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Zinbiel algebras and multiple zeta values

F. Chapoton

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Introduction

Multiple zeta values are the convergent iterated integrals from 0 to 1 of the differential forms $\omega_0 = dt/t$ and $\omega_1 = dt/(1-t)$. They form an algebra over \mathbb{Q} , which has many interesting connections with different domains, including knot theory and perturbative quantum field theory [18, 11]. This algebra is expected to be graded by the weight, and a famous conjecture of Zagier [19] states that the dimensions of homogeneous components are given by the Padovan numbers.

The algebra \mathbf{A}_{MZV} of motivic multiple zeta values is a more subtle construction, in the setting of periods and mixed motives [5, 6, 11]. It can be defined as the quotient of the commutative algebra $\mathbf{A}_{1,0}$, whose elements are seen as formal iterated integrals of ω_0 and ω_1 , by the non-explicit ideal of all relations that can be proved using algebraic geometry. This algebra is known to be graded by the weight and its dimensions are given by the Padovan sequence, by results of Brown [5].

There is a surjective morphism, called the period map, from the motivic algebra \mathbf{A}_{MZV} to the usual algebra of multiple zeta values, defined by taking the numerical value of a formal iterated integral. This period map is expected to be injective, hence an isomorphism.

The aim of this article is to propose an algebraic construction, using the algebraic structures known as zinbiel algebras or dual Leibniz algebras, of some commutative sub-algebras $C_{u,v}$ of $\mathbf{A}_{1,0}$.

The notion of zinbiel algebras was introduced by Loday in relationship with Leibniz algebras [13]. Their name is the reversal of Leibniz, a play of words justified by the Koszul duality of the two corresponding operads. Maybe a better name would be “half-shuffle algebras”, as they are very closely related to shuffle algebras on words.

The algebras $C_{u,v}$, depending on algebraic parameters u, v , have the same graded dimensions as the motivic algebra \mathbf{A}_{MZV} . Our main conjecture is then that the restricted quotient map from $C_{u,v}$ to the motivic algebra \mathbf{A}_{MZV} is generically an isomorphism. The special case when $u + v = 0$ seems to give instead an interesting sub-algebra of \mathbf{A}_{MZV} .

One interest of this construction is that it would provide new bases of the algebra \mathbf{A}_{MZV} indexed by words in 2 and 3. Unlike the known Hoffman basis

indexed by the same set [5, 12], the shuffle product in any of these new bases can be expressed easily in the same basis, as the shuffle product of words. On the negative side, the description of the motivic coproduct in these bases is not clear, and the reduction of standard multiple zeta values as a linear combination is not simple either.

The two special cases with parameters u, v being 1, 0 or 0, 1 are specially interesting, as the conjectural bases obtained are then made of some arborified multiple zeta values, as studied in [15, 7, 17].

The construction of the sub-algebras $C_{u,v}$ is rather simple, as the zinbiel sub-algebras generated inside $A_{1,0}$ by two chosen elements z_2 and z_3 in degrees 2 and 3. To show that they have the correct dimensions, one just needs to prove that they are free as zinbiel algebras. Similar results about freeness of zinbiel sub-algebras of free zinbiel algebras have been obtained in [16].

It is possible that the same kind of ideas could be applied to some variants of multiple zeta values, in particular to the alternating multiple zeta values.

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1 Zinbiel algebras

Some references on zinbiel algebras are [13, 8, 14].

A zinbiel algebra over a commutative ring R is a module L over R endowed with a bilinear product $\prec: L \otimes_R L \rightarrow L$ such that

$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (z \prec y) \quad (1)$$

for all x, y, z in L . It is then convenient to introduce the symmetrized product \mathbb{III} defined by

$$x \mathbb{III} y = x \prec y + y \prec x, \quad (2)$$

for x, y in L . It can be deduced from (1) that \mathbb{III} is always a commutative and associative product on L .

We will always denote by \prec the zinbiel product in a zinbiel algebra.

The free zinbiel algebra on a finite set S over a field k has a very neat and simple description. The underlying vector space has a basis indexed by non-empty words with letters in S . The zinbiel product of two words w and w' is the sum of words in the standard shuffle product of w and w' in which the first letter comes from the first letter of w . Here is one way to remember this rule: the symbol \prec is pointing towards the word whose first letter remains the first letter.

