock price  $S_i^\varepsilon, i = 1, 2, 3, \dots, n$  follow correlated geometric Mixed Modified Fractional Brownian motion with drift  $\mu_i$  and volatility  $\sigma_i$  solution of Eq.(14). Before the derivation of the generalized Mixed-Modified-Fractional-Merton partial differential equation, a counterpart; the Itô's lemma in higher dimensions is needed. Consider a multi-dimensional Mixed Modified Fractional Itô's process of the form

$$dX_{i,t} = \mu_i(X_{i,t}, t)dt + \sigma_i(X_{i,t}, t)dM_{i,t}^{H,\varepsilon}, 1 \leq i \leq n, \quad (18)$$

Therefore, the multi-dimensional Itô's lemma for Eq.(18) is given by

$$df(X_t, t) = \left[ \frac{\partial f(X_t, t)}{\partial t} + \sum_{i=1}^n \mu_i \frac{\partial f(X_t, t)}{\partial x_i} + \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j x_j \frac{\partial^2 f(X_t, t)}{\partial x_i \partial x_j} \right] dt + \sum_{i=1}^n \sigma_i \frac{\partial f(X_t, t)}{\partial x_i} dM_{i,t}^{H,\varepsilon} \quad (19)$$

where  $f(X_t, t) \in \mathcal{C}^{2,1}$ . The derivation of the generalized Mixed-Modified-Fractional-Merton partial differential equation for the European bear spread basket put option is given in the following result.

**Theorem 1.** Let  $f(S_1^\varepsilon, S_2^\varepsilon, S_3^\varepsilon, \dots, S_n^\varepsilon, t) \in \mathcal{C}^{2,1}$  be some functions with multi-assets,  $S_i^\varepsilon$  be the current price of the underlying asset  $i$ ,  $S_j^\varepsilon$  be the current price of the underlying asset  $j$ ,  $K_1$  and  $K_2$  be the strike price of the bear spread option,  $\sigma_i$  be the volatility of asset  $i$ ,  $\sigma_j$  be the volatility of asset  $j$ ,  $\rho_{ij}$  be the correlation coefficient between asset  $i$  and asset  $j$ ,  $T$  be the time to expiry,  $r$  be the risk-free interest rate and  $\cdot$ . Using the Itô's lemma (19), the generalized Mixed-Modified-Fractional-Merton partial differential equation for price of an European bear spread basket put option is obtained as

$$\begin{aligned} & \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \\ & + \lambda \mathbb{E}[f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t)] + \sum_{i=1}^n (r - \lambda \mathbb{E}(Y - 1)) S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \\ & - r f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = 0 \end{aligned} \quad (20)$$

*Proof.* Construct a portfolio consisting of an option  $f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)$  and  $\frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon}$  amount of assets  $S_i^\varepsilon$ . Let the asset  $j$  S be driven by correlated geometric Brownian motions. Using Eq.(19) for some functions  $f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)$  with multi-assets, yields

$$\begin{aligned} df(X_t, t) &= \left[ \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \sum_{i=1}^n \mu_i S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \right] dt \\ &+ \left[ \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \right] dt \\ &+ \sum_{i=1}^n \sigma_i S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} dM_{i,t}^{H,\varepsilon} \\ &+ [f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t)] dN(t) \end{aligned} \quad (21)$$

The value of the portfolio denoted by  $\Pi$  is given by

$$\Pi = f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) - \sum_{i=1}^n S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \quad (22)$$

After one time step  $dt$ , the value of the portfolio changes by

$$d\Pi = df(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) - \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} dS_i^\varepsilon \quad (23)$$

