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▶ **To cite this version:**

Alexandros Leivaditis, Alexandros Singh, Giannos Stamoulis, Dimitrios M. Thilikos, Konstantinos Tsatsanis. Minor obstructions for apex-pseudoforests. Discrete Mathematics, 2021, 344 (10), pp.112529. 10.1016/j.disc.2021.112529 . hal-03327317

HAL Id: hal-03327317
https://hal.science/hal-03327317

Submitted on 22 Oct 2021

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arXiv:1811.06761v4 [math.CO] 24 Jan 2019

Minor-Obstructions for Apex-Pseudoforests*

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Abstract

A graph is called a *pseudoforest* if none of its connected components contains more than one cycle. A graph is an *apex-pseudoforest* if it can become a pseudoforest by removing one of its vertices. We identify 33 graphs that form the minor-obstruction set of the class of apex-pseudoforests, i.e., the set of all minor-minimal graphs that are not apex-pseudoforests.

Keywords: Graph minors, Minor Obstructions

1 Introduction

All graphs in this paper are undirected, finite, and simple, i.e., without loops or multiple edges.

A graph G is a *pseudoforest* if every connected component of G contains at most one cycle. We denote by $\mathcal{P}$ the set of all pseudoforests.

We say that a graph H is a *minor* of G if a graph isomorphic to H can be obtained by some subgraph of G after applying edge contractions. (As in this paper we consider only simple graphs, we always assume that in case multiple edges are created after a contraction, then these edges are automatically suppressed to simple edges.) We say that a graph class $\mathcal{G}$ is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also a member of $\mathcal{G}$. Given a graph class $\mathcal{G}$, its *minor-obstruction set* is defined as the minor-minimal set of all graphs that are not in $\mathcal{G}$ and is denoted by $\mathbf{obs}(\mathcal{G})$. For simplicity, we drop the term "minor" when we refer to an obstruction set as, in this paper, we only consider minor-obstruction sets. We also refer to the members of $\mathbf{obs}(\mathcal{G})$ as *obstructions of* $\mathcal{G}$. Given a set of graphs $\mathcal{H}$ we denote by $\mathbf{exc}(\mathcal{H})$ as the set containing every graph G that excludes all graphs in $\mathcal{H}$ as minors.

Notice that if $\mathcal{G}$ is minor-closed, then a graph G belongs in $\mathcal{G}$ iff none of the graphs in $\mathbf{obs}(\mathcal{G})$ is a minor of G. In this way, $\mathbf{obs}(\mathcal{G})$ can be seen as a complete characterization of $\mathcal{G}$ in terms of forbidden minors, i.e., $\mathcal{G} = \mathbf{excl}(\mathbf{obs}(\mathcal{G}))$.

According to the Roberston and Seymour theorem [37], for every graph class $\mathcal{G}$, the set $\mathbf{obs}(\mathcal{G})$ is finite. The study of $\mathbf{obs}(\mathcal{G})$ for distinct instantiations of minor-closed graph classes is an active topic in graph theory (e.g., see [3–6, 6–9, 11, 13–15, 17, 19, 21, 21–29, 32, 33, 35, 36, 38, 39, 42], see also [1, 31] for related surveys).

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¶Supported by projects DEMOGRAPH (ANR-16-CE40-0028) and ESIGMA (ANR-17-CE23-0010).

It is easy to observe that pseudoforests is a minor-closed graph class. Moreover it holds that $\mathbf{obs}(\mathcal{P}) = \{$ , $\}$, as these two obstructions express the existence of two cycles in the same connected component of a graph.

Given a non-negative integer k and a graph class $\mathcal{G}$, we say that a graph G is a *k-apex of $\mathcal{G}$* if it can be transformed to a member of $\mathcal{G}$ after removing at most k of its vertices. We also use the term *apex of* $\mathcal{G}$ as a shortcut of 1-apex of $\mathcal{G}$. We denote the set of all k-apices of $\mathcal{G}$ by $\mathcal{A}_k(\mathcal{G})$ and, more specifically, we use $\mathcal{P}^{(k)} := \mathcal{A}_k(\mathcal{G})$ for the set of all k-apex-pseudoforests.

The problem of characterizing k-apices of graph classes, has attracted a lot of attention, both from the combinatorial and algorithmic point of view. This problem can be seen as a part of the wider family of *Graph Modification Problems* (where the modification is the removal of a vertex).

It is easy to see that if $\mathcal{G}$ is minor closed, then $\mathcal{A}_k(\mathcal{G})$ is also minor-closed for every $k \geq 0$. Therefore $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$ can be seen as a complete characterization of k-apices of $\mathcal{G}$. The study of $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$ when $\mathcal{G}$ is some minor-closed graph class has attracted some special attention and can generate several known graph invariants. For instance, graphs with a vertex cover of size at most k are the graphs in $\mathcal{A}_k(\mathbf{excl}(\{K_2\}))$, graphs with a vertex feedback set at most k are the graphs in $\mathcal{A}_k(\mathbf{excl}(\{K_3\}))$, and k-apex planar graphs are the graphs in $\mathcal{A}_k(\mathbf{excl}(\{K_5, K_{3,3}\}))$.

The general problem that emerges is, given a finite set of graphs $\mathcal{H}$ and a positive integer k, to identify the set

$$\mathcal{H}^{(k)} := \mathbf{obs}(\mathcal{A}_k(\mathbf{exc}(\mathcal{H}))).$$

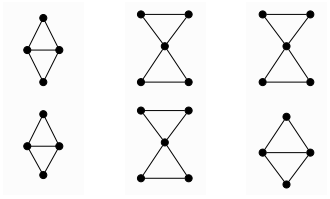
A fundamental result in this direction is that the above problem is computable [2]. Moreover, it was shown in [20] that if $\mathcal{H}$ contains some planar graph, then every graph in $\mathcal{H}^{(k)}$ has $O(k^h)$ vertices, where h is some constant depending (non-constructively) on $\mathcal{H}$. Also, in [43], it was proved that, under the additional assumption that all graphs in $\mathcal{H}$ are connected, this bound becomes linear on k for the intersection of $\mathcal{H}^{(k)}$ with sparse graph classes such as planar graphs or bounded degree graphs. An other structural result in this direction is the characterization of the disconnected obstructions in $\mathcal{H}^{(k)}$ in the case where $\mathcal{H}$ consists only of connected graphs [11] (see Proposition 2.8).

An other direction is to study $\mathcal{H}^{(k)}$ for particular instantiations of $\mathcal{H}$ and k. In this direction $\{K_2\}^{(k)}$ has been identified for $k \in \{1, \ldots, 5\}$ in [7], for $k = 6$ in [15] and for $k = 7$ in [14], while the graphs in $\{K_3\}^{(i)}$ have been identified in [12] for $i \in \{1, 2\}$. Recently, in [10], Ding and Dziobiak identified the 57 graphs in $\{K_4, K_{2,3}\}^{(1)}$, i.e., the obstruction set for apex-outerplanar graphs, and the 25 graphs in $\{$ $\}^{(1)}$, i.e., the obstruction set for apex-cactus graphs (as announced in [18]). Moreover, the problem of identifying $\{K_5, K_{3,3}\}^{(1)}$ (i.e., characterizing 1-apex planar graphs – also simply known as *apex graphs*) has attracted particular attention (see e.g., [29, 31, 42]). In this direction Mattman and Pierce conjectured that $\{K_5, K_{3,3}\}^{(n)}$ consists of the $Y\Delta Y$-families of K_{n+5} and $K_{3^2,2^n}$ and provided evidence on this [32]. Moreover, it has been shown that $|\{K_5, K_{3,3}\}^{(1)}| > 150$, $|\{K_5, K_{3,3}\}^{(2)}| > 82$, $|\{K_5, K_{3,3}\}^{(3)}| > 601$, $|\{K_5, K_{3,3}\}^{(4)}| > 520$, and $|\{K_5, K_{3,3}\}^{(5)}| > 608$ (see [34]). Recently, Jobson and Kézdy [23] identified all 2-connected graphs in $\{K_5, K_{3,3}\}^{(1)}$.

In this paper we identify $\{$ , $\}^{(1)}$, i.e., the obstruction set of apex-pseudoforests. Let $\mathcal{O}^0, \mathcal{O}^1, \mathcal{O}^2, \mathcal{O}^3$ be the sets of graphs depicted in Figures 1, 2, 3, 4, respectively. Notice that, for every $i \in \{0, 1, 2, 3\}$, the graphs in $\mathcal{O}^i$ are all i-connected but not $(i+1)$-connected.

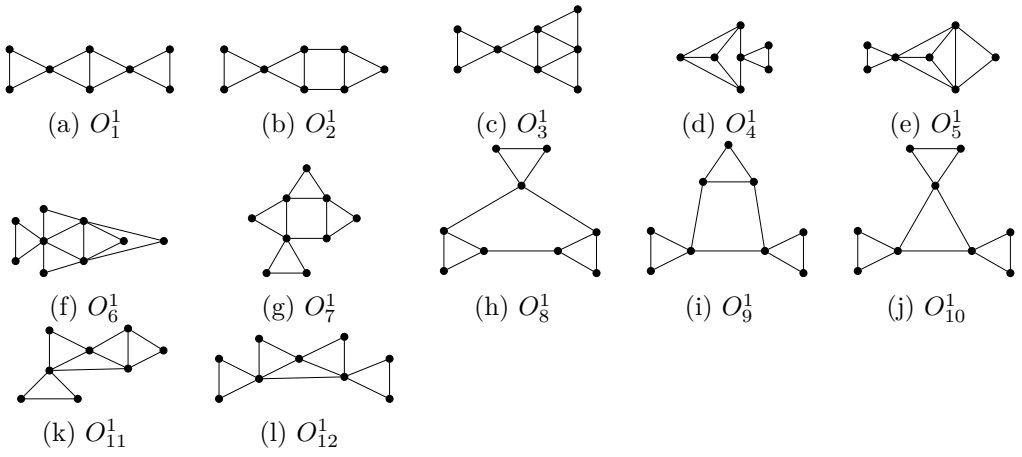
Our main result is the following.

Theorem 1.1. $\mathbf{obs}(\mathcal{P}^{(1)}) = \mathcal{O}^0 \cup \mathcal{O}^1 \cup \mathcal{O}^2 \cup \mathcal{O}^3$.



(a) O_1^0 (b) O_2^0 (c) O_3^0

Figure 1: The set $\mathcal{O}^0$ of obstructions for $\mathcal{P}^{(1)}$ with vertex connectivity 0.

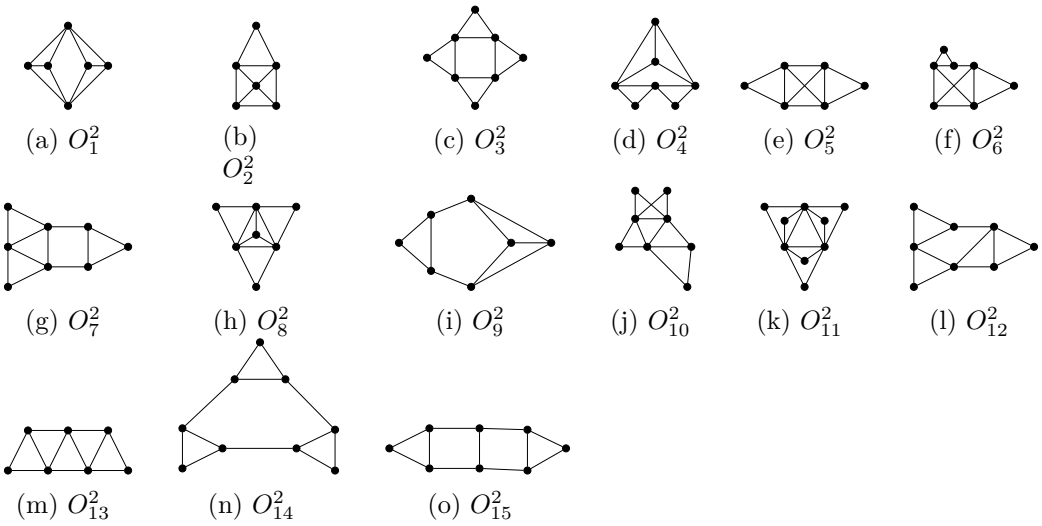


(a) O_1^1 (b) O_2^1 (c) O_3^1 (d) O_4^1 (e) O_5^1

(f) O_6^1 (g) O_7^1 (h) O_8^1 (i) O_9^1 (j) O_{10}^1

(k) O_{11}^1 (l) O_{12}^1

Figure 2: The set $\mathcal{O}^1$ of obstructions for $\mathcal{P}^{(1)}$ of vertex connectivity 1.

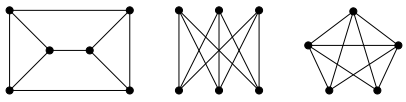


(a) O_1^2 (b) O_2^2 (c) O_3^2 (d) O_4^2 (e) O_5^2 (f) O_6^2

(g) O_7^2 (h) O_8^2 (i) O_9^2 (j) O_{10}^2 (k) O_{11}^2 (l) O_{12}^2

(m) O_{13}^2 (n) O_{14}^2 (o) O_{15}^2

Figure 3: The set $\mathcal{O}^2$ of obstructions for $\mathcal{P}^{(1)}$ with vertex connectivity 2.



(a) O_1^3 (b) O_2^3 (c) O_3^3

Figure 4: The set $\mathcal{O}^3$ obstructions for $\mathcal{P}^{(1)}$ with vertex connectivity 3.

We set $\mathcal{O} = \mathcal{O}^0 \cup \mathcal{O}^1 \cup \mathcal{O}^2 \cup \mathcal{O}^3$. For the proof of Theorem 1.1, we first note, by inspection, that $\mathbf{obs}(\mathcal{P}^{(1)}) \supseteq \mathcal{O}$. As such an inspection might be quite tedious to do manually for all the 33 graphs in $\mathcal{O}$, one may use a computer program that can do this in an automated way (see www.cs.upc.edu/~sedthilk/oapf/ for code in SageMath that can do this). The main contribution of the paper is that $\mathcal{O}$ is a complete list, i.e., that $\mathbf{obs}(\mathcal{P}^{(1)}) \subseteq \mathcal{O}$.

Our proof strategy is to assume that there exists a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ and gradually restrict the structure of G by deriving contradictions to some of the the conditions of the following observation.

Observation 1.2. If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then G satisfies the following conditions:

1. $G \notin \mathcal{P}^{(1)}$,
2. if G' is a minor of G that is different than G, then $G' \in \mathcal{P}^{(1)}$, and
3. none of the graphs in $\mathcal{O}$ is a minor of G.

The rest of the paper is dedicated to the proof of Theorem 1.1 and is organized as follows. In Section 2 we give the basic definitions and some preliminary results. In Section 3 we prove some auxiliary results that restrict the structure of the graphs in $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$. In Section 4 we use the results of Section 3 in order to, first, prove that graphs in $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ are biconnected (Lemma 4.7) and, next, prove that the graphs in $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ are triconnected (Lemma 4.10). The proof of Theorem 1.1 follows from the fact that every triconnected graph either contains a graph in $\mathcal{O}^3$ or it is a graph in $\mathcal{P}^{(1)}$ (Lemma 2.7, proved in Section 2).

