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# Minor-Obstructions for Apex-Pseudoforests\*

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## Abstract

A graph is called a *pseudoforest* if none of its connected components contains more than one cycle. A graph is an *apex-pseudoforest* if it can become a pseudoforest by removing one of its vertices. We identify 33 graphs that form the minor-obstruction set of the class of apex-pseudoforests, i.e., the set of all minor-minimal graphs that are not apex-pseudoforests.

**Keywords:** Graph minors, Minor Obstructions

## 1 Introduction

All graphs in this paper are undirected, finite, and simple, i.e., without loops or multiple edges.

A graph  $G$  is a *pseudoforest* if every connected component of  $G$  contains at most one cycle. We denote by  $\mathcal{P}$  the set of all pseudoforests.

We say that a graph  $H$  is a *minor* of  $G$  if a graph isomorphic to  $H$  can be obtained by some subgraph of  $G$  after applying edge contractions. (As in this paper we consider only simple graphs, we always assume that in case multiple edges are created after a contraction, then these edges are automatically suppressed to simple edges.) We say that a graph class  $\mathcal{G}$  is *minor-closed* if every minor of a graph in  $\mathcal{G}$  is also a member of  $\mathcal{G}$ . Given a graph class  $\mathcal{G}$ , its *minor-obstruction set* is defined as the minor-minimal set of all graphs that are not in  $\mathcal{G}$  and is denoted by  $\mathbf{obs}(\mathcal{G})$ . For simplicity, we drop the term “minor” when we refer to an obstruction set as, in this paper, we only consider minor-obstruction sets. We also refer to the members of  $\mathbf{obs}(\mathcal{G})$  as *obstructions of  $\mathcal{G}$* . Given a set of graphs  $\mathcal{H}$  we denote by  $\mathbf{exc}(\mathcal{H})$  as the set containing every graph  $G$  that excludes all graphs in  $\mathcal{H}$  as minors.

Notice that if  $\mathcal{G}$  is minor-closed, then a graph  $G$  belongs in  $\mathcal{G}$  iff none of the graphs in  $\mathbf{obs}(\mathcal{G})$  is a minor of  $G$ . In this way,  $\mathbf{obs}(\mathcal{G})$  can be seen as a complete characterization of  $\mathcal{G}$  in terms of forbidden minors, i.e.,  $\mathcal{G} = \mathbf{excl}(\mathbf{obs}(\mathcal{G}))$ .

According to the Roberston and Seymour theorem [37], for every graph class  $\mathcal{G}$ , the set  $\mathbf{obs}(\mathcal{G})$  is finite. The study of  $\mathbf{obs}(\mathcal{G})$  for distinct instantiations of minor-closed graph classes is an active topic in graph theory (e.g., see [3–6, 6–9, 11, 13–15, 17, 19, 21, 21–29, 32, 33, 35, 36, 38, 39, 42], see also [1, 31] for related surveys).

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It is easy to observe that pseudoforests is a minor-closed graph class. Moreover it holds that  $\mathbf{obs}(\mathcal{P}) = \{\triangleleft, \bowtie\}$ , as these two obstructions express the existence of two cycles in the same connected component of a graph.

Given a non-negative integer  $k$  and a graph class  $\mathcal{G}$ , we say that a graph  $G$  is a  $k$ -apex of  $\mathcal{G}$  if it can be transformed to a member of  $\mathcal{G}$  after removing at most  $k$  of its vertices. We also use the term *apex of  $\mathcal{G}$*  as a shortcut of 1-apex of  $\mathcal{G}$ . We denote the set of all  $k$ -apices of  $\mathcal{G}$  by  $\mathcal{A}_k(\mathcal{G})$  and, more specifically, we use  $\mathcal{P}^{(k)} := \mathcal{A}_k(\mathcal{G})$  for the set of all  $k$ -apex-pseudoforests.

The problem of characterizing  $k$ -apices of graph classes, has attracted a lot of attention, both from the combinatorial and algorithmic point of view. This problem can be seen as a part of the wider family of *Graph Modification Problems* (where the modification is the removal of a vertex).

It is easy to see that if  $\mathcal{G}$  is minor closed, then  $\mathcal{A}_k(\mathcal{G})$  is also minor-closed for every  $k \geq 0$ . Therefore  $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$  can be seen as a complete characterization of  $k$ -apices of  $\mathcal{G}$ . The study of  $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$  when  $\mathcal{G}$  is some minor-closed graph class has attracted some special attention and can generate several known graph invariants. For instance, graphs with a vertex cover of size at most  $k$  are the graphs in  $\mathcal{A}_k(\mathbf{excl}(\{K_2\}))$ , graphs with a vertex feedback set at most  $k$  are the graphs in  $\mathcal{A}_k(\mathbf{excl}(\{K_3\}))$ , and  $k$ -apex planar graphs are the graphs in  $\mathcal{A}_k(\mathbf{excl}(\{K_5, K_{3,3}\}))$ .

The general problem that emerges is, given a finite set of graphs  $\mathcal{H}$  and a positive integer  $k$ , to identify the set

$$\mathcal{H}^{(k)} := \mathbf{obs}(\mathcal{A}_k(\mathbf{exc}(\mathcal{H}))).$$

A fundamental result in this direction is that the above problem is computable [2]. Moreover, it was shown in [20] that if  $\mathcal{H}$  contains some planar graph, then every graph in  $\mathcal{H}^{(k)}$  has  $O(k^h)$  vertices, where  $h$  is some constant depending (non-constructively) on  $\mathcal{H}$ . Also, in [43], it was proved that, under the additional assumption that all graphs in  $\mathcal{H}$  are connected, this bound becomes linear on  $k$  for the intersection of  $\mathcal{H}^{(k)}$  with sparse graph classes such as planar graphs or bounded degree graphs. An other structural result in this direction is the characterization of the disconnected obstructions in  $\mathcal{H}^{(k)}$  in the case where  $\mathcal{H}$  consists only of connected graphs [11] (see [Proposition 2.8](#)).

An other direction is to study  $\mathcal{H}^{(k)}$  for particular instantiations of  $\mathcal{H}$  and  $k$ . In this direction  $\{K_2\}^{(k)}$  has been identified for  $k \in \{1, \dots, 5\}$  in [7], for  $k = 6$  in [15] and for  $k = 7$  in [14], while the graphs in  $\{K_3\}^{(i)}$  have been identified in [12] for  $i \in \{1, 2\}$ . Recently, in [10], Ding and Dziobiak identified the 57 graphs in  $\{K_4, K_{2,3}\}^{(1)}$ , i.e., the obstruction set for apex-outerplanar graphs, and the 25 graphs in  $\{\triangleleft\}^{(1)}$ , i.e., the obstruction set for apex-cactus graphs (as announced in [18]). Moreover, the problem of identifying  $\{K_5, K_{3,3}\}^{(1)}$  (i.e., characterizing 1-apex planar graphs – also simply known as *apex graphs*) has attracted particular attention (see e.g., [29, 31, 42]). In this direction Mattman and Pierce conjectured that  $\{K_5, K_{3,3}\}^{(n)}$  consists of the  $Y\Delta Y$ -families of  $K_{n+5}$  and  $K_{3,2^{2^n}}$  and provided evidence on this [32]. Moreover, it has been shown that  $|\{K_5, K_{3,3}\}^{(1)}| > 150$ ,  $|\{K_5, K_{3,3}\}^{(2)}| > 82$ ,  $|\{K_5, K_{3,3}\}^{(3)}| > 601$ ,  $|\{K_5, K_{3,3}\}^{(4)}| > 520$ , and  $|\{K_5, K_{3,3}\}^{(5)}| > 608$  (see [34]). Recently, Jobson and Kézdy [23] identified all 2-connected graphs in  $\{K_5, K_{3,3}\}^{(1)}$ .

In this paper we identify  $\{\triangleleft, \bowtie\}^{(1)}$ , i.e., the obstruction set of apex-pseudoforests. Let  $\mathcal{O}^0, \mathcal{O}^1, \mathcal{O}^2, \mathcal{O}^3$  be the sets of graphs depicted in Figures 1, 2, 3, 4, respectively. Notice that, for every  $i \in \{0, 1, 2, 3\}$ , the graphs in  $\mathcal{O}^i$  are all  $i$ -connected but not  $(i + 1)$ -connected.

Our main result is the following.

**Theorem 1.1.**  $\mathbf{obs}(\mathcal{P}^{(1)}) = \mathcal{O}^0 \cup \mathcal{O}^1 \cup \mathcal{O}^2 \cup \mathcal{O}^3$ .

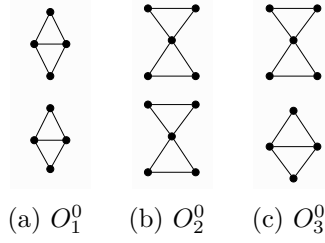


Figure 1: The set  $\mathcal{O}^0$  of obstructions for  $\mathcal{P}^{(1)}$  with vertex connectivity 0.

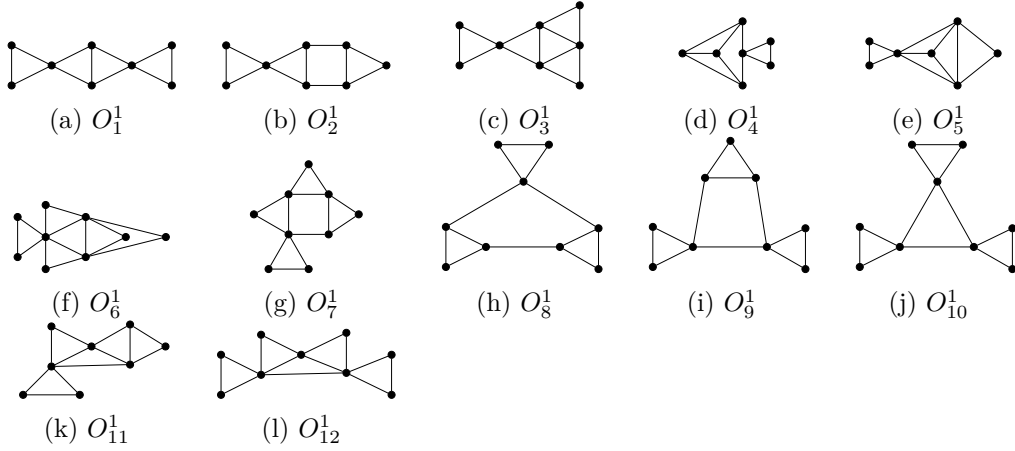


Figure 2: The set  $\mathcal{O}^1$  of obstructions for  $\mathcal{P}^{(1)}$  of vertex connectivity 1.

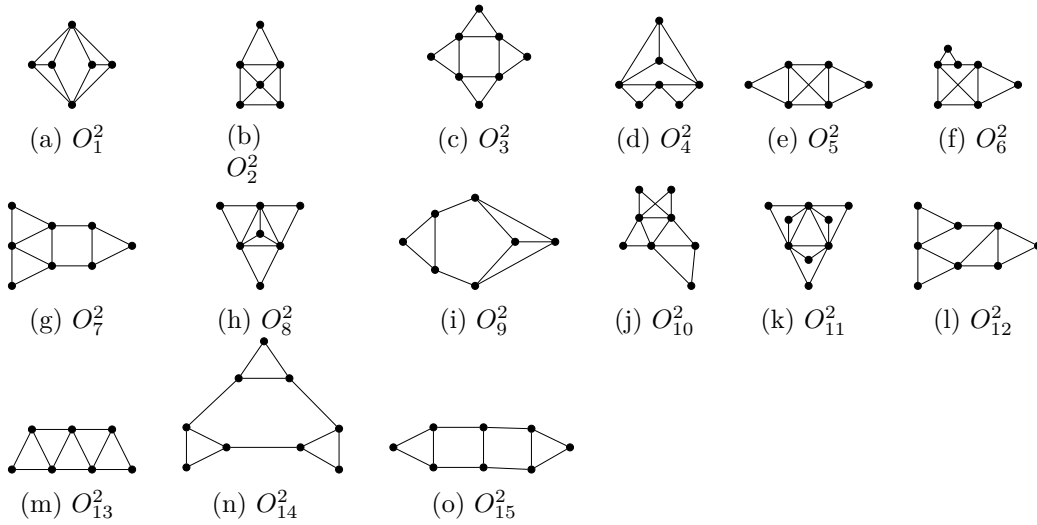


Figure 3: The set  $\mathcal{O}^2$  of obstructions for  $\mathcal{P}^{(1)}$  with vertex connectivity 2.

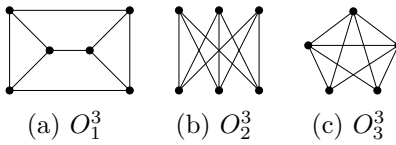


Figure 4: The set  $\mathcal{O}^3$  obstructions for  $\mathcal{P}^{(1)}$  with vertex connectivity 3.

We set  $\mathcal{O} = \mathcal{O}^0 \cup \mathcal{O}^1 \cup \mathcal{O}^2 \cup \mathcal{O}^3$ . For the proof of [Theorem 1.1](#), we first note, by inspection, that  $\mathbf{obs}(\mathcal{P}^{(1)}) \supseteq \mathcal{O}$ . As such an inspection might be quite tedious to do manually for all the 33 graphs in  $\mathcal{O}$ , one may use a computer program that can do this in an automated way (see [www.cs.upc.edu/~sedthilk/oapf/](http://www.cs.upc.edu/~sedthilk/oapf/) for code in SageMath that can do this). The main contribution of the paper is that  $\mathcal{O}$  is a complete list, i.e., that  $\mathbf{obs}(\mathcal{P}^{(1)}) \subseteq \mathcal{O}$ .

Our proof strategy is to assume that there exists a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  and gradually restrict the structure of  $G$  by deriving contradictions to some of the conditions of the following observation.

*Observation 1.2.* If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $G$  satisfies the following conditions:

1.  $G \notin \mathcal{P}^{(1)}$ ,
2. if  $G'$  is a minor of  $G$  that is different than  $G$ , then  $G' \in \mathcal{P}^{(1)}$ , and
3. none of the graphs in  $\mathcal{O}$  is a minor of  $G$ .

The rest of the paper is dedicated to the proof of [Theorem 1.1](#) and is organized as follows. In [Section 2](#) we give the basic definitions and some preliminary results. In [Section 3](#) we prove some auxiliary results that restrict the structure of the graphs in  $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ . In [Section 4](#) we use the results of [Section 3](#) in order to, first, prove that graphs in  $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  are biconnected ([Lemma 4.7](#)) and, next, prove that the graphs in  $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  are triconnected ([Lemma 4.10](#)). The proof of [Theorem 1.1](#) follows from the fact that every triconnected graph either contains a graph in  $\mathcal{O}^3$  or it is a graph in  $\mathcal{P}^{(1)}$  ([Lemma 2.7](#), proved in [Section 2](#)).

## 2 Definitions and preliminary results

**Sets and integers.** We denote by  $\mathbb{N}$  the set of all non-negative integers and we set  $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ . Given two integers  $p$  and  $q$ , we set  $[p, q] = \{p, \dots, q\}$  and given a  $k \in \mathbb{N}^+$  we denote  $[k] = [1, k]$ . Given a set  $A$ , we denote by  $2^A$  the set of all its subsets and we define  $\binom{A}{2} := \{e \mid e \in 2^A \wedge |e| = 2\}$ . If  $\mathcal{S}$  is a collection of objects where the operation  $\cup$  is defined, then we denote  $\mathbf{US} = \bigcup_{X \in \mathcal{S}} X$ .

**Graphs.** Given a graph  $G$ , we denote by  $V(G)$  the set of vertices of  $G$  and by  $E(G)$  the set of the edges of  $G$ . For an edge  $e = \{x, y\} \in E(G)$ , we use instead the notation  $e = xy$ , that is equivalent to  $e = yx$ . Given a vertex  $v \in V(G)$ , we define the *neighborhood* of  $v$  as  $N_G(v) = \{u \mid u \in V(G), \{u, v\} \in E(G)\}$  and the *closed neighborhood* of  $v$  as  $N_G[v] = N_G(v) \cup \{v\}$ . If  $X \subseteq V(G)$ , then we write  $N_G(X) = (\bigcup_{v \in X} N_G(v)) \setminus X$ . The *degree* of a vertex  $v$  in  $G$  is defined as  $\mathbf{deg}_G(v) = |N_G(v)|$ . We define  $\delta(G) = \min\{\mathbf{deg}_G(x) \mid x \in V(G)\}$ . Given two graphs  $G_1, G_2$ , we define the *union* of  $G_1, G_2$  as the graph  $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$  and the *intersection* of  $G_1, G_2$  as the graph  $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$ . A *subgraph* of a graph  $G = (V, E)$  is every graph  $H$  where  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . If  $S \subseteq V(G)$ , the subgraph of  $G$  *induced by*  $S$ , denoted by  $G[S]$ , is the graph  $(S, E(G) \cap \binom{S}{2})$ . We also define  $G \setminus S$  to be the subgraph of  $G$  induced by  $V(G) \setminus S$ . If  $S \subseteq E(G)$ , we denote by  $G \setminus S$  the graph  $(V(G), E(G) \setminus S)$ . Given a vertex  $x \in V(G)$  we define  $G \setminus x = G \setminus \{x\}$  and given an edge  $e \in E(G)$  we define  $G \setminus e = G \setminus \{e\}$ .

**Paths and separators.** If  $s, t \in V(G)$ , an  $(s, t)$ -*path* of  $G$  is any connected subgraph  $P$  of  $G$  with maximum degree 2, where  $\mathbf{deg}_P(s) = 1$  and  $\mathbf{deg}_P(t) = 1$ . The *distance* between  $s$  and  $t$  in  $G$

is the minimum number of edges of an  $(s, t)$ -path in  $G$ . Given a path  $P$ , we say that  $v \in V(P)$  is an *internal vertex* of  $P$  if  $\deg_P(v) = 2$ , while if  $\deg_P(v) = 1$  we say that  $v$  is a *terminal vertex* of  $P$ . We say that two paths  $P_1$  and  $P_2$  in  $G$  are *internally vertex disjoint* if none of the internal vertices of the one is an internal vertex of the other. Given an integer  $k$  and a graph  $G$ , we say that  $G$  is  $k$ -connected if for each  $\{u, v\} \in \binom{V}{2}$ , there exists  $k$  pairwise internally disjoint  $(u, v)$ -paths of  $G$ , say  $P_1, \dots, P_k$ , such that for each  $\{i, j\} \in \binom{[k]}{2}$ ,  $P_i \neq P_j$ ,  $V(P_i) \cap V(P_j) = \{u, v\}$ . We call 2-connected graphs *biconnected* and 3-connected graphs *triconnected*. Given a set  $S \subseteq V(G)$ , we say that  $S$  is a *separator* of  $G$  if  $G$  has less connected components than  $G \setminus S$ . We call a separator of size  $k$  a  $k$ -separator. Notice that, by Menger's theorem a graph is  $k$ -connected iff it does not contain a separator of size  $< k$ . A *block* of a graph  $G$  is a maximal biconnected subgraph.

