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TESTING A CLASS OF TIME VARYING CHARN MODELS

Youssef SALMAN ^{1,2} & Joseph NGATCHOU WANDJI ¹ & Zaher KHRAIBANI ²

¹ *IECL, Lorraine University, France*

² *LM, Lebanese University, Faculty of sciences, Lebanon*

Email: yousef.salman@univ-lorraine.fr

Email: joseph.ngatchou-wandji@univ-lorraine.fr

Email: Zaher.khraibani@ul.edu.lb

Abstract. We study a likelihood ratio test for testing a general class of CHARN models. The LAN property is established for the family of likelihoods under study. The test is proved to be optimal.

Keywords. Times series, changepoint, likelihood ratio test, LAN, optimality.

Résumé. Nous étudions un test du rapport de vraisemblance pour tester une classe générale de modèles CHARN. La propriété LAN est établie pour la famille des vraisemblances considérées. L'optimalité du test est prouvée.

Mots-clés. Séries chronologiques, rupture, test du rapport de vraisemblance, LAN, optimalité.

1 Introduction

We consider X_1, \dots, X_n , observations generated by the CHARN model "Conditional Heteroskedastic Autoregressive Nonlinear" [1]

$$X_t = T(\rho_0 + \gamma \odot \omega(t); \mathbf{X}_{t-1}) + V(\mathbf{X}_{t-1})\varepsilon_t, t \in \mathbb{Z} \quad (1)$$

where $(X_t)_{t \in \mathbb{Z}}$ is a locally stationary ergodic process, $(\varepsilon_t)_{t \in \mathbb{Z}}$ is a standard white noise with knowing density function f , $\mathbf{X}_t = (X_t, \dots, X_{t-d+1})^\top$, T and V are real smooth functions and $V > 0$, $\rho_0^\top \in \mathbb{R}^p$, $\gamma = (\gamma_1^\top, \dots, \gamma_{k+1}^\top)^\top$ and for every $j = 1, \dots, k+1$, $\gamma_j \in \mathbb{R}^p$, $\omega(t) = (\mathbb{1}_{[\tau_0, \tau_1)}(t), \mathbb{1}_{[\tau_1, \tau_2)}(t), \dots, \mathbb{1}_{[\tau_{k-1}, \tau_k)}(t), \mathbb{1}_{[\tau_k, \tau_{k+1})}(t))^\top = (\omega_1, \dots, \omega_{k+1}) \in \{0, 1\}^{k+1}$. We assume that, for every $j = 1, \dots, k$, $n_j(n)$ represents the number of observations in $[\tau_{j-1}, \tau_j)$, $\tau_0 = 1 < \tau_1 < \dots < \tau_{k+1} = n$. We suppose that, as $n \rightarrow +\infty$, $n_j(n) \rightarrow +\infty$ and $\frac{n_j(n)}{n} \rightarrow \alpha_j$. F_j is the distribution function of X_j on $[\tau_{j-1}, \tau_j)$. For every $\Pi = (\Pi_1^\top, \dots, \Pi_{k+1}^\top)^\top \in \mathbb{R}^{p(k+1)}$ and $\Theta = (\Theta_1^\top, \dots, \Theta_{k+1}^\top) \in \mathbb{R}^{p(k+1)}$,

$$\Pi \odot \Theta = \Pi_1 \Theta_1 + \dots + \Pi_{k+1} \Theta_{k+1}$$

and for any β and $\tilde{\beta} \in \mathbb{R}^{p(k+1)}$,

$$\beta \circ \tilde{\beta} = \tilde{\beta} \circ \beta = \begin{pmatrix} \begin{pmatrix} \beta_{1,1} \\ \vdots \\ \beta_{1,p} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \beta_{k+1,1} \\ \vdots \\ \beta_{k+1,p} \end{pmatrix} \end{pmatrix} \circ \begin{pmatrix} \begin{pmatrix} \tilde{\beta}_{1,1} \\ \vdots \\ \tilde{\beta}_{1,p} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \tilde{\beta}_{k+1,1} \\ \vdots \\ \tilde{\beta}_{k+1,p} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \beta_{1,1}\tilde{\beta}_{1,1} \\ \vdots \\ \beta_{1,p}\tilde{\beta}_{1,p} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \beta_{k+1,1}\tilde{\beta}_{k+1,1} \\ \vdots \\ \beta_{k+1,p}\tilde{\beta}_{k+1,p} \end{pmatrix} \end{pmatrix} \in \mathbb{R}^{p(k+1)}$$

stands for the Hadamard product [2].

