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**Dissipative time-domain 1D model for viscothermal acoustic propagation in wind instruments.**

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1 Accurate modeling of acoustic propagation in tubes of varying cross-section in musical  
2 acoustics must include the effect of viscous and thermal boundary layers. Models of  
3 viscothermal losses are classically written in the frequency domain. An approximate  
4 time-domain model is proposed, where all the physical parameters of the instrument  
5 as the bore shape or the wave celerity are explicit coefficients. The model depends on  
6 absolute tabulated constants which only reflect that the pipe is axisymmetric. It can  
7 be seen as a telegrapher's equations augmented by an adjustable number of auxiliary  
8 unknowns. A global energy is dissipated. A time discretization based on variational  
9 approximation is proposed along with numerical experiments and comparisons with  
10 other models.

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## 11 I. INTRODUCTION

12 Time-domain simulations of wind musical instruments require the choice of a balance  
13 between efficiency and accuracy. Real-time synthesis techniques, such as digital waveguide  
14 synthesis with lumped wall loss filter ([Abel \*et al.\*, 2003](#)), ([Mignot \*et al.\*, 2010](#)), modal de-  
15 composition ([Silva \*et al.\*, 2014](#)), and Finite Difference - Time Domain ([Bilbao and Harrison,  
16 2016](#)), favor their performance objective at the expense of model and discretization er-  
17 rors. By contrast, computer aided instrument prototyping requires a fine assessment of the  
18 physical phenomena occurring in the instrument, including dissipation and dispersion effects  
19 on propagative waves inside air filled pipes, caused by viscous and thermal boundary lay-  
20 ers. The Navier-Stokes equations of fluid dynamics indeed do account for all these effects,  
21 but a practical understanding of the macroscopic phenomena from direct numerical simu-  
22 lation requires a tremendous computational effort. In the different characteristic regimes  
23 found in musical instruments, several approximate 3D or 1D models exhibit good accuracy  
24 and performance ; they are briefly reviewed thereafter. More details about the qualitative

25 and quantitative fulfilment of their underlying assumptions can be found in (Thibault and  
26 Chabassier, 2020). Most of them, including the prevalent model initially derived by (Zwikker  
27 and Kosten, 1949) (ZK), are naturally expressed in the frequency domain, which hampers  
28 their use for synthesising the sound of musical instruments by coupling the pipe to a nonlin-  
29 ear embouchure and using time-varying commands. This work focuses on the time domain  
30 formulation of a stable and accurate 1D model, and its numerical implementation based on  
31 a variational approach, aiming at computer aided instrument prototyping.

### 32 **A. Viscous and thermal boundary layers in 3D**

33 The Linearized Navier Stokes (LNS) equations, also called thermoviscoacoustic equations,  
34 describe the evolution of the acoustic density, pressure, temperature and velocity, which are  
35 small perturbations of a background medium assumed steady with no mean flow. They  
36 feature acoustic wave propagation, as well as viscous and thermal diffusion (Tijdeman, 1975).

37 For a domain filled with air bounded with rigid, non-porous isothermal walls, and when  
38 the acoustic wavelength is greater than the viscous and thermal characteristic lengths, the  
39 acoustic velocity and temperature are functions of the acoustic pressure and two potentials  
40 (Kampinga *et al.*, 2011), which are solution to non homogeneous heat equations with Dirich-

41 let boundary conditions. The potentials are close to one in the bulk of the domain, and drop  
 42 to zero near the walls. Then the acoustic pressure satisfies a Helmholtz equation with vari-  
 43 able coefficients that explicitly depend on these potentials. This leads to the emergence of  
 44 viscous and thermal boundary layers affecting the acoustic pressure.

45 When these boundary layers are thin with respect to the radius of the pipe, an asymptotic  
 46 analysis justifies to model them as an effective wall impedance. In the interior domain, the  
 47 acoustic pressure is then solution to a uniform 3D Helmholtz equation (Cremer, 1948),  
 48 (Bruneau *et al.*, 1989), (Jith and Sarkar, 2018), (Berggren *et al.*, 2018).

## 49 B. Model reduction to 1D

In the case of a cylindrical pipe, (Kirchhoff, 1868) derives an implicit analytical dispersion relation to the LNS equations that must be solved iteratively, which does not offer a practical modeling framework. The reduced 1D model derived from an approximation of LNS by (Zwikker and Kosten, 1949) has been shown to be valid for audible frequencies and assuming the reduced frequency  $k = \omega R/c$  is small, where  $\omega$  is the angular frequency,  $R$  the radius of the pipe and  $c$  the celerity of the wave (Tijdeman, 1975). More precisely, the average pressure along a pipe section  $\hat{p}(x, \omega)$ , and the air flow  $\hat{v}(x, \omega)$ , along the longitudinal spatial

variable  $x \in [0, L]$ , are solution to the so-called Zwikker and Kosten (ZK) model:

$$\text{(ZK)} \begin{cases} \frac{d\hat{p}}{dx} + j\omega \frac{\rho}{S} \frac{1}{1 - F_v(\omega)} \hat{v} = 0, & (1a) \\ \frac{d\hat{v}}{dx} + j\omega \frac{S}{\rho c^2} [1 + (\gamma - 1)F_\theta(\omega)] \hat{p} = 0, & (1b) \end{cases}$$

where  $S$  is the pipe section,  $L$  its length,  $\rho$  the static density,  $\gamma$  the ideal gas constant, and coefficients  $F_v$  and  $F_\theta$  describe the contribution of respectively viscous and thermal dissipation, as:

$$F_v(\omega) = \phi \left( R \sqrt{-\frac{j\omega \rho}{\mu}} \right), \quad F_\theta(\omega) = \phi \left( R \sqrt{-\frac{j\omega \rho C_P}{\kappa}} \right),$$

50

$$\text{with } \phi(\alpha) = \frac{2J_1(\alpha)}{\alpha J_0(\alpha)}, \quad (2)$$

51 where  $\mu$  is the gas viscosity,  $\kappa$  the thermal conductivity,  $C_p$  the specific heat with constant  
 52 pressure (see Table I), and where  $J_0$  and  $J_1$  are zeroth- and first-order Bessel functions.

53 Although it has been derived in the context of cylindrical geometries, model (ZK) has been  
 54 used intensively for varying geometries, namely for  $S$  depending on the longitudinal variable  
 55  $x$ . The classical horn equations describing plane wave propagation in an axisymmetric  
 56 lossless pipe can be retrieved from an asymptotic analysis relying on Euler's equations in a  
 57 pipe with varying section (Rienstra, 2005). Model (ZK) can be seen as a perturbation of  
 58 these horn equations, and has been employed for dissipative pipes with varying section for  
 59 instance in (Chaigne and Kergomard, 2016), (Bilbao and Harrison, 2016), (Tournemenne

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Sound velocity:  $c = 331.45\sqrt{T/T_0}$  m s<sup>-1</sup>

Density:  $\rho = 1.2929 T_0/T$  kg m<sup>-3</sup>

Viscosity:  $\mu = 1.708 \times 10^{-5}(1 + 0.0029 t)$  kg m<sup>-1</sup>s<sup>-1</sup>

Thermal conductivity:  $\kappa = 5.77 \times 10^{-3}(1 + 0.0033 t)$  Cal/(ms °C)

Spec. heat with constant p.:  $C_p = 240$  Cal/(kg °C)

Ratio of specific heats:  $\gamma = 1.402$

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TABLE I. Numerical values (Chaigne and Kergomard, 2016) of air constants used in the model.  $t$  is the temperature in Celsius, and  $T$  the absolute temperature with  $T_0 = 273.15$ K.

60 and Chabassier, 2019) in the harmonic regime. Curvature of the wave fronts can occur  
 61 in varying geometries and especially in the instrument bell, which can be modeled by an  
 62 equation similar to (1) (Hélie *et al.*, 2013). The present work focuses on viscothermal effects,  
 63 and will neglect the curvature effects which may be included in future work.

64 The highly nonlinear dependency of  $F_v$  and  $F_\theta$  with  $\omega$  induces a nonlocal formulation  
 65 in the time domain leading to mathematical and numerical intricacy, which motivates the  
 66 present work.



### 67 C. Model approximations and time domain representations

68 Model (ZK) has been approximated for different frequency regimes and/or pipe sizes,  
 69 especially when the Stokes number  $s = R\sqrt{\rho\omega/\mu}$  is large (Tijdeman, 1975), (Keefe, 1984),  
 70 (Stinson, 1991), (Scheichl, 2004). Terms in  $\sqrt{j\omega}$  arise in the derived equations, leading in the  
 71 time domain to fractional derivatives. Note that similar terms arise in the Webster-Lokshin  
 72 1D model which models the acoustic pressure close to the boundary layers using Cremer  
 73 3D effective wall impedance (Hélie *et al.*, 2013). These terms can be treated numerically  
 74 with approximations of diffusive representations (Hélie and Matignon, 2006), (Benjamin  
 75 *et al.*, 2017). Other approaches are based on direct diffusive representations of model (ZK)  
 76 (Thompson *et al.*, 2014), (Bilbao and Harrison, 2016), (Schmutzhard *et al.*, 2017).

### 77 D. Contribution and outline

78 One important aspect of space and time discretization of a model is to control the trade-off  
 79 between accuracy and efficiency, in all targeted applications and configurations.

