

Financial Economics  
&  
Risk Management



Financial Economics  
Meeting Crisis Challenges

# Bull Spread Option pricing using a mixed modified fractional process with stochastic volatility and interest rates

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07/01/2021

# WHAT is the contribution of our paper ?

## Highlights of our paper

- We price options so as to take into account the existence of memory (short or long) characterizing the stochastic processes that generate prices, volatility and interest rates.
- In particular, we propose a model for Bull Spread options in a Mixed Modified Fractional Hull-White-Vasicek stochastic volatility and stochastic interest rate model.
- We propose a specific Bull Spread Vulnerable option pricing based on MMFHWV model.
- Vulnerable option is a contingent claims on defaultable instruments, subject to their issuer's default risk.

## WHY our results are relevant ?

- Since the 1973, Black and Scholes proposed an analytical formula for option pricing.
- Since 1987, vulnerable options pricing has been proposed by Johnson and Stulz [3].
- Financial institutions actively trade derivative contracts with their corporate clients, as well as with other financial institutions in over-the-counter (OTC) markets.
- Klein [4] derived analytic formula for vulnerable option.
- The assumptions of Klein [4] consider the correlation between the option's underlying asset and the default risk of the counter party, and the option issue capital structure.
- Hung and Liu [2] and Yang et al. [5] studied the pricing of vulnerable options in incomplete market.

- These competing models modeled the interest rate of the Black and Scholes model and authors such as Vasicek[3] , Dothan [6] , Cox-Ingersoll-Ross[5] and Black-Derman-Toy[3] have worked on these models.
- The constant volatility assumption of the Black-Scholes model is unreliable, because in the market volatility is in the form of a smile.
- Hence the need to introduce in the market stochastic volatility models which give more realistic results compared to the Black-Scholes model.
- Hull-White[2] , Stein-Stein[9] , Heston [8] and many others authors [10], have propose option taking into account stochastic volatility and interest rates.

## But.....

- All the models mentioned above do not assume the existence of long or short memory in the generators processes of price, volatility and interest rate time series.
- Some authors try to fulfill some limitations of based on recent literature review of option models. We try to contribute in that direction by consider the **rough volatility and rough interest rate**.
- **For the relevance of considering rough volatility in option in risk management, see Rosenbaum and Gatheral (2017) and some references therein**
- **Our aim is to propose a closed form Bull spread option by considering a long or short memory into the time series of price, volatility and interest rate.**

# HOW do we obtain our results ?

- We first present a pricing model for Bull Spread options in a Mixed Modified Fractional Hull-White-Vasicek stochastic volatility and stochastic interest rate model.
- We simulate our model using Milstein discretization scheme and we price bull spread option by using Monte Carlo method.
- We propose the PDE of the bull spread specific case of vulnerable options under stochastic volatility(Hull-white) and stochastic interest rates(Vasicek).
- The double Mellin transform provides an explicit analytic closed form formula for the bull spread specific vulnerable option.

# Illustration

Determining the characteristics of stock prices leads to the definition of statistical procedure using a fractal approach, which is carried out in three steps :

- Identifying the dimension of the series to determine membership in fractal characters;
- It follows from the first determination of  $H$  in order to appreciate the Joseph effect.
- The test the auto-similarity of the time series, by using the Box-counting method based on the box of the VIX-Volatility-Historical-Chart from 1990 to 2021 downloaded from "www.macrotrends.net"

# Application of the fractal characteristics in the CBOE market

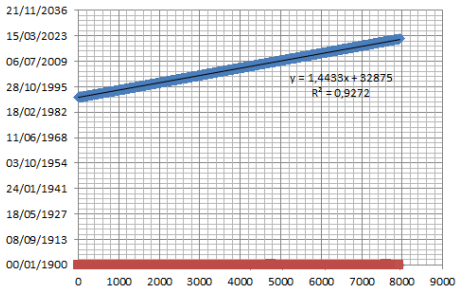


Figure: VIX:1990-2021

Actions	Trend line	Dimension	$H_2$
VIX	$Y = 1.4433X + 32875$	1.44	0.69

Table: Fractal characteristics applied to the VIX from 1990 to 2021.