We aim to test

$$H_0 : \gamma = \gamma_0 \quad \text{against} \quad H_1^{(n)} : \gamma = \gamma_n = \gamma_0 + \beta/\sqrt{n}, \quad \text{where } \gamma_0 \text{ and } \beta \in \mathbb{R}_*^{p(k+1)} \quad (2)$$

In this purpose, we used the likelihood ratio test.

Our primary goal is to verify that τ_j , $j = 1, \dots, k$, are instants of change or not. This work is preliminary to the construction of a method for testing weak changes.

The changepoint theory was started by Page [3]. He used the cumulative sum (CUSUM) to detect the changepoints in the mean of independent observations. Since then, there has been a lot of work on changepoints see, eg. [4], [5], [6], [7].

2 Asymptotics

In the present paper, we investigate the case where the functions T , V and f are known, as well as the nuisance parameter ρ_0 .

2.1 Likelihood ratio test

The log-likelihood ratio test Θ_n for H_0 against $H_1^{(n)}$, can be expressed as follow

$$\Theta_n = \Theta_{1n} - \Theta_{2n} + o_P(1), \quad (3)$$

where

$$\begin{aligned} \bullet \quad \Theta_{1n} &= \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ \frac{1}{V(\mathbf{X}_{t-1})} \beta^\top N(\gamma_0, \mathbf{X}_{t-1}) \phi_f[\varepsilon_t(\gamma_0)] \right\} \\ \bullet \quad \Theta_{2n} &= \frac{1}{2n} \sum_{t=1}^n \left\{ \frac{1}{V^2(\mathbf{X}_{t-1})} \beta^\top M(\gamma_0, \mathbf{X}_{t-1}) \beta \phi_f'[\varepsilon_t(\gamma_0)] - \frac{1}{V(\mathbf{X}_{t-1})} \beta^\top \mathcal{H}(\gamma_0, \mathbf{X}_{t-1}) \beta \phi_f[\varepsilon_t(\gamma_0)] \right\} \end{aligned}$$

- $M(\gamma_0, \mathbf{X}_{t-1}) = N(\gamma_0, \mathbf{X}_{t-1})N^\top(\gamma_0, \mathbf{X}_{t-1})$
- $N(\gamma_0, \mathbf{X}_{t-1}) = \omega(t) \circ D_\gamma[T(\gamma_0, \mathbf{X}_{t-1})]$
- $D_\gamma[T(\gamma_0, \mathbf{X}_{t-1})] = \left(\nabla_{\gamma_1}[T(\gamma_0, \mathbf{X}_{t-1})], \dots, \nabla_{\gamma_{k+1}}[T(\gamma_0, \mathbf{X}_{t-1})] \right)^\top \in \mathbb{R}^{p(k+1)}$
- $\nabla_{\gamma_i}[T(\gamma_0, \mathbf{X}_{t-1})] = \left(\frac{\partial T}{\partial \gamma_{i,1}}(\gamma_0, \mathbf{X}_{t-1}), \frac{\partial T}{\partial \gamma_{i,2}}(\gamma_0, \mathbf{X}_{t-1}), \dots, \frac{\partial T}{\partial \gamma_{i,p}}(\gamma_0, \mathbf{X}_{t-1}) \right)^\top \in \mathbb{R}^p$ is the gradient of T with respect to γ_i
- $\mathcal{H}(\gamma_0, \mathbf{X}_{t-1}) = \omega(t)\omega(t)^\top \circ H_\gamma(\gamma_0, \mathbf{X}_{t-1})$
- $\mathcal{H}(\tilde{\gamma}, \mathbf{X}_{t-1}) = \begin{pmatrix} \mathcal{H}_1(\tilde{\gamma}, \mathbf{X}_{t-1}) & 0 & \dots & 0 \\ 0 & \mathcal{H}_2(\tilde{\gamma}, \mathbf{X}_{t-1}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathcal{H}_{k+1}(\tilde{\gamma}, \mathbf{X}_{t-1}) \end{pmatrix} \in \mathbb{R}^{p(k+1) \times p(k+1)}$
- $\mathcal{H}_i(\tilde{\gamma}, \mathbf{X}_{t-1}) = \omega_i^2 \begin{pmatrix} \frac{\partial^2 T}{\partial \gamma_{i,1}^2}(\tilde{\gamma}, \mathbf{X}_{t-1}) & \dots & \frac{\partial^2 T}{\partial \gamma_{i,p} \partial \gamma_{i,1}}(\tilde{\gamma}, \mathbf{X}_{t-1}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial \gamma_{i,1} \partial \gamma_{i,p}}(\tilde{\gamma}, \mathbf{X}_{t-1}) & \dots & \frac{\partial^2 T}{\partial \gamma_{i,p}^2}(\tilde{\gamma}, \mathbf{X}_{t-1}) \end{pmatrix} \in \mathcal{M}_p(\mathbb{R})$ for $i = 1, \dots, k+1$.

