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ABSTRACT. The paper is devoted to the mathematical analysis of the Cauchy problem for general cross-diffusion systems without any assumption about its entropic structure. A global existence result of nonnegative solutions is obtained by applying a classical Schauder fixed point theorem. The proof is upgraded for enhancing the regularity of the solution, namely its gradient belongs to the space $L^r((0,T)\times\Omega)$ for some $r>2$. To this aim, the Schauder's strategy is coupled with an extension of Meyers regularity result for linear parabolic equations. We show how this approach allows to prove the well-posedness of the problem using only assumptions prescribing and admissibility range for the *ratios* between the diffusion and cross-diffusion coefficients. The results are compared with those that are reachable with an additional regularity assumption on the parabolic operator, namely a small BMO assumption for its coefficients. Finally, the question of the maximal principle is also addressed, especially when source terms are incorporated in the equation in order to ensure the confinement of the solution.

Keywords: cross-diffusion system; quasilinear parabolic equations; global in time existence; uniqueness.

1. INTRODUCTION

A strongly coupled cross-diffusion system for the unknown $u=(u_i)_{i=1,\dots,m}$ reads

$$\partial_t u_i = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\Big(\sum_{j=1}^{m} D^k_{i,j}(u)\frac{\partial u_j}{\partial x_k}\Big) =: \nabla\cdot J_i, \qquad \text{for } i=1,\dots,m. \tag{1.1}$$

Its analysis is known to be difficult because of the coupling of the highest order derivatives terms. Nevertheless, since cross-diffusive models occur in many domains, such as for instance biology, chemistry, ecology and fluid mechanics, there is a wide literature on the subject. Let us give some bibliographical references, without of course trying to be exhaustive.

System (1.1) is studied by Choi, Huan and Lui in [13]. They show a global weak existence result by assuming that each matrix $D^k(u)=(D^k_{i,j}(u))_{1\le i,j\le m}$ is positive definite and that its components are continuous and uniformly bounded with respect to u. The proof is based on Galerkin method and on the application of the Schauder fixed-point theorem to a linearized system. An application is given in a one-dimensional electrochemistry context. According to [3], similar assumptions but with smoother coefficients give the existence of a unique maximal classical solution.

Unfortunately, when the assumptions of positive definiteness and of uniform boundedness for the matrices D^k fall down, proving the existence of solutions for a cross-diffusive problem becomes a tricky question, not to mention an hypothetical uniqueness result. The literature in this context thus essentially reduces to the study of very particular systems. We refer for instance to the reviews [41] and Chapter 4 in [26] for a variety of illustrations. Let us give only two examples, with a linear dependance of D^k on

u (corresponding to the setting considered in the present paper) for emphasizing the importance of the remaining open questions, even for systems involved in primary societal models.

An important particular case of (1.1) is the so-called Shigesada–Kawasaki–Teramoto (SKT) system [45], related to population dynamics and reading, *e.g.* in the case $m = N = 2$, as follows:

$$D^1_{1,1}(u) = D^2_{1,1}(u) = 2\alpha_{11}u_1 + \alpha_{12}u_2 + \delta, \qquad D^1_{1,2}(u) = D^2_{1,2}(u) = \alpha_{12}u_1,$$
$$D^1_{2,1}(u) = D^2_{2,1}(u) = \alpha_{21}u_2, \qquad D^1_{2,2}(u) = D^2_{2,2}(u) = \alpha_{21}u_1 + 2\alpha_{22}u_2 + \delta.$$

The unknown u_i, for $i = 1,2$, stands for the population density of the i^{th} species. The SKT system has been widely studied. The existence of a solution is proved both in the non-degenerate case ($\delta \neq 0$: [9, 27, 28, 32, 42, 43]) and in the degenerate case ($\delta = 0$: [35, 25]), always with some restrictions on the non-negative coefficients $(\alpha_{i,j})_{1\leq i,j\leq 2}$. The perturbation of the flux J_i by a term describing a prescribed environmental potential, in the form $d_iu_i\nabla P$, $i = 1,2$, is considered in [10, 20, 47]. In [48], the authors prove a global existence result for the SKT system with a nonlinear reactive source term of the form $(a_i - \sum_{j=1}^2 b_{ij}u_j)u_i$, $i = 1,2$, while an extension of the SKT model, but with linear reaction terms, is studied in [12].

Another particular case of (1.1) appears in seawater intrusion models, whatever they are based on the sharp-diffuse interface approach, as described and studied in [14, 15, 17], or on the sharp interface approach as in [23, 1, 37]. As mentioned in [1], such models can be described by (1.1) by setting

$$D^1_{1,1}(u) = D^2_{1,1}(u) = (1-\alpha)u_1 + \delta, \qquad D^1_{1,2}(u) = D^2_{1,2}(u) = (1-\alpha)u_1,$$
$$D^1_{2,1}(u) = D^2_{2,1}(u) = (1-\alpha)u_2, \qquad D^1_{2,2}(u) = D^2_{2,2}(u) = u_2 + \delta,$$

where u_1 (resp. u_2) denotes the thickness of the freshwater part (resp. of the saltwater part) and $\alpha \in (0,1)$ is the relative density contrast between freshwater and saltwater. The nonnegative parameter δ characterizes the thickness of the mixing area separating the salt and freshwater on the one hand, and the saturated and unsaturated part of the aquifer on the other hand. Thus, the degenerate case ($\delta = 0$) corresponds to the sharp interface approach. The proof of existence given in [1] is based on an entropy estimate, defined by the Boltzmann entropy density, allowing both the control of the gradients and the statement of the nonnegativity of the solution. This latter point is not proved as a maximum principle result but is a direct consequence of the change of unknown defined by the entropy.

Associating an entropy with a system of parabolic equations is a classical approach. For cross-diffusive problems, the method has been greatly developed by Jüngel and collaborators, see for instance [25], and by Desvillettes et al. in [18]. The entropy decay is a powerful tool for providing an uniform estimate of the solution in the space $L^2(0,T;H^1(\Omega))$ that yields the global in time existence, assuming "only" that the (algebraic) structure of the tensor D allows the definition of an adapted entropy. As already mentioned, the change of unknown defined by the entropy allows sometimes, as a *bonus* result, to prove the boundedness of the solution.

The two latter examples are also emblematic of the very few existing uniqueness results for cross-diffusion systems. The uniqueness of the solution remains an open problem both for the first and third example. For the well-researched SKT system, we refer to the recent works [40] with $\delta \neq 0$ and [11] with $\delta = 0$. Both of them are based on a dual method, coupled moreover with the entropy method of Gajewski in [11]. These methods, the dual and the entropy one, are renowned both for dealing with weak solutions (unlike the semigroup tools that are restricted to mild solutions) and for not being restricted to scalar problems. But, despite the qualities of these methods, uniqueness results are limited by further restrictions on the parameters in (1.1). For instance in [40], the source terms are assumed quadratic and, furthermore, such that $0 < (\alpha_{ij})^2 < 8\alpha_{ii}\alpha_{ji}$, $i \neq j$; this assumption allows in particular the authors to recover a kind of positive definite character for $D(u)$ in the formulation (1.1), namely, for any $\xi \in \mathbb{R}^2$,

$$D(u)\xi \cdot \xi \geq \alpha(u_1 + u_2)|\xi|^2 + \delta|\xi|^2, \tag{1.2}$$

with $0 < \alpha < \min(\alpha_{ij}, \delta)$. Let us already emphasize that in the present paper, we choose a cross-diffusion model that does not satisfy Assumption (1.2).

The present paper aims at breaking away from the usual class of hypothesis mentioned before. Our methodology is not based on a structural, algebraic, assumption on the tensor D like the one allowing entropy methods, but rather on an heterogeneity assumption on the *ratio* between the diffusive and the cross-diffusive part of the operator. More precisely, we do not specify any algebraic equation satisfied by the parameters of D in (1.1); we relax our assumptions into inequalities describing the necessary smallness of some of the *ratios* between the different components of the tensor D. Indeed the latter is often the most accessible information, in particular due to the lack of complete physical data or due to the phenomenological character of the model. In particular our approach may be chosen when the identification of the entropy structure of the system seems difficult. We use classical variational tools and we do not need to restrict the results to smooth solution. We only use a Meyers' type result which ensures that the weak solution in $L^2(H^1)$ of a parabolic problem is actually a little more regular, in $L^r(W^{1,r})$, $r > 2$. The key point, unfortunately the most restricting one, is to characterize r with regard to the data of the problem. Then the method consists in controlling the *ratios* between the components of tensor D for reaching the regularity, namely the value of r, which allows to prove the uniqueness with Gagliardo–Nirenberg type inequalities for handling the nonlinear coupled cross-diffusive terms.

As a consequence, we can give in particular a range for the *ratios* in D ensuring the uniqueness of the bounded solution of (1.1) (completed by boundary conditions). The boundedness of the solution is thus, of course, also of interest. With the (non-entropic) tools used in the present paper, the question corresponds to the proof of weak maximum principles. We thus prove that there exists a source term confining the solution of (1.1) below any prescribed maximal value.

Enhancing the regularity of the solution is a key point in our approach. Here we choose to exploit self-improving properties in the spirit of Meyers regularity theorem. Basically, the greater the ellipticity rate of the operator, the greater the gain in regularity (because at the limit the operator behaves like the Laplacian). It largely explains our smallness assumptions. Another approach for ensuring the excited summability of the gradients consists in making stronger assumptions on the smoothness of the domain and of the coefficients of the equations. Interesting results are those obtained by Krylov in [29] for linear parabolic equations with BMO coefficients and extended by Dong and Kim for small VMO coefficients in [19]. As it will become clear below, such additional assumptions does not prevent some other smallness assumptions and lead to local in time results.

Despite our method applies for the study of any system in the form (1.1), all the results and all the computations of the present paper are done for a particular class of cross-diffusion systems, the one classically modeling the dispersal of two interacting biological species (see for instance [22]). Indeed, it is one of the less cumbersome systems containing all the difficulties inherent to the analysis of a strongly coupled cross-diffusion. Since the model also corresponds to the seawater intrusion model presented above, some issues left open in [15] are also clarified in the present paper. By the way, we emphasize that this paper aims primary at exposing a methodology for the study of cross-diffusion systems, based on assumptions on the parameters rates rather than structural assumptions. This methodology can of course be applied to more general systems, provided that the assumptions are properly adapted. Notice however that if the cross-diffusion parameters do no longer depend linearly on the solution and remain unbounded, the uniqueness proof could become unfeasible. Notice also that, due to the use of an interpolation inequality for handling with the nonlinearity in the cross-diffusive terms, the validity of the uniqueness result may be restricted. Here for instance it only holds true for a one-dimensional or two-dimensional domain (while in [40], for the well-structured SKT system, the result holds true up to dimension 4).

The paper is organized as follows. In Section 2, we introduce the cross-diffusion system and the functional setting, and we remind some auxiliary results that will be used thereafter. In Section 3, we state the main results of the paper, namely the global in time existence of solutions, the uniqueness of solutions and the maximum principle. Section 4 is devoted to the proof of the global existence result. The proof is divided in two steps, namely (i) the existence of solutions to a linearized system and (ii) the nonnegativity of the solution. The uniqueness of the solution is established in Section 5. The proof rests mainly on the fact that the gradient of the solution belongs to L^r for some $r > 2$. More precisely, we generalize to the quasilinear case the regularity result given by Meyers ([36]) in the elliptic case and extended to the parabolic case by A. Bensoussan, J.-L. Lions and G. Papanicolaou (see [6]). Then, if $N = 2$, a version of the Gagliardo-Nirenberg inequality allowing to control the L^4 norm by the L^2 and H^1_0 norms is sufficient for proving the uniqueness result, provided the value $r = 4$ is reached. The results require that the operator satisfies an uniform ellipticity assumption and that its coefficients are L^∞ functions. The last hypothesis requires, in our case, the uniform boundedness of the solution $(u_i)_{i=1}^m$. In Section 6, several considerations on the maximum principle are thus presented. Intuition suggests using "pumping" source terms that are sufficiently large to control the upper limit of the solution. We establish the existence of such source terms by introducing a penalized problem for which we let the penalization blow up. To conclude, we give a physical meaning for this penalization approach to demonstrate the existence of a confined solution.

2. Mathematical setting and auxiliary results

In the present section, we introduce the natural functional setting for addressing the well-posedness of cross-diffusive problems. We also give three auxiliary lemmas about the regularity of parabolic systems.

We consider an open bounded domain Ω of $\mathbb{R}^N$, $N \in \mathbb{N}^*$, $N \le 3$ for practical applications. The boundary of Ω, assumed to be of class $\mathscr{C}^1$, is denoted by Γ. The time interval of interest is $(0,T)$, T being any positive real number. Set $\Omega_T := (0,T) \times \Omega$.

The cross-diffusive system we deal with is a particular case of (1.1), namely

$$\partial_t u_i - \nabla \cdot \big(\delta_i \nabla u_i + u_i \sum_{j=1}^m K_{i,j} \nabla u_j\big) = Q_i(u) \ \text{ in } \Omega_T, \text{ for } i = 1,\dots,m. \tag{2.1}$$

It is completed by the following boundary and initial conditions, for $i = 1,\dots,m$:

$$u_i = u_{i,D} \text{ in } (0,T) \times \Gamma, \quad u_i(0,x) = u_i^0(x) \text{ in } \Omega.$$

We consider the fully non-degenerate setting

$$\delta_i > 0 \quad 1 \le i \le m. \tag{2.2}$$

For any $1 \leq i, j \leq m$, the tensor $K_{i,j}$ is assumed to be bounded and uniformly elliptic. More precisely, there exist two positive real numbers, $0 < K_{i,j}^- \leq K_{i,j}^+$, such that

$$0 < K_{i,j}^-|\xi|^2 \leq K_{i,j}\xi \cdot \xi = \sum_{k,l=1}^{N} (K_{i,j})_{kl}\xi_k\xi_l \leq K_{i,j}^+|\xi|^2, \ \ \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{2.3}$$

System (2.1) is classically used for modeling the dispersal of m interacting species. Turning back to the two examples given in Section 1, the system (2.1) also corresponds to the aquifer model presented on page 2, the tensors $K_{i,j}$ describing the permeability of the underground. On the other hand, the system (2.1) could appear both more general and more simple than the SKT system that reads $m = 2$ and

$$\partial_t u_i - \nabla \cdot \big(\delta_i \nabla u_i + u_i(2\alpha_{ii}\nabla u_i + \alpha_{ij}\nabla u_j) + \alpha_{ij}u_j\nabla u_i\big) = Q_i(u), \ i = 1,2, \ j \neq i.$$

It appears more general –and actually it is– since we use tensors $K_{i,j}$ instead of scalar parameters α_{ij}. It could appear simpler because of the loss of a nonlinearity, namely the one involving $u_j\nabla u_i$, $j \neq i$, in (2.1). It is not the case. We emphasize first that the form chosen for (2.1) does not satisfy the assumption (1.2) used for instance in [39, 40] (see the proof of (2.3) in [39] to be convinced). Furthermore, as already mentioned in Section 1, we claim that (2.1) is the least cumbersome system containing all the difficulties inherent to the analysis of a strongly coupled cross-diffusion and that our methodology may be applied to more complex equations, provided that the assumptions are properly adapted.

Let us now introduce some elements for the functional setting used in the present paper. For the sake of brevity we shall write $H^1(\Omega) = W^{1,2}(\Omega)$ and

$$V = H_0^1(\Omega), \ V' = H^{-1}(\Omega), \ H = L^2(\Omega).$$

The embeddings $V \subset H = H' \subset V'$ are dense and compact. For any $T > 0$, let $W(0,T)$ denote the space

$$W(0,T) := \big\{\omega \in L^2(0,T;V), \ \partial_t\omega \in L^2(0,T;V')\big\}$$

endowed with the Hilbertian norm $\|\omega\|_{W(0,T)}^2 = \|\omega\|_{L^2(0,T;V)}^2 + \|\partial_t\omega\|_{L^2(0,T;V')}^2$. The following embeddings are continuous (see [34] prop. 2.1 and thm 3.1, chapter 1)

$$W(0,T) \subset \mathscr{C}([0,T];[V,V']_{\frac{1}{2}}) = \mathscr{C}([0,T];H)$$

while the embedding

$$W(0,T) \subset L^2(0,T;H) \tag{2.4}$$

is compact thanks to the classical Aubin-Lions' compactness result (see *e.g.* [46]). The first auxiliary result used in the sequel, by F. Mignot (see [21]), is the following.