Let us introduce a convenient notation. For a_1, \dots, a_n elements of a zinbiel algebra, let

$$K(a_1, \dots, a_n) = a_1 \prec K(a_2, \dots, a_n) \quad (3)$$

be their right-parenthesized product, with $K(a_n) = a_n$ by convention.

In the free zinbiel algebra over a finite set S , the basis element indexed by a word (s_1, \dots, s_n) with letters in S is exactly $K(s_1, \dots, s_n)$.

2 Algebra of convergent words in 0, 1

Let A be the free zinbiel algebra on two generators 0 and 1.

By the general description of free zinbiel algebra recalled above, A has a basis indexed by non-empty words in 0, 1. In this basis, the product \prec is one-half of the shuffle product. For example,

$$(10) \prec (10) = (1010) + 2(1100),$$

where we have emphasized the letter that remains the first letter. The associated commutative product \mathbb{III} is the standard shuffle product. The unit for the commutative product must be added as the empty word.

The algebra A is bigraded, by the number of 0 and the number of 1 in a word.

Let $\mathcal{A}_{1,0}$ be the set of words starting with 1 and ending with 0. Let $A_{1,0}$ be the sub-space of A spanned by these words. Then clearly $A_{1,0}$ is a zinbiel sub-algebra of A .

The algebra $A_{1,0}$ is not a free zinbiel algebras, because of the following relations. For every pair of words x and y in $\mathcal{A}_{1,0}$, there holds

$$(1x) \prec y = (1y) \prec x. \quad (4)$$

Indeed, when seen in A , this equality becomes

$$(1 \prec x) \prec y = (1 \prec y) \prec x,$$

which is a consequence of the zinbiel axiom (1).

3 Free sub-algebras

Our aim is to build, inside the algebra $A_{1,0}$, free zinbiel sub-algebras on two generators.

More precisely, let u and v be two parameters, not both zero and define

$$z_2 = (1, 0) \quad \text{and} \quad z_3 = u(1, 0, 0) + v(1, 1, 0) \quad (5)$$

in $A_{1,0}$.

Let $C_{u,v}$ be the zinbiel sub-algebra of $A_{1,0}$ generated by z_2 and z_3 . Note that it only depend on the class of (u, v) in the projective line \mathbb{P}^1 .

As both z_2 and z_3 are homogeneous with respect to the total grading of $A_{1,0}$ by the length of words, $C_{u,v}$ inherits a grading where z_2 has degree 2 and z_3 has degree 3.

For a word $w = (w_1, \dots, w_n)$ in the alphabet $\{2, 3\}$, let us denote

$$K_{u,v}(w) = K(z_{w_1}, \dots, z_{w_n}). \quad (6)$$

Theorem 1. *For all (u, v) not both zero, the sub-algebra $C_{u,v}$ is a free zinbiel algebra on two generators over \mathbb{Q} .*

Proof. Using the grading of $C_{u,v}$, it is enough to prove that the elements $K_{u,v}(w)$ for all words w in 2 and 3 of any given sum are linearly independent in $A_{1,0}$. By using the bigrading of $A_{1,0}$, one can instead prove the linear independence of the leading terms of these elements $K_{u,v}(w)$ with respect to either grading.

If $u \neq 0$, the leading term of $K_{u,v}(w)$ with respect to the number of 0 is a non-zero multiple of the element $K_{1,0}(w)$.

If $v \neq 0$, the leading term of $K_{u,v}(w)$ with respect to the number of 1 is a non-zero multiple of the element $K_{0,1}(w)$.

It is therefore enough to prove the statement in the cases $(u, v) = (1, 0)$ and $(u, v) = (0, 1)$. This is done in the next two sections. \square

Note that these two cases are really distinct, as there is no zinbiel automorphism of $A_{1,0}$ that would exchange them.

Lemma 1. *Let w_1, w_2, \dots, w_n be words in $A_{1,0}$. Then $K(w_1, \dots, w_n)$ is the sum over all shuffles of the words w_i such that the first letters remain in the same order.*

This is easily proved by induction, starting from the definition of \prec .

These shuffles will be called **good-shuffles**. The identity shuffle is always a good-shuffle.