Substituting Eq.(14) and Eq.(25) into Eq.(23) yields

$$\begin{aligned} d\Pi &= df(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) - \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} dS_i^\varepsilon \quad (24) \\ d\Pi &= \left[ \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \sum_{i=1}^n (r - \lambda \mathbb{E}(Y - 1)) S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \right] dt \\ &+ \left[ \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \right] dt \\ &+ \sum_{i=1}^n \sigma_i S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} dM_{i,t}^{H,\varepsilon} - \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} (r - \lambda \mathbb{E}(Y - 1)) S_i^\varepsilon dt \\ &- \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \sigma_i S_i^\varepsilon dM_{i,t}^{H,\varepsilon} - \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} (Y - 1) S_i^\varepsilon dN(t) \\ &+ [f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t)] dN(t) \quad (25) \end{aligned}$$

i.e

$$\begin{aligned} d\Pi &= \left[ \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \right] dt \\ &+ \left[ f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) - \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} (Y - 1) S_i^\varepsilon \right] dN(t) \quad (26) \end{aligned}$$

or  $dN(t) = 1$  with the probability  $\lambda dt$ . Therefore, we have

$$\begin{aligned} d\Pi &= \left[ \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \right] dt \\ &+ \lambda \mathbb{E} \left[ f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) - \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} (Y - 1) S_i^\varepsilon \right] dt \quad (27) \end{aligned}$$

The portfolio is now riskless due to the elimination of  $dM_t^{H,\varepsilon}$  term. It must then earn a return similar to other short term riskless securities such as bank account. Therefore

$$d\Pi = r\Pi dt \quad (28)$$

where  $r$  is the riskless interest rate. Substituting Eq.(22) and Eq.(27) into Eq.(28) gives

$$\begin{aligned} &\left[ \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \right] dt \\ &+ \lambda \mathbb{E} \left[ f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) - \sum_{i=1}^n \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} (Y - 1) S_i^\varepsilon \right] dt \\ &= r \left[ f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) - \sum_{i=1}^n S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \right] dt \quad (29) \end{aligned}$$

Rearranging and solving the last equation further, the multi-dimensional Mixed Modified Fractional Merton partial differential equation for the price of an option  $f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)$  with multi-assets is obtained as

$$\begin{aligned} & \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \\ & + \lambda \mathbb{E} [f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t)] + \sum_{i=1}^n (r - \lambda \mathbb{E}(Y - 1)) S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \\ & - r f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = 0 \end{aligned} \quad (30)$$

□

## 5 Multi-dimensional Mellin transform for the tariffication of the european Put option on a bear spread basket of multi-assets

The following result gives the derivation of the expression for the integral equation for the price of the European bear spread basket put option by means of the Mellin transform in higher dimensions.

**Theorem 2.** *The generalized Mixed-Modified-Fractional-Merton partial differential equation for the price of the European bear spread basket put option is given by*

$$\left\{ \begin{array}{l} \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \\ + \lambda \mathbb{E} [f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t)] + \sum_{i=1}^n (r - \lambda \mathbb{E}(Y - 1)) S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \\ - r f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = 0 \\ 0 < S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon < \infty, 0 \leq t \leq T \end{array} \right. \quad (31)$$

with the initial and boundary conditions are given by

$$f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, T) = 0, \sum_{i=1}^n a_i S_i^\varepsilon \rightarrow \infty, \text{ on } [0, T] \quad (32)$$

$$f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, T) = \phi(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon) = \max(K_2 - \sum_{i=1}^n \alpha_i S_i^\varepsilon, 0) - \max(K_1 - \sum_{i=1}^n \alpha_i S_i^\varepsilon, 0), \text{ on } [0, \infty[ \quad (33)$$

$$f(0, 0, \dots, 0, t) = K_2 e^{-r(T-t)} - K_1 e^{-r(T-t)}, t \in [0, T] \quad (34)$$

Then, the expression for the integral equation for the price of the European put option on a bear spread basket of multi-assets is obtained as

$$\begin{aligned} f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) &= \frac{1}{(2\pi)^n} \int_{a_1 - i\infty}^{a_1 + i\infty} \dots \int_{a_n - i\infty}^{a_n + i\infty} \left[ \frac{B(w_1, \dots, w_n) K_2^{\left(1 + \sum_{i=1}^n w_i\right)} - B(w_1, \dots, w_n) K_1^{\left(1 + \sum_{i=1}^n w_i\right)}}{(w_1 + \dots + w_n)(w_1 + \dots + w_n + 1)} \right] \\ &\times \left( e^{D(w_1, \dots, w_n)} (S_1^\varepsilon)^{-w_1} \dots (S_n^\varepsilon)^{-w_n} \right) dw_1 \dots dw_n \end{aligned} \quad (35)$$