A vertex  $v \in V(G)$  is a *cut-vertex* of  $G$  if  $\{v\}$  is a separator of  $G$ . We also say that  $S$  is a *rich separator* if  $G \setminus S$  has at least 2 more connected components than  $G$ .

**Special graphs.** By  $K_r$  we denote the complete graph on  $r$  vertices. Similarly, by  $K_{r_1, r_2}$  we denote the complete bipartite graph of which one part has  $r_1$  vertices and the other  $r_2$ .

For an  $r \geq 3$ , we denote by  $C_r$  the connected graph on  $r$  vertices of degree 2 (i.e., the cycle on  $r$  vertices). If  $G$  is a graph and  $C$  is a subgraph of  $G$  isomorphic to  $C_r$  for some  $r \geq 3$ , then an edge  $e = \{u, v\} \in E(G) \setminus E(C)$  where  $u, v \in V(C)$  is called *chord* of  $C$ .

For  $r \geq 3$ , the  $r$ -wheel, denoted by  $W_r$ , is the graph obtained by adding a new vertex  $v_{\text{new}}$ , called the *central vertex* of  $W_r$ , to  $C_r$  along with edges, called *spokes*, connecting each vertex of  $C_r$  with  $v_{\text{new}}$ . The subgraph  $W_r \setminus v_{\text{new}}$  is called the *circumference* of  $W_r$ .

A graph  $G$  is *outerplanar* if it can be embedded in the plane so that there's no crossing edges and all its vertices lie on the same face. It is known that the obstruction set of the class of outerplanar graphs is  $\{K_{2,3}, K_4\}$ . The outer face of such an embedding contains every vertex of  $G$ . Thus, we can observe the following:

*Observation 2.1.* If  $G$  is biconnected and outerplanar then  $G$  contains a *Hamiltonian cycle*, i.e. a cycle which contains every vertex of  $G$ .

**Minors.** We define  $G/e$ , the graph obtained from the graph  $G$  by *contracting* an edge  $e = xy \in E(G)$ , to be the graph obtained by replacing the edge  $e$  by a new vertex  $v_e$  which becomes adjacent to all neighbors of  $x, y$  (apart from  $y$  and  $x$ ). Given two graphs  $H$  and  $G$  we say that  $H$  is a *minor* of  $G$ , denoted by  $H \leq G$ , if  $H$  can be obtained by some subgraph of  $G$  after contracting edges.

Given a set  $\mathcal{H}$  of graphs, we write  $\mathcal{H} \leq G$  to denote that  $\exists H \in \mathcal{H} : H \leq G$  and we defined  $\text{excl}(\mathcal{H}) = \{G \mid \mathcal{H} \not\leq G\}$ . If  $H \not\leq G$ , then we say that  $G$  is *H-minor free*, or, in short, *H-free*. Also, given a graph  $G$  and a set of graphs  $\mathcal{H}$  we say that  $G$  is  $\mathcal{H}$ -free if it is  $H$ -free, for each  $H \in \mathcal{H}$ . Given a graph class  $\mathcal{G}$  we say that  $\mathcal{G}$  is *minor-closed* if  $\forall H H \leq G \wedge G \in \mathcal{G} \Rightarrow H \in \mathcal{G}$ . We also define  $\text{obs}(\mathcal{G})$  as the set of all minor-minimal graphs that do not belong in  $\mathcal{G}$  and we call  $\text{obs}(\mathcal{G})$  the *obstruction set* of the class  $\mathcal{G}$ .

If  $e = xy$  is an edge of a graph  $G$  then the operation of replacing  $e$  by a path of length 2, i.e. two edges  $\{x, v_e\}, \{v_e, y\}$ , where  $v_e$  is a new vertex, is called *subdivision* of  $e$ . A graph  $G$  is called a *subdivision* of a graph  $H$  if  $G$  can be obtained from  $H$  by repeatedly subdividing edges, i.e. by replacing some edges of  $H$  with new paths between its endpoints, so that the intersection of any two such paths is either empty or a vertex of  $H$ . The original vertices of  $H$  are called *branch vertices*, while the new vertices are called *subdividing vertices*. If a graph  $G$  contains a subdivision of  $H$  as

a subgraph, then  $H$  is a *topological minor* of  $G$ . It is easy to see that if  $H$  is a topological minor of  $G$  then it is also a minor of  $G$ .

Let  $G$  be a subdivision of some  $W_r$  wheel. In keeping with the notation previously introduced for wheels, we define the *spokes* of  $G$  to be the paths of  $G$  produced by the subdivision of the spokes of  $W_r$  and similarly we define the *circumference* of  $G$  to be the cycle of  $G$  produced by the subdivision of the circumference of  $W_r$ .

The following is an easy consequence of Dirac's Theorem [16], stating that if  $\delta(G) \geq 3$ , then  $G$  contains  $K_4$  as a minor.

**Proposition 2.2.** *Let  $G$  be a biconnected  $K_4$ -free graph. Then there exist at least two vertices of degree 2.*

**Triconnected components.** Let  $G$  be a graph and  $S \subseteq V(G)$  and let  $V_1, \dots, V_q$  be the vertex sets of the connected components of  $G \setminus S$ . We define  $\mathcal{C}(G, S) = \{G_1, \dots, G_q\}$  where, for  $i \in [q]$ ,  $G_i$  is the graph obtained from  $G[V_i \cup S]$  if we add all edges between vertices in  $S$ . We call the members of the set  $\mathcal{C}(G, S)$  *augmented connected components*. Given a vertex  $x \in V(G)$  we define  $\mathcal{C}(G, x) = \mathcal{C}(G, \{x\})$ .

Given a graph  $G$ , the set  $\mathcal{Q}(G)$  of its *triconnected components* is recursively defined as follows:

- If  $G$  is triconnected or a clique of size  $\leq 3$ , then  $\mathcal{Q}(G) = \{G\}$ .
- If  $G$  contains a separator  $S$  where  $|S| \leq 2$ , then  $\mathcal{Q}(G) = \bigcup_{H \in \mathcal{C}(G, S)} \mathcal{Q}(H)$ .

Notice that all graphs in  $\mathcal{Q}(G)$  are either cliques on at most 3 vertices or triconnected graphs (graphs without any separator of less than 3 vertices). The study of triconnected components of plane graphs dates back to the work of Saunders Mac Lane in [30] (see also [41]).

*Observation 2.3.* Let  $G$  be a graph. All graphs in  $\mathcal{Q}(G)$  are topological minors of  $G$ .

Let  $G$  be a graph and  $v \in V(G)$  where  $\mathbf{deg}_G(v) \geq 4$ . Let also  $\mathcal{P}_v = \{A, B\}$  be a partition of  $N_G(v)$  such that  $|A|, |B| \geq 2$ . We define the  $\mathcal{P}_v$ -*split* of  $G$  to be the graph  $G'$  obtained by adding, in the graph  $G \setminus v$ , two new adjacent vertices  $v_A$  and  $v_B$  and making  $v_A$  adjacent to the vertices of  $A$  and  $v_B$  adjacent to the vertices of  $B$ . If  $G'$  can be obtained by some  $\mathcal{P}_v$ -split of  $G$ , we say that  $G'$  is a *splitting* of  $G$ .

*Observation 2.4.* If  $G'$  is a splitting of  $G$  then  $G$  is a minor of  $G'$ .

**Proposition 2.5** (Tutte [40]). *A graph  $G$  is triconnected if and only if there is a sequence of graphs  $G_0, \dots, G_q$  such that  $G_0$  is isomorphic to  $W_r$  for some  $r \geq 3$ ,  $G_q = G$ , and for  $i \in [q]$ ,  $G_i$  is a splitting of  $G_{i-1}$  or  $\exists e \in E(G_i) : G_{i-1} = G_i \setminus e$ .*

The next proposition is a direct consequence of [Observation 2.3](#) and [Proposition 2.5](#).

**Proposition 2.6.** *Let  $G$  be a graph.  $K_4 \not\leq G$  if and only if none of the graphs in  $\mathcal{Q}(G)$  is triconnected.*

**Lemma 2.7.** *If  $G$  is a triconnected graph that is not isomorphic to  $W_r$ , for some  $r \geq 3$ , then  $\mathcal{O}^3 \leq G$ .*

*Proof.* Let  $G$  be a triconnected graph not isomorphic to a wheel. By [Proposition 2.5](#), there exists a sequence of graphs

$$W_r = G_0, G_1, \dots, G_q = G$$

for some  $r \geq 3$ , such that for every  $i \in [q]$ ,  $G_i$  is a splitting of  $G_{i-1}$  or there exists an edge  $e \in E(G_i)$  such that  $G_{i-1} = G_i \setminus e$ . Observe that  $r \geq 4$ , since if  $r = 3$  then  $q = 0$  due to the fact that none of the vertices of  $W_3$  can be split and all of them are adjacent to one another. Also, since  $G \not\cong W_r$  we have that  $q \geq 1$ .

Let  $z$  be the central vertex of  $W_r$  and  $C_r = W_r \setminus z$ . We examine how the graph  $G_1$  may occur from  $W_r$ . For that, we distinguish the following two cases:

*Case 1:* There exists an edge  $e \in E(G_1)$  such that  $W_r = G_1 \setminus e$ . Let  $e = uv$  for some  $u, v \in V(G_1) = V(W_r)$ . Since every vertex in  $V(C_r)$  is in the neighborhood of  $z$  in  $W_r$  then  $u \neq z$  and  $v \neq z$ . Now, since  $u, v$  are not adjacent vertices in  $W_r$  there exist an internal vertex in each of the two  $(u, v)$ -paths of the graph  $C_r$ , say  $x, y$ , respectively (see [Figure 5](#)). Therefore, by contracting each of the  $(x, u), (x, v), (y, v), (y, u)$ -paths of  $C_r$  to an edge we get  $\mathcal{O}_3^3$  as a minor of  $G_1$ . Now, by [Observation 2.4](#),  $G_1$  is a minor of  $G$  and therefore  $\mathcal{O}_3^3 \leq G$ , a contradiction.

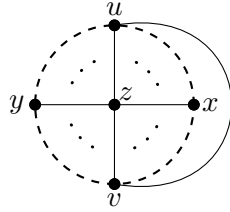


Figure 5: The structure of the graph  $G$  in Case 1.

*Case 2:*  $G_1$  is a splitting of  $W_r$ . Observe that  $G_1$  is a splitting of  $W_r$  obtained by a  $\mathcal{P}_z$ -split. So, let  $\mathcal{P}_z = \{A, B\}$  and  $v_A, v_B$  the new adjacent vertices of  $G_1$ , where  $N_{G_1}(v_A) = A$  and  $N_{G_1}(v_B) = B$ . We have that  $|A|, |B| \geq 2$  and so there exist  $x_1, y_1 \in A$  and  $x_2, y_2 \in B$ . We now distinguish the following subcases:

*Subcase 2.1:* One of the two  $(x_1, y_1)$ -paths in  $C_r$  contains both of  $x_2, y_2$  (see leftmost figure of [Figure 6](#)). This implies that  $\mathcal{O}_1^3 \leq G_1$  and, as in case 1, it follows that  $\mathcal{O}_1^3 \leq G$ , a contradiction.

*Subcase 2.2:* Each one of the two  $(x_1, y_1)$ -paths in  $C_r$  contains exactly one of  $x_2, y_2$  (see rightmost figure of [Figure 6](#)). This implies that  $\mathcal{O}_2^3 \leq G_1$  and, as in case 1, it follows that  $\mathcal{O}_2^3 \leq G$ , a contradiction.

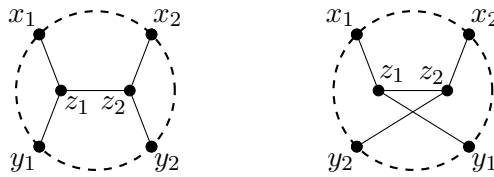


Figure 6: The structure of the graph  $G$  in the two Subcases of Case 2.

Since we have exhausted all possible cases for  $G_1$  we conclude that  $G \in \mathcal{O}^3$ , a contradiction.  $\square$



**Disconnected obstructions.** We need the following result by Dinneen (see [11]).

**Proposition 2.8.** *Let  $\mathcal{H}$  be a set of connected graphs. For every  $k \in \mathbb{N}$ , if  $H \in \mathcal{H}^{(k)}$  and  $H_1, \dots, H_r$  are the connected components of  $H$ , then there is a sequence  $k_1, \dots, k_r$  such that  $\sum_{i \in [r]} k_i = k + 1$  and  $H_i \in \mathcal{H}^{(k_i-1)}$ ,  $i \in [r]$ .*

**Lemma 2.9.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $G$  is connected.*

*Proof.* As both graphs in  $\mathbf{obs}(\mathcal{P}) = \{\triangleleft, \bowtie\}$  are connected, **Proposition 2.8** applies for  $\mathcal{H} = \mathbf{obs}(\mathcal{P})$  and  $k = 1$ . This means that  $\mathcal{O}^{(0)}$  contains all disconnected graphs in  $\mathbf{obs}(\mathcal{P}^{(1)})$ . Therefore every  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  should be a connected graph.  $\square$

### 3 Auxiliary lemmata

By **Lemma 2.9**, we know that a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  should be connected. In this section we prove a series of lemmata that further restrict the structure of the graphs in  $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ .

#### 3.1 General properties of the obstructions

Given a graph  $G$  and a vertex  $v \in V(G)$  we say that  $v$  is *simplicial* if  $G[N_G(v)]$  is isomorphic to  $K_r$  for  $r = \mathbf{deg}_G(v)$ . We say that  $e \in E(G)$  is a *bridge* if  $G$  has less connected components than  $G \setminus e$ . A graph which does not contain any bridge is called *bridgeless*.

Given a graph class  $\mathcal{G}$ , a graph  $G$ , and a vertex  $x$ , where  $G \setminus x \in \mathcal{G}$ , then we say that  $x$  is a  $\mathcal{G}$ -*apex* of  $G$ .

**Lemma 3.1.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then*

1.  $\delta(G) \geq 2$ ,
2.  $G$  is bridgeless and
3. all its vertices of degree 2 are simplicial.

*Proof.* (1) Consider a vertex  $u \in V(G)$  with  $\mathbf{deg}_G(u) < 2$ . If  $G \setminus u \in \mathcal{P}^{(1)}$ , then also  $G \in \mathcal{P}^{(1)}$ , since  $u$  does not participate in a cycle, a contradiction.

(2) Consider an edge  $e = xy$  that is a bridge of  $G$ . By **Lemma 2.9**,  $G$  is connected. Since  $e$  is a bridge, then  $G \setminus e$  contains two connected components  $H_1, H_2$ , such that  $x \in V(H_1)$  and  $y \in V(H_2)$ . Observe that by  $\mathcal{O}^0$ -freeness of  $G$ , one of  $H_1, H_2$ , say  $H_1$ , contains at most one cycle and therefore, due to (1),  $H_1$  is isomorphic to a cycle.

Consider the graph  $G' = G/e$  and let  $v_e$  be the vertex formed by contracting  $e$ . We denote  $H'_1, H'_2$  the graphs obtained from  $H_1, H_2$  by replacing the vertices  $x, y$  with  $v_e$ , respectively. Observe that  $H'_1$  is also isomorphic to a cycle. By minor-minimality of  $G$ , it holds that  $G' \in \mathcal{P}^{(1)}$  and therefore there exists some  $u \in V(G')$  that is a  $\mathcal{P}$ -apex of  $G'$ . So, if  $u \in V(H'_1)$  then  $v_e$  is also a  $\mathcal{P}$ -apex of  $G'$ . Therefore we consider the case that  $u \in V(H'_2)$ . If  $u = v_e$ , then every connected component of  $H'_2 \setminus v_e$  contains at most one cycle. Since,  $H'_2 \setminus v_e = H_2 \setminus y$  then also every connected component of  $G \setminus y$  contains at most one cycle, a contradiction. If  $u \neq v_e$ , then consider the augmented connected component  $Q' \in \mathcal{C}(G', u)$  which contains  $v_e$ . Also, let  $Q$  be the augmented connected component of  $\mathcal{C}(G, u)$  that contains  $e$ . Observe that, since  $Q'$  contains at most one cycle, the same holds for  $Q$ . Hence,  $G \setminus u \in \mathcal{P}$ , a contradiction.

(3) Suppose, to the contrary, that there exists a non-simplicial vertex  $v \in V(G)$  of degree 2, and let  $e \in E(G)$  be an edge incident to  $v$ , i.e.  $e = uv$  for some  $u \in V(G)$ . By minor-minimality of  $G$ , we have that  $G' := G/e \in \mathcal{P}^{(1)}$ . Let  $x$  be an  $\mathcal{P}$ -apex vertex of  $G'$  and  $v_e$  the vertex formed by contracting  $e$ . Observe that, every cycle in  $G$  that contains  $v$  also contains  $u$  and so if  $x = v_e$  then  $u$  is an  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction. Therefore,  $x \neq v_e$  and so  $x \in V(G)$ . Since  $v$  is a non-simplicial vertex, the contraction of  $e$  can only shorten cycles and not destroy them. Hence,  $x$  is an  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction.  $\square$

For a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ , observe that, due to [Lemma 3.1](#), all of its connected components and blocks contain a cycle. Moreover, for such  $G$ , all graphs in  $\mathcal{Q}(G)$  are either triconnected or isomorphic to  $K_3$ .

### 3.2 Properties of obstructions containing a $K_4$

We now prove some Lemmata which will be useful in the main section of the proof.