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $\limsup_{|\lambda|\to+\infty} |f(\lambda)/\lambda| < +\infty$. Let $\omega \in L^2(0,T;H)$ be such that $\partial_t\omega \in L^2(0,T;V')$ and $f(\omega) \in L^2(0,T;V)$. Then*

$$\langle \partial_t\omega, f(\omega)\rangle_{V',V} = \frac{d}{dt}\int_\Omega \Big(\int_0^{\omega(\cdot,y)} f(r)\,dr\Big)\,dy \quad \text{in } \mathscr{D}'(0,T).$$

The second auxiliary lemma is the basis of our proof for the uniqueness result for cross-diffusive problems. It is a parabolic extension of the Meyers regularity theorem [36]. The aim is to obtain a precise estimate of a solution of a parabolic system in $X_p = L^p(0,T;W_0^{1,p}(\Omega))$, $p \geq 2$, endowed with the norm

$$\left(\int_0^T ||v(t)||^p_{W_0^{1,p}(\Omega)} dt\right)^{1/p} := ||\nabla v||_{L^p(\Omega_T)^N}.$$

The proof may be recovered from the arguments in Chapter 2, Theorem 2.2 and Chapter 1, Section 4 of [6]. It is detailed in Appendix. The space $Y_p = L^p(0,T;W^{-1,p}(\Omega))$ is endowed with the norm $||f||_{Y_p} = \inf_{\mathrm{div}_x g = f} ||g||_{(L^p(\Omega_T))^N}$. Given $F \in Y_p$, there is a unique solution $u \in X_p$ of the following initial boundary value problem

$$\partial_t u - \Delta u = F \text{ in } \Omega_T, \quad u = 0 \text{ on } (0,T)\times\Gamma, \quad u(0,x) = 0 \text{ in } \Omega.$$

We set $\Lambda^{-1} = \partial_t - \Delta$, so that $u = \Lambda(F)$. Let g be defined by

$$g(p) := ||\Lambda||_{\mathscr{L}(Y_p;X_p)}.$$

It is well-known that $g(2) = 1$. Now, let $A \in (L^\infty(\Omega))^{N\times N}$ be such that there exists $\alpha > 0$ satisfying

$$\sum_{i,j=1}^N A_{i,j}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \text{ for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^N.$$

We set $\beta := \max_{1\leq i,j\leq n} ||A_{i,j}||_{L^\infty(\Omega)}$ and $\mathscr{A}u = -\sum_{i,j=1}^N \partial_{x_i}\big(A_{i,j}\partial_{x_j}u\big)$. We state the following Lemma (*cf* [6] and Appendix).

Lemma 2. *Let $f \in L^2(0,T;V')$, $u^0 \in H$ and $u \in L^2(0,T;V)$ be the solution of*

$$\partial_t u + \mathscr{A}u = f \text{ in } \Omega_T, \quad u(0) = u^0 \text{ in } \Omega. \tag{2.5}$$

There exists $r > 2$, depending on α, β and Ω, such that if $u^0 \in W_0^{1,r}(\Omega)$ and $f \in Y_r$, then $u \in X_r$. Furthermore, the following estimate holds true

$$||u||_{X_r} \leq C(\alpha,\beta,r)(||f||_{Y_r} + \beta T^{1/r}||u^0||_{W_0^{1,r}(\Omega)}), \tag{2.6}$$

where the constant $C(\alpha,\beta,r) > 0$ depends on Ω, α, β and r (but not on T) as follows:

$$C(\alpha,\beta,r) \leq \frac{g(r)}{(1-k(r))\,(\beta+c)}, \quad k(r) = g(r)(1-\mu+\nu) \tag{2.7}$$

where $\mu = (\alpha+c)/(\beta+c)$, $\nu = (\beta^2+c^2)^{1/2}/(\beta+c)$ and c is any real number such that $c > (\beta^2 - \alpha^2)/2\alpha$. If, moreover, A is symmetric, the estimate (2.7) *holds true with $\mu = \alpha/\beta$ and $\nu = c = 0$.*

Remark 1. *According to* (2.7)*), the value of r depends on the characteristics (α,β) of the elliptic operator $\mathscr{A}$, roughly on the ellipticity rate α/β. Actually the real number r may be chosen in the range*

$$2 < r \leq \sup\{r_0 \in \mathbb{R}; k(r_0) < 1\}. \tag{2.8}$$

Then, the smaller $(1-\mu+\nu)$, the larger r. But, up to our knowledge, there is no estimate of $g(r)$, $r \geq 2$ except when r = 2, which makes the inequality (2.8) *difficult to exploit. In the elliptical case, we can however mention the optimal integrability exponent of the gradient field $r_{opt} = 2Z/(Z-1)$, with $Z = \sqrt{\beta/\alpha}$ when $\mathscr{A}$ is symmetric (cf the work of Astala, Leonetti and Nesi* [4], [31], [38]*). The critical exponent r_{opt} was already highlighted by a counterexample given by Meyers in his seminal paper* [36].

For comparison purposes, we give another regularity result. It is based on an additional hypothesis for the coefficients in A which are assumed to be measurable in time and to have a small BMO (*bounded mean oscillation*, see [24]) semi-norm in space. It is an interesting setting since discontinuous coefficients remain allowed. For any $R > 0$, define the quantity $A_R^\#$ by

$$A_R^\# = \sup_{(t,x)\in\Omega_T} \sup_{\eta<R} \mathrm{osc}_x\big(A,(t,t+\eta^2)\times B_\eta(x)\big),$$

$$\mathrm{osc}_x\big(A,(t,t+\eta^2)\times B_\eta(x)\big) = \eta^{-2}|B_\eta(x)|^{-2}\int_t^{t+\eta^2}\int_{y,z\in B_\eta(x)}|A(s,y)-A(s,z)|dydzds$$

where $B_\eta(x)$ is the ball of radius η centered in x. The following result may be obtained from Theorem 2.2 by Dong and Kim in [19] and using local maps for recovering Ω.

Lemma 3. *Let $u^0 \in W_0^{1,r}(\Omega)$, $f \in L^r(0,T;W^{-1,r}(\Omega))$ and $u \in L^2(0,T;V)$ be the solution of* (2.5)*. Assume further that the coefficients of A satisfy a small BMO property, namely: there exists some $\bar{A}^\# = \bar{A}^\#(\alpha,\beta,r) > 0$ such that if $A_{R_0}^\# \le \bar{A}^\#$ for some $R_0 \in (0,1]$, then $u \in L^r(0,T;W_0^{1,r}(\Omega))$ and the estimate* (2.6) *holds true, but with a constant in the form $C_{BMO}(\alpha,\beta,r,T)$.*

Unlike Lemma 2, the quality of the results obtained by Lemma 3 does not depend on any smallness assumption on the coefficients of A. Note however that the constant $C_{BMO}(\alpha,\beta,r,T)$ depends on T. This constant may be estimated using especially the computations in [29][1]. More precisely, a careful reading shows that it actually depends exponentially on T and that it increases with β/α. It should also be noted that Meyers' counter-example referred to in Note 2 is constructed with an operator whose coefficients do not test the small BMO hypothesis.

Remark 2. *All the results of the present paper are derived for systems completed by Dirichlet type boundary conditions. Lemma 2 is actually proven in the appendix using the arguments in* [6]*, thus the Dirichlet boundary conditions for ensuring that the norm $\|\cdot\|_{W_0^{1,p}} = \|\nabla(\cdot)\|_{(L^p)^N}$ is actually equivalent to the norm in $W^{1,p}$. The extensions of our results to Dirichlet boundary conditions holding solely on a non negligible part of the boundary, $\Gamma_1 \subset \Gamma$, $|\Gamma_1| \neq 0$, is thus straightforward.*

Notice however that all the results may also be extended to settings with Neumann boundary conditions. Indeed, using the smoothness of Γ and local maps, we can construct (by reflexion) an extension outside $\bar{\Omega}$ of the solution $u \in L^2(0,T;H^1(\Omega))$ of

$$\partial_t u + \mathscr{A} u = f \text{ in } \Omega_T, \quad u(0) = u^0 \text{ in } \Omega, \quad \partial_A u = 0 \text{ in } (0,T)\times\Gamma,$$

and recover locally, especially in Ω, the results of Lemma 2. The extension of Lemma 3 to mixed boundary conditions is contained in [19]*. Yet, if, for instance, the boundary condition on Γ is of Neumann's type, some estimates in the remaining of the paper require the use of the Gronwall lemma. Without further assumptions on the source terms and boundary conditions, we would thus be limited to local in time results.*

3. MAIN RESULTS: GLOBAL IN TIME EXISTENCE, UNIQUENESS, MAXIMUM PRINCIPLE

We aim at giving an existence result of physically admissible weak solutions for the cross-diffusive model (2.1) completed by initial and boundary conditions. The mathematical analysis thus should provide at least three kinds of result. The first point is of course to state an existence result of weak solution.

[1]Nevertheless looking for an anlytical expression of this constant seems unreachable.

Next an uniqueness result should comfort us regarding its physical meaning. The third type of result, induced by another physical concern, is the admissible range of values for the solution (bear in mind the examples of chemical models or the ones modeling populations dynamics). The mathematical version of this later point is the statement of a realistic maximum principle.

The present paper is organized by the latter three questionings.

3.1. **Existence result of a weak solution.** The obvious difficulty for the mathematical analysis of cross-diffusion systems is the coupling of the equations by the gradient of their solution. In the particular case of systems in the form (2.1) considered here, an additional and substantial difficulty is due to the fact that the cross-diffusive terms are nonlinear. Since we expect to finally exhibit bounded solutions (see the last section on the maximum principle below), we first introduce an artificial bound on a part of these nonlinear terms. More precisely, for $\ell > 0$, we set

$$T_\ell(u) = \max\big\{0, \min\{u, \ell\}\big\}.$$

We then consider the following problem: for $i = 1, \dots, m$,

$$\partial_t u_i - \nabla \cdot \big(\delta_i \nabla u_i + T_\ell(u_i) \sum_{j=1}^{m} K_{i,j} \nabla u_j\big) = Q_i(u) \quad \text{in } \Omega_T, \tag{3.1}$$

$$u_i = u_{i,D} \text{ in } (0,T) \times \Gamma, \quad u_i(0,x) = u_i^0(x) \text{ in } \Omega. \tag{3.2}$$

The initial and boundary conditions are supposed to satisfy the compatibility conditions

$$u_i^0(x) = u_{i,D}(0,x), \quad x \in \Gamma, \quad 1 \le i \le m, \tag{3.3}$$

when the traces $u^0_{i\,|\Gamma}$ and $u_{i,D}(0,.)$ are meaningful. We assume that there exists a lifting of each boundary function $u_{i,D}$, still denoted the same for convenience, belonging to the space $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$. Due to the smoothness of Γ, such a result is ensured if $u_{i,D} \in L^2(0,T;H^{1/2}(\Gamma)) \cap H^1(0,T;H^{-1/2}(\Gamma)))$ (see [33]). The initial data u_i^0 are assumed to be in H, the source terms $Q_i(v)$ to be in $L^2(\Omega_T)$ for any $v \in (W(0,T))^m$, $1 \le i \le m$. For the sake of simplicity, we set $m = 2$.

The following existence result holds true.

Theorem 1. *Assume that the tensor K satisfies:*

$$\frac{(K^+_{1,2})^2}{K^-_{1,1}} < \frac{4\delta_2}{\ell}, \ \frac{(K^+_{2,1})^2}{K^-_{2,2}} < \frac{4\delta_1}{\ell}. \tag{3.4}$$

Assume that $Q_i \in L^2(0,T;(H^1(\Omega)'))$. Then for any $T > 0$, the problem (3.1)–(3.2) *admits a weak solution $(u_i)_{i=1,2} \in (W(0,T))^2$. Furthermore, if almost everywhere in Ω_T, $0 \le u_i^0$, $0 \le u_{i,D}$ and $Q_i(v) \ge 0$ if $v_i \le 0$, the following relation holds true*

$$0 \le u_i(t,x) \quad \textit{for a.e. } x \in \Omega, \textit{ for all } t \in (0,T),\ i = 1,2.$$

Remark 3. *Notice that, thanks to the nonnegativeness result proved for u_i, $i = 1,2$, Theorem 1 actually gives an existence result for* (3.1)–(3.2) *where T_ℓ only truncates the large values of u_i, that is $T_\ell(u) = u$ for $u \le \ell$ and $T_\ell(u) = \ell$ for $u \ge \ell$.*

Remark 4. *The assumption* (3.4) *may be viewed as a limitation of the ratio between $K_{i,i}$ and $K_{i,j}$, $j \neq i$, $i = 1,2$, which strongly depends on ℓ. In the last section, we will prove that*

$$0 \leq u_i(t,x) \leq \ell \quad \text{for a.e. } x \in \Omega, \text{ for all } t \in (0,T)$$

if $0 \leq u_i^0, u_{i,D} \leq \ell$ a.e. in Ω_T, by introducing "sufficiently pumping" source terms to enforce the boundedness of the solution. Such a maximum principle gives a physical meaning to the parameter ℓ. (Since the function T_l can be removed from (3.1)*, we notice also that Theorem 1 gives the existence of a weak solution to the original problem* (2.1)*.) Otherwise, this parameter could appear artificial and, because we study a global in time solution, it should be expected to tend to ∞, leading to a meaningless assumption* (3.4)*, unless there are no cross-diffusive terms. Another interpretation of* (3.4) *when no maximum principle can be proved for u_i, $i = 1$ or 2, is that we recover with Theorem 1 a local in time existence result for the original problem* (2.1).

Remark 5. *The interested reader will check that the proof of Theorem 1 developed below may be easily adapted for extending the existence result to cross-diffusive systems in the form*

$$\partial_t u_i - \nabla \cdot \left(\delta_i \nabla u_i + \sum_{j=1}^{m} T_{ij}(u_1,u_2) K_{i,j} \nabla u_j\right) = Q_i \quad \text{in } \Omega_T,$$

$$u_i = u_{i,D} \text{ in } (0,T) \times \Gamma, \quad u_i(0,x) = u_i^0(x) \text{ in } \Omega,$$

where $T_{ij}(u)$ are continuous and bounded functions, provided that slight modifications (depending on the T_{ij}'s) are made on Assumption (4.24).

3.2. **Uniqueness result.** As already mentioned in the introduction, proving a uniqueness result for a cross-diffusive system is always a tricky problem. Here we choose to found our results on an additional regularity result. Let us emphasize once again that it does not consist in reducing the analysis to the framework of smooth or mild solutions. We rather prove a Meyer's type property allowing to upgrade the regularity of any solution of the cross-diffusive problem from $L^2(H^1)$ to $L^r(W^{1,r})$, for some $r > 2$. We expose this result in the following proposition, which we believe to be of self-interest.

First we introduce some notations for turning back to the setting of Lemma 2. In the particular case of System (3.1), the tensor $\mathscr{A}$ appearing in Lemma 2 reads $\mathscr{A}_i = (\delta_i + K_{i,i} T_l(\bar{u}_i))\mathrm{Id}$, for $i = 1$ or $i = 2$. With the notations of Lemma 2, $\mathscr{A}$ is characterized by the quantities $\alpha_i = \delta_i$ and $\beta_i = \delta_i + \ell K_{i,i}^+$ for $i = 1,2$. Let $c_i = 0$ if $K_{i,i}$ is symmetric and $c_i > (\beta_i^2 - \alpha_i^2)/2\alpha_i$ if not. Set

$$\mu_i = \frac{\alpha_i + c_i}{\beta_i + c_i} = \frac{\delta_i + c_i}{\delta_i + \ell K_{i,i}^+ + c_i}, \quad {\nu_i}^2 = \frac{\beta_i^2 + c_i^2}{(\beta_i + c_i)^2} = \frac{(\delta_i + \ell K_{i,i}^+)^2 + c_i^2}{(\delta_i + \ell K_{i,i}^+ + c_i)^2}, \tag{3.5}$$

with $\nu_i \geq 0$. In accordance with (2.8), we are going to deal with the greatest real number r such that

$$k_i(r) = g(r)(1 - \mu_i + \nu_i) < 1 \text{ for } i = 1,2. \tag{3.6}$$

One easily checks that $c_i \mapsto 1 - \mu_i + \nu_i$ is a decreasing function. It is thus sufficient to define $r = r(\ell, \delta_1, \delta_2, K_{1,1}^+, K_{2,2}^+)$ by

$$2 < r < \sup\{r_0 > 2;\, k_i^*(r_0) = g(r_0)(1 - \mu_i^* + \nu_i^*) < 1 \text{ for } i = 1,2\}, \tag{3.7}$$

where

$$\mu_i^* = \begin{cases} \dfrac{\alpha_i}{\beta_i} & \text{if } K_{i,i} \text{ is a symmetric tensor,} \\ \dfrac{\alpha_i + (\beta_i^2 - \alpha_i^2)/2\alpha_i}{\beta_i + (\beta_i^2 - \alpha_i^2)/2\alpha_i} & \text{if not;} \end{cases} \tag{3.8}$$

$$\nu_i^* = \begin{cases} 0 & \text{if } K_{i,i} \text{ is a symmetric tensor,} \\ \nu_i^* \geq 0,\ \nu_i^{*2} = \dfrac{\beta_i^2 + (\beta_i^2 - \alpha_i^2)^2/4\alpha_i^2}{(\beta_i + (\beta_i^2 - \alpha_i^2)/2\alpha_i)^2} & \text{if not.} \end{cases} \tag{3.9}$$

Then the following result holds.