# Interpretation

- the first step of the analysis confirms the fractal dimension of stock prices and the CBOE VIX do not tend towards a standard Brownian motion.
- the Hurst index shows the weak fluctuation of prices with a tendency towards unity.
- furthermore, the Hurst exponent obtained is greater than 0.5, hence the confirmation of the presence of long memory in the lessons
- we deduce that the history of the assets of the CBOE strongly influences their fluctuations
- we deduce that the three processes namely the price of the asset, its volatility and its interest rate can be modeled by mixed modified fractional Brownian motion which converge to Brownian motion with different Hurst parameter.
- Hence, the viability of our new model with a memory.

# Notations

- $\mu$  the drift of the process for the stock and for the variance;
- $\sigma_s$  is the volatility of the stock price;
- $\sigma_v$  is the volatility of the variance;
- $\kappa$  the mean reversion speed for the interest rate;
- $\theta$  the mean reversion level for the interest rate;
- $\sigma_r$  the volatility of the interest rate;
- $Q$  is a Risk neutral probability measure and  $P$  is a probability measure;;
- $S_t$  is the dynamic stock price,  $V_t$  is its volatility and  $r_t$  is its interest rate;
- $P$  is the price of a zero-coupon bond at time  $t$  and  $U$  is the price of the bull spread specific vulnerable option at time  $t$ .
- $M_{.,s}^{H_1,\varepsilon}$  and  $M_{.,v}^{H_2,\varepsilon}, M_{.,r}^{H_3,\varepsilon}$  are three Mixed Modified Fractional Brownian motions with Hurst parameters  $H_1, H_2, H_3 \in ]\frac{1}{2}, 1[$

# Assumptions

## Some assumptions of the MMFHWV model

- (a) Consider the probability space  $(\Omega, F, P)$  where  $F = (F_t)_{t \geq 0}$  is a complete and right continuous filtration generated by  $F_t = \sigma \left( M_{t,S}^{H_{1,\varepsilon}}; M_{t,S}^{H_{2,\varepsilon}}; M_{t,S}^{H_{3,\varepsilon}} \right)$  and  $P$  is a probability measure on  $\Omega$ .
- (b) The dynamic of stock price  $S_t$  and its volatility  $V_t$  are given by a Mixed Modified Fractional Hull-White no-arbitrage model, namely :

$$\begin{cases} dS_t = \mu S_t dt + \sigma_S S_t dM_{t,S}^{H_{1,\varepsilon}}, \\ dV_t = \mu V_t dt + \sigma_V V_t dM_{t,V}^{H_{2,\varepsilon}}. \end{cases} \quad (1)$$

where  $M_{.,S}^{H_{1,\varepsilon}}$  and  $M_{.,V}^{H_{1,\varepsilon}}$  are two Mixed Modified Fractional Brownian motions with Hurst parameters  $H_1, H_2 \in ]\frac{1}{2}, 1[$

## Assumptions of the MMFHWV model

- (c) The evolution of interest rate  $r_t$  is given by the Mixed Modified Fractional Vasicek no-arbitrage model, namely :

$$dr_t = \kappa(\theta - r_t)dt + r_t \sigma_r dM_{t,r}^{H_3, \varepsilon}. \quad (2)$$

where  $M_{.,r}^{H_3, \varepsilon}$  is the Mixed Modified Fractional Brownian motion with Hurst parameter  $H_3 \in ]\frac{1}{2}, 1[$