80 In the context of computer aided instrument prototyping, our purpose is to ensure numer-  
 81 ical stability and quantify accuracy with respect to model (ZK). Energy-based methods have

82 proven especially efficient to discretize wave equations in time (Van Der Schaft, 2006),(Bil-  
83 bao, 2009), (Cohen, 2013), (Hélie and Silva, 2017), (Chabassier *et al.*, 2020) (Chatziioannou,  
84 2019). The model of (Bilbao and Harrison, 2016), which satisfies an energy balance identity,  
85 is particularly suitable for physics-based sound synthesis. However, synthesizing sounds of  
86 a specific instrument with this model requires to run an optimization algorithm for every  
87 different value taken by the pipe radius and temperature before running time iterations. The  
88 present work makes this optimization step geometry-independent, by re-writing the model of  
89 (Bilbao and Harrison, 2016) with a new expression of the coefficients. Optimized constants  
90 are given in Table II and are usable directly in synthesis algorithms. The new coefficients  
91 of the model explicitly depend on the non-constant geometrical and physical parameters,  
92 therefore no optimization must be implemented nor launched before time iterations.

93 The model, presented in Sec. II along with its energy balance, is suitable for variational  
94 approximation, therefore naturally compatible for coupling with other energy-based models.  
95 Its mathematical derivation is developed in Sec. III and a practical implementation method  
96 is proposed in Sec. IV along with a space and time discretization which guarantees a discrete  
97 energy identity and an algorithmic strategy for explicit update of the unknowns. A numer-  
98 ical scheme using 1D mixed spectral finite elements is proposed, allowing high-order space

99 discretization leading to arbitrary accuracy. A comparison with other models is proposed in  
100 Sec. V.

101 Boundary conditions model the way waves are introduced at one boundary, or how they  
102 radiate in exterior air at the bell (Rabiner and Schafer, 1978), (Dalmont *et al.*, 2001), (Silva  
103 *et al.*, 2009), (Monteghetti *et al.*, 2018). Their energy-based time domain formulation can be  
104 intricate, and since it is not the scope of the present work, they are considered elementary  
105 (closed pipe, induced flow, or open pipe), and abusively omitted when non necessary.

## 106 II. MAIN RESULT

The acoustic pressure averaged on a pipe section and the volume flow can be modeled by the solutions  $p(x, t)$  and  $v(x, t)$  to the following system for  $x \in ]0, L[$ ,  $t > 0$ , where  $N$  is an

integer and  $v_i(x, t)$ ,  $p_0(x, t)$ ,  $p_i(x, t)$  are  $2N + 1$  auxiliary variables:

$$\left\{ \begin{array}{l} \frac{\rho}{S} \frac{\partial v}{\partial t} + R_0 v + \sum_{i=1}^N R_i (v - v_i) + \frac{\partial p}{\partial x} = 0, \end{array} \right. \quad (3a)$$

$$\left\{ \begin{array}{l} \frac{S}{\rho c^2} \frac{\partial p}{\partial t} + G_0 (p - p_0) + \sum_{i=1}^N G_i (p - p_0 - p_i) \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \frac{\partial v}{\partial x} = 0, \end{array} \right. \quad (3b)$$

$$\left\{ \begin{array}{l} L_i \frac{dv_i}{dt} = R_i (v - v_i), \text{ for } 1 \leq i \leq N, \end{array} \right. \quad (3c)$$

$$\left\{ \begin{array}{l} C_0 \frac{dp_0}{dt} = G_0 (p - p_0) \\ \quad \quad \quad \quad \quad + \sum_{i=1}^N G_i (p - p_0 - p_i), \end{array} \right. \quad (3d)$$

$$\left\{ \begin{array}{l} C_i \frac{dp_i}{dt} = G_i (p - p_0 - p_i) \text{ for } 1 \leq i \leq N. \end{array} \right. \quad (3e)$$

The coefficients of this system are defined as

$$\left\{ \begin{array}{l} R_0(x) = \frac{\pi \mu}{S(x)^2} a_0, \end{array} \right. \quad (4a)$$

$$\left\{ \begin{array}{l} L_i(x) = \frac{\rho}{S(x)} a_i, \quad R_i(x) = \frac{\pi \mu}{S(x)^2} \frac{a_i}{b_i}, \end{array} \right. \quad (4b)$$

$$\left\{ \begin{array}{l} C_0(x) = \frac{S(x)(\gamma - 1)}{\rho c^2}, \quad G_0 = \frac{\pi \kappa (\gamma - 1)}{\rho^2 c^2 C_P} a_0, \end{array} \right. \quad (4c)$$

$$\left\{ \begin{array}{l} C_i(x) = \frac{S(x)(\gamma - 1)}{\rho c^2} a_i, \quad G_i = \frac{\pi \kappa (\gamma - 1)}{\rho^2 c^2 C_P} \frac{a_i}{b_i}, \end{array} \right. \quad (4d)$$

107 where the coefficients  $a_i$  and  $b_i$  are dimensionless constants obtained from an optimization  
 108 procedure described in Section III, see Table II for  $N \in \{2, 4, 8\}$  and supplementary material  
 109 `SuppPub1.txt` up to  $N=16$ <sup>1</sup>. We expect that increasing the value of  $N$  will make the solu-  
 110 tion closer to the original model (1), although it will also increase the number of auxiliary  
 111 unknowns. The form of (3) is the same as the system used in (Bilbao and Harrison, 2016),  
 112 but the values of the coefficients change, because the variable  $v$  represents here the acoustic

113 flow. To conform to their notations, our formulas for  $R_i$  and  $L_i$  should be multiplied by  $S$ ,  
 114 and  $C_i$  and  $G_i$  should be divided by  $S$ .

115

116 This model is shown in Section III to satisfy the following energy balance identity

$$\frac{d}{dt}\mathcal{E} = -Q - p(L)v(L) + p(0)v(0), \quad (5)$$

$$\mathcal{E} = \frac{1}{2} \int_{\Omega} \left[ \frac{\rho}{S} v^2 + \frac{S}{\rho c^2} p^2 + C_0 p_0^2 + \sum_{i=1}^N C_i p_i^2 + \sum_{i=1}^N L_i v_i^2 \right], \quad (6)$$

$$Q = \int_{\Omega} \left[ R_0 v^2 + \sum_{i=1}^N R_i (v - v_i)^2 + G_0 (p - p_0)^2 + \sum_{i=1}^N G_i (p - p_0 - p_i)^2 \right] \geq 0 \quad (7)$$

117 The model hence dissipates energy through the negative term  $-Q$ , and can exchange work  
 118 with other systems via its boundaries  $x = 0$  and  $x = L$ .

### 119 III. MATHEMATICAL DERIVATION

System (1) can be written as

$$\left\{ \begin{array}{l} \frac{d\hat{p}}{dx} + \frac{\rho}{S} \left[ j\omega + \frac{G(\tau_v \omega)}{\tau_v} \right] \hat{v} = 0, \end{array} \right. \quad (8a)$$

$$\left\{ \begin{array}{l} \frac{d\hat{v}}{dx} + \frac{S}{\rho c^2} \left[ j\omega + \frac{\gamma - 1}{\frac{1}{j\omega} + \frac{\tau_{\theta}}{G(\tau_{\theta} \omega)}} \right] \hat{p} = 0, \end{array} \right. \quad (8b)$$

$i$	$a_i$	$b_i$
0	8	
1	$1.023\,15 \times 10^{-1}$	$1.031\,48 \times 10^{-3}$
2	$6.452\,52 \times 10^{-3}$	$4.096\,97 \times 10^{-6}$
Coefficients for $N = 2$ (17 it.)		
$i$	$a_i$	$b_i$
0	8	
1	$2.101\,57 \times 10^{-1}$	$1.046\,29 \times 10^{-2}$
2	$4.075\,43 \times 10^{-2}$	$4.020\,92 \times 10^{-4}$
3	$8.148\,25 \times 10^{-3}$	$1.622\,09 \times 10^{-5}$
4	$1.961\,59 \times 10^{-3}$	$5.688\,60 \times 10^{-7}$
Coefficients for $N = 4$ (23 it.)		
$i$	$a_i$	$b_i$
0	8	
1	$1.864\,11 \times 10^{-1}$	$3.168\,42 \times 10^{-2}$
2	$8.063\,38 \times 10^{-2}$	$5.883\,91 \times 10^{-3}$
3	$3.520\,99 \times 10^{-2}$	$1.112\,01 \times 10^{-3}$
4	$1.533\,51 \times 10^{-2}$	$2.116\,66 \times 10^{-4}$
5	$6.695\,83 \times 10^{-3}$	$4.045\,03 \times 10^{-5}$
6	$2.932\,51 \times 10^{-3}$	$7.735\,96 \times 10^{-6}$
7	$1.328\,25 \times 10^{-3}$	$1.444\,92 \times 10^{-6}$
8	$9.403\,66 \times 10^{-4}$	$1.483\,83 \times 10^{-7}$
Coefficients for $N = 8$ (114 it.)		

TABLE II. Coefficients  $(a_i, b_i)$ , optimized for  $M = 100$  values of  $\zeta$  ranging from 8 to  $2 \times 10^6$ , covering radii of  $1 \times 10^{-3}$  m to 0.1 m and frequencies of 20 Hz to  $2 \times 10^4$  Hz. In parentheses: number of iterations to reach the stagnation threshold  $10^{-8}$ .