Proposition 1. *Let (u_1, u_2) be a solution of Problem* (3.1)–(3.2) *and let r be a real number satisfying* (3.7)*. Assume that $Q_i \in L^r(0,T;W^{-1,r}(\Omega))$, $i = 1,2$, and that $(\ell, \delta_1, \delta_2)$ and the tensor K satisfy*

$$K_{i,j}^+ < \frac{1}{g(r)\ell}\big(1 - g(r)(1 - \mu_i^* + \nu_i^*)\big)(\beta_i^* + c_i^*), \quad i = 1,2, i \neq j, \tag{3.10}$$

and that $(u_1^0, u_2^0) \in (W^{1,r}(\Omega))^2$. Then ∇u_1 and ∇u_2 belong to $(L^r(\Omega_T))^N$ and are bounded as follows:

$$||\nabla u_i||_{(L^r(\Omega_T))^N} \leq C_{r,i}(\ell, \delta_1, \delta_2, K, \|Q_i\|_{L^r(W^{-1,r})}, T^{1/r}\|(u_1^0, u_2^0)\|_{W^{1,r}(\Omega)}). \tag{3.11}$$

Remark 6. *The characterization* (3.7) *of r and the assumption* (3.10) *both depend on the function g which is the norm of the inverse of the Heat operator, $g(p) = ||\Lambda||_{\mathscr{L}(L^p(W^{-1,p});L^p(W_0^{1,p}))}$ and which could appear hard to compute explicitely. We actually have $g(r) \geq 1$ (see the end of the Appendix). Thus* (3.7) *and* (3.10) *require in particular*

$$1 - \mu_i^* + \nu_i^* < 1 \text{ and } K_{i,j}^+ < \frac{1}{\ell}(\mu_i^* - \nu_i^*))(\beta_i^* + c_i^*).$$

One checks with some computations that the first condition is always satisfied. The second one illustrates how restrictive are (3.7) *and* (3.10)*. The explicit form of the bound* (3.11) *is given in* (5.2)*.*

For comparison, we claim and prove the following.

Proposition 2. *Let (u_1, u_2) be a solution of Problem* (3.1)–(3.2)*. Assume $N = 2$, $(u_1^0, u_2^0) \in (W^{1,r}(\Omega))^2$ and that K satisfies the small BMO hypothesis described in Lemma 3. Assume further that K is such that*

$$C_{BMO}(\alpha_i, \beta_i, r, T)\ell K_{i,-i}^+ < 1, \tag{3.12}$$

where we use the notation $(i,-i) = (i,j)$ with $j \neq i$. Then the conclusion of Proposition 1 still hods true, but with $C_{r,i}$ in (3.11) *depending moreover on $C_{BMO}(\alpha_i, \beta_i, r, T)$.*

Note that even if Lemma 3 does not require a smallness assumption for ensuring the invertibility of the linear parabolic operator, handling with the cross-diffusive system reintroduce such a condition, namely (3.12), which is very restrictive for large times due to the exponential dependence of C_{BMO} on T.

The important point is that a precise characterization of the regularity parameter r with regard to the data of the problem is given in (3.7). More precisely, it only depends on the coefficients of the operators and on their L^∞ norms. Conversely, one may attempt to sufficiently restrict the range of these data for reaching a given regularity parameter r. The smaller $1 - \mu_i^* + \nu_i^*$, $i = 1,2$, the bigger r. We use this process in the two-dimensional case for proving that the cross-diffusive problem (3.1)-(3.2) is well-posed. Indeed, if $N = 2$, there exists a version of the Gagliardo–Nirenberg inequality allowing the control of the L^4 norm by the L^2 and H_0^1 norms. It appears that this inequality is sufficient for proving the uniqueness result, provided we reach the value $r = 4$. We thus assume that

$$g(4)(1-\mu_i^* + \nu_i^*) < 1, \; i = 1,2. \tag{3.13}$$

To set the ideas on a simple example, notice that if the tensors $K_{i,i}$, $i = 1,2$, are symmetric, then $1-\mu_i^* + \nu_i^* = \ell K_{i,i}^+/(\delta_i + \ell K_{i,i}^+)$ and the latter assumption reduces to $\ell(g(4)-1)K_{i,i}^+ < \delta_i$, $i = 1,2$. We are now in a position to establish the following uniqueness result, which asserts that our problem is well-posed in the space $W(0,T)$.

Theorem 2. *Set $N = 2$. Assume that the tensor K satisfy* (3.10) *for $r = 4$, $(u_1^0, u_2^0) \in (W^{1,4}(\Omega))^2$ and* (3.13)*. Assume*[2] *that the source terms Q_i in* (3.1) *belong to $L^\infty((0,T)\times\Omega)$ and set $q_{+,i} = \|Q_i\|_\infty/(\delta_i + \ell K_{i,i}^+)$, $i = 1,2$. Assume*

$$\frac{(K_{1,2}^+)^2}{K_{1,1}^-} < \frac{3\delta_2}{\ell} \quad \text{and} \quad \frac{(K_{2,1}^+)^2}{K_{2,2}^-} < \frac{3\delta_1}{\ell}, \tag{3.14}$$

$$c_{4,i}^{1/4} = \frac{g(4)(\delta_i + \ell K_{i,i}^+)}{((1-g(4)(1-\mu_i^*+\nu_i^*))(\beta_i + c_i^*) - g(4)\ell K_{i,-i}^+)} < \frac{1}{2}, \quad i = 1,2. \tag{3.15}$$

Then the solution (u_1, u_2) of Problem (3.1)-(3.2) *is unique in the space $\big(W(0,T)\cap L^4(0,T,W^{1,4}(\Omega))\big)^2$ for any $T > 0$.*

Remark 7. *Notice also that choosing $r > N+2$ in Proposition 1 ensures that (u_1, u_2) is Hölder continuous. Here, with $N = 2$ and $r = 4$, we choose not to reach this regularity.*

Remark 8. *The result based on a small BMO assumption (see Proposition 2) reads as follows: still assuming $N = 2$ and that the tensor K satisfies* (3.14)*, assuming further that K satisfies the assumptions of Proposition 2 for $r = 4$, there exists a real number $C_{bmo,i}$ depending on α_i, β_i, $K_{i,i}^+$, u_i^0, Q_i, $i = 1,2$, ℓ and Ω, such that if*

$$\frac{C_{bmo,i}}{1 - C_{BMO}(\alpha_i, \beta_i, 4, T)\ell K_{i,-i}^+} < 1, \quad i = 1,2,$$

then the solution (u_1, u_2) of Problem (3.1)-(3.2) *is unique in $(W(0,T))^2$. The quality of this result is comparable to that of the theorem but it should be noted that the criterion of smallness is much less explicit.*

3.3. **Maximum principle.** The obvious difference between our original system (2.1) and the one considered in Theorems 1 and 2 is the truncation defined by the function T_ℓ. The uniqueness result in Theorem 2 gives for instance sense to numerical investigations but it remains frustrating. For turning back to the original problem, the first tentation is of course to let $\ell \to \infty$. Unfortunately, such an attempt is useless due the assumption (3.4). This point is not surprising if we carefully look at the structure of (1.1): if the coefficient u_j of the cross-diffusion term is unbounded, there is no other term in (1.1) for controlling its explosion in order to state uniform estimates with regard to ℓ. The setting of System (1.1) is thus very different from the one of the SKT system (notice that the terms $u_j \nabla u_i \cdot \nabla u_i$ and $u_j \nabla u_j \cdot \nabla u_j$

[2]This hypothesis may be alleviated by simply assuming that $Q_i \in L^4(0,T;W^{-1,4}(\Omega))$. We have made this choice to make the condition (3.15) simpler to write.

may be used with the Cauchy-Schwarz and Young inequalities for containing the term $u_j\nabla u_j\cdot\nabla u_i$ when deriving *a priori* estimates for the SKT system) and there is no hope to get similar results as the ones derived for instance in [39].

Here, for stating a complete maximum principle for (1.1), we have to make additional assumptions.

A classical approach consists, once again, in making restrictions on the algebraic structure of the equation. These models are often called volume-filling models because of their physical derivation (see [25]). For instance, in the context of population dynamics, volume limitations lead to a limitation of the population densities. Another classical example appears in chemistry modeling since the sum of the concentrations shall not go above 1. With our notations, it means that there exists $\ell>0$ such that $u_1+u_2\le\ell$. Together with the non-negativity of the solutions, this result induces the boundedness of u_1 and u_2. Subsection 6.1 is further devoted to the maximum principle in volume-filling systems: System (2.1) with $K_{i,j}=K$ for all i,j appears actually to behave like a volume-filling system.

Nevertheless, we prefer avoiding such structural assumption. The common sense then suggests using sufficiently large "pumping" source terms in order to control the upper bound of the solution. We question this idea in Proposition 3 below.

Proposition 3. *Assume the assumptions in Theorem 1 fulfilled. Assume $0\le u_i^0\le\ell$ a.e. in Ω and $0\le u_{i,D}\le\ell$ a.e. in $(0,T)\times\Gamma$. There exists source terms $Q_i\in L^2(0,T;(H^1(\Omega)'))$, $i=1,2$, such that the system* (2.1) *completed by the initial and boundary conditions* (3.2) *admits a weak global solution such that, for any $T>0$, $(u_i-u_{i,D})_{i=1,2}\in W(0,T)^2$ and the following maximum principle holds true:*

$$0\le u_i(t,x)\le\ell\quad \textit{for a.e. } x\in\Omega, \textit{ for all } t\in(0,T) \textit{ and for all } i=1,2.$$

We straightforward infer from the latter proposition the following result which sums up all the results of the paper.

Theorem 3. *Assume the assumptions in Theorem 2 to be fulfilled. Assume $0\le u_i^0\le\ell$ a.e. in Ω and $0\le u_{i,D}\le\ell$ a.e. in $(0,T)\times\Gamma$. There exist source terms $Q_i\in L^2(0,T;(H^1(\Omega)'))$, $i=1,2$, such that the system* (2.1) *completed by the initial and boundary conditions* (3.2) *admits a unique bounded weak global solution.*

Remark 9. *The source term exhibited in the proof of Proposition 3 is more precisely in the form $Q_i=\chi_{\{u_i\ge\ell\}}\nabla\cdot\mathscr{Q}_i+q_i(u)$ with $\mathscr{Q}_i\in(L^2(\Omega_T))^N$ and $q_i(v)$ in $L^2(\Omega_T)$ for any $v\in(W(0,T))^2$, $q_i(v)\ge0$ if $v_i\le0$, $q_i(v)\le0$ if $v_i\ge\ell$.*

Remark 10. *Some elements for the physical interpretation of the penalization process used in the proof of Proposition 2 are provided in Subsection 6.4.*

Remark 11. *The analogous of Theorem 3 may be proven under the assumptions leading to the enhanced regularity of Proposition 1. In this case, we claim the existence of a source term $Q_i\in L^r(0,T;W^{-1,r}(\Omega))$ ensuring the maximum principle. Hence, for short, we have proved that, if the diffusive operator in* (2.1) *is close enough to the Laplacian operator (see the Meyer's type criterion), if the cross-diffusive operator in* (2.1) *is sufficiently small with regard to the diffusive one (see the existence result), then, for well-prepared data (see the regularity of the initial data and the choice of the source term ensuring that the solution is bounded by a real number satisfying* (3.15)*), the problem* (2.1), (3.2) *is well-posed.*

The following sections are devoted to the proof of the three main results in this paper (namely Theorems 1, 2, and 3), regarding respectively the global existence in time, the uniqueness and the maximum's principle.

4. PROOF OF THEOREM 1

For the sake of simplicity, we assume that $Q_i = 0$, $u_{i,D} = 0$, $i = 1,2$. Nevertheless, we emphasize that Theorem 1 is both valid for non homogeneous Dirichlet boundary conditions and for non null source terms. We refer to [17] for more details about. The proof is divided in two steps: proving the existence of a weak solution of (3.1)–(3.2); proving the non-negativity of any weak solution of (3.1)–(3.2).

STEP 1. EXISTENCE OF A WEAK SOLUTION

We aim at finding a weak solution $(u_1,u_2) \in (W(0,T))^2$ of (3.1)–(3.2) in the following sense: for any $w \in L^2(0,T;V)$,

$$\int_0^T \langle \partial_t u_1, w\rangle_{V',V} dt + \int_{\Omega_T} (\delta_1 + K_{1,1}T_\ell(u_1))\nabla u_1 \cdot \nabla w\,dxdt + \int_{\Omega_T} K_{1,2}T_\ell(u_1)\nabla u_2 \cdot \nabla w\,dxdt = 0, \quad (4.1)$$

$$\int_0^T \langle \partial_t u_2, w\rangle_{V',V} dt + \int_{\Omega_T} (\delta_2 + K_{2,2}T_\ell(u_2))\nabla u_2 \cdot \nabla w\,dxdt + \int_{\Omega_T} K_{2,1}T_\ell(u_2)\nabla u_1 \cdot \nabla w\,dxdt = 0. \quad (4.2)$$

In view of the nonlinearity of the problem, we adopt a fixed point strategy.

Definition of the map $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2)$

We define an application $\mathscr{F} : (L^2(0,T;H^1(\Omega)))^2 \to (L^2(0,T;V))^2$ by

$$\mathscr{F}(\bar{u}_1,\bar{u}_2) = \big(\mathscr{F}_1(\bar{u}_1,\bar{u}_2), \mathscr{F}_2(\bar{u}_1,\bar{u}_2)\big) = (u_1,u_2), \quad (4.3)$$

where (u_1,u_2) is the unique solution of the following initial boundary value problem

$$\partial_t u_1 - \nabla\cdot\big((\delta_1 + T_\ell(\bar{u}_1)K_{1,1})\nabla u_1 + T_\ell(\bar{u}_1)K_{1,2}\nabla \bar{u}_2\big) = 0 \text{ in } \Omega_T, \quad (4.4)$$

$$\partial_t u_2 - \nabla\cdot\big((\delta_2 + T_\ell(\bar{u}_2)K_{2,2})\nabla u_2 + T_\ell(\bar{u}_2)K_{2,1}\nabla \bar{u}_1\big) = 0 \text{ in } \Omega_T, \quad (4.5)$$

$$(u_1,u_2) = (0,0) \text{ in } (0,T)\times\Gamma, \quad (4.6)$$

$$(u_1(0,x), u_2(0,x)) = (u_1^0(x), u_2^0(x))\ x \in \Omega. \quad (4.7)$$

Notice that (4.4)-(4.7) can be solved by considering first (4.4) and (4.6)-(4.7) for $i = 1$ (thus a linear system in u_1) and next (4.5) and (4.6)-(4.7) for $i = 2$ (a linear system in u_2). The existence of a unique weak solution $(u_1,u_2) \in (L^2(0,T;V))^2$ for the parabolic problem with bounded coefficients (4.4)-(4.7) is thus obvious. It satisfies, for all $w \in L^2(0,T;V)$,

$$\int_0^T \langle \partial_t u_1, w\rangle_{V',V}\, dt + \int_{\Omega_T} \big((\delta_1 + K_{1,1}T_\ell(\bar{u}_1))\nabla u_1 + K_{1,2}T_\ell(\bar{u}_1)\nabla \bar{u}_2\big)\cdot\nabla w\,dxdt = 0, \quad (4.8)$$

$$\int_0^T \langle \partial_t u_2, w\rangle_{V',V}\, dt + \int_{\Omega_T} \big((\delta_2 + K_{2,2}T_\ell(\bar{u}_2))\nabla u_2 + K_{2,1}T_\ell(\bar{u}_2)\nabla \bar{u}_1\big)\cdot\nabla w\,dxdt = 0. \quad (4.9)$$

We now collect the properties allowing the use of the Schauder's fixed point theorem for $\mathscr{F}$ in some appropriate subset of $(L^2(0,T;H))^2$.

Sequential continuity of $\mathscr{F}_1$ in $(L^2(0,T;H))^2$ when $\mathscr{F}$ is restricted to any bounded subset of $(L^2(0,T;H^1(\Omega)))^2$

Pick a real number $M > 0$, that we will precise later on, and assume that $\mathscr{F}$ is restricted to the set $\{\mathbf{u} \in (L^2(0,T;H^1(\Omega)))^2\,;\, \|\mathbf{u}\|_{(L^2(0,T;H^1(\Omega)))^2} \le M\}$. For proving the sequential continuity of $\mathscr{F}_1$, assume given a sequence $(\bar u_{1,n},\bar u_{2,n})$ in this set and $(\bar u_1,\bar u_2)\in(L^2(0,T;H))^2$ such that

$$(\bar u_{1,n},\bar u_{2,n})\to(\bar u_1,\bar u_2) \text{ in } (L^2(0,T;H))^2.$$

Since $(\bar u_{1,n},\bar u_{2,n})$ is uniformly bounded (by M) in $(L^2(0,T;H^1(\Omega)))^2$, there exists an increasing function φ in $\mathbb{N}$, and a subsequence $(\bar u_{1,\varphi(n)},\bar u_{2,\varphi(n)})$ weakly converging in $(L^2(0,T;H^1(\Omega)))^2$. Due to the uniqueness of the limit we thus have

$$(\bar u_{1,\varphi(n)},\bar u_{2,\varphi(n)})\rightharpoonup(\bar u_1,\bar u_2) \text{ weakly in } (L^2(0,T;H^1(\Omega)))^2,$$
$$(\bar u_{1,\varphi(n)},\bar u_{2,\varphi(n)})\to(\bar u_1,\bar u_2) \text{ in } (L^2(0,T;H))^2 \text{ and a.e. in } \Omega_T$$

and, since $\|\nabla\bar u_i\|_{L^2(0,T;H)}\le\liminf\|\nabla\bar u_{i,\varphi(n)}\|_{L^2(0,T;H)}$,

$$||\nabla\bar u_i||_{(L^2(0,T;H))^N}\le M,\ i=1,2, \tag{4.10}$$

Setting

$$u_{1,n}=\mathscr{F}_1(\bar u_{1,n},\bar u_{2,n}),\quad u_1=\mathscr{F}_1(\bar u_1,\bar u_2),$$

we now aim at showing that $u_{1,n}\rightharpoonup u_1$ weakly in $W(0,T)$ and thus strongly in $L^2(0,T;H)$ thanks to a classical result of Aubin.