2 Definitions and preliminary results

Sets and integers. We denote by $\mathbb{N}$ the set of all non-negative integers and we set $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Given two integers p and q, we set $[p, q] = \{p, \ldots, q\}$ and given a $k \in \mathbb{N}^+$ we denote $[k] = [1, k]$. Given a set A, we denote by 2^A the set of all its subsets and we define $\binom{A}{2} := \{e \mid e \in 2^A \wedge |e| = 2\}$. If $\mathcal{S}$ is a collection of objects where the operation $\cup$ is defined, then we denote $\bigcup\mathcal{S} = \bigcup_{X \in \mathcal{S}} X$.

Graphs. Given a graph G, we denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of the edges of G. For an edge $e = \{x, y\} \in E(G)$, we use instead the notation $e = xy$, that is equivalent to $e = yx$. Given a vertex $v \in V(G)$, we define the *neighborhood* of v as $N_G(v) = \{u \mid u \in V(G), \{u, v\} \in E(G)\}$ and the *closed neighborhood* of v as $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, then we write $N_G(X) = (\bigcup_{v \in X} N_G(v)) \setminus X$. The *degree* of a vertex v in G is defined as $\mathbf{deg}_G(v) = |N_G(v)|$. We define $\delta(G) = \min\{\mathbf{deg}_G(x) \mid x \in V(G)\}$. Given two graphs G_1, G_2, we define the *union* of G_1, G_2 as the graph $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and the *intersection* of G_1, G_2 as the graph $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$. A *subgraph* of a graph $G = (V, E)$ is every graph H where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $S \subseteq V(G)$, the subgraph of G *induced by* S, denoted by $G[S]$, is the graph $(S, E(G) \cap \binom{S}{2})$. We also define $G \setminus S$ to be the subgraph of G induced by $V(G) \setminus S$. If $S \subseteq E(G)$, we denote by $G \setminus S$ the graph $(V(G), E(G) \setminus S)$. Given a vertex $x \in V(G)$ we define $G \setminus x = G \setminus \{x\}$ and given an edge $e \in E(G)$ we define $G \setminus e = G \setminus \{e\}$.

Paths and separators. If $s, t \in V(G)$, an (s, t)-*path* of G is any connected subgraph P of G with maximum degree 2, where $\mathbf{deg}_P(s) = 1$ and $\mathbf{deg}_P(t) = 1$. The *distance* between s and t in G

is the minimum number of edges of an (s,t)-path in G. Given a path P, we say that $v \in V(P)$ is an *internal vertex* of P if $\mathbf{deg}_P(v) = 2$, while if $\mathbf{deg}_P(v) = 1$ we say that v is a *terminal vertex* of P. We say that two paths P_1 and P_2 in G are *internally vertex disjoint* if none of the internal vertices of the one is an internal vertex of the other. Given an integer k and a graph G, we say that G is k-connected if for each $\{u,v\} \in \binom{V}{2}$, there exists k pairwise internally disjoint (u,v)-paths of G, say $P_1, \ldots, P_k$, such that for each $\{i,j\} \in \binom{[k]}{2}$, $P_i \neq P_j$, $V(P_i) \cap V(P_j) = \{u,v\}$. We call 2-connected graphs *biconnected* and 3-connected graphs *triconnected*. Given a set $S \subseteq V(G)$, we say that S is a *separator* of G if G has less connected components than $G \setminus S$. We call a separator of size k a *k-separator*. Notice that, by Menger's theorem a graph is k-connected iff it does not contain a separator of size $< k$. A *block* of a graph G is a maximal biconnected subgraph.

A vertex $v \in V(G)$ is a *cut-vertex* of G if $\{v\}$ is a separator of G. We also say that S is a *rich separator* if $G \setminus S$ has at least 2 more connected components than G.

Special graphs. By K_r we denote the complete graph on r vertices. Similarly, by K_{r_1,r_2} we denote the complete bipartite graph of which one part has r_1 vertices and the other r_2.

For an $r \geq 3$, we denote by C_r the connected graph on r vertices of degree 2 (i.e., the cycle on r vertices). If G is a graph and C is a subgraph of G isomorphic to C_r for some $r \geq 3$, then an edge $e = \{u,v\} \in E(G) \setminus E(C)$ where $u, v \in V(C)$ is called *chord* of C.

For $r \geq 3$, the *r-wheel*, denoted by W_r, is the graph obtained by adding a new vertex v_{new}, called the *central vertex* of W_r, to C_r along with edges, called *spokes*, connecting each vertex of C_r with v_{new}. The subgraph $W_r \setminus v_{\text{new}}$ is called the *circumference* of W_r.

A graph G is *outerplanar* if it can be embedded in the plane so that there's no crossing edges and all its vertices lie on the same face. It is known that the obstruction set of the class of outerplanar graphs is $\{K_{2,3}, K_4\}$. The outer face of such an embedding contains every vertex of G. Thus, we can observe the following:

Observation 2.1. If G is biconnected and outerplanar then G contains a *Hamiltonian cycle*, i.e. a cycle which contains every vertex of G.

Minors. We define G/e, the graph obtained from the graph G by *contracting* an edge $e = xy \in E(G)$, to be the graph obtained by replacing the edge e by a new vertex v_e which becomes adjacent to all neighbors of x, y (apart from y and x). Given two graphs H and G we say that H is a *minor* of G, denoted by $H \leq G$, if H can be obtained by some subgraph of G after contracting edges.

Given a set $\mathcal{H}$ of graphs, we write $\mathcal{H} \leq G$ to denote that $\exists H \in \mathcal{H} : H \leq G$ and we defined $\mathbf{excl}(\mathcal{H}) = \{G \mid \mathcal{H} \not\leq G\}$. If $H \not\leq G$, then we say that G *is H-minor free*, or, in short, *H-free*. Also, given a graph G and a set of graphs $\mathcal{H}$ we say that G is $\mathcal{H}$-free if it is H-free, for each $H \in \mathcal{H}$. Given a graph class $\mathcal{G}$ we say that $\mathcal{G}$ is *minor-closed* if $\forall H \ H \leq G \wedge G \in \mathcal{G} \Rightarrow H \in \mathcal{G}$. We also define $\mathbf{obs}(\mathcal{G})$ as the set of all minor-minimal graphs that do not belong in $\mathcal{G}$ and we call $\mathbf{obs}(\mathcal{G})$ the *obstruction set* of the class $\mathcal{G}$.

If $e = xy$ is an edge of a graph G then the operation of replacing e by a path of length 2, i.e two edges $\{x, v_e\}, \{v_e, y\}$, where v_e is a new vertex, is called *subdivision* of e. A graph G is called a *subdivision* of a graph H if G can be obtained from H by repeatedly subdividing edges, i.e. by replacing some edges of H with new paths between its endpoints, so that the intersection of any two such paths is either empty or a vertex of H. The original vertices of H are called *branch vertices*, while the new vertices are called *subdividing vertices*. If a graph G contains a subdivision of H as

a subgraph, then H is a *topological minor* of G. It is easy to see that if H is a topological minor of G then it is also a minor of G.

Let G be a subdivision of some W_r wheel. In keeping with the notation previously introduced for wheels, we define the *spokes* of G to be the paths of G produced by the subdivision of the spokes of W_r and similarly we define the *circumference* of G to be the cycle of G produced by the subdivision of the circumference of W_r.

The following is an easy consequence of Dirac's Theorem [16], stating that if $\delta(G) \geq 3$, then G contains K_4 as a minor.

Proposition 2.2. *Let G be a biconnected K_4-free graph. Then there exist at least two vertices of degree 2.*

Triconnected components. Let G be a graph and $S \subseteq V(G)$ and let $V_1, \ldots, V_q$ be the vertex sets of the connected components of $G \setminus S$. We define $\mathcal{C}(G, S) = \{G_1, \ldots, G_q\}$ where, for $i \in [q]$, G_i is the graph obtained from $G[V_i \cup S]$ if we add all edges between vertices in S. We call the members of the set $\mathcal{C}(G, S)$ *augmented connected components*. Given a vertex $x \in V(G)$ we define $\mathcal{C}(G, x) = \mathcal{C}(G, \{x\})$.

Given a graph G, the set $\mathcal{Q}(G)$ of its *triconnected components* is recursively defined as follows:

- If G is triconnected or a clique of size ≤ 3, then $\mathcal{Q}(G) = \{G\}$.
- If G contains a separator S where $|S| \leq 2$, then $\mathcal{Q}(G) = \bigcup_{H \in \mathcal{C}(G,S)} \mathcal{Q}(H)$.

Notice that all graphs in $\mathcal{Q}(G)$ are either cliques on at most 3 vertices or triconnected graphs (graphs without any separator of less than 3 vertices). The study of triconnected components of plane graphs dates back to the work of Saunders Mac Lane in [30] (see also [41]).

Observation 2.3. Let G be a graph. All graphs in $\mathcal{Q}(G)$ are topological minors of G.

Let G be a graph and $v \in V(G)$ where $\mathbf{deg}_G(v) \geq 4$. Let also $\mathcal{P}_v = \{A, B\}$ be a partition of $N_G(v)$ such that $|A|, |B| \geq 2$. We define the *$\mathcal{P}_v$-split of G* to be the graph G' obtained by adding, in the graph $G \setminus v$, two new adjacent vertices v_A and v_B and making v_A adjacent to the vertices of A and v_B adjacent to the vertices of B. If G' can be obtained by some $\mathcal{P}_v$-split of G, we say that G' is a *splitting* of G.

Observation 2.4. If G' is a splitting of G then G is a minor of G'.

Proposition 2.5 (Tutte [40])**.** *A graph G is triconnected if and only if there is a sequence of graphs $G_0, \ldots, G_q$ such that G_0 is isomorphic to W_r for some $r \geq 3$, $G_q = G$, and for $i \in [q]$, G_i is a splitting of G_{i-1} or $\exists e \in E(G_i) : G_{i-1} = G_i \setminus e$.*

The next proposition is a direct consequence of Observation 2.3 and Proposition 2.5.

Proposition 2.6. *Let G be a graph. $K_4 \not\leq G$ if and only if none of the graphs in $\mathcal{Q}(G)$ is triconnected.*

Lemma 2.7. *If G is a triconnected graph that is not isomorphic to W_r, for some $r \geq 3$, then $\mathcal{O}^3 \leq G$.*

Proof. Let G be a triconnected graph not isomorphic to a wheel. By Proposition 2.5, there exists a sequence of graphs

$$W_r = G_0, G_1, \ldots, G_q = G$$

for some $r \geq 3$, such that for every $i \in [q]$, G_i is a splitting of G_{i-1} or there exists an edge $e \in E(G_i)$ such that $G_{i-1} = G_i \setminus e$. Observe that $r \geq 4$, since if $r = 3$ then $q = 0$ due to the fact that none of the vertices of W_3 can be split and all of them are adjacent to one another. Also, since $G \ncong W_r$ we have that $q \geq 1$.

Let z be the central vertex of W_r and $C_r = W_r \setminus z$. We examine how the graph G_1 may occur from W_r. For that, we distinguish the following two cases:

Case 1: There exists an edge $e \in E(G_1)$ such that $W_r = G_1 \setminus e$. Let $e = uv$ for some $u, v \in V(G_1) = V(W_r)$. Since every vertex in $V(C_r)$ is in the neighborhood of z in W_r then $u \neq z$ and $v \neq z$. Now, since u, v are not adjacent vertices in W_r there exist an internal vertex in each of the two (u, v)-paths of the graph C_r, say x, y, respectively (see Figure 5). Therefore, by contracting each of the $(x, u), (x, v), (y, v), (y, u)$-paths of C_r to an edge we get O_3^3 as a minor of G_1. Now, by Observation 2.4, G_1 is a minor of G and therefore $O_3^3 \leq G$, a contradiction.

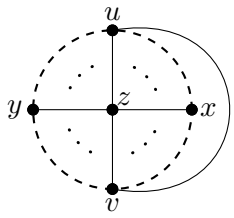


Figure 5: The structure of the graph G in Case 1.

Case 2: G_1 is a splitting of W_r. Observe that G_1 is a splitting of W_r obtained by a $\mathcal{P}_z$-split. So, let $\mathcal{P}_z = \{A, B\}$ and v_A, v_B the new adjacent vertices of G_1, where $N_{G_1}(v_A) = A$ and $N_{G_1}(v_B) = B$. We have that $|A|, |B| \geq 2$ and so there exist $x_1, y_1 \in A$ and $x_2, y_2 \in B$. We now distinguish the following subcases:

Subcase 2.1: One of the two (x_1, y_1)-paths in C_r contains both of x_2, y_2 (see leftmost figure of Figure 6). This implies that $O_1^3 \leq G_1$ and, as in case 1, it follows that $O_1^3 \leq G$, a contradiction.

Subcase 2.2: Each one of the two (x_1, y_1)-paths in C_r contains exactly one of x_2, y_2 (see rightmost figure of Figure 6). This implies that $O_2^3 \leq G_1$ and, as in case 1, it follows that $O_2^3 \leq G$, a contradiction.

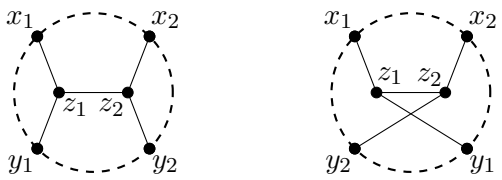


Figure 6: The structure of the graph G in the two Subcases of Case 2.

Since we have exhausted all possible cases for G_1 we conclude that $G \in \mathcal{O}^3$, a contradiction. □

Disconnected obstructions. We need the following result by Dinneen (see [11]).

Proposition 2.8. *Let $\mathcal{H}$ be a set of connected graphs. For every $k \in \mathbb{N}$, if $H \in \mathcal{H}^{(k)}$ and $H_1, \ldots, H_r$ are the connected components of H, then there is a sequence $k_1, \ldots, k_r$ such that $\sum_{i\in[r]} k_i = k + 1$ and $H_i \in \mathcal{H}^{(k_i-1)}, i \in [r]$.*

Lemma 2.9. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then G is connected.*

Proof. As both graphs in $\mathbf{obs}(\mathcal{P}) = \{$  $\}$ are connected, Proposition 2.8 applies for $\mathcal{H} = \mathbf{obs}(\mathcal{P})$ and $k = 1$. This means that that $\mathcal{O}^{(0)}$ contains all disconnected graphs in $\mathbf{obs}(\mathcal{P}^{(1)})$. Therefore every $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ should be a connected graph. □

3 Auxiliary lemmata

By Lemma 2.9, we know that a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ should be connected. In this section we prove a series of lemmata that further restrict the structure of the graphs in $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$.

3.1 General properties of the obstructions

Given a graph G and a vertex $v \in V(G)$ we say that v is *simplicial* if $G[N_G(v)]$ is isomorphic to K_r for $r = \mathbf{deg}_G(v)$. We say that $e \in E(G)$ is a *bridge* if G has less connected components than $G \setminus e$. A graph which does not contain any bridge is called *bridgeless*.

Given a graph class $\mathcal{G}$, a graph G, and a vertex x, where $G \setminus x \in \mathcal{G}$, then we say that x is a *$\mathcal{G}$-apex* of G.

Lemma 3.1. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then*

1. *$\delta(G) \geq 2$,*
2. *G is bridgeless and*
3. *all its vertices of degree 2 are simplicial.*

Proof. (1) Consider a vertex $u \in V(G)$ with $\mathbf{deg}_G(u) < 2$. If $G \setminus u \in \mathcal{P}^{(1)}$, then also $G \in \mathcal{P}^{(1)}$, since u does not participate in a cycle, a contradiction.