120 where the characteristic viscous and thermal times are given by

$$\tau_v(x) := \frac{R(x)^2 \rho}{\mu} \text{ and } \tau_\theta(x) := \frac{R(x)^2 \rho C_P}{\kappa}, \quad (9)$$

121 and the loss coefficients depend on

$$G(\zeta) := \frac{j\zeta \phi(\sqrt{-j\zeta})}{1 - \phi(\sqrt{-j\zeta})}, \quad \forall \zeta \in \mathbb{R}, \quad (10)$$

122 where  $\phi$  is given by (2). The nondimensionalized frequency  $\zeta$  in the viscous (resp. thermal)  
 123 term will take values equal (resp. close to) the squared Stokes number  $s^2$ . Its range of  
 124 interest, for applications in musical acoustics, lies approximately between  $\zeta_{\min} = 8$  (low  
 125 frequency, thin pipes) and  $\zeta_{\max} = 2 \times 10^6$  (high frequency, wide pipes). This corresponds to  
 126 frequencies of 20 Hz to 20 kHz and radii of 1 mm to 100 mm, subject to  $\omega R \ll c$  in order for  
 127 model (1) to be valid. Function  $G$  tends to the constant value 8 as  $\zeta$  tends to zero, while for  
 128 large  $\zeta$ , it is asymptotically equivalent to  $2\sqrt{j\zeta}$ . Replacing  $G$  by its asymptotic expansion  
 129 as  $\zeta \rightarrow \infty$  leads to fractional operators in time coming from the term  $\sqrt{j\omega}$ , which can be  
 130 dealt with in the time domain using diffusive representations (Hélie and Matignon, 2006).

131 Instead, in the same spirit as (Bilbao and Harrison, 2016), function  $G$  is approximated  
 132 in the whole range of interest with a family of functions  $G^N$  of the following form

$$G^N(\zeta) = a_0 + \sum_{i=1}^N \frac{a_i j\zeta}{b_i j\zeta + 1}, \quad (11)$$

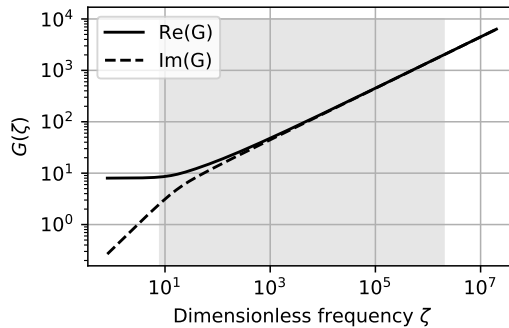


FIG. 1. Dimensionless loss coefficient  $G(\zeta)$ , w.r.t. dimensionless frequency  $\zeta$ . The range of interest  $[\zeta_{\min}, \zeta_{\max}]$  is highlighted in gray.

133 where  $(a_i, b_i)_{i=1\dots N}$  are real coefficients, that must be positive in order to ensure the passivity  
 134 of the resulting time domain model. In contrast to what is used in (Bilbao and Harrison,  
 135 2016; Schmutzhard *et al.*, 2017), given a number of oscillators  $N$ , a single optimization must  
 136 be performed to approximate  $G$  once and for all, rather than one optimization for every  
 137 possible radius.

### 138 A. Optimization of the model coefficients

139 The objective function is chosen to be

$$E = \sum_{k=1}^M \left| \frac{G^N(\zeta_k)}{G(\zeta_k)} - 1 \right|^2, \quad (12)$$



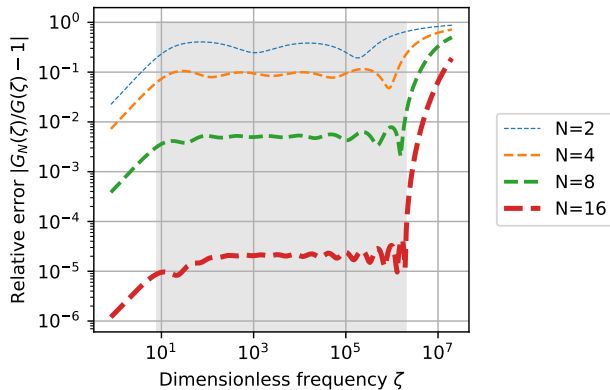


FIG. 2. Relative error between  $G$  and  $G^N$ , as a function of dimensionless frequency  $\zeta$ , for several values of  $N$ . The range of interest  $[\zeta_{\min}, \zeta_{\max}]$  is highlighted in gray.

140 where  $\zeta_k$  are  $M = 100$  exponentially spaced values, spanning the range of interest  $[\zeta_{\min}, \zeta_{\max}]$ .

142

143 The positivity of  $(a_i, b_i)_{i=1\dots N}$  is enforced by expressing them as  $(\exp(a'_i), \exp(b'_i))$ . This  
 144 reparametrization warrants the use of unconstrained optimization algorithms, and is suitable  
 145 to control the model's behavior on frequencies spanning several orders of magnitude. The  
 146 BFGS algorithm (Nocedal and Wright, 2006) is then used to find a minimizer of function  
 147  $E((a'_i, b'_i)_{i=1\dots N})$ . Table II gives values of coefficients  $(a_i, b_i)$  that minimize this  $\ell^2$  error, for  
 148 different values of  $N$ . Note that these coefficients do not depend on the geometry of the  
 149 instrument, and can be readily used as given. The choice of  $N$  is a trade-off between the

150 better precision enabled by using more additional variables, and the higher computing power  
 151 required for simulation (see Sec. [IV D](#)).

## 152 B. Resulting family of models in the frequency domain

153 Replacing  $G$  with  $G^N$  yields the following family of approximate models:

$$\begin{cases} \frac{d\hat{p}}{dx} + \frac{\rho}{S} \left[ j\omega + \frac{a_0}{\tau_v} + \sum_{i=1}^N \frac{a_i j\omega}{b_i \tau_v j\omega + 1} \right] \hat{v} = 0, \\ \frac{d\hat{v}}{dx} + \frac{S}{\rho c^2} \left[ j\omega + \frac{\gamma - 1}{\frac{1}{j\omega} + \frac{1}{\frac{a_0}{\tau_\theta} + \sum_{i=1}^N \frac{a_i j\omega}{b_i \tau_\theta j\omega + 1}}} \right] \hat{p} = 0, \end{cases}$$

154 These equations can be written and represented using an equivalent electronic circuit with  
 155 Foster structure at each abscissa  $x$ , as is done in ([Bilbao and Harrison, 2016](#)), by defining  
 156 the coefficients  $R_i$ ,  $L_i$ ,  $C_i$  and  $G_i$  as in equations (4). Such a representation is useful, not  
 157 only to obtain an explicit choice of state-space representation for the rational functions in  
 158 (13), but also for the derivation of the energy balance.

159

### C. Resulting family of models in the time domain

This family of models can be written in the time domain as follows

$$\begin{cases}
 \frac{\rho}{S} \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} + \Delta = 0, & (14a) \\
 \frac{S}{\rho c^2} \frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} + m = 0, & (14b) \\
 \Delta = \Delta_0 + \sum_{i=1}^N \Delta_i, \quad m = m_0 + \sum_{i=1}^N m_i, & (14c) \\
 \Delta_0 = R_0 v, \quad m_0 = G_0 q_0, & (14d) \\
 \Delta_i = R_i w_i = L_i d_t v_i, \quad v = w_i + v_i, & (14e) \\
 m = C_0 d_t p_0, \quad p = p_0 + q_0, & (14f) \\
 m_i = C_i d_t p_i = G_i q_i, \quad q_0 = p_i + q_i. & (14g)
 \end{cases}$$

160 Eliminating the unknowns  $\Delta$ ,  $m$ ,  $w_i$  and  $q_i$  directly leads to (3), where only  $2N + 1$   
 161 auxiliary variables are necessary.

162

### D. Energy balance

The energy identity comes from the circuit representation (Bilbao and Harrison, 2016), and the associated equations (14). Let us multiply the first equation with  $v$  and the second equation with  $p$ , and integrate both in space over  $[0, L]$ . The terms  $v\Delta$  and  $pm$  that arise can be interpreted as the power brought into the electronic circuit. They can be written,

using the rest of system (14), as

$$\begin{cases} v\Delta = R_0 v^2 + \sum_{i=1}^N \left( R_i w_i^2 + \frac{L_i}{2} \frac{dv_i^2}{dt} \right) \\ p m = \frac{C_0}{2} \frac{dp_0^2}{dt} + G_0 q_0^2 + \sum_{i=1}^N \left( G_i q_i^2 + \frac{C_i}{2} \frac{dp_i^2}{dt} \right) \end{cases}$$

163 Eliminating the same unknowns as before establishes the energy balance identity (5) where  
 164 the energy is defined as (6) and the losses are given by (7).

#### 165 IV. PRACTICAL IMPLEMENTATION

166 Space and time discretization of system (3) can be done with various numerical methods  
 167 depending on the situation. In the context of sound synthesis, it is essential to design stable,  
 168 accurate and efficient numerical schemes that couple the pipe with other elements such as  
 169 the radiation at the bell, junctions with tone holes, or the embouchure, that can behave  
 170 nonlinearly.

171 The current article proposes to use one-dimensional finite elements in space followed by  
 172 an energy-consistent time discretization, in order to ensure numerical stability via an energy  
 173 technique and to provide a numerical method that will easily extend to the presence of  
 174 couplings. For the sake of simplicity, but without loss of generality, it is assumed that the  
 175 outwards pipe flow is equal to given values  $\lambda^-(t)$  and  $\lambda^+(t)$  respectively at the pipe entrance

176 and bell. In the presence of realistic coupling terms at the entrance and bell, these values  
 177 will be unknowns and will require additional equations to be evaluated.