We begin by some uniform estimates for proving that a subsequence of $u_{1,n}$ is actually converging. For any $n\in\mathbb{N}$, $u_{1,n}$ satisfies (4.8). Pick any $\tau\in[0,T]$ and choose $w=u_{1,n}\chi_{(0,\tau)}$ in (4.8), $\chi_{(0,\tau)}$ denoting the characteristic function of $(0,\tau)\subset(0,T)$. We obtain

$$\int_0^\tau\langle\partial_t u_{1,n},u_{1,n}\rangle_{V',V}\,dt+\int_{\Omega_\tau}(\delta_1+K_{1,1}T_\ell(\bar u_{1,n}))\nabla u_{1,n}\cdot\nabla u_{1,n}\,dxdt$$
$$+\int_{\Omega_\tau}K_{1,2}T_\ell(\bar u_{1,n})\nabla\bar u_{2,n}\cdot\nabla u_{1,n}\,dxdt=0. \tag{4.11}$$

Since $u_{1,n}$ belongs to $W(0,T)$, hence to $\mathscr{C}([0,T];L^2(\Omega))$, we use Lemma 1 and write

$$\int_0^\tau\langle\partial_t u_{1,n},u_{1,n}\rangle_{V',V}\,dt=\frac12||u_{1,n}(\cdot,\tau)||_H^2-\frac12||u_1^0||_H^2.$$

On the other hand, we have

$$\int_{\Omega_\tau}\big(\delta_1+K_{1,1}T_\ell(\bar u_{1,n})\big)\nabla u_{1,n}\cdot\nabla u_{1,n}\,dxdt\ge\delta_1||\nabla u_{1,n}||^2_{L^2(0,\tau;H)}$$

and, using the Cauchy-Schwarz and Young inequalities, for any $\eta_1>0$

$$\Big|\int_{\Omega_\tau}K_{1,2}T_\ell(\bar u_{1,n})\nabla\bar u_2^n\cdot\nabla u_{1,n}\,dxdt\Big|\le MK_{1,2}^+\ell||\nabla u_{1,n}||_{L^2(0,\tau;H)}\le\frac{{K_{1,2}^+}^2M^2}{4\eta_1}\ell^2+\eta_1||\nabla u_{1,n}||^2_{L^2(0,\tau;H)}.$$

Using the latter estimates in (4.11), we obtain, for all $\tau\in[0,T]$

$$\frac12||u_{1,n}(\cdot,\tau)||_H^2+(\delta_1-\eta_1)||\nabla u_{1,n}||^2_{L^2(0,\tau;H)}\le\frac{{K_{1,2}^+}^2M^2}{4\eta_1}\ell^2+\frac12||u_{1,0}||_H^2. \tag{4.12}$$

We choose η_1 such that $\delta_1-\eta_1\ge\eta_0$ for some $\eta_0>0$. We infer from (4.12) that the sequence $(u_{1,n})_n$ is uniformly bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$: there exist real numbers $A_M=A_M(\delta,K,u_{1,0},\ell,M)$ and $B_M=B_M(\delta,K,u_{1,0},\ell,M)$ such that

$$||u_{1,n}||_{L^\infty(0,T;H)}\le A_M,\qquad||u_{1,n}||_{L^2(0,T;V)}\le B_M. \tag{4.13}$$

We now prove that $(u_{1,n})_n$ is also bounded in $H^1(0,T;V')$. Using the operational norm for $L^2(0,T;V')$ viewed as the dual space of $L^2(0,T;V)$, we write

$$||\partial_t u_{1,n}||_{L^2(0,T;V')} = \sup_{||w||_{L^2(0,T;V)}\leq 1} \Big| \int_0^T \langle \partial_t u_{1,n}, w\rangle_{V',V}\, dt\Big|$$

$$= \sup_{||w||_{L^2(0,T;V)}\leq 1} \Big| -\int_{\Omega_T} \big(\delta_1 + K_{1,1}T_\ell(\bar{u}_{1,n})\big)\nabla u_{1,n}\cdot\nabla w\, dxdt - \int_{\Omega_T} K_{1,2}T_\ell(\bar{u}_{1,n})\nabla\bar{u}_{2,n}\cdot\nabla w\, dxdt\Big|.$$

Since

$$\Big|\int_{\Omega_T} \big(\delta_1 + K_{1,1}T_\ell(\bar{u}_{1,n})\big)\nabla u_{1,n}\cdot\nabla w\Big| \leq \big(\delta_1 + K_{1,1}^+\ell\big)||u_{1,n}||_{L^2(0,T;V)}||w||_{L^2(0,T;V)},$$

and since $u_{1,n}$ is uniformly bounded in $L^2(0,T;V)$, we have

$$\Big|\int_{\Omega_T} \big(\delta_1 + K_{1,1}T_\ell(\bar{u}_{1,n})\big)\nabla u_{1,n}\cdot\nabla w\, dxdt\Big| \leq \big(\delta_1 + K_{1,1}^+\ell\big)B_M||w||_{L^2(0,T;V)}. \tag{4.14}$$

Furthermore,

$$\Big|\int_{\Omega_T} K_{1,2}T_\ell(\bar{u}_{1,n})\nabla\bar{u}_{2,n}\cdot\nabla w\, dxdt\Big| \leq K_{1,2}^+M\ell||w||_{L^2(0,T;V)}. \tag{4.15}$$

Gathering together (4.14) and (4.15), we conclude that

$$||\partial_t u_{1,n}||_{L^2(0,T;V')} \leq C_M, \quad C_M := \delta_1 B_M + \ell(\max_{j=1,2} K_{1,j}^+)(B_M+M). \tag{4.16}$$

With (4.13) and (4.16), we have proved that the sequence $(u_{1,n})_n$ is uniformly bounded in the space $W(0,T)$. Using Aubin-Lions' lemma, we extract a subsequence $(u_{1,\psi(n)})_n$ from $(u_{1,\varphi(n)})_n$, converging strongly in $L^2(\Omega_T)$, almost everywhere in Ω_T and weakly in $W(0,T)$ to some limit denoted by v_1. From the convergence stated a.e. in Ω_T for $(\bar{u}_{1,\psi(n)})_n \subset (\bar{u}_{1,\varphi(n)})_n$, we see that for all $w \in L^2(0,T;H^1(\Omega))$, $T_\ell(\bar{u}_{1,\psi(n)})\nabla w \to T_\ell(\bar{u}_1)\nabla w$ strongly in $L^2(\Omega_T)$ by dominated convergence. We thus check that v_1 solves (4.4) and (4.6)-(4.7). Due to the uniqueness of the solution of this problem, we conclude first that $v = u_1$, next that the whole sequence $u_{1,\varphi(n)} \rightharpoonup u_1$ weakly in $W(0,T)$ and strongly in $L^2(0,T;H)$. Reiterating the process for any subsequence $(\bar{u}_{1,\varphi(n)}, \bar{u}_{2,\varphi(n)})_n$ extracted from $(\bar{u}_{1,n}, \bar{u}_{2,n})_n$ and using once again the uniqueness of the solution of (4.4) and (4.6)-(4.7), we conclude that the whole sequence $u_{1,n} = \mathscr{F}_1(\bar{u}_{1,n}, \bar{u}_{2,n})$ converges to $u_1 = \mathscr{F}_1(\bar{u}_1, \bar{u}_2)$ in $L^2(0,T;H)$. The sequential continuity of $\mathscr{F}_1$ in $(L^2(0,T;H))^2$ is established.

Sequential continuity of $\mathscr{F}_2$ in $(L^2(0,T;H))^2$ when $\mathscr{F}$ is restricted to any bounded subset of $(L^2(0,T;H^1(\Omega)))^2$

Likewise, we study the sequential continuity of $\mathscr{F}_2$ by setting $u_{2,n} = \mathscr{F}_2(\bar{u}_{1,n}, \bar{u}_{2,n})$, $u_2 = \mathscr{F}_2(\bar{u}_1, \bar{u}_2)$, and showing that $u_{2,n} \to u_2$ in $L^2(0,T;H)$. The key estimates

$$||u_{2,n}||_{L^\infty(0,T;H)} \leq D_M = D_M(\delta_2, K, u_{2,0}, \ell, M), \tag{4.17}$$

$$||u_{2,n}||_{L^2(0,T;V)} \leq E_M = E_M(\delta_2, K, u_{2,0}, \ell, M) \tag{4.18}$$

$$||\partial_t u_{2,n}||_{L^2(0,T;V')} \leq F_M, \quad F_M := \delta_2 E_M + \ell(\max_{j=1,2} K_{2,j}^+)(E_M+M) \tag{4.19}$$

are obtained using the same type of arguments than those in the proof of the sequential continuity of $\mathscr{F}_1$, and the details are thus omitted.

Existence of $\mathscr{C} \subset (L^2(0,T;H^1(\Omega))^2$ such that $\mathscr{F}(\mathscr{C}) \subset \mathscr{C}$

For using the Schauder's fixed point theorem in $(L^2(0,T;H))^2$, we have to look for a nonempty bounded closed convex set of $(L^2(0,T;H))^2$, denoted by $\mathscr{C}$, such that $\mathscr{F}(\mathscr{C}) \subset \mathscr{C}$. We actually are going to construct $\mathscr{C}$ as a bounded subset of $(L^2(0,T;H^1(\Omega))^2$ so that the result $\mathscr{F}(\mathscr{C}) \subset \mathscr{C}$ will imply in particular that there exists a real number $M > 0$, depending on initial data, such that any $(u_1,u_2) = \mathscr{F}(\bar{u}_1,\bar{u}_2)$ satisfies

$$||\nabla u_1||_{L^2(0,T;H)} \leq M \quad \text{and} \quad ||\nabla u_2||_{L^2(0,T;H)} \leq M. \tag{4.20}$$

Hence the former results will apply and $\mathscr{F}$ will be sequentially continuous in $\mathscr{C}$.

Taking $w = u_1 \in L^2(0,T;V)$ (resp. $w = u_2 \in L^2(0,T;V)$) in (4.4) (resp. (4.5)) leads to

$$\int_0^T \langle \partial_t u_1, u_1 \rangle_{V',V}\, dt + \int_{\Omega_T} (\delta_1 + K_{1,1} T_\ell(\bar{u}_1)) \nabla u_1 \cdot \nabla u_1 \, dxdt \\ + \int_{\Omega_T} K_{1,2} T_\ell(\bar{u}_1) \nabla \bar{u}_2 \cdot \nabla u_1 \, dxdt = 0, \tag{4.21}$$

$$\int_0^T \langle \partial_t u_2, u_2 \rangle_{V',V}\, dt + \int_{\Omega_T} (\delta_2 + K_{2,2} T_\ell(\bar{u}_2)) \nabla u_2 \cdot \nabla u_2 \, dxdt \\ + \int_{\Omega_T} K_{2,1} T_\ell(\bar{u}_2) \nabla \bar{u}_1 \cdot \nabla u_2 \, dxdt = 0. \tag{4.22}$$

Applying Lemma 1 to the function $f =$Id, summing up the equations (4.21) and (4.22) and using the elliptic properties of the tensor K, we obtain

$$\frac{1}{2}\int_\Omega u_1(T,x)^2\, dx + \frac{1}{2}\int_\Omega u_2(T,x)^2\, dx - \frac{1}{2}\int_\Omega u_1(0,x)^2\, dx - \frac{1}{2}\int_\Omega u_2(0,x)^2\, dx \\ + \int_{\Omega_T} (\delta_1 + K_{1,1}^- T_\ell(\bar{u}_1)) |\nabla u_1|^2 \, dxdt + \int_{\Omega_T} (\delta_2 + K_{2,2}^- T_\ell(\bar{u}_2)) |\nabla u_2|^2 \, dxdt \\ + \underbrace{\int_{\Omega_T} K_{1,2} T_\ell(\bar{u}_1) \nabla \bar{u}_2 \cdot \nabla u_1 \, dxdt}_{(1)} + \underbrace{\int_{\Omega_T} K_{2,1} T_\ell(\bar{u}_2) \nabla \bar{u}_1 \cdot \nabla u_2 \, dxdt}_{(2)} = 0 \tag{4.23}$$

where

$$\begin{aligned} |(1)| &\leq \int_{\Omega_T} K_{1,1}^- T_\ell(\bar{u}_1) |\nabla u_1|^2 \, dxdt + \frac{\ell K_{1,2}^{+\;2}}{4K_{1,1}^-} \int_{\Omega_T} |\nabla \bar{u}_2|^2 \, dxdt, \\ |(2)| &\leq \int_{\Omega_T} K_{2,2}^- T_\ell(\bar{u}_2) |\nabla u_2|^2 \, dxdt + \frac{\ell K_{2,1}^{+\;2}}{4K_{2,2}^-} \int_{\Omega_T} |\nabla \bar{u}_1|^2 \, dxdt. \end{aligned}$$

Assuming that (3.4) holds true, there exists $p \geq 2$ such that

$$\frac{(K_{1,2}^+)^2}{K_{1,1}^-} \leq \frac{p-1}{p} \times \frac{4\delta_2}{\ell}, \; \frac{(K_{2,1}^+)^2}{K_{2,2}^-} \leq \frac{p-1}{p} \times \frac{4\delta_1}{\ell}. \tag{4.24}$$

Denoting by C_0 the real number such that

$$C_0 = \frac{p}{2}\int_\Omega u_1^0(x)^2\, dx + \frac{p}{2}\int_\Omega u_2^0(x)^2\, dx, \tag{4.25}$$

we infer from (4.23) with (4.24) that

$$\delta_1\|\nabla u_1\|^2_{(L^2(\Omega_T))^N}+\delta_2\|\nabla u_2\|^2_{(L^2(\Omega_T))^N}\le C_0, \tag{4.26}$$

as soon as $\delta_1\|\nabla\bar{u}_1\|^2_{(L^2(\Omega_T))^N}+\delta_2\|\nabla\bar{u}_2\|^2_{(L^2(\Omega_T))^N}\le C_0$. Notice that (4.26) yields $\|\nabla u_i\|_{(L^2(\Omega_T))^N}\le\sqrt{C_0/\delta_i}$, $i=1,2$, and $\sum_i\|u_i\|^2_{L^\infty(L^2)}\le 2C_0$. We thus define the subset $\mathscr{C}$ of $(L^2(0,T;H))^2$ by

$$\begin{aligned}\mathscr{C}:=\{&(u_1,u_2)\in(L^2(0,T;V))^2;\ (u_1(0,.),u_2(0,.))=(u_1^0(.),u_2^0(.)),\\ &\delta_1\|\nabla u_1\|^2_{(L^2(\Omega_T))^N}+\delta_2\|\nabla u_2\|^2_{(L^2(\Omega_T))^N}\le C_0,\\ &\|\partial_t u_1\|_{L^2(0,T,V')}\le C_M,\ \|\partial_t u_2\|_{L^2(0,T,V')}\le F_M\}\end{aligned}\tag{4.27}$$

where C_0 is defined by (4.24)-(4.25), $M=\max(\sqrt{C_0/\delta_1},\ \sqrt{C_0/\delta_2})$ and C_M and F_M are defined by (4.16) and (4.19).

Schauder's fixed point result

The set $\mathscr{C}$ is obviously a convex and bounded (thanks to Poincaré's inequality) subset of $(L^2(0,T;H))^2$. It has been constructed so that $\mathscr{F}(\mathscr{C})\subset\mathscr{C}$. Since $\mathscr{C}$ is also a bounded subset of $(L^2(0,T;H^1(\Omega))^2$, we proved that $\mathscr{F}$ restricted to $\mathscr{C}$ is sequentially continuous in $(L^2(0,T;H))^2$. The sequential compactness of $\mathscr{F}_i(\mathscr{C})$ in $L^2(0,T;H)$ $(i=1,2)$ is straightforward due to the Aubin-Lions' lemma. Since we work in metric spaces, the compactness of $\mathscr{F}(\mathscr{C})$ in $(L^2(0,T;H))^2$ follows.

For using the Schauder's fixed point theorem, it remains to show that the set $\mathscr{C}$ is strongly closed in $(L^2(0,T;H))^2$. Consider a sequence $(u_1^n,u_2^n)_n$ in $\mathscr{C}^2$ and a couple of functions $(\underline{u}_1,\underline{u}_2)\in(L^2(0,T;H))^2$ such that

$$(u_1^n,u_2^n)\to(\underline{u}_1,\underline{u}_2)\text{ in }(L^2(0,T;H))^2.$$

Let us check that $(\underline{u}_1,\underline{u}_2)\in\mathscr{C}^2$. Due to the definition of $\mathscr{C}$, the sequence $\left(u_1^n,u_2^n\right)_n$ is uniformly bounded in the space $(W(0,T))^2$. Thus, we assert that there exists $(u_1,u_2)\in(W(0,T))^2$ such that, up to a subsequence denoted by $(u_1^{n_k},u_2^{n_k})_k$, the following convergence holds true:

$$(u_1^{n_k},u_2^{n_k})\rightharpoonup(u_1,u_2)\text{ weakly in }(W(0,T))^2.$$

Because of the uniqueness of the limit in $(L^2(0,T;H))^2$, $(u_1,u_2)=(\underline{u}_1,\underline{u}_2)$ and, furthermore, we have $\|\nabla\underline{u}_i\|^2_{(L^2(\Omega_T))^N}\le\liminf_{k\to\infty}\|\nabla u_i^{n_k}\|^2_{(L^2(\Omega_T))^N}$ and $\|\partial_t\underline{u}_i\|_{L^2(0,T,V')}\le\liminf_{k\to\infty}\|\partial_t u_i^{n_k}\|_{L^2(0,T,V')}$, meaning $(\underline{u}_1,\underline{u}_2)\in\mathscr{C}^2$. The closeness of $\mathscr{C}$ is proved.