**Lemma 3.2.** *If  $G$  is a biconnected graph such that  $\mathcal{O} \not\subseteq G$ , then there exists at most one triconnected graph in  $\mathcal{Q}(G)$ .*

*Proof.* Suppose, to the contrary, that there are at least two triconnected graphs in  $\mathcal{Q}(G)$  and let  $H_1, H_2$  be two of them. Due to the recursive definition of  $\mathcal{Q}(G)$  and by [Observation 2.3](#), there exists a separator  $S$  such that  $H_1, H_2$  are topological minors of some  $G_1, G_2 \in \mathcal{C}(G, S)$ , respectively. By the biconnectivity of  $G$  we have that  $S$  is a 2-separator of  $G$  and let  $S = \{x, y\}$ . Since  $H_1, H_2$  are triconnected graphs and topological minors of  $G$  then, by [Lemma 2.7](#), we have that each  $H_i$ ,  $i \in [2]$  is isomorphic to a wheel. Thus,  $K_4$  is a topological minor of both  $H_1$  and  $H_2$ . Let  $Q_1, Q_2$  be the subdivisions of  $K_4$  in  $G_1, G_2$ , respectively.

*Claim:* For each  $G_i, i \in \{1, 2\}$ ,  $\triangleleft$  is a topological minor of  $G_i$  such that  $x, y$  are the branch vertices of  $\triangleleft$  of degree 2.

*Proof of Claim:* Let  $i \in \{1, 2\}$ . By Menger's theorem, there exist two disjoint paths from the separator  $S$  to  $Q_i$ . Let  $P_1, P_2$  be the shortest such paths (as in [Figure 7](#)) and for each of  $P_1, P_2$  identify its endpoints.

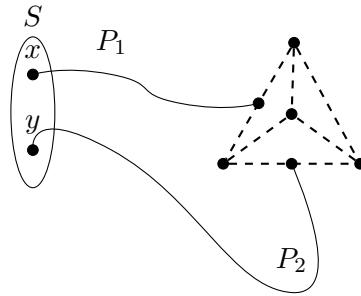


Figure 7: The paths  $P_1, P_2$  from  $S$  to  $Q_i$ .

Therefore, there exists a subdivision  $A$  of  $K_4$  in  $G_i$  such that  $x, y \in V(A)$ . Let  $e = u_x v_x, e' = u_y v_y$  be the subdivided edges of  $K_4$  that contain  $x, y$ , respectively.

Observe that if  $e = e'$  then the desired result holds, while if  $e \neq e'$ , then there exists some  $a \in \{u_x, v_x\}$  and  $b \in \{u_y, v_y\}$  such that  $a \neq b$ . Since  $ab$  is a subdivided edge of  $K_4$ , then by identifying  $x$  with  $a$  and  $y$  with  $b$ , we get the desired result. Claim follows.

Therefore, by applying the above Claim for both  $G_1, G_2$  we get  $O_1^2$  as a minor of  $G$ , a contradiction.  $\square$

The results of [Lemma 2.7](#) and [Lemma 3.2](#), together with [Observation 2.3](#) and [Proposition 2.6](#) imply the following corollary:

**Corollary 3.3.** *Let  $G$  be a biconnected graph such that  $\mathcal{O} \not\leq G$  and  $K_4 \leq G$ . Then there exists a unique triconnected graph  $H$  in  $\mathcal{Q}(G)$  such that:*

- $H$  is isomorphic to an  $r$ -wheel for some  $r \geq 3$  and
- $H$  is a topological minor of  $G$ .

Let  $G$  be a biconnected graph such that  $\mathcal{O} \not\leq G$  and  $K_4 \leq G$  and let  $K$  be a subdivision of the (unique)  $r$ -wheel  $H \in \mathcal{Q}(G)$ , as in [Corollary 3.3](#). We call the pair  $(H, K)$  an  $r$ -wheel-subdivision pair of  $G$ . Notice that there may be many  $r$ -wheel-subdivision pairs in  $G$ , as there might be many possible choices for  $K$  (but there is only one choice for  $H$ ).

**Lemma 3.4.** *Let  $G$  be a biconnected graph such that  $K_4 \leq G$  and  $\mathcal{O} \not\leq G$ . Let  $(H, K)$  be an  $r$ -wheel-subdivision pair of  $G$ . Then for every  $(x, y)$ -path which intersects  $K$  only in its endpoints, there exists an edge  $e \in E(H)$  such that  $x, y$  are both vertices of the subdivision of  $e$  in  $K$ .*

*Proof.* Recall that  $H$  is isomorphic to an  $r$ -wheel for some  $r \geq 3$ , and  $K$  is a subdivision of  $H$ . Consider an  $(x, y)$ -path which intersects  $K$  only in its endpoints. Suppose, to the contrary, that  $x, y$  belong to subdivisions of different edges of  $H$ . We distinguish the following cases:

*Case 1:* One of  $x, y$ , say  $x$ , is a branch vertex on the circumference of  $K$ .

First, we observe the following:

*Observation 1:*  $r \neq 3$ . Indeed, if  $H \cong W_3$ , then since  $y$  belongs to the subdivision of an edge of  $H$  not incident to  $x$ , a subdivision of a bigger wheel would be formed with  $x$  as its central vertex (see [Figure 8](#)), a contradiction to the definition of the triconnected components.

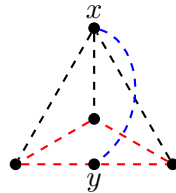


Figure 8: The  $(x, y)$ -path (depicted in blue) where  $y$  belongs to the subdivision of some edge of  $H$  not incident to  $x$  (depicted in red) in the proof of [Observation 1](#).

Suppose then that  $r \geq 4$ . Let  $x_1, x_2$  be the vertices adjacent to  $x$  on the circumference of  $H$ . We distinguish the following subcases:

*Subcase 1.1:*  $y$  belongs to the subdivision of some spoke  $e$  of  $H$ . Then,  $e$  is not incident to  $x$ . If  $y$  is an internal vertex of the subdivision of a spoke incident to either  $x_1$  or  $x_2$  then  $O_1^3 \leq G$  (see leftmost figure of [Figure 9](#)), while if the spoke is not incident to  $x_1$  or  $x_2$  then  $O_3^3 \leq G$  (see central figure of [Figure 9](#)), a contradiction in both cases.

*Subcase 1.2:*  $y$  belongs to some subdivided edge  $e$  of the circumference of  $K$ . Then,  $e$  is different from the subdivided edges corresponding to  $xx_1, xx_2$ . Hence,  $O_3^3 \leq G$  (see rightmost figure of Figure 9), a contradiction.

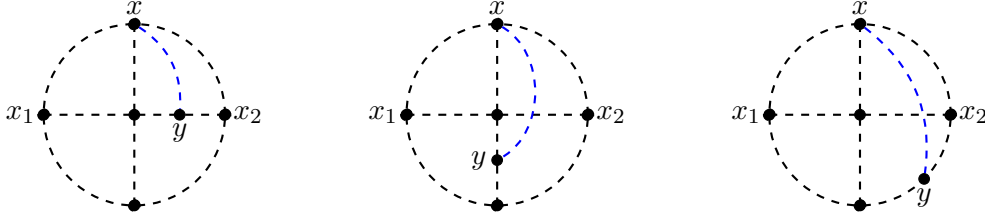


Figure 9: Possible configurations of the  $(x, y)$ -path (depicted in blue) in the proof of Subcases of Case 1.

*Case 2:* One of  $x, y$ , say  $x$ , is a subdividing vertex on the circumference of  $K$ .

Since we have examined the case that one of  $x, y$  is a branch vertex on the circumference of  $K$ , suppose that  $y$  is not such.

Let  $e = uv$  be the edge of  $H$  whose the corresponding subdivision in  $K$  contains  $x$ .

*Observation 2:*  $y$  is not the central vertex of  $H$ . This is because, if otherwise, a subdivision of a bigger wheel would be formed in  $G$  (see Figure 10), which a contradiction to the definition of the triconnected components.

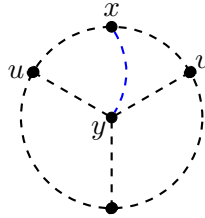


Figure 10: The  $(x, y)$ -path (depicted in blue) in the proof of Observation 2, where  $y$  is the central vertex of  $H$ .

*Subcase 2.1:*  $y$  is an internal vertex of some subdivided edge  $e'$  of a spoke of  $K$ . If  $e'$  is incident to either  $u$  or  $v$ , then  $O_1^3 \leq G$  (see leftmost figure of Figure 11), while if  $e'$  is not incident to either  $u$  or  $v$  then  $O_2^3 \leq G$  (see central figure of Figure 11), a contradiction in both cases.

*Subcase 2.2:*  $y$  is an internal vertex of some subdivided edge of the circumference of  $K$  different from  $e$ . Then  $O_1^3 \leq G$  (see rightmost figure of Figure 11), a contradiction.

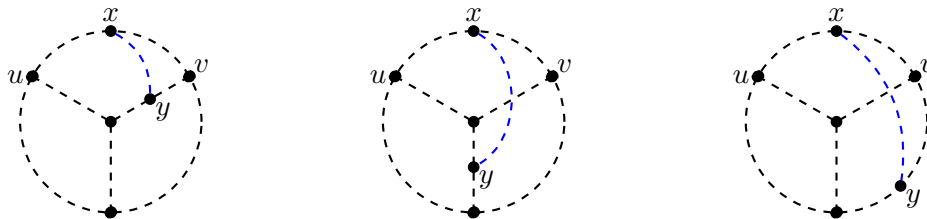


Figure 11: Possible configurations of the  $(x, y)$ -path (depicted in blue) in the proof of Subcases of Case 2.

*Case 3:* Both of  $x, y$  are internal vertices of the subdivisions of some spokes  $e, e'$  of  $W_r$ , respectively.

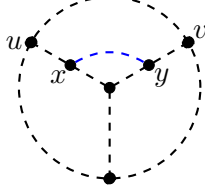


Figure 12: The  $(x, y)$ -path (depicted in blue) in the proof of Case 3.

In this case,  $e, e'$  are distinct and so  $O_1^3 \leq G$  (see Figure 12), a contradiction.  $\square$

We can now define the notion of a *flap*. Let  $G$  be a biconnected graph such that  $\mathcal{O} \not\leq G$  and  $K_4 \leq G$ . Let also  $(H, K)$  be an  $r$ -wheel-subdivision pair of  $G$ . A  $(x, y)$ -flap of  $(H, K)$  corresponding to some separator  $S = \{x, y\} \subseteq V(K)$  of  $G$  is the graph  $\mathbf{U}\{C \in \mathcal{C}(G, S) : K_4 \not\leq C \wedge C \text{ is biconnected}\}$ . Given an  $(x, y)$ -flap  $F$  of  $(H, K)$  and a vertex  $v \in V(F)$ , we say that  $F$  is  $v$ -oriented if every cycle of  $F$  contains  $v$ .

Regarding the arguments in the remaining part of Subsection 3.2, consider a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  such that  $K_4 \leq G$ . Observe that  $K_4$  is a minor of a block  $B$  of  $G$  and therefore we can consider an  $r$ -wheel-subdivision pair of  $B$ .

**Lemma 3.5.** *Let  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  such that  $K_4 \leq G$  and  $(H, K)$  be an  $r$ -wheel-subdivision pair of a block  $B$  of  $G$ . If  $F$  is an  $(x, y)$ -flap of  $(H, K)$ , then it holds that:*

1.  $F$  is biconnected,
2.  $G[V(F)]$  contains a cycle, and
3. there exists an edge  $e \in E(H)$  such that  $x, y$  are both vertices of the subdivision of  $e$  in  $K$ .

*Proof.* Observe that (1) and (2) are direct consequences of the definition of the  $(x, y)$ -flap and Lemma 3.1. To prove (3) suppose, towards a contradiction, that there exists an  $(x, y)$ -flap such that  $x, y$  belong to subdivisions of different edges of  $K$ . Then, from the definition of the  $(x, y)$ -flap, there exists a biconnected graph  $C \in \mathcal{C}(G, S)$  such that  $K_4 \not\leq C$  and hence it contains an  $(x, y)$ -path that intersects  $K$  only in its endpoints. We arrive at a contradiction to the result of Lemma 3.4.  $\square$

We conclude this subsection by proving the next result concerning flaps:

**Lemma 3.6.** *Let  $G$  be a biconnected graph such that  $\mathcal{O} \not\leq G$  and  $K_4 \leq G$  and let  $(H, K)$  be an  $r$ -wheel-subdivision pair of  $G$ . Then every  $(x, y)$ -flap  $F$  of  $(H, K)$  is either  $x$ -oriented or  $y$ -oriented.*

*Proof.* Consider an  $(x, y)$ -flap  $F$  of  $(H, K)$  for which the contrary holds. We distinguish the following cases:

*Case 1:* There exists a cycle  $C$  in  $F$  disjoint to both  $x, y$ . Then, since  $F$  is biconnected, there exist two disjoint paths  $P_1, P_2$  connecting the cycle  $C$  with  $x, y$ , respectively. Hence, by contracting all the edges of  $P_1, P_2$  we form  $O_9^2$  as a minor of  $G$ , a contradiction (see Figure 13).

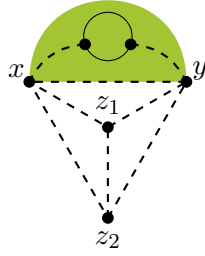


Figure 13: An example of an  $(x, y)$ -flap that contains a cycle disjoint to  $x, y$  in the the proof of Case 1.

*Case 2:* There exists a cycle  $C$  of  $F$  that contains  $x$  but not  $y$  and a cycle  $C'$  that contains  $y$  but not  $x$ . Then, if  $C, C'$  are disjoint,  $O_1^1 \leq G$ , if they share only one vertex,  $O_4^2 \leq G$ , and if they share more than one vertex,  $O_1^2 \leq G$ , a contradiction in all cases (see figure Figure 14).

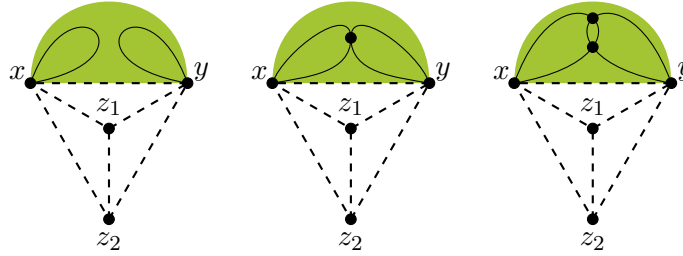


Figure 14: The ways  $C, C'$  may intersect in the proof of Case 2.

Since we arrived at a contradiction in both cases, Lemma follows. □

**Lemma 3.7.** *Let  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  such that  $K_4 \leq G$  and let  $(H, K)$  be an  $r$ -wheel-subdivision pair of a block  $B$  of  $G$ . Then if  $r \geq 4$ , the center of  $K$  is a  $\mathcal{P}$ -apex vertex of  $B$ .*

*Proof.* By Lemma 3.2,  $H$  is the unique triconnected graph in  $\mathcal{Q}(B)$ . By Lemma 3.5, for every  $(x, y)$ -flap of  $(H, K)$  there exists an edge  $e \in E(H)$  such that  $x, y$  are both vertices of the subdivision of  $e$  in  $K$ . Keep in mind that, again by Lemma 3.5, every flap  $F$  of  $(H, K)$  is biconnected and  $G[V(F)]$  contains a cycle.

*Observation:* For every  $(x, y)$ -flap it holds that one of  $x, y$  is the centre of  $K$ . Indeed, if there exists an  $(x, y)$ -flap corresponding to some  $e \in E(H)$  such that neither of  $x, y$  is the center of  $K$ , then if  $e$  is an edge of the circumference of  $H$ ,  $O_2^2 \leq G$  (see left figure of Figure 15), while if  $e$  is a spoke of  $H$ ,  $O_4^1 \leq G$  (see right figure of Figure 15), a contradiction in both cases.

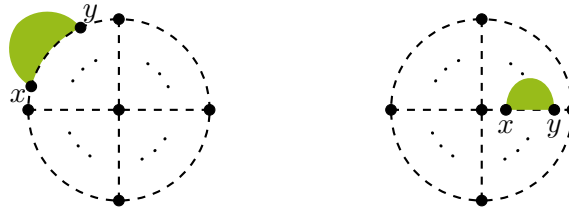


Figure 15: Possible configurations of an  $(x, y)$ -flap (depicted in green) in the proof of Observation.

Let  $z$  be the centre of  $K$ . According to Observation, every flap of  $(H, K)$  is a  $(z, y)$ -flap. Furthermore, notice that every  $(z, y)$ -flap  $F$  is  $z$ -oriented. Indeed, if otherwise, then [Lemma 3.6](#) implies that there would exist a cycle in  $F$  containing  $y$  but not  $z$  and hence  $O_4^1 \leq G$ , which is a contradiction.

Therefore, taking into account that  $B$  is a block of  $G$ , every cycle in  $B$ , except for the circumference of  $K$ , contains  $z$  and so  $z$  is a  $\mathcal{P}$ -apex vertex of  $B$ .  $\square$

### 3.3 Properties of obstructions containing a $K_{2,3}$

The purpose of this section is to prove [Lemma 3.11](#) that gives us some information on the structure of a connected graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  that is  $K_4$ -free and contains  $K_{2,3}$  as a minor.

Let  $S$  be a 2-separator of  $G$  and  $B$  be a block of some  $H \in \mathcal{C}(G, S)$ . We say that  $B$  is an  $S$ -block of  $G$  if  $S \subseteq V(B)$ . We call  $S$  a  $b$ -rich separator if at least three graphs in  $\mathcal{C}(G, S)$  contain  $S$ -blocks. We start with an easy observation.

*Observation 3.8.* Let  $G$  be a  $K_4$ -free graph such that  $K_{2,3} \leq G$ . Then  $G$  contains a  $b$ -rich separator.

Using [Lemma 3.1](#), we easily derive the next observation that will be frequently used in the course of the remaining proofs of this section.

*Observation 3.9.* Let  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  be a graph,  $S$  be a  $b$ -rich separator of  $G$ , and  $H \in \mathcal{C}(G, S)$  that contains an  $S$ -block. Then the graph  $G[V(H)]$  contains a cycle.

**Lemma 3.10.** *Let  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  be a connected  $K_4$ -free graph such that  $K_{2,3} \leq G$ . Then  $G$  contains a unique  $b$ -rich separator.*

*Proof.* We first prove the following claim:

*Claim 1:* If  $S_1, S_2, \dots, S_n$  are the  $b$ -rich separators of  $G$ , then there exists some  $x \in V(G)$  such that  $\bigcap_{i=1}^k S_i = \{x\}$ .