(2) Consider an edge $e = xy$ that is a bridge of G. By Lemma 2.9, G is connected. Since e is a bridge, then $G \setminus e$ contains two connected components H_1, H_2, such that $x \in V(H_1)$ and $y \in V(H_2)$. Observe that by $\mathcal{O}^0$-freeness of G, one of H_1, H_2, say H_1, contains at most one cycle and therefore, due to (1), H_1 is isomorphic to a cycle.

Consider the graph $G' = G/e$ and let v_e be the vertex formed by contracting e. We denote H_1', H_2' the graphs obtained from H_1, H_2 by replacing the vertices x, y with v_e, respectively. Observe that H_1' is also isomorphic to a cycle. By minor-minimality of G, it holds that $G' \in \mathcal{P}^{(1)}$ and therefore there exists some $u \in V(G')$ that is a $\mathcal{P}$-apex of G'. So, if $u \in V(H_1')$ then v_e is also a $\mathcal{P}$-apex of G'. Therefore we consider the case that $u \in V(H_2')$. If $u = v_e$, then every connected component of $H_2' \setminus v_e$ contains at most one cycle. Since, $H_2' \setminus v_e = H_2 \setminus y$ then also every connected component of $G \setminus y$ contains at most one cycle, a contradiction. If $u \neq v_e$, then consider the augmented connected component $Q' \in \mathcal{C}(G', u)$ which contains v_e. Also, let Q be the augmented connected component of $\mathcal{C}(G, u)$ that contains e. Observe that, since Q' contains at most one cycle, the same holds for Q. Hence, $G \setminus u \in \mathcal{P}$, a contradiction.

(3) Suppose, to the contrary, that there exists a non-simplicial vertex $v \in V(G)$ of degree 2, and let $e \in E(G)$ be an edge incident to v, i.e. $e = uv$ for some $u \in V(G)$. By minor-minimality of G, we have that $G' := G/e \in \mathcal{P}^{(1)}$. Let x be an $\mathcal{P}$-apex vertex of G' and v_e the vertex formed by contracting e. Observe that, every cycle in G that contains v also contains u and so if $x = v_e$ then u is an $\mathcal{P}$-apex vertex of G, a contradiction. Therefore, $x \neq v_e$ and so $x \in V(G)$. Since v is a non-simplicial vertex, the contraction of e can only shorten cycles and not destroy them. Hence, x is an $\mathcal{P}$-apex vertex of G, a contradiction. □

For a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$, observe that, due to Lemma 3.1, all of its connected components and blocks contain a cycle. Moreover, for such G, all graphs in $\mathcal{Q}(G)$ are either triconnected or isomorphic to K_3.

3.2 Properties of obstructions containing a K_4

We now prove some Lemmata which will be useful in the main section of the proof.

Lemma 3.2. *If G is a biconnected graph such that $\mathcal{O} \not\leq G$, then there exists at most one triconnected graph in $\mathcal{Q}(G)$.*

Proof. Suppose, to the contrary, that there are at least two triconnected graphs in $\mathcal{Q}(G)$ and let H_1, H_2 be two of them. Due to the recursive definition of $\mathcal{Q}(G)$ and by Observation 2.3, there exists a separator S such that H_1, H_2 are topological minors of some $G_1, G_2 \in \mathcal{C}(G, S)$, respectively. By the biconnectivity of G we have that S is a 2-separator of G and let $S = \{x, y\}$. Since H_1, H_2 are triconnected graphs and topological minors of G then, by Lemma 2.7, we have that each H_i, $i \in [2]$ is isomorphic to a wheel. Thus, K_4 is a topological minor of both H_1 and H_2. Let Q_1, Q_2 be the subdivisions of K_4 in G_1, G_2, respectively.

Claim: For each $G_i, i \in \{1, 2\}$,  is a topological minor of G_i such that x, y are the branch vertices of of degree 2.

Proof of Claim: Let $i \in \{1, 2\}$. By Menger's theorem, there exist two disjoint paths from the separator S to Q_i. Let P_1, P_2 be the shortest such paths (as in Figure 7) and for each of P_1, P_2 identify its endpoints.

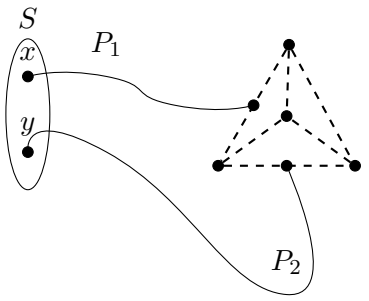


Figure 7: The paths P_1, P_2 from S to Q_i.

Therefore, there exists a subdivision A of K_4 in G_i such that $x, y \in V(A)$. Let $e = u_x v_x, e' = u_y v_y$ be the subdivided edges of K_4 that contain x, y, respectively.

Observe that if $e = e'$ then the desired result holds, while if $e \neq e'$, then there exists some $a \in \{u_x, v_x\}$ and $b \in \{u_y, v_y\}$ such that $a \neq b$. Since ab is a subdivided edge of K_4, then by identifying x with a and y with b, we get the desired result. Claim follows.

Therefore, by applying the above Claim for both G_1, G_2 we get O_1^2 as a minor of G, a contradiction. □

The results of Lemma 2.7 and Lemma 3.2, together with Observation 2.3 and Proposition 2.6 imply the following corollary:

Corollary 3.3. *Let G be a biconnected graph such that $\mathcal{O} \not\leq G$ and $K_4 \leq G$. Then there exists a unique triconnected graph H in $\mathcal{Q}(G)$ such that:*

- *H is isomorphic to an r-wheel for some $r \geq 3$ and*
- *H is a topological minor of G.*

Let G be a biconnected graph such that $\mathcal{O} \not\leq G$ and $K_4 \leq G$ and let K be a subdivision of the (unique) r-wheel $H \in \mathcal{Q}(G)$, as in Corollary 3.3. We call the pair (H, K) an *r-wheel-subdivision pair of G*. Notice that there may be many r-wheel-subdivision pairs in G, as there might be many possible choices for K (but there is only one choice for H).

Lemma 3.4. *Let G be a biconnected graph such that $K_4 \leq G$ and $\mathcal{O} \not\leq G$. Let (H, K) be an r-wheel-subdivision pair of G. Then for every (x, y)-path which intersects K only in its endpoints, there exists an edge $e \in E(H)$ such that x, y are both vertices of the subdivision of e in K.*

Proof. Recall that H is isomorphic to an r-wheel for some $r \geq 3$, and K is a subdivision of H. Consider an (x, y)-path which intersects K only in its endpoints. Suppose, to the contrary, that x, y belong to subdivisions of different edges of H. We distinguish the following cases:

Case 1: One of x, y, say x, is a branch vertex on the circumference of K.

First, we observe the following:

Observation 1: $r \neq 3$. Indeed, if $H \cong W_3$, then since y belongs to the subdivision of an edge of H not incident to x, a subdivision of a bigger wheel would be formed with x as its central vertex (see Figure 8), a contradiction to the definition of the triconnected components.

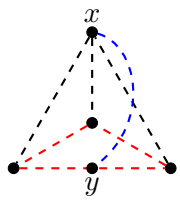


Figure 8: The (x, y)-path (depicted in blue) where y belongs to the subdivision of some edge of H not incident to x (depicted in red) in the proof of Observation 1.

Suppose then that $r \geq 4$. Let x_1, x_2 be the vertices adjacent to x on the circumference of H. We distinguish the following subcases:

Subcase 1.1: y belongs to the subdivision of some spoke e of H. Then, e is not incident to x. If y is an internal vertex of the subdivision of a spoke incident to either x_1 or x_2 then $O_1^3 \leq G$ (see leftmost figure of Figure 9), while if the spoke is not incident to x_1 or x_2 then $O_3^3 \leq G$ (see central figure of Figure 9), a contradiction in both cases.

Subcase 1.2: y belongs to some subdivided edge e of the circumference of K. Then, e is different from the subdivided edges corresponding to xx_1, xx_2. Hence, $O_3^3 \leq G$ (see rightmost figure of Figure 9), a contradiction.

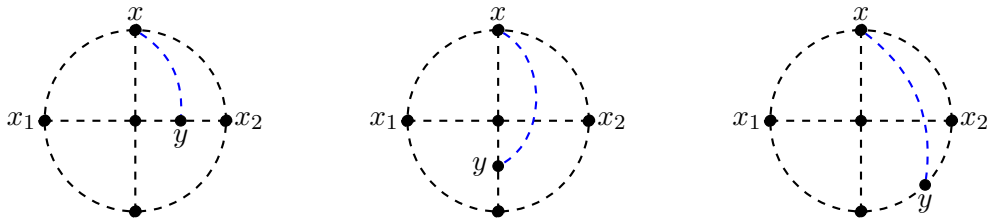


Figure 9: Possible configurations of the (x, y)-path (depicted in blue) in the proof of Subcases of Case 1.

Case 2: One of x, y, say x, is a subdividing vertex on the circumference of K.

Since we have examined the case that one of x, y is a branch vertex on the circumference of K, suppose that y is not such.

Let $e = uv$ be the edge of H whose the corresponding subdivision in K contains x.

Observation 2: y is not the central vertex of H. This is because, if otherwise, a subdivision of a bigger wheel would be formed in G (see Figure 10), which a contradiction to the definition of the triconnected components.

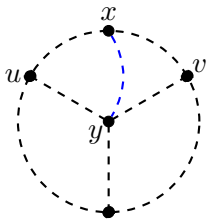


Figure 10: The (x, y)-path (depicted in blue) in the proof of Observation 2, where y is the central vertex of H.

Subcase 2.1: y is an internal vertex of some subdivided edge e' of a spoke of K. If e' is incident to either u or v, then $O_1^3 \leq G$ (see leftmost figure of Figure 11), while if e' is not incident to either u or v then $O_2^3 \leq G$ (see central figure of Figure 11)), a contradiction in both cases.

Subcase 2.2: y is an internal vertex of some subdivided edge of the circumference of K different from e. Then $O_1^3 \leq G$ (see rightmost figure of Figure 11), a contradiction.

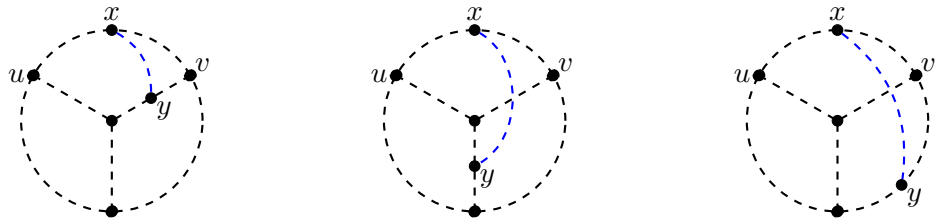


Figure 11: Possible configurations of the (x, y)-path (depicted in blue) in the proof of Subcases of Case 2.

Case 3: Both of x, y are internal vertices of the subdivisions of some spokes e, e' of W_r, respectively.

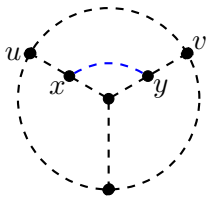


Figure 12: The (x, y)-path (depicted in blue) in the proof of Case 3.

In this case, e, e' are distinct and so $O_1^3 \leq G$ (see Figure 12), a contradiction. □

We can now define the notion of a *flap*. Let G be a biconnected graph such that $\mathcal{O} \not\leq G$ and $K_4 \leq G$. Let also (H, K) be an r-wheel-subdivision pair of G. A (x, y)-*flap* of (H, K) corresponding to some separator $S = \{x, y\} \subseteq V(K)$ of G is the graph $\bigcup\{C \in \mathcal{C}(G, S) : K_4 \not\leq C \wedge C \text{ is biconnected}\}$. Given an (x, y)-flap F of (H, K) and a vertex $v \in V(F)$, we say that F is v-*oriented* if every cycle of F contains v.

Regarding the arguments in the remaining part of Subsection 3.2, consider a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ such that $K_4 \leq G$. Observe that K_4 is a minor of a block B of G and therefore we can consider an r-wheel-subdivision pair of B.

Lemma 3.5. *Let $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ such that $K_4 \leq G$ and (H, K) be an r-wheel-subdivision pair of a block B of G. If F is an (x, y)-flap of (H, K), then it holds that:*

1. *F is biconnected,*
2. *$G[V(F)]$ contains a cycle, and*
3. *there exists an edge $e \in E(H)$ such that x, y are both vertices of the subdivision of e in K.*

Proof. Observe that (1) and (2) are direct consequences of the definition of the (x, y)-flap and Lemma 3.1. To prove (3) suppose, towards a contradiction, that there exists an (x, y)-flap such that x, y belong to subdivisions of different edges of K. Then, from the definition of the (x, y)-flap, there exists a biconnected graph $C \in \mathcal{C}(G, S)$ such that $K_4 \not\leq C$ and hence it contains an (x, y)-path that intersects K only in its endpoints. We arrive at a contradiction to the result of Lemma 3.4. □

We conclude this subsection by proving the next result concerning flaps:

Lemma 3.6. *Let G be a biconnected graph such that $\mathcal{O} \not\leq G$ and $K_4 \leq G$ and let (H, K) be an r-wheel-subdivision pair of G. Then every (x, y)-flap F of (H, K) is either x-oriented or y-oriented.*

Proof. Consider an (x, y)-flap F of (H, K) for which the contrary holds. We distinguish the following cases:

Case 1: There exists a cycle C in F disjoint to both x, y. Then, since F is biconnected, there exist two disjoint paths P_1, P_2 connecting the cycle C with x, y, respectively. Hence, by contracting all the edges of P_1, P_2 we form O_9^2 as a minor of G, a contradiction (see Figure 13).

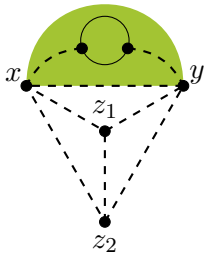


Figure 13: An example of an (x, y)-flap that contains a cycle disjoint to x, y in the the proof of Case 1.

Case 2: There exists a cycle C of F that contains x but not y and a cycle C' that contains y but not x. Then, if C, C' are disjoint, $O_1^1 \leq G$, if the share only one vertex, $O_4^2 \leq G$, and if they share more than one vertex, $O_1^2 \leq G$, a contradiction in all cases (see figure Figure 14).

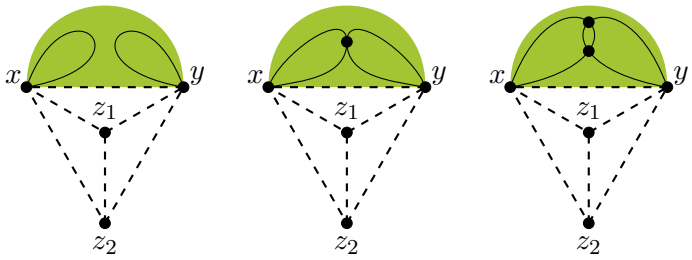


Figure 14: The ways C, C' may intersect in the proof of Case 2.

Since we arrived at a contradiction in both cases, Lemma follows. □

Lemma 3.7. *Let $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ such that $K_4 \leq G$ and let (H, K) be an r-wheel-subdivision pair of a block B of G. Then if $r \geq 4$, the center of K is a $\mathcal{P}$-apex vertex of B.*

Proof. By Lemma 3.2, H is the unique triconnected graph in $\mathcal{Q}(B)$. By Lemma 3.5, for every (x, y)-flap of (H, K) there exists an edge $e \in E(H)$ such that x, y are both vertices of the subdivision of e in K. Keep in mind that, again by Lemma 3.5, every flap F of (H, K) is biconnected and $G[V(F)]$ contains a cycle.