- (d) The hurst parameters  $H_1, H_2$  and  $H_3$  can be verified  $H_3 > H_2 > H_1 > \frac{1}{2}$ .
- (e) The correlation matrix is defined as follows:

$$\left\{ \begin{array}{l} E^P [dM_{t,i}^{H_j, \varepsilon} dM_{t,i}^{H_j, \varepsilon}] = \chi^{H_j} \rho^i dt, \quad \forall j \in \{1, 2, 3\}, \forall i \in \{s, v, r\}, \\ E^P [dM_{t,i}^{H_p, \varepsilon} dM_{t,k}^{H_q, \varepsilon}] = \rho_{ik} \frac{\chi^{H_p, \varepsilon} \chi^{H_q, \varepsilon}}{\chi^{H_p, \varepsilon} \chi^{H_q, \varepsilon}} dt, \\ \forall p \neq q, p, q \in \{1, 2, 3\}; i, k \in \{s, v, r\}, \chi^{\varepsilon} = (a + b\varepsilon^{H_p - \frac{1}{2}})^2. \end{array} \right.$$

# The MMFHWV Model Framework

## MMFHWV Model Framework

In the Mixed Modified Fractional Hull-White model Eq.(1), if we select the parameter  $\mu$  as a stochastic process(see Mixed Modified Fractional Vasicek mode(2)), we obtain the Mixed Modified Fractional Hull-White-Vasicek(MMFHWV) model defined by the trivariate system of stochastic diferential equations(SDEs):

$$\begin{cases} dS_t = r_t S_t dt + \sigma_s S_t dM_{t,S}^{H_{1,\varepsilon}}, \\ dV_t = r_t V_t dt + \sigma_v V_t dM_{t,V}^{H_{2,\varepsilon}}, \\ dr_t = \kappa(\theta - r_t)dt + r_t \sigma_r dM_{t,r}^{H_{3,\varepsilon}}. \end{cases} \quad (3)$$

Where  $M_{t,S}^{H_{1,\varepsilon}}$ ,  $M_{t,V}^{H_{2,\varepsilon}}$ , and  $M_{t,r}^{H_{3,\varepsilon}}$  are three correlated standard Mixed Modified Fractional Brownian motions.

The stock price and variance follow the processes in Eq(1) and the interest rate follow the processes in Eq.(2) under the historical measure  $P$  also called the physical measure. For pricing purposes, however, we need the processes for  $(S_t, V_t, r_t)$  under the risk-neutral measure  $Q$ . In the MMFHWV model Eq.(3), this is done by modifying each SDE in Eq.(1) and Eq.(2) separately by an application of Girsanov's theorem.

The risk-neutral process for the MMFHWV Eq.(3) is defined by

$$\begin{cases} dS_t = r_t S_t dt + \sigma_s S_t d\mathbf{f}M_{t,S}^{H1,\varepsilon}, \\ dV_t = r_t V_t dt + \sigma_v V_t d\mathbf{f}M_{t,v}^{H2,\varepsilon}, \\ dr_t = \kappa^*(\theta^* - r_t)dt + r_t \sigma_r d\mathbf{f}M_{t,r}^{H3,\varepsilon}. \end{cases} \quad (4)$$

With

$$\begin{cases} \mathbf{f}M_{t,S}^{H1,\varepsilon} = M_{t,S}^{H1,\varepsilon} + \frac{\mu-r}{\sigma_s} t \\ \mathbf{f}M_{t,v}^{H2,\varepsilon} = M_{t,v}^{H2,\varepsilon} + \frac{\mu-r}{\sigma_v} t \\ \mathbf{f}M_{t,r}^{H3,\varepsilon} = M_{t,r}^{H3,\varepsilon} + \frac{\mu-r}{\sigma_r \chi^{H3,\varepsilon}} t. \end{cases} \quad (5)$$

The risk-neutral process for the stock price is

$$\begin{cases} dS_t = S_t \left[ r + b\sigma_s \varphi_{t,s}^\lambda \right] dt + \sigma_s \frac{p}{\chi^{H_{1,\varepsilon}}} d\mathbf{B}_{t,s}^{\mathbf{0}}, \\ \mathbf{B}_{t,s} = B_{t,s} + \frac{\sqrt{\mu-r}}{\sigma_s \chi^{H_{1,\varepsilon}}} t. \end{cases} \quad (6)$$