We now have the tools for using the Schauder's fixed point theorem [49, Corollary 9.7]. There exists $(u_1,u_2)\in\mathscr{C}^2$ such that $\mathscr{F}(u_1,u_2)=(u_1,u_2)$. Then (u_1,u_2) is a weak solution of problem (4.1)–(4.2).

STEP 2. NON NEGATIVITY OF THE SOLUTIONS.

Let us solely prove that $0\le u_1(t,x)$ for all $t\in(0,T)$ and for almost every $x\in\Omega$. Showing the non-negativity of u_2 follows the same lines. For the sake of completeness, we reuse the source term $Q_1(u)$ in (4.1). Let $u_m=\sup(0,-u_1)$. The function u_m belongs to $L^2(0,T;V)$, since $u_{1,D}$ is nonnegative, and is such that $\nabla u_m=-\chi_{\{u_1<0\}}\nabla u_1$ (see [5] Lemma 2.1; χ_A denotes the characteristic function of a set A).

We assume that the function Q_1 is such that $Q_1(u)u_m \geq 0$ for any $u = (u_1,u_2)$. Let $\tau \in (0,T)$. Setting $w(t,x) = -u_m(x,t)\chi_{(0,\tau)}(t)$ in (4.1) results in

$$\int_0^\tau \langle \partial_t u_1, -u_m\rangle_{V',V} + \int_0^\tau \int_\Omega \delta_1 \chi_{\{u_1<0\}}|\nabla u_1|^2$$
$$-\int_0^\tau \int_\Omega K_{1,1}T_\ell(u_1)\nabla u_1\cdot\nabla u_m - \int_0^\tau \int_\Omega K_{1,2}T_\ell(u_1)\nabla u_2\cdot\nabla u_m = -\int_0^\tau \int_\Omega Q_1(u)u_m. \quad (4.28)$$

In order to evaluate the first term in the left hand side of (4.28), we apply Lemma 1 with function f defined by $f(\lambda) = \max(0,-\lambda)$, $\lambda \in \mathbb{R}$. Of course $u_m(t,x) \neq 0$ iff $u_1(t,x) < 0$. We have

$$\int_0^\tau \langle \partial_t u_1, -u_m\rangle_{V',V} dt = \frac{1}{2}\int_\Omega \left(u_m^2(\tau,x) - u_m^2(0,x)\right)dx = \frac{1}{2}\int_\Omega u_m^2(\tau,x)\,dx.$$

Since $T_\ell(u_1)\chi_{\{u_1<0\}} = 0$ by definition of T_ℓ, the two last terms in the left hand side of (4.28) are null. Hence, with the assumption on Q_1, (4.28) gives $\int_\Omega u_m^2(\tau,x)\,dx \leq -2\int_0^\tau\int_\Omega \delta_1\chi_{\{u_1<0\}}|\nabla u_1|^2\,dxdt \leq 0$ and $u_m = 0$ a.e. in Ω_T.

This ends the proof of Theorem 1.

5. Additional regularity result and proof of Theorem 2

Theorem 2 is devoted to a uniqueness result for Problem (3.1)–(3.2). The first step towards its proof is an additional regularity result, in the spirit of the Meyer's theorem, proved for the cross-diffusive problem under consideration (in subsection 5.1 below). We aim at upgrading the regularity of the solution exhibited in Theorem 1 from $L^2(0,T;H^1(\Omega))$ to $L^r(0,T;W^{1,r}(\Omega))$ for some $r > 2$. This regularity will allow to handle the nonlinear terms in the system for proving the uniqueness of the solution.

5.1. **Proof of Proposition 1.** We adapt the proof of Theorem 1. We turn back to the construction of the intermediate solution which appears as the fixed point of an application in Step 1 of the proof of Theorem 1. We recall its outline. If $\mathscr{F}$ is the application defined in (4.3) and if $\mathscr{C}$ is the nonempty (strongly) closed convex bounded subset of the space $(L^2(0,T;H))^2$ defined in (4.27), we have shown that $\mathscr{F}(\mathscr{C}) \subset \mathscr{C}$ and that there exists $(u_1,u_2) \in \mathscr{C}$ such that $\mathscr{F}(u_1,u_2) = (u_1,u_2)$. This fixed point for $\mathscr{F}$ is a weak solution of problem (4.1)–(4.2) in $(L^2(0,T;H^1(\Omega))^2$. Now, we prove that, if the assumptions of Proposition 1 are fulfilled, this solution is actually in $L^r(\Omega_T)$, $r > 2$. To this aim, we modify the definition of the convex bounded subset $\mathscr{C}$ by including an estimate in the norm $L^r(0,T;W^{1,r}(\Omega))$ of its elements.

Let M' be a strictly positive real number that we will define later on. We set

$$\begin{aligned}\mathscr{D} := \{ \quad & (u_1,u_2) \in (L^r(0,T;W_0^{1,r}(\Omega)))^2,\, (u_1(0),u_2(0)) = (u_1^0,u_2^0),\\ & \|(u_1;u_2)\|_{(W(0,T))^2} \leq M,\ \|\nabla u_i\|_{(L^r(\Omega_T))^N} \leq M',\, i=1,2\}.\end{aligned} \quad (5.1)$$

Our aim is to check that $\mathscr{F}(\mathscr{D}) \subset \mathscr{D}$ for some appropriate choice of M'. Let $(\bar{u}_1,\bar{u}_2) \in \mathscr{D}$ and $(u_1,u_2) = \mathscr{F}(\bar{u}_1,\bar{u}_2)$. Applying Lemma 2 to (4.8) and (4.9), we deduce that, with the notations of (3.5) and (3.6),

$$\begin{aligned}||\nabla u_i||_{(L^r(\Omega_T))^N} &\leq \frac{g(r)\left(\ell K_{i,-i}^+||\nabla\bar{u}_{-i}||_{(L^r(\Omega_T))^N} + (\delta_i + \ell K_{i,i}^+)T^{1/r}||u_i^0||_{W^{1,r}(\Omega)}\right)}{(1-k_i(r))(\beta_i + c_i)}\\ &\leq \frac{g(r)\left(\ell K_{i,-i}^+||\nabla\bar{u}_{-i}||_{(L^r(\Omega_T))^N} + (\delta_i + \ell K_{i,i}^+)T^{1/r}||u_i^0||_{W^{1,r}(\Omega)}\right)}{(1-g(r)(1-\mu_i^*+\nu_i^*))(\beta_i + c_i^*)}, \quad i=1,2,\end{aligned}$$

where we recall the notation $(i,-i)=(i,j)$ with $j\neq i$. Assume that ℓ, $K^+_{i,-i}$ and δ_i, $i=1,2$, satisfy the condition (3.10) given in Proposition 1. This condition implies that there exists γ, $0<\gamma<1$, such that

$$\frac{g(r)\ell K^+_{i,-i}}{(1-g(r)(1-\mu_i^*+\nu_i^*))(\beta_i+c_i^*)}\leq 1-\gamma.$$

We thus have

$$||\nabla u_i||_{(L^r(\Omega_T))^N}\leq (1-\gamma)M'+\frac{g(r)(\delta_i+\ell K^+_{i,i})T^{1/r}\|u_i^0\|_{W^{1,r}(\Omega)}}{(1-g(r)(1-\mu_i^*+\nu_i^*))(\beta_i+c_i^*)}.$$

Now, we choose the constant M' such that the initial conditions satisfy

$$\frac{g(r)(\delta_i+\ell K^+_{i,i})T^{1/r}\,||u_i^0||_{W^{1,r}(\Omega)}}{(1-g(r)(1-\mu_i^*+\nu_i^*))(\beta_i+c_i^*)}\leq \gamma M',\quad i=1,2.,$$

that is

$$M'=\max_{i=1,2}\Big\{\frac{g(r)(\delta_i+\ell K^+_{i,i})T^{1/r}\,||u_i^0||_{W^{1,r}(\Omega)}}{((1-g(r)(1-\mu_i^*+\nu_i^*))(\beta_i+c_i^*)-g(r)\ell K^+_{i,-i})}\Big\}\tag{5.2}$$

Then, combining the two previous inequalities, we obtain

$$||\nabla u_1||_{(L^r(\Omega_T))^N}\leq M'\quad\text{and}\quad ||\nabla u_2||_{(L^r(\Omega_T))^N}\leq M'.\tag{5.3}$$

We emphasize that the real M' does not depend on the real number M in (5.1).

We have the tools to perform a fixed point analysis similar to the one in the proof of Theorem 1. We have already chosen M' so that the bounded convex $\mathscr{D}$ defined by (5.1) satisfies $\mathscr{F}(\mathscr{D})\subset\mathscr{D}$. Let us show that $\mathscr{D}$ is closed in $L^2(0,T;H)$. We proceed as we did for the set $\mathscr{C}$. In fact, it is sufficient to check that, if (u_1^n,u_2^n) denotes a sequence of functions of $\mathscr{D}^2$ such that

$$(u_1^n,u_2^n)\to(u_1,u_2)\text{ in }L^2(0,T;H),$$

then $\nabla u_i\in(L^r(\Omega_T))^N$ with $\|\nabla u_i\|_{(L^r(\Omega_T))^N}\leq M'$, $i=1,2$. Due to the definition of $\mathscr{D}$, the sequence $(\nabla u_1^n,\nabla u_2^n)_n$ is uniformly bounded in the space $(L^r(\Omega_T))^N$. Thus, there exits $(v_1,v_2)\in(L^r(\Omega_T))^{2N}$ such that, for an appropriate subsequence here characterized by an increasing function φ, the convergence $(\nabla u_1^{\varphi(n)},\nabla u_2^{\varphi(n)})\rightharpoonup(v_1,v_2)$ holds true weakly in $(L^r(\Omega_T))^{2N}$. It means

$$\int_{\Omega_T}\nabla u_i^{\varphi(n)}\cdot\Phi\,dxdt\to\int_{\Omega_T}v_i\cdot\Phi\,dxdt,\quad\forall\Phi\in(L^{r'}(\Omega_T))^N,\ \frac{1}{r}+\frac{1}{r'}=1,\tag{5.4}$$

and besides

$$\|v_i\|_{(L^r(\Omega_T))^N}\leq\liminf_{n\to\infty}\|\nabla u_i^{\varphi(n)}\|_{(L^r(\Omega_T))^N}\leq M'.\tag{5.5}$$

But we know (see the proof of the closeness of $\mathscr{C}$ in $L^2(0,T;H)$) that

$$(u_1^{\varphi(n)},u_2^{\varphi(n)})\rightharpoonup(u_1,u_2)\text{ weakly in }\ L^2(0,T,V)$$

thus in particular

$$\int_{\Omega_T}\nabla u_i^{\varphi(n)}\cdot\Phi\,dxdt\to\int_{\Omega_T}\nabla u_i\cdot\Phi\,dxdt,\quad\forall\Phi\in(L^2(\Omega_T))^N.$$

Since $r>2$, we have $L^2(\Omega_T)\subset L^{r'}(\Omega_T)$ and then we infer from the latter convergence together with (5.4) that $\nabla u_i=v_i$ in $L^{r'}(\Omega_T)$ for $i=1,2$. We conclude the proof thanks to (5.5). In brief, $\mathscr{D}$ is a nonempty convex, bounded closed set in $(L^2(0,T;H))^2$, satisfying $\mathscr{F}(\mathscr{D})\subset\mathscr{D}$.

The remainder of the proof follow the lines of the one of Theorem 1. It follows from Schauder fixed point theorem that there exist $(\tilde{u}_1,\tilde{u}_2)\in\mathscr{D}$ such that $\mathscr{F}(\tilde{u}_1,\tilde{u}_2)=(\tilde{u}_1,\tilde{u}_2)$. This fixed point is a weak solution of problems (4.1)–(4.2) and its gradient is uniformly bounded in the space $(L^r(\Omega_T))^{2N}$. The proof of Proposition 1 is complete.

Remark 12 (Proof of Proposition 2)**.** *Note first that, if $N=2$, the small BMO assumption makes sense in our fixed point strategy. Indeed the solution u_i belongs to $L^2(0,T;V)$ and $V\subset VMO$ (vanishing mean oscillation space, see Sarason* [44]*) thanks to the Poincaré-Wirtinger inequality. The proof in the small BMO case thus only consists in replacing the bound given in* (2.7) *by $C_{BMO}(\alpha,\beta,r,T)$. The constant M' in* (5.2) *then have to be replaced by*

$$\max_{i=1,2}\Big\{\frac{C_{BMO}(\alpha_i,\beta_i,r,T)(\delta_i+\ell K_{i,i}^+)T^{1/r}\,||u_i^0||_{W^{1,r}(\Omega)}}{1-C_{BMO}(\alpha_i,\beta_i,r,T)\ell K_{i,-i}^+}\Big\}.$$

5.2. **Proof of Theorem 2.**

We begin by focusing the study of the well-posedness of (3.1)-(3.2) in small times. To this aim, we introduce a small characteristic time scale, denoted $1/\phi$ for a given positive real number ϕ, and we work in the time interval $(0,T_0):=(0,T/\phi)$. The precise definition of its smallness will be specified at the end of the proof.

We change the time scale by setting $t^*=\phi t$ and $u_i^*(t^*,x)=u_i(t,x)$, $i=1,2$. If (u_1,u_2) and $(\bar{u}_1,\bar{u}_2)$ are two weak solutions of (3.1), the functions $v_i:=u_i^*-\bar{u}_i^*\in W(0,T)$, $i=1,2$, weakly solve the following system in $\Omega\times(0,\phi T_0)=\Omega_T$:

$$\begin{aligned}
\phi\partial_{t^*}v_1-\nabla\cdot\big((\delta_1+K_{1,1}T_\ell(u_1^*))\nabla v_1\big)-\nabla\cdot\big(K_{1,1}(T_\ell(u_1^*)-T_\ell(\bar{u}_1^*))\nabla\bar{u}_1^*\big)&\\
-\nabla\cdot(K_{1,2}T_\ell(u_1^*)\nabla v_2)-\nabla\cdot\big(K_{1,2}(T_\ell(u_1^*)-T_\ell(\bar{u}_1^*))\nabla\bar{u}_2^*\big)&=0,\\
\phi\partial_{t^*}v_2-\nabla\cdot\big((\delta_2+K_{2,2}T_\ell(u_2^*))\nabla v_2\big)-\nabla\cdot\big(K_{2,2}(T_\ell(u_2^*)-T_\ell(\bar{u}_2^*))\nabla\bar{u}_2^*\big)&\\
-\nabla\cdot(K_{2,1}T_\ell(u_2^*)\nabla v_1)-\nabla\cdot\big(K_{2,1}(T_\ell(u_2^*)-T_\ell(\bar{u}_2^*))\nabla\bar{u}_1^*\big)&=0.
\end{aligned}$$

We multiply these equations by, respectively, v_1 and v_2 and we integrate over $(0,t)\times\Omega$ with $0<t\le T$. Using the fact that $v_1(0,.)=v_2(0,.)=0$ a.e. in Ω and the coercivity property of $K_{i,i}$, we get after summing up the two equations:

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega\big(|v_1|^2(t,x)+|v_2|^2(t,x)\big)+\int_{\Omega_t}\big((\delta_1+K_{1,1}^-T_\ell(u_1^*))|\nabla v_1|^2+(\delta_2+K_{2,2}^-T_\ell(u_2^*))|\nabla v_2|^2\big)\\
&\quad+\int_{\Omega_t}\big(T_\ell(u_1^*)-T_\ell(\bar{u}_1^*)\big)\big(K_{1,1}\nabla\bar{u}_1^*+K_{1,2}\nabla\bar{u}_2^*\big)\cdot\nabla v_1+\int_{\Omega_t}\big(K_{1,2}T_\ell(u_1^*)+K_{2,1}T_\ell(u_2^*)\big)\nabla v_1\cdot\nabla v_2\\
&\quad+\int_{\Omega_t}\big(T_\ell(u_2^*)-T_\ell(\bar{u}_2^*)\big)\big(K_{2,1}\nabla\bar{u}_1^*+K_{2,2}\nabla\bar{u}_2^*\big)\cdot\nabla v_2\le 0.
\end{aligned}$$

By the definition of T_ℓ and since $u_i^*,\bar{u}_i^*\ge 0$, we have that $T_\ell(u_i^*)\ge 0$ and

$$T_\ell(u_i^*)-T_\ell(\bar{u}_i^*)=\begin{cases}u_i^*-\bar{u}_i^* & \text{if } 0\le u_i^*,\bar{u}_i^*\le\ell,\\ \ell-\bar{u}_i^* & \text{if } u_i^*\ge\ell,\ 0\le\bar{u}_i^*\le\ell,\\ u_i^*-\ell & \text{if } 0\le u_i^*\le\ell,\ \bar{u}_i^*\ge\ell,\\ 0 & \text{if } u_i^*,\bar{u}_i^*\ge\ell.\end{cases}$$