Observation: For every (x, y)-flap it holds that one of x, y is the centre of K. Indeed, if there exists an (x, y)-flap corresponding to some $e \in E(H)$ such that neither of x, y is the center of K, then if e is an edge of the circumference of H, $O_2^2 \leq G$ (see left figure of Figure 15), while if e is a spoke of H, $O_4^1 \leq G$ (see right figure of Figure 15), a contradiction in both cases.

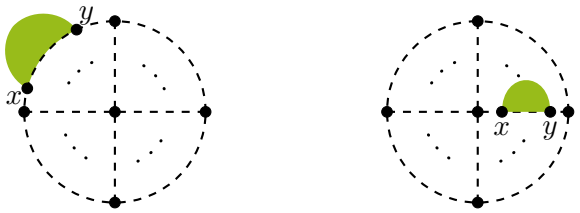


Figure 15: Possible configurations of an (x, y)-flap (depicted in green) in the proof of Observation.

Let z be the centre of K. According to Observation, every flap of (H, K) is a (z, y)-flap. Furthermore, notice that every (z, y)-flap F is z-oriented. Indeed, if otherwise, then Lemma 3.6 implies that there would exist a cycle in F containing y but not z and hence $O_4^1 \leq G$, which is a contradiction.

Therefore, taking into account that B is a block of G, every cycle in B, except for the circumference of K, contains z and so z is a $\mathcal{P}$-apex vertex of B. □

3.3 Properties of obstructions containing a $K_{2,3}$

The purpose of this section is to prove Lemma 3.11 that gives us some information on the structure of a connected graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ that is K_4-free and contains $K_{2,3}$ as a minor.

Let S be a 2-separator of G and B be a block of some $H \in \mathcal{C}(G, S)$. We say that B is an *S-block* of G if $S \subseteq V(B)$. We call S a *b-rich separator* if at least three graphs in $\mathcal{C}(G, S)$ contain S-blocks. We start with an easy observation.

Observation 3.8. Let G be a K_4-free graph such that $K_{2,3} \leq G$. Then G contains a b-rich separator.

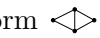
Using Lemma 3.1, we easily derive the next observation that will be frequently used in the course of the remaining proofs of this section.

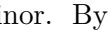
Observation 3.9. Let $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ be a graph, S be a b-rich separator of G, and $H \in \mathcal{C}(G, S)$ that contains an S-block. Then the graph $G[V(H)]$ contains a cycle.

Lemma 3.10. *Let $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ be a connected K_4-free graph such that $K_{2,3} \leq G$. Then G contains a unique b-rich separator.*

Proof. We first prove the following claim:

Claim 1: If $S_1, S_2, \ldots, S_n$ are the b-rich separators of G, then there exists some $x \in V(G)$ such that $\bigcap_{i=1}^{k} S_i = \{x\}$.

Proof of Claim 1: We first prove that every two b-rich separators of G have a non-empty intersection. Suppose, towards a contradiction, that there exist two b-rich separators S_1, S_2 of G such that $S_1 \cap S_2 = \emptyset$. Let $H \in \mathcal{C}(G, S_1)$ be the (unique) augmented component such that $S_2 \subseteq V(H)$. Observe that there exist at least two augmented connected components that contain S-blocks in $\mathcal{C}(G, S_1) \setminus \{H\}$ which by Observation 3.9 contain a cycle and together form  as a minor. By applying the same arguments symmetrically, there exist at least two augmented components in $\mathcal{C}(G, S_2)$ that do not contain S_1, which together also form  as a minor. Then, notice that $O_1^0 \leq G$ (see rightmost figure of Figure 16), a contradiction.

Now, suppose that there exist three b-rich separators S_1, S_2, S_3 with $S_1 \cap S_2 \cap S_3 = \emptyset$. Notice that since $S_2 \cap S_3 \neq \emptyset$, there exists a unique augmented connected component of $\mathcal{C}(G, S_1)$ that contains both S_2, S_3, while there also exist at least two other augmented connected components that contain S-blocks which together form  as a minor. By applying the same argument to $\mathcal{C}(G, S_2), \mathcal{C}(G, S_3)$, we have that for each said separator we can form  as a minor and therefore, $O_{11}^2 \leq G$ (see leftmost figure of Figure 16), a contradiction.

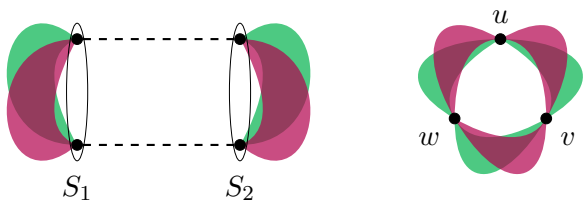


Figure 16: The proof of Claim 1.

Therefore, if $S_1, \ldots, S_k$ are the rich 2-separators of G, then $\bigcap_{i=1}^{k} S_i = \{x\}$ for some $x \in V(G)$. Claim 1 follows.

According to Claim 1, we can consider $S_1 = \{x, u_1\}$, $S_2 = \{x, u_2\}$ as two b-rich separators of G where $u_1 \neq u_2$.

Let

$$H_1 = \bigcup\{H \in \mathcal{C}(G, S_1) : u_2 \notin V(H)\} \quad \text{and} \quad H_2 = \bigcup\{H \in \mathcal{C}(G, S_2) : u_1 \notin V(H)\}.$$

Consider $H = G \setminus ((V(H_1) \cup V(H_2)) \setminus \{x, u_1, u_2\})$.

Since for each $i \in [2]$, $H \setminus x \setminus u_i$ is a connected subgraph of $G \setminus H_i$ which is also connected, we easily derive the following:

Observation 1: If $y \in V(H \setminus x)$, then there exists a (y, u_1)-path in $H \setminus x \setminus u_2$ and a (y, u_2)-path in $H \setminus x \setminus u_1$. In particular, u_1, u_2 are not cut-vertices of $H \setminus x$.

Since $G \setminus x \notin \mathcal{P}^{(1)}$, then it contains two cycles C_1, C_2, which are connected in $G \setminus x$. We will now argue that the following hold:

Claim 2: C_1, C_2 are in $H \setminus x$.

Proof of Claim 2: Keep in mind that $\diamond$ is a minor of H_i, $i \in [2]$. If C_1, C_2 are both in some H_i, $i \in [2]$, say H_1, then $\mathbf{obs}(\mathcal{P}) \leq H_1 \setminus x$ thus implying that $\{O_1^0, O_3^0\} \leq G$, a contradiction. Also, if each C_i belongs to different H_j, $i, j \in [2]$, say $C_1 \subseteq H_1$ and $C_2 \subseteq H_2$, then since C_1, C_2 are connected in $G \setminus x$, we have $O_{12}^1 \leq G$ (see left figure of Figure 17), a contradiction.

Therefore, at least one of C_1, C_2 is in $H \setminus x$. Suppose, without loss of generality, that $C_1 \subseteq H_1 \setminus x$. Then $C_2 \subseteq H \setminus x$. Observation 1 implies that C_2 is connected with u_1 through a path disjoint from u_2 and thus, if $u_2 \notin V(C_2)$, $O_3^0 \leq G$ (see central figure of Figure 17), while if $u_2 \in V(C_2)$, then by contracting all edges in the said path, we get $O_6^1 \leq G$ (see right figure of Figure 17), a contradiction in both cases . Claim 2 follows.

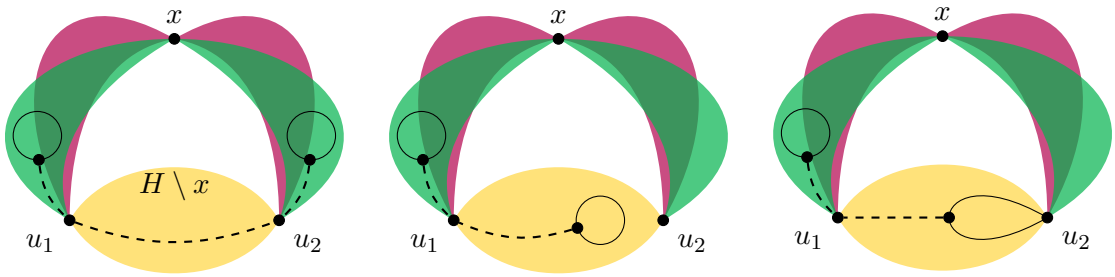


Figure 17: The cycles C_1 and C_2 in the proof of Claim 2.

Observation 2: $\{u_1, u_2\} \subseteq V(C_1 \cup C_2)$. Indeed, if there exists an $i \in [2]$, such that $u_i \notin V(C_1 \cup C_2)$, then by Claim 2, C_1, C_2 are in $H \setminus x$ and Observation 1 implies that there exists a path connecting C_1, C_2 avoiding u_i. Hence, $\{O_1^0, O_3^0\} \leq G$, a contradiction.

Claim 3: There exists a block H' of $H \setminus x$ that contains both C_1, C_2 and is outerplanar.

Proof of Claim 3: We start with the following observation:

Observation 3: There exists some block H' of $H \setminus x$ that contains both u_1, u_2. Indeed, suppose towards a contradiction that there exists some cut-vertex $v \in V(H \setminus x)$ separating u_1, u_2 in $H \setminus x$. Then, there exist $D_1, D_2 \in \mathcal{C}(H \setminus x, v)$ such that $D_1 \neq D_2, u_1 \in V(D_1)$ and $u_2 \in V(D_2)$. By Observation 2, we can assume that $C_1 \subseteq D_1$ and $C_2 \subseteq D_2$, which in turn implies that $O_3^2 \leq G$ (see Figure 18), a contradiction.

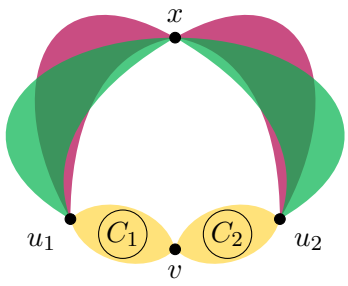


Figure 18: The cycles C_1, C_2 in the proof of Observation 3.

According to Observation 3, let H' be a block of $H \setminus x$ that contains both u_1, u_2. Suppose that some C_i, $i \in [2]$, say C_1, is not in H'. Then, there exists some cut-vertex u of $H \setminus x$ such that C_1 is in some $H'' \in \mathcal{C}(H \setminus x, u) \setminus \{H'\}$. Observation 1 implies that $u \neq u_1, u_2$. Therefore, $O_3^1 \leq G$, a contradiction. Thus, H' contains C_1, C_2.

To prove outerplanarity, we observe the following:

Observation 4: If $K_{2,3} \leq H'$ then H' contains a b-rich separator of G. Indeed, if $K_{2,3} \leq H'$ then by Observation 3.8 H' contains a b-rich separator S. If S is not a b-rich separator of G, then there exist a path connecting some $A, B \in \mathcal{C}(H', S)$ in G thus implying that $K_4 \leq G$, a contradiction.

Observation 4 implies that if $K_{2,3} \leq H'$ then there exists a b-rich separator in G that does not contain x, a contradiction to Claim 1. Therefore, $K_4 \not\leq H'$ (since $K_4 \not\leq G$) and $K_{2,3} \not\leq H'$, i.e. H' is outerplanar. Claim 3 follows.

We now return to the proof of the lemma. According to Claim 3, let H' be a block of $H \setminus x$ that contains both C_1, C_2 and is outerplanar. Also, let C be the Hamiltonian cycle of H' (which exists due to biconnectivity and outerplanarity of H'). Since $C_1, C_2 \subseteq H'$, then there exists some chord e of C. We distinguish the following cases:

Case 1: $u_1u_2 \notin E(C)$. Let P_1, P_2 be the connected components of $C \setminus u_1 \setminus u_2$. Notice that K_4-freeness of G implies that e is incident to vertices of some P_i, $i \in [2]$, say P_1. But then, $O_{11}^2 \leq G$, a contradiction.

Case 2: $u_1u_2 \in E(C)$. Then, the existence of e implies that $O_{10}^2 \leq G$, a contradiction.

This completes the proof of the Lemma. □

Let G be a biconnected graph and S a rich-separator of G, such that every $H \in \mathcal{C}(G, S)$ is outerplanar. For an $H \in \mathcal{C}(G, S)$, we denote C_H the Hamiltonian cycle of H, which exists due

to biconnectivity and outerplanarity of H. Given an $x \in V(G)$, we say that an edge $e \in E(G)$ is an *x-chord* of G, if there exists an $H \in \mathcal{C}(G,S)$ such that e is a chord of C_H incident to x. Also, given two vertices $x, y \in V(G)$, we say that an edge e is an *(x,y)-disjoint chord* if there exists an $H \in \mathcal{C}(G,S)$ such that e is a chord of C_H disjoint to x, y.

Lemma 3.11. *Let G be a K_4-free biconnected graph such that $\mathcal{O} \not\leq G$ and $K_{2,3} \leq G$. Also, let $S = \{x, y\}$ be a b-rich separator of G. Then this b-rich separator is unique and if there exist at least two augmented connected components in $\mathcal{C}(G,S)$ not isomorphic to a cycle, then one of the following holds:*

- *There exists a unique (x,y)-disjoint chord and there do not exist both x-chords and y-chords, or*
- *There do not exist (x,y)-disjoint chords and there exists at most one x-chord or at most one y-chord.*

Proof. Recall that, by Lemma 3.10, S is the unique b-rich separator of G. Suppose that there exist at least two augmented connected components in $\mathcal{C}(G,S)$ not isomorphic to a cycle. Keep in mind that if some $H \in \mathcal{C}(G,S)$ is not isomorphic to a cycle, then C_H contains some chord. Also, observe that K_4-freeness of G implies that $xy \in E(C_H)$.

Claim 1: There exists at most one (x,y)-disjoint chord in G.

Proof of Claim 1: Suppose to the contrary that there exist two (x,y)-disjoint chords, namely e, e'. If e, e' are chords of some $C_H,\ H \in \mathcal{C}(G,S)$, then the existence of some $H' \in \mathcal{C}(G,S)$, different than H that contains a chord implies that $\diamond \leq G \setminus (H \setminus S)$ and thus $\{O_1^0, O_3^0\} \leq G$, a contradiction. Therefore, e, e' are chords of some $C_H, C_{H'}$ (respectively) where $H, H' \in \mathcal{C}(G,S)$ different than H and hence $O_{15}^2 \leq G$ (see Figure 19), a contradiction. Claim 1 follows.

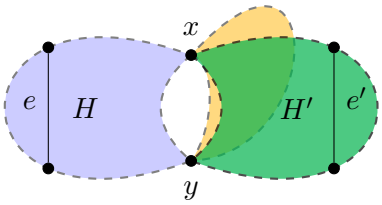


Figure 19: The chords e, e' in the second part of the proof of Claim 1.

We now distinguish the following cases depending on whether there exists an (x,y)-disjoint chord.

Case 1: There exists an (x,y)-disjoint chord.
Let e be an (x,y)-disjoint chord, which is a chord of some $C_H,\ H \in \mathcal{C}(G,S)$. Claim 1 implies that every other chord of some $C_{H'},\ H' \in \mathcal{C}(G,S)$ is either an x-chord or a y-chord.