#### 178 **A. One-dimensional finite elements for space discretization**

179 The finite element method (FEM) relies on a variational formulation (weak form) of  
 180 the entire system in usual infinite dimensional functional spaces ([Brezis, 2010](#)) to which  
 181 belong  $p$ ,  $v$ ,  $p_0$ ,  $p_i$  and  $v_i$ , followed by the definition of finite dimensional spaces in which we  
 182 seek numerically the approximate solutions  $p_h$ ,  $v_h$ ,  $p_{h,0}$ ,  $p_{h,i}$  and  $v_{h,i}$ . One possible choice,  
 183 called Mixed Spectral FEM, is described in ([Tournemenne and Chabassier, 2019](#)) and is  
 184 followed here. It consists in using as finite dimensional spaces the set of piecewise polynomial  
 185 functions of the spatial variable  $x$ , element by element, where jumps across element edges  
 186 are authorized for  $v_h$  and  $v_{h,i}$  but not for  $p_h$ ,  $p_{h,0}$  and  $p_{h,i}$ . These polynomial functions  
 187 are chosen as the Lagrange interpolation polynomials on the Gauss-Lobatto points of each  
 188 element. The order of the polynomial functions is called the order of the FEM and will  
 189 be noted  $r$  thereafter. Finally, the integral terms coming from the variational formulation

190 are evaluated using a quadrature formula based on the same Gauss-Lobatto points, so that  
 191 mass matrices are diagonal (Cohen, 2013).

192 This procedure results in the definition of matrices that replace all the spatial operators  
 193 in (3). The following “semi-discrete system” is obtained. Let  $V_h, V_{h,i}, P_h, P_{h,0}, P_{h,i}$  be the  
 194 vectors of coordinates of resp.  $v_h, v_{h,i}, p_h, p_{h,0}$  and  $p_{h,i}$  in the spanning basis of the finite  
 195 elements, and see (Tournemene and Chabassier, 2019) for the definition of the matrices  
 196  $M_h^V, M_h^P, B_h, E_h$ . The diagonal matrices  $R_{h,0}, R_{h,i}, G_{h,0}, G_{h,i}, C_{h,0}, C_{h,i}$  and  $L_{h,i}$ , are  
 197 obtained via a similar procedure to the diagonal mass matrices  $M_h^V$  and  $M_h^P$ .

$$\left\{ \begin{array}{l} M_h^V \frac{\partial V_h}{\partial t} + R_{h,0} V_h + \sum_{i=1}^N R_{h,i} (V_h - V_{h,i}) - B_h P_h = 0, \quad (16a) \\ M_h^P \frac{\partial P_h}{\partial t} + G_{h,0} (P_h - P_{h,0}) + \sum_{i=1}^N G_{h,i} (P_h - P_{h,0} - P_{h,i}) \\ \quad + B_h^* V_h + \lambda_+ E_h^+ + \lambda_- E_h^- = 0, \quad (16c) \\ L_{h,i} \frac{\partial V_{h,i}}{\partial t} = R_{h,i} (V_h - V_{h,i}), \quad \text{for } 1 \leq i \leq N, \quad (16d) \\ C_{h,0} \frac{\partial P_{h,0}}{\partial t} = G_{h,0} (P_h - P_{h,0}) \\ \quad + \sum_{i=1}^N G_{h,i} (P_h - P_{h,0} - P_{h,i}), \quad (16f) \\ C_{h,i} \frac{\partial P_{h,i}}{\partial t} = G_{h,i} (P_h - P_{h,0} - P_{h,i}), \quad \text{for } 1 \leq i \leq N. \quad (16g) \end{array} \right.$$

198 One advantage of this formulation is the natural treatment of the boundary conditions  $\lambda_{\pm}$ ,  
 199 which can become Lagrange multipliers for coupling with other systems. This semi-discrete

200 system satisfies an analogue of the previous energy identity (5) where the continuous spatial  
 201 norms are replaced with their semi discrete counterparts.

202 Note that the semi-discrete system (16) can be straightforwardly adapted for computation  
 203 in the harmonic regime. This leads to one totally discrete system per value of  $\omega$ , where the  
 204 operator  $\partial_t$  is replaced with a multiplication with  $j\omega$ , which requires a sparse matrix inversion  
 205 per  $\omega$ .

## 206 B. Energy consistent time discretization

207 For time discretization, an interleaved scheme similar to (Bilbao and Harrison, 2016) is  
 208 used. The time step is denoted  $\Delta t$ , and the step number  $n$ , so that  $t = n\Delta t$ . Unknowns  
 209  $P_h, P_{h,0}$  and  $P_{h,i}$  are evaluated at integer times  $\{0, 1, 2, \dots\}$ , whereas  $V_h$  and  $V_{h,i}$  are evaluated  
 210 on a staggered time grid  $\{1/2, 3/2, 5/2, \dots\}$ . Let us define the discrete operators  $\delta$  and  $\mu$  acting  
 211 on any vector sequence  $\{X^n\}_{n \in \mathbb{N}}$  as

$$\delta X^{n+\frac{1}{2}} = \frac{X^{n+1} - X^n}{\Delta t}, \quad \mu X^{n+\frac{1}{2}} = \frac{X^{n+1} + X^n}{2}. \quad (17)$$

Conversely, if  $\{Y^{n+\frac{1}{2}}\}_n$  is a sequence with indices on the staggered grid,  $\{\delta Y^n\}_n$  is defined at integer times. The proposed totally discrete scheme reads

$$\left\{ \begin{array}{l} M_h^V \delta V_h^n + R_{h,0} \mu V_h^n + \sum_{i=1}^N R_{h,i} \mu (V_h - V_{h,i})^n \\ \qquad \qquad \qquad - B_h P_h^n = 0, \end{array} \right. \quad (18a)$$

$$\left\{ \begin{array}{l} M_h^P \delta P_h^{n+\frac{1}{2}} + G_{h,0} \mu (P_h - P_{h,0})^{n+\frac{1}{2}} \\ + \sum_{i=1}^N G_{h,i} \mu (P_h - P_{h,0} - P_{h,i})^{n+\frac{1}{2}} + B_h^* V_h^{n+\frac{1}{2}} \\ \qquad \qquad \qquad + \lambda_+^{n+\frac{1}{2}} E_h^+ + \lambda_-^{n+\frac{1}{2}} E_h^- = 0, \end{array} \right. \quad (18b)$$

$$L_{h,i} \delta V_{h,i}^n = R_{h,i} \mu (V_h - V_{h,i})^n, \quad \forall i, \quad (18c)$$

$$\left\{ \begin{array}{l} C_{h,0} \delta P_{h,0}^{n+\frac{1}{2}} = G_{h,0} \mu (P_h - P_{h,0})^{n+\frac{1}{2}} \\ \qquad \qquad \qquad + \sum_{i=1}^N G_{h,i} \mu (P_h - P_{h,0} - P_{h,i})^{n+\frac{1}{2}}, \end{array} \right. \quad (18d)$$

$$\left\{ \begin{array}{l} C_{h,i} \delta P_{h,i}^{n+\frac{1}{2}} = G_{h,i} \mu (P_h - P_{h,0} - P_{h,i})^{n+\frac{1}{2}}, \quad \forall i. \end{array} \right. \quad (18e)$$

212 This system can be interpreted as an interleaved leap-frog scheme for pressure and flow,  
 213 combined with an implicit midpoint resolution of the electronic circuit identities. It satisfies  
 214 the following discrete equivalent of the energy balance (5) (see Appendix):

$$\delta \mathcal{E}_h^{n+\frac{1}{2}} = - \left[ \mu Q_{h,\text{visc}}^{n+\frac{1}{2}} + Q_{h,\text{therm}}^{n+\frac{1}{2}} \right] + \mathcal{S}_h^{n+\frac{1}{2}} \quad (19)$$

215 where the discrete energy is defined as

$$\mathcal{E}_h^n = \mathcal{E}_{h,V}^n + \mathcal{E}_{h,P}^n + \mu \mathcal{E}_{h,\text{visc}}^n + \mathcal{E}_{h,\text{therm}}^n + \mathfrak{e}_h^n. \quad (20)$$



The different terms of energy are defined as

$$\begin{aligned}
 \mathcal{E}_{h,V}^n &:= \frac{1}{2} \|\mu V_h^n\|_{M_h^V}^2, & \mathcal{E}_{h,P}^n &:= \frac{1}{2} \|P_h^n\|_{\widetilde{M}_h^P}^2 \\
 \mathcal{E}_{h,\text{visc}}^{n+\frac{1}{2}} &:= \sum_{i=1}^N \frac{1}{2} \|V_{h,i}^{n+\frac{1}{2}}\|_{L_{h,i}}^2 \\
 \\
 \mathcal{E}_{h,\text{therm}}^n &:= \frac{1}{2} \|P_{h,0}^n\|_{C_{h,0}}^2 + \sum_{i=1}^N \frac{1}{2} \|P_{h,i}^n\|_{C_{h,i}}^2 \\
 \mathfrak{e}_h^n &:= \frac{\Delta t^2}{8} \left\| \mu V_h^n + \sum_{i=1}^N R_{h,0}^{-1} R_{h,i} \mu (V_h - V_{h,i})^n \right\|_{\widetilde{R}_{h,0}}^2 \\
 Q_{h,\text{visc}}^n &:= \|\mu V_{h,0}^n\|_{R_{h,0}}^2 + \sum_{i=1}^N \|\mu (V_h - V_{h,i})^n\|_{R_{h,i}}^2 \\
 \\
 Q_{h,\text{therm}}^{n+\frac{1}{2}} &:= \left\| \mu (P_h - P_{h,0})^{n+\frac{1}{2}} \right\|_{G_{h,0}}^2 \\
 &\quad + \sum_{i=1}^N \left\| \mu (P_h - P_{h,0} - P_{h,i})^{n+\frac{1}{2}} \right\|_{G_{h,i}}^2 \\
 S_h^{n+\frac{1}{2}} &:= -\lambda_+^{n+\frac{1}{2}} (E_h^+)^* \mu P_h^{n+\frac{1}{2}} - \lambda_-^{n+\frac{1}{2}} (E_h^-)^* \mu P_h^{n+\frac{1}{2}}
 \end{aligned}$$

where

$$\widetilde{R}_{h,0} = R_{h,0} (M_h^V)^{-1} R_{h,0},$$

216 and for any time series of vectors  $\{X^n\}_n$ ,  $\{Y^n\}_n$ , their scalar product is defined as

217  $(X^n, Y^m) = \sum_k X_k^n Y_k^m$  and the weighted norm, for any non negative matrix A, is  $\|X^n\|_A =$

218  $(AX^n, X^n)^{1/2}$ . This energy is positive as soon as the modified mass matrix  $\widetilde{M}_h^P$ , defined as

219  $\widetilde{M}_h^P = M_h^P - \frac{\Delta t^2}{4} B_h^* (M_h^V)^{-1} B_h$  is a positive quadratic form. This leads to the following

220 stability condition:

$$\Delta t \leq 2 \left[ \rho \left( (M_h^P)^{-1} B_h^* (M_h^V)^{-1} B_h \right) \right]^{-\frac{1}{2}} \quad (21)$$

221 which is the same condition as for solving the lossless wave equation with the classical leap-  
 222 frog scheme (Chabassier and Imperiale, 2013). Showing the convergence of the discrete  
 223 scheme, as  $\Delta t$  and  $h$  tend to zero, is out of the scope of the present work but it is expected  
 224 that the dissipation of a positive energy will enable such a result (Chabassier and Imperiale,  
 225 2017).