Thus, in all the cases, $|T_\ell(u_i^*) - T_\ell(\bar{u}_i^*)| \le |u_i^* - \bar{u}_i^*| = |v_i|$. For notational convenience, let $K_{i,+} = \max_{j=1,2} |K_{i,j}^+|$, $i = 1,2$. We have

$$\left| \int_{\Omega_t} (T_\ell(u_1^*) - T_\ell(\bar{u}_1^*)) \big(K_{1,1}\nabla \bar{u}_1^* + K_{1,2}\nabla \bar{u}_2^*\big) \cdot \nabla v_1 \, dxds \right| \le \int_0^t \int_\Omega K_{1,+} |v_1| \big(|\nabla \bar{u}_1^*| + |\nabla \bar{u}_2^*|\big) |\nabla v_1| \, dxds.$$

Next, we compute

$$\int_0^t \int_\Omega K_{i,+} |v_i| \big(|\nabla \bar{u}_i^*| + |\nabla \bar{u}_{-i}^*|\big) |\nabla v_i| \, dxds$$
$$\le \int_0^t K_{i,+} \Big(\int_\Omega |v_i|^4 \, dx\Big)^{1/4} \Big((\int_\Omega (|\nabla \bar{u}_i^*|^4 \, dx)^{1/4} + (\int_\Omega |\nabla \bar{u}_{-i}^*|^4 \, dx)^{1/4} \Big) \Big(\int_\Omega |\nabla v_i|^2 \, dx\Big)^{1/2} ds. \quad (5.6)$$

The analogous of Proposition 1 can be proved for u_i^* and $\bar{u}_i^*$ (note that the proof of Lemma 2 given in Annex shows that the result in Prop. 1 does not depend on ϕ). It ensures the existence of C_4 defined by (5.2) with an obvious modification for including the source term Q_i such that

$$\|\nabla u_i^*\|_{(L^4((0,T)\times\Omega))^2} \le C_4, \quad \text{i=1,2}.$$

More precisely, we have

$$\begin{aligned} \|\nabla u_i^*\|_{(L^4((0,T)\times\Omega))^2} &= \phi^{1/4} \|\nabla u_i\|_{(L^4((0,T/\phi)\times\Omega))^2} \\ &\le \phi^{1/4} \frac{g(4)(\delta_i + \ell K_{i,i}^+)(T/\phi)^{1/4} (q_{+,i} + ||u_i^0||_{W^{1,4}(\Omega)})}{((1 - g(4)(1 - \mu_i^* + \nu_i^*))(\beta_i + c_i^*) - g(r)\ell K_{i,-i}^+)} = C_4. \end{aligned} \quad (5.7)$$

Hence

$$(\int_{\Omega_T} (|\nabla \bar{u}_i^*|^4 \, dx)^{1/4} + (\int_{\Omega_T} |\nabla \bar{u}_{-i}^*|^4 \, dx)^{1/4} \le 2C_4.$$

On the other hand, by the Gagliardo–Nirenberg inequality, we have

$$\Big(\int_\Omega |u|^4 \, dx\Big)^{1/4} \le C_G ||u||_{L^2(\Omega)}^{1/2} ||\nabla u||_{(L^2(\Omega))^2}^{1/2}, \quad \forall u \in H_0^1(\Omega).$$

Then, combining the Hölder and Young inequalities, we obtain

$$\int_{\Omega_t} K_{i,+} |v_i| (|\nabla \bar{u}_i^*| + |\nabla \bar{u}_{-i}^*|) |\nabla v_i| \, dxds$$
$$\le K_{i,+} C_G \Big(\int_0^t ||v_i||_{L^2(\Omega)}^{2/3} ||\nabla v_i||_{(L^2(\Omega))^2}^2 ds\Big)^{3/4} \Big((\int_{\Omega_t} (|\nabla \bar{u}_i^*|^4 \, dx)^{1/4} + (\int_{\Omega_t} |\nabla \bar{u}_{-i}^*|^4 \, dx)^{1/4} \Big)$$
$$\le 2K_{1,+} C_G C_4 \max_{(0,t)} ||v_i||_{L^2(\Omega)}^{1/2} \Big(\int_{\Omega_t} |\nabla v_i|^2\Big)^{3/4} \le \frac{27}{4} \frac{K_{i,+}^4 C_G^4 C_4^4}{\varepsilon_i^3} \max_{(0,t)} ||v_i||_{L^2(\Omega)}^2 + \varepsilon_i \int_{\Omega_t} |\nabla v_i|^2 \, dxds,$$

for any arbitrary given $\varepsilon_i > 0$, $i = 1,2$. Finally, using once again the Cauchy-Schwarz and Young inequalities, we get for any arbitrary $\varepsilon_{i+1} > 0$, $i = 1,2$:

$$\begin{aligned} \left| \int_0^t \int_{\Omega_t} K_{i,-i} T_\ell(u_i^*) \nabla v_i \cdot \nabla v_{-i} \right| &\le \ell^{1/2} K_{i,-i}^+ \Big(\int_{\Omega_t} |\nabla v_{-i}|^2\Big)^{1/2} \Big(\int_{\Omega_t} T_\ell(u_i^*) |\nabla v_i|^2\Big)^{1/2} \\ &\le \frac{\ell (K_{i,-i}^+)^2}{4\varepsilon_{i+1}} (\int_{\Omega_t} |\nabla v_{-i}|^2) + \varepsilon_{i+1} (\int_{\Omega_t} T_\ell(u_i^*) |\nabla v_i|^2). \end{aligned}$$

By combining all the inequalities above, we obtain that

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega (v_1^2+v_2^2)(t,x)\,dx + (\delta_1-\varepsilon_1-\frac{\ell(K_{2,1}^+)^2}{4\varepsilon_4})\int_{\Omega_t}|\nabla v_1|^2\,dxds\\
&+(\delta_2-\varepsilon_2-\frac{\ell(K_{1,2}^+)^2}{4\varepsilon_3})\int_{\Omega_t}|\nabla v_2|^2\,dxds + (K_{1,1}^- -\varepsilon_3)\int_{\Omega_t}T_\ell(u_1^*)|\nabla v_1|^2\,dxds\\
&+(K_{2,2}^- -\varepsilon_4)\int_{\Omega_t}T_\ell(u_2^*)|\nabla v_2|^2\,dxds\\
&\le \frac{27}{4}\frac{K_{1,+}^4C_G^4C_4^4}{\varepsilon_1^3}\max_{(0,T)}(\int_\Omega |v_1|^2(t,x)\,dx) + \frac{27}{4}\frac{K_{2,+}^4C_G^4C_4^4}{\varepsilon_2^3}\max_{(0,T)}(\int_\Omega |v_2|^2(t,x)\,dx).
\end{aligned} \tag{5.8}$$

Pick $\varepsilon_1=\delta_1/4$, $\varepsilon_2=\delta_2/4$, $\varepsilon_3=K_{1,1}^-$ and $\varepsilon_4=K_{2,2}^-$. We get

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega (v_1^2+v_2^2)(t,x)\,dx + \big(\frac{3\delta_1}{4}-\frac{\ell(K_{2,1}^+)^2}{4K_{2,2}^-}\big)\int_{\Omega_t}|\nabla v_1|^2\,dxds + \big(\frac{3\delta_2}{4}-\frac{\ell(K_{1,2}^+)^2}{4K_{1,1}^-}\big)\int_{\Omega_t}|\nabla v_2|^2\,dxds\\
&\le \frac{3^3\,4^2\,K_{1,+}^4C_G^4C_4^4}{\delta_1^3}\max_{(0,T)}(\int_\Omega |v_1|^2(t,x)\,dx) + \frac{3^3\,4^2\,K_{2,+}^4C_G^4C_4^4}{\delta_2^3}\max_{(0,T)}(\int_\Omega |v_2|^2(t,x)\,dx).
\end{aligned}$$

Finally, assuming (3.14), the maximum of the left hand side of the latter relation for $t\in(0,T)$ satisfies

$$\sum_{i=1,2}\big(\frac{\phi}{2}-\frac{3^3\,4^2\,K_{i,+}^4C_G^4C_4^4}{\delta_i^3}\big)\max_{(0,T)}\int_\Omega |v_i|^2(t,x)\,dx\le 0. \tag{5.9}$$

If ϕ satisfies

$$\frac{\phi}{2}-\frac{3^3\,4^2\,K_{1,+}^4C_G^4C_4^4}{\delta_1^3}\ge 0 \quad\text{and}\quad \frac{\phi}{2}-\frac{3^3\,4^2\,K_{2,+}^4C_G^4C_4^4}{\delta_2^3}\ge 0, \tag{5.10}$$

and if (ℓ,δ_1,δ_2) and the tensor K satisfy (3.14), then (5.9) implies that

$$\int_{\Omega_T}|\nabla v_i|^2\,dxds,\quad i=1,2,$$

and so $v_i=0$, that is $u_i^*=\bar u_i^*$ almost everywhere in Ω_T.

Turning back to the original time scale, it means that the solution $u=(u_1,u_2)$ of (3.1)-(3.2) is unique in $(0,\hat t_0)\times\Omega$ with

$$\hat t_0=\min\Big\{\frac{T\delta_1^3}{3^32^5\times K_{1,+}^4C_G^4C_4^4},\frac{T\delta_2^3}{3^32^5\times K_{2,+}^4C_G^4C_4^4},T\Big\}.$$

Indeed, choosing

$$\phi=\phi_0=T/\hat t_0$$

ensures the validity of (5.10).

We now aim at propagating this uniqueness result to the whole interval of interest. The important point is a precise computation of the real number C_4 that characterizes the size of the interval $(0,\hat t_0)$ where the uniqueness is ensured. According to (5.2),

$$C_4:=c_{4,i}^{1/4}T^{1/4}\big(q_{+,i}+\|u_i^0\|_{W^{1,4}(\Omega)}\big),$$

with $c_{4,i}$ given in Theorem 2. According to Lusin's theorem, since $(u_i^*, \bar{u}_i^*) \in L^4(0,T,W^{1,4}(\Omega))^2$ for $i = 1,2$, for any given $\varepsilon > 0$, there exists some closed interval $I \subset (\phi_0\hat{t}_0/2, \phi_0\hat{t}_0) \subset (0,T)$ such that $|I| \geq \phi_0\hat{t}_0/2 - \varepsilon$ and the restriction of $(u_i^*, \bar{u}_i^*)$ in I is a continuous in time function. Of course, we have $|I| \min_{t\in I} \|u_i^*\|^4_{W^{1,4}(\Omega)} \leq \|u_i^*\|^4_{L^4(0,T;W^{1,4}(\Omega))} \leq C_4^4$. Similar computations may be done with $\bar{u}_i^*$. We can therefore pick some $t_0^* \in I$ such that

$$\|(u_i^*(t_0^*), \bar{u}_i^*(t_0^*))\|^4_{(W^{1,4}(\Omega))^2} \leq \frac{1}{\phi_0\hat{t}_0/2 - \varepsilon} \times \|u_i^*\|^4_{L^4(0,T;W^{1,4}(\Omega))} \leq \frac{C_4^4}{\phi_0\hat{t}_0/2 - \varepsilon}.$$

Let $\gamma \in (0,1/2)$ such that $\phi_0\hat{t}_0/2 - \varepsilon = \gamma T$. The latter estimate then reads

$$\|(u_i^*(t_0^*), \bar{u}_i^*(t_0^*))\|^4_{(W^{1,4}(\Omega))^2} \leq \frac{C_4^4}{\gamma T}. \tag{5.11}$$

We now consider the solutions u_i and $\bar{u}_i$ as starting from t_0^* and we try to follow the previous lines for proving the uniqueness in a new interval in the form $[t_0^*, t_0^* + \hat{t}_1]$. To this aim, we set $t^* = \phi_1(t - t_0^*)$ and $u_i^*(t^*, x) = u_i(t,x)$, $i = 1,2$. Using (5.11) in (5.2)–(5.3), we obtain the following analogous for (5.7):

$$\|\nabla u_i^*\|^4_{(L^4((0,T)\times\Omega))^2} \leq c_{4,i} T \left(q_{+,i} + \frac{C_4}{(\gamma T)^{1/4}} \right)^4 =: (C_4^1)^4. \tag{5.12}$$

Next, we follow the previous lines and show that $v_i = u_i^* - \bar{u}_i^* = 0$ a.e. in $[t_0^*, t_0^* + T)$, the time scaling factor ϕ_1 still being defined by the condition (5.10) but with C_4^4 replaced by $(C_4^1)^4$. Turning back to the original time scale, it means that the solution of (3.1)-(3.2) is unique in $(0, t_0^* + \hat{t}_1)$ with

$$\hat{t}_1 = \min\Big\{ \min_{i=1,2} \Big\{ \frac{T\delta_i^3}{3^3 2^5 K_{i,+}^4 C_G^4 (C_4^1)^4} \Big\}, T \Big\}.$$

For propagating this uniqueness result to the whole interval of interest, it is sufficient to ensure that the sequence $(C_4^n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} C_4^0 = C_4, \\ C_4^{n+1} = c_{4,i}^{1/4} \big(q_{+,i} T^{1/4} + \gamma^{-1/4} C_4^n \big), \end{cases}$$

is such that $\sum_{n\geq 0} (C_4^n)^{-4} = \infty$. This result is especially ensured if $\lim_{n\to\infty} C_4^n < \infty$ thus if $c_{4,i}^{1/4}/\gamma^{1/4} < 1$. Further γ may be chosen arbitrarily close to $1/2$. We thus obtain the criterion $c_{4,i} < 1/2$. The proof of Theorem 2 is complete. □

Remark 13. *The previous strategy may be repeated for the proof in the small BMO setting. Estimate* (5.7) *is modified in view of the expression of M' given in Remark 2. The analogous of the condition* (5.10) *is in the form*

$$\phi/T \geq e^{4C_i T/\phi} \; 3^3 \, 2^5 \, K_{i,+}^4 C_G^4 C_{bmo,i}^4 / \delta_i^3 (1 - C_{BMO}(\alpha_i, \beta_i, 4, T/\phi) \ell K_{i,-i}^+)^4,$$

where $C_{bmo,i}$ and C_i are such that $C_{BMO}(\alpha_i, \beta_i, 4, T)(\delta_i + \ell K_{i,i}^+)(\|u_i^0\|_{W^{1,4}} + q_{+,i}) = C_{bmo,i} e^{C_i T}$. But, since $\phi > 4T$ (otherwise the result is obvious), such a condition is fulfilled if

$$\phi \geq T \max_{i=1,2} \Big\{ e^{C_i} \frac{3^3 \, 2^5 \, K_{i,+}^4 C_G^4 C_{bmo,i}^4}{\delta_i^3 (1 - C_{BMO}(\alpha_i, \beta_i, 4, T) \ell K_{i,-i}^+)^4} \Big\}.$$

Hence, the rest of the computations can be reproduced as is, of course by changing the value of the constants.

6. About the maximum principle and proof of Theorem 3

In the present section, we look for additional assumptions allowing the proof of a complete maximum principle for (2.1). We first consider an example of the simplest setting. We study a particular case of the system which behaves like a volume-filling model. As emphasized by the proof, the algebraic structure of the system "naturally" ensures a maximum principle: the boundedness of the solution is proved with classical arguments. Notice that models with segregation properties (*e.g.* [7]) also inherits naturally of maximum principle properties. As already mentioned, the aim of this paper is to avoid as far as possible this kind of structural assumption. Nevertheless, we are aware that another kind of assumption is necessary. Indeed, as emphasized by Le and Nguyen in [30] (Theorem 1.5), there even may exists a classical solution (in the case of a cross-diffusion system with smooth coefficients) changing sign. If the maximum principle is not induced by the structure of the system, we choose to deal with "sufficiently pumping" source terms for enforcing the boundedness of the solution. In the second subsection, we thus prove the existence of source terms confining the solution under any prescribed value. The proof consists in introducing in the original system a penalizing term that we let blow up. Since this method introduces additional nonlinearity in the problem, we give another existence proof, still based on a fixed point argument, but using the Brouwer's topological degree method instead of the Schauder's theorem. Finally, in the last subsection, we show how the penalization method introduced for proving the existence of confining source terms may be interpreted from the physical point of view.

6.1. **A volume-filling model: classical weak maximum principle.** In the present subsection, we give an example of the simplest setting ensuring a maximum principle, namely a volume-filling algebraic structure. We consider the following particular case of system (2.1) with $K_{i,j} = K$, $1 \le i, j \le 2$:

$$\partial_t u_1 - \delta\Delta u_1 - \nabla\cdot(Ku_1\nabla u_1 + Ku_1\nabla u_2) = Q(u)u_1, \tag{6.1}$$

$$\partial_t u_2 - \delta\Delta u_2 - \nabla\cdot(Ku_2\nabla u_2 + Ku_2\nabla u_1) = Q(u)u_2, \tag{6.2}$$

completed by (3.2). We aim showing that this problem has a volume-filling structure, that is that we can exhibit a solution such that

$$0 \le u_1 + u_2 \le \ell \text{ a.e. in } \Omega_T$$

provided that the initial and Dirichlet data satisfy the same relation. The non-negativity of the solutions has already been proved in the general setting. We thus simply check, using formal *a priori* estimates, that one may expect a solution such that $u_\ell := \max(u_1 + u_2 - \ell, 0) = 0$ almost everywhere in Ω_T. Assume $u_1^0 + u_2^0 \le \ell$ and $Q(v) \le 0$ for any $v = (v_1, v_2)$ such that $v_1 + v_2 \le 1$. For the sake of simplicity, set $u_{i,D} = 0$. Summing up (6.1) and (6.2), we obtain

$$\partial_t(u_1+u_2) - \delta\Delta(u_1+u_2) - \nabla\cdot\big(K(u_1+u_2)\nabla(u_1+u_2)\big) = Q(u)(u_1+u_2).$$

We multiply this equation by u_ℓ and integrate by parts over Ω. We obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\ell|^2\,dx + \int_\Omega \big(\delta + (u_1+u_2)K\big)\nabla u_\ell\cdot\nabla u_\ell\,dx - \int_\Omega Q(u)(u_1+u_2)u_\ell\,dx = 0.$$

Using the assumption on Q, the non-negativity of u_i and the coercivity of K, we infer from the latter relation that $\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\ell|^2\,dx + \delta\int_\Omega |\nabla u_\ell|^2\,dx \le 0$ a.e. in $(0,T)$. It follows that $u_\ell \le 0$ and thus $u_1 + u_2 \le \ell$ a.e. in Ω_T.