Recall that there exists some $H' \in \mathcal{C}(G,S)$ different than H that is not isomorphic to a cycle. Therefore, $C_{H'}$ contains some chord e' that is either an x-chord or a y-chord, say x-chord. We prove the following claim:

Claim 2: Every edge of G that is a chord of some $C_{H''},\ H'' \in \mathcal{C}(G,S)$ different from e is an x-chord.

Proof of Claim 2: Suppose to the contrary that there exists an edge e'' of G that is a chord of some $C_{H''},\ H'' \in \mathcal{C}(G,S)$ different from e and is not an x-chord. Claim 1 implies that e'' is a y-chord.

Observe that $H'' \in \{H, H'\}$, because otherwise $O_8^1 \leq G$, a contradiction. Therefore, if $H'' = H$, then $\{O_{13}^2, O_{12}^2\} \leq G$ (see left and central figure of Figure 20), while if $H'' = H'$, $O_7^2 \leq G$ (see right figure of Figure 20), a contradiction in both cases. Claim 2 follows.

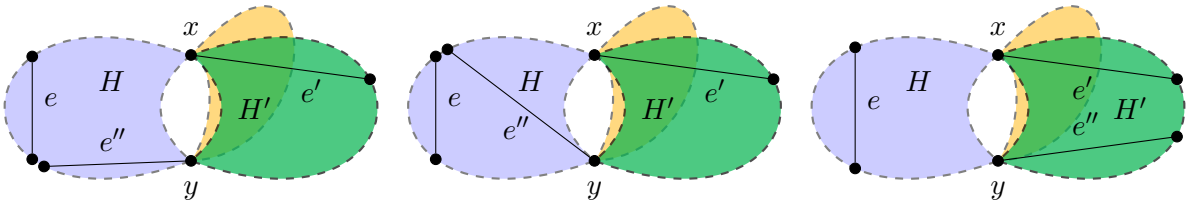


Figure 20: The possible configurations of the chords e, e', e'' in the proof of Claim 2

Case 2: There do not exist (x, y)-disjoint chords.

We will prove that there exists at most one x-chord and at most one y-chord.

Suppose, towards a contradiction, that there exist two x-chords, namely e_1^x and e_2^x, and two y-chords, namely e_1^y and e_2^y.

We say that a pair of edges (e, e'), where $e, e' \in \{e_1^x, e_2^x, e_1^y, e_2^y\}$ is *homologous* if there exists some $H \in \mathcal{C}(G, S)$ such that $e, e' \in E(H)$.

We now distinguish the following subcases:

Subcase 2.1: (e_1^x, e_2^x) is not homologous and (e_1^y, e_2^y) is not homologous. Then, $\{O_2^0, O_{12}^1, O_3^2\} \leq G$ depending whether there exist 0, 1 or 2 homologous pairs (e, e') , where $e \in \{e_1^x, e_2^x\}, e' \in \{e_1^y, e_2^y\}$. In any case we have a contradiction (see Figure 21).

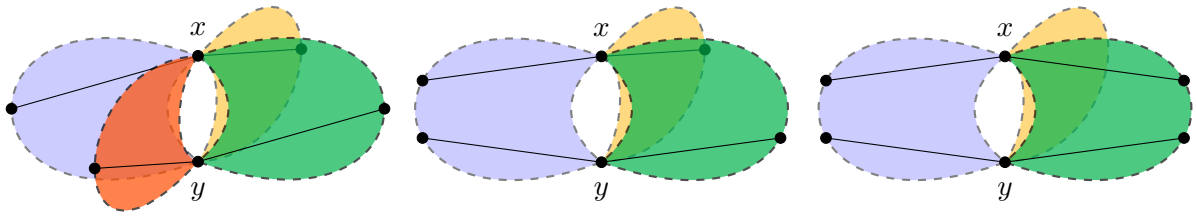


Figure 21: The possible configurations of the edges e_1^y, e_2^y, e_1^x, and e_2^x in the proof of Subcase 2.1.

Subcase 2.2: (e_1^x, e_2^x) is homologous and (e_1^y, e_2^y) is not homologous.

Let $H \in \mathcal{C}(G, S)$ such that $e_1^x, e_2^x \in E(H)$. Notice that since (e_1^y, e_2^y) is not homologous, at most one of e_1^y, e_2^y is in $E(H)$. If none of e_1^y, e_2^y is in $E(H)$, then $O_3^0 \leq G$, while if some of e_1^y, e_2^y is in $E(H)$, then $O_6^1 \leq G$, a contradiction (see Figure 22).

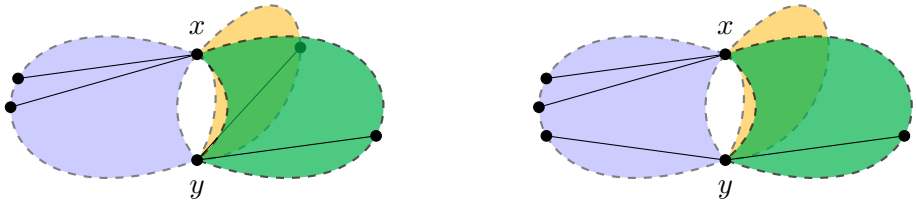


Figure 22: The possible configurations of the edges e_1^y, e_2^y, e_1^x, and e_2^x in the proof of Subcase 2.2.

Subcase 2.3: (e_1^x, e_2^x) is not homologous and (e_1^y, e_2^y) is homologous. This case is symmetric to the previous one.

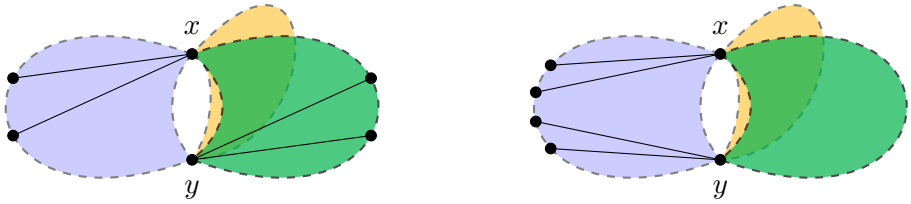


Figure 23: The possible configurations of the cycles e_1^y, e_2^y, e_1^x, and e_2^x in the proof of Subcase 2.4.

Subcase 2.4: (e_1^x, e_2^x) is homologous and (e_1^y, e_2^y) is homologous. Let $H \in \mathcal{C}(G, S)$ such that $e_1^x, e_2^x \in E(H)$ and $H' \in \mathcal{C}(G, S)$ such that $e_1^y, e_2^y \in E(H')$. If $H \neq H'$, then $O_1^0 \leq G$, while if $H = H'$, then $O_{11}^2 \leq G$. In both cases we have a contradiction (see Figure 23). □

4 Confining connectivity

In this section we further restrict the structure of a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$. The first step is to prove that G is biconnected (Lemma 4.7) and the second one is to prove that G is triconnected (Lemma 4.10).

4.1 Proving biconnectivity

In this section we prove that every graph in $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ is biconnected (Lemma 4.7). For this we prove a series of lemmata that gradually restrict the structure of such a graph.

We begin by making two observations. Since Lemma 3.1 implies that every block of a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ has a cycle then by the O_{10}^1-freeness of such a graph we derive the following:

Observation 4.1. If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then every block of G contains at most 2 cut-vertices.

Also, for a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ we have that $G \notin \mathcal{P}^{(1)}$ and this implies the following:

Observation 4.2. If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ is a connected graph, then for every cut-vertex $v \in V(G)$ there exists an $H \in \mathcal{C}(G, v)$ such that $\mathbf{obs}(\mathcal{P}) \leq H \setminus x$.

Lemma 4.3. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$, then G cannot have more than 1 cut-vertex.*

Proof. Recall that, from Lemma 2.9, G is connected. Suppose, towards a contradiction, that G has at least 2 cut-vertices. Then, there exists a block B containing 2 cut-vertices u_1, u_2. Let

$$\mathcal{C}_1 = \bigcup \{H \in \mathcal{C}(G, u_1) : u_2 \notin V(B)\} \quad \text{and} \quad \mathcal{C}_2 = \bigcup \{H \in \mathcal{C}(G, u_2) : u_1 \notin V(B)\}.$$

We now prove a series of claims:

Claim 1: Both $\mathcal{C}_1, \mathcal{C}_2$ are isomorphic to K_3.
Proof of Claim 1: Suppose, to the contrary, that one of $\mathcal{C}_1, \mathcal{C}_2$, say $\mathcal{C}_1$, contains 2 cycles, which is equivalent to $\mathbf{obs}(\mathcal{P}) \leq \mathcal{C}_1$, since $\mathcal{C}_1$ is connected. Let $H \in \mathcal{C}(G, u_1)$ be the component which contain u_2. We distinguish two cases:
Case 1: $\mathcal{C}_1 \setminus u_1 \in \mathcal{P}$. Then, by Observation 4.2, $\mathbf{obs}(\mathcal{P}) \leq H \setminus u_1$ and therefore, since $V(\mathcal{C}_1) \cap V(H \setminus u_1) = \emptyset$, we have that $\mathcal{O}^0 \leq G$, a contradiction.

Case 2: $\mathcal{C}_1 \setminus u_1 \notin \mathcal{P}$, or equivalently $\mathbf{obs}(\mathcal{P}) \leq \mathcal{C}_1 \setminus u_1$. Observe that, since H contains the cut-vertex u_2, there exist 2 blocks H_1, H_2 of G in H such that $V(H_1) \cap V(H_2) = \{u_2\}$. Then, since, by Lemma 3.1, each block of G contains a cycle, we have that $\bowtie \leq H$. Hence, $\{O_2^0, O_3^0\} \leq G$, a contradiction.

Therefore, both of $\mathcal{C}_1, \mathcal{C}_2$ contain at most one cycle and, since both are non-empty, by Lemma 3.1, Claim 1 follows.

Claim 2: Every cycle in B contains either u_1 or u_2.
Proof of Claim 2: Suppose, to the contrary, that there exists a cycle C containing none of u_1, u_2. By Menger's Theorem there exist two internally disjoint (u_1, u_2)-paths P_1, P_2. We distinguish the following cases:

Case 1: Both of P_1, P_2 intersect C.

Let z_1, z_2 be the vertices where P_1, P_2 meet C for the first time, respectively. Let, also, w_1, w_2 be the vertices that P_1, P_2 meet C for the last time, respectively. Since $V(P_1) \cap V(P_2) = \{u_1, u_2\}$ we have that $\{z_1, w_1\} \cap \{z_2, w_2\} = \emptyset$. If $z_1 \neq w_1$ or $z_2 \neq w_2$, say $z_1 \neq w_1$, then by contracting the edges in the (w_1, u_2)-subpath of P_1 we form O_1^1 as a minor of G, a contradiction (see left figure of Figure 24). Therefore, we have that $z_1 = w_1$ and $z_2 = w_2$, in which case we have again O_1^1 as a minor of G, a contradiction (see right figure of Figure 24).

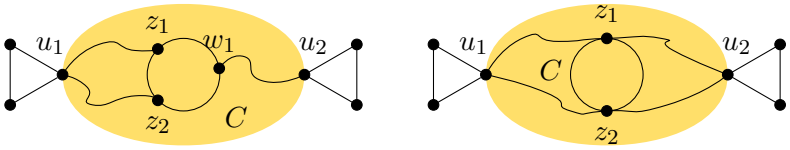


Figure 24: The paths P_1 and P_2 in Case 1 of Claim 2.

Case 2: Either P_1 or P_2 is disjoint to C.

Say, without loss of generality, that $V(P_1) \cap V(C) = \emptyset$. Let $v_{new} \notin V(G)$ and consider the graph B' obtained by adding v_{new} to B and making it adjacent to u_1, u_2. Observe that B' is also biconnected. Then, by Menger's theorem, there exist two paths from v_{new} to C intersecting only in v_{new}. Therefore, in B there exists a (x, u_1)-path Q_1 and a (y, u_2)-path Q_2 such that $x, y \in V(C)$ and $V(Q_1) \cap V(Q_2) = \emptyset$. Let z_1, z_2 be the vertices where Q_1, Q_2 meet P_1 for the first time, respectively (starting from x and y). Then, suppose that, without loss of generality, z_1 is closest to u_1 in P_1 than z_2. Hence, by contracting all the edges of the (z_1, u_1)-,(z_2, u_2)-subpaths of P_1 we form O_9^1 as a minor of G, a contradiction (see Figure 25). Claim 2 follows.

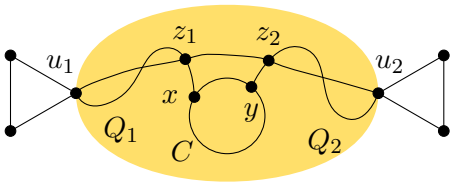


Figure 25: The paths P_1, Q_1, Q_2 in Case 2 of Claim 2.

We return to the proof of the Lemma. Notice that, by Observation 4.1, u_1, u_2 are the only

cut-vertices of G contained in B. Therefore, Claim 1 implies that $\mathcal{C}_1, B, \mathcal{C}_2$ are the only blocks of G.

Let $H_1 \in \mathcal{C}(G, u_2)$ such that $u_1 \in H_1$ and $H_2 \in \mathcal{C}(G, u_1)$ such that $u_2 \in H_2$. By Observation 4.2, we have that $\mathbf{obs}(\mathcal{P}) \leq H_1 \setminus u_2, H_2 \setminus u_1$, or equivalently $H_1 \setminus u_2, H_2 \setminus u_1 \notin \mathcal{P}$. Therefore, since $H_1 \setminus u_2, H_2 \setminus u_1$ are connected, Claim 2 implies that there exist two cycles C_1, C_2 in B such that $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$. If $V(C_1) \cap V(C_2) = \emptyset$ then $O_2^0 \leq G$ and if $|V(C_1) \cap V(C_2)| \geq 2$ then $O_1^1 \leq G$, a contradiction in both cases. Hence, $V(C_1) \cap V(C_2) = \{x\}$ for some $x \in V(B)$. As x is not a cut-vertex of B, there exists a (u_1, u_2)-path P in B such that $x \notin V(P)$.

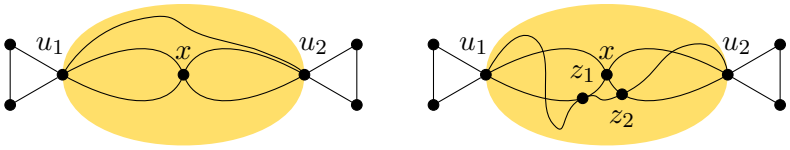


Figure 26: The cycles C_1, C_2 and the path P in the end of the proof of the Lemma.

If $V(P) \cap V(C_1 \cup C_2) = \{u_1, u_2\}$ then $O_{12}^1 \leq G$, a contradiction (see the leftmost figure of Figure 26). Therefore P intersects, without loss of generality, C_1 at a vertex different from u_1. Let $z_1 \in V(C_1)$ be the vertex that P meets C_1 for the last time. Also, let $z_2 \in V(C_2)$ be the vertex that the (z_1, u_2)-subpath of P meets C_2 for the first time. Then, the cycle C_1, the (z_1, z_2)-subpath of P, the (z_2, u_2)-path in C_2 that does not contain x and the (x, u_2)-path of C_2 that does not contain z_2, along with $\mathcal{C}_1, \mathcal{C}_2$ form O_1^1 as a minor of G, a contradiction (see the rightmost figure of Figure 26). □

Lemma 4.4. *Let $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ be a graph that contains a cut-vertex x. Then $\mathcal{C}(G, x) = \{B, K_3\}$ for some biconnected graph B.*

Proof. Recall that by Lemma 2.9, G is connected. By Observation 4.2 there exists some $B \in \mathcal{C}(G, x)$ such that $\mathbf{obs}(\mathcal{P}) \leq B \setminus x$. By Lemma 4.3 x is the only cut-vertex of G and therefore B is biconnected.