### 226 C. Explicit update of the unknowns

227 One of the advantages of the of spectral finite elements is that  $M_h^P$ ,  $M_h^V$ ,  $L_{h,i}$ ,  $C_{h,0}$  and  $C_{h,i}$   
 228 are diagonal matrices, making them trivial to invert. Thanks to an algorithmic elimination  
 229 strategy, which can also be interpreted as a Schur complement, an explicit update of the  
 230 unknowns is possible and is given in Appendix B. This approach generalizes the one given  
 231 in Table I of (Bilbao and Harrison, 2016) for finite differences, to finite elements of arbitrary  
 232 order in matrix notations.

233 **D. Accuracy and efficiency**

234 Model (18) is applied to a simplified natural trumpet of total length 1.335 m. The pipe  
 235 comprises two parts: a cylinder, followed by a “Bessel horn”:

$$R(x) = \begin{cases} 6 \times 10^{-3} \text{ m} & \text{if } 0 < x < 0.716, \\ \gamma(x - x_P)^{-\alpha} & \text{if } 0.716 < x < 1.335, \end{cases}$$

236 where  $\alpha = 0.7$  is the parameter of the Bessel horn. Constants  $\gamma = 4.40437 \times 10^{-3} \text{ m}$  and  
 237  $x_P = 1.35897 \text{ m}$  are chosen such that the radius is continuous at the junction, and that the  
 238 bell radius is 60 mm, see Fig. 3. For simplicity, we assume a zero-pressure condition at the  
 239 bell.

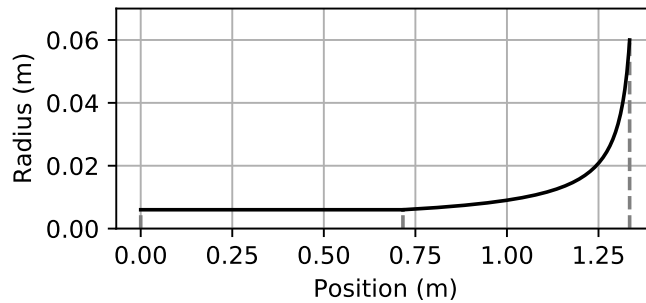


FIG. 3. The simplified natural trumpet : radius w.r.t. position along bore

240 An impulse response is computed by setting the input boundary term to a prescribed  
 241 flow  $-\lambda_-^{n+\frac{1}{2}} = v_0 \left( (n + \frac{1}{2}) \Delta t \right)$ , with

$$v_0(t) = \begin{cases} \frac{8V_0}{3t_1} \sin^4 \left( \pi \frac{t}{t_1} \right) & \text{if } 0 < t < t_1, \\ 0 & \text{otherwise,} \end{cases} \quad (22)$$

242 where the duration of the impulse is  $t_1 = 4 \times 10^{-4}$  s and the total injected volume is  
 243  $V_0 = 1 \times 10^{-7}$  m<sup>3</sup>. At the bell an elementary open condition  $p = 0$  is applied: the gen-  
 244 eralization to more realistic radiation impedances is a natural possible extension of the  
 245 present work.

246 The numerical scheme (18) is used to compute the impulse response at the temperature  
 247  $t = 20$  °C with 34 elements of order 10 and a time step  $\Delta t = 3.185 \times 10^{-6}$  s, which is the  
 248 largest value satisfying the stability condition (21) . The final time is set to  $T = 0.2$  s.

249 Fig. 4 displays the computed impulse response, i.e. the evolution of the pressure at the  
 250 entrance of the simplified trumpet with respect to time, for  $N \in \{2, 4, 8, 16\}$ , between 0 s  
 251 and 0.2 s (top), 0.13 s and 0.18 s (bottom). The main reflections from the pipe ends can be  
 252 observed, as well as partial reflections coming from the radius variation in the Bessel part of  
 253 the pipe. Despite the lack of dissipation at the pipe ends, the impulse response is decaying

254 due to the viscothermal losses. As  $N$  increases, the numerical solutions change until they  
 255 cannot be visually distinguished between  $N = 8$  and  $N = 16$ .

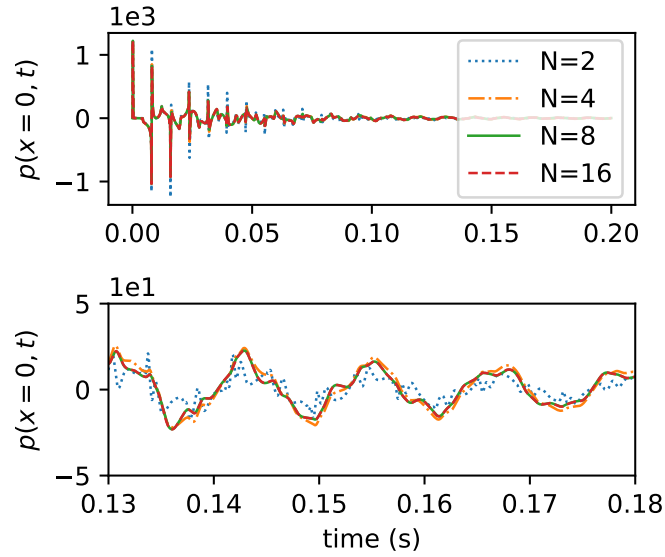


FIG. 4. Impulse response: evolution of the pressure at the entrance of the simplified trumpet with respect to time, for  $N \in \{2, 4, 8, 16\}$ , between 0s and 0.2s (top), 0.13s and 0.18s (bottom). The curves for  $N = 8$  and  $N = 16$  cannot be visually distinguished.

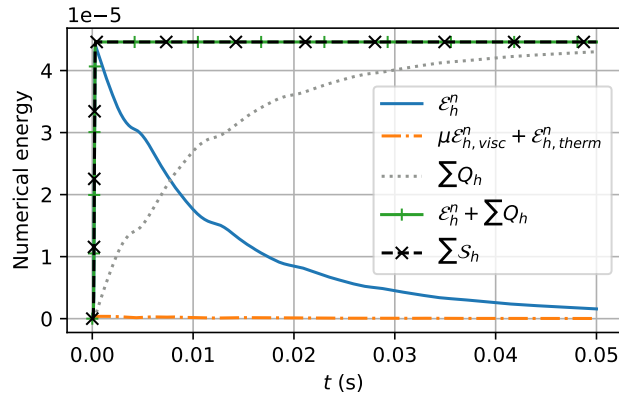


FIG. 5. Energy distribution w.r.t. time

256 Fig 5 displays the energy distribution with respect to time, according to the definitions  
 257 given in Sec. IV B. The total energy in the pipe, brought by the impulse input flow, is globally  
 258 decaying after  $t_1$ , and is temporarily stored in the auxiliary variables through the viscous  
 259 and thermal energy terms. The energy identity (19) is satisfied up to machine precision as  
 260 illustrated in Fig. 6.

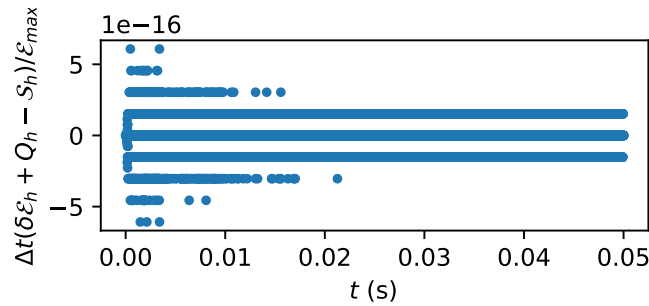


FIG. 6. Energy relative deviation w.r.t. time

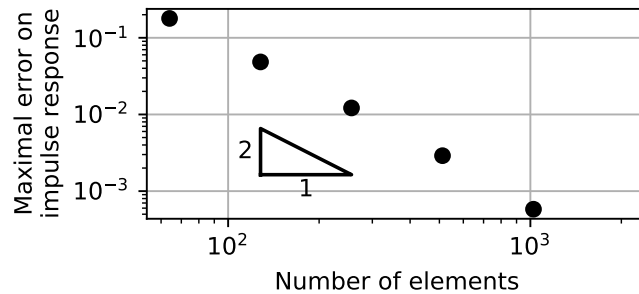


FIG. 7. Consecutive relative  $L^\infty([0, T], x = 0)$  error w.r.t. the number of finite elements of order  
 10.

261 Finally Fig. 7 shows the consecutive relative  $L^\infty$ -error in time of the impulse response  
 262 with respect to the number of finite elements. As the spatial discretization of the pipe  
 263 is refined, the time step is decreased by choosing the largest possible value given by the  
 264 stability condition (21). The obtained impulse response is interpolated on a fixed time grid  
 265 and compared. The resulting space-time convergence curve displays a second order rate of  
 266 convergence.