6.2. **An example of explicit admissible source term when $\delta_i = 0$.** In the present paper, we question the possibility of tuning up the value of the source term for ensuring a maximum principle. A first example follows in the simplest setting when $\delta_i = 0$, $i = 1,2$. We remind that, when $\delta_i = 0$, the decay of the entropy enables to obtain a bound (depending only on the data) for the $L^2(\Omega_T)$-norm of the gradient of u_i $(i = 1,2)$ (*cf.* [1]).

Assume the source terms Q_i equal to $\dfrac{-C_i(t,x)}{\eta + (u_i - \ell)^+}$, where C_i is a non negative function of $L^2(\Omega_T)$, η is some positive real number (chosen smaller than 1). Assume also that the initial and boundary data satisfy $0 \le u_i^0 \le \ell$ and $u_{i,D} = 0$.

Let us solely prove that $u_1(t,x) \le \ell$ for all $t \in (0,T)$ and for almost every $x \in \Omega$. Showing this result for u_2 follows the same lines. Let $u_M = \eta + (u_1 - \ell)^+$. The function $u_M - \eta$ belongs to $L^2(0,T;V)$ and is such that $\nabla u_M = \chi_{\{u_1 > \ell\}} \nabla u_1$. Let $\tau \in (0,T]$. Setting $w(t,x) = u_M(t,x)\chi_{(0,\tau)}(t)$ in (4.1) (with $\delta_1 = 0$) results in

$$\int_0^\tau \langle \partial_t u_1, u_M \rangle_{V',V} + \int_{\Omega_\tau} C_1(t,x)\,dx\,dt + \int_{\Omega_\tau} T_\ell(u_1) K_{1,1} \nabla u_M \cdot \nabla u_M = -\int_{\Omega_\tau} T_\ell(u_1) K_{1,2} \nabla u_2 \cdot \nabla u_M. \quad (6.3)$$

Since the function T_ℓ is extended continuously and constantly outside the interval $(0,\ell)$, we deduce from (6.3)

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big((u_1-\ell)^+\big)^2(\tau,x)dx + \eta\,\Big(\int_\Omega u_1(\tau,x)dx - \int_\Omega u_1^0(x)dx\Big) + \int_{\Omega_\tau} C_1(t,x)dx\,dt \\ +\quad &\ell K_{1,1}^- \int_{\Omega_\tau} |\nabla u_M|^2\,dx\,dt \le -\ell \int_{\Omega_\tau} K_{1,2} \nabla u_2 \cdot \nabla u_M := J_0. \end{aligned} \quad (6.4)$$

Then, using the regularity result for the gradient of u_2 established in [1] (thus the constant C_0 below), we estimate J_0 as follows

$$|J_0| \quad\le\quad K_{1,2}^+ \ell \times \Big(\int_{\Omega_\tau} |\nabla u_2|^2\Big)^{1/2} \times \Big(\int_{\Omega_\tau} |\nabla u_M|^2\Big)^{1/2} \le \ell^2 + \frac{(K_{1,2}^+ C_0)^2}{4} \int_{\Omega_\tau} |\nabla u_M|^2. \quad (6.5)$$

Combining (6.5) with (6.4), we obtain for all $\tau \in (0,T]$

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big((u_1-\ell)^+\big)^2(\tau,x)dx + \frac{4\ell K_{1,1}^- - (K_{1,2}^+ C_0)^2}{4} \int_{\Omega_\tau} |\nabla u_M|^2\,dx\,dt \\ +\quad &\eta \int_\Omega u_1(\tau,x)dx + \Big(\int_{\Omega_\tau} C_1(t,x)\,dx\,dt - \ell^2\Big) \;\le\; \eta \int_\Omega u_1^0(x)dx. \end{aligned}$$

Now, assume that ℓ is sufficiently large so that

$$K_{1,2}^+ < \frac{2\sqrt{\ell K_{1,1}^-}}{C_0}.$$

Then, if the source term is large enough, namely $\int_{\Omega_T} C_1(t,x)\,dx\,dt \ge \ell^2$, by taking $\tau = T$ in the previous inequality, we deduce that $\int_\Omega u_1(T,x)dx \le \int_\Omega u_1^0(x)dx$. If the pumping is stronger, namely if we impose *e.g.* $\eta \le 1$ and

$$\int_{\Omega_T} C_1(t,x)\,dx\,dt \ge \ell^2 + \int_\Omega u_1^0(x)dx$$

we ensure that $|\nabla(u-\ell)^+|^2 = 0$ a.e. in Ω_T. Moreover $(u-\ell)^+(T,x) = 0$ a.e. $x \in \Omega$ and then $(u-\ell)^+ = 0$ a.e. in Ω_T. We also get that $u_1(T,.) = 0$ a.e. in Ω.

6.3. **Proof of Theorem 3.** Here we turn back to the general setting of the cross-diffusion model. We prove that there exist "sufficiently pumping" source terms for enforcing the boundedness of the solution, namely Theorem 3. We only have to prove Proposition 3.

STEP 1. EXISTENCE OF A WEAK SOLUTION FOR A PENALIZED PROBLEM

Let the function U_ℓ defined in $\mathbb{R}$ by

$$U_\ell(x) = \max\{\ell, x\}.$$

Let $\varepsilon > 0$. Consider the following penalized problem

$$\partial_t u_i^\varepsilon - \nabla \cdot \big(\delta_i \nabla u_i^\varepsilon + T_\ell(u_i^\varepsilon) \sum_{j=1}^{2} K_{i,j} \nabla u_j^\varepsilon\big) - \frac{1}{\varepsilon} \Delta U_\ell(u_i^\varepsilon) = Q_i(u^\varepsilon) \text{ in } \Omega_T, \tag{6.6}$$

$$u_i^\varepsilon = u_{i,D}, \quad \text{in } (0,T)\times\Gamma, \quad u_i^\varepsilon(0,x) = u_i^0(x) \quad \text{in } \Omega. \tag{6.7}$$

Once again we use a fixed point strategy for proving the existence of a weak solution of (6.6)-(6.7). But in view of the new nonlinearity introduced in the system, we rather use a topological degree argument. Problem (6.6)-(6.7) is rewritten as

$$\begin{aligned}
&u_i^\varepsilon - u_{i,D} \in W(0,T),\\
&\langle \partial_t(u_i^\varepsilon - u_{i,D}), v\rangle_{L^2(0,T;V')\times L^2(0,T;V)} + \delta_i \int_{\Omega_T} \nabla u_i^\varepsilon \cdot \nabla v \, dxdt = \langle F_i(u^\varepsilon), v\rangle_{L^2(0,T;V')\times L^2(0,T;V)}
\end{aligned}$$

where $F_i(u) \in L^2(0,T;V')$ for any $u = (u_1,u_2) \in (L^2(0,T;H^1(\Omega)))^2$ is defined by

$$\begin{aligned}
\langle F_i(u), v\rangle_{L^2(0,T;V')\times L^2(0,T;V)} &= -\int_{\Omega_T} \big(T_\ell(u_i)(K_{i,i}\nabla u_i + K_{i,j}\nabla u_j) + \frac{1}{\varepsilon}\nabla U_\ell(u_i)\big)\cdot \nabla v \, dxdt\\
&+ \int_{\Omega_T} Q_i(u) v \, dxdt - \langle \partial_t u_{i,D}, v\rangle_{L^2(0,T;V')\times L^2(0,T;V)}.
\end{aligned}$$

Notice that the function $F_i : (L^2(0,T;V))^2 \to L^2(0,T;V')$ is continuous. Next, denote by L_i the operator from $L^2(0,T;V')$ into $L^2(0,T;V)$ defined by $L_i(S) = u_i - u_{i,D}$ where u_i is the unique solution of

$$\begin{aligned}
&u_i - u_{i,D} \in W(0,T),\\
&\langle \partial_t(u_i - u_{i,D}), v\rangle_{L^2(0,T;V')\times L^2(0,T;V)} + \delta_i \int_{\Omega_T} \nabla u_i \cdot \nabla v \, dxdt = \langle S, v\rangle_{L^2(0,T;V')\times L^2(0,T;V)}.
\end{aligned}$$

Now solving (6.6)-(6.7) consists in solving $u^\varepsilon - u_D = (L_1(F_1(u^\varepsilon)), L_2(F_2(u^\varepsilon)))$. For any $s \in [0,1]$, we set $d(s,u) = (sL_1(F_1(u)), sL_2(F_2(u)))$. For $M > 0$, let $\mathscr{B}_M = \{u \in (L^2(0,T;H^1(\Omega)))^2;\ \|u\|_{(L^2(0,T;H^1(\Omega)))^2} < M\}$. If the following conditions are fulfilled

(i) $\exists M > 0;\ \big(u - d(s,u) = 0,\ s \in [0,1] \text{ and } u \in (L^2(0,T;H^1(\Omega)))^2\big) \Rightarrow u \in \mathscr{B}_M$
(ii) the function d is continuous from $[0,1]\times\overline{\mathscr{B}_M}$ into $\overline{\mathscr{B}_M}$
(iii) the set $\{d(s,u),\ s \in [0,1],\ u \in \overline{\mathscr{B}_M}\}$ is relatively compact in $L^2(\Omega_T)$

there is no solution of the equation $u - d(s,u) = 0$ on the boundary of $\mathscr{B}_M$ and we can define the topological degree ([8]) $deg(\mathrm{Id} - d(s,\cdot), \mathscr{B}_M, 0)$. It does not depend on s. Thus

$$deg(\mathrm{Id} - d(s,\cdot), \mathscr{B}_M, 0) = deg(\mathrm{Id} - d(0,\cdot), \mathscr{B}_M, 0) = deg(\mathrm{Id}, \mathscr{B}_M, 0) = 1.$$

It follows that there exists $u^\varepsilon \in \mathscr{B}_M$ such that $u^\varepsilon - d(1,u^\varepsilon) = 0$, that is

$$u^\varepsilon - u_D = (L_1(F_1(u^\varepsilon)), L_2(F_2(u^\varepsilon))).$$

All the elements for checking points (i)-(iii) have already been exposed in the proof of Theorem 1. It is clear that the only new terms, namely $\varepsilon^{-1}\nabla U_\ell(u_i)$, have a diffusive form that does not perturb the estimates. We thus do not detail their proof.

Remark 14. *Notice that this proof can be adapted for ensuring more regularity to the solutions and obtaining the analogous of Proposition 1. Indeed, the computations performed in Subsection 5.1 allow to restrict properly the operators L_i, $i = 1,2$, to $L^r(0,T;W_0^{1,r}(\Omega))$, $r > 2$.*

STEP 2. UNIFORM ESTIMATES OF ANY SOLUTION OF THE PENALIZED PROBLEM

Clearly any solution u^ε of (6.6)-(6.7) lies in the set $\mathscr{C}$ defined in (4.27). The following uniform estimates thus hold true

$$\|\partial_t u_i^\varepsilon\|_{L^2(0,T;V')} + \|u_i^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C, \quad i = 1,2, \tag{6.8}$$

where we denote by C a generic constant that does not depend on ε. Nevertheless, we have to look at the influence of the penalization on the behavior of u_i^ε above ℓ. To this aim, we compute once again energy estimates. Set $u_{i,D} = 0$ for the sake of simplicity. We multiply (6.6) by u_i^ε, integrate by parts over Ω and sum up the results for $i = 1,2$. Using the coercivity of $K_{i,i}$ and $Q_i(u^\varepsilon) \in L^2(\Omega_T)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega ((u_1^\varepsilon)^2 + (u_2^\varepsilon)^2)\,dx + \sum_{i=1}^{2}\int_\Omega \Big(\delta_i + K_{i,i}^- + \frac{1}{\varepsilon}\chi_{\{u_i^\varepsilon \ge \ell\}}\Big)|\nabla u_i^\varepsilon|^2\,dx \\
&\quad + \int_\Omega \big(T_\ell(u_1^\varepsilon)K_{1,2}\nabla u_2^\varepsilon \cdot \nabla u_1^\varepsilon + T_\ell(u_2^\varepsilon)K_{2,1}\nabla u_1^\varepsilon \cdot \nabla u_2^\varepsilon\big)\,dx \\
&\quad \le \int_\Omega \big(Q_1(u^\varepsilon)u_1^\varepsilon + Q_2(u^\varepsilon)u_2^\varepsilon\big)\,dx \le C(t) + \int_\Omega ((u_1^\varepsilon)^2 + (u_2^\varepsilon)^2)\,dx
\end{aligned} \tag{6.9}$$

where $C(t)$ belongs to $L^1(0,T)$. Using the Cauchy-Schwarz and Young inequalities we write for $i = 1,2$, $j \neq i$,

$$\Big|\int_\Omega T_\ell(u_i^\varepsilon)K_{i,j}\nabla u_j^\varepsilon \cdot \nabla u_i^\varepsilon\,dx\Big| \le \varepsilon_i \int_\Omega T_\ell(u_i^\varepsilon)K_{i,i}^-|\nabla u_i^\varepsilon|^2\,dx + \frac{(K_{i,j}^+)^2\ell}{4K_{ii}^-\varepsilon_i}\int_\Omega |\nabla u_j^\varepsilon|^2\,dx,$$

for any $\varepsilon_i > 0$. Inserting this result in (6.9), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega ((u_1^\varepsilon)^2 + (u_2^\varepsilon)^2)\,dx + \sum_{i\ge 1, j=-i}^{2}\int_\Omega \Big((\delta_i - \frac{(K_{j,i}^+)^2\ell}{4K_{j,j}^-\varepsilon_j}) + K_{i,i}^-(1-\varepsilon_i) + \frac{1}{\varepsilon}\chi_{\{u_i^\varepsilon \ge \ell\}}\Big)|\nabla u_i^\varepsilon|^2\,dx \\
&\quad \le C(t) + \int_\Omega ((u_1^\varepsilon)^2 + (u_2^\varepsilon)^2)\,dx.
\end{aligned} \tag{6.10}$$

The assumptions (3.4) ensure the existence of $0 < \varepsilon_i < 1$ such that $\delta_i - ((K_{j,i}^+)^2\ell)/4K_{j,j}^-\varepsilon_j > 0$. From (6.10), the Gronwall lemma gives

$$\sqrt{\varepsilon}\|\nabla u_i^\varepsilon\|_{(L^2(\Omega_T))^N} + \|\chi_{\{u_i^\varepsilon \ge \ell\}}\nabla u_i^\varepsilon\|_{(L^2(\Omega_T))^N} \le C\sqrt{\varepsilon}.$$

This direct estimate can be improved. We rather multiply (6.6) by $u^{\varepsilon}_{i,\ell} := \sup(u^{\varepsilon}_i - \ell, 0)$. Bear in mind that

$$\Big|\int_\Omega T_\ell(u^\varepsilon_i)K_{i,j}\nabla u^\varepsilon_j \cdot \nabla u^\varepsilon_{i,\ell}\,dx\Big| \le \frac{1}{2\varepsilon}\int_\Omega |\nabla u^\varepsilon_{i,\ell}|^2\,dx + C\varepsilon\int_\Omega |\nabla u^\varepsilon_j|^2\,dx \le \frac{1}{2\varepsilon}\int_\Omega |\nabla u^\varepsilon_{i,\ell}|^2\,dx + C(t)\varepsilon$$

where $C(t)$ belongs to $L^1(0,T)$. Then, assuming that the source term Q_i is such that $Q_i(u)u_{i,\ell} \le 0$, we get the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega ((u^\varepsilon_{1,\ell})^2 + (u^\varepsilon_{2,\ell})^2)\,dx + \sum_{i\ge 1, j=-i}^{2}\int_\Omega \Big(\delta_i + K^-_{i,i} + \frac{1}{2\varepsilon}\Big)|\nabla u^\varepsilon_{i,\ell}|^2\,dx \le C(t)\varepsilon. \tag{6.11}$$

Since $u^\varepsilon_{i,\ell}(0,x) = 0$ a.e. in Ω, the estimate (6.11) gives with the Gronwall lemma and the Poincaré inequality

$$\|\chi_{\{u^\varepsilon_i \ge \ell\}} u^\varepsilon_i\|_{L^2(0,T;V)} \le C\varepsilon. \tag{6.12}$$

STEP 3. LETTING THE PENALIZATION BLOW UP

We let $\varepsilon \to 0$. In view of estimates (6.8)-(6.12), there is a subsequence of (u^ε), not relabeled for convenience, and $(u,Q) \in (W(0,T))^2 \times (L^2(0,T;(H^1(\Omega))'))^2$ such that

$$u^\varepsilon_i \rightharpoonup u_i \text{ weakly in } W(0,T),$$
$$\frac{1}{\varepsilon}\nabla U_\ell(u^\varepsilon_i) = \frac{\chi_{\{u^\varepsilon_i \ge \ell\}}}{\varepsilon}\nabla u^\varepsilon_i \rightharpoonup \mathscr{Q}_i \text{ weakly in } (L^2(\Omega_T))^N,\ Q := (\nabla\cdot\mathscr{Q}_1, \nabla\cdot\mathscr{Q}_2)$$

and moreover, thanks to a compactness argument of Aubin's type,

$$u^\varepsilon_i \to u_i \text{ a.e. in } \Omega_T.$$

Letting $\varepsilon \to 0$ in (6.6)-(6.7), we conclude that u_i is a nonnegative solution of

$$\partial_t u_i - \nabla\cdot\big(\delta_i \nabla u_i + T_\ell(u_i)\sum_{j=1}^m K_{i,j}\nabla u_j\big) = Q_i(u) + \nabla\cdot\mathscr{Q}_i \text{ in } \Omega_T,$$
$$u_i = u_{i,D} \text{ in } (0,T)\times\Gamma, \quad u_i(0,x) = u^0_i(x) \text{ in } \Omega.$$

Moreover, due to (6.12), $u_i(t,x) \le \ell$ almost everywhere in Ω_T. Proposition 3 is proved. It remains to notice that Remark 9 comes straightforward from the construction of $\mathscr{Q}_i$.