*Proof of Claim 1:* We first prove that every two  $b$ -rich separators of  $G$  have a non-empty intersection. Suppose, towards a contradiction, that there exist two  $b$ -rich separators  $S_1, S_2$  of  $G$  such that  $S_1 \cap S_2 = \emptyset$ . Let  $H \in \mathcal{C}(G, S_1)$  be the (unique) augmented component such that  $S_2 \subseteq V(H)$ . Observe that there exist at least two augmented connected components that contain  $S$ -blocks in  $\mathcal{C}(G, S_1) \setminus \{H\}$  which by [Observation 3.9](#) contain a cycle and together form  $\diamond$  as a minor. By applying the same arguments symmetrically, there exist at least two augmented components in  $\mathcal{C}(G, S_2)$  that do not contain  $S_1$ , which together also form  $\diamond$  as a minor. Then, notice that  $O_1^0 \leq G$  (see rightmost figure of [Figure 16](#)), a contradiction.

Now, suppose that there exist three  $b$ -rich separators  $S_1, S_2, S_3$  with  $S_1 \cap S_2 \cap S_3 = \emptyset$ . Notice that since  $S_2 \cap S_3 \neq \emptyset$ , there exists a unique augmented connected component of  $\mathcal{C}(G, S_1)$  that contains both  $S_2, S_3$ , while there also exist at least two other augmented connected components that contain  $S$ -blocks which together form  $\diamond$  as a minor. By applying the same argument to  $\mathcal{C}(G, S_2), \mathcal{C}(G, S_3)$ , we have that for each said separator we can form  $\diamond$  as a minor and therefore,  $O_{11}^2 \leq G$  (see leftmost figure of [Figure 16](#)), a contradiction.

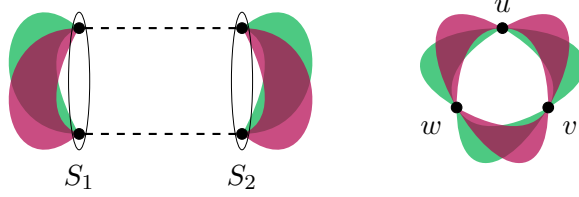


Figure 16: The proof of Claim 1.

Therefore, if  $S_1, \dots, S_k$  are the rich 2-separators of  $G$ , then  $\bigcap_{i=1}^k S_i = \{x\}$  for some  $x \in V(G)$ . Claim 1 follows.

According to Claim 1, we can consider  $S_1 = \{x, u_1\}$ ,  $S_2 = \{x, u_2\}$  as two b-rich separators of  $G$  where  $u_1 \neq u_2$ .

Let

$$H_1 = \bigcup \{H \in \mathcal{C}(G, S_1) : u_2 \notin V(H)\} \quad \text{and} \quad H_2 = \bigcup \{H \in \mathcal{C}(G, S_2) : u_1 \notin V(H)\}.$$

Consider  $H = G \setminus ((V(H_1) \cup V(H_2)) \setminus \{x, u_1, u_2\})$ .

Since for each  $i \in [2]$ ,  $H \setminus x \setminus u_i$  is a connected subgraph of  $G \setminus H_i$  which is also connected, we easily derive the following:

*Observation 1:* If  $y \in V(H \setminus x)$ , then there exists a  $(y, u_1)$ -path in  $H \setminus x \setminus u_2$  and a  $(y, u_2)$ -path in  $H \setminus x \setminus u_1$ . In particular,  $u_1, u_2$  are not cut-vertices of  $H \setminus x$ .

Since  $G \setminus x \notin \mathcal{P}^{(1)}$ , then it contains two cycles  $C_1, C_2$ , which are connected in  $G \setminus x$ . We will now argue that the following hold:

*Claim 2:*  $C_1, C_2$  are in  $H \setminus x$ .

*Proof of Claim 2:* Keep in mind that  $\triangleleft$  is a minor of  $H_i$ ,  $i \in [2]$ . If  $C_1, C_2$  are both in some  $H_i$ ,  $i \in [2]$ , say  $H_1$ , then  $\text{obs}(\mathcal{P}) \leq H_1 \setminus x$  thus implying that  $\{O_1^0, O_3^0\} \leq G$ , a contradiction. Also, if each  $C_i$  belongs to different  $H_j$ ,  $i, j \in [2]$ , say  $C_1 \subseteq H_1$  and  $C_2 \subseteq H_2$ , then since  $C_1, C_2$  are connected in  $G \setminus x$ , we have  $O_{12}^1 \leq G$  (see left figure of Figure 17), a contradiction.

Therefore, at least one of  $C_1, C_2$  is in  $H \setminus x$ . Suppose, without loss of generality, that  $C_1 \subseteq H \setminus x$ . Then  $C_2 \subseteq H \setminus x$ . Observation 1 implies that  $C_2$  is connected with  $u_1$  through a path disjoint from  $u_2$  and thus, if  $u_2 \notin V(C_2)$ ,  $O_3^0 \leq G$  (see central figure of Figure 17), while if  $u_2 \in V(C_2)$ , then by contracting all edges in the said path, we get  $O_6^1 \leq G$  (see right figure of Figure 17), a contradiction in both cases. Claim 2 follows.

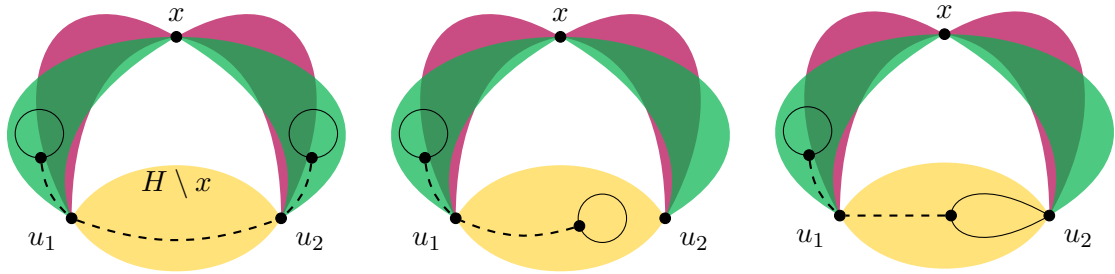


Figure 17: The cycles  $C_1$  and  $C_2$  in the proof of Claim 2.



*Observation 2:*  $\{u_1, u_2\} \subseteq V(C_1 \cup C_2)$ . Indeed, if there exists an  $i \in [2]$ , such that  $u_i \notin V(C_1 \cup C_2)$ , then by Claim 2,  $C_1, C_2$  are in  $H \setminus x$  and Observation 1 implies that there exists a path connecting  $C_1, C_2$  avoiding  $u_i$ . Hence,  $\{O_1^0, O_3^0\} \leq G$ , a contradiction.

*Claim 3:* There exists a block  $H'$  of  $H \setminus x$  that contains both  $C_1, C_2$  and is outerplanar.

*Proof of Claim 3:* We start with the following observation:

*Observation 3:* There exists some block  $H'$  of  $H \setminus x$  that contains both  $u_1, u_2$ . Indeed, suppose towards a contradiction that there exists some cut-vertex  $v \in V(H \setminus x)$  separating  $u_1, u_2$  in  $H \setminus x$ . Then, there exist  $D_1, D_2 \in \mathcal{C}(H \setminus x, v)$  such that  $D_1 \neq D_2, u_1 \in V(D_1)$  and  $u_2 \in V(D_2)$ . By Observation 2, we can assume that  $C_1 \subseteq D_1$  and  $C_2 \subseteq D_2$ , which in turn implies that  $O_3^2 \leq G$  (see Figure 18), a contradiction.

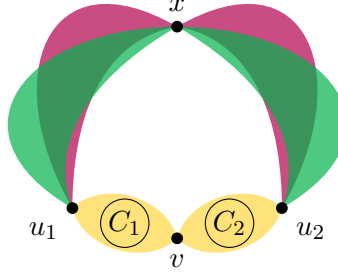


Figure 18: The cycles  $C_1, C_2$  in the proof of Observation 3.

According to Observation 3, let  $H'$  be a block of  $H \setminus x$  that contains both  $u_1, u_2$ . Suppose that some  $C_i, i \in [2]$ , say  $C_1$ , is not in  $H'$ . Then, there exists some cut-vertex  $u$  of  $H \setminus x$  such that  $C_1$  is in some  $H'' \in \mathcal{C}(H \setminus x, u) \setminus \{H'\}$ . Observation 1 implies that  $u \neq u_1, u_2$ . Therefore,  $O_3^1 \leq G$ , a contradiction. Thus,  $H'$  contains  $C_1, C_2$ .

To prove outerplanarity, we observe the following:

*Observation 4:* If  $K_{2,3} \leq H'$  then  $H'$  contains a b-rich separator of  $G$ . Indeed, if  $K_{2,3} \leq H'$  then by Observation 3.8  $H'$  contains a b-rich separator  $S$ . If  $S$  is not a b-rich separator of  $G$ , then there exist a path connecting some  $A, B \in \mathcal{C}(H', S)$  in  $G$  thus implying that  $K_4 \leq G$ , a contradiction.

Observation 4 implies that if  $K_{2,3} \leq H'$  then there exists a b-rich separator in  $G$  that does not contain  $x$ , a contradiction to Claim 1. Therefore,  $K_4 \not\leq H'$  (since  $K_4 \not\leq G$ ) and  $K_{2,3} \not\leq H'$ , i.e.  $H'$  is outerplanar. Claim 3 follows.

We now return to the proof of the lemma. According to Claim 3, let  $H'$  be a block of  $H \setminus x$  that contains both  $C_1, C_2$  and is outerplanar. Also, let  $C$  be the Hamiltonian cycle of  $H'$  (which exists due to biconnectivity and outerplanarity of  $H'$ ). Since  $C_1, C_2 \subseteq H'$ , then there exists some chord  $e$  of  $C$ . We distinguish the following cases:

*Case 1:*  $u_1 u_2 \notin E(C)$ . Let  $P_1, P_2$  be the connected components of  $C \setminus u_1 \setminus u_2$ . Notice that  $K_4$ -freeness of  $G$  implies that  $e$  is incident to vertices of some  $P_i, i \in [2]$ , say  $P_1$ . But then,  $O_{11}^2 \leq G$ , a contradiction.

*Case 2:*  $u_1 u_2 \in E(C)$ . Then, the existence of  $e$  implies that  $O_{10}^2 \leq G$ , a contradiction.

This completes the proof of the Lemma.  $\square$

Let  $G$  be a biconnected graph and  $S$  a rich-separator of  $G$ , such that every  $H \in \mathcal{C}(G, S)$  is outerplanar. For an  $H \in \mathcal{C}(G, S)$ , we denote  $C_H$  the Hamiltonian cycle of  $H$ , which exists due

to biconnectivity and outerplanarity of  $H$ . Given an  $x \in V(G)$ , we say that an edge  $e \in E(G)$  is an  $x$ -chord of  $G$ , if there exists an  $H \in \mathcal{C}(G, S)$  such that  $e$  is a chord of  $C_H$  incident to  $x$ . Also, given two vertices  $x, y \in V(G)$ , we say that an edge  $e$  is an  $(x, y)$ -disjoint chord if there exists an  $H \in \mathcal{C}(G, S)$  such that  $e$  is a chord of  $C_H$  disjoint to  $x, y$ .

**Lemma 3.11.** *Let  $G$  be a  $K_4$ -free biconnected graph such that  $\mathcal{O} \not\leq G$  and  $K_{2,3} \leq G$ . Also, let  $S = \{x, y\}$  be a  $b$ -rich separator of  $G$ . Then this  $b$ -rich separator is unique and if there exist at least two augmented connected components in  $\mathcal{C}(G, S)$  not isomorphic to a cycle, then one of the following holds:*

- *There exists a unique  $(x, y)$ -disjoint chord and there do not exist both  $x$ -chords and  $y$ -chords, or*
- *There do not exist  $(x, y)$ -disjoint chords and there exists at most one  $x$ -chord or at most one  $y$ -chord.*

*Proof.* Recall that, by [Lemma 3.10](#),  $S$  is the unique  $b$ -rich separator of  $G$ . Suppose that there exist at least two augmented connected components in  $\mathcal{C}(G, S)$  not isomorphic to a cycle. Keep in mind that if some  $H \in \mathcal{C}(G, S)$  is not isomorphic to a cycle, then  $C_H$  contains some chord. Also, observe that  $K_4$ -freeness of  $G$  implies that  $xy \in E(C_H)$ .

*Claim 1:* There exists at most one  $(x, y)$ -disjoint chord in  $G$ .

*Proof of Claim 1:* Suppose to the contrary that there exist two  $(x, y)$ -disjoint chords, namely  $e, e'$ . If  $e, e'$  are chords of some  $C_H, H \in \mathcal{C}(G, S)$ , then the existence of some  $H' \in \mathcal{C}(G, S)$ , different than  $H$  that contains a chord implies that  $\triangleleft \triangleright \leq G \setminus (H \setminus S)$  and thus  $\{O_1^0, O_3^0\} \leq G$ , a contradiction. Therefore,  $e, e'$  are chords of some  $C_H, C_{H'}$  (respectively) where  $H, H' \in \mathcal{C}(G, S)$  different than  $H$  and hence  $O_{15}^2 \leq G$  (see [Figure 19](#)), a contradiction. Claim 1 follows.

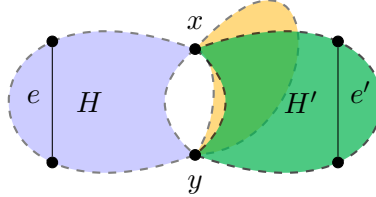


Figure 19: The chords  $e, e'$  in the second part of the proof of Claim 1.

We now distinguish the following cases depending on whether there exists an  $(x, y)$ -disjoint chord.

*Case 1:* There exists an  $(x, y)$ -disjoint chord.

Let  $e$  be an  $(x, y)$ -disjoint chord, which is a chord of some  $C_H, H \in \mathcal{C}(G, S)$ . Claim 1 implies that every other chord of some  $C_{H'}, H' \in \mathcal{C}(G, S)$  is either an  $x$ -chord or a  $y$ -chord.

Recall that there exists some  $H' \in \mathcal{C}(G, S)$  different than  $H$  that is not isomorphic to a cycle. Therefore,  $C_{H'}$  contains some chord  $e'$  that is either an  $x$ -chord or a  $y$ -chord, say  $x$ -chord. We prove the following claim:

*Claim 2:* Every edge of  $G$  that is a chord of some  $C_{H''}, H'' \in \mathcal{C}(G, S)$  different from  $e$  is an  $x$ -chord.

*Proof of Claim 2:* Suppose to the contrary that there exists an edge  $e''$  of  $G$  that is a chord of some  $C_{H''}, H'' \in \mathcal{C}(G, S)$  different from  $e$  and is not an  $x$ -chord. Claim 1 implies that  $e''$  is a  $y$ -chord.

Observe that  $H'' \in \{H, H'\}$ , because otherwise  $O_8^1 \leq G$ , a contradiction. Therefore, if  $H'' = H$ , then  $\{O_{13}^2, O_{12}^2\} \leq G$  (see left and central figure of Figure 20), while if  $H'' = H'$ ,  $O_7^2 \leq G$  (see right figure of Figure 20), a contradiction in both cases. Claim 2 follows.

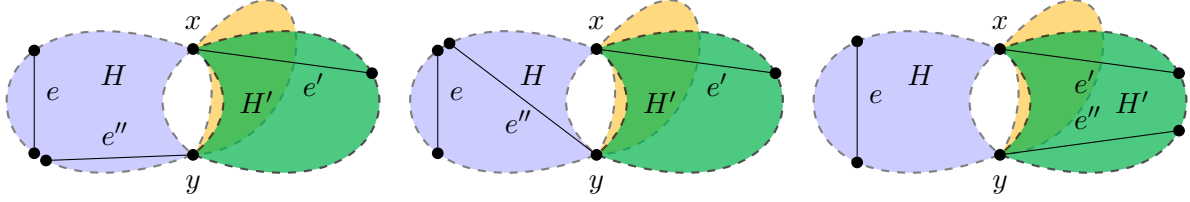


Figure 20: The possible configurations of the chords  $e, e', e''$  in the proof of Claim 2

*Case 2:* There do not exist  $(x, y)$ -disjoint chords.

We will prove that there exists at most one  $x$ -chord and at most one  $y$ -chord.

Suppose, towards a contradiction, that there exist two  $x$ -chords, namely  $e_1^x$  and  $e_2^x$ , and two  $y$ -chords, namely  $e_1^y$  and  $e_2^y$ .

We say that a pair of edges  $(e, e')$ , where  $e, e' \in \{e_1^x, e_2^x, e_1^y, e_2^y\}$  is *homologous* if there exists some  $H \in \mathcal{C}(G, S)$  such that  $e, e' \in E(H)$ .

We now distinguish the following subcases:

*Subcase 2.1:*  $(e_1^x, e_2^x)$  is not homologous and  $(e_1^y, e_2^y)$  is not homologous. Then,  $\{O_2^0, O_{12}^1, O_3^2\} \leq G$  depending whether there exist 0, 1 or 2 homologous pairs  $(e, e')$ , where  $e \in \{e_1^x, e_2^x\}$ ,  $e' \in \{e_1^y, e_2^y\}$ . In any case we have a contradiction (see Figure 21).

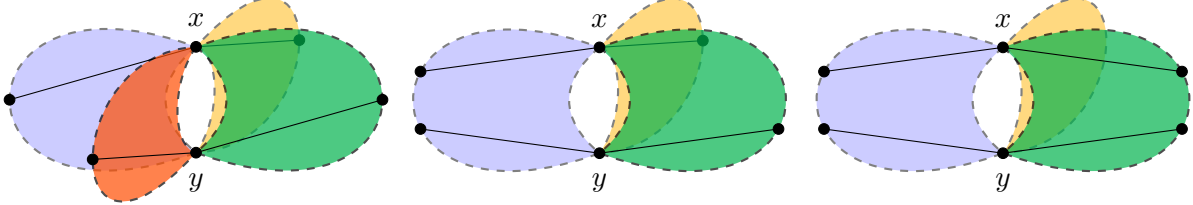


Figure 21: The possible configurations of the edges  $e_1^y, e_2^y, e_1^x$ , and  $e_2^x$  in the proof of Subcase 2.1.

*Subcase 2.2:*  $(e_1^x, e_2^x)$  is homologous and  $(e_1^y, e_2^y)$  is not homologous.