*Proof.* It is assumed that the underlying assets and follow geometric Mixed Modified Fractional Brownian motion. From Eq.(31)

$$\left\{ \begin{array}{l} \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial t} + \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j S_i^\varepsilon S_j^\varepsilon \frac{\partial^2 f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial x_i \partial x_j} \\ + \lambda \mathbb{E} [f(S_1^\varepsilon y, \dots, S_n^\varepsilon y, t) - f(S_1^\varepsilon, \dots, S_n^\varepsilon, t)] + \sum_{i=1}^n (r - \lambda \mathbb{E}(Y - 1)) S_i^\varepsilon \frac{\partial f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)}{\partial S_i^\varepsilon} \\ - r f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = 0 \\ 0 < S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon < \infty, 0 \leq t \leq T \end{array} \right. \quad (36)$$

To use the multi-dimensional Mellin transform, assume that  $f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)$  satisfies the condition

$$f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = \begin{cases} O(1), \text{ if } S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon \rightarrow 0^+ \\ O(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon), \text{ if } S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon \rightarrow +\infty \end{cases} \quad (37)$$



Equation Eq.(37) guarantees the existence of the multi-dimensional Mellin transform. Let  $\widehat{f}(w_1, w_2, \dots, w_n, t)$  denote the multi-dimensional Mellin transform of  $f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)$  given by

$$\widehat{f}(w_1, w_2, \dots, w_n, t) = \int_0^\infty \dots \int_0^\infty f(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) (S_1^\varepsilon)^{w_1-1} (S_2^\varepsilon)^{w_2-1} \dots (S_n^\varepsilon)^{w_n-1} dS_1^\varepsilon dS_2^\varepsilon \dots dS_n^\varepsilon \quad (38)$$

where the complex variable  $w_1, w_2, \dots, w_n$  exists in an appropriate domain of convergence  $\mathbb{C}^n$ . Conversely, the multi-dimensional Mellin transform inversion formula for the European put option on a bear spread basket of multi-assets is given by

$$f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) = \frac{1}{(2\pi)^n} \int_{a_1-i\infty}^{a_1+i\infty} \dots \int_{a_n-i\infty}^{a_n+i\infty} \widehat{f}(w_1, w_2, \dots, w_n, t) (S_1^\varepsilon)^{-w_1} (S_2^\varepsilon)^{-w_2} \dots (S_n^\varepsilon)^{-w_n} dw_1 \dots dw_n \quad (39)$$

Thus to find the multi-dimensional Mellin transform of Eq.(31), applying the definition given by Eq.(38) to Eq.(31), yields

$$\begin{aligned} & \frac{d\widehat{f}(w_1, w_2, \dots, w_n, t)}{dt} + \left[ \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j w_i w_j \right] \widehat{f}(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) \\ & + \left[ \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2 w_i - \sum_{i=1}^n (r - \lambda \mathbb{E}(Y - 1)) w_i - r + \lambda G(w_1, w_2, \dots, w_n) \right] \widehat{f}(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = 0 \end{aligned} \quad (40)$$

where

$$G(w_1, w_2, \dots, w_n) = (w_1 + w_2 + \dots + w_n) \mathbb{E}[Y] + \mathbb{E}[Y^{-w_1-w_2-\dots-w_n}]. \quad (41)$$

We recall that, for the jump size distribution, different distributions are commonly used in financial literature, prominent candidates are among others (Log-normally distributed jumps, double exponentially distributed jumps, Gamma distributed jumps and One-sided exponentially distributed jumps). The choice of the distribution of  $Y$  is crucial in order to solve the problem analytically in the  $(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t)$ -domain. In this paper, the explicit expression for  $G(w_1, w_2, \dots, w_n)$  is given in the case for  $Y$  follows the Log-normally distributed jumps given by Merton[15]. We obtain :

$$\begin{cases} \mathbb{E}(Y) = e^{\mu + \frac{1}{2}\delta^2} \\ \mathbb{E}[Y^{-w_1-w_2-\dots-w_n}] = e^{\mu(w_1+w_2+\dots+w_n) + \frac{1}{2}(w_1^2+w_2^2+\dots+w_n^2)\delta^2}. \end{cases} \quad (42)$$