6.4. **Concept of confined solution.** Another way for stating Theorem 3 consists in introducing a concept of confined solution for the problem.

Definition 1. *The problem* (1.1) *completed by appropriate boundary and initial conditions admits a confined solution if there exists a source term* $Q \in (L^2(0,T;(H^1(\Omega))'))^m$ *and* $u \in (W(0,T))^m$ *such that* u_i *solves*

$$\partial_t u_i - \nabla\cdot J_i = Q_i \text{ in } \Omega_T$$

and u_i *is bounded almost everywhere in* Ω_T, $i = 1,..,m$.

The advantage of this definition is that the term 'confined' clearly corresponds to the construction of the solution which is forced to remain bounded by the penalization method. Another asset is that it sometimes corresponds to a physical interpretation of the confinement. Let us turn back to the aquifer model presented on page 2. Define the depths h, h_1 and h_2 so that $u_1 = h - h_1$ and $u_2 = h_2 - h$ (see Figure

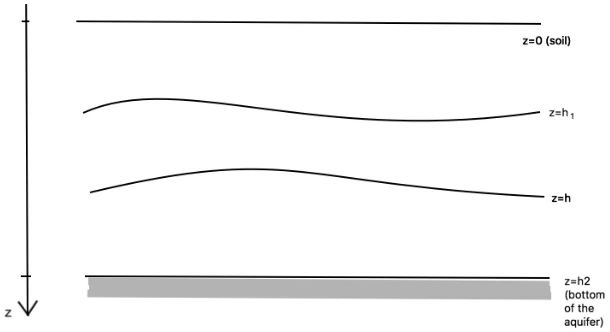

FIGURE 1. Aquifers modeling

1). Assuming the necessary conditions for Theorem 1, namely $\ell = h_2$ and $1-4\delta/h_2 < \alpha \le 1$, we can prove the existence of a weak solution $u=(u_1,u_2)$, with nonnegative components, and thus of h and h_1 solving

$$\partial_t h - \delta\Delta h + \alpha\nabla\cdot\big((h_2-h)\nabla h)\big) - \nabla\cdot\big((1-\alpha)(h_2-h)\nabla h_1\big) = 0, \tag{6.13}$$

$$\partial_t h_1 - \delta\Delta h_1 - \nabla\cdot\big((1-\alpha)(h_2-h_1)\nabla h_1)\big) - \alpha\nabla\cdot\big((h_2-h)\nabla h\big) = 0, \tag{6.14}$$

in Ω_T completed by initial and Dirichlet boundary conditions with the hierarchy of interface depths, $h_1 \le h \le h_2$ a.e. in Ω_T. One retrieves the formulation of the aquifer model of [14]. We now aim proving the existence of a confined solution for the problem above. The physical intuition consists in trying to prove that $0 \le h_1$, that is $u_1+u_2 \le h_2$ a.e. in Ω_T. Let us address this question with our penalization method.

Since our purpose concerns the boundedness of u_1+u_2, we change the set of unknowns, replacing the pair (u_1,u_2) by the pair $(u_1, s=u_1+u_2)$. We thus consider the following form of the cross-diffusion system:

$$\partial_t u_1 - \delta\Delta u_1 - \nabla\cdot\big((1-\alpha)u_1\nabla s\big) = 0,$$
$$\partial_t s - \delta\Delta s - \nabla\big((s-\alpha u_1)\nabla s\big) - \alpha\nabla\cdot\big((u_1-s)\nabla u_1\big) = 0.$$

We now penalize properly the second equation. By properly we mean that we bear in mind that we have to preserve the non-negativity of the functions u_1 and $u_2 = s-u_1$. We thus set $U_0(x) = \max(0,x)$ and we introduce the following penalized system.

$$\partial_t u_1^\varepsilon - \delta\Delta u_1^\varepsilon - \nabla\cdot\big((1-\alpha)U_0(u_1^\varepsilon)\nabla s^\varepsilon\big) = 0, \tag{6.15}$$

$$\begin{aligned}\partial_t s^\varepsilon - \delta\Delta s^\varepsilon &- \nabla\Big(\big(U_0(s^\varepsilon-u_1^\varepsilon) + (1-\alpha)U_0(u_1^\varepsilon)\big)\nabla s^\varepsilon\Big)\\ &-\alpha\nabla\cdot\big(U_0(u_1^\varepsilon - s^\varepsilon)\nabla u_1^\varepsilon\big) - \frac{1}{\varepsilon}\nabla\cdot\big(U_0(s^\varepsilon-u_1^\varepsilon)\nabla U_0(s^\varepsilon-h_2)\big) = 0.\end{aligned} \tag{6.16}$$

One may check that, following the lines of the proof of Proposition 3 in Subsection 6.3, we obtain at the limit $\varepsilon\to 0$ the existence of a bounded solution (u_1,s), such that $u_1 \ge 0$, $s-u_1 \ge 0$ and $s \le h_2$ a.e. in Ω_T, of the following system

$$\partial_t u_1 - \delta\Delta u_1 - \nabla\cdot\big((1-\alpha)u_1\nabla s\big) = 0,$$
$$\partial_t s - \delta\Delta s - \nabla\big((s-\alpha u_1)\nabla s\big) - \alpha\nabla\cdot\big((u_1-s)\nabla u_1\big) - \nabla\cdot\mathscr{Q} = 0.$$

Here we denoted by $\mathscr{Q}$ the weak limit in $(L^2(\Omega_T))^N$ of $\varepsilon^{-1}(s-u_1)\nabla U_0(s^\varepsilon - h_2)$, which satisfies moreover $(s-h_2)\mathscr{Q} = 0$ almost everywhere in Ω_T. Turning back to the interfaces depths, this means that we have exhibited a mathematically confined solution (h_1,h) of (6.13)-(6.14) with $0 \le h_1 \le h \le h_2$ a.e. in Ω_T, which appears as the weak solution of

$$\partial_t h - \delta\Delta h + \alpha\nabla\cdot\big((h_2-h)\nabla h\big) - \nabla\cdot\big((1-\alpha)(h_2-h)\nabla h_1\big) - \nabla\cdot\mathscr{Q} = 0, \tag{6.17}$$

$$\partial_t h_1 - \delta\Delta h_1 - \nabla\cdot\big((1-\alpha)(h_2-h_1)\nabla h_1)\big) - \alpha\nabla\cdot\big((h_2-h)\nabla h\big) - \nabla\cdot\mathscr{Q} = 0, \tag{6.18}$$

in Ω_T completed by initial and Dirichlet boundary conditions, where $\mathscr{Q}$ is such that

$$h_1\mathscr{Q} = 0 \text{ a.e. in } \Omega_T.$$

The interesting point is that there exists a physical interpretation of the latter penalization process. With the penalization term in (6.15)-(6.16), we assume that the aquifer is highly permeable above the depth $z=0$, thus the very high averaged permeability, namely equal to ε^{-1}, when the thickness u_1+u_2 of the water exceeds h_2. At the first order, this very conductive layer acts like a confining layer, as emphasized by the bound $u_1+u_2 \le h_2$ at the limit $\varepsilon \to 0$.

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APPENDIX: PROOF OF LEMMA 2

Let $f \in L^2(0,T;V')$, $u^0 \in H$ and $u \in L^2(0,T;V)$ be the solution of

$$\phi \partial_t u + \mathscr{A} u = f \text{ in } \Omega_T, \quad u(0,x) = u^0 \text{ in } \Omega.$$

We aim at proving that, if f and u^0 are sufficiently smooth, there exists $r > 2$ such that $u - u^0$ belongs to $L^r(0,T;W_0^{1,r}(\Omega))$. For the sake of simplicity, the proof is presented for $u^0 = 0$. Of course, u also satisfies $\phi(\beta+c)^{-1}\partial_t u + (\beta+c)^{-1}\mathscr{A}u = (\beta+c)^{-1}f$, where $(\beta+c)^{-1} \le 1$ for any $c \ge 0$. Next some computations, detailed in [6], allow to transform the equation in the form

$$\phi(\beta+c)^{-1}\partial_t u - \mathrm{div}((A_1+A_2)\nabla u) = (\beta+c)^{-1} f$$

where A_1 is a symmetric matrix such that the operator $\mathscr{A}_1 = -\mathrm{div}(A_1\nabla)$ is uniformly elliptic, while $\mathscr{A}_2 = -\mathrm{div}(A_2\nabla)$ is bounded. More precisely, using μ and ν defined in Lemma 2, we have

$$\sum_{i,j=1}^{N} A_{1i,j}\xi_i\xi_j \ge \mu|\xi|^2 \ \forall \xi \in \mathbb{R}^N, \ \|A_1\|_2 \le 1 \text{ and } \|A_2\|_2 \le \nu \text{ a.e. in } (0,T)\times\Omega,$$

where $0 < \mu \le 1$ (the case $\mu = 1$ corresponds to the case where $\mathscr{A}_1 = -\beta\Delta$) and $\nu < \mu$, $\|A_i\|_2 := \sup_{\xi\in\mathbb{R}^N\setminus\{0\}} |A_i\xi|/|\xi|$. If A is symmetric, we set $A = A_1$, $A_2 = 0$ and thus $\mu = \alpha/\beta \le 1$ (that is $c = 0$), $\nu = 0$. Setting $g^*(t,x) = g(t/\phi(\beta+c),x)$ for any function g involved in the problem and using the operator $\Lambda^{-1} = \partial_{t^*} - \Delta$, the problem now reads: find $u^* \in L^2(0,\tau;V)$ such that

$$u^* + (\Lambda(\mathscr{A}_1^* + \Delta))u^* + (\Lambda\mathscr{A}_2^*)u^* = (\beta+c)^{-1}\Lambda f^* \text{ in } \Omega_\tau, \quad u^*(0,x) = u^0 \text{ in } \Omega,$$

with $\tau = (\beta+c)T/\phi$ and $\Omega_\tau = (0,\tau)\times\Omega$. For simplicity, we still write $X_r = L^r(0,\tau;W_0^{1,r}(\Omega))$ and $Y_r = L^r(0,\tau;W^{-1,r}(\Omega))$. Now, it is sufficient for our purpose to prove that there exists $r > 2$ such that

$$\mathrm{n}(r) := \|\Lambda(\mathscr{A}_1^* + \Delta) + \Lambda\mathscr{A}_2^*\|_{\mathscr{L}(X_r;X_r)} < 1. \tag{6.19}$$

Indeed, the later estimate ensures that the operator Id $+\Lambda(\mathscr{A}_1^* + \Delta) + \Lambda\mathscr{A}_2^*$ is invertible (Id denoting the identity) and thus the existence of $u^* \in X_r$ defined by

$$u^* = (\beta+c)^{-1}\big(\mathrm{Id} + \Lambda(\mathscr{A}_1^* + \Delta) + \Lambda\mathscr{A}_2^*\big)^{-1}\Lambda f^*. \tag{6.20}$$

We first write

$$\begin{aligned} \mathrm{n}(r) &\le \|\Lambda\|_{\mathscr{L}(Y_r;X_r)}\big(\|\mathscr{A}_1^* + \Delta\|_{\mathscr{L}(X_r;Y_r)} + \|\mathscr{A}_2^*\|_{\mathscr{L}(X_r;Y_r)}\big) \\ &= g(r)\big(\|\mathscr{A}_1^* + \Delta\|_{\mathscr{L}(X_r;Y_r)} + \|\mathscr{A}_2^*\|_{\mathscr{L}(X_r;Y_r)}\big). \end{aligned} \tag{6.21}$$

An important point is that $g(r)$ does not depend on T (use a scaling argument for the proof). We notice that, for any given $k \in W^{1,r}(\Omega)$, if $h = (\mathrm{Id} - A_1^*)\nabla k$ then $(\mathscr{A}_1^* + \Delta)k = \mathrm{div}(h)$ and, thanks to Lemma 4 below,

$$\|(\mathscr{A}_1^* + \Delta)k\|_{Y_r} \le \|h\|_{L^r(\Omega_\tau)} = \|(\mathrm{Id} - A_1^*)\nabla k\|_{(L^r(\Omega_\tau))^N} \le (1-\mu)\|\nabla k\|_{(L^r(\Omega_\tau))^N}.$$

It follows that

$$\|\mathscr{A}_1^* + \Delta\|_{\mathscr{L}(X_r;Y_r)} \leq 1-\mu.$$

Proving that

$$\|\mathscr{A}_2^*\|_{\mathscr{L}(X_r;Y_r)} \leq \nu$$

is straightforward. The two latter estimates give in (6.21)

$$\mathrm{n}(r) \leq g(r)(1-\mu+\nu) = k(r). \tag{6.22}$$

We know that $g(2)=1$. Since $0<1-\mu+\nu<1$, $k(2)<1$. It is proved in [6] (relation (2.98) in Chapter 2) that, according to Riesz-Thorin's theorem, there exists a continuous function ρ, defined in $[2,\infty)$, such that $g \leq \rho$ and $\rho(2)=1$. It follows that there exists $r>2$ such that $\rho(p)(1-\mu+\nu)<1$ for any $p \in [2,r]$. In particular, r is such that (6.19) is fulfilled. Definition (6.20) thus makes sense.

It remains to prove the estimate (2.7). Turning back to (6.20), we write

$$\begin{aligned}\|u^*\|_{X_r} &\leq (\beta+c)^{-1}\|\left(\mathrm{Id}+\Lambda(\mathscr{A}_1^*+\Delta)+\Lambda\mathscr{A}_2^*\right)^{-1}\|_{\mathscr{L}(X_r;X_r)}\|\Lambda\|_{\mathscr{L}(Y_r;X_r)}\|f^*\|_{Y_r}\\ &\leq \frac{g(r)}{(\beta+c)(1-k(r))}\|f^*\|_{Y_r}.\end{aligned}$$

Bearing in mind that u^* and f^* correspond to u and f after a rescaling in time, the latter estimate is exactly (2.7).

Lemma 2 is proved. All that remains is to show the following technical lemma, which was used in the latter proof.

Lemma 4. *Let A_1 be a symmetric definite matrix such that $\|A_1\|_2 \leq 1$ and such that $0 \leq \mu \leq 1$ where $\mu = \sup\{\alpha \in \mathbb{R}_+; \sum_{i,j=1}^N A_{1ij}\xi_i\xi_j \geq \alpha|\xi|^2$ for any $\xi \in \mathbb{R}^N\}$. Then $\|\mathrm{Id}-A_1\|_2 \leq 1-\mu$.*

Proof. Since $\mathrm{Id}-A_1$ is a symmetric matrix, we choose the following definition of its spectral norm:

$$\|\mathrm{Id}-A_1\|_2 = \max_{\lambda \in Sp(\mathrm{Id}-A_1)} |\lambda|$$

where $Sp(\mathrm{Id}-A_1)$ is the set of eigenvalues of the matrix $\mathrm{Id}-A_1$. Let $\lambda \in Sp(Id-A_1)$ and ξ_λ an associated eigenvector: $(\mathrm{Id}-A_1)\xi_\lambda = \lambda\xi_\lambda$. The scalar product of the latter relation by ξ_λ gives $(1-\lambda)|\xi_\lambda|^2 = A_1\xi_\lambda\cdot\xi_\lambda$. On the first hand, thanks to the definition of μ, we infer from the latter relation that $(1-\lambda)|\xi_\lambda|^2 \geq \mu|\xi_\lambda|^2$. An eigenvector being non null, it follows that $\lambda \leq 1-\mu$. On the other hand, $\lambda|\xi_\lambda|^2 = (\mathrm{Id}-A_1)\xi_\lambda\cdot\xi_\lambda \geq 0$ since $\mu|\xi_\lambda|^2 \leq A_1\xi_\lambda\cdot\xi_\lambda \leq |A_1\xi_\lambda||\xi_\lambda| \leq |\xi_\lambda|^2$. Thus $\lambda \geq 0$. It follows that $\|\mathrm{Id}-A_1\|_2 \leq 1-\mu$. □

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