where

$$\varepsilon_t(\gamma) = \frac{X_t - T(\rho_0 + \gamma \odot \omega(t), \mathbf{X}_{t-1})}{V(\mathbf{X}_{t-1})}.$$

2.2 LAN property

After finding the expression of the likelihood ratio test, we must find its distribution. To compute the distribution of the test under the null hypothesis, we need to establish the LAN property stated in [5]. To do this, we proved that, under H_0 ,

$$\Theta_{2n} \xrightarrow[n \rightarrow +\infty]{} \frac{\eta(\gamma_0, \beta)}{2} \quad (4)$$

and

$$\Theta_{1n} \xrightarrow[n \rightarrow +\infty]{} \mathcal{N}(0, \eta(\gamma_0, \beta)), \quad (5)$$

where

$$\begin{aligned}\eta(\gamma_0, \beta) &= \sum_{j=1}^{k+1} \alpha_j \sum_{1 \leq h \leq m \leq p} \beta_{j,h} \beta_{j,m} \eta_{j,2}(\gamma_0), \\ \eta_{j,2}(\gamma_0) &= I(f) \int_{\mathbb{R}} \left(\frac{1}{V(x)} \right)^2 \frac{\partial T}{\partial \gamma_{j,h}}(\gamma_0, x) \frac{\partial T}{\partial \gamma_{j,m}}(\gamma_0, x) dF_j(x) < \infty, \\ I(f) &= \int_{\mathbb{R}} \phi_f^2(x) f(x) dx < \infty.\end{aligned}$$

We considered the central sequence $\Pi_n(\gamma_0, \beta)$ expressed as follow

$$\Pi_n(\gamma_0, \beta) = \Theta_{1n} = \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ \frac{1}{V(\mathbf{X}_{t-1})} \beta^\top N(\gamma_0, \mathbf{X}_{t-1}) \phi_f[\varepsilon_t(\gamma_0)] \right\}. \quad (6)$$

Based on the previous results and under H_0 , we can write

$$\begin{pmatrix} \Pi_n(\gamma_0, \beta) \\ \Theta_n \end{pmatrix} \xrightarrow{law} \mathcal{N} \left(\begin{pmatrix} 0 \\ -\frac{\eta(\gamma_0, \beta)}{2} \end{pmatrix}, \begin{pmatrix} \eta(\gamma_0, \beta) & \eta(\gamma_0, \beta) \\ \eta(\gamma_0, \beta) & \eta(\gamma_0, \beta) \end{pmatrix} \right). \quad (7)$$