The risk-neutral process for the variance of stock price is

$$\begin{cases} dV_t = V_t \left[ r + b\sigma_v \varphi_{t,v}^\lambda \right] dt + \sigma_v \frac{p}{\chi^{H_{2,\varepsilon}}} d\mathbf{B}_{t,v}^{\mathbf{0}}, \\ \mathbf{B}_{t,v} = B_{t,v} + \frac{\sqrt{\mu-r}}{\sigma_v \chi^{H_{2,\varepsilon}}} t. \end{cases} \quad (7)$$

The risk-neutral process for the interest rate is

$$\begin{cases} dr_t = \kappa^* (\theta^* - r_t) + \sigma_r \varphi_{t,r}^\lambda dt + \sigma_r \frac{p}{\chi^{H_{3,\varepsilon}}} d\mathbf{B}_{t,r}^{\mathbf{0}}, \\ \mathbf{B}_{t,r} = B_{t,r} + \frac{\sqrt{\lambda r_t}}{\sigma_r \chi^{H_{3,\varepsilon}}} t. \end{cases} \quad (8)$$

with  $\kappa^* = \kappa + \frac{\sqrt{\lambda}}{\chi^{H_{3,\varepsilon}}}$  and  $\theta^* = \frac{\kappa\theta}{\kappa + \frac{\sqrt{\lambda}}{\chi^{H_{3,\varepsilon}}}}$ , are the

risk-neutral parameter and  $\lambda$  the interest rate risk parameter which is given by Breeden[4].



## Definition 1

(MMFHWV model) In the Mixed Modified Fractional Hull-White model Eq.(6) and Eq.(7), if we select the parameter  $\mu$  as a stochastic process( see Mixed Modified Fractional Vasicek model Eq.(8)), then we will obtain the new model appointed Mixed-Modified-Fractional-Hull-White-Vasicek (MMFHWV) model and then, under the risk-neutral measure  $Q$ , the dynamics of  $S_t$ ,  $V_t$  and  $r_t$  are given by the SDEs:

$$\begin{cases} dS_t = S_t \left[ r + b\sigma_s \varphi_{t,s}^\lambda dt + \sigma_s \frac{p}{\chi^{H_{1,\varepsilon}}} d\mathcal{B}_{t,s} \right], \\ dV_t = V_t \left[ r + b\sigma_v \varphi_{t,v}^\lambda dt + \sigma_v \frac{p}{\chi^{H_{2,\varepsilon}}} d\mathcal{B}_{t,v} \right], \\ dr_t = \kappa^* (\theta^* - r_t) + \sigma_r \varphi_{t,r}^\lambda dt + \sigma_r \frac{p}{\chi^{H_{3,\varepsilon}}} d\mathcal{B}_{t,r}. \end{cases} \quad (9)$$

where  $\mathcal{B}_{t,s}$ ,  $\mathcal{B}_{t,v}$  and  $\mathcal{B}_{t,r}$  are three Brownians motions.

# Advantages of our model

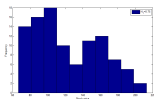
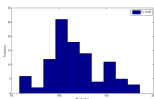
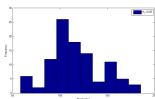
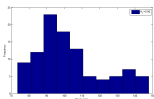
Our paper looks at these two families of models (stochastic interest rate and stochastic volatility). But our inspiration comes from the fact that we model the two families of models by assuming that their random parts are described by mixed modified fractional Brownian motions in such a way that our market is without arbitrage. The combination of the stochastic interest rate model (Vasicek[3] ) and the stochastic volatility model (Hull-White[2] ) allows us to define a three-factors model.

Nammed MMFHWV model for

”Mixed-Modified-Fractional-Hull-White -Vasiseck ”, this new model is a combination of the Hull and White[2] and the Vasicek[3] model. In this model, the volatility process and asset model are not correlated, while the interest rate process and asset model, the volatility process and interest rates process are correlated, with each other and they are controlled by a distinct diffusion. In MMFHWV model, the existence of the mean reversion process causes the adjustment of the volatility process and the interest rates behavior in the financial markets and it is a benefit of the MMFHWV model.