Therefore

$$G(w_1, w_2, \dots, w_n) = (w_1 + w_2 + \dots + w_n) e^{\mu + \frac{1}{2}\delta^2} + e^{\mu(w_1+w_2+\dots+w_n) + \frac{1}{2}(w_1^2+w_2^2+\dots+w_n^2)\delta^2}. \quad (43)$$

By setting

$$\begin{aligned} D(w_1, w_2, \dots, w_n) &= \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{j,i=1}^n \rho_{ij} \sigma_i \sigma_j w_i w_j + \frac{1}{2} (a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2 w_i \\ &- \sum_{i=1}^n (r - \lambda \mathbb{E}(Y - 1)) w_i - r + \lambda G(w_1, w_2, \dots, w_n) \end{aligned} \quad (44)$$

Eq.(40) becomes

$$\frac{d\widehat{f}(w_1, w_2, \dots, w_n, t)}{dt} + D(w_1, w_2, \dots, w_n) \widehat{f}(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = 0 \quad (45)$$

The solution to Eq.(45) is obtained as

$$\widehat{f}(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = J(w_1, w_2, \dots, w_n) e^{-D(w_1, w_2, \dots, w_n)t} \quad (46)$$

where  $J(w_1, w_2, \dots, w_n)$  is the constant of integration given by

$$J(w_1, w_2, \dots, w_n) = \widehat{\phi}(w_1, w_2, \dots, w_n) e^{D(w_1, w_2, \dots, w_n)T} \quad (47)$$

The multi-dimensional Mellin transform of the final condition

$$\phi(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon) = \max(K_2 - \sum_{i=1}^n \alpha_i S_i^\varepsilon, 0) - \max(K_1 - \sum_{i=1}^n \alpha_i S_i^\varepsilon, 0) \quad (48)$$

in Eq.(33) is obtained as

$$\widehat{\phi}(w_1, w_2, \dots, w_n) = \frac{B(w_1, \dots, w_n) K_2^{\left(1 + \sum_{i=1}^n w_i\right)} - B(w_1, \dots, w_n) K_1^{\left(1 + \sum_{i=1}^n w_i\right)}}{(w_1 + \dots + w_n)(w_1 + \dots + w_n + 1)} \quad (49)$$

where

$$B(w_1, \dots, w_n) = \frac{\prod_{i=1}^n \Gamma(w_i)}{\Gamma\left(\sum_{i=1}^n w_i\right)} \quad (50)$$

Equation Eq.(50) is called the multinomial beta function of n-variables. Substituting Eq.(49) into Eq.(47) yields

$$J(w_1, w_2, \dots, w_n) = \frac{B(w_1, \dots, w_n) K_2^{\left(1 + \sum_{i=1}^n w_i\right)} - B(w_1, \dots, w_n) K_1^{\left(1 + \sum_{i=1}^n w_i\right)}}{(w_1 + \dots + w_n)(w_1 + \dots + w_n + 1)} e^{D(w_1, w_2, \dots, w_n)T} \quad (51)$$

Therefore, Eq.(46) becomes

$$\widehat{f}(S_1^\varepsilon, S_2^\varepsilon, \dots, S_n^\varepsilon, t) = \frac{B(w_1, \dots, w_n) K_2^{\left(1 + \sum_{i=1}^n w_i\right)} - B(w_1, \dots, w_n) K_1^{\left(1 + \sum_{i=1}^n w_i\right)}}{(w_1 + \dots + w_n)(w_1 + \dots + w_n + 1)} e^{D(w_1, w_2, \dots, w_n)(T-t)} \quad (52)$$