By returning to testing H_0 against $H_1^{(n)}$, we consider the following statistic of test:

$$\mathcal{T}_n(\gamma_0, \beta) = \frac{\Pi_n(\gamma_0, \beta)}{\widehat{\pi}_n(\gamma_0, \beta)} \quad (8)$$

where

- $\widehat{\pi}_n(\gamma_0, \beta) = \sqrt{\widehat{\eta}_n(\gamma_0, \beta)}$,
- $\widehat{\eta}_n(\gamma_0, \beta) = \sum_{j=1}^{k+1} \widehat{\alpha}_j \sum_{1 \leq h \leq m \leq p} \widehat{\beta}_{j,h} \widehat{\beta}_{j,m} \widehat{\eta}_{j,2}(\gamma_0)$,
- $\widehat{\eta}_{j,2}(\gamma_0) = I(f) \int_{\mathbb{R}} \frac{1}{V^2(x)} \frac{\partial T}{\partial \gamma_{j,h}}(\gamma_0, x) \frac{\partial T}{\partial \gamma_{j,m}}(\gamma_0, x) dF_j(x)$
- $\widehat{\eta}_{j,2}(\gamma_0)$ is an estimator of $\eta_{j,2}(\gamma_0)$ and $\widehat{\alpha}_j$ is an estimator of $\lim_{n \rightarrow +\infty} \frac{n_j(n)}{n}$.

As $n \rightarrow +\infty$, under H_0 , we have

$$\mathcal{T}_n(\gamma_0, \beta) \xrightarrow{law} \mathcal{N}(0, 1).$$

From (7), by the Le Cam's third lemma (proposition 4.2 in [5]), under $H_1^{(n)}$, we can write

$$\Pi_n(\gamma_0, \beta) \xrightarrow{\text{law}} \mathcal{N}(\eta(\gamma_0, \beta), \eta(\gamma_0, \beta)).$$

We show that, when $n \rightarrow +\infty$, under H_0 ,

$$\widehat{\pi}_n(\gamma_0, \beta) \longrightarrow \pi(\gamma_0, \beta).$$

This convergence remains true under $H_1^{(n)}$ by contiguity.

By using Le cam's third lemma, we conclude that under $H_1^{(n)}$, as $n \rightarrow +\infty$,

$$\frac{\Pi_n(\gamma_0, \beta)}{\pi(\gamma_0, \beta)} \xrightarrow{\text{law}} \mathcal{N}(\pi(\gamma_0, \beta), 1).$$

Then, under $H_1^{(n)}$, as $n \rightarrow +\infty$, we have

$$\frac{\Pi_n(\gamma_0, \beta)}{\widehat{\pi}_n(\gamma_0, \beta)} \xrightarrow{\text{law}} \mathcal{N}(\pi(\gamma_0, \beta), 1).$$

We used in this section many mathematical tools in order to analyse the asymptotic behavior of our test under the null and the alternatives hypotheses.

2.3 Power of the test

To calculate the power of our test statistic, we need to derive the asymptotic cumulative distribution of $\frac{\Pi_n(\gamma_0, \beta)}{\widehat{\pi}_n(\gamma_0, \beta)}$.

From simple computation, one has:

$$\lim_{n \rightarrow +\infty} P\left(\frac{\Pi_n(\gamma_0, \beta)}{\widehat{\pi}_n(\gamma_0, \beta)} > z_\alpha \mid H_1^{(n)}\right) = 1 - \Phi(z_\alpha - \pi(\gamma_0, \beta)),$$

where Φ is the cumulative distribution function of a standard Gaussian distribution and z_α is its $(1 - \alpha)$ -quantile, $\alpha \in [0, 1]$.

By using section 4.4.3 of [5], the test based on $\mathcal{T}_n(\gamma_0, \beta)$ is locally asymptotic optimal.

3 Conclusion

The aim of the present work is a preliminary study to a new changepoint detection method that we are currently investigating as a generalisation of Ngatchou-Wandji and Ltaifa [8]. The likelihood ratio test studied here is optimal. The LAN property is an important tool to find the distribution of the test under the local alternatives.

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