# Simulation of MMFHWV Model

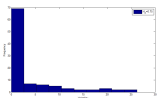
In fig.2, 3 and 4, we illustrate changes in the value of stock path, volatility of the stock's path and interest rate path under modified Hurst parameter. The result indicate that: By increasing the value of  $H_1$ , (see Fig.2), the value of the stock price is reduced

(a)  $H_1 = 0.75$ (b)  $H_1 = 0.80$ (c)  $H_1 = 0.85$ (d)  $H_1 = 0.90$ 

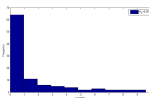
**Figure:** The Stock price paths under MMFHWV model with  $S_0 = 115$ ,  $\gamma = 0.01$ ,  $\sigma = 0.9$ ,  $\kappa = 0.2$ ,  $\theta = 0.2$ ,  $\sigma = 0.8$

# Simulation of MMFHWV Model

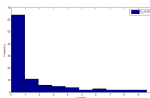
By increasing the value of  $H_2$ , (see Fig.3), the value of the volatility decrease and more, quickly convergence to zero.



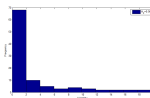
(a)  $H_2 = 0.75$



(b)  $H_2 = 0.80$



(c)  $H_2 = 0.85$

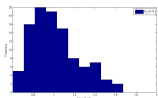


(d)  $H_2 = 0.90$

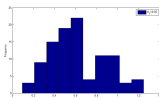
**Figure:** The volatility paths under MMFHWV model with  $S_0 = 115$ ,  $\gamma = 0.01$ ,  $\theta = 0.9$ ,  $\kappa = 0.2$ ,  $\theta = 0.2$ ,  $\sigma = 0.8$

# Simulation of MMFHWV Model

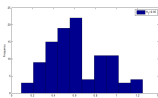
By increasing the value of  $H_3$ , (see Fig.4, the value of the Interest rate increase.



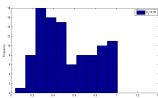
(a)  $H_3 = 0.75$



(b)  $H_3 = 0.80$



(c)  $H_3 = 0.85$



(d)  $H_3 = 0.90$

**Figure:** The Interest rate paths under MMFHWV model with  $S_0 = 115$ ,  $\nu = 0.01$ ,  $\rho = 0.9$ ,  $\kappa = 0.2$ ,  $\theta = 0.2$ ,  $\sigma = 0.8$

# Value of the bull spread option under MMFHWV model

By applying the monte carlo metho, the value of the bull spread option under MMFHWV model is given as follow.

	a=1,b=1		a=2,b=1	
	MMFHWV	CI	MMFHWV	CI
n=1	6.1533	[5.6466 ; 6.1637]	08.3262	[04.2898 ; 8.4086]
n=2	6.6704	[6.6600 ; 7.1771]	12.4450	[12.3626 ; 16.4814]

**Table:** Prices of Bull spread call options under MMFHWV model.

# Bond price formula

By the Feynman-Kac formula, we have the following pricing formula.

## Proposition 4.1

*In the fixed MMFHWV model, the price of a zero-coupon bond at time  $t$  is given by:*

$$P(\tau, r, T) = A(\tau) e^{-B(\tau)r}, \quad (10)$$

with  $\tau = T - t$ .  $A(\tau)$  and  $B(\tau)$  are given by

$$\begin{cases} A(\tau) = \exp \frac{\chi^H \mathbb{3}, \epsilon \sigma_r^2}{2\kappa^2(\tau + \frac{2}{\kappa^*} (e^{-\kappa^* \tau} - 1)) - \frac{1}{2\kappa^*} (e^{-\kappa^* \tau} - 1)} \\ - \int_0^{\tau} \kappa^* \theta^*(T-z) B(z) dz, \\ B(\tau) = \frac{1 - e^{-\kappa^* \tau}}{\kappa^*} \end{cases} \quad (11)$$