## 267 V. COMPARISON WITH OTHER MODELS

268 Model (3) is compared with several models of the literature, in time domain and in  
 269 frequency domain. The comparison includes two reduced 1D models: model (ZK) (1) and  
 270 Webster-Loskhin model (WL) developed in (Hélie *et al.*, 2013), without the hypothesis of  
 271 curved wavefronts, which is out of the scope of the present article. They are compared to  
 272 3D linear acoustic equations in the air column, associated with effective wall impedance  
 273 boundary conditions of two types: from (Cremer, 1948) with an incident angle of  $\pi/2$  noted  
 274 (Cr), and from (Berggren *et al.*, 2018) noted (BBN). All 3D solutions are obtained using  
 275 finite elements for the spatial discretization using the software Montjoie<sup>2</sup>, in an axisymmetric  
 276 configuration. The numerical solution relies on a curved mesh of the simplified natural

277 trumpet. The input boundary condition is a constant pressure set at the extremity of an  
 278 cylindrical adaptation part, and the output boundary condition is a zero acoustic pressure  
 279 on a plane surface which closes the pipe. The reader is referred to (Thibault and Chabassier,  
 280 2020) for a discussion of the domain of validity of each model, a detailed description of the  
 281 3D simulations (Sec. 3) and the implementation of model (WL) (Sec. 4.4). All 1D solutions  
 282 are obtained using the software OpenWiND<sup>3</sup>.

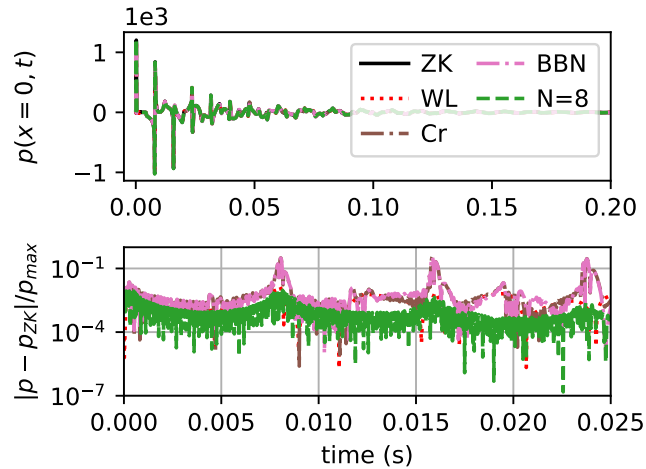


FIG. 8. Impulse response: evolution of the pressure at the entrance of the simplified trumpet with respect to time, for different models : ZK, WL, scheme (18) with  $N=8$ , Cr and BBN, between 0s and 0.2s (top). Relative difference of each impulse response with ZK between 0s and 0.025s (bottom).

283 A first comparison concerns the impulse response obtained with each model. Models  
 284 (ZK), (WL), (Cr) and (BBN) are all formulated in the frequency domain, and are suited



285 to calculating an input impedance  $Z_{in}(\omega)$  where  $\omega = 2\pi f$  is the dual variable to the time  
 286 variable  $t$ . The response is calculated by taking the fast Fourier transform of the input signal  
 287 (22), multiplying at each frequency by the impedance, and applying the inverse fast Fourier  
 288 transform to the result. The sampling frequency is chosen to be 52 747.2 Hz, so that the error  
 289 on the input signal due to spectral folding is less than  $1 \times 10^{-5}$ . The computation is done  
 290 with  $n = 52747$  samples (1.0 s), and requires to evaluate the impedance of each model at  
 291 26373 frequencies (negative frequencies are obtained by conjugate symmetry). The resulting  
 292 signals are displayed in Fig. 8. They decay below  $2 \times 10^{-6}$  of their maximal value after 0.9 s,  
 293 meaning the error due to periodization is negligible.

294 A second comparison is done on the input impedance. System (16) is solved, where the  
 295 operator  $\partial_t$  has been replaced with  $j\omega$ , for  $\omega \in \{\omega_j\}_{1 \leq j \leq M}$ ,  $M$  linearly spaced pulsations  
 296 in the range  $[\omega^-, \omega^+]$ . Model (3) is first solved with  $N = 2, 4, 8, 16$  and compared with the  
 297 other models. Fig 9 displays a focus around the second impedance peak of the simplified  
 298 natural trumpet. As  $N$  increases, the second impedance peak shifts in frequency and in  
 299 quality factor, and reaches values that can not be visually distinguished from model (ZK)  
 300 for  $N = 8$  and  $N = 16$ . Models (Cr) and (BBN) are very close to the 1D models, (Cr) being

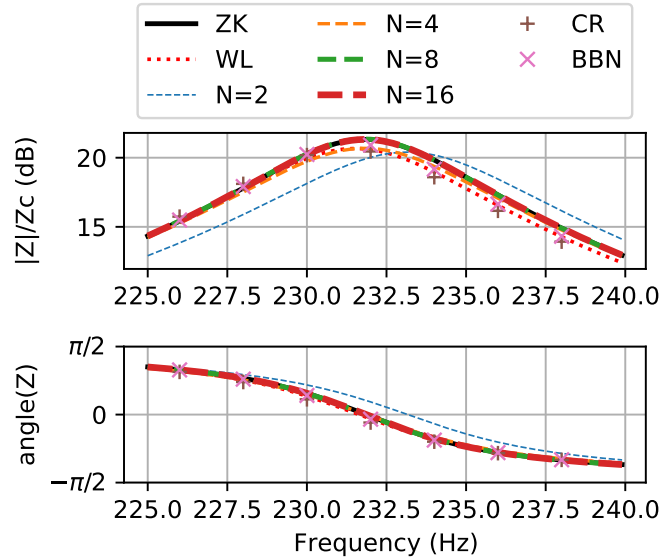


FIG. 9. Amplitude (top) and angle (bottom) of the input impedance computed of the simplified natural trumpet with model (3) for  $N = 2, 4, 8, 16$  (dashed lines of increasing width), compared with modes (ZK) (thick black line), (Cr) (plus marker) and (BBN) (x marker). Focus around the second impedance peak.

301 closer to (WL) which indeed relies on Cremer wall impedances, and (BBN) to (ZK). It can  
 302 be noted that the difference in resonance frequency (up to 1% i.e. 17 cents) between models  
 303 is of the same order as the differential pitch sensitivity of the ear (around 4 cents) (Micheyl  
 304 *et al.*, 2006).

305 Fig. 10 quantitatively compares the models. With model (ZK) taken as a reference, the  
 306 largest error of the other models on the impulse response is calculated. This error amounts  
 307 to 1.5% for (WL). For the numerical schemes, the discrepancy results from two kinds of

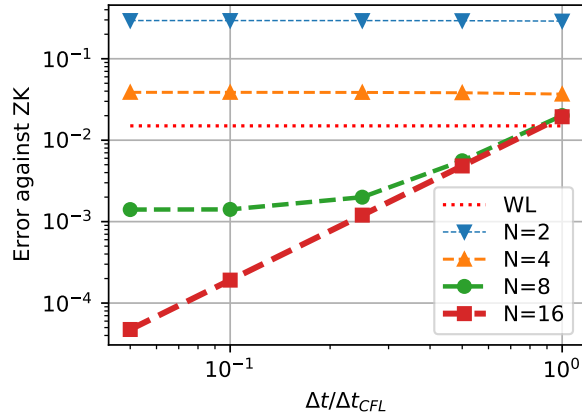


FIG. 10. Relative  $L^\infty$  error on the impulse response computed with  $N \in 2, 4, 8, 16$  (markers) and reconstructed from WL (no marker) with respect to ZK model.

308 error: model error, and discretization error. The former is due to approximation (11) and  
 309 can be reduced by increasing  $N$ ; the latter is due to the numerical scheme (18) and can  
 310 be reduced by decreasing the time step  $\Delta t$ , as done in Fig 10. For  $N = 8$  the model error  
 311 compared to (ZK) is observed to be about 0.14%, and for  $N = 16$  it is less than 0.02%.

## 312 VI. CONCLUSIONS AND PROSPECTS

313 This work presents a 1D model for viscothermal wave propagation suitable for time-  
 314 domain simulation, which makes use of an adjustable number of auxiliary unknowns. It is  
 315 derived from an approximation of the loss coefficients of the Zwicker–Kosten model with  
 316 rational functions over the whole range of frequencies and radii of musical acoustics. The

317 proposed improvement upon previous work is that the coefficient optimization procedure  
318 has been done only once, as the objective function does not depend on temperature or  
319 pipe radius. It leads to a closed-form for the resulting model, where the user only needs  
320 to specify the instrument geometry and physical constants. Numerical comparison with  
321 3D models show that the approximation error is smaller than the discrepancies between  
322 different models. The use of auxiliary variables induces a numerical burden which was to  
323 be expected for accounting for viscothermal effects. The model satisfies an energy identity  
324 and is therefore suitable for time-domain coupling with other models as sound radiation, or  
325 reed evolution, which are a possible extension of this work. This work could be extended to  
326 include additional forms of acoustic losses, such as wall admittance due to porosity, or to  
327 better justify its use in tubes with variable cross-sectional area.