Let $D = \bigcup\{H \in \mathcal{C}(G, x) : H \neq B\}$, that is $G \setminus (B \setminus x)$. We will prove that $D \cong K_3$. Suppose, towards a contradiction, that D contains more than one cycle. Then, since D is connected, we have that $\mathbf{obs}(\mathcal{P}) \leq D$ and so $D \cap (B \setminus x) = \emptyset$ implies that $\mathcal{O}^0 \leq G$, a contradiction. Therefore, D contains at most one cycle. But since x is a cut-vertex we have that $D \neq \emptyset$ and hence, Lemma 3.1 implies that $D \cong K_3$ which concludes the proof of the Lemma. □

Lemma 4.5. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then $K_4 \not\leq G$ or G is biconnected.*

Proof. Keep in mind that, by Lemma 2.9, G is connected. Suppose, to the contrary, that $K_4 \leq G$ and G is not biconnected. By Lemma 4.4 we have that $\mathcal{C}(G, x) = \{B, K_3\}$ where B is a biconnected graph. Observe that $K_4 \leq B$ and let (H, K) be an r-wheel-subdivision pair of B.

We argue that the following holds:

Claim 1: x is a branch vertex of K.
Proof of Claim 1: If $x \in V(K)$ but x is not a branch vertex of K then $O_4^1 \leq G$, a contradiction.

Suppose then that $x \notin V(K)$. Notice that, since $x \in V(B)$ and B is a biconnected, then there exist two paths P_1, P_2 from x to some vertex of K, respectively, such that $V(P_1) \cap V(P_2) = \{x\}$.

Let u_1, u_2 be the first time P_1 and P_2 meet K, respectively. Let P_1' be the (x, u_1)-subpath of P_1 and P_2' be the (x, u_2)-subpath of P_2. Then, the (u_1, u_2)-path $P_1' \cup P_2'$ intersects K only in its endpoints and by Lemma 3.4 there exists an edge $e \in E(H)$ such that u_1, u_2 are both vertices of the subdivision of e in K (see Figure 27).

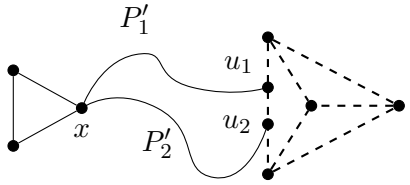
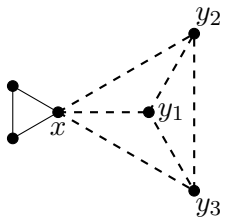


Figure 27: The paths P_1', P_2' from x to K in the proof of Claim 1.

Let P be the path corresponding to the subdivision of e in K. Let, also, z_1 be the branch vertex of K incident to e that is closest to u_1 in P and z_2 be the other branch vertex of K incident to e. Then, by contracting all the edges in the (u_1, z_1)-,(u_2, z_2)-subpaths of P we form O_4^1 as a minor of G, a contradiction. Claim 1 follows.

We now prove that H is isomorphic to K_4. Suppose to the contrary, that H is isomorphic to an r-wheel, where $r \geq 4$. By Claim 1, x is a branch vertex of K. Then, x is the center of K, otherwise $O_4^1 \leq G$, which is a contradiction. Since B is a block of G, by Lemma 3.7, we have that x is a $\mathcal{P}$-apex vertex of B and so the fact that $\mathcal{C}(G, x) = \{B, K_3\}$ implies that x is also a $\mathcal{P}$-apex vertex of G, a contradiction. Hence, $r = 3$, i.e. H is isomorphic to K_4.

According to Claim 1, we have that x is a branch vertex of K. Let $y_i, i \in [3]$, be the three other branch vertices of K, as in the following figure:



Keep in mind that, by Lemma 3.5, for every (s, t)-flap F of (H, K) it holds that F is biconnected, $G[V(F)]$ contains a cycle, and there exists an edge $e \in E(H)$ such that s, t belong both to the subdivision of e in K.

Claim 2: Every flap of (H, K) is x-oriented.

Proof of Claim 2: We first prove that every flap of (H, K) is an (x, t)-flap. Observe that if there exists an (s, t)-flap where s and t belong to the subdivision of a $y_i y_j$ edge of H, then $O_5^1 \leq G$ (see left figure of Figure 28), while if there exists an (s, t)-flap where s and t belong to the subdivision of an xy_i edge of H and $s, t \neq x$, then $O_1^1 \leq G$ (see right figure of Figure 28), a contradiction in both cases.

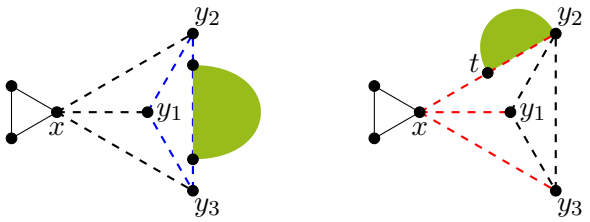


Figure 28: Left figure: An (s,t)-flap in the subdivision of some y_iy_j edge of H (depicted in blue) in the proof of Claim 2. Right figure: An (s,t)-flap in the subdivision of some xy_i edge of H (depicted in red) in the proof of Claim 2.

Therefore, every flap of (H,K) is an (x,t)-flap. Observe now that every (x,t)-flap F of (H,K) is x-oriented. Indeed, if otherwise then Lemma 3.6 implies that there exists a cycle in F containing t but not z and hence $O_1^1 \leq G$, a contradiction. Claim 2 follows.

Therefore, since $\mathcal{C}(G,x) = \{B, K_3\}$ and by Claim 2 every flap of (H,K) is x-oriented, then x is a $\mathcal{P}$-apex vertex of G and so $G \in \mathcal{P}^{(1)}$, a contradiction. □

Lemma 4.6. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then $K_{2,3} \not\leq G$ or G is biconnected.*

Proof. Suppose, to the contrary, that G has a, unique due to Lemma 4.3, cut-vertex u and contains $K_{2,3}$ as a minor. Since G contains a cut-vertex, then by Lemma 4.5 it is K_4-free. Notice that, by Lemma 4.4, the augmented connected components of $\mathcal{C}(G,u)$ are some $T \cong K_3$ and a biconnected graph B. By Lemma 3.10, we have that there exists a unique b-rich separator $S = \{x,y\}$. We distinguish two cases, based on whether u belongs to S or not:

Case 1: The cut-vertex u is neither of x, y.

Let $H \in \mathcal{C}(G,S)$ such that $u \in V(H)$. Let $\hat{H} = H \setminus (T \setminus u)$. Observe that $B = (G \setminus H) \cup \hat{H}$ and keep in mind that $\hat{H} \in \mathcal{C}(B,S)$. Since B is a block, every $H \in \mathcal{C}(B,S)$ is biconnected and outerplanar. For every $H \in \mathcal{C}(B,S)$, let C_H be the Hamiltonian cycle of H. Recall that $xy \in E(C_H)$. Also, keep in mind that since B is biconnected, for every $H' \in \mathcal{C}(G,S) \setminus \{H\}$, $G[V(H')]$ contains a cycle.

Observation 1: B does not contain (x,y)-disjoint chords. Indeed, if H contains a (x,y)-disjoint chord, then $\{O_1^0, O_3^0\} \leq G$, while if some $H' \in \mathcal{C}(B,S) \setminus \{H\}$ contains a (x,y)-disjoint chord, then $O_1^2 \leq G$ (see Figure 29), a contradiction.

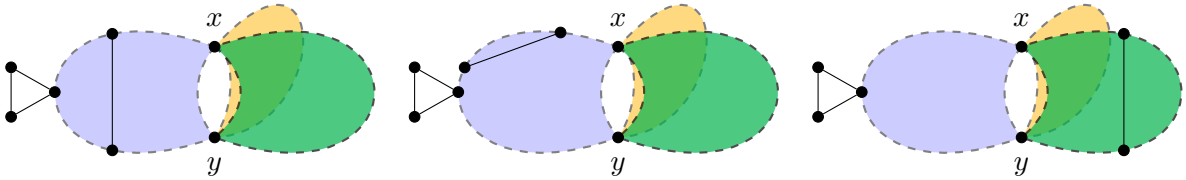


Figure 29: The (x,y)-disjoint chord in the proof of Observation 1

Claim 1: $B \setminus (\hat{H} \setminus \{x,y\})$ does not contain both x-chords and y-chords.

Proof of Claim 1: Suppose to the contrary that there exist an x-chord and a y-chord in $B \setminus (\hat{H} \setminus \{x,y\})$, namely e_x and e_y, respectively. If there exists some $H' \in \mathcal{C}(B,S) \setminus \{\hat{H}\}$ such that $e_x, e_y \in E(H')$, then $O_7^2 \leq G$, while if they are in different augmented connected components in $\mathcal{C}(G,S) \setminus \{\hat{H}\}$ then $O_9^1 \leq G$, a contradiction in both cases (see Figure 30). Claim 1 follows.

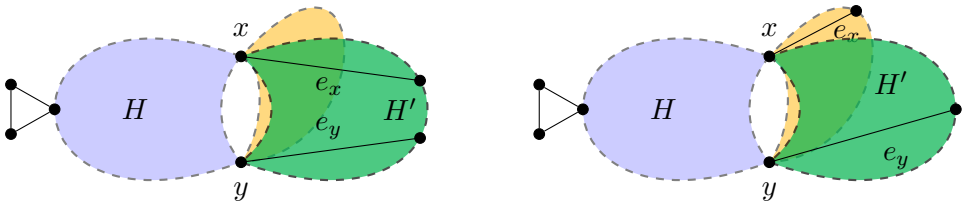


Figure 30: The chords e_x, e_y in the proof of Claim 1

According to Claim 1, suppose that $B \setminus (\hat{H} \setminus \{x, y\})$ contains x-chords but not y-chords.

We also observe the following:

Observation 2: Every chord of $C_{\hat{H}}$ is a y-chord. Indeed, if there exist an x-chord and a y-chord in $\hat{H}$, then $O_6^1 \leq G$, a contradiction (see Figure 31). Also, if all chords of $C_{\hat{H}}$ are x-chords, then x is a $\mathcal{P}$-apex of G, a contradiction.

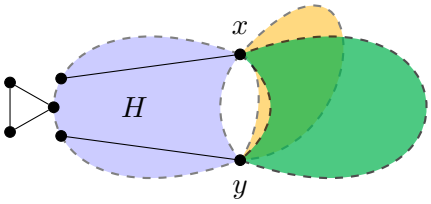


Figure 31: An x-chord and a y-chord of C in Observation 2 of Case 1.

Claim 2: Either every $C_{H'},\ H' \in \mathcal{C}(B, S) \setminus \{\hat{H}\}$ is chordless or $C_{\hat{H}}$ is chordless.

Proof of Claim 2: Suppose to the contrary that there exist a chord e of some $C_{H'},\ H' \in \mathcal{C}(B, S) \setminus \{\hat{H}\}$ and a chord e' of $C_{\hat{H}}$. Claim 1 implies that e is an x-chord, while Observation 1 implies that e' is a y-chord. Thus, $O_{12}^1 \leq G$, a contradiction. Claim 2 follows.

According to Claim 2, either every $C_{H'},\ H' \in \mathcal{C}(B, S) \setminus \{\hat{H}\}$ is chordless or $C_{\hat{H}}$ is chordless which implies that either y or x, respectively, is a $\mathcal{P}$-apex of G, a contradiction.

Case 2: The cut-vertex u is either x or y, say y.

We first prove the following:

Claim 3: There exists a unique augmented connected component in $\mathcal{C}(B, S)$ that is not isomorphic to a cycle.

Proof of Claim 3: First, notice that if each augmented connected component in $\mathcal{C}(B, S)$ is isomorphic to a cycle, then $G \in \mathcal{P}^{(1)}$, a contradiction. Therefore there exists an augmented connected component in $\mathcal{C}(B, S)$ that is not isomorphic to a cycle. Suppose towards a contradiction that $\mathcal{C}(B, S)$ contains two augmented connected components not isomorphic to a cycle. We distinguish the following subcases:

Subcase 2.1: B contains an (x, y)-disjoint chord, namely e.

Then Lemma 3.11 implies that e is the unique (x, y)-disjoint chord of B and B does not contain both x-chords and y-chords. Let $H \in \mathcal{C}(B, S)$ such that $e \in E(H)$.

Recall that there exists some $H' \in \mathcal{C}(B, S) \setminus \{H\}$ that is not isomorphic to a cycle. Therefore, there exists some chord e' of $C_{H'}$. If e' is an x-chord, then $O_9^1 \leq G$ (see Figure 32), while if e' is a

y-chord, there does not exist an x-chord and therefore y is a $\mathcal{P}$-apex of G, a contradiction in both cases.

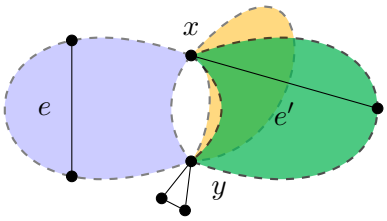


Figure 32: The chords e, e' in the first part the proof of Subcase 2.1.

Subcase 2.2: B does not contain an (x, y)-disjoint chord.

Then Lemma 3.11 implies that H contains at most one x-chord or at most one y-chord. If there exists at most one x-chord, then y is a $\mathcal{P}$-apex of G, a contradiction. Therefore there exists at most one y-chord.

Suppose that there exists a y-chord, namely e_y, and let $H \in \mathcal{C}(B, S)$ such that $e_y \in E(H)$. Recall that there exists some $H' \in \mathcal{C}(B, S) \setminus \{H\}$ that is not isomorphic to a cycle. Therefore, there exists some chord of $C_{H'}$, namely e'. Observe that e' is an x-chord and that there exists some x-chord e'', $e \neq e'$, otherwise y would be a $\mathcal{P}$-apex of G. If $e'' \notin E(H)$, then $\{O_2^0, O_3^0\} \leq G$, while if $e'' \in E(H)$, then $O_{12}^1 \leq G$ (see Figure 33), a contradiction in both cases. Claim 3 follows.

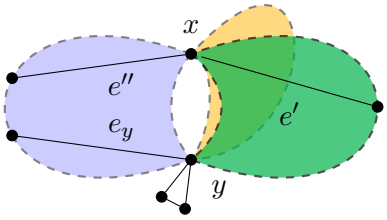
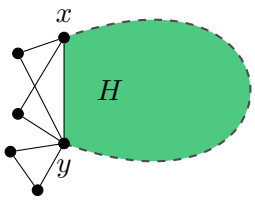


Figure 33: The chords e_y, e', e'' in the last part of the proof of Subcase 2.2.

We now proceed with the proof of Case 2 of the Lemma. According to Claim 3, let H be the unique augmented connected component of B that is not isomorphic to a cycle. Therefore, due to Lemma 3.1, every $H' \in \mathcal{C}(B, S) \setminus \{H\}$ is isomorphic to K_3.