# Pricing for Bull Spread Option under MMFHWV

The Bull spread of vulnerable option, with maturity time  $T$ , and strikes price  $K_1$  and  $K_2$ , the payoff function is

$$h(S_T, V_T) = g(S_T) - \alpha \mathbf{1}_{\{V_T \geq \gamma^*\}} + (1 - \alpha) \frac{V_T}{D} \mathbf{1}_{\{V_T \leq \gamma^*\}}, \quad (12)$$

where  $g(S_T) = (S_T - K_1)^+ - (S_T - K_2)^+$ , for the Bull spread option,  $\alpha = 1$  and  $\gamma^*$  is a critical value such that a credit loss occurs if the value of the option writer's assets and the no-arbitrage price of the option is given by

$$U(t, s, v, r) = E \left[ e^{-\int_t^T r_s ds} h(S_T, V_T) \mid S_t = s, V_t = v, r_t = r \right]. \quad (13)$$

and let consider  $\Omega$  be the domain defined by:

$$\Omega = \{(S, V, r, t) : 0 < S < +\infty, 0 < V < +\infty, -\infty < r < +\infty, 0 < t < T\} \quad (14)$$

# PDE of Bull Spread option pricing under the MMFHWV model

## Theorem 2

*Under the MMFHWV model, the option value function  $U$  satisfies in the domain (14) the following partial differential equation is given by:*

$$\begin{aligned}
 & \frac{\partial U}{\partial t} + \frac{1}{2} \chi^{H_{1,\varepsilon}} \sigma_s^2 s^2 \frac{\partial^2 U}{\partial s^2} + \frac{1}{2} \chi^{H_{2,\varepsilon}} \sigma_v^2 v_t^2 \frac{\partial^2 U}{\partial v^2} + \frac{1}{2} \chi^{H_{3,\varepsilon}} \sigma_r^2 \frac{\partial^2 U}{\partial r^2} \\
 & + \rho_{sv} \sigma_s \sigma_v s v \rho \frac{\partial^2 U}{\chi^{H_{1,\varepsilon}} \chi^{H_{2,\varepsilon}} \partial s \partial v} + \rho_{sr} \sigma_s \sigma_r s \rho \frac{\partial^2 U}{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}} \partial s \partial r} \\
 & + \rho_{vr} \sigma_v \sigma_r v \rho \frac{\partial^2 U}{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}} \partial v \partial r} + r s \frac{\partial U}{\partial s} + v \frac{\partial U}{\partial v} \\
 & + \kappa^*(\theta^* - r) \frac{\partial U}{\partial r} - rU = 0
 \end{aligned} \tag{15}$$

# Option formula of Bull Spread under the MMFHWV model

with the terminal condition  $U(T, s, v, r) = h(s, v)$  and

$$\chi^{H_i, \varepsilon} = (a + b\varepsilon^{H_i - \frac{1}{2}})^2, i = 1.2.3.$$

## Theorem 3

*The price of specific vulnerable Bull Spread option with Mixed-Modified-Fractional-Hull-White-Vasicek model, defined by Eq.(13) is given by*

$$U(t, s, v, r) = sN(ba_1, b_2, \xi) - K_1B(t, r, T)N(b_1, b_2, \xi) - sN(ba_3, b_4, \xi) + K_2B(t, r, T)N(b_3, b_4, \xi).$$

with  $\xi = \frac{\sqrt{\bar{B}(\tau)}}{2 A(\tau)B_1(\tau)}$

where

$$\hat{a}_1 = \frac{\ln \frac{s}{K_1} - \ln P(t, r, T) + \frac{1}{2} \int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\beta_r \int_0^p \overline{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\beta_r \int_0^p \overline{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \quad (16)$$