328 **APPENDIX A: DISCRETE ENERGY BALANCE**

 329 **1. Useful identities**

Using discrete-time operators  $\mu$  and  $\delta$  defined in (17), the following identities, implicitly centered in  $t^{n+\frac{1}{2}}$ , hold

$$\left\{ \begin{array}{l} (\mu X, \mu Y) + \frac{\Delta t^2}{4} (\delta X, \delta Y) = \mu(X, Y), \\ \mu\mu X = X + \frac{\Delta t^2}{4} \delta\delta X, \\ (\mu X, \delta X) = \delta \left( \frac{1}{2} \|X\|^2 \right). \end{array} \right. \quad \begin{array}{l} \text{(A1a)} \\ \text{(A1b)} \\ \text{(A1c)} \end{array}$$

 330 **2. Notations**

In this appendix, for the sake of brevity, subscripts  $h$  and time integers  $n$  and  $n + 1/2$  are omitted. We denote

$$\begin{aligned} \Delta_0 &= R_0 \mu V, & m_0 &= G_0 \mu (P - P_0), \\ \Delta_i &= R_i \mu (V - V_i), & m_i &= G_i \mu (P - P_0 - P_i), \\ \Delta &= \Delta_0 + \sum_i \Delta_i, & m &= m_0 + \sum_i m_i. \end{aligned}$$

331 Notice that  $\Delta_i \stackrel{(18c)}{\equiv} L_i$ ,  $\delta V_i$ ,  $m_i \stackrel{(18e)}{\equiv} C_i \delta P_i$ ,  $m \stackrel{(18d)}{\equiv} C_0 \delta P_0$ . Moreover we use the notations  
 332 defined in section IV B.

### 3. Energy derivation

Taking the scalar product of  $\mu(18a)$  with  $\mu\mu V$  yields  $(M^V\mu\delta V + \mu\Delta, \mu\mu V) - (B\mu P, \mu\mu V) =$

0. The scalar product of (18b) with  $\mu P$  gives

$$(M^P\delta P + m, \mu P) + (B^*V, \mu P) + \underbrace{(\lambda_-E^- + \lambda_+E^+, \mu P)}_{-S_h^{n+\frac{1}{2}}} = 0. \quad (A2)$$

Then sum the two previous results and use identity (A1c) twice

$$\delta\frac{1}{2}\|\mu V\|_{M^V}^2 + \delta\frac{1}{2}\|P\|_{M^P}^2 + \underbrace{(\mu\Delta, \mu\mu V)}_{\textcircled{A}} + \underbrace{(m, \mu P)}_{\textcircled{B}} + \underbrace{(\mu P, B^*(V - \mu\mu V))}_{\textcircled{C}} = S_h^{n+\frac{1}{2}}. \quad (A3)$$

334 Terms  $\textcircled{A}$ ,  $\textcircled{B}$  and  $\textcircled{C}$  are treated separately.

$$\textcircled{A} \stackrel{(A1a)}{=} \mu(\Delta, \mu V) - \frac{\Delta t^2}{4}(\delta\Delta, \delta\mu V). \quad (A4)$$

Now, the discrete work  $(\Delta, \mu V)$  writes

$$\begin{aligned} (\Delta, \mu V) &= (\mu V, \Delta_0 + \sum_i \Delta_i) \\ &= (\mu V, R_0\mu V) + \sum_i (\mu V_i, \overbrace{\Delta_i}^{L_i\delta V}) \\ &\quad + \sum_i (\mu(V - V_i), \underbrace{\Delta_i}_{R_i\mu(V-V_i)}) \\ &= \|\mu V\|_{R_0}^2 + \sum_i \left[ \delta\frac{1}{2}\|V_i\|_{L_i}^2 + \|\mu(V - V_i)\|_{R_i}^2 \right] \end{aligned}$$

using (A1c) again. Hence

$$\begin{aligned} \textcircled{A} &= \mu \|\mu V\|_{R_0}^2 - \frac{\Delta t^2}{4} (\delta \Delta, \delta \mu V) \\ &\quad + \mu \sum_i \left[ \delta \frac{1}{2} \|V_i\|_{L_i}^2 + \|\mu(V - V_i)\|_{R_i}^2 \right] \quad (\text{A5}) \end{aligned}$$

Moreover, the discrete work  $(m, \mu P)$  writes

$$\begin{aligned} \textcircled{B} &= \underbrace{(m, \mu P_0)}_{C_0 \delta P_0} + \underbrace{(m, \mu(P - P_0))}_{m_0 + \sum_i m_i} \\ &= \delta \frac{1}{2} \|P_0\|_{C_0}^2 + \underbrace{(m_0, \mu(P - P_0))}_{G_0 \mu(P - P_0)} \\ &\quad + \sum_i \left[ \underbrace{(m_i, \mu(P - P_0 - P_i))}_{G_i \mu(P - P_0 - P_i)} + \underbrace{(m_i, \mu P_i)}_{C_i \delta P_i} \right] \\ &= \delta \frac{1}{2} [\|P_0\|_{C_0}^2 + \sum_i \|P_i\|_{C_i}^2] + \|\mu(P - P_0)\|_{G_0}^2 \\ &\quad + \sum_i \|\mu(P - P_0 - P_i)\|_{G_i}^2 \end{aligned}$$

335 Finally, using (A1b) and commutativity of  $\mu$  and  $\delta$ ,

$$\begin{aligned} \textcircled{C} &= -\frac{\Delta t^2}{4} (\mu B P, \delta \delta V) \\ &\stackrel{(18a)}{=} -\frac{\Delta t^2}{4} (\mu B P, (M^V)^{-1} \delta [B P - \Delta]) \\ &= \frac{\Delta t^2}{4} \left[ -\delta \frac{1}{2} \|P\|_{B^*(M^V)^{-1} B}^2 + \underbrace{(\mu B P, (M^V)^{-1} \delta \Delta)}_{M^V \delta V + \Delta} \right] \\ &= \frac{\Delta t^2}{4} \left[ -\delta \frac{1}{2} \|P\|_{B^*(M^V)^{-1} B}^2 + (\delta \mu V, \delta \Delta) \right. \\ &\quad \left. + \delta \frac{1}{2} \|\Delta\|_{(M^V)^{-1}}^2 \right] \end{aligned}$$

336 Using (A3) and the values of (A), (B) and (C), and replacing  $\Delta$  with its value, leads to the  
 337 expected relation (19).

### 338 APPENDIX B: EXPLICIT UPDATE OF THE UNKNOWNNS

Assume  $P_h^n, P_{h,0}^n, P_{h,i}^n, V_h^{n+1/2}, V_{h,i}^{n+1/2}$  are known. Define

$$\begin{aligned}\tilde{G}_h &= G_{h,0} + \sum_i G_{h,i}, \quad \tilde{R}_h = R_{h,0} + \sum_i R_{h,i}, \\ \tilde{C}_{h,0} &= C_{h,0} + \frac{\Delta t}{2} \tilde{G}_h.\end{aligned}$$

The next iterates  $P_h^{n+1}, P_{h,0}^{n+1}, P_{h,i}^{n+1}, V_h^{n+3/2}, V_{h,i}^{n+3/2}$  can be computed as follows, where all the matrices to invert are diagonal, beginning with  $P_h^{n+1}$ :

$$\begin{aligned}\left[ M_h^P + \frac{\Delta t}{2} \tilde{C}_{h,0}^{-1} C_{h,0} \tilde{G}_h \right] P_h^{n+1} &= \\ \left[ M_h^P - \frac{\Delta t}{2} \tilde{C}_{h,0}^{-1} C_{h,0} \tilde{G}_h \right] P_h^n &+ \Delta t \tilde{C}_{h,0}^{-1} C_{h,0} \tilde{G}_h P_{h,0}^n \\ + \Delta t \sum_i \tilde{C}_{h,0}^{-1} (G_{h,i}^{-1} + \frac{\Delta t}{2} C_{h,i}^{-1})^{-1} C_{h,0} P_{h,i}^n & \\ - \Delta t \left[ B_h^* V_h^{n+\frac{1}{2}} + \lambda_+^{n+\frac{1}{2}} E_h^+ + \lambda_-^{n+\frac{1}{2}} E_h^- \right]. &\end{aligned}$$

Now that  $P_h^{n+1}$  is known, we compute

$$\begin{aligned}\left[ C_{h,0} + \frac{\Delta t}{2} \tilde{G}_h \right] P_{h,0}^{n+1} &= \left[ C_{h,0} - \frac{\Delta t}{2} \tilde{G}_h \right] P_{h,0}^n \\ + \frac{\Delta t}{2} \tilde{G}_h (P_h^n + P_h^{n+1}) &- \Delta t \sum_i (G_{h,i}^{-1} + \frac{\Delta t}{2} C_{h,i}^{-1})^{-1} P_{h,i}^n.\end{aligned}$$

Now that  $P_{h,0}^{n+1}$  is known, we compute

$$\begin{aligned}\left[ C_{h,i} + \frac{\Delta t}{2} G_{h,i} \right] P_{h,i}^{n+1} &= \left[ C_{h,i} - \frac{\Delta t}{2} G_{h,i} \right] P_{h,i}^n \\ + \frac{\Delta t}{2} G_{h,i} (P_h^n + P_h^{n+1} - P_{h,0}^n - P_{h,0}^{n+1}). &\end{aligned}$$



339 Since  $P_h^{n+1}$  is known, we compute

$$\begin{aligned} \left[ M_h^V + \frac{\Delta t}{2} \tilde{R}_h \right] V_h^{n+\frac{3}{2}} &= \left[ M_h^V - \frac{\Delta t}{2} \tilde{R}_h \right] V_h^{n+\frac{1}{2}} \\ &+ \Delta t \sum_i (R_{h,i}^{-1} + \frac{\Delta t}{2} L_{h,i}^{-1})^{-1} V_{h,i}^{n+\frac{1}{2}} + \Delta t B P_h^{n+1} \end{aligned}$$

Finally,  $V_{h,i}^{n+\frac{3}{2}}$  is obtained as

$$\begin{aligned} \left[ L_{h,i} + \frac{\Delta t}{2} R_{h,i} \right] V_{h,i}^{n+\frac{3}{2}} &= \left[ L_{h,i} - \frac{\Delta t}{2} R_{h,i} \right] V_{h,i}^{n+\frac{1}{2}} \\ &+ \frac{\Delta t}{2} R_{h,i} (V_h^{n+\frac{1}{2}} + V_h^{n+\frac{3}{2}}). \end{aligned}$$

340 <sup>1</sup>See supplementary material at <https://asa.scitation.org/doi/suppl/10.1121/10.0005537> for the ta-  
341 bles of coefficients  $(a_i, b_i)$  for  $N = 0$  to 16.