We have that H is outerplanar and due to Lemma 4.4, H is also biconnected. Let C be the Hamiltonian cycle of H. Keep in mind that $xy \in E(C)$. Then the graph G is as in the following figure:



Observe that every (x, y)-disjoint chord, x-chord, and y-chord of B is a chord of C.

Claim 4: There do not exist (x, y)-disjoint chords in B.

Proof of Claim 4: Suppose, towards a contradiction, that there exists a (x, y)-disjoint chord in B, namely e. We observe that the following holds:

Observation 3: There exists at most one (x, y)-disjoint chord in B. Indeed, if the contrary holds then $\{O_1^0, O_3^0\} \leq G$, a contradiction. (see Figure 34)

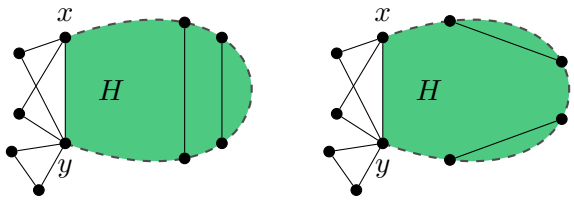


Figure 34: The two (x, y)-disjoint chords in Observation 1 of Case 2.

Observation 3 implies that e is the unique (x, y)-disjoint chord in B. If there exists some x-chord in B then $\{O_7^1, O_{11}^1\} \leq G$, a contradiction (see Figure 35). Therefore every edge $e' \in E(H), e' \neq e$ that is a chord of C_H, is a y-chord and thus y is a $\mathcal{P}$-apex vertex of G, a contradiction. Claim 4 follows.

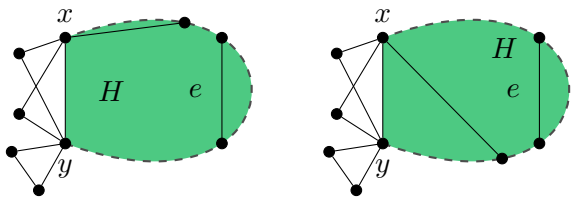


Figure 35: A (x, y)-disjoint chord e and an x-chord in the proof of Claim 4 of Case 2.

We now conclude the proof of the Lemma. Since x is not an $\mathcal{P}$-apex vertex of G, there exists some chord of C not incident to x, namely e. By Claim 4, e is a y-chord.

Observation 4: There exists at most one x-chord. Indeed if there exist two x-chords, we have $O_{11}^1 \leq G$, a contradiction (see Figure 36).

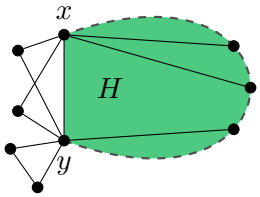
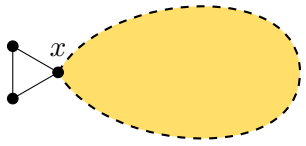


Figure 36: A y-chord and two x-chords in the last part of the proof of Case 2.

Therefore, Observation 4 implies that y is a $\mathcal{P}$-apex vertex of G, a contradiction. This completes the proof of the Lemma. □

Lemma 4.7. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then G is biconnected.*

Proof. Suppose to the contrary that G has a cut-vertex x. Due to Lemma 4.3, Lemma 4.5, and Lemma 4.6 we have that $\mathcal{C}(G, x) = \{T, H\}$ where $T \cong K_3$ and H is outerplanar and biconnected. Let C be the Hamiltonian cycle of H. Therefore, the structure of G is as in the following figure:



Claim: There exists a vertex $u \in V(C)$ such that every chord of C not incident to x is incident to u.

Proof of Claim: To prove the Claim we make a series of observations.

Observation 1: Every two chords not incident to x share a vertex. Indeed, suppose that there exist two disjoint chords $e_1 = u_1v_1, e_2 = u_2v_2$ of C not incident to x. Then $\{O_2^1, O_3^1\} \leq G$ (see Figure 37), a contradiction.

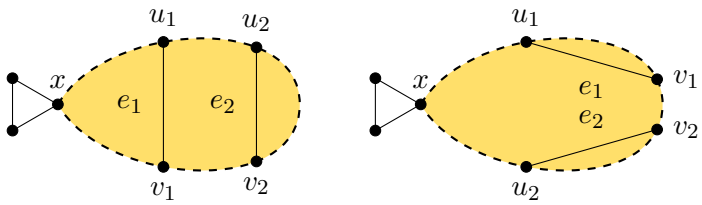


Figure 37: The two chords of C disjoint to x, y in the proof of Observation 1.

Observation 2: Every three chords not incident to x share a vertex. Indeed, suppose to the contrary, that there exist three chords e_1, e_2, e_3 not incident to x that do not share a vertex. Observation 1 implies that every two of e_1, e_2, e_3 share a vertex and hence, $O_3^1 \leq G$, a contradiction (see Figure 38).

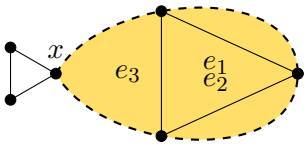


Figure 38: The proof of Observation 2.

Claim follows from the above two observations.

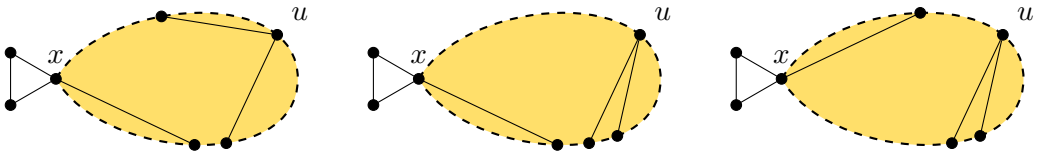


Figure 39: The possible configurations of the chords that conclude the proof of Lemma 4.7.

We now continue with the proof of the Lemma. Since x is not a $\mathcal{P}$-apex vertex of G there exist two chords e_1, e_2 not incident to x. By the above Claim, both e_1, e_2 share a vertex u. Since u is not a $\mathcal{P}$-apex vertex of G there exists a chord e not incident to u which, again by the above Claim, is incident to x. Hence, $\{O_7^1, O_{11}^1, O_3^0\} \leq G$, a contradiction (see Figure 39). □

4.2 Proving triconnectivity

The purpose of this subsection is to prove that all graphs in $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ are triconnected (Lemma 4.10).

Lemma 4.8. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then $K_4 \not\leq G$ or G is triconnected.*

Proof. Suppose, to the contrary, that G is not triconnected and $K_4 \leq G$, which, by Lemma 4.5, implies that G is biconnected. Let (H, K) be an r-wheel-subdivision pair of G. Since G is biconnected, Lemma 3.7 implies that H is isomorphic to K_4.

Keep in mind that, by Lemma 3.5, for every (u, v)-flap F of (H, K) there exists an edge $e \in E(H)$ such that u, v belong both to the subdivision of e in K and $G[V(F)]$ contains a cycle.

Keep in mind that for every (u, v)-flap of (H, K) it holds that one of u, v is a branch vertex of K, otherwise $O_9^2 \leq G$, a contradiction (see Figure 40).

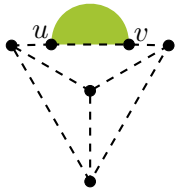


Figure 40: An example of a (u, v)-flap such that both u, v are subdividing vertices of the corresponding path.

We distinguish the following two cases:

Case 1: There exists an (x, u)-flap F, where x is a branch vertex of K and u is a subdividing vertex in the subdivision of an edge $e = xy \in E(H)$.

Let z_1, z_2 be the two other branch vertices of K (as in Figure 41).

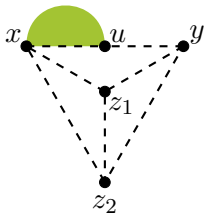


Figure 41: The structure of G in Case 1.

Claim 1: All flaps of (H, K) are (x, w)-flaps where w is a vertex in the subdivision of an edge $e' \in \{xy, xz_1, xz_2\}$ in K.

Proof of Claim 1: Suppose, to the contrary, that Claim 1 does not hold. We distinguish the following subcases:

Subcase 1.1: There exists a (y, v)-flap F' where v is a subdividing vertex in the subdivision of e in K. If $u = v$ then $V(F) \cap V(F') = \{u\}$ and so $O_4^2 \leq G$, while if $u \neq v$ then $V(F) \cap V(F') = \emptyset$ and so $O_1^1 \leq G$, a contradiction in both cases (see Figure 42).

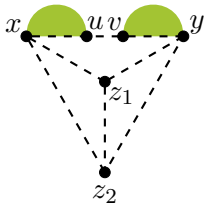


Figure 42: The configuration of the flaps of (H, K) in Subcase 1.1 of Claim 1.

Subcase 1.2: There exists a flap in the subdivision of the edges yz_1 or yz_2. But, then $O_6^2 \leq G$ (see Figure 43), a contradiction.

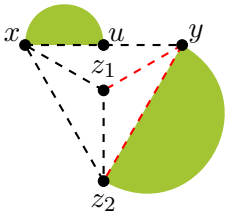


Figure 43: A flap in the subdivision of some edge $yz_i,\ i \in [2]$ (depicted in red) in Subcase 1.2 of Claim 1.

Subcase 1.3: There exists a flap in the subdivision of the edge z_1z_2. But, then $O_5^2 \leq G$ (see Figure 44), a contradiction.

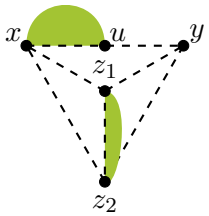


Figure 44: A flap in the subdivision of z_1z_2 in Subcase 1.3 of Claim 1.

Subcase 1.4: There exists a (z_i, v)-flap such that v is a subdividing vertex in the subdivision of the edge z_ix, $i \in [2]$. But, then $O_2^1 \leq G$ (see Figure 45), a contradiction.

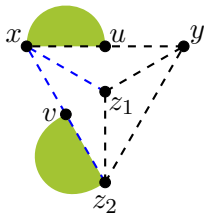


Figure 45: A flap in the subdivision of some edge $z_ix,\ i \in [2]$ (depicted in blue) in Subcase 1.4 of Claim 1.

Since we have exhausted all possible cases, Claim 1 follows.

To conclude Case 1 we prove the following:

Claim 2: Every flap of (H, K) is x-oriented.

Proof of Claim 2: Suppose, towards a contradiction, that there exists a flap F' that is not x-oriented. Then, by Claim 1, this is an (x, w)-flap where w is some vertex in the subdivision of an edge $e' \in \{xy, xz_1, xz_2\}$. Since, F' is not x-oriented then Lemma 3.6 implies exists a cycle C in F' that contains w but not x. If $e' = xz_i$, for some $i \in \{1, 2\}$, then $O_2^1 \leq G$, a contradiction (see the left figure of Figure 46). Thus, $e' = xy$. Then, if $w \neq y$ we have that $O_4^1 \leq G$ (see the central figure of Figure 46). Therefore, $w = y$ and so $F' \neq F$ and $V(F) \cup V(F') = \{x\}$. Therefore, by contracting F to a cycle we get a cycle containing x but not y disjoint to C. Hence $O_1^1 \leq G$, a contradiction (see the right figure of Figure 46). Claim 2 follows.

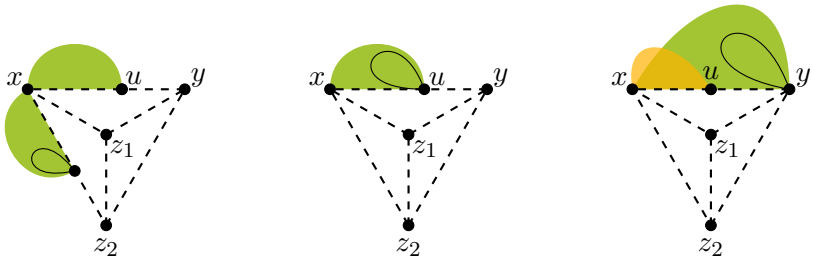


Figure 46: The possible configurations of the non x-oriented flap F' in the proof of Claim 2.

Claim 2 implies that x is a $\mathcal{P}$-apex vertex of G and so we arrive at a contradiction.

Case 2: Every (u, v)-flap is such that u, v are branch vertices of K.

We argue that the following holds:

Claim 3: All flaps of (H, K) share a branch vertex of K.

Proof of Claim 3: To prove Claim 3 we make the following two observations.

Observation 1: Every two flaps of (H, K) share a branch vertex of K. Indeed, suppose, to the contrary, that there exist two flaps, an (x, y)-flap and an (x', y')-flap, such that $xy, x'y'$ are two non-incident edges of H. But then, $O_5^2 \leq G$, a contradiction (see Figure 47).

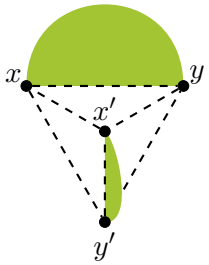


Figure 47: The configuration of the flaps of G in the proof of Observation 1.

Observation 2: Every three flaps of (H, K) share a branch vertex of K. Indeed, suppose to the contrary that there exist three flaps F_1, F_2, F_3 such that $V(F_1) \cap V(F_2) \cap V(F_3) = \emptyset$. Observation 1 implies that $V(F_1) \cap V(F_2) = \{x\}$, $V(F_1) \cap V(F_2) = \{y\}$, and $V(F_1) \cap V(F_2) = \{z\}$, where x, y, z are three branch vertices of K. Hence, $O_8^2 \leq G$, a contradiction (see Figure 48).

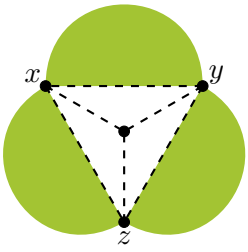


Figure 48: The configuration of the flaps of G in the proof of Observation 2.

Claim 3 is an immediate result of the above two Observations.

According to Claim 3, all flaps of (H, K) contain a branch vertex of K, namely x.

Claim 4: Every flap of (H, K) is x-oriented.

Proof of Claim 4: Recall that, by Claim 3, every flap of (H, K) is an (x, y)-flap, where y is a branch vertex of K different from x. Suppose, towards a contradiction, that there exists an (x, y)-flap F of (H, K) that is not x-oriented. Then, Lemma 3.6 implies that F is y-oriented and that there exists a cycle in F which contains y but not x.

We will prove that F is the only flap of (H, K). Indeed, if there exists another flap, then, by Claim 1, it is an (x, y')-flap, where y' is a branch vertex of K different from both x and y. Thus, $O_5^1 \leq G$, a contradiction (see Figure 49).

Therefore, F is the only flap of (H, K) and since it is y-oriented, y is a $\mathcal{P}$-apex vertex of G, a contradiction. Claim 4 follows.

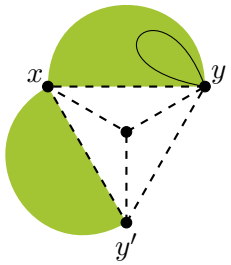


Figure 49: An example of the (x, y)-flap and the (x, y')-flap in the proof of Claim 4.