Let  $H \in \mathcal{C}(G, S)$  such that  $e_1^x, e_2^x \in E(H)$ . Notice that since  $(e_1^y, e_2^y)$  is not homologous, at most one of  $e_1^y, e_2^y$  is in  $E(H)$ . If none of  $e_1^y, e_2^y$  is in  $E(H)$ , then  $O_3^0 \leq G$ , while if some of  $e_1^y, e_2^y$  is in  $E(H)$ , then  $O_6^1 \leq G$ , a contradiction (see Figure 22).

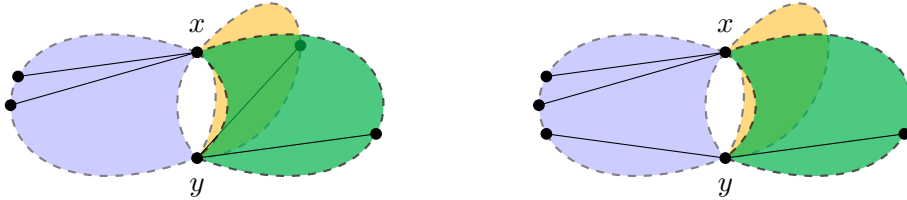


Figure 22: The possible configurations of the edges  $e_1^y, e_2^y, e_1^x$ , and  $e_2^x$  in the proof of Subcase 2.2.

*Subcase 2.3:*  $(e_1^x, e_2^x)$  is not homologous and  $(e_1^y, e_2^y)$  is homologous. This case is symmetric to the previous one.

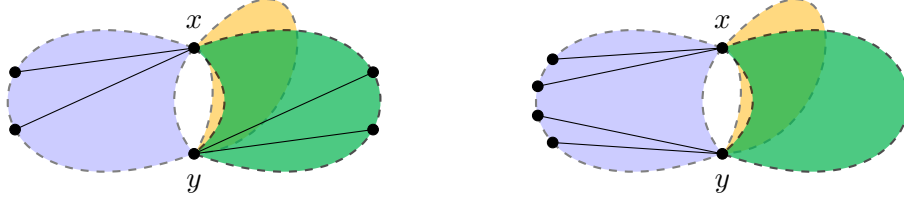


Figure 23: The possible configurations of the cycles  $e_1^y, e_2^y, e_1^x,$  and  $e_2^x$  in the proof of Subcase 2.4.

*Subcase 2.4:*  $(e_1^x, e_2^x)$  is homologous and  $(e_1^y, e_2^y)$  is homologous. Let  $H \in \mathcal{C}(G, S)$  such that  $e_1^x, e_2^x \in E(H)$  and  $H' \in \mathcal{C}(G, S)$  such that  $e_1^y, e_2^y \in E(H')$ . If  $H \neq H'$ , then  $\mathcal{O}_1^0 \leq G$ , while if  $H = H'$ , then  $\mathcal{O}_{11}^2 \leq G$ . In both cases we have a contradiction (see Figure 23).  $\square$

## 4 Confining connectivity

In this section we further restrict the structure of a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ . The first step is to prove that  $G$  is biconnected (Lemma 4.7) and the second one is to prove that  $G$  is triconnected (Lemma 4.10).

### 4.1 Proving biconnectivity

In this section we prove that every graph in  $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  is biconnected (Lemma 4.7). For this we prove a series of lemmata that gradually restrict the structure of such a graph.

We begin by making two observations. Since Lemma 3.1 implies that every block of a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  has a cycle then by the  $\mathcal{O}_{10}^1$ -freeness of such a graph we derive the following:

*Observation 4.1.* If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then every block of  $G$  contains at most 2 cut-vertices.

Also, for a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  we have that  $G \notin \mathcal{P}^{(1)}$  and this implies the following:

*Observation 4.2.* If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  is a connected graph, then for every cut-vertex  $v \in V(G)$  there exists an  $H \in \mathcal{C}(G, v)$  such that  $\mathbf{obs}(\mathcal{P}) \leq H \setminus x$ .

**Lemma 4.3.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ , then  $G$  cannot have more than 1 cut-vertex.*

*Proof.* Recall that, from Lemma 2.9,  $G$  is connected. Suppose, towards a contradiction, that  $G$  has at least 2 cut-vertices. Then, there exists a block  $B$  containing 2 cut-vertices  $u_1, u_2$ . Let

$$\mathcal{C}_1 = \bigcup \{H \in \mathcal{C}(G, u_1) : u_2 \notin V(B)\} \quad \text{and} \quad \mathcal{C}_2 = \bigcup \{H \in \mathcal{C}(G, u_2) : u_1 \notin V(B)\}.$$

We now prove a series of claims:

*Claim 1:* Both  $\mathcal{C}_1, \mathcal{C}_2$  are isomorphic to  $K_3$ .

*Proof of Claim 1:* Suppose, to the contrary, that one of  $\mathcal{C}_1, \mathcal{C}_2$ , say  $\mathcal{C}_1$ , contains 2 cycles, which is equivalent to  $\mathbf{obs}(\mathcal{P}) \leq \mathcal{C}_1$ , since  $\mathcal{C}_1$  is connected. Let  $H \in \mathcal{C}(G, u_1)$  be the component which contain  $u_2$ . We distinguish two cases:

*Case 1:*  $\mathcal{C}_1 \setminus u_1 \in \mathcal{P}$ . Then, by Observation 4.2,  $\mathbf{obs}(\mathcal{P}) \leq H \setminus u_1$  and therefore, since  $V(\mathcal{C}_1) \cap V(H \setminus u_1) = \emptyset$ , we have that  $\mathcal{O}^0 \leq G$ , a contradiction.

*Case 2:*  $\mathcal{C}_1 \setminus u_1 \notin \mathcal{P}$ , or equivalently  $\mathbf{obs}(\mathcal{P}) \leq \mathcal{C}_1 \setminus u_1$ . Observe that, since  $H$  contains the cut-vertex  $u_2$ , there exist 2 blocks  $H_1, H_2$  of  $G$  in  $H$  such that  $V(H_1) \cap V(H_2) = \{u_2\}$ . Then, since, by [Lemma 3.1](#), each block of  $G$  contains a cycle, we have that  $\bowtie \leq H$ . Hence,  $\{O_2^0, O_3^0\} \leq G$ , a contradiction.

Therefore, both of  $\mathcal{C}_1, \mathcal{C}_2$  contain at most one cycle and, since both are non-empty, by [Lemma 3.1](#), Claim 1 follows.

*Claim 2:* Every cycle in  $B$  contains either  $u_1$  or  $u_2$ .

*Proof of Claim 2:* Suppose, to the contrary, that there exists a cycle  $C$  containing none of  $u_1, u_2$ . By Menger's Theorem there exist two internally disjoint  $(u_1, u_2)$ -paths  $P_1, P_2$ . We distinguish the following cases:

*Case 1:* Both of  $P_1, P_2$  intersect  $C$ .

Let  $z_1, z_2$  be the vertices where  $P_1, P_2$  meet  $C$  for the first time, respectively. Let, also,  $w_1, w_2$  be the vertices that  $P_1, P_2$  meet  $C$  for the last time, respectively. Since  $V(P_1) \cap V(P_2) = \{u_1, u_2\}$  we have that  $\{z_1, w_1\} \cap \{z_2, w_2\} = \emptyset$ . If  $z_1 \neq w_1$  or  $z_2 \neq w_2$ , say  $z_1 \neq w_1$ , then by contracting the edges in the  $(w_1, u_2)$ -subpath of  $P_1$  we form  $O_1^1$  as a minor of  $G$ , a contradiction (see left figure of [Figure 24](#)). Therefore, we have that  $z_1 = w_1$  and  $z_2 = w_2$ , in which case we have again  $O_1^1$  as a minor of  $G$ , a contradiction (see right figure of [Figure 24](#)).

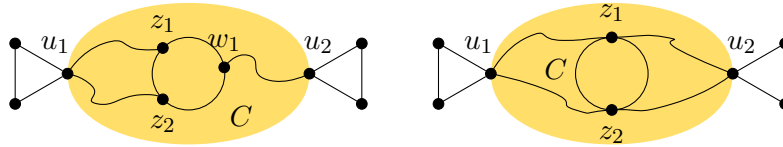


Figure 24: The paths  $P_1$  and  $P_2$  in Case 1 of Claim 2.

*Case 2:* Either  $P_1$  or  $P_2$  is disjoint to  $C$ .

Say, without loss of generality, that  $V(P_1) \cap V(C) = \emptyset$ . Let  $v_{new} \notin V(G)$  and consider the graph  $B'$  obtained by adding  $v_{new}$  to  $B$  and making it adjacent to  $u_1, u_2$ . Observe that  $B'$  is also biconnected. Then, by Menger's theorem, there exist two paths from  $v_{new}$  to  $C$  intersecting only in  $v_{new}$ . Therefore, in  $B$  there exists a  $(x, u_1)$ -path  $Q_1$  and a  $(y, u_2)$ -path  $Q_2$  such that  $x, y \in V(C)$  and  $V(Q_1) \cap V(Q_2) = \emptyset$ . Let  $z_1, z_2$  be the vertices where  $Q_1, Q_2$  meet  $P_1$  for the first time, respectively (starting from  $x$  and  $y$ ). Then, suppose that, without loss of generality,  $z_1$  is closest to  $u_1$  in  $P_1$  than  $z_2$ . Hence, by contracting all the edges of the  $(z_1, u_1)$ -,  $(z_2, u_2)$ -subpaths of  $P_1$  we form  $O_9^1$  as a minor of  $G$ , a contradiction (see [Figure 25](#)). Claim 2 follows.

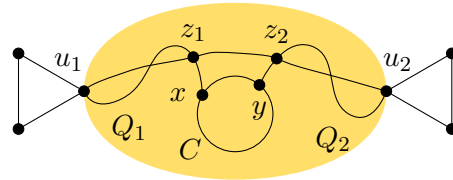


Figure 25: The paths  $P_1, Q_1, Q_2$  in Case 2 of Claim 2.

We return to the proof of the Lemma. Notice that, by [Observation 4.1](#),  $u_1, u_2$  are the only

cut-vertices of  $G$  contained in  $B$ . Therefore, Claim 1 implies that  $C_1, B, C_2$  are the only blocks of  $G$ .

Let  $H_1 \in \mathcal{C}(G, u_2)$  such that  $u_1 \in H_1$  and  $H_2 \in \mathcal{C}(G, u_1)$  such that  $u_2 \in H_2$ . By [Observation 4.2](#), we have that  $\mathbf{obs}(\mathcal{P}) \leq H_1 \setminus u_2, H_2 \setminus u_1$ , or equivalently  $H_1 \setminus u_2, H_2 \setminus u_1 \notin \mathcal{P}$ . Therefore, since  $H_1 \setminus u_2, H_2 \setminus u_1$  are connected, Claim 2 implies that there exist two cycles  $C_1, C_2$  in  $B$  such that  $u_1 \in V(C_1)$  and  $u_2 \in V(C_2)$ . If  $V(C_1) \cap V(C_2) = \emptyset$  then  $O_2^0 \leq G$  and if  $|V(C_1) \cap V(C_2)| \geq 2$  then  $O_1^1 \leq G$ , a contradiction in both cases. Hence,  $V(C_1) \cap V(C_2) = \{x\}$  for some  $x \in V(B)$ . As  $x$  is not a cut-vertex of  $B$ , there exists a  $(u_1, u_2)$ -path  $P$  in  $B$  such that  $x \notin V(P)$ .

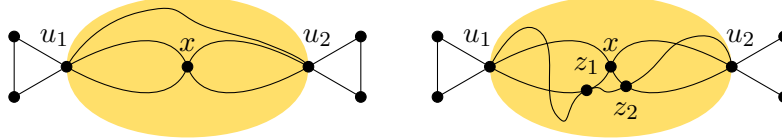


Figure 26: The cycles  $C_1, C_2$  and the path  $P$  in the end of the proof of the Lemma.

If  $V(P) \cap V(C_1 \cup C_2) = \{u_1, u_2\}$  then  $O_{12}^1 \leq G$ , a contradiction (see the leftmost figure of [Figure 26](#)). Therefore  $P$  intersects, without loss of generality,  $C_1$  at a vertex different from  $u_1$ . Let  $z_1 \in V(C_1)$  be the vertex that  $P$  meets  $C_1$  for the last time. Also, let  $z_2 \in V(C_2)$  be the vertex that the  $(z_1, u_2)$ -subpath of  $P$  meets  $C_2$  for the first time. Then, the cycle  $C_1$ , the  $(z_1, z_2)$ -subpath of  $P$ , the  $(z_2, u_2)$ -path in  $C_2$  that does not contain  $x$  and the  $(x, u_2)$ -path of  $C_2$  that does not contain  $z_2$ , along with  $C_1, C_2$  form  $O_1^1$  as a minor of  $G$ , a contradiction (see the rightmost figure of [Figure 26](#)).  $\square$

**Lemma 4.4.** *Let  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  be a graph that contains a cut-vertex  $x$ . Then  $\mathcal{C}(G, x) = \{B, K_3\}$  for some biconnected graph  $B$ .*

*Proof.* Recall that by [Lemma 2.9](#),  $G$  is connected. By [Observation 4.2](#) there exists some  $B \in \mathcal{C}(G, x)$  such that  $\mathbf{obs}(\mathcal{P}) \leq B \setminus x$ . By [Lemma 4.3](#)  $x$  is the only cut-vertex of  $G$  and therefore  $B$  is biconnected.

Let  $D = \bigcup \{H \in \mathcal{C}(G, x) : H \neq B\}$ , that is  $G \setminus (B \setminus x)$ . We will prove that  $D \cong K_3$ . Suppose, towards a contradiction, that  $D$  contains more than one cycle. Then, since  $D$  is connected, we have that  $\mathbf{obs}(\mathcal{P}) \leq D$  and so  $D \cap (B \setminus x) = \emptyset$  implies that  $\mathcal{O}^0 \leq G$ , a contradiction. Therefore,  $D$  contains at most one cycle. But since  $x$  is a cut-vertex we have that  $D \neq \emptyset$  and hence, [Lemma 3.1](#) implies that  $D \cong K_3$  which concludes the proof of the Lemma.  $\square$

**Lemma 4.5.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $K_4 \not\leq G$  or  $G$  is biconnected.*

*Proof.* Keep in mind that, by [Lemma 2.9](#),  $G$  is connected. Suppose, to the contrary, that  $K_4 \leq G$  and  $G$  is not biconnected. By [Lemma 4.4](#) we have that  $\mathcal{C}(G, x) = \{B, K_3\}$  where  $B$  is a biconnected graph. Observe that  $K_4 \leq B$  and let  $(H, K)$  be an  $r$ -wheel-subdivision pair of  $B$ .

We argue that the following holds:

*Claim 1:*  $x$  is a branch vertex of  $K$ .

*Proof of Claim 1:* If  $x \in V(K)$  but  $x$  is not a branch vertex of  $K$  then  $O_4^1 \leq G$ , a contradiction.

Suppose then that  $x \notin V(K)$ . Notice that, since  $x \in V(B)$  and  $B$  is a biconnected, then there exist two paths  $P_1, P_2$  from  $x$  to some vertex of  $K$ , respectively, such that  $V(P_1) \cap V(P_2) = \{x\}$ .

Let  $u_1, u_2$  be the first time  $P_1$  and  $P_2$  meet  $K$ , respectively. Let  $P'_1$  be the  $(x, u_1)$ -subpath of  $P_1$  and  $P'_2$  be the  $(x, u_2)$ -subpath of  $P_2$ . Then, the  $(u_1, u_2)$ -path  $P'_1 \cup P'_2$  intersects  $K$  only in its endpoints and by [Lemma 3.4](#) there exists an edge  $e \in E(H)$  such that  $u_1, u_2$  are both vertices of the subdivision of  $e$  in  $K$  (see [Figure 27](#)).

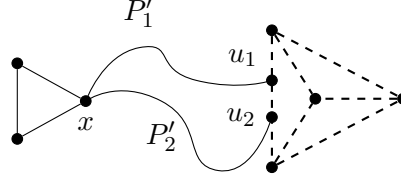
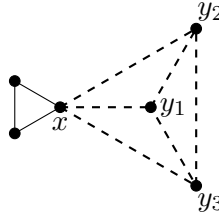


Figure 27: The paths  $P'_1, P'_2$  from  $x$  to  $K$  in the proof of Claim 1.

Let  $P$  be the path corresponding to the subdivision of  $e$  in  $K$ . Let, also,  $z_1$  be the branch vertex of  $K$  incident to  $e$  that is closest to  $u_1$  in  $P$  and  $z_2$  be the other branch vertex of  $K$  incident to  $e$ . Then, by contracting all the edges in the  $(u_1, z_1)$ -,  $(u_2, z_2)$ -subpaths of  $P$  we form  $O_4^1$  as a minor of  $G$ , a contradiction. Claim 1 follows.

We now prove that  $H$  is isomorphic to  $K_4$ . Suppose to the contrary, that  $H$  is isomorphic to an  $r$ -wheel, where  $r \geq 4$ . By Claim 1,  $x$  is a branch vertex of  $K$ . Then,  $x$  is the center of  $K$ , otherwise  $O_4^1 \leq G$ , which is a contradiction. Since  $B$  is a block of  $G$ , by [Lemma 3.7](#), we have that  $x$  is a  $\mathcal{P}$ -apex vertex of  $B$  and so the fact that  $\mathcal{C}(G, x) = \{B, K_3\}$  implies that  $x$  is also a  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction. Hence,  $r = 3$ , i.e.  $H$  is isomorphic to  $K_4$ .

According to Claim 1, we have that  $x$  is a branch vertex of  $K$ . Let  $y_i, i \in [3]$ , be the three other branch vertices of  $K$ , as in the following figure:



Keep in mind that, by [Lemma 3.5](#), for every  $(s, t)$ -flap  $F$  of  $(H, K)$  it holds that  $F$  is biconnected,  $G[V(F)]$  contains a cycle, and there exists an edge  $e \in E(H)$  such that  $s, t$  belong both to the subdivision of  $e$  in  $K$ .

*Claim 2:* Every flap of  $(H, K)$  is  $x$ -oriented.