Taking the inverse multi-dimensional Mellin transform of Eq.(52), the expression for the integral equation for the price of the European put option on a bear spread basket of multi-assets is obtained as

$$\begin{aligned} f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) &= \frac{1}{(2\pi)^n} \int_{a_1 - i\infty}^{a_1 + i\infty} \dots \int_{a_n - i\infty}^{a_n + i\infty} \left[ \frac{B(w_1, \dots, w_n) K_2^{\left(1 + \sum_{i=1}^n w_i\right)} - B(w_1, \dots, w_n) K_1^{\left(1 + \sum_{i=1}^n w_i\right)}}{(w_1 + \dots + w_n)(w_1 + \dots + w_n + 1)} \right] \\ &\times \left( e^{D(w_1, \dots, w_n)} (S_1^\varepsilon)^{-w_1} \dots (S_n^\varepsilon)^{-w_n} \right) dw_1 \dots dw_n \end{aligned} \quad (53)$$

□

We complete this section by showing explicitly that Eq. (39) with the special choices of  $f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) = E_p(S_1^\varepsilon, \dots, S_n^\varepsilon, t)$  and log-normally distributed jumps is equivalent to Mixed Modified Fractional Merton's infinite series solution for a European put in the bear spread basket option.

**Proposition 1.** *If  $f(S_1^\varepsilon, \dots, S_n^\varepsilon, t) = E_p(S_1^\varepsilon, \dots, S_n^\varepsilon, t)$  and jumps are distributed log-normally with parameters  $\mu$  and  $\delta$ ,  $Y \sim \ln(\mu, \delta)$ , then Eq.(39) is equivalent to Mixed Modified Fractional Merton's solution which is formulated as an infinite series of Mixed Modified Fractional Black Schole Merton- prices(MMFBSM):*

$$E_p(S_1^\varepsilon, \dots, S_n^\varepsilon, t) = \sum_{n=0}^{\infty} \frac{\lambda Y (T-t)^n}{n!} e^{-\lambda Y (T-t)} \times f^{MMFBSM}(S_1^\varepsilon, \dots, S_n^\varepsilon, K_1, K_2, r, T, t) \quad (54)$$

Where  $f^{MMFBSM}(S_1^\varepsilon, \dots, S_n^\varepsilon, K_1, K_2, r, T, t)$  denotes the European put in the bear spread basket option due to the Mixed Modified Fractional Black Schole Merton and equals

$$\begin{aligned} f^{MMFBSM}(S_1^\varepsilon, \dots, S_n^\varepsilon, K_1, K_2, r, T, t) &= K_2 e^{-r(T-t)} \mathcal{N}(-d_2) - \sum_{i=1}^n \alpha_i S_{i,t}^\varepsilon \mathcal{N}(-d_1) \\ &- K_1 e^{-r(T-t)} \mathcal{N}(-d_4) - \sum_{i=1}^n \alpha_i S_{i,t}^\varepsilon \mathcal{N}(-d_3). \end{aligned} \quad (55)$$

with

$$\left\{ \begin{array}{l} d_1 = \frac{\ln\left(\frac{\sum_{i=1}^n \alpha_i S_{i,t}^{\varepsilon_i}}{K_2}\right) + \left[r(T-t) - \lambda(Y-1)(T-t) + n \ln(Y) + \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2(T-t) + n\delta^2\right]}{\sqrt{(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2(T-t) + n\delta^2}} \\ d_2 = d_1 - \sqrt{(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2(T-t) + n\delta^2} \\ d_3 = \frac{\ln\left(\frac{\sum_{i=1}^n \alpha_i S_{i,t}^{\varepsilon_i}}{K_1}\right) + \left[r(T-t) - \lambda(Y-1)(T-t) + n \ln(Y) + \frac{1}{2}(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2(T-t) + n\delta^2\right]}{\sqrt{(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2(T-t) + n\delta^2}} \\ d_4 = d_3 - \sqrt{(a + b\varepsilon^{H-\frac{1}{2}})^2 \sum_{i=1}^n \sigma_i^2(T-t) + n\delta^2}. \end{array} \right. \quad (56)$$

The proof of this Proposition 1 is obvious and as indication, we can use an approach given by Frontczak [8].

## 6 Conclusion

In this paper, we have studied the generalized Mixed-Modified-Fractional-Merton like partial differential equation with multi-assets under mixed modified fractional geometric Brownian motion was derived. We have used the multidimensional Mellin transform was applied to derive the integral equation for the price of the European put option on a bear spread basket of multi-assets.

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