Claim 4, taking account that G is biconnected, implies that x is a $\mathcal{P}$-apex vertex of G, which is a contradiction. That proves Lemma 4.8. □

Lemma 4.9. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then $K_{2,3} \not\leq G$ or G is triconnected.*

Proof. Suppose, to the contrary, that G is not triconnected and $K_{2,3} \leq G$, which, by Lemma 4.6, implies that G is biconnected. Also, since G is not triconnected, by Lemma 4.8, it is K_4-free and so, by Lemma 3.10, there exists a unique b-rich separator $S = \{x, y\}$ in G. We argue that the following holds:

Claim 1: There exists a unique augmented component in $\mathcal{C}(G, S)$ not isomorphic to a cycle.

Proof of Claim 1: Observe that there exists an augmented connected component in $\mathcal{C}(G, S)$ not isomorphic to a cycle, otherwise $G \in \mathcal{P}^{(1)}$. Suppose towards a contradiction that there exist two augmented connected components in $\mathcal{C}(G, S)$ that are not isomorphic to a cycle.

Lemma 3.11 implies that one of the following holds:

- There exists a unique (x, y)-disjoint chord and there do not exist both x-chords and y-chords in G. But then, if there do not exist x-chords (or y-chords), then y (or x, respectively) is a $\mathcal{P}$-apex of G, a contradiction.
- There do not exist (x, y)-disjoint chords and there exists at most one x-chord or at most one y-chord. But then, if there exists at most one x-chord (or y-chord), then y (or x, respectively) is a $\mathcal{P}$-apex of G, a contradiction.

Notice that each of the above implies Claim 1.

According to Claim 1, let H be the unique augmented connected component in $\mathcal{C}(G, S)$ that is not isomorphic to K_3 and C be the Hamiltonian cycle of H (which exists due to biconnectivity and outerplanarity of H). Keep in mind that $xy \in E(C)$.

Claim 2: Every chord of C is either an x-chord or a y-chord.

Proof of Claim 2: Suppose, towards a contradiction, that there exists some (x, y)-disjoint chord uv. Observe that there is a unique such chord, since otherwise $\{O_1^0, O_3^0\} \leq G$.

Suppose, without loss of generality, that in $C \setminus xy$ there exists an (x, u)-path which does not contain v – we denote this path by P_1. Let P_2 be the (y, v)-path in $C \setminus xy$ (as shown in Figure 50).

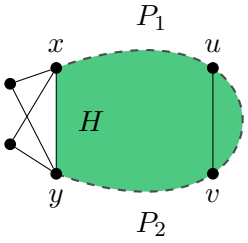


Figure 50: The chord uv and paths P_1, P_2 as in the proof of Claim 1.

We argue that the following holds:

Subclaim: All chords of C, other than uv, are incident to the same vertex of the separator.

Proof of Subclaim: Suppose, towards a contradiction, that there exists an x-chord xx' and a y-chord yy'. Then, it cannot be the case that $x' \in V(P_1)$ and $y' \in V(P_2)$, otherwise $O_3^2 \leq G$. On the other hand, it also cannot be the case that both x' and y' belong to the same $P_i, i \in [2]$ since that implies $O_{10}^2 \leq G$ (see Figure 51). Subclaim follows.

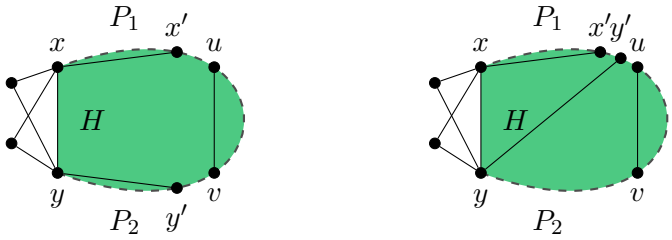


Figure 51: Possible configurations of chords xx', yy', as in the proof of Subclaim.

According to the Subclaim, all chords of C, other than uv, are incident to the same vertex of the separator, say x - but then x is a $\mathcal{P}$-apex vertex of G, a contradiction. Therefore, an (x, y)-disjoint chord cannot exist and this concludes the proof of Claim 2.

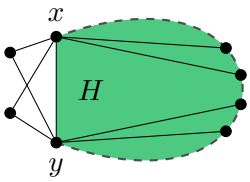


Figure 52: An example of H having at least two x-chords and at least two y-chords.

Now, since x is not an $\mathcal{P}$-apex there exist two chords of C not incident to x. By Claim 2, these are y-chords. Symmetrically, there exist two x-chords. Therefore, we have that $O_{13}^2 \leq G$, a contradiction (as shown in Figure 52). □

Lemma 4.10. *If $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ then G is triconnected.*

Proof. Suppose, to the contrary, that G is not triconnected. Then, by Lemma 4.7, Lemma 4.8, and Lemma 4.9, G is biconnected and outerplanar and so it contains a Hamiltonian cycle, namely C.

Observe that G has at most 3 vertices of degree 2. Indeed, if there exist 4 vertices $v_1, v_2, v_3, v_4 \in V(G)$ of degree 2, then, by Lemma 3.1, they are simplicial and by contracting all edges of C, except those that are incident to v_1, v_2, v_3, v_4, we can form O_3^2 as a minor of G (as depicted in Figure 53), a contradiction.

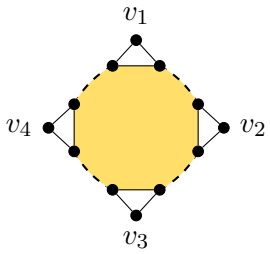


Figure 53: An outerplanar graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$ having 4 vertices of degree 2.

On the other hand, by Proposition 2.2, G has at least two vertices of degree 2. Thus, we distinguish the following two cases:

Case 1: G has exactly 2 vertices u, v of degree 2, which, by Lemma 3.1, are simplicial.

Observe that $C \setminus u \setminus v$ is the union of two vertex disjoint paths P_1, P_2. Let u_1, u_2 be the neighbors of u in P_1, P_2, respectively and v_1, v_2 be the neighbors of v in P_1, P_2, respectively. Therefore, the structure of the graph G is as follows:

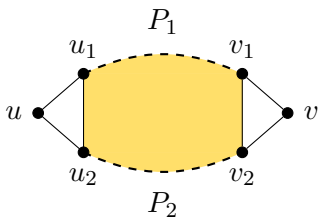


Figure 54: The structure of the graph G in Case 1 of Lemma 4.10.

We prove the following claim concerning the chords of C.

Claim 1: Every chord of C is between a vertex of P_1 and a vertex of P_2.

Proof of Claim 1: Suppose to the contrary that there exists an edge connecting non-consecutive vertices of P_1 or P_2, say P_1, and let $e = xy$ be such an edge whose endpoints have the smallest possible distance in P_1.

Let P_1' be the subpath of P_1 between x and y. Since e is a chord of C, P_1' contains an internal vertex w. Note then that, since u, v are the only vertices of G of degree 2, w is of degree greater than 2 and so there exists a neighbor of w, denoted by z, such that z, w are not adjacent in P_1. Observe that z is a vertex of P_1', since otherwise $K_4 \leq G$. But then wz is an edge connecting non-consecutive vertices of P_1 whose endpoints have smaller distance (in P_1) than that of x, y, a contradiction to the minimality of P_1' (see Figure 55). This concludes the proof of Claim 1.

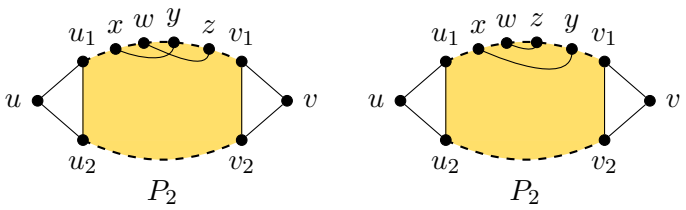


Figure 55: The chord wz in the proof of Claim 1.

We now make a series of observations:

Observation 1: Every internal vertex of P_1 and P_2 is incident to a chord. Indeed, it is obvious from the fact that u, v are the only vertices of degree 2.

Observation 2: Every internal vertex of P_1 is adjacent to u_2 or v_2. Respectively, every internal vertex of P_2 is adjacent to u_1 or v_1. Indeed, if there exists an internal vertex x of P_1 or P_2, say P_1, not incident to u_2 or v_2 then by Observation 1 and Claim 1 x is adjacent to an internal vertex of P_2 and hence $O_{15}^2 \leq G$, a contradiction (as shown in Figure 56).

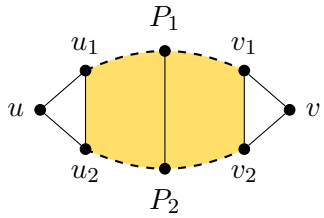


Figure 56: A chord connecting two internal vertices of P_1, P_2 in the proof of Observation 2 of Case 1.

Observation 3: One of P_1, P_2 must be of length at most 1. Indeed, suppose to the contrary, that both P_1, P_2 is of length at least 2. Then, both of P_1, P_2 contain an internal vertex, namely x_1, x_2, respectively. Then, by Observation 1, they are both incident to some chord of C. By Observation 2, x_1 is adjacent to u_2 or v_2, say u_2. Then, again by Observation 2 and K_4-freeness of G, x_2 is adjacent to v_1, as shown in Figure 57. But then, $O_1^0 \leq G$, a contradiction.

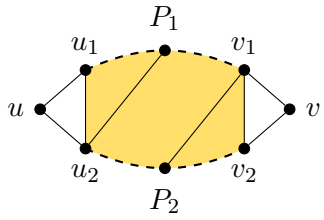


Figure 57: An example of the graph G in Observation 3 of Case 1.

Observation 4: Either u_1 or v_1 is adjacent to every internal vertex of P_2. Respectively, either u_2 or v_2 is adjacent to every internal vertex of P_1. Indeed, if otherwise, then Observations 1 and 2 imply, without loss of generality, that there exist $x, y \in V(P_1) \setminus \{u_1, v_1\}$ such that x, y are adjacent to u_2, v_2, respectively. Hence, $O^2_{13} \leq G$ (see Figure 58).

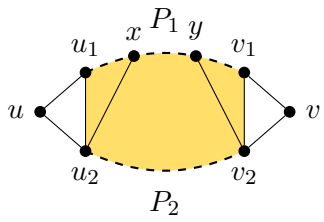


Figure 58: Two chords xu_2, yv_2 where x, y are internal vertices of P_1 in Observation 4 of Case 1.

By Observation 3 we can assume that P_2 is of length $j \leq 1$. Then, by Observation 4, u_2 or v_2, say u_2, is adjacent to every internal vertex of P_1. This implies that every chord of C, except for v_1v_2 (if $v_2 \neq u_2$), is incident to u_2 and hence u_2 is a $\mathcal{P}$-apex vertex of G, a contradiction.

Case 2: G has exactly 3 vertices u, v, w of degree 2, which, by Lemma 3.1, are simplicial.

Note that if all three vertices have pairwise disjoint closed neighborhoods, then $O^2_{14} \leq G$, which is a contradiction.

Therefore, at least two of them, say u and v, have non-disjoint neighborhoods. We argue that the following holds:

Claim 2: $N_G(u) \cap N_G(v) = \{x\}$ for some $x \in V(G)$.

Proof of Claim 2: Since u, v have non-disjoint neighborhoods, then either $N_G(u) \cap N_G(v) = e$ for some edge $e \in E(G)$ or $N_G(u) \cap N_G(v) = \{x\}$ for some vertex $x \in V(G)$.

Suppose that $N_G(u) \cap N_G(v) = \{a, b\}$ for some edge $e = ab \in E(G)$ and consider any two internally vertex disjoint paths (which exist due to biconnectivity of G) from w to, say, v. Observe that one of the paths contains a and the other contains b. Therefore $K_{2,3} \leq G$, with $\{a, b\}$ forming one part of the $K_{2,3}$ minor and $\{u, v, w\}$ forming the other, a contradiction. Claim 2 follows.

By Claim 2, $C \setminus \{u, v, w, x\}$ is the union of two vertex disjoint paths R_1, R_2. We can assume that u has a neighbor u_1 in R_1, v has a neighbor v_2 in R_2, and $N_G(w) = \{w_1, w_2\}$, where $w_1 \in R_1$ and $w_2 \in R_2$. By arguments identical to the proof of Claim 1 in Case 1, we have that every chord of C is between a vertex in $R_1 \cup \{x\}$ and a vertex in $R_2 \cup \{x\}$. Therefore, the structure of the graph G is as follows (Figure 59):

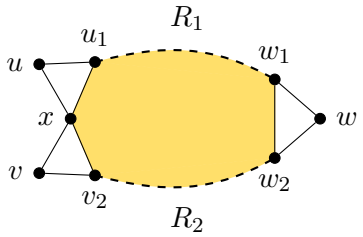


Figure 59: The structure of the graph G in Case 2 of Lemma 4.10.

We now observe the following about the paths R_1, R_2:

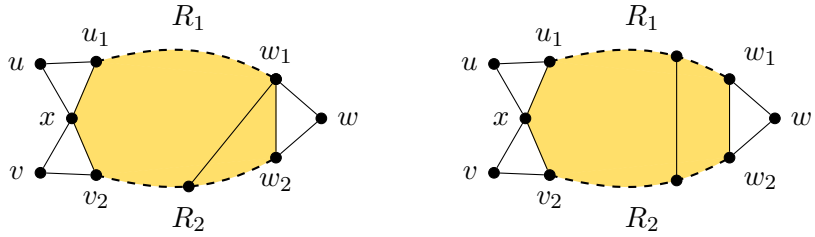


Figure 60: The two configurations of the chord not incident to x in the proof of Observation 5 of Case 2.

Observation 5: One of R_1, R_2 is of length 0. Indeed, suppose, towards a contradiction, that both of R_1, R_2 is of length at least 1. Since x is not a $\mathcal{P}$-apex vertex of G then there exists a chord e between a vertex in R_1 and a vertex in R_2 such that $e \neq w_1w_2$. Then, if e is incident to w_1w_2 we have that $O^2_{12} \leq G$, while if e is disjoint from w_1w_2 we have that $O^2_7 \leq G$, a contradiction in both cases (see Figure 60).

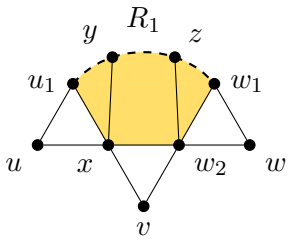


Figure 61: An example of the graph G in the last part of the proof of Lemma 4.10.

By Observation 5, we can assume that R_2 is of length 0, i.e. $v_2 = w_2$. Then, every chord of C, except for xw_2, is between a vertex of R_1 and a vertex in $\{x, w_2\}$. Then, since x and w_2 are not $\mathcal{P}$-apex vertices of G there exist y, z (possibly with $y = z$) internal vertices of R_1 incident to x and w_2, respectively. Hence, $O^2_{13} \leq G$ (see Figure 61), a contradiction. The proof of Lemma 4.10 is complete. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. As we mentioned in the end of Section 1, $\mathbf{obs}(\mathcal{P}^{(1)}) \supseteq \mathcal{O}$ and what remains is to prove that $\mathcal{O} \supseteq \mathbf{obs}(\mathcal{P}^{(1)})$ or alternative that $\mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O} = \emptyset$. For this assume, towards a contradiction, that there exists a graph $G \in \mathbf{obs}(\mathcal{P}^{(1)}) \setminus \mathcal{O}$. From Lemma 4.10, G should be triconnected. Therefore, from Lemma 2.7, either $\mathcal{O}^3 \leq G$, a contradiction, or G is isomorphic to W_r, for some $r \geq 3$, again a contradiction, as $W_r \in \mathcal{P}^{(1)}$ for all $r \geq 3$. □

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