$$\begin{aligned} \hat{b}_1 = & \frac{\ln \frac{v}{y^*} - \ln P(t, r, T) - \frac{1}{2} \int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\beta_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\beta_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \\ & + \frac{2\xi \int_0^{R_t} \sigma_s^2(v) dv + \int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\beta_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\beta_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} dv \end{aligned} \quad (17)$$

$$\hat{a}_2 = \frac{\ln \frac{s}{K_1} - \ln P(t, r, T) - \frac{1}{2} \int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_s \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_r \int_0^p \overline{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_s \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \quad (18)$$

$$\hat{b}_2 = \frac{\ln \frac{v}{Y^*} - \ln P(t, r, T) - \frac{1}{2} \int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_v \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_v \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \quad (19)$$

$$\hat{a}_3 = \frac{\ln \frac{s}{K_2} - \ln P(t, r, T) + \frac{1}{2} \int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\beta_r \int_0^p \overline{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_s^2 + 2\beta_r \int_0^p \overline{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \quad (20)$$

$$\hat{b}_3 = \frac{\ln \frac{v}{y^*} - \ln P(t, r, T) - \frac{1}{2} \int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\beta_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\beta_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} + \frac{2\xi \int_0^{R_t} \sigma_s^2(v) dv}{\int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\beta_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(v) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \quad (21)$$

$$\hat{a}_4 = \frac{\ln \frac{s}{K_2} - \ln P(t, r, T) - \frac{1}{2} \int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_r \int_0^p \overline{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_s \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_r \int_0^p \overline{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_s \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \quad (22)$$

$$\hat{b}_4 = \frac{\ln \frac{v}{Y^*} - \ln P(t, r, T) - \frac{1}{2} \int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_v \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)}{\int_0^{R_t} \chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_r \int_0^p \overline{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}} \sigma_v \sigma_r L(v)} + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(v)} \quad (23)$$

# Interpretation

- The result given by the Theorem 2 is an elliptical partial differential equation.
- Its solution is found under the Black Scholes form.
- The obtained result given by the Theorem 3 is comparable to that obtained by Black Scholes in the parabolic case.
- Therefore, its implementation becomes simple and easy in the rooms of market
- The originality of this paper comes from the fact that, we differentiated the values of the Hurst parameter for the three factors namely the price of the underlying, its volatility and its stochastic interest rate which is not always the case for models found in the current literature.



# Conclusion

In this paper, we have presented a pricing model for Bull Spread options in a Mixed Modified Fractional Hull-White-Vasicek stochastic volatility and stochastic interest rate model. We have discretize the stochastic process with Milstein discretization scheme and we have priced bull spread option by using Monte Carlo algorithm. We have used the double Mellin transform to study bull spread specific case of vulnerable bull spread options under stochastic volatility(Hull-white) and stochastic interest rates(Vasicek).

**Thank  
for you  
kind attention.**



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## The basic properties of double Mellin transforms

### Definition 4

The double Mellin transform is defined by

$$M_{xy}(f(x, y), w_1, w_2) = \int_0^{\infty} \int_0^{\infty} f(x, y) x^{w_1-1} y^{w_2-1} dx dy \quad (24)$$

where  $f(x, y)$  is a locally Lebesgue integrable function and  $w_1$  and  $w_2$  are complex numbers. Also, if  $a < \text{Re}(w_1)$ ,  $\text{Re}(w_2) < b$  and if  $c_1$  and  $c_2$  are such that  $a < c_1 < b$  and  $a < c_2 < b$ , then the inverse of the double Mellin transform is given by

$$M_{xy}^{-1}(f(w_1, w_2)) = \frac{1}{(2\pi)^2} \int_{a_1-i\infty}^{a_1+i\infty} \int_{a_2-i\infty}^{a_2+i\infty} f(w_1, w_2) x^{-w_1} y^{-w_2} dw_1 dw_2 \quad (25)$$