3.1 Words 10 and 100

Let $C_{10,100}$ be the zinbiel sub-algebra of $A_{1,0}$ generated by the words 10 and 100. Let us denote in this section the word 10 by 2 and the word 100 by 3. The algebra $C_{10,100}$ is bigraded by the number of 2 and the number of 3, as 10 and 100 are homogeneous in $A_{1,0}$ with linearly independent bidegrees.

For a word w in the alphabet $\{2, 3\}$, let $\mathbf{cat} w$ be the word in $A_{1,0}$ obtained from w by the substitution $2 \mapsto 10$ and $3 \mapsto 100$.

By Lemma 1, for a word $w = w_1 \dots w_n$ in $\{2, 3\}$, the expansion of $K(w) = K(w_1, \dots, w_n)$ is a sum of words in $\{0, 1\}$ with the following property: the letters 1 (one in each w_i) remain in the same order. Moreover, every letter 0 coming from w_i is placed somewhere on the right of the letter 1 coming from w_i .

Theorem 2. *The zinbiel algebra $C_{10,100}$ is free over \mathbb{Z} .*

Proof. Because $C_{10,100}$ is generated by 2 and 3, its dimensions are bounded by the dimensions of the free zinbiel algebra in two generators.

It is therefore enough to prove that the dimension of the homogeneous component of any given bidegree (k, ℓ) with respect to 2 and 3 is at least the number of words with k letters 2 and ℓ letters 3.

Let us fix (k, ℓ) and consider the set S of words with k letters 2 and ℓ letters 3. Let us endow S with the lexicographic order induced by the ordering of letters $2 < 3$.

Let M be the square matrix with rows and columns indexed by S with coefficient $M_{w,w'}$ being the number of occurrences of the word $\mathbf{cat} w'$ in the expansion of $K(w)$ as a sum of words.

Let us prove that M is upper triangular with 1 on the diagonal.

Lemma 2. *Let $w, w' \in S$ such that $M_{w,w'} \neq 0$ and w shares a prefix with w' . Then the restriction to the common prefix of any good-shuffle that maps $\mathbf{cat} w$ to $\mathbf{cat} w'$ is the identity.*

Proof. This is proved by induction on the length i of the common prefix, starting from the empty prefix. Let us assume the induction hypothesis before w_i and that moreover $w_i = w'_i$. The 1 in w'_i must come from the 1 in w_i because the 1's remain in the same order. Then the 0 (one or two) from w'_i must be to the right of their associated 1, which must therefore be the 1 coming from w_i . The only possible way is that these 0 are not shuffled and remain at their initial positions in w_i . \square

In the case $w = w'$, this lemma implies that the diagonal of M is made of 1's.

Now consider $w \neq w' \in S$ such that $M_{w,w'} \neq 0$ and the first different letter happens at position i , where $w_i \neq w'_i$. Assume by contradiction that $w_i = 2$ and $w'_i = 3$. By the lemma 2, any good-shuffle that maps $\mathbf{cat} w$ to $\mathbf{cat} w'$ is the identity on the common prefix of w and w' . Therefore the letter 1 in w'_i comes from the letter 1 in w_i . But then the two 0 in w'_i need to have their associated 1 on their left, which is not possible. \square

3.2 Words 10 and 110

Let us now turn to the other case, slightly more complicated.

Let $C_{10,110}$ be the zinbiel sub-algebra of $A_{1,0}$ generated by the words 10 and 110. Let us denote in this section the word 10 by 2 and the word 110 by 3. The algebra $C_{10,110}$ is bigraded by the number of 2 and the number of 3, as 10 and 110 are homogeneous in $A_{1,0}$ with linearly independent bidegrees.

For a word w in the alphabet $\{2, 3\}$, let $\mathbf{cat} w$ be the word in $A_{1,0}$ obtained from w by the substitution $2 \mapsto 10$ and $3 \mapsto 110$.

Theorem 3. *The zinbiel algebra $C_{10,110}$ is free over \mathbb{Q} .*

Proof. Because $C_{10,110}$ is generated by 2 and 3, its dimensions are bounded by the dimensions of the free zinbiel algebra in two generators.

It is therefore enough to prove that the dimension of the homogeneous component of any given bidegree (k, ℓ) with respect to 2 and 3 is at least the number of words with k letters 2 and ℓ letters 3.

Let us fix (k, ℓ) and consider the set S of words with k letters 2 and ℓ letters 3. Let us endow S with the lexicographic order induced by the ordering of letters $2 < 3$.