*Proof of Claim 2:* We first prove that every flap of  $(H, K)$  is an  $(x, t)$ -flap. Observe that if there exists an  $(s, t)$ -flap where  $s$  and  $t$  belong to the subdivision of a  $y_i y_j$  edge of  $H$ , then  $O_5^1 \leq G$  (see left figure of [Figure 28](#)), while if there exists an  $(s, t)$ -flap where  $s$  and  $t$  belong to the subdivision of an  $x y_i$  edge of  $H$  and  $s, t \neq x$ , then  $O_1^1 \leq G$  (see right figure of [Figure 28](#)), a contradiction in both cases.

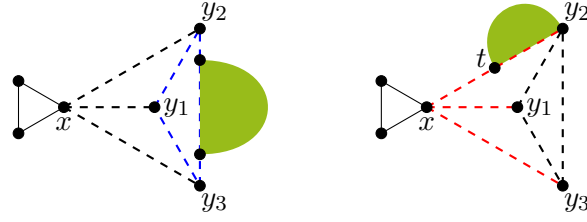


Figure 28: Left figure: An  $(s, t)$ -flap in the subdivision of some  $y_i y_j$  edge of  $H$  (depicted in blue) in the proof of Claim 2. Right figure: An  $(s, t)$ -flap in the subdivision of some  $x y_i$  edge of  $H$  (depicted in red) in the proof of Claim 2.

Therefore, every flap of  $(H, K)$  is an  $(x, t)$ -flap. Observe now that every  $(x, t)$ -flap  $F$  of  $(H, K)$  is  $x$ -oriented. Indeed, if otherwise then Lemma 3.6 implies that there exists a cycle in  $F$  containing  $t$  but not  $z$  and hence  $O_1^1 \leq G$ , a contradiction. Claim 2 follows.

Therefore, since  $\mathcal{C}(G, x) = \{B, K_3\}$  and by Claim 2 every flap of  $(H, K)$  is  $x$ -oriented, then  $x$  is a  $\mathcal{P}$ -apex vertex of  $G$  and so  $G \in \mathcal{P}^{(1)}$ , a contradiction.  $\square$

**Lemma 4.6.** *If  $G \in \text{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $K_{2,3} \not\leq G$  or  $G$  is biconnected.*

*Proof.* Suppose, to the contrary, that  $G$  has a, unique due to Lemma 4.3, cut-vertex  $u$  and contains  $K_{2,3}$  as a minor. Since  $G$  contains a cut-vertex, then by Lemma 4.5 it is  $K_4$ -free. Notice that, by Lemma 4.4, the augmented connected components of  $\mathcal{C}(G, u)$  are some  $T \cong K_3$  and a biconnected graph  $B$ . By Lemma 3.10, we have that there exists a unique b-rich separator  $S = \{x, y\}$ . We distinguish two cases, based on whether  $u$  belongs to  $S$  or not:

*Case 1:* The cut-vertex  $u$  is neither of  $x, y$ .

Let  $H \in \mathcal{C}(G, S)$  such that  $u \in V(H)$ . Let  $\hat{H} = H \setminus (T \setminus u)$ . Observe that  $B = (G \setminus H) \cup \hat{H}$  and keep in mind that  $\hat{H} \in \mathcal{C}(B, S)$ . Since  $B$  is a block, every  $H \in \mathcal{C}(B, S)$  is biconnected and outerplanar. For every  $H \in \mathcal{C}(B, S)$ , let  $C_H$  be the Hamiltonian cycle of  $H$ . Recall that  $xy \in E(C_H)$ . Also, keep in mind that since  $B$  is biconnected, for every  $H' \in \mathcal{C}(G, S) \setminus \{H\}$ ,  $G[V(H')]$  contains a cycle.

*Observation 1:*  $B$  does not contain  $(x, y)$ -disjoint chords. Indeed, if  $H$  contains a  $(x, y)$ -disjoint chord, then  $\{O_1^0, O_3^0\} \leq G$ , while if some  $H' \in \mathcal{C}(B, S) \setminus \{H\}$  contains a  $(x, y)$ -disjoint chord, then  $O_1^2 \leq G$  (see Figure 29), a contradiction.

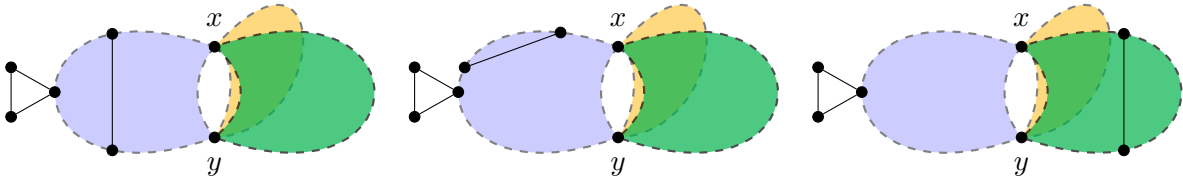


Figure 29: The  $(x, y)$ -disjoint chord in the proof of Observation 1

*Claim 1:*  $B \setminus (\hat{H} \setminus \{x, y\})$  does not contain both  $x$ -chords and  $y$ -chords.

*Proof of Claim 1:* Suppose to the contrary that there exist an  $x$ -chord and a  $y$ -chord in  $B \setminus (\hat{H} \setminus \{x, y\})$ , namely  $e_x$  and  $e_y$ , respectively. If there exists some  $H' \in \mathcal{C}(B, S) \setminus \{\hat{H}\}$  such that  $e_x, e_y \in E(H')$ , then  $O_7^2 \leq G$ , while if they are in different augmented connected components in  $\mathcal{C}(G, S) \setminus \{\hat{H}\}$  then  $O_9^1 \leq G$ , a contradiction in both cases (see Figure 30). Claim 1 follows.



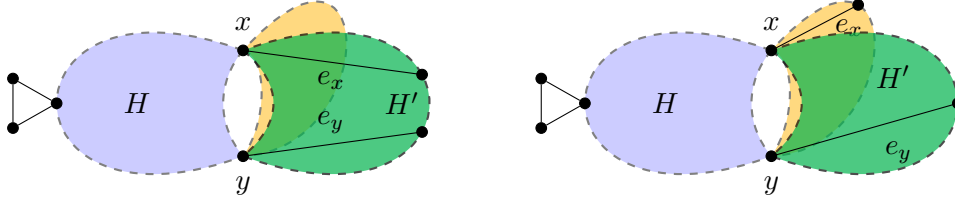


Figure 30: The chords  $e_x, e_y$  in the proof of Claim 1

According to Claim 1, suppose that  $B \setminus (\hat{H} \setminus \{x, y\})$  contains  $x$ -chords but not  $y$ -chords.

We also observe the following:

*Observation 2:* Every chord of  $C_{\hat{H}}$  is a  $y$ -chord. Indeed, if there exist an  $x$ -chord and a  $y$ -chord in  $\hat{H}$ , then  $O_6^1 \leq G$ , a contradiction (see Figure 31). Also, if all chords of  $C_{\hat{H}}$  are  $x$ -chords, then  $x$  is a  $\mathcal{P}$ -apex of  $G$ , a contradiction.

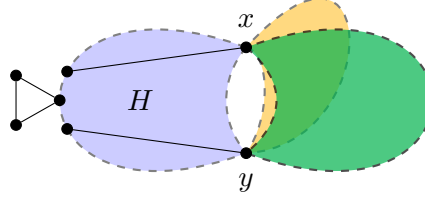


Figure 31: An  $x$ -chord and a  $y$ -chord of  $C$  in Observation 2 of Case 1.

*Claim 2:* Either every  $C_{H'}$ ,  $H' \in \mathcal{C}(B, S) \setminus \{\hat{H}\}$  is chordless or  $C_{\hat{H}}$  is chordless.

*Proof of Claim 2:* Suppose to the contrary that there exist a chord  $e$  of some  $C_{H'}$ ,  $H' \in \mathcal{C}(B, S) \setminus \{\hat{H}\}$  and a chord  $e'$  of  $C_{\hat{H}}$ . Claim 1 implies that  $e$  is an  $x$ -chord, while Observation 1 implies that  $e'$  is a  $y$ -chord. Thus,  $O_{12}^1 \leq G$ , a contradiction. Claim 2 follows.

According to Claim 2, either every  $C_{H'}$ ,  $H' \in \mathcal{C}(B, S) \setminus \{\hat{H}\}$  is chordless or  $C_{\hat{H}}$  is chordless which implies that either  $y$  or  $x$ , respectively, is a  $\mathcal{P}$ -apex of  $G$ , a contradiction.

*Case 2:* The cut-vertex  $u$  is either  $x$  or  $y$ , say  $y$ .

We first prove the following:

*Claim 3:* There exists a unique augmented connected component in  $\mathcal{C}(B, S)$  that is not isomorphic to a cycle.

*Proof of Claim 3:* First, notice that if each augmented connected component in  $\mathcal{C}(B, S)$  is isomorphic to a cycle, then  $G \in \mathcal{P}^{(1)}$ , a contradiction. Therefore there exists an augmented connected component in  $\mathcal{C}(B, S)$  that is not isomorphic to a cycle. Suppose towards a contradiction that  $\mathcal{C}(B, S)$  contains two augmented connected components not isomorphic to a cycle. We distinguish the following subcases:

*Subcase 2.1:*  $B$  contains an  $(x, y)$ -disjoint chord, namely  $e$ .

Then Lemma 3.11 implies that  $e$  is the unique  $(x, y)$ -disjoint chord of  $B$  and  $B$  does not contain both  $x$ -chords and  $y$ -chords. Let  $H \in \mathcal{C}(B, S)$  such that  $e \in E(H)$ .

Recall that there exists some  $H' \in \mathcal{C}(B, S) \setminus \{H\}$  that is not isomorphic to a cycle. Therefore, there exists some chord  $e'$  of  $C_{H'}$ . If  $e'$  is an  $x$ -chord, then  $O_9^1 \leq G$  (see Figure 32), while if  $e'$  is a

$y$ -chord, there does not exist an  $x$ -chord and therefore  $y$  is a  $\mathcal{P}$ -apex of  $G$ , a contradiction in both cases.

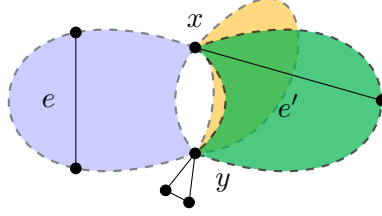


Figure 32: The chords  $e, e'$  in the first part the proof of Subcase 2.1.

*Subcase 2.2:*  $B$  does not contain an  $(x, y)$ -disjoint chord.

Then [Lemma 3.11](#) implies that  $H$  contains at most one  $x$ -chord or at most one  $y$ -chord. If there exists at most one  $x$ -chord, then  $y$  is a  $\mathcal{P}$ -apex of  $G$ , a contradiction. Therefore there exists at most one  $y$ -chord.

Suppose that there exists a  $y$ -chord, namely  $e_y$ , and let  $H \in \mathcal{C}(B, S)$  such that  $e_y \in E(H)$ . Recall that there exists some  $H' \in \mathcal{C}(B, S) \setminus \{H\}$  that is not isomorphic to a cycle. Therefore, there exists some chord of  $C_{H'}$ , namely  $e'$ . Observe that  $e'$  is an  $x$ -chord and that there exists some  $x$ -chord  $e''$ ,  $e \neq e'$ , otherwise  $y$  would be a  $\mathcal{P}$ -apex of  $G$ . If  $e'' \notin E(H)$ , then  $\{O_2^0, O_3^0\} \leq G$ , while if  $e'' \in E(H)$ , then  $O_{12}^1 \leq G$  (see [Figure 33](#)), a contradiction in both cases. Claim 3 follows.

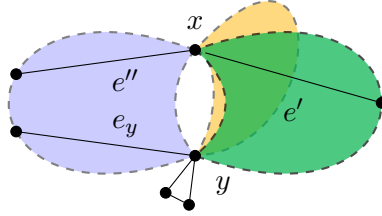
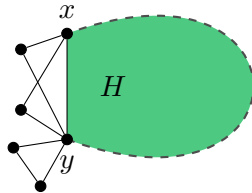


Figure 33: The chords  $e_y, e', e''$  in the last part of the proof of Subcase 2.2.

We now proceed with the proof of Case 2 of the Lemma. According to Claim 3, let  $H$  be the unique augmented connected component of  $B$  that is not isomorphic to a cycle. Therefore, due to [Lemma 3.1](#), every  $H' \in \mathcal{C}(B, S) \setminus \{H\}$  is isomorphic to  $K_3$ .

We have that  $H$  is outerplanar and due to [Lemma 4.4](#),  $H$  is also biconnected. Let  $C$  be the Hamiltonian cycle of  $H$ . Keep in mind that  $xy \in E(C)$ . Then the graph  $G$  is as in the following figure:



Observe that every  $(x, y)$ -disjoint chord,  $x$ -chord, and  $y$ -chord of  $B$  is a chord of  $C$ .

*Claim 4:* There do not exist  $(x, y)$ -disjoint chords in  $B$ .

*Proof of Claim 4:* Suppose, towards a contradiction, that there exists a  $(x, y)$ -disjoint chord in  $B$ , namely  $e$ . We observe that the following holds:

*Observation 3:* There exists at most one  $(x, y)$ -disjoint chord in  $B$ . Indeed, if the contrary holds then  $\{O_1^0, O_3^0\} \leq G$ , a contradiction. (see [Figure 34](#))

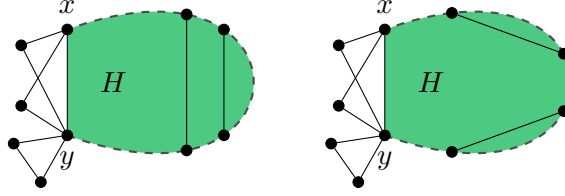


Figure 34: The two  $(x, y)$ -disjoint chords in Observation 1 of Case 2.

Observation 3 implies that  $e$  is the unique  $(x, y)$ -disjoint chord in  $B$ . If there exists some  $x$ -chord in  $B$  then  $\{O_7^1, O_{11}^1\} \leq G$ , a contradiction (see [Figure 35](#)). Therefore every edge  $e' \in E(H), e' \neq e$  that is a chord of  $C_H$ , is a  $y$ -chord and thus  $y$  is a  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction. Claim 4 follows.

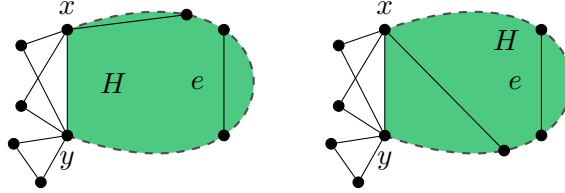


Figure 35: A  $(x, y)$ -disjoint chord  $e$  and an  $x$ -chord in the proof of Claim 4 of Case 2.

We now conclude the proof of the Lemma. Since  $x$  is not an  $\mathcal{P}$ -apex vertex of  $G$ , there exists some chord of  $C$  not incident to  $x$ , namely  $e$ . By Claim 4,  $e$  is a  $y$ -chord.

*Observation 4:* There exists at most one  $x$ -chord. Indeed if there exist two  $x$ -chords, we have  $O_{11}^1 \leq G$ , a contradiction (see [Figure 36](#)).

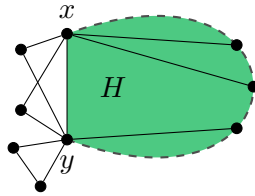
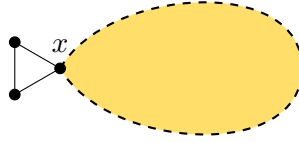


Figure 36: A  $y$ -chord and two  $x$ -chords in the last part of the proof of Case 2.

Therefore, Observation 4 implies that  $y$  is a  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction. This completes the proof of the Lemma.  $\square$

**Lemma 4.7.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $G$  is biconnected.*

*Proof.* Suppose to the contrary that  $G$  has a cut-vertex  $x$ . Due to [Lemma 4.3](#), [Lemma 4.5](#), and [Lemma 4.6](#) we have that  $\mathcal{C}(G, x) = \{T, H\}$  where  $T \cong K_3$  and  $H$  is outerplanar and biconnected. Let  $C$  be the Hamiltonian cycle of  $H$ . Therefore, the structure of  $G$  is as in the following figure:



*Claim:* There exists a vertex  $u \in V(C)$  such that every chord of  $C$  not incident to  $x$  is incident to  $u$ .

*Proof of Claim:* To prove the Claim we make a series of observations.

*Observation 1:* Every two chords not incident to  $x$  share a vertex. Indeed, suppose that there exist two disjoint chords  $e_1 = u_1v_1, e_2 = u_2v_2$  of  $C$  not incident to  $x$ . Then  $\{O_2^1, O_3^1\} \leq G$  (see Figure 37), a contradiction.

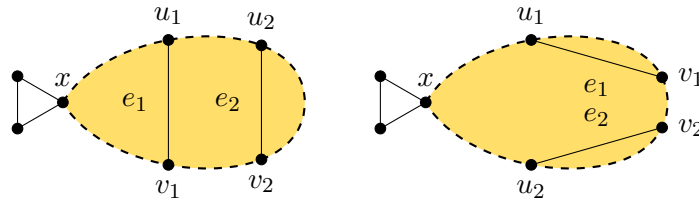


Figure 37: The two chords of  $C$  disjoint to  $x, y$  in the proof of Observation 1.

*Observation 2:* Every three chords not incident to  $x$  share a vertex. Indeed, suppose to the contrary, that there exist three chords  $e_1, e_2, e_3$  not incident to  $x$  that do not share a vertex. Observation 1 implies that every two of  $e_1, e_2, e_3$  share a vertex and hence,  $O_3^1 \leq G$ , a contradiction (see Figure 38).

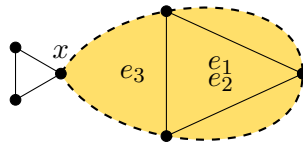


Figure 38: The proof of Observation 2.

Claim follows from the above two observations.

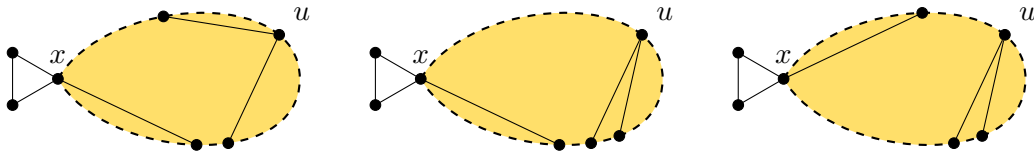


Figure 39: The possible configurations of the chords that conclude the proof of Lemma 4.7.