## Properties for derivative

We have the following properties for derivative

$$M_{xy} \left( x^2 \frac{\partial f(x, y)}{\partial x^2}, w_1, w_2 \right) = w_1(w_1 + 1) \rho(w_1, w_2) \quad (26)$$

$$M_{xy} \left( y^2 \frac{\partial f(x, y)}{\partial y^2}, w_1, w_2 \right) = w_2(w_2 + 1) \rho(w_1, w_2) \quad (27)$$

## Lemma 5

Given complex numbers  $\alpha$  and  $\beta$  with  $\text{Re}(\alpha) \geq 0$ , let

$$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(w) x^{-w} dw,$$

where  $\Gamma(w) x^{-w} = e^{\alpha(w+\beta)^2}$ . Then  $f(x) = \frac{1}{2} \sqrt{\frac{1}{\pi\alpha}} x^\beta e^{\frac{1}{4\alpha} (\ln x)^2}$  holds.



## Lemma 6

Let  $f$  and  $g$  be functions from  $\mathbb{R}_+^n$  into  $\mathbb{C}$ . If  $f(w_1, w_2)$  and  $g(w_1, w_2)$  are the double Mellin transform of  $f(x, y)$  and  $g(x, y)$ , respectively, given by

$$f(w_1, w_2) = \int_0^\infty \int_0^\infty f(x, y) x^{w_1-1} y^{w_2-1} dx dy. \quad (28)$$

Then the double Mellin convolution of  $f$  and  $g$  is given by inverse double Mellin transform of  $f(w_1, w_2)g(w_1, w_2)$  as follows

$$\begin{aligned} f(x, y) * g(x, y) &= M^{-1}_{w_1 w_2} f(w_1, w_2)g(w_1, w_2); x, y \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty f\left(\frac{x}{u}, \frac{y}{w}\right) g(u, w) u^{-1} w^{-1} du dw \end{aligned} \quad (29)$$

## Definition 7

The Hull-White[2] model assumes that the underlying stock price,  $S_t$  and variance  $V_t$  follow a Black-Scholes-type stochastic process. Hence, the Hull-White model is represented by the bivariate system of stochastic differential equations (SDEs):

$$\begin{aligned} dS_t &= \mu S_t dt + S_t \sigma_s dB_{t,s} , \\ dV_t &= \mu V_t dt + \sigma_v V_t dB_{t,v} . \end{aligned} \quad (30)$$

Where  $E^P [dB_{t,s} dB_{t,v}] = \rho_{sv} dt$ .  $B_{t,s}$  and  $B_{t,v}$  are the wiener processes;  $\rho_{sv} \in [-1; 1]$  is the correlation between  $W_{t,s}$  and  $W_{t,v}$ .

## Definition 8

The Vasicek[3] model is a stochastic interest rate model and the corresponding Stochastic Differential Equation (SDE) as follows,

$$dr_t = \kappa(\theta - r_t)dt + r_t\sigma_r dB_{t,r}, \quad (31)$$

where  $B_{.,r}$  is the wiener process.

## Djeutcha et al.[1].

A Mixed Modified Fractional Brownian Motion process whose parameters  $a, b, \varepsilon$  and  $H$  is a linear combination of Brownian Motion  $B_t$  and independent semimartingale process  $B_t^{H,\varepsilon}$ , defined on a probability space  $(\Omega, F, P)$  by:

$$M_t^{H,a,b,\varepsilon} = M_t^{H,\varepsilon} = aB_t + bB_t^{H,\varepsilon}, \quad \forall (a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+, \quad t \in [0, T]. \quad (32)$$

Where  $H \in ]\frac{1}{2}, 1[$  is the Hurst parameter.

- Let  $H \in ]\frac{1}{2}, 1[$ ; For every  $\varepsilon > 0$ , the process  $M_t^{H,\varepsilon}$  is a continuous  $F_t$ -semimartingale.
- The process  $M_t^{H,\varepsilon}$  converge to  $M_t^H$  in  $L^2(\Omega)$  space for all  $t \in [0; T, ]$  when  $\varepsilon \rightarrow 0$ .