Let M be the square matrix with rows and columns indexed by S with coefficient $M_{w,w'}$ being the number of occurrences of the word $\mathbf{cat} w'$ in the expansion of $K(w)$ as a sum of words.

Let us prove that M is lower triangular with no zero on the diagonal.

Let us first remark that the diagonal coefficient for a word w in S counts the number of good-shuffles that preserves $\mathbf{cat} w$. But this set always contains the identity shuffle.

Lemma 3. *Let $w, w' \in S$ such that $M_{w,w'} \neq 0$ and w shares a prefix $w_1 \dots w_i$ with w' . Let σ be any good-shuffle that maps $\mathbf{cat} w$ to $\mathbf{cat} w'$. Then*

- (i) *either σ stabilizes the common prefix,*
- (ii) *or the following statements hold:*
 - *There exists a letter 3 in the prefix. Let w_k be the rightmost such letter. The shuffle σ stabilizes the prefix before w_k .*
 - *The letter w_{i+1} is 2.*
 - *Between w_k and w_{i+1} , the shuffle σ acts like this:*

$$\begin{array}{r} \dots \\ w \\ w' \end{array} \left| \begin{array}{cccc} w_k = 3 & 2 & \dots & w_i = 2 \\ \mathbf{110} & 10 & \dots & 10 \\ \mathbf{110} & 10 & \dots & 10 \end{array} \right| \begin{array}{l} w_{i+1} = 2 \\ 10 \\ 10 \end{array}$$

where each 10 (from a letter 2) displayed in the line w is sent to the final 10 (from a letter 3 or 2) in the previous term of the line w' . The first 1 of w_k is sent to the first 1 of w'_k and the final 10 of w_k is sent in the suffix of w' after w'_i .

Proof. This is proved by induction on the length of the common prefix, starting with the empty prefix.

Assume first the induction hypothesis with condition (i) before w_i and moreover $w_i = w'_i$. Necessarily, the first 1 in w'_i must come from the first 1 in w_i , as the order is preserved on the first letters.

If w_i is 2, the 0 in w'_i must come from the 0 in w_i , as the only available 1 to its left is that of w'_i . So condition (i) holds for the extended prefix.

If w_i is 3, and if the second 1 in w'_i comes from w_i too, one finds that condition (i) holds for the extended prefix, for the same reason as in the previous case.

Otherwise, the second 1 in w'_i must come from w_{i+1} . And the 0 in w'_i must be preceded by all the associated 1's. This implies that this 0 comes from w_{i+1} and that $w_{i+1} = 2$. All this gives condition (ii) for the extended prefix, in the special situation where $k = i$.

Assume now the induction hypothesis with condition (ii) before w_i and moreover $w_i = w'_i$. Then w_i is a 2.

Assume first that σ sends the second 1 of w_k to the 1 in w'_i . Then the 0 in w_k must be sent to the 0 in w'_i . Therefore in this case, one obtains condition (i) for the extended prefix.

Otherwise, σ sends the second 1 of w_k somewhere in the suffix of w' after w'_i . Then the first 1 of w_{i+1} must be sent to the first one of w'_i . And the 0 in w'_i must be preceded by all the associated 1's. This implies that this 0 comes from w_{i+1} and that $w_{i+1} = 2$.

The first statement of condition (ii) holds by induction. We just proved the two other statements, so condition (ii) holds for the extended prefix. \square

Now consider $w \neq w' \in S$ such that $M_{w,w'} \neq 0$ and the first different letter happens at position i where $w_i \neq w'_i$. Assume by contradiction that $w_i = 3$ and $w'_i = 2$. Let σ be any good-shuffle that maps $\mathbf{cat} w$ to $\mathbf{cat} w'$.

By the lemma 3, either condition (i) or condition (ii) holds for σ .

Condition (ii) cannot hold because $w_i = 3$.

Therefore condition (i) holds. The unique letter 1 in w'_i comes from the first letter 1 in w_i . But then the 0 in w'_i either comes from $w_i = 3$ or from some w_j with $j > i$. In both cases, this 0 has not enough 1's on its left. \square

Remark: One can deduce from the proof of lemma 3 a more precise description of the diagonal coefficients of the matrix M . The coefficient of a word w in 2 and 3 is the product of the lengths of the blocks in the unique factorization of w into blocks $322 \dots 2$, omitting the possible initial sequence of 2. For example, for $(2, 3, 3, 2, 3, 2)$, one gets $4 = 1 \times 2 \times 2$.