We now continue with the proof of the Lemma. Since  $x$  is not a  $\mathcal{P}$ -apex vertex of  $G$  there exist two chords  $e_1, e_2$  not incident to  $x$ . By the above Claim, both  $e_1, e_2$  share a vertex  $u$ . Since  $u$  is not a  $\mathcal{P}$ -apex vertex of  $G$  there exists a chord  $e$  not incident to  $u$  which, again by the above Claim, is incident to  $x$ . Hence,  $\{O_7^1, O_{11}^1, O_3^0\} \leq G$ , a contradiction (see Figure 39).  $\square$

## 4.2 Proving triconnectivity

The purpose of this subsection is to prove that all graphs in  $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  are triconnected (Lemma 4.10).

**Lemma 4.8.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $K_4 \not\leq G$  or  $G$  is triconnected.*

*Proof.* Suppose, to the contrary, that  $G$  is not triconnected and  $K_4 \leq G$ , which, by Lemma 4.5, implies that  $G$  is biconnected. Let  $(H, K)$  be an  $r$ -wheel-subdivision pair of  $G$ . Since  $G$  is biconnected, Lemma 3.7 implies that  $H$  is isomorphic to  $K_4$ .

Keep in mind that, by Lemma 3.5, for every  $(u, v)$ -flap  $F$  of  $(H, K)$  there exists an edge  $e \in E(H)$  such that  $u, v$  belong both to the subdivision of  $e$  in  $K$  and  $G[V(F)]$  contains a cycle.

Keep in mind that for every  $(u, v)$ -flap of  $(H, K)$  it holds that one of  $u, v$  is a branch vertex of  $K$ , otherwise  $O_9^2 \leq G$ , a contradiction (see Figure 40).

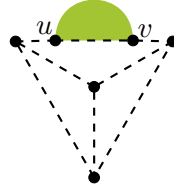


Figure 40: An example of a  $(u, v)$ -flap such that both  $u, v$  are subdividing vertices of the corresponding path.

We distinguish the following two cases:

*Case 1:* There exists an  $(x, u)$ -flap  $F$ , where  $x$  is a branch vertex of  $K$  and  $u$  is a subdividing vertex in the subdivision of an edge  $e = xy \in E(H)$ .

Let  $z_1, z_2$  be the two other branch vertices of  $K$  (as in Figure 41).

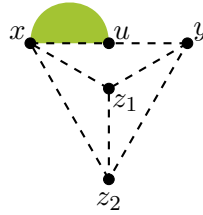


Figure 41: The structure of  $G$  in Case 1.

*Claim 1:* All flaps of  $(H, K)$  are  $(x, w)$ -flaps where  $w$  is a vertex in the subdivision of an edge  $e' \in \{xy, xz_1, xz_2\}$  in  $K$ .

*Proof of Claim 1:* Suppose, to the contrary, that Claim 1 does not hold. We distinguish the following subcases:

*Subcase 1.1:* There exists a  $(y, v)$ -flap  $F'$  where  $v$  is a subdividing vertex in the subdivision of  $e$  in  $K$ . If  $u = v$  then  $V(F) \cap V(F') = \{u\}$  and so  $O_4^2 \leq G$ , while if  $u \neq v$  then  $V(F) \cap V(F') = \emptyset$  and so  $O_1^1 \leq G$ , a contradiction in both cases (see Figure 42).

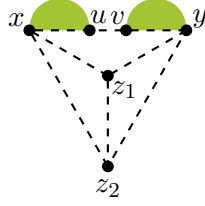


Figure 42: The configuration of the flaps of  $(H, K)$  in Subcase 1.1 of Claim 1.

*Subcase 1.2:* There exists a flap in the subdivision of the edges  $yz_1$  or  $yz_2$ . But, then  $O_6^2 \leq G$  (see Figure 43), a contradiction.

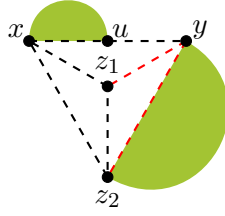


Figure 43: A flap in the subdivision of some edge  $yz_i$ ,  $i \in [2]$  (depicted in red) in Subcase 1.2 of Claim 1.

*Subcase 1.3:* There exists a flap in the subdivision of the edge  $z_1z_2$ . But, then  $O_5^2 \leq G$  (see Figure 44), a contradiction.

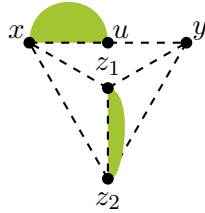


Figure 44: A flap in the subdivision of  $z_1z_2$  in Subcase 1.3 of Claim 1.

*Subcase 1.4:* There exists a  $(z_i, v)$ -flap such that  $v$  is a subdividing vertex in the subdivision of the edge  $z_ix$ ,  $i \in [2]$ . But, then  $O_2^1 \leq G$  (see Figure 45), a contradiction.

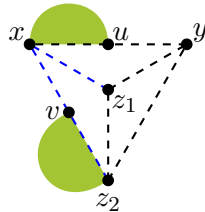


Figure 45: A flap in the subdivision of some edge  $z_ix$ ,  $i \in [2]$  (depicted in blue) in Subcase 1.4 of Claim 1.

Since we have exhausted all possible cases, Claim 1 follows.

To conclude Case 1 we prove the following:

*Claim 2:* Every flap of  $(H, K)$  is  $x$ -oriented.

*Proof of Claim 2:* Suppose, towards a contradiction, that there exists a flap  $F'$  that is not  $x$ -oriented. Then, by Claim 1, this is an  $(x, w)$ -flap where  $w$  is some vertex in the subdivision of an edge  $e' \in \{xy, xz_1, xz_2\}$ . Since,  $F'$  is not  $x$ -oriented then Lemma 3.6 implies exists a cycle  $C$  in  $F'$  that contains  $w$  but not  $x$ . If  $e' = xz_i$ , for some  $i \in \{1, 2\}$ , then  $O_2^1 \leq G$ , a contradiction (see the left figure of Figure 46). Thus,  $e' = xy$ . Then, if  $w \neq y$  we have that  $O_4^1 \leq G$  (see the central figure of Figure 46). Therefore,  $w = y$  and so  $F' \neq F$  and  $V(F) \cup V(F') = \{x\}$ . Therefore, by contracting  $F$  to a cycle we get a cycle containing  $x$  but not  $y$  disjoint to  $C$ . Hence  $O_1^1 \leq G$ , a contradiction (see the right figure of Figure 46). Claim 2 follows.

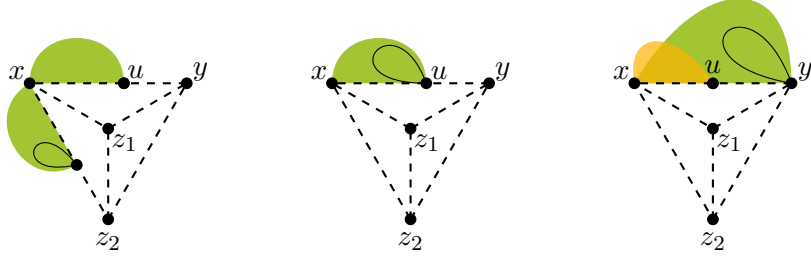


Figure 46: The possible configurations of the non  $x$ -oriented flap  $F'$  in the proof of Claim 2.

Claim 2 implies that  $x$  is a  $\mathcal{P}$ -apex vertex of  $G$  and so we arrive at a contradiction.

*Case 2:* Every  $(u, v)$ -flap is such that  $u, v$  are branch vertices of  $K$ .

We argue that the following holds:

*Claim 3:* All flaps of  $(H, K)$  share a branch vertex of  $K$ .

*Proof of Claim 3:* To prove Claim 3 we make the following two observations.

*Observation 1:* Every two flaps of  $(H, K)$  share a branch vertex of  $K$ . Indeed, suppose, to the contrary, that there exist two flaps, an  $(x, y)$ -flap and an  $(x', y')$ -flap, such that  $xy, x'y'$  are two non-incident edges of  $H$ . But then,  $O_5^2 \leq G$ , a contradiction (see Figure 47).

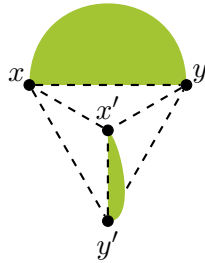


Figure 47: The configuration of the flaps of  $G$  in the proof of Observation 1.

*Observation 2:* Every three flaps of  $(H, K)$  share a branch vertex of  $K$ . Indeed, suppose to the contrary that there exist three flaps  $F_1, F_2, F_3$  such that  $V(F_1) \cap V(F_2) \cap V(F_3) = \emptyset$ . Observation 1 implies that  $V(F_1) \cap V(F_2) = \{x\}$ ,  $V(F_1) \cap V(F_3) = \{y\}$ , and  $V(F_2) \cap V(F_3) = \{z\}$ , where  $x, y, z$  are three branch vertices of  $K$ . Hence,  $O_8^3 \leq G$ , a contradiction (see Figure 48).

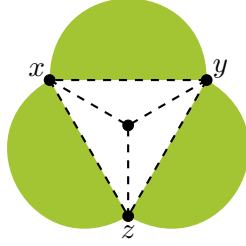


Figure 48: The configuration of the flaps of  $G$  in the proof of Observation 2.

Claim 3 is an immediate result of the above two Observations.

According to Claim 3, all flaps of  $(H, K)$  contain a branch vertex of  $K$ , namely  $x$ .

*Claim 4:* Every flap of  $(H, K)$  is  $x$ -oriented.

*Proof of Claim 4:* Recall that, by Claim 3, every flap of  $(H, K)$  is an  $(x, y)$ -flap, where  $y$  is a branch vertex of  $K$  different from  $x$ . Suppose, towards a contradiction, that there exists an  $(x, y)$ -flap  $F$  of  $(H, K)$  that is not  $x$ -oriented. Then, Lemma 3.6 implies that  $F$  is  $y$ -oriented and that there exists a cycle in  $F$  which contains  $y$  but not  $x$ .

We will prove that  $F$  is the only flap of  $(H, K)$ . Indeed, if there exists another flap, then, by Claim 1, it is an  $(x, y')$ -flap, where  $y'$  is a branch vertex of  $K$  different from both  $x$  and  $y$ . Thus,  $O_5^1 \leq G$ , a contradiction (see Figure 49).

Therefore,  $F$  is the only flap of  $(H, K)$  and since it is  $y$ -oriented,  $y$  is a  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction. Claim 4 follows.

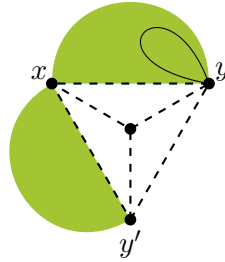


Figure 49: An example of the  $(x, y)$ -flap and the  $(x, y')$ -flap in the proof of Claim 4.

Claim 4, taking account that  $G$  is biconnected, implies that  $x$  is a  $\mathcal{P}$ -apex vertex of  $G$ , which is a contradiction. That proves Lemma 4.8.  $\square$

**Lemma 4.9.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $K_{2,3} \not\leq G$  or  $G$  is triconnected.*

*Proof.* Suppose, to the contrary, that  $G$  is not triconnected and  $K_{2,3} \leq G$ , which, by Lemma 4.6, implies that  $G$  is biconnected. Also, since  $G$  is not triconnected, by Lemma 4.8, it is  $K_4$ -free and so, by Lemma 3.10, there exists a unique b-rich separator  $S = \{x, y\}$  in  $G$ . We argue that the following holds:

*Claim 1:* There exists a unique augmented component in  $\mathcal{C}(G, S)$  not isomorphic to a cycle.

*Proof of Claim 1:* Observe that there exists an augmented connected component in  $\mathcal{C}(G, S)$  not isomorphic to a cycle, otherwise  $G \in \mathcal{P}^{(1)}$ . Suppose towards a contradiction that there exist two augmented connected components in  $\mathcal{C}(G, S)$  that are not isomorphic to a cycle.

Lemma 3.11 implies that one of the following holds:



- There exists a unique  $(x, y)$ -disjoint chord and there do not exist both  $x$ -chords and  $y$ -chords in  $G$ . But then, if there do not exist  $x$ -chords (or  $y$ -chords), then  $y$  (or  $x$ , respectively) is a  $\mathcal{P}$ -apex of  $G$ , a contradiction.
- There do not exist  $(x, y)$ -disjoint chords and there exists at most one  $x$ -chord or at most one  $y$ -chord. But then, if there exists at most one  $x$ -chord (or  $y$ -chord), then  $y$  (or  $x$ , respectively) is a  $\mathcal{P}$ -apex of  $G$ , a contradiction.

Notice that each of the above implies Claim 1.

According to Claim 1, let  $H$  be the unique augmented connected component in  $\mathcal{C}(G, S)$  that is not isomorphic to  $K_3$  and  $C$  be the Hamiltonian cycle of  $H$  (which exists due to biconnectivity and outerplanarity of  $H$ ). Keep in mind that  $xy \in E(C)$ .

*Claim 2:* Every chord of  $C$  is either an  $x$ -chord or a  $y$ -chord.

*Proof of Claim 2:* Suppose, towards a contradiction, that there exists some  $(x, y)$ -disjoint chord  $uv$ . Observe that there is a unique such chord, since otherwise  $\{O_1^0, O_3^0\} \leq G$ .

Suppose, without loss of generality, that in  $C \setminus xy$  there exists an  $(x, u)$ -path which does not contain  $v$  – we denote this path by  $P_1$ . Let  $P_2$  be the  $(y, v)$ -path in  $C \setminus xy$  (as shown in [Figure 50](#)).

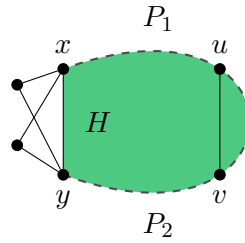


Figure 50: The chord  $uv$  and paths  $P_1, P_2$  as in the proof of Claim 1.

We argue that the following holds:

*Subclaim:* All chords of  $C$ , other than  $uv$ , are incident to the same vertex of the separator.

*Proof of Subclaim:* Suppose, towards a contradiction, that there exists an  $x$ -chord  $xx'$  and a  $y$ -chord  $yy'$ . Then, it cannot be the case that  $x' \in V(P_1)$  and  $y' \in V(P_2)$ , otherwise  $O_3^2 \leq G$ . On the other hand, it also cannot be the case that both  $x'$  and  $y'$  belong to the same  $P_i, i \in [2]$  since that implies  $O_{10}^2 \leq G$  (see [Figure 51](#)). Subclaim follows.

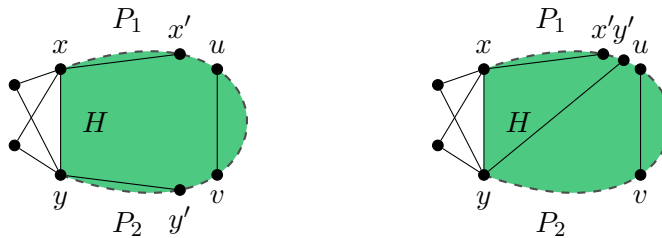


Figure 51: Possible configurations of chords  $xx', yy'$ , as in the proof of Subclaim.

According to the Subclaim, all chords of  $C$ , other than  $uv$ , are incident to the same vertex of the separator, say  $x$  – but then  $x$  is a  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction. Therefore, an  $(x, y)$ -disjoint chord cannot exist and this concludes the proof of Claim 2.

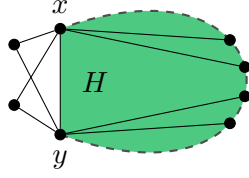


Figure 52: An example of  $H$  having at least two  $x$ -chords and at least two  $y$ -chords.

Now, since  $x$  is not an  $\mathcal{P}$ -apex there exist two chords of  $C$  not incident to  $x$ . By Claim 2, these are  $y$ -chords. Symmetrically, there exist two  $x$ -chords. Therefore, we have that  $O_{13}^2 \leq G$ , a contradiction (as shown in Figure 52).  $\square$

**Lemma 4.10.** *If  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  then  $G$  is triconnected.*

*Proof.* Suppose, to the contrary, that  $G$  is not triconnected. Then, by Lemma 4.7, Lemma 4.8, and Lemma 4.9,  $G$  is biconnected and outerplanar and so it contains a Hamiltonian cycle, namely  $C$ .

Observe that  $G$  has at most 3 vertices of degree 2. Indeed, if there exist 4 vertices  $v_1, v_2, v_3, v_4 \in V(G)$  of degree 2, then, by Lemma 3.1, they are simplicial and by contracting all edges of  $C$ , except those that are incident to  $v_1, v_2, v_3, v_4$ , we can form  $O_3^2$  as a minor of  $G$  (as depicted in Figure 53), a contradiction.

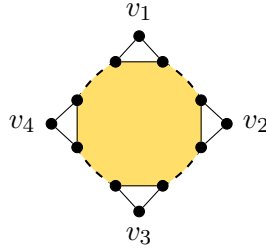


Figure 53: An outerplanar graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$  having 4 vertices of degree 2.

On the other hand, by Proposition 2.2,  $G$  has at least two vertices of degree 2. Thus, we distinguish the following two cases:

*Case 1:*  $G$  has exactly 2 vertices  $u, v$  of degree 2, which, by Lemma 3.1, are simplicial.

Observe that  $C \setminus u \setminus v$  is the union of two vertex disjoint paths  $P_1, P_2$ . Let  $u_1, u_2$  be the neighbors of  $u$  in  $P_1, P_2$ , respectively and  $v_1, v_2$  be the neighbors of  $v$  in  $P_1, P_2$ , respectively. Therefore, the structure of the graph  $G$  is as follows:

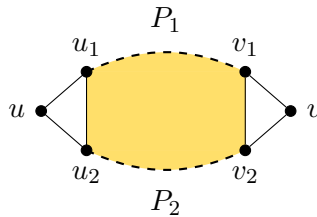


Figure 54: The structure of the graph  $G$  in Case 1 of Lemma 4.10.

We prove the following claim concerning the chords of  $C$ .

*Claim 1:* Every chord of  $C$  is between a vertex of  $P_1$  and a vertex of  $P_2$ .

*Proof of Claim 1:* Suppose to the contrary that there exists an edge connecting non-consecutive vertices of  $P_1$  or  $P_2$ , say  $P_1$ , and let  $e = xy$  be such an edge whose endpoints have the smallest possible distance in  $P_1$ .