342 <sup>2</sup><http://montjoie.gforge.inria.fr>

343 <sup>3</sup><http://openwind.gitlabpages.inria.fr/web>

344

345 Abel, J., Smyth, T., and O. Smith, J. (2003). “A simple, accurate wall loss filter for acoustic  
346 tubes,” International Conference on Digital Audio Effects 2003 Proceedings, London, UK  
347 2003, 53–57.

- 348 Berggren, M., Bernland, A., and Noreland, D. (2018). “Acoustic boundary layers as bound-  
349 ary conditions,” *Journal of Computational Physics* **371**, 633 – 650.
- 350 Berjamin, H., Lombard, B., Vergez, C., and Cottanceau, E. (2017). “Time-domain nu-  
351 merical modeling of brass instruments including nonlinear wave propagation, viscothermal  
352 losses, and lips vibration,” *Acta Acustica united with Acustica* **103**(1), 117–131.
- 353 Bilbao, S. (2009). “Direct simulation of reed wind instruments,” *Computer Music Journal*  
354 **33**(4), 43–55.
- 355 Bilbao, S., Harrison, R., Kergomard, J., Lombard, B., and Vergez, C. (2015). “Passive  
356 models of viscothermal wave propagation in acoustic tubes,” *The Journal of the Acoustical*  
357 *Society of America* **138**(2), 555–558.
- 358 Bilbao, S., and Harrison, R. (2016). “Passive time-domain numerical models of viscothermal  
359 wave propagation in acoustic tubes of variable cross section,” *The Journal of the Acoustical*  
360 *Society of America* **140**(1), 728–740.
- 361 Brezis, H. (2010). *Functional analysis, Sobolev spaces and partial differential equations*  
362 (Springer Science & Business Media).
- 363 Bruneau, M., Herzog, P., Kergomard, J., and Polack, J. (1989). “General formulation of  
364 the dispersion equation in bounded visco-thermal fluid, and application to some simple

- 365 geometries,” *Wave motion* **11**(5), 441–451.
- 366 Chabassier, J., Diaz, J., and Imperiale, S. (2020). “Construction and analysis of fourth  
367 order, energy consistent, family of explicit time discretizations for dissipative linear wave  
368 equations,” *ESAIM: Mathematical Modelling and Numerical Analysis* **54**(3), 845–878.
- 369 Chabassier, J., and Imperiale, S. (2013). “Introduction and study of fourth order theta  
370 schemes for linear wave equations,” *Journal of Computational and Applied Mathematics*  
371 **245**, 194–212.
- 372 Chabassier, J., and Imperiale, S. (2017). “Space/time convergence analysis of a class of  
373 conservative schemes for linear wave equations,” *Comptes Rendus Mathematique* **355**(3),  
374 282–289.
- 375 Chaigne, A., and Kergomard, J. (2016). *Acoustics of musical instruments* (Springer).
- 376 Chatziioannou, V., Schmutzhard, S., Pàmies-Vilà, M., and Hofmann, A. (2019). “Inves-  
377 tigating clarinet articulation using a physical model and an artificial blowing machine,”  
378 *Acta Acustica United with Acustica* **105**(4), 682–694.
- 379 Cohen, G. (2013). *Higher-order numerical methods for transient wave equations* (Springer  
380 Science & Business Media).

- 381 Cremer, L. (1948). “On the acoustic boundary layer outside a rigid wall,” Arch. Elektr.  
382 Uebertr **2**, 235.
- 383 Dalmont, J.-P., Nederveen, C. J., and Joly, N. (2001). “Radiation impedance of tubes  
384 with different flanges: numerical and experimental investigations,” Journal of sound and  
385 vibration **244**(3), 505–534.
- 386 Hélie, T., Hézard, T., Mignot, R., and Matignon, D. (2013). “One-dimensional acoustic  
387 models of horns and comparison with measurements,” Acta acustica united with Acustica  
388 **99**(6), 960–974.
- 389 Hélie, T., and Matignon, D. (2006). “Diffusive representations for the analysis and simula-  
390 tion of flared acoustic pipes with visco-thermal losses,” Mathematical Models and Methods  
391 in Applied Sciences **16**(4), 503–536.
- 392 Hélie, T., and Silva, F. (2017). “Self-oscillations of a vocal apparatus: a port-hamiltonian  
393 formulation,” in *International Conference on Geometric Science of Information*, Springer,  
394 pp. 375–383.
- 395 Jith, J., and Sarkar, S. (2018). “Boundary layer impedance model to analyse the visco-  
396 thermal acousto-elastic interactions in centrifugal compressors,” Journal of Fluids and  
397 Structures **81**, 179–200.

- 398 Kampinga, W., Wijnant, Y. H., and de Boer, A. (2011). “An efficient finite element model  
399 for viscothermal acoustics,” *Acta Acustica united with Acustica* **97**(4), 618–631.
- 400 Keefe, D. H. (1984). “Acoustical wave propagation in cylindrical ducts: Transmission line  
401 parameter approximations for isothermal and nonisothermal boundary conditions,” *The*  
402 *Journal of the Acoustical Society of America* **75**(1), 58–62.
- 403 Kirchhoff, G. (1868). “Ueber den einfluss der wärmeleitung in einem gase auf die schallbe-  
404 wegung,” *Annalen der Physik* **210**(6), 177–193.
- 405 Micheyl, C., Delhommeau, K., Perrot X., , and Oxenham, A. J. (2006). “Influence of musical  
406 and psychoacoustical training on pitch discrimination,” *Hearing research* **219**(1-2), 36–47.
- 407 Mignot, R., Hélie, T., and Matignon, D. (2010). “Digital Waveguide Modeling for Wind In-  
408 struments: Building a State–Space Representation Based on the Webster–Lokshin Model,”  
409 *IEEE transactions on audio, speech, and language processing* **18**(4), 843–854.
- 410 Monteghetti, F., Matignon, D., and Piot, E. (2018). “Energy analysis and discretization of  
411 nonlinear impedance boundary conditions for the time-domain linearized euler equations,”  
412 *Journal of Computational Physics* **375**, 393–426.

- 413 Nocedal, J., and Wright, S. (2006). *Numerical optimization* (Springer Science & Business  
414 Media).
- 415 Rabiner, L. R., and Schafer, R. W. (1978). *Digital processing of speech signals*, 100  
416 (Prentice-hall Englewood Cliffs, NJ).
- 417 Rienstra, S. W. (2005). “Webster’s horn equation revisited,” *SIAM Journal on Applied*  
418 *Mathematics* 65(6), 1981–2004.
- 419 Scheichl, S. (2004). “On the calculation of the transmission line parameters for long tubes  
420 using the method of multiple scales,” in *The Journal of the Acoustical Society of America*  
421 115(2), 534–555.
- 422 Schmutzhard, S., Chatziioannou, V., and Hofmann, A. (2017). “Parameter optimisation  
423 of a viscothermal time-domain model for wind instruments,” in *Proceedings of the 2017*  
424 *International Symposium on Musical Acoustics*, pp. 27–30.
- 425 Silva, F., Guillemain, P., Kergomard, J., Mallaroni, B., and Norris, A. N. (2009). “Approx-  
426 imation formulae for the acoustic radiation impedance of a cylindrical pipe,” *Journal of*  
427 *Sound and Vibration* 322(1-2), 255–263.
- 428 Silva, F., Vergez, C., Guillemain, P., Kergomard, J., and Debut, V. (2014). “MoReeSC: a  
429 framework for the simulation and analysis of sound production in reed and brass instru-

- 430 ments,” *Acta Acustica united with Acustica* **100**(1), 126–138.
- 431 Stinson, M. R. (1991). “The propagation of plane sound waves in narrow and wide circular  
432 tubes, and generalization to uniform tubes of arbitrary cross-sectional shape,” *The Journal*  
433 *of the Acoustical Society of America* **89**(2), 550–558.
- 434 Thibault, A., and Chabassier, J. (2020). “Viscothermal models for wind musical instru-  
435 ments,” Inria Research Report , <https://hal.inria.fr/hal-02917351>.
- 436 Thompson, S. C., Gabrielson, T. B., and Warren, D. M. (2014). “Analog model for ther-  
437 moviscous propagation in a cylindrical tube,” *The Journal of the Acoustical Society of*  
438 *America* **135**(2), 585–590.
- 439 Tijdeman, H. (1975). “On the propagation of sound waves in cylindrical tubes,” *Journal of*  
440 *Sound and Vibration* **39**(1), 1–33.
- 441 Tournemenne, R., and Chabassier, J. (2019). “A comparison of a one-dimensional finite  
442 element method and the transfer matrix method for the computation of wind music in-  
443 strument impedance,” *Acta Acustica united with Acustica* **105**(5), 838–849.
- 444 Van Der Schaft, A. (2006). “Port-hamiltonian systems: an introductory survey,” in *Pro-*  
445 *ceedings of the international congress of mathematicians*, Citeseer, Vol. 3, pp. 1339–1365.

<sup>446</sup> Zwikker, C., and Kosten, C. W. (1949). *Sound Absorbing Materials* (Elsevier Publ. Comp.).