3.3 Remarks and questions

One could ask the same question of freeness about several larger zinbiel sub-algebras of $A_{1,0}$.

Sometimes the answer is clearly negative, for the same reasons as for $A_{1,0}$. This is for instance the case of the sub-algebra generated by the words 10, 110, 1110 which contains one relation (4).

The following cases may be free, as these algebras do not contain any obvious relation of this kind.

- (A) the sub-algebra generated by words 10, 100, 110,
- (B) the sub-algebra generated by words of the shape 10^ℓ ,
- (C) the sub-algebra generated by 110 and words of the shape 10^ℓ ,
- (D) the sub-algebra generated by words not starting by 11,
- (E) the sub-algebra generated by 110 and words not starting by 11.

Some closely related questions have been answered in [16, §4].

4 Quotient map to motivic multiple zeta values

We will use the following convention for formal iterated integrals:

$$I(\varepsilon_1, \dots, \varepsilon_k) = \int_{0 < t_1 < \dots < t_k < 1} \dots \int \omega_{\varepsilon_1}(t_1) \dots \omega_{\varepsilon_k}(t_k), \quad (7)$$

where each ε_i is either 0 or 1, with $\varepsilon_1 = 1$ and $\varepsilon_k = 0$.

We will denote motivic multiple zeta values $\zeta(n_1, \dots, n_k)$, with the convention

$$\zeta(k_1, \dots, k_r) = I(1, 0^{k_1-1}, 1, 0^{k_2-1}, \dots, 1, 0^{k_r-1}) \quad (8)$$

for $r \geq 1$. Here a power of 0 means a repeated 0.

Let us denote by Π the surjective quotient map from $A_{1,0}$ to A_{MZV} , whose kernel is the ideal of motivic relations between formal iterated integrals.

For (u, v) not both zero, the space $C_{u,v}$ is a zinbiel sub-algebra of $A_{1,0}$, hence also a commutative sub-algebra of $A_{1,0}$ for the symmetrized product, which is just the shuffle product.

For every choice of parameters (u, v) not both zero, one can therefore restrict Π to this commutative sub-algebra $C_{u,v}$ of $A_{1,0}$. This gives a morphism of commutative graded algebras Π from $C_{u,v}$ to A_{MZV} . Note that these two graded algebras have the same generating series $F = 1/(1 - x^2 - x^3)$.

One can therefore wonder if the morphism Π could be an isomorphism, under some conditions on (u, v) . As the algebra $C_{u,v}$ itself, this property only depend on the projective class of (u, v) in \mathbb{P}^1 .

Let us consider first the very special case where $u + v = 0$. In this case, the image of the word z_3 in $C_{u,v}$ is given by $\Pi(z_3) = u\zeta(3) + v\zeta(1, 2)$. But it is known that $\zeta(3) = \zeta(1, 2)$ in A_{MZV} , hence $\Pi(z_3) = 0$. Therefore Π is not surjective in this case.

Using a computer, one can compute the first few graded dimensions of the image.

Conjecture 1. *When $u + v = 0$, the image of $C_{u,v}$ by Π is a sub-algebra of A_{MZV} with generating series $1 + x^2F$.*

This is the generating series of the quotient algebra of A_{MZV} by $\zeta(3)$, because

$$1 + x^2F = (1 - x^3)F. \quad (9)$$

One can wonder what could be, for this sub-algebra, an analog of the famous conjecture of Broadhurst-Kreimer describing the dimensions of A_{MZV} according to both weight and depth.

Note also the similarity with the quotient algebra of A_{MZV} by $\zeta(2)$, which has generating series

$$1 + x^3F = (1 - x^2)F \quad (10)$$

and appears in the motivic coproduct and in the study of p -adic multiple zeta values [9, 10].

Excluding from now on the special case $u + v = 0$, one can assume without loss of generality that $u + v = 1$ and set $v = 1 - u$. Then $\Pi(z_3) = \zeta(3)$.

When using u as a formal parameter, one expects the following statement.

Conjecture 2. *The morphism Π from $C_{u,1-u}$ to A_{MZV} is generically an isomorphism of commutative algebras.*

Let us say that a value of u is **non-singular** if this statement holds at u , and **singular** otherwise. For every non-singular u , the isomorphism Π defines a bigrading of the algebra A_{MZV} . Moreover, one gets a basis in A_{MZV} from the basis of $C_{u,1-u}$ made of words in z_2 and z_3 .