Let  $P'_1$  be the subpath of  $P_1$  between  $x$  and  $y$ . Since  $e$  is a chord of  $C$ ,  $P'_1$  contains an internal vertex  $w$ . Note then that, since  $u, v$  are the only vertices of  $G$  of degree 2,  $w$  is of degree greater than 2 and so there exists a neighbor of  $w$ , denoted by  $z$ , such that  $z, w$  are not adjacent in  $P_1$ . Observe that  $z$  is a vertex of  $P'_1$ , since otherwise  $K_4 \leq G$ . But then  $wz$  is an edge connecting non-consecutive vertices of  $P_1$  whose endpoints have smaller distance (in  $P_1$ ) than that of  $x, y$ , a contradiction to the minimality of  $P'_1$  (see Figure 55). This concludes the proof of Claim 1.

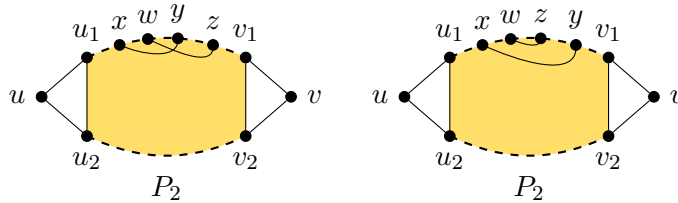


Figure 55: The chord  $wz$  in the proof of Claim 1.

We now make a series of observations:

*Observation 1:* Every internal vertex of  $P_1$  and  $P_2$  is incident to a chord. Indeed, it is obvious from the fact that  $u, v$  are the only vertices of degree 2.

*Observation 2:* Every internal vertex of  $P_1$  is adjacent to  $u_2$  or  $v_2$ . Respectively, every internal vertex of  $P_2$  is adjacent to  $u_1$  or  $v_1$ . Indeed, if there exists an internal vertex  $x$  of  $P_1$  or  $P_2$ , say  $P_1$ , not incident to  $u_2$  or  $v_2$  then by Observation 1 and Claim 1  $x$  is adjacent to an internal vertex of  $P_2$  and hence  $O_{15}^2 \leq G$ , a contradiction (as shown in Figure 56).

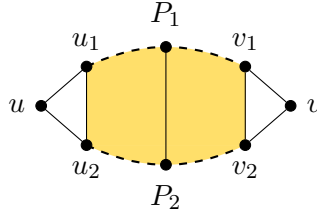


Figure 56: A chord connecting two internal vertices of  $P_1, P_2$  in the proof of Observation 2 of Case 1.

*Observation 3:* One of  $P_1, P_2$  must be of length at most 1. Indeed, suppose to the contrary, that both  $P_1, P_2$  is of length at least 2. Then, both of  $P_1, P_2$  contain an internal vertex, namely  $x_1, x_2$ , respectively. Then, by Observation 1, they are both incident to some chord of  $C$ . By Observation 2,  $x_1$  is adjacent to  $u_2$  or  $v_2$ , say  $u_2$ . Then, again by Observation 2 and  $K_4$ -freeness of  $G$ ,  $x_2$  is adjacent to  $v_1$ , as shown in Figure 57. But then,  $O_1^0 \leq G$ , a contradiction.

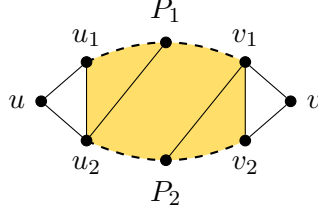


Figure 57: An example of the graph  $G$  in Observation 3 of Case 1.

*Observation 4:* Either  $u_1$  or  $v_1$  is adjacent to every internal vertex of  $P_2$ . Respectively, either  $u_2$  or  $v_2$  is adjacent to every internal vertex of  $P_1$ . Indeed, if otherwise, then Observations 1 and 2 imply, without loss of generality, that there exist  $x, y \in V(P_1) \setminus \{u_1, v_1\}$  such that  $x, y$  are adjacent to  $u_2, v_2$ , respectively. Hence,  $O_{13}^2 \leq G$  (see Figure 58).

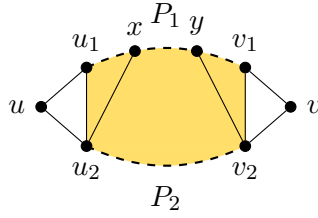


Figure 58: Two chords  $xu_2, yv_2$  where  $x, y$  are internal vertices of  $P_1$  in Observation 4 of Case 1.

By Observation 3 we can assume that  $P_2$  is of length  $j \leq 1$ . Then, by Observation 4,  $u_2$  or  $v_2$ , say  $u_2$ , is adjacent to every internal vertex of  $P_1$ . This implies that every chord of  $C$ , except for  $v_1v_2$  (if  $v_2 \neq u_2$ ), is incident to  $u_2$  and hence  $u_2$  is a  $\mathcal{P}$ -apex vertex of  $G$ , a contradiction.

*Case 2:*  $G$  has exactly 3 vertices  $u, v, w$  of degree 2, which, by Lemma 3.1, are simplicial.

Note that if all three vertices have pairwise disjoint closed neighborhoods, then  $O_{14}^2 \leq G$ , which is a contradiction.

Therefore, at least two of them, say  $u$  and  $v$ , have non-disjoint neighborhoods. We argue that the following holds:

*Claim 2:*  $N_G(u) \cap N_G(v) = \{x\}$  for some  $x \in V(G)$ .

*Proof of Claim 2:* Since  $u, v$  have non-disjoint neighborhoods, then either  $N_G(u) \cap N_G(v) = e$  for some edge  $e \in E(G)$  or  $N_G(u) \cap N_G(v) = \{x\}$  for some vertex  $x \in V(G)$ .

Suppose that  $N_G(u) \cap N_G(v) = \{a, b\}$  for some edge  $e = ab \in E(G)$  and consider any two internally vertex disjoint paths (which exist due to biconnectivity of  $G$ ) from  $w$  to, say,  $v$ . Observe that one of the paths contains  $a$  and the other contains  $b$ . Therefore  $K_{2,3} \leq G$ , with  $\{a, b\}$  forming one part of the  $K_{2,3}$  minor and  $\{u, v, w\}$  forming the other, a contradiction. Claim 2 follows.

By Claim 2,  $C \setminus \{u, v, w, x\}$  is the union of two vertex disjoint paths  $R_1, R_2$ . We can assume that  $u$  has a neighbor  $u_1$  in  $R_1$ ,  $v$  has a neighbor  $v_2$  in  $R_2$ , and  $N_G(w) = \{w_1, w_2\}$ , where  $w_1 \in R_1$  and  $w_2 \in R_2$ . By arguments identical to the proof of Claim 1 in Case 1, we have that every chord of  $C$  is between a vertex in  $R_1 \cup \{x\}$  and a vertex in  $R_2 \cup \{x\}$ . Therefore, the structure of the graph  $G$  is as follows (Figure 59):

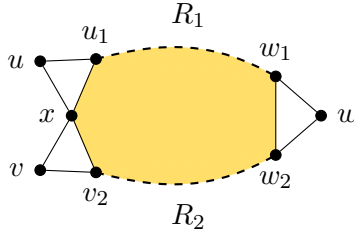


Figure 59: The structure of the graph  $G$  in Case 2 of [Lemma 4.10](#).

We now observe the following about the paths  $R_1, R_2$ :

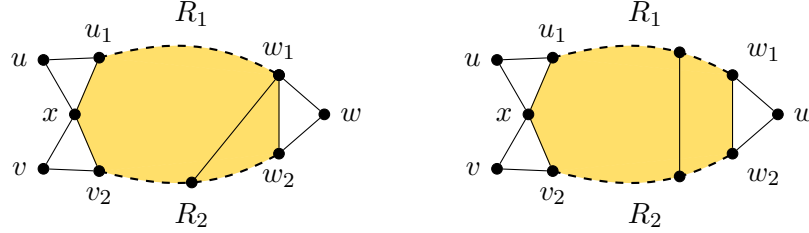


Figure 60: The two configurations of the chord not incident to  $x$  in the proof of [Observation 5](#) of Case 2.

*Observation 5:* One of  $R_1, R_2$  is of length 0. Indeed, suppose, towards a contradiction, that both of  $R_1, R_2$  is of length at least 1. Since  $x$  is not a  $\mathcal{P}$ -apex vertex of  $G$  then there exists a chord  $e$  between a vertex in  $R_1$  and a vertex in  $R_2$  such that  $e \neq w_1w_2$ . Then, if  $e$  is incident to  $w_1w_2$  we have that  $\mathcal{O}_{12}^2 \leq G$ , while if  $e$  is disjoint from  $w_1w_2$  we have that  $\mathcal{O}_7^2 \leq G$ , a contradiction in both cases (see [Figure 60](#)).

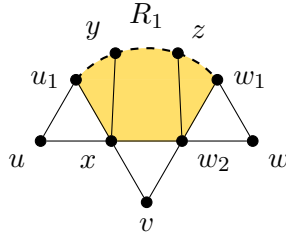


Figure 61: An example of the graph  $G$  in the last part of the proof of [Lemma 4.10](#).

By [Observation 5](#), we can assume that  $R_2$  is of length 0, i.e.  $v_2 = w_2$ . Then, every chord of  $C$ , except for  $xw_2$ , is between a vertex of  $R_1$  and a vertex in  $\{x, w_2\}$ . Then, since  $x$  and  $w_2$  are not  $\mathcal{P}$ -apex vertices of  $G$  there exist  $y, z$  (possibly with  $y = z$ ) internal vertices of  $R_1$  incident to  $x$  and  $w_2$ , respectively. Hence,  $\mathcal{O}_{13}^2 \leq G$  (see [Figure 61](#)), a contradiction. The proof of [Lemma 4.10](#) is complete.  $\square$

We are now in position to prove [Theorem 1.1](#).

*Proof of [Theorem 1.1](#).* As we mentioned in the end of [Section 1](#),  $\mathbf{obs}(\mathcal{P}^{(1)}) \supseteq \mathcal{O}$  and what remains is to prove that  $\mathcal{O} \supseteq \mathbf{obs}(\mathcal{P}^{(1)})$  or alternative that  $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O} = \emptyset$ . For this assume, towards a contradiction, that there exists a graph  $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ . From [Lemma 4.10](#),  $G$  should be triconnected. Therefore, from [Lemma 2.7](#), either  $\mathcal{O}^3 \leq G$ , a contradiction, or  $G$  is isomorphic to  $W_r$ , for some  $r \geq 3$ , again a contradiction, as  $W_r \in \mathcal{P}^{(1)}$  for all  $r \geq 3$ .  $\square$

## References

- [1] Isolde Adler. Open problems related to computing obstruction sets. Manuscript, September 2008.
- [2] Isolde Adler, Martin Grohe, and Stephan Kreutzer. Computing excluded minors. In *nineteenth annual ACM-SIAM symposium on Discrete algorithms*, SODA '08, pages 641–650. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008.
- [3] Dan Archdeacon. A kuratowski theorem for the projective plane. *Journal of Graph Theory*, 5:243 – 246, 10 2006.
- [4] Stefan Arnborg, Andrzej Proskurowski, and Derek G. Corneil. Forbidden minors characterization of partial 3-trees. *Discrete Mathematics*, 80(1):1 – 19, 1990.
- [5] D. Bienstock and N. Dean. On obstructions to small face covers in planar graphs. *J. Comb. Theory Ser. B*, 55(2):163–189, 1992.
- [6] Hans L. Bodlaender and Dimitrios M. Thilikos. Graphs with branchwidth at most three. *J. Algorithms*, 32(2):167–194, 1999.
- [7] Kevin Cattell, Michael J. Dinneen, Rodney G. Downey, Michael R. Fellows, and Michael A. Langston. On computing graph minor obstruction sets. *Theor. Comput. Sci.*, 233:107–127, February 2000.
- [8] Janka Chlebíková. The structure of obstructions to treewidth and pathwidth. *Discrete Appl. Math.*, 120(1-3):61–71, 2002.
- [9] Bruno Courcelle, Rodney G. Downey, and Michael R. Fellows. A note on the computability of graph minor obstruction sets for monadic second order ideals. In *First Japan-New Zealand Workshop on Logic in Computer Science (Auckland, 1997)*, volume 3, pages 1194–1198 (electronic). Springer, 1997.
- [10] Guoli Ding and Stan Dziobiak. Excluded-minor characterization of apex-outerplanar graphs. *Graphs and Combinatorics*, 32(2):583–627, Mar 2016.
- [11] Michael J. Dinneen. Too many minor order obstructions (for parameterized lower ideals). In *First Japan-New Zealand Workshop on Logic in Computer Science (Auckland, 1997)*, volume 3(11), pages 1199–1206 (electronic). Springer, 1997.
- [12] Michael J. Dinneen, Kevin Cattell, and Michael R. Fellows. Forbidden minors to graphs with small feedback sets. *Discrete Math.*, 230(1-3):215–252, 2001. Paul Catlin memorial collection (Kalamazoo, MI, 1996).
- [13] Michael J. Dinneen and Rongwei Lai. Properties of vertex cover obstructions. *Discrete Mathematics*, 307(21):2484–2500, 2007.
- [14] Michael J. Dinneen and Ralph Versteegen. Obstructions for the graphs of vertex cover seven. Technical Report CDMTCS-430, University of Auckland, 2012. Technical report.

- [15] Michael J. Dinneen and Liu Xiong. Minor-order obstructions for the graphs of vertex cover 6. *Journal of Graph Theory*, 41(3):163–178, 2002.
- [16] Gabriel A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *Journal of the London Mathematical Society*, s1-27(1):85–92, 1952.
- [17] Zdeněk Dvořák, Archontia C. Giannopoulou, and Dimitrios M. Thilikos. Forbidden graphs for tree-depth. *Eur. J. Comb.*, 33(5):969–979, July 2012.
- [18] Stan Dziobiak and Gioli Ding. Obstructions of apex classes of graphs. Unpublished results (see <http://msdiscretemath.org/2013/dziobiak.pdf>).
- [19] Michael R. Fellows and Bart M. P. Jansen. Fpt is characterized by useful obstruction sets. In Andreas Brandstädt, Klaus Jansen, and Rüdiger Reischuk, editors, *Graph-Theoretic Concepts in Computer Science*, pages 261–273, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.
- [20] Fedor V. Fomin, Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. Planar f-deletion: Approximation, kernelization and optimal FPT algorithms. In *53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012*, pages 470–479, 2012.
- [21] Andrei Gagarin, Wendy Myrvold, and John Chambers. The obstructions for toroidal graphs with no  $k_3, 3$ 's. *Discrete Mathematics*, 309(11):3625 – 3631, 2009. 7th International Colloquium on Graph Theory.
- [22] Arvind Gupta and Russell Impagliazzo. Bounding the size of planar intertwiners. *SIAM J. Discrete Math.*, 10(3):337–358, 1997.
- [23] Adam S. Jobson and André E. Kézdy. All minor-minimal apex obstructions with connectivity two. *CoRR*, abs/1808.05940, 2018.
- [24] Valentine Kabanets. Recognizability equals definability for partial  $k$ -paths. In Pierpaolo Degano, Roberto Gorrieri, and Alberto Marchetti-Spaccamela, editors, *Automata, Languages and Programming*, pages 805–815, Berlin, Heidelberg, 1997. Springer Berlin Heidelberg.
- [25] N.G. Kinnersley and Michael A. Langston. Obstruction set isolation for the gate matrix layout problem. *Discrete Applied Mathematics*, 54:169–213, 1994.
- [26] Athanassios Koutsonas, Dimitrios M. Thilikos, and Koichi Yamazaki. Outerplanar obstructions for matroid pathwidth. *Discrete Mathematics*, 315-316:95 – 101, 2014.
- [27] Kazimierz Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.*, 15:271–283, 1930.
- [28] J. Lagergren. Upper bounds on the size of obstructions and intertwiners. *J. Comb. Theory, Ser. B*, 73:7–40, 1998.
- [29] Max Lipton, Eoin Mackall, Thomas W. Mattman, Mike Pierce, Samantha Robinson, Jeremy Thomas, and Ilan Weinschelbaum. Six variations on a theme: almost planar graphs. [arxiv.org/abs/1608.01973](http://arxiv.org/abs/1608.01973), 2016.

- [30] Saunders Mac Lane. A structural characterization of planar combinatorial graphs. *Duke Math. J.*, 3(3):460–472, 09 1937.
- [31] Thomas W. Mattman. Forbidden minors: Finding the finite few. *CoRR*, abs/1608.04066, 2016.
- [32] Thomas W. Mattman and Mike Pierce. The  $k_{n+5}$  and  $k_{3^2, 1^n}$  families are obstructions to  $n$ -apex. *CoRR*, abs/1603.00885, 2016.
- [33] Bojan Mohar and Petr Škoda. Obstructions of connectivity two for embedding graphs into the torus. *Canadian Journal of Mathematics*, 66, 07 2012.
- [34] Mike Pierce. *Searching for and classifying the finite set of minor-minimal non-apex graphs*. PhD thesis, Chicago State University, Available at <http://www.csuchico.edu/~tmattman>, 2014.
- [35] Siddharthan Ramachandramurthi. The structure and number of obstructions to treewidth. *SIAM J. Discrete Math.*, 10(1):146–157, 1997.
- [36] Neil Robertson, Paul D. Seymour, and Robin Thomas. Sachs’ linkless embedding conjecture. *J. Comb. Theory, Ser. B*, 64(2):185–227, 1995.
- [37] Neil Robertson and P.D. Seymour. Graph minors. XX. wagner’s conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325 – 357, 2004. Special Issue Dedicated to Professor W.T. Tutte.
- [38] Juanjo Rué, Konstantinos S. Stavropoulos, and Dimitrios M. Thilikos. Outerplanar obstructions for a feedback vertex set. *Eur. J. Comb.*, 33(5):948–968, 2012.
- [39] Dimitrios M. Thilikos. Algorithms and obstructions for linear-width and related search parameters. *Discrete Applied Mathematics*, 105:239–271, 2000.
- [40] W. T. Tutte. A theory of 3-connected graphs. *Nederl. Akad. Wetensch. Proc. Ser. Indag. Math.*, 23:441–455, 1961.
- [41] William Thomas Tutte. *Connectivity in graphs*. University of Toronto Press, 1966.
- [42] Yaming Yu. More forbidden minors for wye-delta-wye reducibility. *Electr. J. Comb.*, 13(1), 2006.
- [43] Dimitris Zoros. *Obstructions and Algorithms for Graph Layout Problems*. PhD thesis, National and Kapodistrian University of Athens, Department of Mathematics, July 2017.