More and more singular values appear when considering the restriction of Π at increasing weights. One could still hope that some specific values of u are non-singular, for example $u = 0$ and $u = 1$. So far, no non-singular value of u is known.

Let us describe the first few polynomials whose zeroes are singular values, in their order of apparition, where n is the weight.

| n | p |
|-----|--|
| 5 | $5u - 6$ |
| 7 | $14u + 51$ |
| 8 | $27u^2 - 26u + 10$ |
| 9 | $865u - 4164$ |
| 10 | $2011u^2 - 3381u + 1581$ |
| 11 | $461516u^4 - 3721029u^3 + 7046644u^2 - 6169912u + 2357966$ |
| 12 | $207786u^4 - 185687u^3 - 1076020u^2 + 1483088u - 562680$ |

One can check that 0 and 1 are not roots of any of these polynomials.

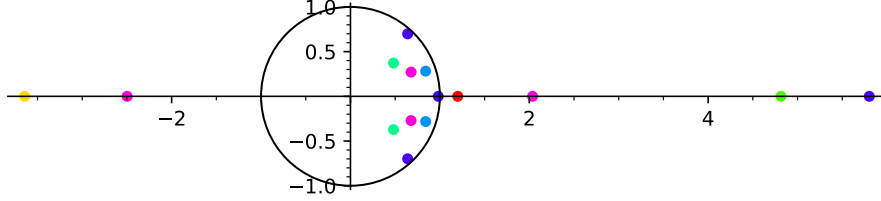


Figure 1: Singular values of u in weight at most 12.

Let us give the first few images by the morphism Π from $C_{u,1-u}$ to A_{MZV} of short words in z_2 and z_3 .

$$B(2) = \zeta(2) \tag{11}$$

$$B(3) = \zeta(3) \tag{12}$$

$$B(2, 2) = 2\zeta(1, 3) + \zeta(2, 2) \tag{13}$$

$$B(2, 3) = (u + 2)\zeta(1, 4) + \zeta(2, 3) + (-u + 1)\zeta(3, 2) \tag{14}$$

$$B(3, 2) = (-u + 4)\zeta(1, 4) + 2\zeta(2, 3) + u\zeta(3, 2). \tag{15}$$

One can then check that $B(2)B(3) = B(2, 3) + B(3, 2)$, as a simple example of the general rule that the product in the conjectural bases is given by the shuffle product.

5 Variants of multiple zeta values

There are some other situations where one could try to apply the same ideas.

A first example is given by the algebra of alternating multiple zeta values, defined as the iterated integrals of the following three 1-forms:

$$\omega_0 = \frac{dt}{t}, \omega_{-1} = \frac{dt}{t-1}, \omega_1 = \frac{dt}{t+1}. \quad (16)$$

A conjecture due to Broadhurst (see [1, 4]) states that the graded dimensions of this algebra are given by the Fibonacci numbers, with generating series $1/(1-x-x^2)$.

One could therefore consider the zinbiel sub-algebra of the free zinbiel algebra on $\{-1, 0, 1\}$ generated by the abstract iterated integrals $I(-1)$ and $I(1, 0)$. Is this a free zinbiel algebra on these generators ?

A similar case is the algebra of multiple Landen values, defined in [2] as iterated integrals of the following 1-forms: $A = dx/x$, $B = dx/(1-x)$, $F = dx/(1-\rho^2x)$ and $G = dx/(1-\rho)$ where ρ is the golden ratio.

Broadhurst conjectured in [2] that the generating series for this algebra is $1/(1-x-x^2-x^3)$, whose coefficients are tribonacci numbers. The exact same formula is also expected to give the graded dimensions of the sub-algebra of iterated integrals of A and G .

The first few terms are

$$1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, \dots$$

and the expected bases, as words in A and G are $\{G\}$ in degree 1; $\{AG, GG\}$ in degree 2 and $\{AGG, AAG, GAG, GGG\}$ in degree 3.

In this setting, one could look for a free zinbiel sub-algebra on three generators of degrees 1, 2 and 3 inside the free zinbiel algebra on A and G .

Another interesting case to consider would be multiple Watson values [3].

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